ON LATTICES WITH FINITE RENORMALIZED COULOMBIAN INTERACTION ENERGY IN THE PLANE

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Abstract. We present criteria for the coulombian interaction energy of infinitely many points in $\mathbb{R}^d$, $d \geq 1$, with a uniformly charged background introduced in [26, 24] to be finite, as well as examples. We also show that in this unbounded setting, it is not always possible to project an L^2_{loc} vector field onto the set of gradients in a way that reduces its average L^2 norm on large balls.

1. Introduction

In [26], S.Serfaty and the second author introduced an energy describing the coulombian interaction energy of infinitely many unit positive charges in the plane with a uniformly negatively charged background. This energy was dubbed W after a similar energy which arose in [2] as a sharp (codimension 2) interface limit of a vector Modica-Mortola type functional, in a bounded domain. The definition in [26] was later modified and generalized in [24] to coulombian interaction energies in any dimensions, and in [25] for Riesz potential interactions.

It turns out that W, appears naturally in several variational settings. This is the case of the Ginzburg-Landau model of superconductivity ([26]) or the Ohta-Kawasaki model for diblock co-polymers ([10]), other natural candidates would be superfluids modeled by the Gross-Pitaevski functional or certain models for dislocations where the analysis of vortices is already well advanced ([14, 15, 8] for example, the literature on these subjects is already quite large). In all these contexts there exist limits when the characteristic size of a vortex goes to zero and the number of vortices tends to $+\infty$, simultaneously. Then the vortices of minimizers will be described at the macro-scale by a certain optimal density and at the micro-scale by discrete subsets of $\mathbb{R}^2$ which minimize W, this energy accounting for the interaction of individual vortices with the field generated by the density, the latter being constant at the microscale if the optimal density is well-behaved.

Another context in which W appears is the case of weighted Fekete N-sets. The Fekete N-sets are N-tuples of points in the plane which minimize

$$w_N(x_1, \ldots, x_N) = -\sum_{i \neq j} \log |x_i - x_j| + N \sum_i V(x_i),$$

where V is a confining potential. In the aforementionned models, the Fekete N sets would arise when the sharp interface limit is taken but the number of vortices remains equal to N. When N tends to $+\infty$, again W governs the arrangement of the points at the microscale ([27], [28]). Note that this arrangement has been studied by other methods as well (see [1] and the references therein).

Also, the energy W plays a role in the Coulomb gas model in statistical mechanics, for which the probability law density of N particles is $P_N = \frac{1}{Z_{n,\beta}} e^{-\beta w_n}$, where β is the inverse of temperature and $Z_{n,\beta}$ is the partition function. There ([27], [28]) the minimum of W appears in the asymptotic expansion of $\log Z_{N,\beta}$ as $N \to +\infty$ for large β, i.e. small temperature, and more recently in the large deviation analysis at finite temperature of [20]. In the context of point processes, it is possible to define the energy of a point process P as the expectation of W under P, but alternative definitions are given in [6, 18].

Finally W coincides with the jellium energy, at least in some situations. In physics, jellium, also known as the uniform electron gas or homogeneous electron gas, is a quantum mechanical model of interacting electrons in a solid where the positive charges are assumed to be uniformly distributed in space whence the electron density is a uniform quantity as well in space. The jellium Hamiltonian consists of three parts: electronic Hamiltonian consisting of the kinetic and electron-electron repulsion terms, the Hamiltonian of the positive background charge interacting electrostatically with itself, and the electron-background interaction Hamiltonian. In particular, in jellium model, a Wigner crystal is the solid (crystalline) phase of electrons first predicted by Eugene Wigner in 1934 in [33]. A gas of electrons moving in 2 dimension or 3 dimension in a uniform, inert, neutralizing background will crystallize and form a lattice if the electron density is less than a critical value. At temperature $T = 0$, when the kinetic energy is equal to zero, the renormalized energy W is exactly the jellium Hamiltonian for the Bravais lattices. Identifying W to the jellium energy in general is a more delicate matter that we do not pursue, see [21] for a precise definition of the jellium energy and some issues related to it.

Until now, some basic but useful facts are known about W (see [26]): It is bounded below, admits minimizers, and minimizers may be approximated by doubly periodic configurations of points. It is also known that among perfect (Bravais) lattices, the triangular lattice is the unique minimizer of W ([26]). The minimal value of W is not known, even though it can be used to express other quantities as in the aforementionned expansion of $logZ_{N,\beta}$, but also the energy of weighted Fekete N-sets (see [23], or [3] in the case of the sphere). Finding the minimum of W seems to be a challenging problem, even though such results exist for energies that similarly measure the distance of a discrete subset of $\mathbb{R}^2$ to the uniform measure ([7]). In the context of point processes, the value of W for some classical processes has been computed in [6], and the minimum of W of one-dimensional stationary point processes has been determined in [19] to be uniquely achieved for the uniform probability on translates of $\mathbb{Z}$.

In this paper we focus on the natural question of which discrete subsets $\Lambda \subset \mathbb{R}^d$ are such that $W(\Lambda) < +\infty$. We find sufficient conditions for this to hold, in terms of discrepancy estimates. Note that necessary conditions can be found in [20], in terms of discrepancies as well, either for point configurations or processes. It turns out also that this connects to problems of independent interest and which to our knowledge have not been addressed in the literature. The first one is that of a Hodge decomposition for L^2_{loc} instead of L^2 vector fields. More precisely is it possible to project an L^2_{loc} vector field on the set of gradients in a way that reduces its average L^2 norm on large balls ? We provide a counter-example in Remark 4.

2. Main results

Let us now define W, following [25] (see also [24]). It is shown in [24] that this definition agrees with the earlier one given in [26] for the case of 2 dimensions, for lattices Λ for which the distance between two distinct points is bounded below by a positive number, what we denote below *uniformly separated* lattices.

Following [25], for any integer $d \geq 2$ we let $g_d : \mathbb{R}^d \to \mathbb{R}$ be defined by

$$g_d(x) = \begin{cases} \dfrac{1}{|x|^{d-2}} & \text{if } d > 2 \\ -\log|x| & \text{if } d = 2. \end{cases} \tag{1}$$

Then we have

$$-\Delta g_d(x) = c_d \delta, \tag{2}$$

where δ denotes the Dirac mass and

$$c_d = d(d-2)|B^d| \text{ if } d \neq 2 \text{ and } c_2 = 2\pi. \tag{3}$$

Given $\eta > 0$ we let $g_d^{(\eta)}(x) = \min(g_d(x), g_d(\eta))$ and define $\delta^{(\eta)}$ by the following equality

$$-\Delta g_d^{(\eta)}(x) = c_d \delta^{(\eta)}. \tag{4}$$

We also let $\delta_p^{(\eta)}(\cdot) = \delta^{(\eta)}(\cdot - p)$. It holds (see [25]) that $\delta^{(\eta)}$ is a positive measure of total mass 1 supported on $\partial B^d(0,\eta)$.

We give the following definitions.

Definition 1. *Let Λ be a discrete subset of $\mathbb{R}^d$ and m be a nonnegative number. Assume $E : \mathbb{R}^d \to \mathbb{R}^d$ satisfies*

$$-\operatorname{div} E = c_d \left(\nu - m\right), \quad \textit{where } \nu = \sum_{p \in \Lambda} \delta_p. \tag{5}$$

The mollified vector-field E_η is defined for any $\eta > 0$ by

$$E_\eta(x) = E(x) + \sum_{p \in \Lambda} \nabla \left(g_d^{(\eta)} - g_d \right)(x - p). \tag{6}$$

Let K_R denote the square $(-R/2, R/2)^d$. The renormalized energy of E is defined as

$$\mathcal{W}(E) = \liminf_{\eta \to 0} \mathcal{W}_\eta(E), \quad \textit{where} \quad \mathcal{W}_\eta(E) = \limsup_{R \to +\infty} \fint_{K_R} |E_\eta|^2 - m c_d g_d(\eta). \tag{7}$$

Note that in the above definition, the number m used to defined $\mathcal{W}(E)$ is the same as the one in (5), hence depends on E.

