COMPACTNESS OF CONFORMALLY COMPACT EINSTEIN MANIFOLDS IN DIMENSION 4

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ABSTRACT. In this paper, we establish some compactness results of conformally compact Einstein metrics on 4-dimensional manifolds. Our results were proved under assumptions on the behavior of some local and non-local conformal invariants, on the compactness of the boundary metrics at the conformal infinity, and on the topology of the manifolds.

1. INTRODUCTION

In this paper we study the compactness of a set of conformally compact Einstein metrics on some manifold X of dimension four with three dimensional boundary ∂X. We introduce a class of conformally invariant quantities on X and on its boundary. We aim to establish a compactness result that under suitable conditions on the size of these invariants, the compactness of a class of metrics on the boundary would imply the compactness of the corresponding conformal structures in the interior. To be more precise, we consider on $X = X^4$ a set of conformally compact Einstein metrics $g = \rho^2 g^+$ where g^+ is an asymptotically hyperbolic Einstein metric on X and ρ is a smooth defining function of the boundary such that g extends to a smooth metric on the closure of X. To state our results, we first introduce a class of 2-tensor S on the boundary which is pointwisely conformally invariant. The definition of S is motivated by the Gauss-Bonnet formula on 4-manifolds with boundary $(X, \partial X, g)$, with g defined on X and extended smoothly to ∂X. On such a manifold $(X, \partial X, g)$, we consider the functional on $(X, \partial X, g)$

$$g \to \int_X |W|_g^2 dv_g + 8 \oint_{\partial X} W_{\alpha n \beta n} L^{\alpha \beta} d\sigma_g,$$

where L is the second fundamental form and W the Weyl tensor, n is the outwards unit normal vector on the boundary, the greek indices $\alpha, \beta, \cdots$ represent the tangential indices letter and $i, j, k \cdots$ are the full indices. Both terms of the functional are conformally invariant, i.e., under conformal change of metric $\hat{g} = e^{2w} g$ for a smooth function w on the closure of X, the value of the above functional stays the same. Critical metrics of the functional under variation of the metric g satisfy in the interior the well-known condition $B_{ij} = 0$ where B_{ij} is the Bach tensor, and on the boundary (more details in Section 2 below)

$$S_{\alpha\beta} := \nabla^i W_{i\alpha n\beta} + \nabla^i W_{i\beta n\alpha} - \nabla^n W_{n\alpha n\beta} + \frac{4}{3} H W_{n\alpha n\beta} = 0,$$

where H denotes the mean curvature. When the boundary is totally geodesic (i.e. $L = 0$), as in the case of a conformally compact Einstein metric, it turns out

$$S_{\alpha\beta} = \frac{1}{2} \nabla^n R_{\alpha\beta} - \frac{1}{12} \nabla^n R g_{\alpha\beta},$$

where $R_{\alpha\beta}$ (or Ric) is the Ricci tensor and R is the scalar curvature. S is a non-local tensor for conformally compact Einstein manifolds and is a conformal invariant in the sense of Lemma 2.1

below, that is $S(e^{2w}g) = e^{-w}S(g)$.

For any four-dimensional Riemannian manifold (X^4, g) with or without boundary, the Q-curvature Q_4 is defined as:

$$(1.1) \qquad Q_4 := -\frac{1}{6}\triangle R - \frac{1}{2}|Ric|^2 + \frac{1}{6}R^2.$$

In the study of conformal geometry, Q is naturally related to a 4th-order differential operator, called the Paneitz ([42]) operator (which is a special case of some general class of GJMS[31] operators) and is a 4-th order generalization of the conformal Laplacian operator defined as:

$$(1.2) \qquad P := (\triangle)^2 - \text{div}\,[(\frac{2}{3}Rg - 2Ric)\nabla]$$

Throughout this paper, we denote by $[g] = \{e^{2w}g|\ w : X \to \mathbb{R}$ is a regular function$\}$ the class of metrics conformal to g. Under the conformal change $g_w := e^{2w}g$, the associate Q-curvature for g_w metric, denote by $Q_4(g_w)$, is related to $Q_4(g)$ by the PDE:

$$(1.3) \qquad P(g)w + Q_4(g) = Q_4(g_w)e^{4w}.$$

The 4th order operator Paneitz operator P and its corresponding Q curvature have been extensively studied in the recent literature, here we will just cite a few of them ([11], [17], [32], [25], [24], [34]).

There are two non-local curvature tensors of order three defined on the boundary ∂X; one is the T curvature defined on the boundary of any compact four manifolds (see [14], [15]), the other is the conformally invariant Q_3 curvature defined on the boundary of confomally compact asymptotically hyperbolic manifolds in ([25]). Without going into details of their respective definitions, here we will just cite that the result that in the special case when the boundary is totally geodesic, it turns out the curvatures T and Q_3 agree (see [15], Lemma 2.2) and in this case

$$T := \frac{1}{12}\frac{\partial R}{\partial n}$$

On a four manifold $(X, \partial X, g)$ with boundary, the Q-curvature on X and the T-curvature on the boundary ∂X are related by the Chern-Gauss-Bonnet formula [14]

$$\chi(X) = \frac{1}{32\pi^2}\int_X (|W|^2 + 4Q)dvol + \frac{1}{4\pi^2}\oint_{\partial X}(\mathscr{L} + T)d\sigma$$

where $\chi(X)$ is the Euler characteristic number of X and $\mathscr{L}d\sigma$ is a pointwise conformal invariant on ∂X.

As a consequence, $\int_X Q + 2\oint_{\partial X} T$ is independent of the choice of metrics in the conformal class $[g]$ since $|W|^2 dvol$ is also a pointwise conformal invariant term on X.

Let us now recall briefly the Yamabe invariants on compact 4-manifolds (X, g) with boundary ∂X. We consider the Yamabe energy functional

$$Y(g) := \frac{1}{6}\int_X R_g + \int_{\partial X} H_g,$$

where R_g is the scalar curvature of the metric g and H_g is the mean curvature on the boundary ∂X. Note that when (X, g) is with totally geodesic boundary ∂X (or more generally, the mean curvature vanishes on the boundary) for a conformal metric $\widetilde{g} = U^2 g \in [g]$, we can rewrite

$$Y(\widetilde{g}) = \int_X |\nabla U|^2 + \frac{1}{6}R_g U^2.$$

We now denote the first Yamabe constant as

$$Y(X, M, [g]) := \inf_{\widetilde{g} \in [g]} \frac{Y(\widetilde{g})}{vol(X, \widetilde{g})^{1/2}} = \inf_{U \in C^1 \setminus \{0\}} \frac{\int_X |\nabla U|^2 + \frac{1}{6} R_g U^2}{\left(\int_X U^4 \right)^{\frac{1}{2}}},$$

and the second Yamabe constant as

$$Y_b(X, M, [g]) := \inf_{\widetilde{g} \in [g]} \frac{Y(\widetilde{g})}{vol(\partial X, \widetilde{g})^{2/3}} = \inf_{U \in C^1 \setminus \{0\}} \frac{\int_X |\nabla U|^2 + \frac{1}{6} R_g U^2}{\left(\oint_M U^3 \right)^{\frac{2}{3}}}.$$

Here $\oint$ denotes the integral on the boundary, and $vol(X, \widetilde{g})$ (resp. $vol(\partial X, \widetilde{g})$) is the volume of X (resp. ∂X) under the metric $\widetilde{g}$.

On $(X, \partial X, g^+)$ a four-dimensional oriented manifold, we say the manifold is conformally compact if there exists some defining function $\rho > 0$ on X so that $\rho^2 g^+$ is a compact metric defined on $\bar{X} =: X \cup \partial X$. In the case when g^+ is a Poincare Einstein metric which we normalized so that $Ricci_{g^+} = -3g^+$, we say that $(X, \partial X, g^+)$ is a conformally compact Einstein manifold (abbreviated as CCE) and we say ∂X the conformal infinity of X. Note that since the choice of the defining functions are by no means unique but a multiple of each other, their corresponding compactified metrics are conformal to each other and so are their restriction to ∂X. Thus the boundary metric on ∂X is unique up to a conformal class.

Throughout this paper, we will choose a special compactification of g^+. This special compactification was first introduced in the paper by Fefferman-Graham [25] (to study the renormalized volume of CCE manifolds with odd dimensional boundary). Here we will restrict our attention to the special case when ∂X is of dimension 3. To define this special compactification, given any boundary metric $h \in [\rho^2 g^+|_{\partial X}]$, one solves the partial differential equation

$$(1.4) \qquad\qquad -\triangle_{g^+} w = 3$$

We denote the metric $g = e^{2w} g^+$ with $g|_{\partial X} = h$ and we name it as the Fefferman-Graham (abbreviated FG) compactification with the boundary h. Later in this paper, we will further derive other relevant properties (e.g. Lemma 2.4, Lemma 4.2) of this compactification; but here we will point out one key property which leads us to think the metric is the most suitable representative metric among the conformal compactification metrics of g^+. The property, which was pointed out and applied to derive a formula of the renormalized volume in the earlier paper by Chang-Qing-Yang [15]; is that, for this choice of compactification, the Q-curvature $Q_4(g)$ on X vanishes identically. To see this, we notice that for the Einstein metric g^+, by our normalization of it being Poincare Einstein, the Paneitz operator can be written as

$$(1.5) \qquad\qquad P_{g^+} = \triangle_{g^+}^2 + 2\triangle_{g^+},$$

so that applying equation (1.5), we find

$$Q_4(g) = e^{-4w}(P_{g^+} w + 6) = 0.$$

In this paper, we will always choose the Yamabe metric on the boundary as representative in the conformal infinity $[g|_{TM}]$ and take the corresponding FG compactification.

Through the whole paper, we assume X is 4-dimensional oriented CCE, and the boundary $M = \partial X = \mathbb{S}^3$ is 3-sphere and the boundary Yamabe metric $\hat{g}$ in the conformal infinity is non-negative type, that is, the scalar curvature of $\hat{g}$ is an non-negative constant; and we denote denote the corresponding FG compactification.

Our main compactness results are as follows.

Theorem 1.1. *Let $\{X, \partial X = \mathbb{S}^3, g_i^+\}$ be a family of 4-dimensional oriented CCE on X with boundary ∂X. We assume the boundary Yamabe metric $\hat{g}_i$ in conformal infinity is of non-negative type and denote g_i be the corresponding FG compactification. Assume*

   (1) The boundary Yamabe metrics $\hat{g}_i$ form a compact family in C^{k+3} norm with $k \geq 2$;

   (2) There is no concentration of S-tensor in L^1 norm for the g_i metric on ∂X in the following sense,

$$\lim_{r \to 0} \sup_i \sup_x \fint_{B(x,r)} |S_i| = 0$$

   (3) $H_1(X, \mathbb{Z}) = H_2(X, \mathbb{Z}) = 0$.

   (4) there exists some positive constant $C_1 > 0$ such that the first Yamabe constant for the compactified metric g_i is bounded uniformly from below by C_1

$$Y(X, M, [g_i]) \geq C_1;$$

   (5) there exists some positive constant $C_2 > 0$ such that the second Yamabe constant for the metric g_i is bounded uniformly from below by C_2

$$Y_b(X, M, [g_i]) \geq C_2;$$

Then, the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0, 1)$ up to a diffeomorphism fixing the boundary.

As consequences of the main theorem, we can establish the following corollaries.

Corollary 1.2. *Under the assumptions (1) and (3)-(5) as in Theorem 1.1, suppose $\{S_i\}$ is a relatively weakly compact family in L^1, that is, the closure of $\{S_i\}$ is compact in the weak topology generated by all linear continuous maps on L^1. Then the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0, 1)$, up to a diffeomorphism fixing the boundary, provided $k \geq 2$.*

Corollary 1.3. *Under the assumptions (1) and (3)-(5) as in Theorem 1.1, suppose there is some constant $C_4 > 0$ such that for some $+\infty \geq q > 1$ one has*

$$\fint |S_i|^q \leq C_4.$$

Then the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0, 1)$, up to a diffeomorphism fixing the boundary, provided $k \geq 2$.

Corollary 1.4. *Under the assumptions (1) and (3)-(5) as in Theorem 1.1,*

   (2')   there is some small constant $\varepsilon_1 > 0$ (possibly depending on C_1, C_2 and C^{k+3} norm bound of the boundary metric and also on the topology of X) such that if moreover, for all i one has

$$\fint |S_i| \leq \varepsilon_1,$$

then the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geq 2$.

Remark 1.5. *In the statement of Theorem 1.1, the conditions (4) and (5) are conformally invariant conditions but condition (2) is not. A more natural conformally invariant condition would be the uniform boundedness of the L^1 norm of the S tensor for the family of metrics, but the authors are so far not able to establish Theorem 1.1 under this more natural assumption. Instead we can establish the compactness result under the stronger assumption (2), which implies the uniform bound of L^1 norm of the S tensor; or we can establish the compactness result under the conformally invariant condition (2′) . We remak that, by Dunford-Pettis Theorem, the condition (2) in Theorem 1.1 is equivalent to the compactness of S-tensor under the weak topology in L^1.*

Remark 1.6. *For the unit ball $X = B^4$ (more generally, when X is a homology sphere removed a 4-ball), the topological conditions (3) in Theorem 1.1 are satisfied.*

Another version of our main theorem is to replace condition on S tensor by the curvature tensor T.

Theorem 1.7. *Under the assumptions (1) and (3)-(5) as in Theorem 1.1, suppose the T curvature on the boundary $T_i = \frac{1}{12} \frac{\partial R_i}{\partial n}$ satisfies the following condition*

$$\liminf_{r \to 0} \inf_i \inf_{x \in M} \fint_{B(x,r)} T_i \geq 0$$

Then the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geq 5$.

Remark 1.8. *In Theorem 1.1 and Theorem 1.7, if the boundary Yamabe metric $\hat{g}$ in conformal infinity is of positive type, we can drop the condition $H_1(X, \mathbb{Z}) = 0$ in the condition (3). To see this, by a result due to Wittten and Yau [48], we know, under the assumptions that the conformal infinity is of positive type and that the conformal infinity is simply connected, then $H_1(X, \mathbb{Z}) = 0$.*

We remark we have assumed the stronger regularity on Theorem 1.7 for $k \geq 5$ than $k \geq 2$ due to a technical reason that in the proof of the theorem we have taken a power series expansion of the metric g_i for up to order 7 (see (4.13) in the proof of Lemma 4.5 below).

Some parallel direct consequences of Theorem 1.7 can be stated as follows.

Corollary 1.9. *Under the assumptions (1) and (3)-(5) as in Theorem 1.1, suppose $\{\max(-T_i, 0)\}$ is a relatively weakly compact family in L^1. Then, the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$ up to a diffeomorphism fixing the boundary, provided $k \geq 5$.*

Corollary 1.10. *Under the assumptions (1) and (3)-(5) as in Theorem 1.1, suppose there is some constant $C_5 > 0$ and some $+\infty \geq q > 1$ independent of i such that for all i one has*

$$\fint_M (\max(-T_i, 0))^q \leq C_5$$

Then the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0,1)$, up to a diffeomorphism fixing the boundary, provided $k \geq 5$.

Corollary 1.11. *Under the assumptions (1) and (3)-(5) as in Theorem 1.1, there is some small constant $\varepsilon_2 > 0$ (possibly depending on C_1, C_2 and C^{k+3} norm bound of the boundary metric and*

also on the topology of X) such that if moreover for all i one has

$$\oint_M (\max(-T_i, 0)) \leq \varepsilon_2,$$

then the family of the Fefferman-Graham compactified metrics (X, g_i) is compact in $C^{k+2,\alpha}$ norm for any $\alpha \in (0, 1)$, up to a diffeomorphism fixing the boundary, provided $k \geq 5$.

Remark 1.12. *Although the non-local terms S and T appears to be independent from each other in their definitions, it turns out their behavior are coupled in the setting of conformally compact Einstein manifolds. As we will show in section 4.2 of the paper, for the limiting metric of a class of conformal compact Einstein manifolds, when the Yamabe invariant on the boundary is non-negative, the limiting metric of the blow-up metrics $T \equiv 0$ is equivalent to $S \equiv 0$.*

The paper is organized as follows: in section 2, we provide some background and some basic calculations; in section 3, we prove a ε-regularity result for our Q_4 flat metrics. In section 4, which is the main part of the paper, we do the blow-up analysis. First we rule out the boundary blow up by our boundedness assumptions on the boundary metrics and the condition on the S-tensors or T curvature in section 4.2, we then rule out the interior blow up based on our assumption that the ∂X is topologically S^3 and the condition (3) in the statement of the theorems by some topological arguments in section 4.3. This permits us to establish the uniform boundedness of the L^2 norm of the curvature tensor of the sequence of Fefferman-Graham's compactified metrics. From there, we apply the ε-regularity argument to jerk up the order of the regularity in section 4.4. Finally in section 5, We estimate some geometric quantity including the diameter of the metrics and show they are uniformly bounded and establish the desired compactness results claimed in section 1.

2. SOME BASIC CALCULUS ON THE BOUNDARY

We use the greek indices $\alpha, \beta, \gamma \cdots$ to represent the tangential indices, n is the unit normal vector on the boundary and letter $i, j, k \cdots$ are full indices. $A = \frac{1}{2}(Ric - \frac{R}{6}g)$ is the Schouten tensor in X and W is the Weyl tensor in X. Denote by $\hat{\nabla}$ the connection on the boundary M and by ∇ the connection in X. Similarly, we denote by $\hat{Ric}, \hat{R}$ the Ricci curvature and scalar curvature on the boundary $M = \partial X$, and $\hat{A} = \hat{Ric} - \frac{\hat{R}}{4}\hat{g}$ is the Schouten tensor on the boundary M. Recall the Cotten tensor in X (resp. on M) is defined by $C_{\alpha\beta\gamma} = A_{\alpha\beta,\gamma} - A_{\alpha\gamma,\beta}$ (resp. $\hat{C}_{\alpha\beta\gamma} = \hat{A}_{\alpha\beta,\gamma} - \hat{A}_{\alpha\gamma,\beta}$). Moreover, we denote by L the second fundamental form on M and H the mean curvature of the boundary M.

Let $T_{i_1 \cdots i_k}$ be a tensor defined on X. Then the Ricci identity

$$(2.1) \qquad T_{i_1 \cdots i_k, jl} = T_{i_1 \cdots i_k, lj} - \sum_{s=1}^{k} R_{m i_s l j} T_{i_1 \cdots i_{s-1} m i_{s+1} \cdots i_k}$$

gives the formula for exchanging derivatives. The curvature tensor is decomposed as

$$R_{ijkl} = W_{ijkl} + A_{ik}g_{jl} + A_{il}g_{ik} - A_{il}g_{jk} - A_{jk}g_{il}.$$

Recall that $C_{ijk} = A_{ij,k} - A_{ik,j} = -W_{lijk,l}$. The second Bianchi identity can be expressed as

$$(2.2) \quad W_{ijkl,m} + W_{ijmk,l} + W_{ijlm,k} + C_{ikm}g_{jl} + C_{jlm}g_{ik} + C_{iml}g_{jk} + C_{jmk}g_{il} + C_{jkl}g_{im} + C_{ilk}g_{jm} = 0.$$

We now recall some facts about the Bach tensor and the Q-curvature in dimension 4. It is known the Bach tensor (see [47])

$$
\begin{aligned}
B_{ij} &= \nabla^k\nabla^l W_{ikjl} + \frac{1}{2}R^{kl}W_{ikjl} = \triangle A_{ij} - \nabla^k\nabla_i A_{jk} + \frac{1}{2}R^{kl}W_{ikjl} \\
&= \triangle A_{ij} - \frac{1}{6}\nabla_i\nabla_j R + R_{ikjp}A^{pk} - R_{ip}A_j{}^p + \frac{1}{2}R^{kl}W_{ikjl}
\end{aligned}
$$

Thus, the Bach-flat equation is

$$(2.3) \qquad \nabla^k\nabla^l W_{ikjl} + A^{kl}W_{ikjl} = 0$$

or equivalently

$$(2.4) \qquad \triangle A_{ij} - \frac{1}{6}\nabla_i\nabla_j R + R_{ikjp}A^{pk} - R_{ip}A_j{}^p + A^{kl}W_{ikjl} = 0$$

since $W_{ik}{}^i{}_l = 0$. Using (2.1), (2.2) and (2.3), we infer

$$(2.5) \qquad \triangle W_{ijkl} + \nabla_l C_{kji} + \nabla_k C_{lij} + \nabla_i C_{jkl} + \nabla_j C_{ilk} = W * Rm + g * W * A$$

since

$$
\begin{aligned}
W_{ijmk,l}{}^m &= W_{ijmk,}{}^m{}_l + W * Rm = C_{kji,l} + W * Rm \\
W_{ijlm,k}{}^m &= W_{ijlm,}{}^m{}_k + W * Rm = C_{lij,k} + W * Rm \\
C_{ikm,}{}^m g_{jl} = \nabla^m\nabla^h W_{himk}g_{jl} = g * W * A, &\quad C_{jlm,}{}^m g_{ik} = g * W * A \\
C_{iml,}{}^m g_{jk} = g * W * A, &\quad C_{jmk,}{}^m g_{il} = g * W * A.
\end{aligned}
$$

Now we recall the Q-curvature $Q = \frac{1}{6}(-\triangle R + R^2 - 3|Ric|^2)$ so that Q-flat metric can be interpreted as

$$(2.6) \qquad \triangle R = R^2 - 3|Ric|^2.$$

The following two lemmas regard basic properties of the tensor S and the relation between S and the behavior of the Weyl tensor on the boundary.

Lemma 2.1.   *(1) S tensor $S_{\alpha\beta} = \nabla^i W_{i\alpha n\beta} + \nabla^i W_{i\beta n\alpha} - \nabla^n W_{n\alpha n\beta} + \frac{4}{3} H W_{\alpha\beta}{}^n$ is a symmetric 2-tensor;*

(2) $Tr(S) = 0$;

(3) S is a conformally invariant tensor in the sense that $S(\varphi^2 g) = \varphi^{-1} S(g)$;

(4) We have

$$S_{\alpha\beta} = \begin{aligned} &-A_{\alpha n,\beta} + A_{\beta n,\alpha} + A_{\alpha\beta,n} + \hat{\triangle} L_{\alpha\beta} - L_{\gamma\alpha,\hat{\beta}\hat{\gamma}} + L_{\beta\gamma} R_{\gamma n\alpha n} \\ &- H R_{\alpha n\beta n} + L_{\gamma\delta} R_{\beta\gamma\alpha\delta} - A_{\gamma n,\gamma} g_{\alpha\beta} + \frac{4}{3} H W_{\alpha n\beta n} \end{aligned}$$

(5) if the boundary is totally geodesic, then

$$S_{\alpha\beta} = A_{\alpha\beta,n} = \frac{1}{2} R_{\alpha\beta,n} - \frac{R_{,n}}{12} g_{\alpha\beta};$$

Proof. (1) and (2) follow by definition: $Tr(S) = 2\nabla^i W_{i\alpha n\alpha} = 2\nabla^i W_{ilnl} = 0$.