Now, we consider the set $\mathcal{F}_{\Lambda,m}$ of vector fields in $L^2_{\mathrm{loc}}(\mathbb{R}^d \setminus \Lambda, \mathbb{R}^d)$ satisfying (5) for a given Λ and m, and the subset $\mathcal{P}_{\Lambda,m}$ of curl-free vector fields in $\mathcal{F}_{\Lambda,m}$, or equivalently the set of those elements in $\mathcal{F}_{\Lambda,m}$ which are gradients. We may now define

$$\mathbb{W}_m(\Lambda) := \inf_{\nabla U \in \mathcal{P}_{\Lambda,m}} \mathcal{W}(\nabla U), \quad \widetilde{\mathbb{W}}_m(\Lambda) := \inf_{E \in \mathcal{F}_{\Lambda,m}} \mathcal{W}(E). \tag{8}$$

We wil see that $\mathbb{W}_m(\Lambda)$ and $\widetilde{\mathbb{W}}_m(\Lambda)$ can be finite for at most one value of m, which is the asymptotic density of Λ if it exists.

Remark 1. *It is clear that* $\widetilde{\mathbb{W}}_m \le \mathbb{W}_m$, *since* $\mathcal{P}_{\Lambda,m} \subset \mathcal{F}_{\Lambda,m}$.

Remark 2. *From a physical point of view, it is natural to consider all curl-free electric vector fields. However, from a mathematical point of view, relaxing the condition to be a gradient leads to nicer properties, as in* [31] *for the calculation of the transportation distance of measures. We can expect in some cases the two definitions to coincide, but it is not clear wether this is the case whenever* $\mathbb{W}_m$ *is finite.*

In [26, 24, 25], only $\mathbb{W}_m$ is considered. One could think at first that $\mathbb{W}_m$ and $\widetilde{\mathbb{W}}_m$ are equal, the argument being the following: Since $\mathcal{W}(E)$ may be seen as the average of $|E|^2$ over $\mathbb{R}^2$ (with the infinite part due to the Dirac masses in (5) removed), then projecting onto the set of curl-free fields would reduce this quantity, so that the infimum of $\mathcal{W}(E)$ over $\mathcal{F}_{\Lambda,m}$ would in fact be achieved by some $E \in \mathcal{P}_{\Lambda,m}$, proving that $\mathbb{W}_m(\Lambda) = \widetilde{\mathbb{W}}_m(\Lambda)$. It turns out however that this is not the case and in fact we prove (see Theorem 1 below) that,

Theorem A. In the case $d = 2$, we have, for $m = 0$, $\mathbb{W}_m(\mathbb{N}) = +\infty$ and $\widetilde{\mathbb{W}}_m(\mathbb{N}) < +\infty$. Note that the asymptotic density of $\mathbb{N}$ in $\mathbb{R}^2$ is zero, which is why we take $m = 0$ here.

Remark 3. *In* [5], *Blanc, Le Bris and Lions have studied infinite sets of points charges in* $\mathbb{R}^d$ *and defined similar energies even in nonlinear models where the background is optimized rather than fixed, with an additional penalizing term. They gave a list of sufficient conditions on the lattices* $\mathcal{L}$ *that yields a finite energy. See* [4], [5].

The rest of the paper is devoted to giving sufficient conditions on Λ for $\widetilde{\mathbb{W}}_m$ and/or $\mathbb{W}_m$ to be finite. There are roughly two factors which can make $\mathbb{W}_m$ or $\widetilde{\mathbb{W}}_m$ infinite. First, there is the coulombic interaction between pairs of points, which can be made infinite by bringing points very close to each other: we will not consider this factor here and to rule it out we restrict ourselves to *uniformly separated* Λ's in the following sense.

Definition 2. *Given a discrete set* $\Lambda \subset \mathbb{R}^d$, *we say that it* uniformly separated *if*

$$\inf_{p \ne q \in \Lambda} |p - q| > 0.$$

The second factor which can make $\mathbb{W}_m$ or $\widetilde{\mathbb{W}}_m$ infinite is the interaction with the background. If we restrict ourselves to uniformly separated Λ's, then for a given m the quantities $\mathbb{W}_m(\Lambda)$ or $\widetilde{\mathbb{W}}_m(\Lambda)$ measure how close $c_d \sum_{p\in\Lambda} \delta_p$ is to a uniform density m. Our second main result shows that this can be measured by simply counting the number of points of Λ in any given ball (see Theorems 2 and 5). In particular we have

Theorem B. Assume that Λ is uniformly separated and that there exists $m, C \ge 0$ and $\varepsilon \in (0,1)$ such that for any $x \in \mathbb{R}^d$ and $R > 1$ we have, denoting $\sharp A$ the number of elements in A,

$$\left| \sharp \left(B^d(x,R) \cap \Lambda \right) - m|B^d|R^d \right| \le CR^{d-1-\varepsilon} \tag{9}$$

Then $\mathbb{W}_m(\Lambda) < +\infty$.

This criterion for finiteness is optimal in the sense that if we replace the right-hand side in (9) by $CR^{d-1+\varepsilon}$, then it is not difficult to construct Λ's satisfying (9) and having infinite renormalized energies (see Proposition 5). This criterion can be relaxed a bit in the case of $\widetilde{\mathbb{W}}_m$ (see Theorem 5).

This leaves open the case $\varepsilon = 0$ (in which case $\mathbb{N}$ and $\mathbb{Z}$ satisfy (9) with $m = 0$). In this case we have

Theorem C. Assume $d = 2$. Let $V \subset \mathbb{Z}^2$ and $\Lambda := \mathbb{Z}^2 \setminus V$. Assume there exists some constant $C > 0$ such that for all $x \in \mathbb{R}^2$ and $R > 1$ we have

$$\sharp\,(V \cap B(x, R)) \leq CR.$$

Then $\widetilde{\mathbb{W}}_1(\Lambda) < +\infty$.

The proof of this theorem is based on the fact that under the above hypothesis there exists a bijection between $\mathbb{Z}^2 \setminus A$ and $\mathbb{Z}^2$ under which points are moved at uniformly bounded distances. This follows from a result of M.Laczkovich, [17]. Its conclusion cannot be improved to $\mathbb{W}_1(\Lambda) < +\infty$, see Proposition 4.

The criterion $\left|\sharp\left(B^d(x, R) \cap \Lambda\right) - m|B^d|R^d\right| \leq CR^{d-1-\varepsilon}$ is satisfied by perfect (or Bravais) lattices, or more generally by periodic lattices (see [12]) — even though in this case (see below) the conclusion of Theorem B is almost trivial. However we are not aware that this is known for quasi-cristalline lattices, and thus we give a construction similar to that of Theorem B which allows us to conclude for an example of Penrose-type lattice Λ that $\widetilde{\mathbb{W}}_m(\Lambda) < +\infty$. We have not sought generality in this direction, and refer to Section 7 for the construction of Λ and the proof that $\widetilde{\mathbb{W}}_m(\Lambda)$ is finite.

3. Properties of $\mathbb{W}_m$, $\widetilde{\mathbb{W}}_m$

We recall some facts from [26, 27, 24, 25].

The density of points is m. (See [25, Lemma 2.1]) If $\eta > 0$ and if $\mathcal{W}(E_\eta)$ is finite, where E satisfies (5) and E_η is defined in (6), then

$$m = \lim_{R \to +\infty} \frac{\nu(K_R)}{R^d}. \tag{10}$$

Structure of $\mathcal{P}_{\Lambda,m}$. If $\mathbb{W}_m(\Lambda)$ is finite, then the set $\{\nabla U \in \mathcal{P}_{\Lambda,m} \mid \mathcal{W}(\nabla U) < +\infty\}$ is a d-dimensional affine space. Any two gradients in this set differ by a constant vector.

Indeed if ∇U and ∇V both belong to $\mathcal{P}_{\Lambda,m}$ then $U - V$ is a harmonic function. Thus if it is not linear, it grows at least quadratically, from which it is not difficult to deduce that if $\mathcal{W}(\nabla V)$ is finite, then $\mathcal{W}(\nabla U)$ must be infinite. Thus two gradients in $\mathcal{P}_{\Lambda,m}$ with finite renormalized energy differ by a constant vector.

Minimization. (See [25, Proposition 1.4]) For any given m, the function $\Lambda \to \mathbb{W}_m(\Lambda)$ is bounded from below and admits a minimizer.

Bravais lattices. (See [26]) Assume $\Lambda = \mathbb{Z}\vec{u} \oplus \mathbb{Z}\vec{v}$ where $(\vec{u}, \vec{v})$ is a basis of $\mathbb{R}^2$ satisfying the normalized volume condition $|\vec{u} \wedge \vec{v}| = 1$. Then, the minimum of $\mathbb{W}_1$ among lattices of this type is achieved by the triangular lattice

$$\Lambda_1 := \sqrt{\frac{2}{\sqrt{3}}}\left((1,0)\mathbb{Z} \oplus \left(\frac{1}{2}, \frac{\sqrt{3}}{2}\right)\mathbb{Z}\right).$$

An important fact when investigating the finiteness of $\mathcal{W}(E)$ is the following fact, the derivation of which is similar to that of [25], Proposition 2.4.

Proposition 1. *Assume E satisfies* (5) *where Λ is uniformly separated. Then for any $\eta > 0$, we have*

$$\mathcal{W}(E) < +\infty \iff \mathcal{W}(E_\eta) < +\infty.$$

Proof. First we claim that for any $\alpha, \eta > 0$ we have $\mathcal{W}(E_\alpha) < +\infty \iff \mathcal{W}(E_\eta) < +\infty$. Indeed

$$E_\alpha(\cdot) = E_\eta(\cdot) + \sum_{p\in\Lambda} \nabla(g_d^{(\alpha)} - g_d^{(\eta)})(\cdot - p).$$

Since $\nabla(g_d^{(\alpha)} - g_d^{(\eta)})$ is supported in the ball of radius $\max(\alpha, \eta)$, for any $x \in \mathbb{R}^d$ the number of p's such that $\nabla(g_d^{(\alpha)} - g_d^{(\eta)})(x - p)$ is nonzero is bounded by the number of points in the ball centered at x with radius $\max(\alpha, \eta)$, which is bounded independantly of x since Λ is uniformly separated. Thus $\|E_\alpha - E_\eta\|_\infty \le c_{\alpha,\eta}$, from which the claim follows.

It follows from the above that if $\mathcal{W}(E)$ is finite, then $\mathcal{W}(E_\alpha) < +\infty$ for every $\alpha > 0$.

To show that $\mathcal{W}(E_\eta) < +\infty \implies \mathcal{W}(E) < +\infty$ we assume that $\mathcal{W}(E_\eta)$ is finite. Then $\mathcal{W}(E_\alpha)$ is finite for every $\alpha > 0$, thus we may assume that $\mathcal{W}(E_\eta) < +\infty$ for some $\eta \in (0, \delta/2)$, where δ is the minimal distance between two points in the lattice. Then to prove the result it suffices to show that for any $\alpha \in (0, \eta]$ we have $\mathcal{W}(E_\alpha) < \mathcal{W}(E_\eta) + c_\eta$, where c_η is independent of α.

To do this we estimate for $R > 1$ the integral over K_R of $|E_\alpha|^2 - |E_\eta|^2$, which we write as $(E_\alpha + E_\eta) \cdot (E_\alpha - E_\eta)$. Since this involves integrating by parts, we first need to find a good radius. Since $\mathcal{W}(E_\eta)$ is finite, we have

$$\int_{K_R} |E_\eta|^2 < CR^d.$$

A mean-value argument then shows that for any $M \in (0, R)$ there exists $t \in (R, R + M)$ such that $\|E_\eta\|^2_{L^2(\partial K_t)} \le CR^d/M$, and then that

$$\int_{\partial K_t} |E_\eta| \le C \frac{R^{d-\frac{1}{2}}}{M^{\frac{1}{2}}}. \tag{11}$$

Moreover, since Λ is uniformly separated, the number of p's in Λ such that the support of $\nabla(g_d^{(\alpha)} - g_d^{(\eta)})(x - p)$ intersects $K_t \setminus K_R$ is bounded above by $c_{\alpha,\eta}|K_t \setminus K_R|$, hence by $c_{\alpha,\eta} M R^{d-1}$. We choose $M = R^{\frac{1}{3}}$ and deduce, using the fact that the supports of the functions $\nabla(g_d^{(\alpha)} - g_d^{(\eta)})(\cdot - p)$ are disjoint if $\alpha, \eta < \delta/2$ and the Cauchy-Schwarz inequality, that

(12)
$$\left| \int_{K_t\setminus K_R} |E_\alpha|^2 - |E_\eta|^2 \right| = \left| \sum_{p\in\Lambda} \int_{K_t\setminus K_R} \left(\nabla(g_d^{(\alpha)} - g_d^{(\eta)})(\cdot - p) \right) \cdot (E_\alpha + E_\eta) \right| \le c_{\alpha,\eta} R^{d-\frac{1}{3}}.$$

To estimate the left-hand side we integrate by parts to find,

$$\int_{K_t} |E_\alpha|^2 - |E_\eta|^2 = -\sum_{p\in\Lambda}\int_{K_t} \mathrm{div}(E_\alpha + E_\eta)(g_d^{(\alpha)} - g_d^{(\eta)})(\cdot - p) + \mathrm{Bdry} \tag{13}$$

$$= c_d \sum_{p,p'\in\Lambda}\int_{K_t}(g_d^{(\alpha)} - g_d^{(\eta)})(x-p)\,d(\delta_{p'}^{(\alpha)}+\delta_{p'}^{(\eta)})(x) - 2c_d m\sum_{p\in\Lambda}\int_{K_t}(g_d^{(\alpha)} - g_d^{(\eta)})(x-p)\,dx + \mathrm{Bdry},$$

where, in view of (11), the boundary term Bdry satisfies the bound

$$\mathrm{Bdry} \le c_{\alpha,\eta} R^{d-\frac{2}{3}}. \tag{14}$$

Then, since the function $(g_d^{(\alpha)} - g_d^{(\eta)})(\cdot - p)$ is equal to 0 outside $B^d(p,\eta)$, is equal to $g_d(\alpha) - g_d(\eta)$ on $\partial B^d(p,\alpha)$ and since the balls $B^d(p,\eta)$ are disjoint, we find

$$\sum_{p,p'\in\Lambda}\int_{K_t}(g_d^{(\alpha)} - g_d^{(\eta)})(x-p)\,d(\delta_{p'}^{(\alpha)}+\delta_{p'}^{(\eta)})(x) = c_d(g_d(\alpha) - g_d(\eta))\sum_{p\in\Lambda}\delta_p^{(\alpha)}(K_t).$$

The right-hand side differs from $\#\{\Lambda\cap K_R\}$ by at most $CR^{d-\frac{2}{3}}$, which bounds the number of points in $K_{t+\alpha}\setminus K_R$ and, on the other hand,

$$\sum_{p\in\Lambda}\int_{K_t}(g_d^{(\alpha)} - g_d^{(\eta)})(x-p)\,dx \le Ct^d\int_{B_\eta^d}|g_d| \le CR^d c_\eta,$$

where $c_\eta = \|g_d\|_{L^1(B_\eta^d)}$, so that indeed $c_\eta \to 0$ as $\eta \to 0$. Inserting the two previous equations in (13) and in view of (12), (14) we find that

$$\left|\fint_{K_R}|E_\alpha|^2 - c_d m g_d(\alpha) - \fint_{K_R}|E_\eta|^2 + c_d m g_d(\eta)\right| \le C\left(c_{\alpha,\eta}R^{-\frac{1}{3}} + c_{\alpha,\eta}\frac{|\#\{\Lambda\cap K_R\} - mR^d|}{R^d} + c_\eta\right).$$

Taking the limit $R\to+\infty$ the right-hand side converges to Cc_η, since m is the asymptotic density of Λ, and we are left with

$$|\mathcal{W}(E_\alpha) - \mathcal{W}(E_\eta)| \le Cc_\eta.$$

□

4. Examples with finite or infinite energy.

We begin by showing that moving the points in $\mathbb{Z}^d$ at a bounded distance yields a Λ with finite energy, assuming Λ is uniformly separated.