3) Let $\widetilde{g} = e^{2f} g$ be a conformal change. Denote ∇ (resp. $\widetilde{\nabla}$) the Levi-Civita connection with respect to the metric g (resp. $\widetilde{g}$). We write T (resp. $\widetilde{T}$) a tensor with respect to the metric g (resp. $\widetilde{g}$). Let Γ_{ij}^k (resp. $\widetilde{\Gamma}_{ij}^k$) be the Christoffel symbols with respect to the metric g (resp. $\widetilde{g}$). We write

$$\widetilde{\Gamma}_{ij}^k = \Gamma_{ij}^k + U_{ij}^k$$

where $U_{ij}^k = \delta^k{}_j \nabla_i f + \delta^k{}_i \nabla_j f - g_{ij} \nabla^k f$ is a $(2,1)$ tensor. Under the conformal change, we know $\widetilde{W} = W$ as a $(3,1)$ tensor and the Cotton tensor (recall $n = 4$)

$$\widetilde{C}_{ijk} = C_{ijk} + \nabla_l f W_{jki}{}^l.$$

Moreover, the mean curvature can be changed as follows

$$\widetilde{H} = e^{-f}(H - 3\nabla_n f)$$

On the other hand, we know $\nabla^i W_{ijkl} = -C_{jkl}$ so that

$$S_{\alpha\beta} = -C_{\alpha n\beta} - C_{\beta n\alpha} - \nabla^n W_{n\alpha n\beta} + \frac{4}{3} H W_{\alpha n\beta}{}^n$$

Gathering these relations, we deduce (recall the unit normal $\widetilde{n}$ (resp. n) for $\widetilde{g}$ (resp. g) satisfying $\widetilde{n} = e^{-f} n$)

$$\begin{aligned} \widetilde{S}_{\alpha\beta} &= e^{-f}(-\widetilde{C}_{\alpha n\beta} - \widetilde{C}_{\beta n\alpha} - e^{-2f} g^{ni} \widetilde{\nabla}_i (e^{2f} W_{n\alpha n\beta}) + \frac{4}{3} H W_{\alpha n\beta}{}^n - 4\nabla_n f W_{\alpha n\beta}{}^n) \\ &= e^{-f}(-C_{\alpha n\beta} - C_{\beta n\alpha} - \nabla_i f (W_{\alpha n\beta}^i + W_{\beta n\alpha}^i) - 2\nabla^n f W_{n\alpha n\beta} - \nabla^n W_{n\alpha n\beta}) \\ &\quad + e^{-f}(U_{nn}^i W_{i\alpha n\beta} + U_{n\alpha}^i W_{nin\beta} + U_{n\beta}^i W_{n\alpha ni} + U_{nn}^i W_{n\alpha i\beta} + \frac{4}{3} H W_{\alpha n\beta}{}^n - 4\nabla_n f W_{\alpha n\beta}{}^n) \end{aligned}$$

Together with

$$U_{nn}^i W_{i\alpha n\beta} = 2\nabla^n f W_{n\alpha n\beta} - \nabla^i f W_{i\alpha n\beta}$$
$$U_{nn}^i W_{n\alpha i\beta} = 2\nabla^n f W_{n\beta n\alpha} - \nabla^i f W_{i\beta n\alpha}$$
$$U_{n\alpha}^i W_{nin\beta} = \nabla_n f W_{n\alpha n\beta}$$
$$U_{n\beta}^i W_{n\alpha ni} = \nabla_n f W_{n\alpha n\beta}$$

the desired result follows, that is, S is a pointwise conformal invariant.

(4) and (5). For the rest of lemma, we use the Fermi coordinates. The Christoffel symbols

satisfy $\Gamma_{\alpha\beta}^n = L_{\alpha\beta}$, $\Gamma_{\alpha n}^\beta = -L_{\alpha\beta}$, $\Gamma_{\alpha n}^n = 0$ on the boundary ([21], P.8). Therefore,

$$
\begin{aligned}
R_{\alpha\beta\gamma n, \alpha} &= \partial_\alpha R_{\alpha\beta\gamma n} - \Gamma_{\alpha\alpha}^l R_{l\beta\gamma n} - \Gamma_{\alpha\beta}^l R_{\alpha l\gamma n} - \Gamma_{\alpha\gamma}^l R_{\alpha\beta l n} - \Gamma_{\alpha n}^l R_{\alpha\beta\gamma l} \\
&= R_{\alpha\beta\gamma n, \hat\alpha} - \Gamma_{\alpha\alpha}^n R_{n\beta\gamma n} - \Gamma_{\alpha\beta}^n R_{\alpha n\gamma n} - \Gamma_{\alpha n}^\delta R_{\alpha\beta\gamma\delta} \\
&= R_{\alpha\beta\gamma n, \hat\alpha} + H R_{\beta n\gamma n} - L_{\beta\alpha} R_{\alpha n\gamma n} + L_{\alpha\delta} R_{\alpha\beta\gamma\delta}
\end{aligned}
$$

By the Codazzi equation,

$$
R_{\alpha\beta\gamma n, \alpha} = L_{\alpha\gamma, \hat\beta\hat\alpha} - \hat\Delta L_{\beta\gamma} + H R_{\beta n\gamma n} - L_{\beta\alpha} R_{\alpha n\gamma n} + L_{\alpha\delta} R_{\alpha\beta\gamma\delta}.
$$

Hence, by curvature decomposition and the above formula

$$
\begin{aligned}
S_{\alpha\beta} &= \nabla^i W_{i\alpha n\beta} + \nabla^\gamma W_{\gamma\beta n\alpha} + \frac{4}{3} H W_{\alpha n\beta n} \\
&= (-A_{\alpha n, \beta} + A_{\alpha\beta, n}) + (\nabla^\gamma R_{\gamma\beta n\alpha} - A_{\gamma n, \gamma} g_{\alpha\beta} + A_{\beta n, \alpha}) + \frac{4}{3} H W_{\alpha n\beta n} \\
&= -A_{\alpha n, \beta} + A_{\beta n, \alpha} + A_{\alpha\beta, n} + \hat\Delta L_{\alpha\beta} - L_{\gamma\alpha, \hat\beta\hat\gamma} + L_{\beta\gamma} R_{\gamma n\alpha n} - H R_{\alpha n\beta n} + L_{\gamma\delta} R_{\beta\gamma\alpha\delta} \\
&\quad - A_{\gamma n, \gamma} g_{\alpha\beta} + \frac{4}{3} H W_{\alpha n\beta n}.
\end{aligned}
$$

When the boundary is totally geodesic, then $L = 0$ and $R_{\alpha n} = 0$. It follows that $S_{\alpha\beta} = A_{\alpha\beta, n}$. This proves the lemma. $\qquad\square$

Remark 2.2. *We remark that in the conformal compact Einstein setting (X, M, g^+), with a Poincare Einstein metric g^+ with $Ric(g^+) = -3g^+$ and $(M, \hat g)$ as conformal infinity; if we choose a special defining function r associated with $\hat g$ (c.f. [29]), with $|\nabla_{g^+} r| = 1$ near the boundary M and*

$$
r^2 g^+ =: g = ds^2 + g_r \ \text{ and } \ g_r = \hat g + g^{(2)} r^2 + g^{(3)} r^3 + O(r^4) g^{(4)}_{\alpha\beta}
$$

Recall $tr(g^{(3)}) = 0$. Thus a straight forward computation gives

$$
\nabla^n R_{\alpha\beta} = -3 g^{(3)}_{\alpha\beta}, \quad \text{ and } \nabla^n R = 0.
$$

Thus in this case, applying properties of the S tensor as above, we get for any compactified metric g on X with totally geodesic boundary,

$$
\tag{2.7} S_{\alpha\beta} = -\frac{3}{2} g^{(3)}_{\alpha\beta}.
$$

Lemma 2.3. *Suppose the boundary is totally geodesic and $W|_M = 0$. Then on the boundary, we have*

(1) $\nabla_n W_{\alpha\beta\gamma\delta} = -S_{\alpha\gamma} g_{\beta\delta} - S_{\beta\delta} g_{\alpha\gamma} + S_{\alpha\delta} g_{\beta\gamma} + S_{\beta\gamma} g_{\alpha\delta}$;
(2) $\nabla_n W_{\alpha\beta\gamma n} = -\hat C_{\gamma\beta\alpha} = -C_{\gamma\beta\alpha}$;
(3) $\nabla_n W_{n\alpha n\beta} = S_{\alpha\beta}$;
(4) $\nabla_n \nabla_n W_{\alpha\beta\gamma\delta} = \hat\nabla_\delta \hat C_{\gamma\alpha\beta} - \hat\nabla_\gamma \hat C_{\delta\alpha\beta} - \hat\nabla_\lambda \hat C_{\alpha\lambda\gamma} g_{\beta\delta} - \hat\nabla_\lambda \hat C_{\beta\lambda\delta} g_{\alpha\gamma} + \hat\nabla_\lambda \hat C_{\alpha\lambda\delta} g_{\beta\gamma} + \hat\nabla_\lambda \hat C_{\beta\lambda\gamma} g_{\alpha\delta}$;
(5) $\nabla_n \nabla_n W_{\alpha\beta\gamma n} = 2 \hat\nabla_\beta S_{\alpha\gamma} - 2 \hat\nabla_\alpha S_{\beta\gamma}$;
(6) $\nabla_n \nabla_n W_{n\alpha n\beta} = -\hat\nabla_\gamma \hat C_{\alpha\beta\gamma} - \hat\nabla_\gamma \hat C_{\beta\alpha\gamma}$;

Proof. (1) Note that

$$
\tag{2.8} C_{\alpha\beta n} = A_{\alpha\beta, n} - A_{\alpha n, \beta} = A_{\alpha\beta, n} = S_{\alpha\beta}.
$$

Now by (2.2),

$$
\begin{aligned}
\nabla_n W_{\alpha\beta\gamma\delta} &= -W_{\alpha\beta n\gamma,\delta} - W_{\alpha\beta\delta n,\gamma} - (C_{\alpha\gamma n}g_{\beta\delta} + C_{\beta\delta n}g_{\alpha\beta} + C_{\alpha n\delta}g_{\beta\gamma} + C_{\beta n\gamma}g_{\alpha\delta}) \\
&= -C_{\alpha\gamma n}g_{\beta\delta} - C_{\beta\delta n}g_{\alpha\beta} - C_{\alpha n\delta}g_{\beta\gamma} - C_{\beta n\gamma}g_{\alpha\delta} \\
&= -S_{\alpha\gamma}g_{\beta\delta} - S_{\beta\delta}g_{\alpha\gamma} + S_{\alpha\delta}g_{\beta\gamma} + S_{\beta\gamma}g_{\alpha\delta},
\end{aligned}
$$

where in the second equality we use $W|_{\partial X} = 0$ and hence $\nabla_\alpha W|_{\partial X} = 0$, and the last equality is by (2.8).

(2) We first prove that $\hat{C}_{\alpha\beta\gamma} = C_{\alpha\beta\gamma}$. By Gauss equation, $R = \hat{R} + 2R_{nn}$. Therefore, $R_{,\gamma} = \hat{R}_{,\hat{\gamma}} + 2R_{nn,\gamma}$. On the other hand, by curvature decomposition and the fact that $\nabla_\alpha W|_{\partial X} = 0$, we have $R_{\alpha n\beta n.\gamma} = W_{\alpha n\beta n,\gamma} + A_{\alpha\beta,\gamma} + A_{nn,\gamma}g_{\alpha\beta} = A_{\alpha\beta,\gamma} + A_{nn,\gamma}g_{\alpha\beta}$. Using above information, we obtain

$$
\begin{aligned}
A_{\alpha\beta,\gamma} &= \frac{1}{2}R_{\alpha\beta,\gamma} - \frac{R_{,\gamma}}{12}g_{\alpha\beta} = \frac{1}{2}\hat{R}_{\alpha\beta,\hat{\gamma}} + \frac{1}{2}R_{\alpha n\beta n.\gamma} - \frac{R_{,\gamma}}{12}g_{\alpha\beta} \\
&= \frac{1}{2}\hat{R}_{\alpha\beta,\hat{\gamma}} + \frac{1}{2}A_{\alpha\beta,\gamma} + \frac{1}{2}A_{nn,\gamma}g_{\alpha\beta} - \frac{R_{,\gamma}}{12}g_{\alpha\beta}.
\end{aligned}
$$

Hence,

$$
\begin{aligned}
\frac{1}{2}A_{\alpha\beta,\gamma} &= \frac{1}{2}\hat{R}_{\alpha\beta,\hat{\gamma}} + \frac{1}{2}A_{nn,\gamma}g_{\alpha\beta} - \frac{R_{,\gamma}}{12}g_{\alpha\beta} \\
&= \frac{1}{2}\hat{R}_{\alpha\beta,\hat{\gamma}} + \frac{1}{8}(2R_{nn,\gamma} - R_{,\gamma})g_{\alpha\beta} = \frac{1}{2}\hat{R}_{\alpha\beta,\hat{\gamma}} - \frac{\hat{R}_{,\hat{\gamma}}}{8}g_{\alpha\beta} = \frac{1}{2}\hat{A}_{\alpha\beta,\hat{\gamma}}.
\end{aligned}
$$

Therefore, $\hat{C}_{\alpha\beta\gamma} = \hat{A}_{\alpha\beta,\hat{\gamma}} - \hat{A}_{\alpha\gamma,\hat{\beta}} = A_{\alpha\beta,\gamma} - A_{\alpha\gamma,\beta} = C_{\alpha\beta\gamma}$.

Now, $\nabla_n W_{\alpha\beta\gamma n} = -C_{\gamma\beta\alpha} - \nabla_\delta W_{\alpha\beta\gamma\delta} = -C_{\gamma\beta\alpha}$.

(3) $\nabla_n W_{n\alpha n\beta} = -C_{\beta n\alpha} - \nabla_\gamma W_{n\alpha\gamma\beta} = -C_{\beta n\alpha}$. Using (2.8), we get $\nabla_n W_{n\alpha n\beta} = S_{\alpha\beta}$.

(4) By (2.2),

$$
\begin{aligned}
\nabla_n\nabla_n W_{\alpha\beta\gamma\delta} &= -(W_{\alpha\beta n\gamma,\delta} + W_{\alpha\beta\delta n,\gamma})_n - (C_{\alpha\gamma n}g_{\beta\delta} + C_{\beta\delta n}g_{\alpha\gamma} + C_{\alpha n\delta}g_{\beta\gamma} + C_{\beta n\gamma}g_{\alpha\delta})_n \\
&= -W_{\alpha\beta n\gamma,n\delta} - W_{\alpha\beta\delta n,n\gamma} - C_{\alpha\gamma n,n}g_{\beta\delta} - C_{\beta\delta n,n}g_{\alpha\gamma} - C_{\alpha n\delta,n}g_{\beta\gamma} - C_{\beta n\gamma,n}g_{\alpha\delta}
\end{aligned}
$$

$$
\text{(2.9)} \qquad = \hat{\nabla}_\delta\hat{C}_{\gamma\alpha\beta} - \hat{\nabla}_\gamma\hat{C}_{\delta\alpha\beta} - C_{\alpha\gamma n,n}g_{\beta\delta} - C_{\beta\delta n,n}g_{\alpha\gamma} - C_{\alpha n\delta,n}g_{\beta\gamma} - C_{\beta n\gamma,n}g_{\alpha\delta},
$$

where in the second quality, we use the Ricci identity (2.1) $W_{\alpha\beta n\gamma,\delta n} = W_{\alpha\beta n\gamma,n\delta}$ by noting that $W|_{\partial X} = 0$, and the last equality is by Lemma 2.1 (2).

Now by the Bach-flat equation $\nabla_l\nabla_k W_{kilj} = 0$ and Lemma 2.1 (2),

$$
C_{\alpha n\beta,n} = C_{\alpha l\beta,l} - C_{\alpha\gamma\beta,\gamma} = -\nabla_l\nabla_k W_{k\alpha l\beta} - C_{\alpha\gamma\beta,\gamma} = -C_{\alpha\gamma\beta,\gamma} = -\hat{C}_{\alpha\gamma\beta,\hat{\gamma}}.
$$

Inserting the above formula into (2.9) gives (4).

(5) By (2.2),

$$
\begin{aligned}
\nabla_n\nabla_n W_{\alpha\beta\gamma n} &= -(W_{\gamma nn\alpha,\beta} + W_{\gamma n\beta n,\alpha})_n - (C_{n\beta n}g_{\gamma\alpha} + C_{nn\alpha}g_{\gamma\beta} + C_{\gamma\beta\alpha}g_{nn})_n \\
&= -W_{\gamma nn\alpha,n\beta} - W_{\gamma n\beta n,n\alpha} - C_{n\beta n,n}g_{\gamma\alpha} - C_{nn\alpha,n}g_{\gamma\beta} - C_{\gamma\beta\alpha,n} \\
&= \hat{\nabla}_\beta S_{\alpha\gamma} - \hat{\nabla}_\alpha S_{\beta\gamma} - C_{n\beta n,n}g_{\gamma\alpha} - C_{nn\alpha,n}g_{\gamma\beta} - C_{\gamma\beta\alpha,n},
\end{aligned}
$$

where in the second quality, we use Ricci identity (2.1) $W_{\gamma nn\alpha,\beta n} = W_{\gamma nn\alpha,n\beta}$ and the last equality is by Lemma 2.1 (3).

Using the Bach-flat equation gives $C_{nn\alpha,n} = C_{nl\alpha,l} - C_{n\beta\alpha,\beta} = -\nabla_l\nabla_k W_{knl\alpha} - C_{n\beta\alpha,\beta} = -C_{n\beta\alpha,\beta}$. Since the boundary is totally geodesic, $C_{n\beta\alpha,\beta} = (A_{n\beta,\alpha} - A_{n\alpha,\beta})_\beta = 0$. Therefore, $C_{nn\alpha,n} = 0$ and

$$
\nabla_n\nabla_n W_{\alpha\beta\gamma n} = \hat{\nabla}_\beta S_{\alpha\gamma} - \hat{\nabla}_\alpha S_{\beta\gamma} - C_{\gamma\beta\alpha,n}.
$$

To compute $C_{\gamma\beta\alpha,n}$, by Ricci identity (2.1)

$$
\begin{aligned}
C_{\gamma\beta\alpha,n} &= A_{\gamma\beta,\alpha n} - A_{\gamma\alpha,\beta n} \\
&= A_{\gamma\beta,n\alpha} - A_{\gamma\alpha,n\beta} - R_{l\gamma n\alpha}A_{l\beta} - R_{l\beta n\alpha}A_{l\gamma} + R_{l\gamma n\beta}A_{l\alpha} + R_{l\alpha n\beta}A_{l\gamma} \\
&= A_{\gamma\beta,n\alpha} - A_{\gamma\alpha,n\beta} = (A_{\gamma n,\beta} + C_{\gamma\beta n})_\alpha - (A_{\gamma n,\alpha} + C_{\gamma\alpha n})_\beta = C_{\gamma\beta n,\alpha} - C_{\gamma\alpha n,\beta},
\end{aligned}
$$

where we use $A_{n\alpha} = R_{n\alpha\beta\gamma} = 0$ because the boundary is totally geodesic. Finally, by (2.8)

$$(2.10) \qquad C_{\gamma\beta\alpha,n} = C_{\gamma\beta n,\alpha} - C_{\gamma\alpha n,\beta} = \hat{\nabla}_\alpha S_{\beta\gamma} - \hat{\nabla}_\beta S_{\alpha\gamma},$$

which finishes the proof of (5).

(6) By the Bach-flat equation,

$$
\begin{aligned}
\nabla_n\nabla_n W_{\alpha n\beta n} &= \nabla_l\nabla_k W_{k\alpha l\beta} - \nabla_\gamma\nabla_\delta W_{\alpha\gamma\beta\delta} - \nabla_n\nabla_\gamma W_{\alpha n\beta\gamma} - \nabla_\gamma\nabla_n W_{\alpha\gamma\beta n} \\
&= -W_{\alpha n\beta\gamma,n\gamma} - W_{\alpha\gamma\beta n,n\gamma} = -\hat{\nabla}_\gamma\hat{C}_{\alpha\beta\gamma} - \hat{\nabla}_\gamma\hat{C}_{\beta\alpha\gamma},
\end{aligned}
$$

where in the second equality we use the Ricci identity (2.1) and the last equality is by Lemma 2.1 (2). $\qquad\square$

We now recall the following result.

Lemma 2.4. *Under the Fefferman-Graham's compactification with the Yamabe metric on the boundary, we have $R = 3\hat{R}$ on M.*

Proof. From Corollary 6.6 in [12], we have $J_g = 2J_{\hat{g}}$ where $J_g = \frac{R}{6}$ (resp. $J_{\hat{g}} = \frac{\hat{R}}{4}$) is the trace of the Schouten tensor of the metric g (resp. $\hat{g}$). Therefore we get the desired result. $\qquad\square$

The following result is well known (see [26]).

Lemma 2.5. *Under any compactification, we have $W|_M = 0$ on M.*

We split the tangent bunlde on the boundary $T_xX = \mathbb{R}\vec{\nu} \oplus T_xM$ for all $x \in M$, where $\vec{\nu}$ is unit normal vector on the boundary. Given a tensor T, we decompose tensor $\nabla^{(k)}T$ on X along M that are related to the splitting $T_xX = \mathbb{R}\vec{\nu} \oplus T_xM$: let us denote by $\nabla^{(k)}_{odd}T$ (resp. $\nabla^{(k)}_{even}T$) the normal component $\vec{\nu}$ appeared odd time (resp. even time) in the tensor $\nabla^{(k)}T$.

Lemma 2.6. *Suppose the boundary is totally geodesic and $W|_M = 0$. We have on the boundary M for any $k \le 1$*

$$\nabla^{(k+1)}_n W = L(\hat{\nabla}^{(k)}S, \hat{\nabla}^{(k)}\hat{C})$$

where L is some linear function. As a consequence, we have on M for any $k \le 1$

$$\nabla^{(k+1)}W = L(\hat{\nabla}^{(k)}S, \hat{\nabla}^{(k)}\hat{C}).$$

More precisely, we have

$$\nabla^{(k+1)}_{odd}W = L(\hat{\nabla}^{(k)}S)$$

and

$$\nabla^{(k+1)}_{even}W = L(\hat{\nabla}^{(k)}\hat{C})$$

Proof. The first part of Lemma comes from Lemma 2.3. Recall $W = 0$ on M. Thus

$$\nabla_\alpha W = 0$$

Also it follows from the Ricci identity (2.1),

$$\nabla_i\nabla_j W = \nabla_j\nabla_i W$$

Thus the desired result follows from Lemma 2.3 again and the lemma is proved. $\qquad\square$

Lemma 2.7. *Suppose the boundary is totally geodesic and $W|_M = 0$ for some Q-flat and Bach-flat metric. Then on the boundary M*

(1) *We have*

$$A_{\alpha\beta,\gamma} = \hat{A}_{\alpha\beta,\gamma}, A_{\alpha\beta,n} = S_{\alpha\beta}, A_{n\alpha,\beta} = A_{\alpha n,\beta} = 0, A_{nn,n} = \frac{1}{6}R_{,n},$$

$$A_{nn,\alpha} = \frac{1}{6}\hat{\nabla}_\alpha R - \hat{A}_{\beta\beta,\alpha}, A_{n\alpha,n} = A_{\alpha n,n} = \frac{1}{6}\hat{\nabla}_\alpha R - \hat{A}_{\alpha\beta,\beta}$$

(2) *We have*

$$A_{\alpha\beta,\gamma\lambda} = \hat{A}_{\alpha\beta,\gamma\lambda}, \;\; A_{\alpha\beta,\gamma n} = A_{\alpha\beta,n\gamma} = \hat{\nabla}_\gamma S_{\alpha\beta},$$

$$A_{\alpha\beta,nn} = \frac{1}{6}\hat{\nabla}_\alpha\hat{\nabla}_\beta R - \hat{A}_{\alpha\beta,\gamma\gamma} - |A|^2 g_{\alpha\beta} + 4A_{\alpha p}A_\beta{}^p$$

$$A_{n\alpha,\beta\gamma} = A_{\alpha n,\beta\gamma} = 0, \;\; A_{n\alpha,nn} = A_{\alpha n,nn} = A_{nn,n\alpha} = A_{nn,\alpha n} = \frac{1}{6}R_{,n\alpha},$$

$$A_{n\alpha,n\beta} = A_{\alpha n,n\beta} = \frac{1}{6}\hat{\nabla}_\beta\hat{\nabla}_\alpha R - \hat{A}_{\alpha\gamma,\gamma\beta},$$

$$A_{n\alpha,\beta n} = A_{\alpha n,\beta n} = \frac{1}{6}\hat{\nabla}_\beta\hat{\nabla}_\alpha R - \hat{A}_{\alpha\gamma,\gamma\beta} - (A_{nn})^2 g_{\alpha\beta} + A_{\alpha\gamma}A_\alpha{}^\gamma$$

$$A_{nn,\alpha\gamma} = \frac{1}{6}\hat{\nabla}_\alpha\hat{\nabla}_\gamma R - \hat{A}_{\beta\beta,\alpha\gamma}, \;\; R_{,nn} = R^2 - 3|Ric|^2 - \hat{\nabla}_\alpha\hat{\nabla}_\alpha R$$

Moreover, we have

$$A_{nn,nn} = -\frac{1}{3}R_{,\alpha\alpha} + \frac{1}{6}(R^2 - 3|Ric|^2) - A^{\alpha\beta}A_{\alpha\beta} + 3(A_{nn})^2 + \frac{1}{4}\hat{\triangle}\hat{R}.$$

In particular, when $\hat{R}$ is constant on M, then

$$A_{nn,nn} = -\frac{1}{3}R_{,\alpha\alpha} + \frac{1}{6}(R^2 - 3|Ric|^2) - A^{\alpha\beta}A_{\alpha\beta} + 3(A_{nn})^2$$

(3) *We have*

$$\nabla^{(2)}_{odd}R = L(\hat{\nabla}\nabla_n R), \; \nabla^{(2)}_{even}R = L(\hat{\nabla}^{(2)}R) + A*A,$$

$$\nabla^{(1)}_{odd}A = L(S, \nabla_n R), \; \nabla^{(1)}_{even}A = L(\hat{\nabla}\hat{A}, \hat{\nabla}R),$$

$$\nabla^{(2)}_{odd}A = L(\hat{\nabla}S, \hat{\nabla}\nabla_n R), \; \nabla^{(2)}_{even}A = L(\hat{\nabla}^{(2)}\hat{A}, \hat{\nabla}^{(2)}R) + A*A.$$

(4) *There holds*

$$A_{\alpha\beta} = \hat{A}_{\alpha\beta} + (A_{nn} + \frac{\hat{R}}{4} - \frac{R}{6})g_{\alpha\beta}$$

(5) *Under the Fefferman-Graham's compactification with the Yamabe metric on the boundary, we have*

$$A_{nn} = \frac{\hat{R}}{4}$$

$$A_{\alpha\beta} = \hat{A}_{\alpha\beta}$$

Proof. (1) The first two equalities are proved in the proof of Lemmas 2.1 and 2.3. Recall $A_{\alpha n} = 0$ on M from the Codazzi equations so that the third one comes. From the relation $tr(A) = \frac{R}{6}$, we have on the boundary

$$A_{\alpha\alpha,n} + A_{nn,n} = \frac{R_{,n}}{6}$$

On the other hand, by Lemma 2.1

$$A_{\alpha\alpha,n} = tr(S) = 0$$

so that

$$A_{nn,n} = \frac{R_{,n}}{6}$$

Similarly,

$$A_{\beta\beta,\alpha} + A_{nn,\alpha} = \frac{R_{,\alpha}}{6}$$

so that

$$A_{nn,\alpha} = \frac{1}{6}\hat{\nabla}_\alpha R - \hat{A}_{\beta\beta,\alpha}.$$

since $A_{\beta\beta,\alpha} = \hat{A}_{\beta\beta,\alpha}$. By the second Bianchi identity, we obtain

$$A_{\alpha\beta,\beta} + A_{\alpha n,n} = \frac{R_{,\alpha}}{6}$$

which implies

$$A_{n\alpha,n} = A_{\alpha n,n} = \frac{1}{6}\hat{\nabla}_\alpha R - \hat{A}_{\alpha\beta,\beta}$$

since A is symmetric and $A_{\alpha\beta,\beta} = \hat{A}_{\alpha\beta,\beta}$.

(2) It follows from $A_{\alpha\beta,\gamma} = \hat{A}_{\alpha\beta,\gamma}$ that $A_{\alpha\beta,\gamma\lambda} = \hat{A}_{\alpha\beta,\gamma\lambda}$.

From the Ricci identity (2.1) and the codazzi equations $R_{\alpha\beta\gamma n} = 0$ on M, we get $A_{\alpha\beta,\gamma n} = A_{\alpha\beta,n\gamma}$
so that $A_{\alpha\beta,\gamma n} = A_{\alpha\beta,n\gamma} = \hat{\nabla}_\gamma S_{\alpha\beta}$.

By the Bach flat equation (2.4) for $A_{\alpha\beta}$ and the decomposition of curvature tensor $Rm = W + A \owedge g$, we infer

$$A_{\alpha\beta,nn} = \triangle A_{\alpha\beta} - \hat{A}_{\alpha\beta,\gamma\gamma} = \frac{1}{6}\hat{\nabla}_\alpha\hat{\nabla}_\beta R - \hat{A}_{\alpha\beta,\gamma\gamma} - R_{\alpha k\beta p}A^{pk} + R_{\alpha p}A_\beta{}^p$$

$$= \frac{1}{6}\hat{\nabla}_\alpha\hat{\nabla}_\beta R - \hat{A}_{\alpha\beta,\gamma\gamma} - |A|^2 g_{\alpha\beta} + 4A_{\alpha p}A_\beta{}^p$$

From the fact $A_{\alpha n} = A_{n\alpha} = 0$, we have $A_{n\alpha,\beta\gamma} = A_{\alpha n,\beta\gamma} = 0$. There holds

$$A_{n\alpha,nn} = A_{\alpha n,nn} = \triangle A_{\alpha n} - A_{\alpha n,\beta\beta} = \triangle A_{\alpha n} = \frac{1}{6}R_{,n\alpha}$$

since $A_{\alpha n} = R_{\alpha\beta\gamma n} = 0$ and Bach flat equation (2.4) for $A_{\alpha n}$. We have

$$A_{nn,\alpha n} = \frac{1}{6}R_{,\alpha n} - A_{\beta\beta,\alpha n} = \frac{1}{6}R_{,\alpha n} - A_{\beta\beta,n\alpha} = \frac{1}{6}R_{,\alpha n} - \hat{\nabla}_\alpha S_{\beta\beta} = \frac{1}{6}R_{,\alpha n}$$

since $tr(S) = 0$. It follows from (2) $A_{nn,n} = \frac{1}{6}R_{,n}$ there holds

$$A_{nn,n\alpha} = \frac{1}{6}R_{,\alpha n}$$

Using $A_{n\alpha,n} = A_{\alpha n,n} = \frac{1}{6}\hat{\nabla}_\alpha R - \hat{A}_{\alpha\gamma,\gamma}$, there holds

$$A_{n\alpha,n\beta} = A_{\alpha n,n\beta} = \frac{1}{6}\hat{\nabla}_\beta\hat{\nabla}_\alpha R - \hat{A}_{\alpha\gamma,\gamma\beta}$$

With the Ricci identity (2.1), we deduce

$$\begin{aligned}
A_{n\alpha,\beta n} &= A_{\alpha n,\beta n} = A_{\alpha n,n\beta} - R_{m\alpha n\beta}A^m{}_n - R_{mnn\beta}A_\alpha{}^m \\
&= \frac{1}{6}\hat{\nabla}_\beta\hat{\nabla}_\alpha R - \hat{A}_{\alpha\gamma,\gamma\beta} - (A_{nn})^2 g_{\alpha\beta} + A_{\alpha\gamma}A_\alpha{}^\gamma
\end{aligned}$$

By $A_{nn,\alpha} = \frac{1}{6}\hat{\nabla}_\alpha R - \hat{A}_{\gamma\gamma,\alpha}$, we infer

$$A_{nn,\alpha\beta} = \frac{1}{6}\hat{\nabla}_\beta\hat{\nabla}_\alpha R - \hat{A}_{\gamma\gamma,\alpha\beta}$$

By Q-flat condition (2.6), we have

$$R_{,nn} = \triangle R - \hat{\nabla}_\alpha\hat{\nabla}_\alpha R = R^2 - 3|Ric|^2 - \hat{\nabla}_\alpha\hat{\nabla}_\alpha R$$

From Bach flat condition (2.5), Q-flat condition (2.6) and from the decomposition of curvature tensor $Rm = W + A \owedge g$, we calculate

$$\begin{aligned}
A_{nn,nn} &= \triangle A_{nn} - A_{nn,\alpha\alpha} = \triangle A_{nn} - (\tfrac{R}{6} - A_{\beta\beta})_{,\alpha\alpha} \\
&= \frac{1}{6}R_{,nn} - A^{\alpha\beta}A_{\alpha\beta} + 3(A_{nn})^2 - \frac{1}{6}R_{,\alpha\alpha} + A_{\beta\beta,\alpha\alpha} \\
&= \frac{1}{6}(\triangle R - R_{,\alpha\alpha}) - A^{\alpha\beta}A_{\alpha\beta} + 3(A_{nn})^2 - \frac{1}{6}R_{,\alpha\alpha} + A_{\beta\beta,\alpha\alpha} \\
&= \frac{1}{6}(R^2 - 3|Ric|^2) - A^{\alpha\beta}A_{\alpha\beta} + 3(A_{nn})^2 - \frac{1}{3}R_{,\alpha\alpha} + A_{\beta\beta,\alpha\alpha}
\end{aligned}$$

On the other hand

$$A_{\beta\beta,\alpha\alpha} = \hat{A}_{\beta\beta,\alpha\alpha} = \frac{1}{4}\hat{R}_{,\alpha\alpha} = \frac{1}{4}\triangle\hat{R}$$

since $tr(\hat{A}) = \frac{1}{4}\hat{R}$. This yields the desired result.

(3) It is just the result from (1) and (2).

(4)

$$2A_{\alpha\beta} = Ric_{\alpha\beta} - \frac{R}{6}g_{\alpha\beta} = R_{\alpha\gamma\beta\gamma} + R_{\alpha n\beta n} - \frac{R}{6}g_{\alpha\beta}$$

By Gauss equation $R_{\alpha\gamma\beta\gamma} = \hat{R}_{\alpha\gamma\beta\gamma}$ and the decomposition of curvature tensor $Rm = W + A\owedge g$,

$$2A_{\alpha\beta} = \hat{A}_{\alpha\beta} + A_{\alpha\beta} + A_{nn}g_{\alpha\beta} - \frac{R}{6}g_{\alpha\beta} + \frac{\hat{R}}{4}g_{\alpha\beta},$$

which implies the desired result.

(5) From Gauss-Codazzi equation, we have $R = \hat{R} + 2R_{nn}$. Together with Lemma 2.4, we infer $R_{nn} = \hat{R}$ and $A_{nn} = \frac{\hat{R}}{4}$. Combining this with the result in (4), we infer $A_{\alpha\beta} = \hat{A}_{\alpha\beta}$. Finally, we prove the result. □

Lemma 2.8. *Suppose the boundary is totally geodesic and $W|_M = 0$ for some Q-flat and Bach-flat metric. We have on the boundary M for any $k \geq 2$*

$$\nabla^{(k)}_{odd}R = L(\hat{\nabla}^{(k-1)}\nabla_n R) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm * \nabla^{(k-2-l)}A$$

$$\nabla^{(k)}_{odd}A = L(\hat{\nabla}^{(k-1)}S, \hat{\nabla}^{(k-1)}\nabla_n R) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm * \nabla^{(k-2-l)}A$$

$$\nabla^{(k)}_{odd}W = L(\hat{\nabla}^{(k-1)}S, \hat{\nabla}^{(k-1)}\nabla_n R) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm * \nabla^{(k-2-l)}Rm$$

where L is some linear function.

Proof. We prove it by the induction. When $k = 1, 2$, it is clear for A and W from lemmas 2.6 and 2.7. Suppose it is true for k. For $k + 1$, we consider first R. The terms $\nabla_{odd}^{(k+1)} R$ are in three cases.

a) $\nabla_{odd}^{(k+1)} R = \nabla_n \nabla_\alpha \nabla_{even}^{(k-1)} R$.

In such case, by the Ricci identity (2.1)

$$\nabla_{odd}^{(k+1)} R = \nabla_\alpha \nabla_n \nabla_{even}^{(k-1)} R + Rm * \nabla^{(k-1)} R = \nabla_\alpha \nabla_{odd}^{(k)} R + Rm * \nabla^{(k-1)} A$$

By the assumptions of the induction, we get the result.

b) $\nabla_{odd}^{(k+1)} R = \nabla_n \nabla_n \nabla_{odd}^{(k-1)} R$.

We write by the Ricci identity (2.1)

$$\nabla_{odd}^{(k+1)} R = \triangle \nabla_{odd}^{(k-1)} R - \nabla_\alpha \nabla_\alpha \nabla_{odd}^{(k-1)} R = \nabla_{odd}^{(k-1)} \triangle R - \nabla_\alpha \nabla_\alpha \nabla_{odd}^{(k-1)} R + \sum_{l=0}^{k-1} \nabla^{(l)} Rm * \nabla^{(k-2-l)} A$$

By flat Q_4 curvature condition (2.6) and using the assumption of the induction, we get the result.

c) $\nabla_{odd}^{(k+1)} R = \nabla_\alpha \nabla_{odd}^{(k)} R$.

This is an easier case. It follows from the assumptions of the induction.

Now we consider the terms $\nabla_{odd}^{(k+1)} A$. Similarly, we consider them in three cases.

a) $\nabla_{odd}^{(k+1)} A = \nabla_n \nabla_\alpha \nabla_{even}^{(k-1)} A$.

In such case, by the Ricci identity (2.1)

$$\nabla_{odd}^{(k+1)} A = \nabla_\alpha \nabla_n \nabla_{even}^{(k-1)} A + Rm * \nabla^{(k-1)} A = \nabla_\alpha \nabla_{odd}^{(k)} A + Rm * \nabla^{(k-1)} A$$

Thus, we get the result.

b) $\nabla_{odd}^{(k+1)} A = \nabla_n \nabla_n \nabla_{odd}^{(k-1)} A$.

We write

$$\nabla_{odd}^{(k+1)} A = \triangle \nabla_{odd}^{(k-1)} A - \nabla_\alpha \nabla_\alpha \nabla_{odd}^{(k-1)} A$$

From the Bach flat equation (2.4), we get

$$\triangle \nabla^{(k-1)} A - \nabla^{(k+1)} \frac{R}{6} + \sum_{l=0}^{k-1} \nabla^{(l)} Rm * \nabla^{(k-1-l)} A = 0$$

which implies

$$\nabla_{odd}^{(k+1)} A = \nabla_{odd}^{(k+1)} \frac{R}{6} - \nabla_\alpha \nabla_\alpha \nabla_{odd}^{(k-1)} A + \sum_{l=0}^{k-1} \nabla^{(l)} Rm * \nabla^{(k-1-l)} A$$

By the assumption of the induction and the above equation, we could write

$$\nabla_{odd}^{(k+1)} A = L(\hat{\nabla}^{(k)} S, \hat{\nabla}^{(k)} \nabla_n R) + \nabla_{odd}^{(k+1)} \frac{R}{6} + \sum_{l=0}^{k-1} \nabla^{(l)} Rm * \nabla^{(k-1-l)} A$$

which implies by the result for the scalar curvature

$$\nabla_{odd}^{(k+1)} A = L(\hat{\nabla}^{(k)} S, \hat{\nabla}^{(k)} \nabla_n R) + \sum_{l=0}^{k-1} \nabla^{(l)} Rm * \nabla^{(k-1-l)} A$$

c) $\nabla_{odd}^{(k+1)} A = \nabla_\alpha \nabla_{odd}^{(k)} A$.

It follows from the assumptions of the induction.

The proof for the Weyl tensor is quite similar as the schouten tensor A. We divide into 3 cases as above

a) $\nabla_{odd}^{(k+1)}W = \nabla_n\nabla_\alpha\nabla_{even}^{(k-1)}W = \nabla_\alpha\nabla_n\nabla_{even}^{(k-1)}W + Rm*\nabla^{(k-1)}A = \nabla_\alpha\nabla_{odd}^{(k)}W + Rm*\nabla^{(k-1)}W$

b) $\nabla_{odd}^{(k+1)}W = \nabla_n\nabla_n\nabla_{odd}^{(k-1)}W = \triangle\nabla_{odd}^{(k-1)}W - \nabla_\alpha\nabla_\alpha\nabla_{odd}^{(k-1)}W$

From the Bach flat equation (2.5) and the Ricci identity (2.1), we get

$$\nabla_{odd}^{(k+1)}W = L(\nabla_{odd}^{(k+1)}A) - \nabla_\alpha\nabla_\alpha\nabla_{odd}^{(k-1)}W + \sum_{l=0}^{k-1}\nabla^{(l)}Rm*\nabla^{(k-1-l)}Rm.$$

c) $\nabla_{odd}^{(k+1)}W = \nabla_\alpha\nabla_{odd}^{(k)}W.$

In the above 3 cases, we can prove the result by the assumptions of the induction and the results for A. We therefore have established the proof of Lemma 2.8. $\square$

Lemma 2.9. *Suppose the boundary is totally geodesic and $W|_M = 0$ for some Q-flat and Bach-flat metric. For any $k \geq 2$, we have*

$$\nabla_{even}^{(k)}R = L(\hat{\nabla}^{(k)}R) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm*\nabla^{(k-2-l)}A$$

$$\nabla_{even}^{(k)}A = L(\hat{\nabla}^{(k)}\hat{A}, \hat{\nabla}^{(k)}R) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm*\nabla^{(k-2-l)}A$$

$$\nabla_{even}^{(k)}W = L(\hat{\nabla}^{(k)}\hat{A}, \hat{\nabla}^{(k)}R) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm*\nabla^{(k-2-l)}Rm$$

where L is some linear function. In particular, when the restriction of R on M is constant, we have

$$\nabla_{even}^{(k)}R = \sum_{l=0}^{k-2}\nabla^{(l)}Rm*\nabla^{(k-2-l)}A$$

$$\nabla_{even}^{(k)}A = L(\hat{\nabla}^{(k)}\hat{A}) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm*\nabla^{(k-2-l)}A$$

$$\nabla_{even}^{(k)}W = L(\hat{\nabla}^{(k)}\hat{A}) + \sum_{l=0}^{k-2}\nabla^{(l)}Rm*\nabla^{(k-2-l)}Rm$$

Proof. We prove the result by induction. For $k = 1, 2$, the results follow from Lemmas 2.6 and 2.7. As before, we treat the three cases. First we consider the scalar curvature.

a) $\nabla_{even}^{(k+1)}R = \nabla_n\nabla_\alpha\nabla_{odd}^{(k-1)}R.$

In such case, by the Ricci identity (2.1)

$$\nabla_{even}^{(k+1)}R = \nabla_\alpha\nabla_n\nabla_{odd}^{(k-1)}R + Rm*\nabla^{(k-1)}R = \nabla_\alpha\nabla_{even}^{(k)}R + Rm*\nabla^{(k-1)}A$$

Thus, we get the result by the assumptions of the induction.

b) $\nabla_{even}^{(k+1)}R = \nabla_n\nabla_n\nabla_{even}^{(k-1)}R.$

We write by the Ricci identity (2.1)

$$\nabla_{even}^{(k+1)}R = \triangle\nabla_{even}^{(k-1)}R - \nabla_\alpha\nabla_\alpha\nabla_{even}^{(k-1)}R = \nabla_{even}^{(k-1)}\triangle R - \nabla_\alpha\nabla_\alpha\nabla_{even}^{(k-1)}R + \sum_{l=0}^{k-1}\nabla^{(l)}Rm*\nabla^{(k-2-l)}A$$

By flat Q_4 curvature condition (2.6), we get the result by the induction.

c) $\nabla_{even}^{(k+1)} R = \nabla_\alpha \nabla_{even}^{(k)} R$.

It is clear by the induction argument.

Now we consider the terms $\nabla_{even}^{(k+1)} A$. Similarly, we consider them in three cases.

a) $\nabla_{even}^{(k+1)} A = \nabla_n \nabla_\alpha \nabla_{old}^{(k-1)} A$.

In such case, by the Ricci identity (2.1)

$$\nabla_{even}^{(k+1)} A = \nabla_\alpha \nabla_n \nabla_{odd}^{(k-1)} A + Rm * \nabla^{(k-1)} A = \nabla_\alpha \nabla_{even}^{(k)} A + Rm * \nabla^{(k-1)} A$$

We get the result by the induction.

b) $\nabla_{even}^{(k+1)} A = \nabla_n \nabla_n \nabla_{even}^{(k-1)} A$.

We write

$$\nabla_{even}^{(k+1)} A = \triangle \nabla_{even}^{(k-1)} A - \nabla_\alpha \nabla_\alpha \nabla_{even}^{(k-1)} A$$

We get

$$\nabla_{even}^{(k+1)} A = \triangle \nabla_{even}^{(k-1)} A - \nabla_\alpha \nabla_\alpha \nabla_{even}^{(k-1)} A = \nabla_{even}^{(k-1)} \triangle A - \nabla_\alpha \nabla_\alpha \nabla_{even}^{(k-1)} A + \sum_{l=0}^{k-1} \nabla^{(l)} Rm * \nabla^{(k-1-l)} A$$

From the Bach flat equation (2.4), we could write

$$\nabla_{even}^{(k+1)} A = \frac{1}{6} \nabla_{even}^{(k+1)} R - \nabla_\alpha \nabla_\alpha \nabla_{even}^{(k-1)} A + \sum_{l=0}^{k-1} \nabla^{(l)} Rm * \nabla^{(k-1-l)} A$$

It follows from the result for R and from the assumptions of the induction.

c) $\nabla_{even}^{(k+1)} A = \nabla_\alpha \nabla_{even}^{(k)} A$.

It is clear in this case by the induction.

The proof for the Weyl tensor is quite similar as the schouten tensor A and the scalar curvature R as in the proof of Lemma 2.8. We omit the details. Thus we have established the lemma.