Proposition 2. *Let Λ satisfy $\inf_{x,y\in\Lambda,x\neq y}|x-y|>0$ and assume there exists a bijective map $\Phi:\Lambda\to\mathbb{Z}^d$ such that $\sup_{p\in\Lambda}|\Phi(p)-p|<\infty$. Then $\widetilde{\mathbb{W}}_1(\Lambda)<+\infty$.*

Proof. Assume the hypothesis are satisfied. We choose $\eta>0$. Then from the previous proposition it suffices to find E satisfying (5) with $m=1$ and such that $\mathcal{W}(E_\eta)<+\infty$ to prove that $\widetilde{\mathbb{W}}_1(\Lambda)$ is finite.

Let $R_1 = 2\sup_{p\in\Lambda}|\Phi(p)-p|$. Then for every $p\in\Lambda$, we solve

$$\begin{cases} -\Delta U_p &= c_d\left(\delta_p^{(\eta)} - \delta_{\Phi(p)}^{(\eta)}\right) & \text{in } B(p,R_1)\\ \frac{\partial U_p}{\partial\nu} &= 0 & \text{on } \partial B(p,R_1)\end{cases}$$

where ν is the outer unit normal on the boundary. From elliptic regularity, ∇U_p is bounded in L^∞ by a constant depending on η but independent of p.

Let V be the $\mathbb{Z}^d$-periodic solution — which is unique modulo an additive constant — of

$$-\Delta V \quad = \quad c_d \left(\sum_{p \in \mathbb{Z}^d} \delta_p^{(\eta)} - 1 \right) \quad \text{in } \mathbb{R}^d$$

Then by periodicity ∇V is bounded in $L^\infty(\mathbb{R}^d)$. We define $E : \mathbb{R}^d \to \mathbb{R}$ by

$$E = \nabla V + \sum_{p \in \Lambda} \nabla U_p + \sum_{p \in \Lambda} \nabla (g_d - g_d^{(\eta)})(\cdot - p),$$

where ∇U_p is extended by 0 outside of $B(p, R_1)$ and is thus defined on the whole of $\mathbb{R}^d$. From the assumptions on Λ and Φ the sum above is finite on any compact set and thus E is well defined and we have

$$-\operatorname{div} E = c_d \left(\sum_{p \in \Lambda} \delta_p - 1 \right) \text{ in } \mathbb{R}^d, \quad E_\eta = \nabla V + \sum_{p \in \Lambda} \nabla U_p.$$

On the other hand, ∇U_p is supported in $B(p, R_1)$ and bounded independently of p. It follows that $E_\eta \in L^\infty(\mathbb{R}^d)$, since the number of p's such that $x \in B(p, R_1)$ is bounded uniformly with respect to $x \in \mathbb{R}^d$, from the uniform separation of Λ. It follows immediately from the definition that $\mathcal{W}(E_\eta) < +\infty$. $\square$

We will prove below that the conclusion in the above proposition cannot be improved to $\mathbb{W}_1(\Lambda) < +\infty$.

A consequence of Proposition 2 is

Corollary 1. *We have, with $m = 1$,*

$$\widetilde{\mathbb{W}}_1(\mathbb{Z}^2 \setminus \mathbb{Z}) < \infty, \ \widetilde{\mathbb{W}}_1(\mathbb{Z}^2 \setminus \mathbb{N}) < \infty$$

Proof. We define the bijective map $\Phi : \mathbb{Z}^2 \setminus \mathbb{Z} \to \mathbb{Z}^2$ by letting

$$\Phi(p_1, p_2) = \begin{cases} (p_1, p_2 - 1) & \text{if } p_2 \geq 1 \\ (p_1, p_2) & \text{if } p_2 < 0 \end{cases}$$

The fact that $\widetilde{\mathbb{W}}_1(\mathbb{Z}^2 \setminus \mathbb{Z}) < \infty$ then follows from the above proposition. The proof for $\mathbb{Z}^2 \setminus \mathbb{N}$ is similar, in this case we let $\Phi(p_1, p_2) = (p_1, p_2)$ if $p_2 < 0$ or $p_1 < 0$, and $\Phi(p_1, p_2) = (p_1, p_2 - 1)$ otherwise. $\square$

A second tool for constructing E's with finite energy is

Proposition 3. *Assume E_1 (resp. E_2) satisfy* (5) *with some Λ_1, m_1 (resp. Λ_2, m_2) which is uniformly separated. Assume also that the union $\Lambda = \Lambda_1 \cup \Lambda_2$ is disjoint and is uniformly separated. The following hold.*

(1) If $\mathcal{W}(E_1) < \infty$ and $\mathcal{W}(E_2) < \infty$ then $\mathcal{W}(E_1 + E_2) < \infty$.

(2) If $\mathcal{W}(E_1 + E_2) < \infty$ and $\mathcal{W}(E_2) < \infty$ for the background m_2, then $\mathcal{W}(E_1) < \infty$.

Proof. From Proposition 1, we may work with the η regularizations of the fields for some arbitrary $\eta > 0$.

Since
$$(E_1 + E_2)_\eta = (E_1)_\eta + (E_2)_\eta,$$
we have for any $R > 0$ that
$$\int_{K_R} |(E_1 + E_2)_\eta|^2 \le 2\int_{K_R} |(E_1)_\eta|^2 + |(E_2)_\eta|^2\,, \int_{K_R} |(E_1)_\eta|^2 \le 2\int_{K_R} |(E_1 + E_2)_\eta|^2 + |(E_2)_\eta|^2\,,$$
from which the proposition follows at once. □

Corollary 2.
$$\widetilde{\mathbb{W}}_0(\mathbb{Z}) < +\infty,\ \widetilde{\mathbb{W}}_0(\mathbb{N}) < +\infty$$

Proof. There exists $E \in \mathcal{F}_{\mathbb{Z}^2,1}$ and from Corollary 1 there exists $E_2 \in \mathcal{F}_{\mathbb{Z}^2\setminus\mathbb{Z},1}$ such that $\mathcal{W}(E)$ and $\mathcal{W}(E_2)$ are both finite. Let $E_1 = E - E_2$. Then, by Proposition 3 we have $\mathcal{W}(E_1) < +\infty$ and since $-\operatorname{div}(E_1) = \sum_{p\in\mathbb{Z}} \delta_p$, and $\mathbb{Z}$ is uniformly separated, we have $E_1 \in \mathcal{F}_{\mathbb{Z},0}$, hence $\widetilde{\mathbb{W}}_0(\mathbb{Z}) < +\infty$. The proof for $\mathbb{N}$ is identical, taking instead $E_2 \in \mathcal{F}_{\mathbb{Z}^2\setminus\mathbb{N},1}$ such that $\mathcal{W}(E_2)$ is finite, which exists from Corollary 1. □

Proposition 4. *In the case $d = 2$ we have*
$$\mathbb{W}_0(\mathbb{Z}) = 0,\ \mathbb{W}_0(\mathbb{N}) = +\infty$$

The case of $\mathbb{Z}$. We need to exhibit U such that $-\Delta U = 2\pi \sum_{p\in\mathbb{Z}} \delta_p$, and such that $\mathcal{W}(\nabla U)$ is finite. A natural candidate would be
$$U(x, y) = -\frac{1}{2}\sum_{k\in\mathbb{Z}} \log((x - k)^2 + y^2),$$
but this series is divergent. However we may consider the series of gradients, and group together the k'th and $-k$'th term to get a series
$$\begin{aligned} E(x, y) &= -\left(\frac{(x, y)}{x^2 + y^2} + \sum_{k=1}^{+\infty} \frac{(x - k, y)}{(x - k)^2 + y^2} + \frac{(x + k, y)}{(x + k)^2 + y^2}\right) \\ &= -\left(\frac{(x, y)}{x^2 + y^2} + \sum_{k=1}^{+\infty} \frac{2(x, y)(x^2 + y^2 - k^2) + 4(0, 1)yk^2}{(x^2 - k^2)^2 + y^4 + 2y^2(x^2 + k^2)}\right), \end{aligned}$$
which is easily seen to be convergent for any (x, y) such that $x \notin \mathbb{Z}$ or $y \neq 0$. Then it is straightforward to check that $\operatorname{curl} E = 0$ in the sense of distributions , so that $E = \nabla U$ for some function U defined in $\mathbb{R}^2 \setminus \mathbb{Z}$ and then that
$$-\Delta U = -\operatorname{div} E = 2\pi \sum_{p\in\mathbb{Z}} \delta_p.$$
Using periodicity, to prove that $\mathcal{W}(\nabla U) = 0$ it suffices to check that, for some $\eta > 0$,
$$\int_{[-\frac{1}{2},\frac{1}{2}]\times\mathbb{R}\cap K_R} |E_\eta|^2 < CR,$$
where E_η is the regularization of ∇U. The regularization takes care of the singularity at the origin and modifies ∇U on a compact set only, hence the only issue is the integrability

of $|\nabla U|^2$ as $y \to \infty$ when x is fixed and the periodicity of ∇U in x-direction, which clearly holds. $\square$