$\square$

3. ε-REGULARITY

Theorem 3.1. *Suppose the assumptions (4) and (5) in Theorem 1.1 are satisfied and assume $\|\hat{Rm}\|_{C^{k+1}(M)}$ and $\|S\|_{L_{loc}^1(M)}$ are bounded, the metric is Q-flat and Bach-flat, the boundary M is totally geodesic, $W|_M = 0$ and the restriction of the scalar curvature $R|_M$ is some positive bounded constant. Assume further there exists some positive constant $C_3 > 0$ such that for any $r < 1$ and for any p, we have*

$$vol(B(p,r)) \leq C_4 r^4.$$

Then There exists constants $\varepsilon > 0$ (independent of k) and $C > 0$ (depending on k, $\|\hat{Rm}\|_{C^{k+1}(M)}$, $\|S\|_{L_{loc}^1(M)}$ and C_1, C_2 in the assumptions (4) and (5) in Theorem 1.1) such that if

$$\|Rm\|_{L^2(B(p,r))} \leq \varepsilon$$

then for all $r < 1$

$$\left\{ \int_{B(p,r/2)} |\nabla^k A|^4 dV_g \right\}^{1/2} \leq \frac{C}{r^{2k+2}} \left(\int_{B(p,r)} |A|^2 dV_g + \oint_{B(p,r) \cap M} |S| + r^4 \right)$$

$$\int_{B(p,r/2)} |\nabla^{k+1} A|^2 dV_g \leq \frac{C}{r^{2k+2}} \left(\int_{B(p,r)} |A|^2 dV_g + \oint_{B(p,r) \cap M} |S| + r^4 \right)$$

$$\left\{ \int_{B(p,r/2)} |\nabla^k Rm|^4 dV_g \right\}^{1/2} \leq \frac{C}{r^{2k+2}} \left(\int_{B(p,r)} |A|^2 dV_g + \oint_{B(p,r)\cap M} |S| + r^4 \right)$$

$$\int_{B(p,r/2)} |\nabla^{k+1} Rm|^2 dV_g \leq \frac{C}{r^{2k+2}} \left(\int_{B(p,r)} |A|^2 dV_g + \oint_{B(p,r)\cap M} |S| + r^4 \right)$$

Proof. We now begin the proof of the theorem by considering the case $k = 0$ first. Let η be some cut-off function such that $\eta = 1$ on $B(p, \frac{3r}{4})$ and $\eta = 0$ outsides $B(p, r)$ and $|\nabla \eta| \leq C/r$. Taking the test tensor $\eta^2 A$ in (2.4), we have

$$
\begin{aligned}
\int \eta^2 |\nabla A|^2 &= -\int \eta^2 \langle \triangle A, A \rangle - 2 \int \eta \langle \nabla A, \nabla \eta \otimes A \rangle + \oint \eta^2 \langle \nabla_n A, A \rangle \\
&= -\int \eta^2 \langle \triangle A - \frac{1}{6}\nabla^2 R, A \rangle - 2 \int \eta \langle \nabla A, \nabla \eta \otimes A \rangle + \oint \eta^2 \langle \nabla_n A, A \rangle \\
&\quad - \frac{1}{6} \int \eta^2 \langle \nabla^2 R, A \rangle \\
&\leq C \int |\nabla A||A|\eta|\nabla \eta| + \left| \frac{1}{36} \int \eta^2 R \triangle R \right| + \left| \int \eta^2 (\triangle A_{ij} - \frac{1}{6}\nabla_i \nabla_j R) A_{ij} \right| \\
&\quad + \left| \oint \eta^2 (\langle \nabla_n A, A \rangle - \frac{1}{6}\langle \nabla R, A(n, \cdot) \rangle \right| + \left| \frac{1}{36} \oint \eta^2 R \nabla_n R \right| \\
&\leq C \int |\nabla A||A|\eta|\nabla \eta| + C \int \eta^2 |Rm||A|^2 \\
&\quad + \left| \oint \eta^2 (\langle \nabla_n A, A \rangle - \frac{1}{6}\langle \nabla R, A(n, \cdot) \rangle \right| + \left| \frac{1}{36} \oint \eta^2 R \nabla_n R \right|
\end{aligned}
$$

(3.1)

Here we use the Q flat condition (2.6) and second Bianchi idendity $A_{ij,j} = \frac{1}{6}R_{,i}$ and also Bach flat equation (2.4).

Now we want to estimate the boundary terms. From Lemmas 2.1 and 2.7, we have $A_{n\alpha} = 0$, $A_{nn,n} - \frac{1}{6}R_{,n} = 0$ and $A_{\alpha\beta} - \hat{A}_{\alpha\beta} = \lambda g_{\alpha\beta}$, since we also have $trS = 0$, we obtain

$$(3.2) \qquad \left| \oint \eta^2 (\langle \nabla_n A, A \rangle - \frac{1}{6}\langle \nabla R, A(n, \cdot) \rangle \right| = \left| \oint \eta^2 \langle S, \hat{A} \rangle \right| \leq \oint \eta^2 |S||\hat{A}|$$

On the other hand, since by our assumption on $(M, \hat{g})$, $\hat{A}$ is bounded in C^0 norm, we obtain

$$\oint \eta^2 |S||\hat{A}| \leq C \oint \eta^2 |S|.$$

Recall $R = 3\hat{R}$ is a bounded constant on the boundary by our assumption, R is uniformly bounded on the boundary. Therefore, we could bound from Q-flat condition (2.6)

$$
\begin{aligned}
\left| \frac{1}{36} \oint \eta^2 R \nabla_n R \right| &= \left| \frac{R}{36} \int \operatorname{div}(\eta^2 \nabla R) \right| \\
&\leq \frac{1}{8} \int_X |\eta \nabla A|^2 + C \int_X \eta^2 |A|^2 + C \int |\nabla \eta|^2 \\
&\leq \frac{1}{8} \int_X |\eta \nabla A|^2 + C \int_X \eta^2 |A|^2 + \frac{C}{r^2} vol(B(p,r))
\end{aligned}
$$

(3.3)

Claim. For any Lipschitz function $f \in C^{1,0}(X)$ and for any regular function η vanishing on $X \setminus B(p, r)$, we have

$$(3.4) \qquad \|\eta f\|_{L^4(X)}^2 \leq C' \int_X |\nabla(\eta f)|^2,$$

and

$$\|\eta f\|_{L^3(M)}^2 \le C' \int_X |\nabla(\eta f)|^2, \tag{3.5}$$

provided ε is small.

To see the claim, we have from the assumption condition (4) on the Yamabe constants in Theorem 1.1, we get

$$\begin{aligned}\|\eta f\|_{L^4}^2 &\le C_1 \int_X |\nabla(\eta f)|^2 + \frac{R}{6}|\eta f|^2 \le C_1\Big(\int_X |\nabla(\eta f)|^2 + \|\eta f\|_{L^4}^2 \|Rm\|_{L^2(B(p,r))}\Big) \\ &\le C_1\Big(\int_X |\nabla(\eta f)|^2 + \|\eta f\|_{L^4}^2 \varepsilon\Big)\end{aligned}$$

so that

$$\|\eta f\|_{L^4}^2 \le 2C_1 \int_X |\nabla(\eta f)|^2$$

provided $2C_1\varepsilon \le 1$. Similarly, by the assumption condiiton (5) on the boundary type Yamabe constants in Theorem 1.1 and (3.4), we infer

$$\|\eta f\|_{L^3(\partial X)}^2 \le C_2\Big(\int_X |\nabla(\eta f)|^2 + \|\eta f\|_{L^4}^2 \varepsilon\Big) \le 2C_2(1 + C'\varepsilon)\int_X |\nabla(\eta f)|^2 \le 4C_2 \int_X |\nabla(\eta f)|^2,$$

provided $C'\varepsilon \le 1$. This proves the claim.

Now we apply the Cauchy-Schwarz inequality and (3.4)

$$\begin{aligned} & C\int |\nabla A||A|\eta|\nabla\eta| + C\int \eta^2|Rm||A|^2 \\ \le\ & \frac{1}{8}\int_X |\eta\nabla A|^2 + C'\Big(\int_X |\nabla\eta|^2|A|^2 + \|\eta A\|_{L^4}^2 \|Rm\|_{L^2(B(p,r))}\Big) \\ \le\ & \frac{1}{8}\int_X |\eta\nabla A|^2 + C'\Big(\int_X |\nabla\eta|^2|A|^2 + \|\nabla(\eta|A|)\|_{L^4}^2 \|Rm\|_{L^2(B(p,r))}\Big) \end{aligned}$$

Together with (3.1)-(3.3), we infer

$$\begin{aligned} & \int \eta^2|\nabla A|^2 \\ \le\ & \frac{1}{2}\int \eta^2|\nabla A|^2 + C\Big(r^2 + \int(\eta^2 + |\nabla\eta|^2)|A|^2 + \|\nabla(\eta|A|)\|_{L^2}^2 \|Rm\|_{L^2(B(p,r))}\Big) + C\oint \eta^2|S| \\ \le\ & \Big(\frac{1}{2} + 2C\|Rm\|_{L^2(B(p,r))}\Big)\int \eta^2|\nabla A|^2 + C\oint \eta^2|S| \\ & + C\Big(r^2 + \int(\eta^2 + |\nabla\eta|^2)|A|^2 + 2\||\nabla\eta||A|\|_{L^2}^2 \|Rm\|_{L^2(B(p,r))}\Big) \end{aligned}$$

Therefore, when $2C\varepsilon < \frac{1}{4}$, we get

$$\int \eta^2|\nabla A|^2 dV_g \le \frac{C}{r^2}\Big(\int_{B(p,r)} |A|^2 dV_g + \oint_{B(p,r)\cap M} |S| + r^4\Big)$$

Here we use $|\nabla|A|| \le |\nabla A|$. Again from the Sobolev inequality (3.4), we deduce

$$\Big\{\int \eta^2|A|^4 dV_g\Big\}^{1/2} \le \frac{C}{r^2}\Big(\int_{B(p,r)} |A|^2 dV_g + \oint_{B(p,r)\cap M} |S| + r^4\Big)$$

Recall from the Bach flat equation for the Weyl tensor (2.5) and the Ricci identity (2.1)

$$\triangle W_{ijkl} = 2(\nabla_i\nabla_k A_{jl} - \nabla_i\nabla_l A_{jk} - \nabla_j\nabla_k A_{il} + \nabla_j\nabla_l A_{ik}) + A*A + A*W + W*W$$

As before, we take $\eta^2 W$ as the test tensor to the above equation. We remark on the boundary $W = 0$. Thus

$$
\begin{aligned}
&\int \eta^2 |\nabla W|^2 \\
\leq\ & C \int (|\nabla W| + |\nabla A|)|W|\eta|\nabla\eta| + C \int \eta^2 |Rm|(|W|^2 + |A|^2) \\
&+ 2|\int \eta^2 (\nabla_k A_{jl}\nabla_i W_{ijkl} - \nabla_l A_{jk}\nabla_i W_{ijkl} - \nabla_k A_{il}\nabla_j W_{ijkl} + \nabla_l A_{ik}\nabla_j W_{ijkl})| \\
\leq\ & C \int (|\nabla W| + |\nabla A|)|W|\eta|\nabla\eta| + \int \eta^2 |Rm|(|W|^2 + |A|^2) + \int \eta^2 |\nabla W||\nabla A| \\
\leq\ & \frac{1}{2} \int \eta^2 |\nabla W|^2 + C \int |W|^2 |\nabla\eta|^2 + (\|\eta W\|_{L^4}^2 + \|\eta A\|_{L^4}^2)\|Rm\|_{L^2(B(p,r))} + \int \eta^2 |\nabla A|^2
\end{aligned}
$$

With the similar arguments as above, we infer

$$
\int \eta^2 |\nabla W|^2 \leq \frac{C}{r^2}\left(\int_{B(p,r)} |Rm|^2 dV_g + r^4\right)
$$

Again from the Sobolev inequality (3.4), we get the desired inequalitiy

$$
\left\{\int (\eta|W|)^4\right\}^{\frac{1}{2}} \leq \frac{C}{r^2}\left(\int_{B(p,r)} |Rm|^2 dV_g + r^4\right)
$$

Now use the relation

$$
|\nabla^{(l)} Rm|^2 = |\nabla^{(l)} W|^2 + |\nabla^{(l)} A \owedge g|^2
$$

Therefore, we obtain the corresponding inequalities for Rm. Thus we have finished the part for $k = 0$ of the theorem.

We now prove for the high $k \geq 1$ by induction. For each k, let η_k be some cut-off function such that $\eta_k = 1$ on $B(p, \frac{r}{2} + \frac{r}{2^{k+2}})$ and $\eta_k = 0$ outsides $B(p, \frac{r}{2} + \frac{r}{2^{k+1}})$ and $|\nabla\eta_k| \leq C/r$. But for simplicity of the notation, we denote all such cut function as η and skip the index k. First we treat the estimates for the Schouten tensor A. From the Bach flat equation (2.4), we obtain

$$
\triangle\nabla^{(k)} A - \nabla^{(2)}\nabla^{(k)}\frac{R}{6} + \sum_{l=0}^{k} \nabla^{(l)} Rm * \nabla^{(k-l)} A = 0
$$

As above, we take $\eta^2 \nabla^{(k)} A$ as the test tensor, integrate the equality of the Bach equation, we obtain

$$
\begin{aligned}
& \int \eta^2 |\nabla^{(k+1)} A|^2 \\
=\ & -\int \eta^2 \langle \triangle \nabla^{(k)} A, \nabla^{(k)} A \rangle - 2 \int \eta \langle \nabla^{(k+1)} A, \nabla \eta \otimes \nabla^{(k)} A \rangle + \oint \eta^2 \langle \nabla_n \nabla^{(k)} A, \nabla^{(k)} A \rangle \\
\leq\ & C \int |\nabla^{(k+1)} A| |\nabla^{(k)} A| \eta |\nabla \eta| + \left| \frac{1}{36} \int \eta^2 \langle \nabla^{(k)} R, \nabla^{(k)} \triangle R \rangle \right| \\
& + C \sum_{l=0}^{k} \left| \int \eta^2 \nabla^{(l)} Rm * \nabla^{(k-l)} A * \nabla^{(k)} A \right| \\
& + C \left| \oint \eta^2 \langle \nabla_n \nabla^{(k)} A, \nabla^{(k)} A \rangle \right| + C \left| \oint \eta^2 \langle \nabla^{(k+1)} R, \nabla^{(k)} A(n, \cdot) \rangle \right| \\
& + \left| \frac{1}{36} \oint \eta^2 \langle \nabla^{(k)} R, \nabla_n \nabla^{(k)} R \rangle \right| + C \sum_{l=0}^{k-1} \oint \eta^2 |\nabla^{(l)} Rm| |\nabla^{(k-1-l)} A| |\nabla^{(k)} R| \\
\leq\ & C \int |\nabla^{(k+1)} A| |\nabla^{(k)} A| \eta |\nabla \eta| + C \sum_{l=0}^{k} \left| \int \eta^2 \nabla^{(l)} Rm * \nabla^{(k-l)} A * \nabla^{(k)} A \right| \\
& + C \left| \oint \eta^2 \langle \nabla_n \nabla^{(k)} A, \nabla^{(k)} A \rangle \right| + C \left| \oint \eta^2 \langle \nabla^{(k+1)} R, \nabla^{(k)} A(n, \cdot) \rangle \right| \\
& + \left| \frac{1}{36} \oint \eta^2 \langle \nabla^{(k)} R, \nabla_n \nabla^{(k)} R \rangle \right| + C \sum_{l=0}^{k-1} \oint \eta^2 |\nabla^{(l)} Rm| |\nabla^{(k-1-l)} A| |\nabla^{(k)} R|
\end{aligned}
$$

Here we use the Q flat condition (2.6) and second Bianchi idendity $A_{ij,j} = \frac{1}{6} R_{,i}$. We need just to consider the boundary term

$$
\langle \nabla_n \nabla^{(k)} A, \nabla^{(k)} A \rangle \ (\text{resp. } \langle \nabla^{(k+1)} R, \nabla^{(k)} A(n, \cdot) \rangle, \text{ or } \langle \nabla^{(k)} R, \nabla_n \nabla^{(k)} R \rangle)
$$

Our basic observation is that in all these products, one is an odd term and another one an even term, where odd and even is defined as in the proof of the Lemmas 2.8 and 2.9; we also deduce from these lemmas that

$$
\langle \nabla_n \nabla^{(k)}_{odd} A, \nabla^{(k)}_{old} A \rangle = O(|\nabla^{(k)} A| + \sum_{l=0}^{k-1} |\nabla^{(k)} A| |\nabla^{(l)} Rm| |\nabla^{(k-1-l)} A|)
$$

and

$$
\begin{aligned}
& \oint \eta^2 \langle \nabla_n \nabla^{(k)}_{even} A, \nabla^{(k)}_{even} A \rangle \\
=\ & \oint \eta^2 \langle \sum c_\alpha \nabla_\alpha \nabla^{(k)}_{odd} A, \nabla^{(k)}_{even} A \rangle + O(\sum_l \oint \eta^2 |\nabla^{(k)} A| |\nabla^{(l)} Rm| |\nabla^{(k-1-l)} A|)
\end{aligned}
$$

where c_α is some constant. By the integration by parts, we infer

$$
\begin{aligned}
& \oint \sum c_\alpha \eta^2 \langle \nabla_\alpha \nabla^{(k)}_{odd} A, \nabla^{(k)}_{even} A \rangle \\
=\ & -2 \oint \eta \sum c_\alpha \nabla_\alpha \eta \langle \nabla^{(k)}_{odd} A, \nabla^{(k)}_{even} A \rangle - \oint \eta^2 \langle \nabla^{(k)}_{odd} A, \sum c_\alpha \nabla_\alpha \nabla^{(k)}_{even} A \rangle
\end{aligned}
$$

Thus, we could estimate from Lemma 2.9

$$
\oint \sum c_\alpha \eta^2 \langle \nabla_\alpha \nabla_{odd}^{(k)} A, \nabla_{even}^{(k)} A \rangle = O\left(\oint \frac{\eta}{r}(|\nabla^{(k)}A|(1 + \sum_l |\nabla^{(l)}Rm||\nabla^{(k-2-l)}A|))\right)
$$
$$
+ O\left(\oint \eta^2(|\nabla^{(k)}A|(1 + \sum_l |\nabla^{(l)}Rm||\nabla^{(k-1-l)}A|))\right)
$$

Our basic observation is that

$$
\oint \eta^2(|\nabla^{(k)}A|(1 + \sum_l |\nabla^{(l)}Rm||\nabla^{(k-1-l)}A|))
$$
$$
\leq Cr^2\|\eta\nabla^{(k)}A\|_{L^3} + \sum_l \|\eta\nabla^{(k)}A\|_{L^3}\|\nabla^{(l)}Rm\|_{L^3(B(p,r_l)\cap M)}\|\eta|\nabla^{(k-1-l)}A|\|_{L^3}
$$

and

$$
\oint \frac{\eta}{r}(|\nabla^{(k)}A|(1 + \sum_l |\nabla^{(l)}Rm||\nabla^{(k-2-l)}A|))
$$
$$
\leq Cr\|\eta\nabla^{(k)}A\|_{L^3} + \sum_l \frac{1}{r}\|\eta\nabla^{(k)}A\|_{L^3}\|\nabla^{(l)}Rm\|_{L^3(B(p,r_l)\cap M)}\|\nabla^{(k-2-l)}A\|_{L^3}
$$

where $r_l = r/2 + r/2^{l+2}$. By the Sobolev trace inequality (3.5) and from the induction, we get for any $l < k$

$$
\|\nabla^{(l)}Rm\|_{L^3(B(p,r_l)\cap M)}^2 \leq \frac{C}{r^{2l+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

$$
\|\eta\nabla^{(k)}A\|_{L^3(M)}^2 \leq C\|\eta\nabla^{(k+1)}A\|_{L^2(X)}^2 + C\|\nabla\eta \otimes \nabla^{(k)}A\|_{L^2(X)}^2
$$
$$
\leq C\|\eta\nabla^{(k+1)}A\|_{L^2(X)}^2 + \frac{C}{r^{2k+2}}(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4)
$$

which implies from the Cauchy-Schwarz inequality

$$
\|\eta\nabla^{(k)}A\|_{L^3}\|\nabla^{(l)}Rm\|_{L^3}\|\eta|\nabla^{(k-1-l)}A|\|_{L^3}
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k)}A\|_{L^3}^2 + \frac{\gamma}{4}(\|\nabla^{(l)}Rm\|_{L^3}\|\eta|\nabla^{(k-1-l)}A|\|_{L^3})^2
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k+1)}A\|_{L^2(X)}^2 + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

and

$$
r\|\eta\nabla^{(k)}A\|_{L^3} \leq Cr^2 + \frac{1}{\gamma}\|\eta\nabla^{(k+1)}A\|_{L^2(X)}^2 + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

and also

$$
\frac{1}{r}\|\eta\nabla^{(k)}A\|_{L^3}\|\nabla^{(l)}Rm\|_{L^3}\|\eta|\nabla^{(k-2-l)}A|\|_{L^3}
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k+1)}A\|_{L^2(X)}^2 + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

On the other hand, we have

$$
\int |\nabla^{(k+1)}A||\nabla^{(k)}A|\eta|\nabla\eta|
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} + C\int |\nabla^{(k)}A|^2|\nabla\eta|^2
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

and

$$
\sum_{l=1}^{k-1}\int \eta^2|\nabla^{(l)}Rm||\nabla^{(k-l)}A||\nabla^{(k)}A|
$$
$$
\leq \sum_{l=1}^{k-1}\|\nabla^{(l)}Rm\|_{L^4(B(p,r_l))}\|\eta\nabla^{(k-l)}A\|_{L^4}\|\eta\nabla^{(k)}A\|_{L^2}
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

Here the constant C depends also on the γ. It remains to treat $\int \eta^2\nabla^{(k)}Rm * A * \nabla^{(k)}A$ and $\int \eta^2 Rm * \nabla^{(k)}A * \nabla^{(k)}A$. For the term $\int \eta^2\nabla^{(k)}Rm * A * \nabla^{(k)}A$, using the Sobolev inequality (3.4), Hölder's inequality and Cauchy-Schwarz inequality

$$
\int \eta^2|\nabla^{(k)}Rm * A * \nabla^{(k)}A)|
$$
$$
\leq \|\eta\nabla^{(k)}Rm\|_{L^2}\|\eta\nabla^{(k)}A\|_{L^4}\|A\|_{L^4(B(p,r_1))}
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k)}A\|^2_{L^4(X)} + C\|\eta\nabla^{(k)}Rm\|^2_{L^2}\|A\|^2_{L^4(B(p,r_1))}
$$
$$
\leq \frac{C}{\gamma}\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

Similarly, we have

$$
\int \eta^2|Rm * \nabla^{(k)}A * \nabla^{(k)}A|
$$
$$
\leq \|Rm\|_{L^4(B(p,r_1))}\|\eta\nabla^{(k)}A\|_{L^4}\|\eta\nabla^{(k)}A\|_{L^2}
$$
$$
\leq \frac{1}{\gamma}\|\eta\nabla^{(k)}A\|^2_{L^4} + C\|Rm\|^2_{L^4(B(p,r_1))}\|\eta\nabla^{(k)}A\|^2_{L^2}
$$
$$
\leq \frac{C}{\gamma}\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

Gathering all these estimates together, we deduce

$$
\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} \leq \frac{1}{2}\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} + \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

provided γ is a sufficiently large constant. Therefore

$$
\|\eta\nabla^{(k+1)}A\|^2_{L^2(X)} \leq \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)
$$