The case of $\mathbb{N}$. We must prove that no $\nabla U \in \mathcal{P}_{\mathbb{N},0}$ is such that $\mathcal{W}(\nabla U) < +\infty$. Our strategy is to construct $\nabla H_1 \in \mathcal{P}_{\mathbb{N},0}$ such that $\mathcal{W}(\nabla H_1) = +\infty$ and such that, letting E_η denote the regularization of ∇H_1, for any fixed $\eta > 0$ we have

$$\int_{K_R} |E_\eta|^2 \leq CR^2 \log^2 R. \tag{15}$$

To construct H_1 we use the Weierstass construction for a holomorphic function in the plane with a simple zero at each $p \in \mathbb{N}$ to define

$$H(z) := \Pi_{k\in\mathbb{N}}(1 - \frac{z}{k})e^{\frac{z}{k}}.$$

Then we let

$$H_1(z) = -\log|H(z)|.$$

It is straightforward to check that the product in the definition of H converges uniformly on any compact subset of $\mathbb{C} \setminus \mathbb{N}$ and that

$$-\Delta H_1 = 2\pi \sum_{k\in\mathbb{N}} \delta_k \text{ in } \mathbb{R}^2$$

and for all $z \in \mathbb{C} = \mathbb{R}^2$

$$|H_1(z)| \leq \sum_{k\in\mathbb{N}} \left|\log(1 - \frac{z}{k}) + \frac{z}{k}\right| \tag{16}$$

and

$$|\nabla H_1(z)| = \left|\sum_{k\in\mathbb{N}} \frac{z}{k(k-z)}\right|. \tag{17}$$

Next, rather than proving (15), we prove the following stronger pointwise estimates, from which (15) clearly follows:

$$|\nabla H_1(z)| \leq C(\log(|z|+1)+1), \quad \text{outside } \cup_{k\in\mathbb{N}} B(k, \frac{1}{4}), \tag{18}$$

$$\left|\nabla H_1(z) + \frac{1}{z-k}\right| \leq C(\log(|z|+1)+1), \quad \text{in } B(k, \frac{1}{4}). \tag{19}$$

For (18), take any $z \in \mathbb{C} \setminus \cup_{k\in\mathbb{N}} B(k, \frac{1}{4})$, it follows from (17) that

$$|\nabla H_1(z)| \leq \sum_{1\leq k\leq[2|z|+1]} \left(\left|\frac{1}{k-z}\right| + \left|\frac{1}{k}\right|\right) + \sum_{k>[2|z|+1]} \left|\frac{z}{k(k-z)}\right| := I + II,$$

where $[\cdot]$ denotes the integer part of a real number. We have

$$II \leq \sum_{k>[2|z|+1]} \frac{|z|}{(k-|z|)^2} \leq |z| \int_{|z|}^{+\infty} \frac{dt}{t^2} \leq 1,$$

$$\sum_{1\leq k\leq[2|z|+1]} \frac{1}{k} \leq 1 + \int_1^{2|z|+1} \frac{dt}{t} \leq 2(\log(|z|+1)+1).$$

On the other hand,

$$\sum_{1\le k\le[2|z|+1]}\left|\frac{1}{k-z}\right| \le \sum_{1\le k\le[2|z|+1]}\left|\frac{1}{\mathcal{R}e(k-z)}\right| \le 5+2\int_1^{2|z|+1}\frac{dt}{t} \le 5(\log(|z|+1)+1).$$

Therefore, for any $z\in\mathbb{C}\setminus\cup_{k\in\mathbb{N}}B(k,\frac14)$, we have $|\nabla H_1(z)|\le 8(\log(|z|+1)+1)$, and therefore (18) holds.

Now we prove (19). Let $z=x+iy\in B(k,\frac14)$ for some $k\in\mathbb{N}$. As above

$$\left|\nabla H_1(z)+\frac{(x-k,-y)}{|z-k|^2}\right| \le 8(\log(|z|+1)+1)+\frac{1}{k} \le 9(\log(|z|+1)+1),$$

or equivalently,

$$\left|\nabla H_1(z)+\frac{1}{z-k}\right| \le 8(\log(|z|+1)+1)+\frac{1}{k} \le 9(\log(|z|+1)+1),$$

since $z\in\mathbb{C}\setminus\cup_{i\neq k\in\mathbb{N}}B(i,\frac14)$. This proves (19)

We now turn to the proof that $\mathcal{W}(\nabla H_1)=+\infty$. This is done by computing a lower bound for $|\nabla H_1(z)|$, where $z=x+iy$. More precisely we prove that or any $\varepsilon>0$, there exists some positive constant C_1 depending on ε such that

$$|\nabla H_1(z)| \ge (\log(|z|+1)-C_1), \quad \text{if } |y|\ge\varepsilon|z|+1. \tag{20}$$

For this purpose we consider the meromorphic function

$$f(z):=\sum_{k\in\mathbb{N}}\frac{z}{k(k-z)}.$$

If $|y|\ge\varepsilon|z|+1$, then $z\in\mathbb{C}\setminus\cup_{k\in\mathbb{N}}B(k,\frac14)$. Thus

$$\left|f(z)-\sum_{1\le k\le[2|z|+1]}\left(\frac{1}{k-z}-\frac{1}{k}\right)\right| \le II \le 1,$$

so that

$$\begin{aligned}\left|f(z)+\sum_{1\le k\le[2|z|+1]}\frac{1}{k}\right| &\le 1+\sum_{1\le k\le[2|z|+1]}\left|\frac{1}{k-z}\right|\\ &\le 1+\sum_{1\le k\le[2|z|+1]}\left|\frac{1}{y}\right|\\ &\le 1+\sum_{1\le k\le[2|z|+1]}\frac{1}{|y|}\\ &\le 1+\frac{2|z|+1}{|y|}\\ &\le 1+2/\varepsilon.\end{aligned}$$

On the other hand, we have

$$\sum_{1\le k\le[2|z|+1]}\frac{1}{k} \ge \log(|z|+1),$$

hence (20) follows. We claim that this implies that $\mathcal{W}(\nabla H_1)=+\infty$.

To see this, we choose $\eta > 0$ small so that if we let E_η denote the regularization of ∇H_1, then $E_\eta = \nabla H_1$ on $\{z = x + iy \in B_{R-1} \mid |y| \geq \varepsilon|z| + 1\}$. Then, integrating (20) there proves that $\mathcal{W}_\eta(\nabla H_1) = +\infty$, hence $\mathcal{W}(\nabla H_1) = +\infty$.

We may now argue by contradiction to prove the proposition. Assume that there exists $H_2 \in \mathcal{P}_{\mathbb{N},0}$ such that $\mathcal{W}(\nabla H_2) < +\infty$. Then $H = H_2 - H_1$ is a harmonic function over $\mathbb{R}^2$. Let E_η denote as above the regularization of ∇H_1, then we have $\nabla H = (\nabla H_2)_\eta - E_\eta$. It follows from (15) and the finiteness of $\mathcal{W}((\nabla H_2)_\eta)$ that

$$\int_{K_R} |\nabla H|^2 \leq CR^2 \log^2 R.$$

This together with the harmonicity of H implies that ∇H is constant, which in turn implies that $\mathcal{W}(E_\eta)$ is finite, a contradiction. □

We summarize the content of this section in the following

Theorem 1. *We have*

$$\widetilde{\mathbb{W}}_0(\mathbb{Z}) = 0,\ \widetilde{\mathbb{W}}_0(\mathbb{N}) < +\infty,\ \widetilde{\mathbb{W}}_1(\mathbb{Z}^2 \setminus \mathbb{Z}) < +\infty,\ \widetilde{\mathbb{W}}_1(\mathbb{Z}^2 \setminus \mathbb{N}) < +\infty \tag{21}$$

$$\mathbb{W}_0(\mathbb{Z}) = 0,\ \mathbb{W}_1(\mathbb{Z}^2) < +\infty,\ \mathbb{W}_1(\mathbb{Z}^2 \setminus \mathbb{Z}) < +\infty \tag{22}$$

$$\mathbb{W}_0(\mathbb{N}) = +\infty,\ \mathbb{W}_1(\mathbb{Z}^2 \setminus \mathbb{N}) = +\infty \tag{23}$$

Proof. The result comes from Corollary 1, Corollary 2, Proposition 3 and Proposition 4. Indeed, it follows from Proposition 3 that

$$\mathbb{W}_0(\mathbb{Z}) < +\infty \Leftrightarrow \mathbb{W}_1(\mathbb{Z}^2 \setminus \mathbb{Z}) < +\infty$$
$$\mathbb{W}_0(\mathbb{N}) = +\infty \Leftrightarrow \mathbb{W}_1(\mathbb{Z}^2 \setminus \mathbb{N}) = +\infty$$

Therefore, (22) and (23) follows from Proposition 4 and the clear fact that $\mathbb{W}_1(\mathbb{Z}^2) < +\infty$. And (21) comes from Corollary 1 and Corollary 2. □

Remark 4. *The fact that* $\widetilde{\mathbb{W}}_0(\mathbb{N}) < +\infty$ *and* $\mathbb{W}_0(\mathbb{N}) = +\infty$ *means that there exists a vector field* E_η *such that*

$$\fint_{K_R} |E_\eta|^2$$

is bounded independently of R*, while this is the case for no gradient having the same divergence as* E_η*.*

5. Sufficient conditions for finite renormalized energy

Theorem 2. *Given a discrete* Λ*, assume there exists* $m \geq 0$ *and* $\varepsilon \in (0,1)$*,* $C > 0$ *such that for any* $x \in \mathbb{R}^d$ *and for* $R > 1$*, we have*

$$\left| \sharp \left(B(x,R) \cap \Lambda\right) - m|B_R^d| \right| \leq CR^{d-1-\varepsilon} \tag{24}$$

and

$$\inf_{x,y \in \Lambda, x \neq y} |x - y| > 0 \tag{25}$$

Then $\mathbb{W}_m(\Lambda) < +\infty$*.*

Remark 5. *For a Bravais lattice, the assumptions in the above theorem are satisfied. It was proved by Landau (1915) — see* [12] *for a more general statement — that the first assumption holds with* $\varepsilon = 1-2/(d+1)$*, see* [11] *for references on more recent developments.*

We recall a technical lemma.

Lemma 1. *(Theorem 8.17 in* [9]*) Assume* $q > d$ *and* $p > 1$*. There exists a constant* $C > 0$ *such that the following holds.*

If u *satisfies*

$$-\Delta u = g + \sum_i \partial_i f_i$$

in some domain $\Omega \subset \mathbb{R}^d$ *such that* $B(0, 2R) \subset \Omega$*, then*

$$\|u\|_{L^\infty(B(0,R))} \leq C(R^{-\frac{d}{p}} \|u\|_{L^p(B(0,2R))} + R^{1-\frac{d}{q}} \|f\|_{L^q(B(0,2R))} + R^{2-\frac{2d}{q}} \|g\|_{L^{q/2}(B(0,2R))}).$$

Proof of Theorem 2. Assume Λ satisfies (24) and (25). The proof consists in constructing $E \in \mathcal{F}_{\Lambda,m}$ such that $\mathcal{W}(E) < +\infty$, which is done by successive approximations constructing first some U^1, then a correction U^2 to U^1, then a correction U^3 to $U^1 + U^2$, etc... In this construction, the U^k's are functions, and the sum of their gradients will converge to E.

Let $R_n = 2^{n-1}$. For all $p \in \Lambda$, we let U^1_p be the solution to

$$\begin{cases} -\Delta U^1_p(y) & = & c_d \left(\delta^{(\eta)}_p(y) - \dfrac{\mathbf{1}_{B(p,R_1)}(y)}{|B(0,R_1)|} \right) & \text{in } B(p, R_1) \\ U^1_p(y) = \dfrac{\partial U^1_p}{\partial \nu}(y) & = & 0 & \text{on } \partial B(p, R_1) \end{cases}$$

where $\mathbf{1}_{B(x,r)}$ is the indicator function of the ball $B(x, r)$. The existence of a solution with Neumann boundary conditions follows from the fact that $\delta^{(\eta)}_p - \frac{\mathbf{1}_{B(p,R_1)}}{\pi {R_1}^2}$ has zero integral, and the radial symmetry of the solution implies U^1_p is constant on the boundary, and the constant can be taken equal to zero. In fact, extending U^1_p by zero outside $B(p, R_1)$, we get a solution of

$$-\Delta U^1_p(y) = c_d \left(\delta^{(\eta)}_p(y) - \frac{\mathbf{1}_{B(p,R_1)}(y)}{|B(0,R_1)|} \right)$$

in $\mathbb{R}^d$, which is supported in $B(p, R_1)$.

We let

$$U^1(y) := \sum_{p\in\Lambda} U^1_p(y).$$

This sum is well defined since, Λ being discrete, it is locally finite. Moreover U^1 solves

$$(26) \qquad -\Delta U^1(y) = c_d \left(\sum_{p\in\Lambda} \delta^{(\eta)}_p - n_1(y) \right), \quad \text{where} \quad n_1(y) := \frac{\sharp\,(\Lambda \cap B(y, R_1))}{|B(0, R_1)|}$$

Then we proceed by induction. For any $k \geq 2$ we let U^k be the solution to

$$(27) \qquad \begin{cases} -\Delta U^k_p(y) & = & c_d \left(\dfrac{\mathbf{1}_{B(p,R_{k-1})}(y)}{|B(0,R_{k-1})|} - \dfrac{\mathbf{1}_{B(p,R_k)}(y)}{|B(0,R_k)|} \right) & \text{in } B(p, R_k) \\ U^k_p(y) = \frac{\partial U^k_p}{\partial \nu}(y) & = & 0 & \text{on } \partial B(p, R_k), \end{cases}$$

and we let $U_p^k = 0$ outside $B(p, R_k)$. We let $U^k(y) := \sum_{p\in\Lambda} U_p^k(y)$, so that

$$-\Delta U^k(y) = c_d \left(n_{k-1}(y) - n_k(y)\right),$$

where, for any $k \in \mathbb{N}$,

$$n_k(y) := \frac{\sharp\left(\Lambda \cap B(y, R_k)\right)}{|B(0, R_k)|}.$$

Now we study the convergence of $\sum_{k=1}^\infty \nabla U^k$.

First we note that since $R_k = 2R_{k-1}$ we have for any $k \geq 2$ that

$$U_p^k(y) = R_k^{2-d} v\left(|y-p|/R_k\right), \tag{28}$$

where v is a bounded C^1 function on $\mathbb{R}_+$ supported in $[0, 1]$ and independent of k. It follows, since $\|\nabla U_p^k\|_\infty \leq C R_k^{1-d}$ and the sum defining U^k has at most $C R_k^d$ non zero terms, that

$$\|\nabla U^k\|_\infty \leq C R_k. \tag{29}$$

Second we estimate $\|U^k\|_\infty$. We claim that

$$\forall k \geq 2,\ \exists C_k \in \mathbb{R} \text{ such that } \|U^k(y) - C_k\|_\infty = O(R_k^{1-\varepsilon}). \tag{30}$$

Indeed, let $a_y(r) := \sharp\left(B(y, r) \cap \Lambda\right)$. Then we have for any $y \notin \Lambda$

$$U^k(y) = R_k^{2-d} \sum_{p\in B(y,R_k)\cap\Lambda} v\left(\frac{|p-y|}{R_k}\right) = R_k^{2-d} \int_0^{R_k} v\left(\frac{t}{R_k}\right) a_y'(t)dt = -R_k^{2-d} \int_0^{R_k} \frac{1}{R_k} v'\left(\frac{t}{R_k}\right) a_y(t)dt.$$

But, using (24), we have $a_y(t) = m|B(0,t)| + O(t^{d-1-\varepsilon})$, hence

$$U^k(y) = -m R_k^{2-d} \int_0^{R_k} \frac{1}{R_k} v'\left(\frac{t}{R_k}\right) |B(0,t)|dt + O(R_k^{1-\varepsilon}).$$

The first term is independent of y, we call it C_k. This proves (30).