By the Sobolev inequality (3.4), we get

$$\|\eta\nabla^{(k)}A\|_{L^4(X)}^2 \leq \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right).$$

It is similar for the Weyl tensor. From the Bach flat equation (2.5), we have

$$
(3.6)\quad
\begin{aligned}
\triangle\nabla^{(k)}W_{ijml} &= 2(\nabla_i\nabla_m\nabla^{(k)}A_{jl} - \nabla_i\nabla_l\nabla^{(k)}A_{jm} - \nabla_j\nabla_m\nabla^{(k)}A_{il} + \nabla_j\nabla_l\nabla^{(k)}A_{im}) \\
&\quad + \sum_{l=0}^{k}(\nabla^{(l)}W * \nabla^{(k-l)}W + \nabla^{(l)}W * \nabla^{(k-l)}A + \nabla^{(l)}A * \nabla^{(k-l)}A)
\end{aligned}
$$

As before, we take $\eta^2\nabla^{(k)}W$ as test tensor and integrate the equality. Thus, we have

$$
\begin{aligned}
\int \eta^2|\nabla^{(k+1)}W|^2 &\leq C\int(|\nabla^{(k+1)}A| + |\nabla^{(k+1)}W|)|\nabla^{(k)}W|\eta|\nabla\eta| \\
&+ C\sum_{l=0}^{k}\left[\left|\int \eta^2\nabla^{(l)}W * \nabla^{(k-l)}A * \nabla^{(k)}W\right| + \left|\int \eta^2\nabla^{(l)}W * \nabla^{(k-l)}A * \nabla^{(k)}A\right|\right. \\
&+ \left|\int \eta^2\nabla^{(l)}A * \nabla^{(k-l)}A * \nabla^{(k)}A\right| + \left|\int \eta^2\nabla^{(l)}W * \nabla^{(k-l)}W * \nabla^{(k)}A\right| \\
&+ \left.\left|\int \eta^2\nabla^{(l)}W * \nabla^{(k-l)}W * \nabla^{(k)}W\right| + \left|\int \eta^2\nabla^{(l)}A * \nabla^{(k-l)}A * \nabla^{(k)}W\right|\right] \\
&+ C\left|\oint \eta^2\langle\nabla_n\nabla^{(k)}W, \nabla^{(k)}W\rangle\right| + C\left|\oint \eta^2\langle\nabla_n\nabla^{(k)}A, \nabla^{(k)}W\rangle\right| \\
&+ C\left|\oint \eta^2\langle\nabla^{(k)}A, \nabla_n\nabla^{(k)}W\rangle\right|
\end{aligned}
$$

Here we use the above two Bach flat equations (2.3) and (3.6) and Ricci identity (2.1). With the similar arguments, we can bound the boundary terms as above

$$
\begin{aligned}
&C\left|\oint \eta^2\langle\nabla_n\nabla^{(k)}W, \nabla^{(k)}W\rangle\right| + C\left|\oint \eta^2\langle\nabla_n\nabla^{(k)}A, \nabla^{(k)}W\rangle\right| + C\left|\oint \eta^2\langle\nabla^{(k)}A, \nabla_n\nabla^{(k)}W\rangle\right| \\
&\leq \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right) + \frac{1}{4}\int \eta^2|\nabla^{(k+1)}W|^2
\end{aligned}
$$

And also from the induction and results for A and Hölder's and Cauchy-Schwarz inequalities

$$
\begin{aligned}
&C\int(|\nabla^{(k+1)}A| + |\nabla^{(k+1)}W|)|\nabla^{(k)}W|\eta|\nabla\eta| \\
&\leq \frac{1}{4}\int \eta^2|\nabla^{(k+1)}W|^2 + C\int \eta^2|\nabla^{(k+1)}A|^2 + C\int|\nabla\eta^2|^2|\nabla^{(k)}W|^2 \\
&\leq \frac{C}{r^{2k+2}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right) + \frac{1}{4}\int \eta^2|\nabla^{(k+1)}W|^2
\end{aligned}
$$

and

$$
\begin{aligned}
C \sum_{l=0}^{k} & \left[\left| \int \eta^2 \nabla^{(l)} W * \nabla^{(k-l)} A * \nabla^{(k)} W \right| + \left| \int \eta^2 \nabla^{(l)} W * \nabla^{(k-l)} A * \nabla^{(k)} A \right| \right. \\
& + \left| \int \eta^2 \nabla^{(l)} A * \nabla^{(k-l)} A * \nabla^{(k)} A \right| + \left| \int \eta^2 \nabla^{(l)} W * \nabla^{(k-l)} W * \nabla^{(k)} A \right| \\
& \left. + \left| \int \eta^2 \nabla^{(l)} W * \nabla^{(k-l)} W * \nabla^{(k)} W \right| + \left| \int \eta^2 \nabla^{(l)} A * \nabla^{(k-l)} A * \nabla^{(k)} W \right| \right] \\
\leq \quad & C \sum_{l=1}^{k-1} [\|\nabla^{(l)} Rm\|_{L^4} \|\eta \nabla^{(k-l)} Rm\|_{L^4} \|\eta \nabla^{(k)} Rm\|_{L^2}] \\
& + C \|Rm\|_{L^4} \|\eta \nabla^{(k)} Rm\|_{L^2} (\|\eta \nabla^{(k)} W\|_{L^4} + \|\eta \nabla^{(k)} A\|_{L^4}) \\
\leq \quad & \frac{C}{r^{2k+2}} \left(\int_{B(p,r)} |Rm|^2 dV_g + \oint_{B(p,r) \cap M} |S| + r^4 \right) + \frac{1}{4} \int \eta^2 |\nabla^{(k+1)} W|^2
\end{aligned}
$$

Finally, we infer

$$
\int \eta^2 |\nabla^{(k+1)} W|^2 \leq \frac{C}{r^{2k+2}} \left(\int_{B(p,r)} |Rm|^2 dV_g + \oint_{B(p,r) \cap M} |S| + r^4 \right)
$$

which implies from the Sobolev inequality (3.4)

$$
\|\eta \nabla^{(k)} W\|_{L^4(X)}^2 \leq \frac{C}{r^{2k+2}} \left(\int_{B(p,r)} |Rm|^2 dV_g + \oint_{B(p,r) \cap M} |S| + r^4 \right)
$$

We have thus finished the proof of Theorem 3.1. $\qquad \square$

Theorem 3.2. *Under the same assumptions as Theorem 3.1, we have the estimates for L^∞ norm, that is, there exists constants ε (independent of k) and C (depending on k) such that if*

$$
\|Rm\|_{L^2(B(p,r))} \leq \varepsilon
$$

then for any $r < 1$

$$
\text{(3.7)} \qquad \sup_{B(p,r/2)} |\nabla^{k-2} Rm| \leq \frac{C}{r^k} \left(\int_{B(p,r)} |Rm|^2 dV_g + \oint_{B(p,r) \cap M} |S| + r^4 \right)^{\frac{1}{2}}
$$

We recall a technique result, which can be found in [46, 10, 38].

Lemma 3.3. *Under the assumption (4) as in Theorem 1.1, there exists constants ε and C such that if*

$$
\|Rm\|_{L^2(B(p,r))} \leq \varepsilon
$$

then for any Lipschitz function f with the compact support in $B(p,r)$

$$
\text{(3.8)} \qquad \left(\int_X f^4 dV_g \right)^{1/2} \leq C \int_X |\nabla f|^2 dV_g
$$

Moreover, we have

$$
\text{(3.9)} \qquad \|f\|_{L^p} \leq C' \|f\|_{L^m}^{1-\gamma} \|\nabla f\|_{L^q}^{\gamma}
$$

where $\gamma = \frac{\frac{1}{m}-\frac{1}{p}}{\frac{1}{4}-\frac{1}{q}+\frac{1}{m}}$. *Here, when $q > 4$, we can let $+\infty \geq p \geq m \geq 2$ and C' is some constant depending on m, p, q, C; when $q = 4$, we can let $+\infty > p \geq m \geq 2$ and C' is some constant depending on m, p, C.*

Proof of Theorem 3.2. Recall a basic fact that for any tensor T, we have

$$|\nabla|T|| \leq |\nabla T|$$

We will show first $\nabla^{k-1}Rm \in L^8(B(p, r/2))$. Let η be some cut-off function such that $supp(\eta) \subset B(p, 3r/4)$ and $\eta \equiv 1$ on $B(p, r/2)$ and $|\nabla\eta| \leq C/r$. From Lemma 3.3

$$\|\eta|\nabla^{k-1}Rm|\|_{L^8}^2 \leq C\|\eta|\nabla^{k-1}Rm|\|_{L^4}\|\nabla(\eta|\nabla^{k-1}Rm|)\|_{L^4}$$

Applying Theorem 3.1, we have

$$\|\eta|\nabla^k Rm|\|_{L^4} \leq \frac{C}{r^{1+k}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)^{\frac{1}{2}}$$

$$\|\nabla(\eta|\nabla^{k-1}Rm|)\|_{L^4} \leq C\left(\frac{1}{r}\|\nabla^{k-1}Rm\|_{L^4(B(p,3r/4))} + \|\eta\nabla^k Rm\|_{L^4}\right)$$

$$\leq \frac{C}{r^{1+k}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)^{\frac{1}{2}}$$

which yields

$$\|\eta|\nabla^{k-1}Rm|\|_{L^8}^2 \leq \frac{C}{r^{1+2k}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)$$

By the same argument,

$$\|\eta|\nabla^{k-2}Rm|\|_{L^8}^2 \leq \frac{C}{r^{2k-1}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)$$

Again from Lemma 3.3

$$\|\eta|\nabla^{k-2}Rm|\|_{L^\infty}^2 \leq C\|\eta|\nabla^{k-1}Rm|\|_{L^8}\|\eta|\nabla^{k-2}Rm|\|_{L^8}$$

$$\leq \frac{C}{r^{2k}}\left(\int_{B(p,r)}|Rm|^2 dV_g + \oint_{B(p,r)\cap M}|S| + r^4\right)$$

This gives the desired estimate (3.7), which establishes Theorem 3.2. $\square$

We now derive a better regularity result in the interior of the manifold.

Theorem 3.4. *Suppose the assumption (4) in Theorem 1.1 are satisfied and the metric is Q-flat and Bach-flat. Assume for some $r > 0$ and for some p with $B(p, r) \subset \bar{X} \setminus M$, then there exist constants ε (independent of k) and C (depending on k) such that if*

$$\|Rm\|_{L^2(B(p,r))} \leq \varepsilon$$

then

$$\left\{\int_{B(p,r/2)}|\nabla^k A|^4 dV_g\right\}^{1/2} \leq \frac{C}{r^{2k+2}}\int_{B(p,r)}|A|^2 dV_g$$

$$\int_{B(p,r/2)}|\nabla^{k+1}A|^2 dV_g \leq \frac{C}{r^{2k+2}}\int_{B(p,r)}|A|^2 dV_g$$

$$\left\{ \int_{B(p,r/2)} |\nabla^k Rm|^4 dV_g \right\}^{1/2} \leq \frac{C}{r^{2k+2}} \int_{B(p,r)} |Rm|^2 dV_g$$

$$\int_{B(p,r/2)} |\nabla^{k+1} Rm|^2 dV_g \leq \frac{C}{r^{2k+2}} \int_{B(p,r)} |Rm|^2 dV_g$$

Moreover, we have

$$\sup_{B(p,r/2)} |\nabla^{k-2} Rm| \leq \frac{C}{r^k} \left(\int_{B(p,r)} |Rm|^2 dV_g \right)^{\frac{1}{2}} \leq \frac{C\varepsilon}{r^k}$$

Proof. The proof is as same as the one of Theorems 3.1 and 3.2. We just remark that there is no boundary term in the estimates of the corresponding inequalities now.   □

4. BLOW-UP ANALYSIS

4.1. **Statement of the results.**

In this section, we will do blow up analysis both on the boundary and in the interior. Recall the Fefferman-Graham's compactification $g_i = e^{2w_i} g_i^+ = v_i^{-2} g_i^+$ where g_i^+ is a conformally compact Einstein metric and the defining function w_i solves the equation (1.4). We aim to prove the curvature tensor for the FG metric is uniformly bounded, namely,

Theorem 4.1. *Under the same assumptions as in Theorem 1.1 (or Theorem 1.7), there exists some positive constant $C > 0$ such that for all index i, we have*

$$(4.1) \qquad\qquad\qquad\qquad \|Rm_{g_i}\|_{C^{k-2}} \leq C$$

Moreover, we have

$$(4.2) \qquad\qquad\qquad\qquad \|Rm_{g_i}\|_{C^{k+1}} \leq C$$

   Our strategy to prove Theorem 4.1 as follows: to get the uniform C^1 bound of the curvature tensor of metrics g_i, we will first prevent the boundary blow up in section 4.2. We then use this fact to help us to rule out the interior blow-up in section 4.3. After that we apply Theorems 3.2 and 3.4 to get uniform C^{k-2} of the curvature tensor and in section 4.4 applying the Bach flat and Q-flat equations to improve the regularity of the curvature tensor and get the uniform boundedness of their C^{k+1} norm.

   A property of Q-flat metrics g_i in our setting is that, under the assumption that the scalar curvature of their boundary metric $\hat{g}_i$ is non-negative, the scalar curvature of g_i is positive. This fact was first proved in Chang-Case [12]. For the convenience of readers, here we present also the proof of the result.

Lemma 4.2. [12] *Let $(X, \partial X, g^+)$ be a 4-dimensional conformal compact Einstein manifold and $g = e^{2w} g^+$ be the Fefferman-Graham's compactification, that is, $-\triangle_{g^+} w = 3$. Assume the representative of conformal infinity $h = g|_{T\partial X}$ has non-negative scalar curvature. Then the scalar curvature $R_g > 0$ is positive in X.*

Proof. Let $n = 3$ be the dimension of the boundary of X. As in [32], for any $s \in \mathbb{C}$ with $\text{Re}(s) > n/2$ and $s \notin n/2 + \mathbb{N}/2$ and for any given $f \in C^\infty(\partial X)$, we consider the following

Poisson equations

$$\begin{cases} -\triangle_{g^+} v_s - s(n-s)v_s = 0 & \text{in } X \\ v_s = F_s r^{n-s} + G_s r^s, & F_s, G_s \in C^\infty(X) \\ F_s = f & \text{on } \partial X \end{cases}$$

where r is some special defining function with respect to h some representative of the conformal boundary. Thus the unique Poisson operators can be defined as $\mathcal{P}(s)f := v_s$ provided $s(n-s) \notin \sigma_{pp}(-\triangle_{g^+})$ the essential spectrum of $-\triangle_{g^+}$. $\mathcal{P}(s)$ is meromorphic and could extend holomorphically across $s \notin n/2 + \mathbb{N}/2$ provided $s(n-s) \notin \sigma_{pp}(-\triangle_{g^+})$. Here we are interested in $f \equiv 1$ on ∂X. For $s = n+1$, the solution v_{n+1} satisfies (see [40]):
1) $v = r^{-1} + R_h r/4n(n-1) + O(r^2)$ is positive;
2) $v_{n+1}^2 - |\nabla_{g^+} v_{n+1}|^2 = R_h/n(n-1)$ on ∂X;
3) $-\triangle_{g^+}(v_{n+1}^2 - |\nabla_{g^+} v_{n+1}|^2) = -2|\nabla_{g^+}^2 v_{n+1} - v_{n+1} g^+|^2 \leq 0$;
where $R_h = \hat{R}$ is the scalar curvature of the metric h on the boundary. We call the compactified metric $g_{n+1}^* := v_{n+1}^{-2} g^+$ the Jack Lee's metric. Moreover, by the observation in [16], we have the scalar curvature $R_{g_{n+1}^*} = n(n+1)(v_{n+1}^2 - |\nabla_{g^+} v_{n+1}|^2)$. Assume $R_h \geq 0$ on the boundary, using the Maximum principle, Jack Lee [40] proved $v_{n+1}^2 - |\nabla_{g^+} v_{n+1}|^2 \geq 0$ and the first eigenvalue $\lambda_1(-\triangle_{g^+}) \geq \frac{n^2}{4}$, that is, the spectrum of $-\triangle_{g^+}$ is bounded below by $\frac{n^2}{4}$. Together with the observation in [16], we have also $R_{g_{n+1}^*} \geq 0$. As a consequence, using the result due to Graham-Zworski [32], v_s is holomorphic in $\text{Re}(s) > n/2$. Moreover, we have $v_s > 0$ provided $s \in (n/2, +\infty)$. For any $s \in (n/2, n) \cup (n, n+1]$, we consider the compactified metrics $g_s^* := y_s^2 g^+$ where $y_s := (v_s)^{1/(n-s)}$. For $s = n+1$, g_s^* is just Jack Lee's metric. When $s = n = 3$, we define $g_s^* := e^{2w} g^+$ where $w := -\frac{d}{ds} v_s|_{s=3}$. We note $v_3 \equiv 1$ in X. It follows from the Poisson equations that

$$-\triangle_{g^+} w = 3 \text{ in } X$$

Thus, g_s^* is an analytic family of metrics for $s \in (n/2, +\infty)$. Recall $J_{g_s^*} = \frac{R_{g_s^*}}{6}$ the trace of Schouten tensor in X and $J_h = \frac{R_h}{4}$ the one on the boundary ∂X. Set $\gamma = s - n/2$. The direct calculations lead to for any $s \in (n/2, n) \cup (n, \infty)$

$$J_{g_s^*} = \frac{1-2\gamma}{2} y_s^{-2}(|\nabla_{g_s^*} y_s|^2 - 1) \text{ and } J_{g_s^*}|_{\partial X} = \frac{2\gamma - 1}{2(\gamma - 1)} J_h$$

When $s = 3$, $J_{g_s^*} = e^{-2w}(1 - |\nabla_{g^+} w|^2)$ and $J_{g_s^*}|_{\partial X} = 2J_h$ (we set $y_s = e^w$). Moreover, we have the expansion near the boundary ∂X, $w = \log r + A + Br^3$ where $A = -\frac{1}{2} J_h r^2 + O(r^4)$ and $B = B_0 + B_2 r^2 + O(r^4)$ are even expansions. Here B_0 is Q_3 curvature on the boundary up to a multiple (see [25]). That is, g_3^* is just Fefferman-Graham compactification. We set $m = 3 - 2\gamma$. By the formula (6.6) in [12], we have the following equations for $J_s := J_{g_s^*}$ for $s > n/2$ and $s \neq 3$

(4.3) $y_s^{-m} \text{div}_{g_s^*}(y_s^{-m} \nabla_{g_s^*} J_s) = -c_1(s)|E|_{g_s^*}^2 + c_2(s)J_s^2$

where $E = Ric_{g_s^*} - \frac{R_{g_s^*}}{4} g_s^*$ is traceless Ricci tensor, $c_1(s) := \frac{2s-n-1}{(n-1)^2}$ and $c_2(s) := \frac{4s(n+1-s)}{(n+1)(2s-n-1)}$. When $s = 3$, such equation can be read as

(4.4) $\triangle_{g_s^*} J_s = -\frac{1}{2}|E|_{g_s^*}^2 + \frac{3}{2} J_s^2$

Recall $Q_4(g_s^*) \equiv 0$ when $s = 3$. By the continuous method, Case-Chang [12] show $J_s > 0$ in X provided $J_h \geq 0$ when $s > \frac{5}{2}$. For this purpose, we define the set $I = \{s \in (\frac{5}{2}, 4], J_t > 0 \text{ in } X \ \forall t \in [s, 4]\}$. Using the Lee's result ([40], see also Lemma 2.2[33]) $J_4 > 0$ in X. Thus $4 \in I$. As g_s^* is an analytic family, we claim that I is open. Indeed, assume $s \in I$, there exists

some $\varepsilon > 0$ such that for any $t \in (s - \varepsilon, s)$, $J_t > 0$ in X. Otherwise, there exists a increasing sequence $\{t_n\} \subset (\frac{5}{2}, 4]$ such that $t_n \uparrow s$ and a sequence of points $\{x_n\} \subset X$ such that $J_{t_n}(x_n) \leq 0$. Since $\bar{X}$ is compact, up to a subsequence, we have $x_n \to y \in \bar{X}$. As g_s^* is an analytic family, we have $J_s(y) \leq 0$. Hence $J_s(y) = 0$ and $y \in \partial X$ since $J_s > 0$ in X. Again from analyticity of g_s^*, we have some uniform Fermi coordinates $(z, r) \in \partial X \times [0, \varepsilon)$ for any metric $g_{t_n}^*$ around the boundary ∂X, that is $d_{g_{t_n}^*}((z, r), \partial X) = r$. Moreover, such geodesic tube $\partial X \times [0, \varepsilon)$ contains a common neighborhood of ∂X for any $g_{t_n}^*$. Set $V = \partial_r$ the unitary vector field in $\partial X \times [0, \varepsilon)$ whose restriction on the boundary ∂X is just insides normal vector field of the boundary. We write $x_n = (z_n, r_n)$ so that $(z_n, 0)$ is the orthogonal projection of x_n on the boundary ∂X. By the relation $J_s = \frac{2\gamma - 1}{2(\gamma - 1)} J_h$, we have $J_h(y) = 0$. Note when $n = 3$ and $s \in (\frac{5}{2}, 4]$, we have $c_1(s) \geq 0$ and $c_2(s) \geq 0$. Using the equations (4.3) and strong maximum principle, we have $V J_s(y) > 0$. On the other hand, $J_{t_n}(x_n) < 0$ and $J_{t_n}(z_n, 0) \geq 0$ since $J_h \geq 0$ on the boundary. Thus, there exists $\bar{r}_n \in (0, r_n)$ such that $V J_{t_n}(z_n, \bar{t}_n) < 0$. It is clear that $(z_n, \bar{t}_n) \to y$. By the analyticity of J_s in s and in space variables, we infer $V J_s(y) \leq 0$ which contradicts the fact $V J_s(y) > 0$. Therefore, I is open.

We use the idea in [33] to prove I is also closed. Let $s_n \in I \to s \in (\frac{5}{2}, 4]$. Again from the fact g_s^* is an analytic family, it follows that $J_s \geq 0$ in X. Using the equations (4.3) and strong maximum principle, we have $J_s > 0$ in X or $J_s \equiv 0$ in X. The idea due to [33] can rule out the latter case. More precisely, the latter case implies $|\nabla_{g_s^*} y_s| \equiv 1$ or equivalently, $|\nabla_{g^+} y_s| \equiv y_s$. Hence the positive function $\phi := (y_s)^s \in L^2$ satisfies

$$-\triangle_{g^+}\phi = s(n - s)\phi$$

which contradicts $\lambda(-\triangle_{g^+}) \geq \frac{n^2}{4}$. As a consequence, $I = (\frac{5}{2}, 4]$. Finally, we prove the desired result. $\qquad\square$

We call some **basic facts** for the conformal metrics $g_i = e^{2w_i} g_i^+ = v_i^{-2} g_i^+$.