On the other hand, from (24) we have

$$\|n_k - m\|_\infty \leq C R_k^{-1-\varepsilon}, \tag{31}$$

so that

$$\|\Delta U^k\|_\infty = O(R_k^{-1-\varepsilon}). \tag{32}$$

Now, we claim that (30) and (32) imply that

$$\|\nabla U^k\|_\infty = O(R_k^{-\varepsilon}). \tag{33}$$

To see this we use the elliptic estimate of Lemma 1. For all $y \in \mathbb{R}^d$ we have

$$\int_{B(y,R_k)} |\nabla U^k|^2 = \int_{B(y,R_k)} |\nabla(U^k - C_k)|^2 \tag{34}$$

$$= -\int_{B(y,R_k)} \Delta U^k (U^k - C_k) + \int_{\partial B(y,R_k)} \frac{\partial U^k}{\partial \nu}(U^k - C_k) \leq C R_k^d \left(R_k^{-2\varepsilon} + \|\nabla U^k\|_\infty R_k^{-\varepsilon}\right)$$

Now we apply Lemma 1. We have

$$-\Delta\left(\nabla U^k\right) = c_d \nabla\left(n_{k-1} - n_k\right),$$

therefore for any $q > d$ and $p > 1$,

$$\|\nabla U^k\|_{L^\infty(B_{R_k/2})} \le C\left(R_k^{-\frac{d}{p}}\|\nabla U^k\|_{L^p(B_{R_k})} + R_k^{1-\frac{d}{q}}\|n_{k-1}-n_k\|_{L^q(B_{R_k})}\right).$$

Then, taking $p = 2$ and noting that (31) implies $\|n_{k-1}-n_k\|_q \le CR_k^{\frac{d}{q}-(1+\varepsilon)}$, we find using (34) that

$$\|\nabla U^k\|_{L^\infty(B_{R_k/2})} \le C\left(R_k^{-2\varepsilon} + R_k^{-\varepsilon}\|\nabla U^k\|_{L^\infty(B_{R_k})}\right)^{\frac{1}{2}} + CR_k^{-\varepsilon}.$$

This proves (33).

Now (33) implies that the sum $\sum_{k\ge 2}\nabla U^k$ converges, and if we let $E_\eta = \nabla U_1 + \sum_{k\ge 2}\nabla U^k$, then $-\operatorname{div} E_\eta = c_d(\sum_{p\in\Lambda}\delta_p^{(\eta)} - m)$, using (26), (27) and (31). Moreover E_η is a gradient since it is a sum of gradients, thus it is the regularization of some $E \in \mathcal{P}_{\Lambda,m}$.

To conclude, summing (33) we find that E_η is bounded in L^∞ and therefore that $\mathcal{W}(E_\eta) < \infty$, which implies that $\mathcal{W}(E) < +\infty$, and then that $\mathbb{W}_m(\Lambda) < +\infty$. $\square$

For $\widetilde{\mathbb{W}}_m$ the hypothesis of Theorem 2 can be slightly relaxed.

Theorem 2′. *Assume there exists some non-negative number $m \ge 0$ and some positive numbers $\varepsilon \in (0,1)$, $C > 0$ and a increasing sequence $\{R_n\}$ tending to $+\infty$ such that for any $x \in \mathbb{R}^d$ and for any $n \in \mathbb{N}$, we have*

$$|\sharp\,(B(x,R_n)\cap\Lambda) - m|B(0,R_n)|| \le CR_n^{d-1-\varepsilon},$$

and such that

$$\sum_n R_n^{-\varepsilon} < +\infty$$

and

$$\inf_{x,y\in\Lambda, x\ne y}|x-y| > 0.$$

Then $\widetilde{\mathbb{W}}_m(\Lambda) < +\infty$.

To prove Theorem 2', we will first use the following simple estimate.

Lemma 2. *Let u be a solution of the following problem in $B(0,R) \subset \mathbb{R}^d$.*

$$\begin{cases} -\Delta u &= f \quad \text{in } B(0,R) \\ \frac{\partial u}{\partial \nu} &= 0 \quad \text{on } \partial B(0,R) \end{cases}$$

Then

$$\int_{B(0,R)} |\nabla u|^2 \le CR^{d+2}\|f\|_\infty^2$$

where C is a constant independent of R.

Proof. We have

$$\int_{B(0,R)}|\nabla u|^2 = -\int_{B(0,R)} u\Delta u = \int_{B(0,R)} fu \le \|u\|_1\|f\|_\infty \le \sqrt{|B(0,R)|}\|u\|_2\|f\|_\infty$$

Without loss of generality, we assume $\int u = 0$. By Poincaré inequality,

$$\|u\|_2 \le CR\|\nabla u\|_2.$$

Inserting in the previous inequality, the desired result follows. $\square$

Proof of Theorem $2'$. Define

$$m_0 = \sum_{p\in\Lambda} \delta_p^{(\eta)}$$

and, for any integer $k \geq 1$,

$$I_k = \frac{\mathbf{1}_{B_{R_k}}}{|B_{R_k}|},$$

where $\mathbf{1}_{B_{R_k}}$ is the indicator function of the ball $B(0, R_k)$.

We let for any $k \geq 1$

$$m_k = m_{k-1} * I_k * I_k$$

and, for any $y \in \mathbb{R}^d$, we let the function $U_{k,y}$ be the solution to

$$\begin{cases} -\Delta U_{k,y}(x) &= c_d\,(m_{k-1}(x) - m_{k-1} * I_k(y))\,\mathbf{1}_{B_{R_k}}(x-y) & \text{in } B(y, 2R_k) \\ \frac{\partial U_{k,y}}{\partial \nu}(x) &= 0 & \text{on } \partial B(y, 2R_k). \end{cases}$$

Then we let

$$E_k(x) = \frac{1}{|B(0,R_k)|}\int_{\mathbb{R}^d} \nabla U_{k,y}(x)\,dy.$$

We have

$$\begin{aligned} -\operatorname{div} E_k(x) &= c_d\left(m_{k-1}(x) - \frac{1}{|B(0,R_k)|}\int_{\mathbb{R}^d} m_{k-1} * I_k(y)\mathbf{1}_{B_{R_k}}(x-y)\,dy\right) \\ &= c_d\,(m_{k-1}(x) - m_{k-1} * I_k * I_k(x)) \\ &= c_d\,(m_{k-1}(x) - m_k(x))\,. \end{aligned} \tag{35}$$

We claim that

$$m_k(y) = m + O(R_k^{-1-\varepsilon}). \tag{36}$$

To see this, it suffices to note that from the commutativity of the convolution we have

$$m_k - m = (m_0 * I_k - m) * (I_k * I_{k-1} * I_{k-1} * \cdots * I_1 * I_1),$$

where we also used the fact that, since m is a constant, $m * I_j = m$. Then, from our first assumption, we have $|m_0 * I_k - m| \leq CR_k^{-(1+\varepsilon)}$, which implies (36) since every I_k is a positive function with integral 1, and thus convoluting a function with it does not increase the L^∞ norm.

From (36) it then follows that $|m_{k-1}(.) - m_{k-1} * I_k(y)|_\infty < CR_k^{-1-\varepsilon}$. Then, from (5) and using Lemma 2 we find

$$\|\nabla U_y^k\|^2_{L^2(B(y,2R_k))} \leq CR_k^{d+2}R_{k-1}^{-2-2\varepsilon} \leq CR_k^{d-2\varepsilon}.$$

Finally, using Lemma 1 as in the proof of Theorem 2 with $q > d$ arbitrary, we obtain

$$\begin{aligned} \|\nabla U_y^k\|_{L^\infty(B(y,R_k))} &\leq C\left(R_k^{-\frac{d}{2}}\|\nabla U_y^k\|_{L^2(B(y,2R_k))} + R_k^{1-\frac{d}{q}}\|m_{k-1} - m_{k-1} * I_k\|_{L^q(B(y,R_k))}\right) \\ &\leq C\left(R_k^{-\frac{d}{2}}R_k^{\frac{d}{2}-\varepsilon} + R_k^{1-\frac{d}{q}}R_k^{-1-\varepsilon}R_k^{\frac{d}{q}}\right) \leq CR_k^{-\varepsilon}. \end{aligned}$$

It follows that $\|E_k\|_\infty \leq CR_k^{-\varepsilon}$ and thus $E_\eta := \sum_k E_k$ is well defined and satisfies $-\operatorname{div} E_\eta = c_d(\sum_{p\in\Lambda}\delta_p^{(\eta)} - m)$. Moreover it is bounded in L^∞ hence $\mathcal{W}(E_\eta) < +\infty$. Thus

E_η is the regularization of some E such that $-\operatorname{div} E = c_d(\sum_{p\in\Lambda}\delta_p - m)$ and $\mathcal{W}(E) < +\infty$, which proves that $\widetilde{\mathbb{W}}_m(\Lambda)$ is finite. □

The conditions in Theorem 2 are optimal in the following sense. We state this in the case $d = 2$ and $m = 0$ for convenience, but a similar construction works for any $d \geq 2$ and $m \geq 0$.