Corollary 4.3. *We have*

$$(4.5) \qquad Ric_{g_i^+} = Ric_{g_i} + 2v_i \nabla^2_{g_i}(v_i^{-1}) + (v \triangle_{g_i}(v_i^{-1}) - 3v_i^2 |\nabla_{g_i}(v_i^{-1})|^2)g_i;$$

$$(4.6) \qquad -12 = R_{g_i^+} = v_i^{-2}(R_{g_i} + 6v_i \triangle_{g_i}(v_i^{-1}) - 12v^2 |\nabla_{g_i}(v_i^{-1})|^2);$$

$$(4.7) \qquad R_{g_i} = 6v_i^2(1 - |\nabla_{g_i}(v_i^{-1})|^2);$$

$$(4.8) \qquad Ric_{g_i} = -2v_i \nabla^2_{g_i}(v_i^{-1}).$$

$$(4.9) \qquad -R_{g_i} = 2v_i \triangle_{g_i}(v_i^{-1}).$$

Moreover, there holds

$$(4.10) \qquad \|\nabla_{g_i}(v_i^{-1})\| \leq 1.$$

Proof. The equalities (4.5) and (4.6) come from the conformal change. The equalities (4.7) and (4.9) are the results of (1.4). In fact, (1.4) is equivalent to

$$(4.11) \qquad v_i \triangle_{g_i^+} v_i^{-1} - |\nabla_{g_i} v_i^{-1}|^2 = -3.$$

On the other hand, we have

$$\triangle_{g_i} v_i^{-1} = v_i^2 \triangle_{g_i^+} v_i^{-1} + 2v_i^3 |\nabla_{g_i^+} v_i^{-1}|^2 = v_i^2 \triangle_{g_i^+} v_i^{-1} + 2v_i |\nabla_{g_i} v_i^{-1}|^2$$

Thus, we infer

$$(4.12) \qquad\qquad v_i^{-1} \triangle_{g_i} v_i^{-1} = -3 + 3|\nabla_{g_i} v_i^{-1}|^2$$

Going back to (4.6), we get (4.7) and (4.9).

From Lemma 4.2, we know $R_{g_i} \geq 0$. Hence, by (4.7), we get (4.10) and finish the proof of Corollary 4.3. $\qquad\qquad\square$

We now outline the proof of Theorem 4.1. For the sequence of boundary Yamabe metric $\hat{g}_i$ which is C^{k+3} compact, we will first establish the $C^{k-1,\alpha}$ compactness of the corresponding FG metrics g_i on the interior X; we will then apply a bootstrapping argument in section 4.4 to establish the $C^{k+2,\alpha}$ compactness of the metrics g_i.

We now notice that by the ε regularity result established in Theorem 3.2, to establish the $C^{k-1,\alpha}$ compactness of the metrics g_i, it suffices to prove the family is C^1 bounded. For this purpose, we see that the uniform boundedness of C^1 norm for the curvature $\|Rm_{g_i}\|_{C^1}$ induces the uniform boundedness of L^2 norm for the curvature Rm_{g_i} on the ball $B(p, r)$ and also the uniform boundedness of L^1 norm for the S-tensor S_i on the boundary $M \cap B(p, r)$. On the other hand, thanks to Bishop-Gromov volume comparison Theorem, we have the estimate $vol(B(p, r)) \leq Cr^4$ once the curvature is uniformly bounded for the metrics g_i. We will now begin to establish this assertion by a contradiction argument. Assume the family g_i is not C^1 bounded, or equivalently, the the C^1 norm of its curvature tends to infinity as i tends to infinity, that is,

$$\|Rm_{g_i}\|_{C^1} \to \infty \qquad \text{as } i \to \infty$$

We rescale the metric

$$\bar{g}_i = K_i^2 g_i$$

where there exists some point $p_i \in X$ such that

$$K_i^2 = \max\{\sup_X |Rm_{g_i}|, \sup_X |\nabla Rm_{g_i}|^{2/3}\} = |Rm_{g_i}|(p_i)(\text{ or } |\nabla Rm_{g_i}|^{2/3}(p_i))$$

We mark the point p_i as $0 \in X$. Thus, we have

$$|Rm_{\bar{g}_i}|(0) = 1 \text{ or } |\nabla Rm_{\bar{g}_i}|(0) = 1$$

We denote the corresponding defining function $\bar{v}_i^{-1} = K_i e^{w_i}$, that is, $\bar{g}_i = \bar{v}_i^{-2} g_i^+$.

We observe C^1 norm for the curvature $\|Rm_{\bar{g}_i}\|_{C^1}$ is uniformly bounded. Applying ε-regularity (Theorems 3.2 and 3.4), we obtain C^{k-2} norm for the curvature $\|Rm_{\bar{g}_i}\|_{C^{k-2}}$ is also uniformly bounded.

On the other hand, we claim there is no collapse for the family of metrics $\{\bar{g}_i\}$. To see this, we observe first the Sobolev inequality (3.4) implies $B_{\bar{g}_i}(x, r) \geq cr^4$ for all $x \in \bar{X}$ and $r < 1$. As the boundary M is totally geodesic, we can use the doubling argument along the boundary M to get a compact manifold without boundary $Y := X \bigcup_M X_1$ where X_1 is a copy of X with opposite orientation. On the closed manifold Y, we have a natural metric $(Y, \widetilde{g}_i)$ extending g_i on X. $(Y, \widetilde{g}_i)$ is a $C^{2,\alpha}$ closed Riemannian manifold with any $\alpha \in (0, 1)$. Thanks to a result of Cheeger-Gromov-Taylor [19], we have the uniform lower bound for any closed simple geodesic on $(Y, \widetilde{g}_i)$, which yields that both the interior injectivity radius and the boundary injectivity radius on X are uniformly bounded from below; thus we have proved the claim. We now notice by a version of Cheeger-Gromov-Hausdorff's compactness theorem for the manifolds with boundary (for the convenience of readers, we will give more details in the Appendix in Lemma A.2), modulo a subsequence and modulo diffeomorphism group, $\{\bar{g}_i\}$ converges in pointed Gromov-Hausdorff's sense for $C^{k-1,\alpha}$ norm to a non-flat limit metric $\bar{g}_\infty$ with totally geodesic boundary whose doubling metric is complete.

We now have two types of blow-up.

Type (I) : Boundary blow-up

If there exists two positive constants $C, \Lambda > 0$ and a sequence of points $p_i \in X$ such that $d_{\bar{g}_i}(\partial X, p_i) \leq C$ and $\liminf_i |Rm_{\bar{g}_i}(p_i)| + |\nabla Rm_{\bar{g}_i}(p_i)| = \Lambda > 0$. In such case, we translate p_i to 0 and do blow up analysis. Modulo a subsequence, we can assume

$$\lim_i |Rm_{\bar{g}_i}(0)| + |\nabla Rm_{\bar{g}_i}(0)| = \Lambda < \infty.$$

In section 4.2 below, we will do analysis on this type of blow-up and show that it does not occur.

Type (II) : Interior blow up

If blow -up of type I does not occur; that is, $d_{\bar{g}_i}(\partial X, p_i)$ tends to infinity as i tends to infinity. By applying some result in section 4.2 , together with the topological assumptions we have made in the statements of Theorems 1.1 and 1.7. ; we will show in section 4.3 below this type of blow up cannot occur either.

Finally in section 4.4, we will combine the results in section 4.2 and 4.3 , together with some bootstrapping arguments to finish the proof of Theorems 1.1 and 1.7.

We remark the basic facts stated in Corollary 4.3 for the pair (v_i, g_i) continue to hold for the pair $(\bar{v}_i, \bar{g}_i)$ w.r.t to the same base metric g_i^+.

4.2. **blow-up analysis on the boundary.**

Under the assumptions of Theorem 1.1, if we suppose $K_i \to \infty$, we have asserted in section 4.1 that $(X, \bar{g}_i)$ converges to (X_∞, g_∞) in $C^{k-1,\alpha}$ norm in Gromov-Hausdorff sense for some $k \geq 2$ and for all $\alpha \in (0, 1)$, which is a manifold with totally geodesic boundary.

The following Lemma is the main part of this section, actually the key argument in this paper.

Lemma 4.4. *Under the assumptions of Theorem 1.1 and assuming that $\bar{g}_i$ has the type I boundary blow up, g_∞ is conformal to hyperbolic space form.*

Proof. We divide the proof in several steps.

Step 1. Claim: There exists some $C > 0$ such that $\bar{v}_i^{-1} \geq C$ provided $d_{\bar{g}_i}(x, \partial X) \geq A$ and $\bar{v}_i(x)d_{\bar{g}_i}(x, \partial X) \leq C$ provided $0 \leq d_{\bar{g}_i}(x, \partial X) \leq A$ where $d_{\bar{g}_i}$ is the Riemann distance function w.r.t. the metric $\bar{g}_i$. Here A is some uniform constant smaller than the boundary injectivity radius.

Without loss of generality, we assume $A = 1$. Let us denote by $r_i(x) := d_{\bar{g}_i}(x, \partial X)$ the distance function to the boundary. From the Chapter 3 in [18], we have the elementary properties $|\nabla_{\bar{g}_i} r_i| = 1$ and $|\triangle_{\bar{g}_i} r_i|$ is bounded in the tube neighborhood of the boundary $\{x, r_i(x) \leq 1\}$ since the boundary is totally geodesic and $\bar{g}_i$ has the bounded curvature. We could take a cut-off function $\eta \in C_0^2(B(x, r_i(x)))$ such that $\eta \equiv 1$ in $B(x, r_i(x)/2)$, $\eta \equiv 0$ off $B(x, 3r_i(x)/4)$, $\|\nabla\eta\| \leq \frac{4}{r_i(x)}$ and $\|\triangle\eta\| \leq \frac{C}{r_i^2(x)}$. In fact, we fix a non-negative regular function ξ such that $\xi(t) = 1$ if $t \leq \frac{1}{2}$ and $\xi(t) = 0$ if $t \geq \frac{3}{4}$. Set $\eta(\cdot) = \xi(\frac{d_{\bar{g}_i}(\cdot, x)}{r_i(x)})$. Let z be a maximal point of $\bar{v}_i r_i \eta$.

At that point we have $\nabla_{\bar{g}_i} \bar{v}_i r_i \eta(z) = 0$ and $0 \geq \triangle_{\bar{g}_i}(\bar{v}_i r_i \eta)(z)$. We recall the equation

$$-\triangle_{\bar{g}_i} \bar{v}_i + \frac{R_{\bar{g}_i}}{6}\bar{v}_i = -2\bar{v}_i^3$$

which implies at the point z

$$
\begin{aligned}
0 &\geq \triangle_{\bar{g}_i}(\bar{v}_i r_i \eta)(z) = (2\bar{v}_i^3 + \frac{R_{\bar{g}_i}}{6}\bar{v}_i)r_i\eta - 2\frac{|\nabla_{\bar{g}_i}(r_i\eta)|^2 \bar{v}_i}{r_i\eta} + \bar{v}_i\triangle_{\bar{g}_i}(r_i\eta) \\
&\geq 2\bar{v}_i^3 r_i\eta - 2\frac{|\nabla_{\bar{g}_i}(r_i\eta)|^2 \bar{v}_i}{r_i\eta} + \bar{v}_i\triangle_{\bar{g}_i}(r_i\eta)
\end{aligned}
$$

Here we use the fact $R_{\bar{g}_j} \geq 0$. On the other hand, $|\nabla_{\bar{g}_i}(r_i\eta)|(z) \leq 5$ and $|\triangle_{\bar{g}_i}(r_i\eta)|(z) \leq \frac{C}{r_i(z)}$. Hence at the point z, we infer

$$
\begin{aligned}
0 &\geq 2(\bar{v}_i r_i\eta)^3(z) - (50(\bar{v}_i r_i\eta)(z)(r_i\eta)(z) + C(\bar{v}_i r_i\eta)(z)\eta(z)) \\
&\geq 2(\bar{v}_i r_i\eta)^3(z) - (50(\bar{v}_i r_i\eta)(z) + C(\bar{v}_i r_i\eta)(z))
\end{aligned}
$$

Hence, $\bar{v}_i r_i\eta(z) \leq C$ and as a consequence $\bar{v}_i(x)r_i(x) \leq C$.

As a consequence, we claim there exists some $C > 0$ such that $\bar{v}_i^{-1} \geq C$ provided $d_{\bar{g}_i}(x, \partial X) \geq 1$. Recall $-R_{\bar{g}_j}\bar{v}_i^{-1} = 2\triangle_{\bar{g}_i}(\bar{v}_i^{-1}) \leq 0$ since $R_{\bar{g}_j} \geq 0$. It follows from the maximal principle that $\bar{v}_i^{-1}$ atteint son minimum on the boundary in the set $\{x, r_i(x) \geq 1\}$. Hence, the desired claim yields.

Step 2. The limit metric g_∞ is conformal to A.H. Einstein manifold.

Assume $\bar{g}_i$ converge to some complete non compact manifold (X_∞, g_∞) with a boundary in the pointed Gromov Hausdorff sense (cf. [22] Theorem 6.35). Indeed, by the Sobolev inequality (the assumption (4) in Theorem 1.1), there exists some constant $c > 0$ and $r_0 > 0$ such that for any point p, $vol(B(p,r)) \geq cr^4$ for all $r \in (0, r_0)$. By the result due to Cheeger-Gromov-Taylor [19] (see also [22] Theorem 5.42), we have the lower bound estimate for the injectivity radius. Thus, the desired convergence follows from Gromov Hausdorff convergence. For this limiting metric, due to our assumption that $\hat{g}_i$ is a compact family, the boundary is $\mathbb{R}^3$ endowed with Euclidean metric. We denote by x_1 the geodesic defining function for the limiting AH metric g_∞^+ with respect to the boundary metric $\mathbb{R}^3$. Indeed, g_∞^+ is complete since (X_∞, g_∞) is complete and the defining function $f = \lim_i \bar{v}_i^{-1}$ satisfying $\|\nabla_{g_\infty} f\| \leq 1$ and $f(x) \geq C_1 \min(d(x, \partial X), 1)$. We now claim the metric $g_\infty^+ = f^{-2}g_\infty$ is Einstein with negative scalar curvature. For this purpose, we take the limit in equations (4.8) and (4.12) and get that

$$
\begin{aligned}
f\triangle_{g_\infty} f &= -3 + 3|\nabla_{g_\infty} f|^2 \\
Ric_{g_\infty} &= -2f^{-1}\nabla_{g_\infty}^2(f).
\end{aligned}
$$

Again from conformal change, it follows

$$Ric_{g_\infty^+} = Ric_{g_\infty} + 2f^{-1}\nabla_{g_\infty}^2(f) + (f^{-1}\triangle_{g_\infty}(f) - 3f^{-2}|\nabla_{g_\infty}(f)|^2)g_\infty$$

Together with the two previous relations, we infer

$$Ric_{g_\infty^+} = -3g_\infty^+.$$

Therefore, the desired claim follows.

Moreover, we note that (4.12), (4.11) and (1.4) are equivalent between them. We have seen (4.12) is true for the limit metric g_∞ by replacing v^{-1} by f. Hence (1.4) is also true for the limit metric g_∞^+, that is

$$-\triangle_{g_\infty^+} \log f = 3$$

Hence, g_∞ is Fefferman-Graham's compactification of g_∞^+. On the other hand, recall the S-tensor is a pointwise conformal invariant. From our assumption we have

$$\limsup_{r \to 0} \sup_i \sup_x \fint_{B(x,r)} |S_i| = 0,$$

we infer for the limiting metric and for any $r > 0$

$$\fint_{B(x,r)} |S_\infty| = 0$$

As a consequence, S-tensor for the limiting metric $S_\infty = 0$ on the boundary. Near the boundary, the limiting metric g_∞^+ on X_∞ is a locally hyperbolic space. This is a result due to Biquard [7] and Biquard-Herzlich [8] since the boundary metric is flat and the S-tensor vanishes.

Step 3. g_∞^+ is a locally hyperbolic space.
To see this, we work for the Einstein metric g_∞^+. By (2.5), the Weyl tensor satisfies

$$\triangle_{g_\infty^+} W = Rm * W$$

since the cotton tensor $C = 0$. Therefore,

$$|\triangle_{g_\infty^+} W| \leq C(x)|W|$$

where $C = C(x)$ is some regular function. Set

$$A := \{x \in X_\infty | \ \exists r > 0 \text{ such that } W \equiv 0 \text{ on } B(x,r)\}.$$

It is clear that A is an open set. From the step 1, we know the Weyl tensor W vanishes in a neighborhood of the boundary so that A is not empty. As $C(x)$ is a regular function, we can always bound it locally by some positive constant from above. Applying the well known unique continuation principle for this strong elliptic system (see [41]), A is also closed. As a consequence, $A = X_\infty$ since X_∞ is connected, that is, W is identically equal to 0. Hence, g_∞^+ is a hyperbolic space form.

Thus, we have finished the proof of Lemma 4.4. $\qquad\qquad\qquad\qquad\qquad\qquad\square$

We will now prove the analogue of Lemma 4.4 under the assumptions of Theorem 1.7 instead of Theorem 1.1, where the assumptions on S-tensor is replaced by the assumptions of T-curvature. To do so, we will first explore the relation between the S tensor and the T curvature.

The non-local terms S-tensor and T-curvature come from different considerations: the former one from boundary metric of Bach flat equations; the latter one from boundary term of the Gauss-Bonnet integrand. However, we will show they are linked in the sense that for the limiting metric g_∞, $T_\infty \equiv 0$ is equivalent to $S_\infty \equiv 0$. For conformally compact Einstein manifolds, when the boundary metric is Ricci flat, the fact that $T_\infty \equiv 0$ implies $S_\infty \equiv 0$, was proved earlier in [12, 13]. We will apply the same strategy of proof there, but as our limiting metric g_∞ is now defined on the non-compact manifold X_∞, we need to do some more careful analysis.

Recall under the assumptions of Theorem 1.7, if we suppose $K_i \to \infty$, we have asserted in section 4.1 that $(X, \bar{g}_i)$ converges to (X_∞, g_∞) in $C^{k-1,\alpha}$ norm in Gromov-Hausdorff sense for some $k \geq 5$ and for all $\alpha \in (0,1)$, which is a manifold with totally geodesic boundary (we have convergence in $C^{k+2,\alpha}$ norm in Gromov-Hausdorff sense by a bootstrapping argument in section 4.4). We now state our result.

Lemma 4.5. *For the limiting metric g_∞ is our setting, $T_\infty \equiv 0$ iff $S_\infty \equiv 0$.*

Proof. We will first show that $T_\infty \equiv 0$ implies $S_\infty \equiv 0$.

Recall the blow up boundary metric $h = \hat{g}_\infty$ on the boundary is flat. Let r be the special defining function related to h. Therefore, in a collar neighborhood $V = [0, \varepsilon) \times B$ where $B \subset M$ is some compact subset of conformal infinity, we have the expansion (see [29]) for $g_\infty^+ = r^{-2}(dr^2 + g_r)$

$$g_r = h + \kappa r^3 + o(r^3)$$

where g_r is a family of metrics on M and $\kappa = -\frac{2}{3}S$ by Remark 2.2. We also recall $\mathrm{tr}_h\kappa = 0$. If there is no confusion, we drop the index r for g_r. Moreover, we have (see [30])

$$0 = rg^{ij}\partial_r^2 g_{ij} - g^{ij}\partial_r g_{ij} - \frac{r}{2}g^{li}g^{mj}\partial_r g_{lm}\partial_r g_{ij}$$
$$0 = r\partial_r^2 g_{ij} - 2\partial_r g_{ij} - rg^{lm}\partial_r g_{il}\partial_r g_{jm} + \frac{r}{2}g^{lm}\partial_r g_{lm}\partial_r g_{ij} - (g^{lm}\partial_r g_{lm})g_{ij} - 2rRic(g_r)_{ij}$$

As shown in [13], by differentiating up to 5 times and evaluating at $r = 0$ for these relations, we infer

$$g_r = h + \kappa r^3 - \frac{\triangle_h\kappa}{10}r^5 + \frac{1}{2}(k_1 + \frac{5}{24}|\kappa|^2 h)r^6 + O(r^7)$$

where k_1 is the traceless part of the composition $(\kappa^2)_{ij} = \kappa_{im}\kappa_j^m$. The direct calculations lead to

$$\sqrt{\det g_\infty^+} = r^{-4}\sqrt{\det h}(1 - \frac{3}{32}|\kappa|^2 r^6 + o(r^6))$$
$$(g_r)^{ij} = h^{ij} - r^3\kappa^{ij} + O(r^5)$$

so that

(4.13) $$\triangle_{g_\infty^+} = (r\partial_r)^2 - 3r\partial_r - \frac{9}{16}r^6|\kappa|^2 r\partial_r + r^2\triangle_h - r^5\mathrm{div}\,(\kappa(\nabla_h\cdot,\cdot)) + O(r^7)$$

Here $O(r^7)$ means the bounded operators like $r\partial_r$ or in M with the coefficients in $O(r^7)$. Using the relation (4.8), the set of functions $\bar{v}_i^{-1}$ is compact in $C^{k+2,\alpha}$ space. Recall the set of metrics $\bar{g}_i$ is relatively compact in $C^{k+2,\alpha}$ space. Using the construction in [40], the special defining functions r_i related to g_i^+ forment a relatively compact set locally in $C^{k+2,\alpha}(V)$ space provided $k \geq 1$. Here ε is independent of i. We know (see[25])

$$\bar{v}_i^{-1} = r_i\exp(A + Br_i^3)$$

where $A = -\frac{R_{h_i}}{8}r_i^2 + O(r_i^4)$ and $B = c_3T(\bar{g}_i) + B_2r_i^2 + O(r_i^4)$ with c_3 an universal constant. We state both $\bar{v}_i^{-1}$ and r_i are relatively compact in $C^{k-1,\alpha}(V)$. By taking the limit as $i \to \infty$, we infer

$$f = r(1 + O(r^4))$$

since $R_{h_i} \to 0$ and under our assumption $T(\bar{g}_i) \to T_\infty \equiv 0$. Recall f solves (1.4), that is,

(4.14) $$-\triangle_{g_\infty^+}\log f = 3$$

We have the following expansion for $\log f$

$$\log f = \log r + f_4r^4 + f_5r^5 + f_6r^6 + f_7r^7$$

where $f_4, f_5, f_6 \in C^2(M \cap V)$ and $f_7 \in C^2(V)$. Together with (4.13) and (4.14), we deduce

$$\triangle_{g_\infty^+}\log f = -3 - \frac{9}{16}r^6|\kappa|^2 + 4f_4r^4 + r^6\triangle_h f_4 + 10f_5r^5 + 18f_6r^6 + O(r^7) = -3$$

As a consequence, we get

$$f_4 = f_5 = 0, f_6 = \frac{1}{32}|\kappa|^2$$

so that

$$f = r + \frac{1}{32}|\kappa|^2 r^6 + O(r^7)$$

Now we can calculate the scalar curvature for the metric $g_\infty = f^2 g_\infty^+$

$$R_{g_\infty} = 6(f^{-2} - |\nabla_{g_\infty^+} f^{-1}|^2) = 6(f^{-2} - (r\partial_r f^{-1})^2 - r^2|\nabla_{g_r} f|^2) = -\frac{9}{4}|\kappa|^2 r^4 + o(r^4)$$

that is, the scalar curvature is negative in some neighborhood of the boundary provided $\kappa \neq 0$. On the other hand

$$R_{g_\infty} = \lim R_{\bar{g}_i} \geq 0$$

Thus, $\kappa = 0$ in $V \cap M$. Now arguing as in the proof of the previous lemma, we prove E is hyperbolic space. As we need to do the expansion of g_r up to order 7, $\bar{g}_i$ should be in C^7. We know $\bar{g}_i$ is compact in $C_{loc}^{k+2,\alpha}$ norm as in section 4.4 below. Hence, we require $k + 2 \geq 7$, that is, $k \geq 5$. Thus $T_\infty \equiv 0$ implies $S_\infty \equiv 0$. Conversely, by lemma 4.4, $S_\infty \equiv 0$ implies g_∞ is conformal to hyperbolic space form. Using the proof in Proposition 4.8 below, g_∞ is flat. Hence $T_\infty \equiv 0$. Thus we have finished the proof of Lemma 4.5. $\qquad\square$

Lemma 4.6. *Under the assumptions of Theorem 1.7 and assuming that $\bar{g}_i$ has the type I boundary blow up, g_∞ is conformal to hyperbolic space form.*

Proof. We will show under the assumption on T curvature on Theorem 1.7, T_∞ vanishes, hence by the Lemma 4.5, the S_∞-tensor on the boundary also vanishes and thus the same proof as lemma 4.4 can be applied.

To see T_∞ vanishes, we go through the similar proof as lemma 4.4. Denote 0 the marked point. Fixing $r > 0$, we have

$$\fint_{B_{\bar{g}_i}(0,r)} T(\bar{g}_i) = \fint_{B_{g_i}(0,K_i^{-1}r)} T(g_i)$$

By our assumption on T, we infer

$$\fint_{B_{g_\infty}(0,r)} T_\infty = \lim_i \fint_{B_{\bar{g}_i}(0,r)} T(\bar{g}_i) \geq 0$$

since $K_i^{-1}r \to 0$. On the other hand, the scalar curvature for the limiting metric g_∞ is non-negative and vanishes on the boundary which implies that T-curvature is non-positive, that is, $T_\infty = \frac{1}{12}\frac{\partial R_\infty}{\partial n} \leq 0$. Thus, the integral of T_∞-curvature over the geodesic ball $B(0,r)$ vanishes for any $r > 0$. As a consequence, the T_∞ on the boundary for the limiting metric g_∞ is equal to zero. $\qquad\square$

Remark 4.7. *As in Theorem 1.1 and Corollaries 1.2-1.4, we have S-tensor S_i is uniformly bounded in L^1. Thus, we can normalize in C^0 norm for the curvature and going through the blow up analysis. That is, we can assume $k \geq 1$ instead of $k \geq 2$ to reach the same results in Theorem 1.1 and Corollaries 1.2-1.4. Also in Theorem 1.7 and Corollaries 1.9-1.11, if we assume in addition that S-tensor S_i is uniformly bounded in L^1, we can assume $k \geq 4$ in the place of $k \geq 5$ to conclude the same results there.*

We now conclude this session by showing the boundary blow up can not occur by establishing a Liouville type theorem that, under the assumptions of theorems 1.1 or 1.7, g_∞ is the flat metric.

Proposition 4.8. *Under the same assumptions as in Theorem 1.1 or Theorem 1.7, there is no blow up on the boundary.*

Proof. From Lemma 4.4, we know the limiting manifold (X_∞, g_∞^+) is locally hyperbolic space. We now work on the limit metric g_∞. For simplicity, we will omit the index ∞. We denote $\widetilde{g}^+$ standard hyperbolic space with the upper half space model. As $\widetilde{g}^+ = g^+$ in a neighborhood of the boundary $\{x_1 = 0\}$, we can extend this local isometry to a covering map $\pi : \widetilde{g}^+ \to g^+$. We write

$$g_1 = e^{2w_1}\widetilde{g}^+ \text{ and } g_2 = e^{2\widetilde{w}_2}g^+$$

where g_1 is the standard euclidean metric and $g_2 = g_\infty$ the limit FG metric. With the help of the covering map π, we have $\pi^*g_2 = e^{2w_2}g^+$ where $w_2 = \widetilde{w}_2 \circ \pi$. We have for $i = 1, 2$

$$-\triangle_{\widetilde{g}^+}w_i = 3$$

and

$$w_1 = \log x_1$$

Remind x_1 is the geodesic defining function w.r.t. the flat boundary metric. We write $\pi^*g_2 = e^{2w_2}\widetilde{g}^+ = e^{2w_2 - 2w_1}g_1 := e^{2u}g_1$ where $u = w_2 - w_1$. The semi-compactified metric g_2 (or π^*g_2) has flat Q_4 and the boundary metric of g_2 is the euclidean 3-space and totally geometric. Thus u satisfies the following conditions

(4.15)
$$\begin{cases} \triangle^2 u = 0 & \text{in } \mathbb{R}^4_+ \\ -\triangle u - |\nabla u|^2 \geq 0 & \text{in } \mathbb{R}^4_+ \\ u = \nabla u = \triangle u = 0 & \text{on } \partial\mathbb{R}^4_+ \end{cases}$$

The first equation comes from the flat Q_4 curvature and second one from the non-negative scalar curvature. As g_2 on the boundary is euclidean, u on the boundary is a constant 0. On the other hand, we know by properties of g_1 and g_2 that

(4.16)
$$x_1 \triangle u = 2\partial_1 u.$$

since $\triangle_{g^+}u = 0$, that is, $x_1^2 \sum_{i=1}^4 \partial_{x_i}(\frac{1}{x_1^2}\partial_{x_i}u) = 0$. Hence on the boundary, $\partial_1 u = 0$ so that $\nabla u = 0$. On the other hand, it follows from Lemma 2.3 the restriction of the scalar curvature vanishes on the boundary so that $-\triangle u - |\nabla u|^2 = 0$. This yields $\triangle u = 0$ on the boundary. As $-\triangle u$ is harmonic and non-negative, it follows from a result due to H.P.Boas and R.P. Boas [9],

$$-\triangle u = ax_1$$

with some $a > 0$. Hence, by (4.16), $\partial_1 u = -ax_1^2/2$ so that $|\nabla u|^2 \geq |\partial_1 u|^2 = a^2 x_1^4/4$. Hence $-\triangle u - |\nabla u|^2 = ax_1 - a^2 x_1^4/4 \geq 0$ if and only if $a = 0$. From (4.15), $|\nabla u|^2 \leq -\triangle u = 0$, u is a constant. Together with the boundary condition, $u \equiv 0$. That is, g_2 is a flat metric. However, by the blow up arguments, $g_2 = g_\infty$ is not flat since either its curvature or the derivative of its curvature has been normalized to be one at a point in X, which yields the desired contradiction. Thus there is no boundary blow up and we have finished the proof of Proposition 4.8. $\square$

4.3. blow-up analysis in the interior. Now we will do the blow-up analysis in the interior and want to prevent this to happen with the help of Proposition 4.8.

Recall for the interior blow up, we have
(1) $\max\{|Rm_{\bar{g}_i}|(0), |\nabla Rm_{\bar{g}_i}|(0)\} = 1$;
(2) $\lim_i d_i(0, \partial X) = \infty$;
We now claim we also have
(3) For any $C > 0$, $\sup_{d_i(x,\partial X)\leq C} |Rm_{\bar{g}_i}(x)| + |\nabla Rm_{\bar{g}_i}(x)| \to 0$.

We now prove the claim by a contradiction argument. Assume otherwise, then for some fixed $C > 0$, modulo a subsequence, we have

$$\sup_{d_i(x,\partial X) \leq C} |Rm_{\bar{g}_i}(x)| + |\nabla Rm_{\bar{g}_i}(x)| \to \alpha > 0$$

Let $p_i \in \{d_i(x, \partial X) \leq C\}$ such that

$$|Rm_{\bar{g}_i}(p_i)| + |\nabla Rm_{\bar{g}_i}(p_i)| = \sup_{d_i(x,\partial X) \leq C} |Rm_{\bar{g}_i}(x)| + |\nabla Rm_{\bar{g}_i}(x)|$$

We mark the point p_i as the origin. By pointed Hausdorff-Gromov's convergence, $(X, \bar{g}_i, p_i)$ converges to a non-flat manifold with the totally geodesic boundary whose doubling manifold is complete. This is a boundary blow up, which contradicts Proposition 4.8. Therefore, the desired property (3) follows.

Lemma 4.9. *Under the assumptions as in either Theorem 1.1 or Theorem 1.7, we assume $K_i \to \infty$ and $\bar{g}_i$ has the type II interior blow up. Then the limiting metric g_∞ is Ricci free and non-flat.*

Proof. We divide the proof into 3 steps.

Step 1. Let $i_\partial(\bar{g}_i)$ be the boundary injectivity radius. We claim $i_\partial(\bar{g}_i) \to \infty$.

We fix a large $B > 1$ and consider the scaling metric $\widetilde{g}_i = B^{-2}\bar{g}_i$. Thus, $\sup_{d_{\widetilde{g}_i}(x,\partial X) \leq 1} |Rm_{\widetilde{g}_i}(x)| \to 0$. We argue as before: we state the Sobolev inequality (3.4) implies $B(x, r) \geq cr^4$ for all $x \in \{d_{\widetilde{g}_i}(x, \partial X) \leq 1\}$ and $r < 1$. As the boundary M is totally geodesic, we can use the doubling argument along the boundary M to get a compact manifold without boundary $Y := X \bigcup_M X_1$ where X_1 is a copy of X with opposite orientation. On the closed manifold Y, thanks to a result of Cheeger-Gromov-Taylor [19], we have the uniform lower bound for any closed simple geodesic on Y, which yields $i_\partial(\widetilde{g}_i) \geq \alpha > 0$ for some $\alpha > 0$ independent of i. Hence $i_\partial(\bar{g}_i) = Bi_\partial(\widetilde{g}_i) \geq B\alpha$. This gives the desired claim.

Step 2. We claim that there exists some positive constant $C > 0$ such that for any $A > 1$ there exists some positive entire number $N \in \mathbb{N}$ such that for all $j > N$ for all $x \in X$ with $r_i(x) = d_{\bar{g}_i}(x, \partial X) < A$, we have

$$\bar{v}_i^{-1}(x) \geq Cr_i(x)$$

The proof of the claim is as same as that one of the step 1 in Lemma 4.4.

Step 3. The limiting metric g_∞ is of Ricci flat.

To see so, we first claim $\lim \bar{v}_i^{-1}(0) = \infty$. For this purpose, we note first by equation (4.8), $2\triangle_{\bar{g}_i}(v_i^{-1}) = -R_{\bar{g}_j}v_i^{-1} \leq 0$, thus by maximal principle, for each fixed $A > 1$ and for all sufficiently large j one has $\bar{v}_i^{-1}(0) \geq \min_{d_{\bar{g}_i}(x,\partial X)=A} \bar{v}_i^{-1} \geq CA$. Thus, $\liminf \bar{v}_i^{-1}(0) \geq CA$. As A is arbitrary, we infer $\liminf \bar{v}_i^{-1}(0) = \infty$ which yields the desired result. Thus for any compact set $B \subset X_\infty$, $\bar{v}_i^{-1}$ is uniformly unbounded as $\|\nabla_{\bar{g}_i}(\bar{v}_i^{-1})\| \leq 1$. Apply (4.7), we have conclude the scalar curvature of g_∞ is zero, since g_∞ is also Q flat (2.6), we conculde g_∞ is also Ricci flat. Thus we have established Lemma 4.9.

$\square$

We also now apply a recent result due to Cheeger-Naber [20], to show the Weyl tensor of the limiting metric g_∞ also vanishes. First, we recall the result of Cheeger-Naber (Theorem 1.13 in [20]).

Lemma 4.10. *There exists $C = C(v)$ such that if 4-dimensional Riemannian manifolds X^4 satisfies $|Ric_{X^4}| \leq 3$ and $Vol(B_1(p)) > v > 0$ for some point $p \in X$, then*

$$\int_{B_1(p)} |Rm|^2 \leq C(v)$$

As a direct consequence, we have the following result.

Lemma 4.11. *Let E be a 4-dimensional complete non-compact Ricci flat manifold. Assume*

$$V(B(o,t)) \geq Ct^4, \ \forall t > 0$$

for some positive constant $C > 0$. Then we have

$$\int_E |W|^2 < \infty.$$

Proof. Let g be the metric on E. We consider the scaling metric $g_i = i^{-2}g$ for all $i \in \mathbb{N}$. It is clear that g_i is still a complete Ricci flat metric and $Vol(B_1(o)) \geq C$ for each g_i. Using Cheeger-Naber's result, we deduce for the metric g

$$\int_{B_1(o)} |W|_{g_i}^2 = \int_{B_i(o)} |W|_g^2 \leq C$$

where C is a constant independent of i. Letting $i \to \infty$, the desired estimate follows. $\qquad\square$

We now recall a result due to Shen-Sormani [44].

Lemma 4.12. *Let E be a 4-dimensional complete non-compact Ricci flat manifold. Assume E is not flat and oriented. Then for any abelian group G, $H_3(E,G)$ is trivial. Moreover, for any abelian group G*

$$Tor(H_2(E,\mathbb{Z}),G) = 0.$$

In particular, $H_2(E,\mathbb{Z})$ has no elements of finite order.

We will also use the following result on the topology of X.

Lemma 4.13. *If we glue X and the unit 4-ball along the boundary $\mathbb{S}^3$ and denote $\widetilde{X} := X \bigcup_{\mathbb{S}^3} B^4$. Then $\widetilde{X}$ is a homology 4-sphere.*

Proof. We note X is a non-compact manifold. By Proposition 3.29 in [35], we infer that $H_4(X,\mathbb{Z}) = 0$. We use Mayer-Vietoris exact sequences for B^4 and $\overline{X}$

$$\cdots \to H_i(\mathbb{S}^3,\mathbb{Z}) \to H_i(B^4,\mathbb{Z}) \oplus H_i(\overline{X},\mathbb{Z}) \to H_i(\widetilde{X},\mathbb{Z}) \to \cdots$$

We know $H_1(\mathbb{S}^3,\mathbb{Z}) = H_2(\mathbb{S}^3,\mathbb{Z}) = 0, H_0(\mathbb{S}^3,\mathbb{Z}) = H_3(\mathbb{S}^3,\mathbb{Z}) = \mathbb{Z}$ and $H_i(B^4,\mathbb{Z}) = 0$ for $1 \leq i \leq 4$ and $H_i(X,\mathbb{Z}) = 0$ for $i = 1,2,4$ so that $H_i(\widetilde{X},\mathbb{Z}) = 0$ for $i = 1,2$ and $H_4(\widetilde{X},\mathbb{Z}) = \mathbb{Z}$. On the other hand, $\widetilde{X}$ is a connected 4-manifold. Thus $H_0(\widetilde{X},\mathbb{Z}) = \mathbb{Z}$. As X is oriented, $\widetilde{X}$ is closed and oriented. Using the universal coefficient theorem for cohomology (Theorem 3.2 in [35]) , we infer

$$H^1(\widetilde{X},\mathbb{Z}) = 0.$$

Applying the Poincaré duality theorem (Theorem 3.30 in [35]), we get

$$H_3(\widetilde{X},\mathbb{Z}) = 0.$$

Hence, it follows from Theorem 3.26 in [35] that $H_0(\widetilde{X}, \mathbb{Z}) = H_4(\widetilde{X}, \mathbb{Z}) = \mathbb{Z}$ and $H_i(\widetilde{X}, \mathbb{Z}) = 0$ for $i \neq 0, 4$, that is, $\widetilde{X}$ ia a homology 4-sphere. Thus we have finished the proof of Lemma 4.13. $\qquad\square$

We recall a result due to Crisp-Hillman ([23] Theorem 2.2).

Lemma 4.14. *Let X be a closed oriented homology 4-sphere and $\mathbb{S}^3/\Gamma$ be a spherical 3-manifold with Γ some finite group of $SO(4)$. Assume $\mathbb{S}^3/\Gamma$ is embedded in X. Then $\Gamma = \{1\}$ or $\Gamma = Q_8$(quaternion group) or Γ the perfect group (that is, $\mathbb{S}^3/\Gamma$ is a homology 3-sphere).*

Now we will rule out the interior blow-up.

Proposition 4.15. *Under the same assumptions as in Theorem 1.1, there is no blow up in the interior.*

Proof. We argue by contradiction. We assume there is blow up in the interior. Let $\bar{g}_i$ be renormalized metrics as before and we denote $X_i = (X, \bar{g}_i)$ for simplicity. We now choose $E = X_\infty$ the blow up metric in the interior. Hence, it is Ricci flat. Moreover, on each X_i, and for any Lipschitz function with compact support f

$$\|f\|_{L^4}^2 \leq C(\|\nabla f\|_{L^2}^2 + \frac{1}{6}\int R f^2)$$

As C is uniformly bounded from below, we can pass in the limit, that is,

$$\|f\|_{L^4(X_\infty)}^2 \leq C(X_\infty)\|\nabla f\|_{L^2(X_\infty)}^2$$

since the limiting metric is Ricci flat. This means there is no collapse on E.

We divide the proof in two cases.

Case 1: $H_2(E, \mathbb{R}) \neq 0$.

We use mayer-Vietoris exact sequences for E and $\overline{X \setminus E}$

$$\cdots \to H_2(\mathbb{S}^3/\Gamma, \mathbb{R}) \to H_2(E, \mathbb{R}) \oplus H_2(\overline{X \setminus E}, \mathbb{R}) \to H_2(X, \mathbb{R}) \to \cdots$$

We know $H_2(\mathbb{S}^3/\Gamma, \mathbb{R}) = 0 = H_2(X, \mathbb{R})$. This contradicts the fact that the above sequence is exact.

Case 2: $H_2(E, \mathbb{R}) = 0$.

From the result in [6], E is then ALE of order 3, that is, at the infinity, $\partial_\infty E = \mathbb{S}^3/\Gamma$ is spherical 3-manifold where Γ is some finite group of $SO(4)$. By Lemma 4.12, the third betti number $b_3 = 0$. Let $\widetilde{E}$ be the universal cover of E. Thus $\widetilde{E}$ is also Ricci flat. By Bishop-Gromov volume comparison theorem, $V(B(o, r)) \leq C_1 r^4$. As a consequence, the fundamental group of E is finite. Since $H_1(E, \mathbb{Z})$ is the abelization of the fundamental group, $H_1(E, \mathbb{Z})$ is finite and the first betti number $b_1 = 0$.

We note $H_2(E, \mathbb{R}) = 0$. From Lemma 4.12, $H_3(E, \mathbb{Z}) = 0$. On the other hand, it is clear $H_4(E, \mathbb{Z}) = 0$ since E is an open manifold (see [35] Proposition 3.29). Thus, the second, third and forth betti numbers vanish $b_2 = b_3 = b_4 = 0$, and the Euler characteristic number $\chi(E) = 1$.

We know a spherical 3-manifold is Seifert fibred 3-manifold and $\widetilde{X}$ is a homology 4-sphere. By the result due to Crisp-Hillman Lemma 4.14, we know the boundary of E at the infinity is S^3/Γ with $\Gamma = \{1\}$ or $\Gamma = Q_8$ or Γ the perfect group.

From Gauss-Bonnet formula, we have

$$\frac{1}{8\pi^2}\int_E (|W_+|^2 + |W_-|^2) = \chi(E) - \frac{1}{|\Gamma|} = 1 - \frac{1}{|\Gamma|}$$

On the other hand, we have the signature of E is trivial since $H_2(E,\mathbb{R}) = 0$. Using the signature formula

$$0 = -\tau(E) = \frac{1}{12\pi^2}\int_E (|W_+|^2 - |W_-|^2) - \eta(\mathbb{S}^3/\Gamma)$$

where $\eta(\mathbb{S}^3/\Gamma)$ is the eta invariant.

When $\Gamma = \{1\}$, it follows from the Gauss-Bonnet formula that

$$W = 0$$

Hence E is flat since E is Ricci flat. This contradicts the non-flatness of E. Or alternatively, Bishop-Gromov comparison theorem yields that a ALE Ricci flat manifold asymptotic to $\mathbb{R}^4$ is flat, which yields the desired result.

When $\Gamma = Q_8$, we know $\eta(\mathbb{S}^3/\Gamma) = \frac{3}{4}$. Hence, we have

$$\frac{1}{8\pi^2}\int_E (|W_+|^2 + |W_-|^2) = \frac{7}{8}$$

and

$$\frac{1}{8\pi^2}\int_E (|W_+|^2 - |W_-|^2) = \frac{9}{8}$$

The above two equalities leads to a contradiction.

When Γ is the perfect group, that is, the binary icosohedral group of order 120, then $\eta(\mathbb{S}^3/\Gamma) = -1 + \frac{1}{|\Gamma|} + \frac{1079}{360}$ ([27]). Similarly, from the above two formulas, we get

$$\frac{1}{8\pi^2}\int_E (|W_+|^2 + |W_-|^2) = \frac{119}{120}$$

and

$$\frac{1}{8\pi^2}\int_E (|W_+|^2 - |W_-|^2) = \frac{361}{120}$$

This is a contradiction.

Thus all cases can not happen, thus there is no interior blow up and we have finished the proof of Proposition 4.15. $\square$

Remark 4.16. *The same method permits us to do blow up analysis on 4-dimensional homological sphere manifolds without boundary. In particular, one could get the compactness result of Einstein metrics (or the other canonical metrics) on such manifolds under suitable assumptions.*

4.4. **Proof of Theorem 4.1.**

Proof of Theorem 4.1. Propositions 4.8 and 4.15 yields the boundness of $\|Rm_{g_i}\|_{C^1}$. Together with Theorem 3.2, we prove the C^{k-2} norm for the curvature is uniformly bounded. To handle the C^{k+1} norm of the curvature, we recall first the curvature tensor satisfies some elliptic PDE with the Dirichlet boundary conditions

(4.17)
$$\begin{cases} \triangle R = R^2 - 3|Ric|^2 & \text{in } X \\ R = 3\hat{R} & \text{on } M \\ \triangle A - \frac{1}{6}\nabla^2 R = Rm * A & \text{in } X \\ A_{\alpha\beta} = \hat{A}_{\alpha\beta}, A_{\alpha n} = 0, A_{n\alpha} = 0, A_{nn} = \frac{\hat{R}}{4} & \text{on } M \\ \triangle W = Rm * W + g * W * A & \text{in } X \\ W = 0 & \text{on } M \end{cases}$$

The desired result follows from the classical elliptic regularity theory and Cheeger-Gromov-Hausdorff convergence theory (Lemma A.2 in the Appendix). This concludes the proof of Theorem 4.1. □

5. Proof of the results in Section 1

Proof of Theorem 1.1. We have already proved the family of metrics g_i has the bounded curvature in C^{k+1} norm in Section 4. The uniform Sobolev inequality holds for the family metric g_i by the assumptions in Theorem 1.1, which implies, as in Section 4, for all i, for all $x \in \bar{X}$, we have $vol(B_{g_i}(x, 1)) \geq C > 0$ for some constant $C > 0$ independent of i, x, that is, there is non-collapse for the volume. Working on the doubling manifold, it follows from a result of Cheeger-Gromov-Taylor [19], we have the uniform lower bound for any closed simple geodesic on the doubling manifolds, which yields that both the interior injectivity radius and the boundary injectivity radius uniformly lower bound on X. By Cheeger-Gromov-Hausdorff compactness theory, to prove the compactness of metrics g_i, it suffices to prove their diameters are uniformly bounded from above. We will prove this fact by contradiction. We divide the proof in 4 steps.

Step 1. Without loss of generality, we suppose the boundary injectivity radius is bigger than 1. There exists some $C > 0$ such that $v_i^{-1} \geq C$ provided $d_{g_i}(x, \partial X) \geq 1$ and $v_i(x)d_{g_i}(x, \partial X) \leq C$ provided $0 \leq d_{g_i}(x, \partial X) \leq 1$. Thus the limit metric is conformal to an asymptotic hyperbolic Einstein manifold. The claim can be proved in the same way as in the proof of Lemma 4.4.

Assume the diameter of g_i is unbounded, then g_i converges to some non-compact metric g_∞ on manifold X_∞ with totally geodesic boundary in the Cheeger-Gromov-Hausdorff sense, whose doubling is complete. Note that by our assumption of Theorem 1.1 that the boundary metric $\{\hat{g}_i\}$ of this family $\{g_i\}$ is a compact family.

Step 2. There exists some constant $C > 0$ independent of i such that $\int |Rm_{g_i}|^2 \leq C$. From the relation (4.9), we infer

$$\int_X v_i^{-1} R_{g_i} = -2 \int_X \triangle_{g_i}(v_i^{-1}) = -2 \oint_{\partial X} \frac{\partial v_i^{-1}}{\partial n} = 2vol(\partial X, \hat{g}_i)$$

since $\frac{\partial_i v_i^{-1}}{\partial n} = -1$ on the boundary. Thus, with the help of step 1, we get for some given constant $C > 0$

$$\int_{d_{g_i}(x, \partial X) \geq 1} R_{g_i} \leq C \tag{5.1}$$

On the other hand, the boundary metric $(\partial X, h_i)$ is a compact family and also g_i has uniform bound for the curvature tensor. Hence

$$\int_{d_{g_i}(x, \partial X) \leq 1} R_{g_i} \leq C \tag{5.2}$$

Combining (5.1) and (5.2), we obtain

$$\int_X R_{g_i} \leq C.$$

which implies

$$\int_X R_{g_i}^2 \leq C \tag{5.3}$$

since the scalar curvature R_{g_i} is uniformly nounded. Using the free Q-curvature condition (2.6), we deduce

$$\int_X |E_{g_i}|^2 = \frac{1}{12}\int_X R_{g_i}^2 - \frac{1}{3}\oint_{\partial X} \frac{\partial R_{g_i}}{\partial n}$$

As we have the compactness of boundary metric and the curvature tensor is bounded in C^k norm, we get the uniform bound for L^2 norm with the Ricci tensor. Now, thanks of the Gauss-Bonnet-Chern formula, we have

$$\chi(X) = \frac{1}{32\pi^2}\int_X \left(|W_{g_i}|^2 + \frac{1}{6}R_{g_i}^2 - 2|E_{g_i}|^2\right)$$

since the boundary is totally geodesic. This yields the L^2 bound for the Weyl tensor. Therefore, the desired result yields.

Step 3. We denote $f = \lim v_i^{-1}$, assuming X_∞ is non-compact, we claim

$$\lim_{x\to\infty} \frac{f(x)}{d_{g_\infty}(x,\partial X_\infty)} = 1 \tag{5.4}$$

To prove the claim, we fix a point $0 \in \partial X_\infty$ and use the distance function $r = d_{g_\infty}(x,0)$. Note ∂X_∞ is a compact set so that $\displaystyle\lim_{x\to\infty} \frac{r(x)}{d_{g_\infty}(x,\partial X_\infty)} = 1$. We remark on the limiting metric one has always the Sobolev inequality, that is, for any compactly supported Lipschitz function U

$$\left(\int_{X_\infty} U^4\right)^{1/2} \le C\int_{X_\infty} \left(|\nabla_{g_\infty}U|^2 + \frac{1}{6}R_\infty U^2\right)$$

From step 1, $\int R_\infty^2 < \infty$. Thus, we can find r_0 such that

$$\int_{\{x,r(x)\ge r_0\}} R_\infty^2 \le \varepsilon,$$

where ε is a small constant appeared in Theorem 3.4. Recall from Theorem 3.4 and from the non-negative scalar curvature and also the relations (A.1) and (4.7)

$$\|\nabla_{g_\infty}f\| \le 1 \tag{5.5}$$

$$|\nabla_{g_\infty}(f)|^2 = 1 - o(1) \tag{5.6}$$

$$(\nabla_{g_\infty}^2(f))_i^{\ j} = o(r^{-1}) \tag{5.7}$$

In fact, we have from (5.5) that $f(x) \le r(x)$. On the other hand, it follows from Theorem 3.4

$$Rm = o(r^{-2})$$

Together with the relations (4.7) and (4.8), we obtain (5.6) and (5.7).

Now we consider vector field $-\nabla_{g_\infty}f(x)$. For any $\varepsilon > 0$, there exists some $A > 0$ such that for all $r(x) > A$

$$1 - \varepsilon \le |\nabla_{g_\infty}(f)| \le 1$$

Set $S_t := \{x; r(x) = t\}$. Define $m(t) := \inf_{S_t} f$. We remark $m(t) \ge 0$ since $f \ge 0$. We consider along the flow F_t of $-\nabla_{g_\infty}f(x)$. We have $F_r(S_{r+A}) \subset B(0,2r+A)\setminus B(0,A)$ since $|\nabla_{g_\infty}(f)| \le 1$. On the other hand

$$m(2r+A) \le \inf_{S_{r+A}} f(F_r(S_{r+A})) \le \inf_{S_{r+A}} f(S_{r+A}) - r(1-\varepsilon),$$

which implies

$$m(r + A) - r(1 - \varepsilon) \geq m(2r + A) \geq 0$$

However, $f(x) \leq r$ for all $x \in S_r$ since $|\nabla_{\bar{g}_\infty}(f)| \leq 1$. Gathering the above facts, we get the desired claim.

Step 4. A contradiction.

We know the Sobolev inequality is still true for the limiting metric, that is, for any Lipschitz function U compactly supported in $\{x, r(x) \geq r_0\}$, we have

$$(\int_{X_\infty} U^4)^{1/2} \leq C \int_{X_\infty} |\nabla_{g_\infty} U|^2$$

For large s, we fix a point x with $r(x) = r_0 + s$ and consider the function $U(y) = s - d(y, x)$ if $d(y, x) \leq s$, otherwise $U(y) = 0$. From the above inequality, we get

$$\frac{s}{2} vol(B(x, \frac{s}{2}))^{\frac{1}{4}} \leq vol(B(x, s))^{\frac{1}{2}}$$

so that

$$vol(B(0, 2(r_0 + s))) \geq vol(B(x, \frac{s}{2})) \geq C(2(r_0 + s)))^4$$

We denote $s_1 = 2(r_0 + s)$. By Proposition 3.4 in [18], Proposition 4.5 in [47] and the Courant-Lebesgue Lemma, there exists some $s_2 \in (s_1/2, s_1)$ such that

$$vol(\partial B(0, s_2)) \geq c s_1^3$$

and

$$vol(\partial B(0, s_2)) = vol(\partial B(0, s_2) \setminus N),$$

where N is the set of cut-locus w.r.t. 0. Set $w = s_2$. Integrating on $B(0, w)$, we get

$$0 \geq \int_{B(0,w)} \triangle_{g_\infty}(f) = \oint_{\partial B(0,w)} \langle \nabla_{g_\infty}(f), \nabla_{g_\infty} r \rangle = \oint_{S_w} \langle \nabla_{g_\infty}(f), \nabla_{g_\infty} r \rangle - vol(\partial X_\infty)$$

Hence we could find a regular point x on the sphere $\partial S(0, w)$ such that

$$\langle \nabla_{g_\infty} f, \nabla_{g_\infty} r \rangle \leq \varepsilon$$

for some small ε since $vol(\partial B(0, w)) \geq c w^3$. Taking normalized radial geodesic connecting this point $\gamma(t)$ to the boundary ∂X, we set $l(t) = f(\gamma(t))$. Then

$$l'(s) = \langle \nabla_{g_\infty} f, \nabla_{g_\infty} r \rangle, \text{ and, } l''(s) = \nabla^2_{g_\infty} f(\nabla_{g_\infty} r, \nabla_{g_\infty} r) = o(t^{-1})$$

provided $s \in [t/2, t]$ with the large t. Thus, for any small $\varepsilon > 0$ and for all $s \in [t/2, t]$ (t depending on ε) one has

$$l'(s) \leq 2\varepsilon$$

limit so that

$$l(t) - l(t/2) \leq t\varepsilon$$

This contradicts of the claim (5.4) of the step 3 for the large t. Hence, we have finished the proof of Theorem 1.1. □

Remark 5.1. *We can get more informations at the infinity by the strategy as in* [47].

Proof of Corollary 1.2 . By the assumption (4), $\hat{g}_i$ is a compact family so that

$$\lim_{r \to 0} \sup_i \sup_x Vol(B(x, r)) = 0$$

Now the desired result follows from Theorem 1.1 and the Dunford-Pettis' Theorem on the weak compactness in L^1. In fact, by Dunford-Pettis' Theorem, a subset in L^1 is relatively weakly compact if and only if it is uniformly integrable. Hence, relatively weak compactness of the family $\{S_i\}$ implies the condition (2) in Theorem 1.1. □

Proof of Corollary 1.3 . For $1 < q < \infty$, we know the bounded set in L^q is weakly compact in L^q since L^q is reflexive. Thus such set is also weakly compact in L^1 and the desired result follows from Corollary 1.2. When $q = \infty$, a bounded set in L^q is also bounded in L^2 since $\{(M, \hat{g}_i)\}$ is compact. Therefore, we prove the result. □

Proof of Corollary 1.4 . We argue by contradiction. Otherwise, we could find a sequence of conformally compact oriented Einstein metrics (X, g_i^+) which satisfies the assumptions (1) and (3-5) as in Theorem 1.1, $\oint_X |S_i| \to 0$ and whose compacitified metrics (X, g_i) would blow up. Now S-tensor converge in L^1 and thus it is strongly compact in L^1. Therefore, it is weakly compact in L^1. It follows from Corollary 1.2 that it is a compact family of the compacitified metrics (X, g_i). This contradiction gives the desired result. □

Proof of Theorem 1.7. The proof is almost same as the one of Theorem 1.1. The only difference is to replace Lemma 4.4 by Lemma 4.6 for the boundary blow-up setting. □

Proof of Corollary 1.9 . By the assumption (4), $\hat{g}_i$ is a compact family so that

$$\lim_{r \to 0} \sup_i \sup_x Vol(B(x, r)) = 0$$

Now the desired result follows from Theorem 1.7 and the Dunford-Pettis' Theorem on the weak compactness in L^1. □

Proof of Corollary 1.10 . By the assumption T_i is uniformly bounded from below, By the Hölder's inequality, we infer for any $x \in M$, $r > 0$ and i

$$\oint_{B(x,r)} (T_i)_- \leq Vol(B(x,r))^{\frac{q}{q-1}} (\oint_{B(x,r)} ((T_i)_-)^q)^{1/q} \leq C_6^{1/q} Vol(B(x,r))^{\frac{q}{q-1}}$$

Here $(T_i)_-$ is the negative part of the Q_3 curvature. From the compactness of $\hat{g}_i$, we deduce that

$$\liminf_{r \to 0} \inf_i \inf_{x \in M} \int_{B(x,r)} T_i \geq 0.$$

Finally, the desired result follows from Theorem 1.7. Therefore, we prove the result. □

Proof of Corollary 1.11 . This is a direct result of Theorem 1.7. The proof is similar to the one of Corollary 1.4. □

APPENDIX A. CHEEGER-GROMOV-HAUSDORFF THEORY FOR MANIFOLDS WITH BOUNDARY

There is Cheeger-Gromov-Hausdorff theory for manifolds with boundary in the literature cf[5, 36, 37] etc. For the convenience of readers, we give a description.
Let (X, g) be a Riemannian manifold with boundary, for $p \in X \backslash \partial X$ define the interior injectivity radius of p, $i_{int}(p)$, to be the supremum over all $r > 0$ such that all unitary geodesics $\gamma : [0, t_\gamma] \to$

X that start at $\gamma(0) = p$ are minimizing from 0 to $\min\{t_\gamma, r\}$, where t_γ is the first time the geodesic γ intersects ∂X. The interior injectivity radius of M is defined as

$$i_{int}(g) = \inf\{i_{int}(p)|p \in X \setminus \partial X\}.$$

The boundary injectivity radius of $p \in \partial X$ is defined by

$$i_\partial(p) = \inf\{t|\gamma_p \text{ stops minimizing at } t\},$$

where γ_p is the geodesic in M such that $\gamma'_p(0)$ is the inward unitary normal tangent vector at p. The boundary injectivity radius of X is defined by

$$i_\partial(g) = \inf\{i_\partial(p)|p \in \partial X\}.$$

Let us recall some definition about the harmonic radius for the manifolds with the boundary. Assume X is a complete 4-dimensional manifold with the boundary ∂X. The local coordinates (x_0, x_1, x_2, x_3) in D around some interior point $p \in X \setminus \partial X$ is called harmonic if $\triangle x^i = 0$ for all $0 \le i \le 3$. When $p \in \partial X$, we need the coordinates x_i harmonic and also $x_0|_{D \cap \partial X} \equiv 0$ and $(x_1, x_2, x_3)|_{D \cap \partial X}$ are also harmonic coordinates on the boundary. Given $\alpha \in (0,1)$ and $Q \in (1,2)$, we define the harmonic radius $r^{1,\alpha}(Q)$ to be the biggest number r satisfying the following properties:
1)if $dist(p, \partial X) > r$, there is a neighborhood D of p in $X \setminus \partial X$ and a coordinate chart $\varphi : B_{r/2}(0) \to D$ such that, in these coordinates for any tangent vector $\eta \in T_x X$

$$(A.1) \qquad\qquad Q^{-2}|\eta|^2 \le g_{jk}(x)\eta^j\eta^k \le Q^2|\eta|^2$$

and

$$(A.2) \qquad\qquad r^{1+\alpha}\sup|x-y|^{-\alpha}|\partial g_{jk}(x) - \partial g_{jk}(y)| \le Q - 1$$

2)if $dist(p, \partial X) \le r$, there is a neighborhood D of p in $\bar{X}$ and a coordinate chart $\varphi : B^+_{4r}(0) \to D$ such that $\{x^0 = 0\}$ maps to ∂X and relations (A.1) and (A.2) hold in these coordinates.

We recall a compactness result due to Anderson-Katsuda-Kurylev-Lassas-Taylor. Given $R_0, i_0, S_0, d_0 \in (0, \infty)$, let us denote $\mathcal{M}(R_0, i_0, S_0, d_0)$ the class of compact, connected, 4-dimensional Riemannian manifolds with the boundary $(\bar{X}, g)$, with smooth metric tensor satisfying the following conditions:

$$(A.3) \qquad\qquad \|Ric_X\|_{L^\infty(X)}\| \le R_0, \|Ric_{\partial X}\|_{L^\infty(\partial X)} \le R_0$$

where Ric is the Ricci tensor;

$$(A.4) \qquad\qquad i_{int}(X) \ge i_0, i(\partial X) \ge i_0, i_\partial(X) \ge 2i_0$$

where $i(\partial X)$ is the injectivity radius of the boundary manifolds ∂X with the induced metric;

$$(A.5) \qquad\qquad \|H\|_{Lip(\partial X)} \le S_0$$

where H is the mean curvature on the boundary;

$$(A.6) \qquad\qquad diam(\bar{X}) \le d_0$$

Let us denote $\mathcal{N}_*(\alpha, \rho, Q)$ the class of pointed manifolds $(\bar{X}, g, p)$ with $p \in \bar{X}$ such that the harmonic radius $r^{1,\alpha}(Q) \ge \rho$.

Lemma A.1. *(Anderson-Katsuda-Kurylev-Lassas-Taylor)*
$\mathcal{M}(R_0, i_0, S_0, d_0)$ is precompact in the $C^{1,\alpha}$ topology for any $\alpha \in (0,1)$. Moreover, there exists $\rho > 0$ depending on α such that

$$\mathcal{M}(R_0, i_0, S_0, d_0) \subset \mathcal{N}_*(\alpha, \rho, Q).$$

The second assertion in above Lemma still holds without the diameter assumption if we work with the pointed complete connected manifolds.

We now give the high order regularity of metric tensors for the manifold with the boundary, provided the boundary metric and curvature tensor are regular, that is,

Lemma A.2. *Given $\alpha \in (0,1)$, let (X,g) be a complete regular n-dimensional Riemmannian metric with $C^{k+2,\alpha}$ totally geodesic boundary ∂X for some $k \geq 1$. Assume (A.4) holds and there exists some positive constant R_1 such that*

$$(A.7) \qquad \|Ric\|_{C^{k,\alpha}(X)} \leq R_1, \ \|\hat{g}\|_{C^{k+2,\alpha}(\partial X)} \leq R_1$$

Then there exists some positive constant $C = C(i_0, R_1, n)$ such that

$$\|g\|_{C^{k+2,\alpha}(X)} \leq C$$

For simplycity, we just prove the result for 4-dimensional case. This part is some adaption of the result in [5]. For the convenience of readers, we give the proof in details.

Proof. Let (x_0, x_1, x_2, x_3) be some harmonic coordinates on a neighborhood D of some point $p \in \bar{X}$. From Lemma A.1, we know the harmonic radius is bounded from below since we have lower bound for the injectivity radius and boundness of Ricci curvature. We work in such coordinates. Let $(g_{ij})_{0 \leq i,j \leq 3}$ be metric matrix and its inverse matrix $(g^{ij})_{0 \leq i,j \leq 3}$. Denote $g = \det g_{ij}$ and A_{ij} the determinant of 3×3 matrix formed by omitting column i and row j from the matrix (g_{ij}). We have some elliptic PDE for metric tensor

$$\triangle g_{ij} = -2Ric_{ij} + \mathcal{P}_{ij}(g, \partial g)$$

where $\mathcal{P}(g, \partial g)$ is a quadratic form in ∂g with coefficients that are rational functions of g_{ij}. Recall in harmonic coordinates, we could write Laplace-Betrami operator as follows

$$\triangle u = g^{ij} \partial_i \partial_j u$$

Again by Lemma A.1, metric matrix (g_{ij}) and its invers (g^{ij}) are bounded in the Hölder space $C^{1,\alpha}$ so that the terms on the right hand side are bounded in the Hölder space $C^{0,\alpha}$. Let p be some interior point on X. By the classical interior estimates à priori (see [28] Theorem 6.2), we get the boundness of the metric matrix (g_{ij}) in the Hölder space $C^{2,\alpha}$. Iterating the above procedure, we get the interior $C^{k+2,\alpha}$ estimates à priori. Now we treat the boundary case. Assume $p \in \partial X$. We use the above elliptic equations for the indexes $1 \leq i = \gamma, j = \beta \leq 3$. We note the boundary is totally geodesic so that $g_{\gamma\beta} = \hat{g}_{\gamma\beta}$ on $D \cap \partial X$. Moreover, $g_{\gamma\beta} \in C^{k+2,\alpha}(D \cap \partial X)$. Thus, we get the elliptic system with Dirichlet boundary conditions

$$\begin{cases} \triangle g_{\gamma\beta} & = & -2Ric_{\gamma\beta} + \mathcal{P}_{\gamma\beta}(g, \partial g) & \text{in } D \\ g_{\gamma\beta} & = & \hat{g}_{\gamma\beta} & \text{on } D \cap \partial X \end{cases}$$

By Theorem 6.6 [28], we infer the boundness of $(g_{\gamma\beta})$ in the Hölder space $C^{2,\alpha}$. Now we write elliptic PDE for g^{0i} with Neumann boundary conditions. Let us denote $N = \frac{\nabla x_0}{|\nabla x_0|}$ the unit normal vector on the boundary ∂X. In local coordinates, we can write on the boundary ∂X

$$N(u) = (g^{00})^{-1/2} g^{0j} \partial_j u$$

Note the boundary ∂X is totally geodesic so that on the boundary $D \cap \partial X$

$$N(g^{00}) = -2Hg^{00} = 0$$

and

$$N(g^{0\gamma}) = -Hg^{0\gamma} + \frac{1}{2}(g^{00})^{-1/2} g^{\gamma j} \partial_j g^{00} = \frac{1}{2}(g^{00})^{-1/2} g^{\gamma j} \partial_j g^{00}$$

We can write Ricci equation for the components g^{0i} with Neumann boundary conditions

$$\begin{cases} \triangle g^{0i} & = & -2Ric^{0i} + \mathcal{P}^{0i}(g, \partial g) & \text{in } D \\ N(g^{00}) & = & 0 & \text{on } D \cap \partial X \\ N(g^{0\gamma}) & = & \frac{1}{2}(g^{00})^{-1/2}g^{\gamma j}\partial_j g^{00} & \text{on } D \cap \partial X \end{cases}$$

Using Theorem 6.30 [28], we deduce the boundness of (g_{00}) in the Hölder space $C^{2,\alpha}$. Going back the equation, we have $N(g^{0\gamma}) \in C^{1,\alpha}(D \cap \partial X)$. Again from Theorem 6.30 [28], we deduce the boundness of $(g^{0\gamma})$ in the Hölder space $C^{2,\alpha}$. To see this, recall in harmonic coordinates, we have

$$\frac{1}{2}(\delta_{ij}) \le (g_{ij}) \le 2(\delta_{ij}), \ \frac{1}{2}(\delta_{ij}) \le (g^{ij}) \le 2(\delta_{ij})$$

Thus for any $i, j \in \{0, 1, 2, 3\}$, we have $\frac{1}{2} \le g_{ii}, g^{ii} \le 2$ and $|g_{ij}| \le 2$. Hence the coefficients in the boundary derivative N are all in $C^{1,\alpha}(D \cap \partial X)$ and $(g^{00})^{-1/2}g^{00} \ge \sqrt{1/2}$. On the other hand, the coefficients g^{ij} in the Laplace-Betrami operator $\triangle = g^{ij}\partial_i\partial_j$ are in $C^{0,\alpha}(D)$. Thus the desired uniform estimates follows. From the fact $g = \det(g_{\gamma\beta})/g^{00}$, we obtain $g \in C^{2,\alpha}(D)$. Therefore

$$A_{0i} = gg^{0i} \in C^{2,\alpha}(D)$$

Now we denote $(h_{\gamma\beta}) = (g_{\gamma\beta})$ the 3×3 matrix and $(h^{\gamma\beta})$ the inverse matrix of $(h_{\gamma\beta})$, and $h = \det(h_{\gamma\beta})$. We remark for any $1 \le \gamma \le 3$

$$g_{\gamma 0} = (-1)^{3+\beta}\frac{A_{0\beta}}{hh^{\beta\gamma}} \in C^{2,\alpha}(D)$$

Finally, we have

$$g_{00} = \frac{1 - g_{\gamma 0}g^{\gamma 0}}{g^{00}} \in C^{2,\alpha}(D)$$

Now, iterating the above procedure, we prove the desired result $g \in C^{k+2,\alpha}(D)$ since the coefficients in the boundary derivative N are all in $C^{k+1,\alpha}(D \cap \partial X)$ and the coefficients g^{ij} in the Laplace-Betrami operator $\triangle = g^{ij}\partial_i\partial_j$ and the terms on the right hand side in Ricci equations are in $C^{k,\alpha}(D)$. Thus, we finish the proof. $\square$

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