Proposition 5. *There exists $\Lambda \subset \mathbb{R}^2$ such that $\widetilde{\mathbb{W}}_0(\Lambda) = +\infty$ and for any $x \in \mathbb{R}^2$ and any $R > 1$*

$$\sharp\,(B(x,R)\cap\Lambda) \leq CR^{1+\varepsilon}. \tag{37}$$

Proof. The counter-example is as follows. Given $\varepsilon > 0$, we choose $1 < \alpha < 2$ such that $2/\alpha < 1+\varepsilon$ and we let

$$\Lambda = \{(k^\alpha, l^\alpha) \mid k, l \in \mathbb{Z}\}.$$

Assume $-\operatorname{div} E = \sum_{p\in\Lambda}\delta_p$. We need to prove that Λ satisfies(37) and that $\mathcal{W}(E) = +\infty$.

To see that (37) is satisfied, note that since $\alpha > 1$, the density of points in Λ decreases as one moves away from the origin. Thus it suffices to check that $\sharp\,(B(0,R)\cap\Lambda) \leq CR^{1+\varepsilon}$. But $B(0,R) \subset K_R$, where $K_R = [-R,R]\times[-R,R]$, and if $R = n^\alpha$ then we can compute explicitely

$$\sharp\,(K_R\cap\Lambda) = (2n+1)^2 = (2R^{\frac{1}{\alpha}}+1)^2 \leq CR^{\frac{2}{\alpha}} \leq CR^{1+\varepsilon}.$$

Therefore (37) holds.

To prove that $\mathcal{W}(E) = +\infty$ it suffices to prove that $\mathcal{W}(E_\eta) = +\infty$ for some fixed arbitrary $\eta > 0$, where E_η is the regularization of E, which satisfies

$$-\operatorname{div} E_\eta = \sum_{p\in\Lambda}\delta_p^{(\eta)}.$$

For this we compute

$$\begin{aligned}
\int_{K_R}|E_\eta|^2 &\geq \int_0^R\left(\int_{\partial K_t}(E_\eta\cdot\nu)^2\right)dt\\
&\geq \int_0^R\frac{1}{|\partial K_t|}\left(\int_{\partial K_t}(E_\eta\cdot\nu)\right)^2dt\\
&= \int_0^R\frac{1}{|\partial K_t|}\left(\int_{K_t}\operatorname{div} E_\eta\right)^2dt\\
&\geq c\int_0^R\frac{1}{t}(t^{\frac{2}{\alpha}})^2\,dt = cR^{\frac{4}{\alpha}},
\end{aligned}\tag{38}$$

where c is a generic small positive number independant of R.

Since $\alpha < 2$, we deduce that $⨍_{K_R}|E_\eta|^2$ tends to $+\infty$ as $R \to +\infty$, and then that $\mathcal{W}(E_\eta) = +\infty$. □

6. Critical case

In view of Theorem 2 and Proposition 5, the critical discrepancy between $\sum_{p\in\Lambda}\delta_p$ and the uniform measure $m\,dx$ is when $|\sharp\,(B(x,R)\cap\Lambda) - m|B(0,R)|| = O(R^{d-1})$. This includes the cases $\Lambda = \mathbb{Z}\subset\mathbb{R}^2$ or $\mathbb{N}\subset\mathbb{R}^2$. As shown by Theorem 1, we cannot expect

$W(\Lambda)$ to be finite under such an assumption. However something can be said for $\tilde{W}$, which relies on Proposition 2 and a result of M.Laczkovich [16, 17], at the basis of which lies a duality argument which can be generalized to measures not necessarily of the form $\sum_{p\in\Lambda}\delta_p$ (see M.Sodin-B.Tsirelson [31], or G.Strang [32]).

We cite the following theorem from [17]. Here $\mathrm{Per}(A)$ denotes the $(d-1)$-dimensional Hausdorff measure of $\partial A \subset \mathbb{R}^d$, whereas $\mathrm{Per}_1(A)$ is a variant of perimeter defined as the d-dimensional Lebesgue measure of the set of $x \in \mathbb{R}^d$ such that $d(x, \partial A) < 1$.

Theorem 3. *Let Λ be a discrete subset of $\mathbb{R}^d$ and $m > 0$. Then the following statements are equivalent:*

i) *There exists $C > 0$ such that for any finite union of unit cubes A it holds that*
$$|\#(\Lambda \cap A) - m|A|| \le C\,\mathrm{Per}(A),$$
ii) *There exists $C > 0$ such that for bounded measurable A it holds that*
$$|\#(\Lambda \cap A) - m|A|| \le C\,\mathrm{Per}_1(A),$$
iii) *There exists a bijection from $\varphi : \Lambda \to m^{-\frac{1}{d}}\mathbb{Z}^d$ such that $\sup_{p\in\Lambda} |\varphi(p) - p| < +\infty$.*

Together with Proposition 2 this implies immediately that

Theorem 4. *Let $\Lambda \subset \mathbb{R}^d$ be discrete, uniformly separated, and such that either*

i) *There exists $C > 0$ such that for any finite union of unit cubes A it holds that*
$$|\#(\Lambda \cap A) - m|A|| \le C\,\mathrm{Per}(A),$$
or

ii) *There exists $C > 0$ such that for bounded measurable A it holds that*
$$|\#(\Lambda \cap A) - m|A|| \le C\,\mathrm{Per}_1(A),$$
then $\widetilde{\mathbb{W}}_m(\Lambda) < +\infty$.

In the case $d = 2$ and if Λ is a subset of $\mathbb{Z}^2$ we have a simpler statement

Corollary 3. *Assume $\Lambda = \mathbb{Z}^2 \setminus V$, where V is such that there exists $C > 0$ such that for $x \in \mathbb{R}^2$ and any $R > 1$ it holds that*
$$\#(V \cap B(x, R)) \le CR,$$
then $\widetilde{\mathbb{W}}_1(\Lambda) < +\infty$.

Proof. We show that the inequality
$$|\#(\Lambda \cap A) - |A|| \le C\,\mathrm{Per}(A) \tag{39}$$
holds for any finite union of cubes A. Let $A_1, \dots, A_k$ denote the connected components of A. Then each A_i may be included in a ball B_i of radius R_i such that $R_i \le C\,\mathrm{Per}(A_i)$ for some universal constant $C > 0$. This is of course specific to the dimension 2. Applying the hypothesis of the corollary in B_i we find that $V \cap B_i$ has cardinal at most CR_i, hence $\#(V \cap A_i) \le C\,\mathrm{Per}(A_i)$. On the other hand $|\#(\mathbb{Z}^2 \cap A_i) - |A_i|| < C\,\mathrm{Per}(A_i)$ (a way to see this is to apply Theorem 3 to $\Lambda = \mathbb{Z}^2$).

It follows that $|\#(\Lambda \cap A_i) - |A_i|| < C\,\mathrm{Per}(A_i)$, and summing over i we deduce (39). □

Another easy consequence of Proposition 2, for which we omit a proof is

Corollary 4. *Let $\Lambda \subset \mathbb{R}^{d+1}$ be discrete and uniformly separated, and of the form $\Lambda = \Lambda_1 \times \mathbb{Z}^d$, where $\Lambda_1 \subset \mathbb{R}$.*

If there exists $C > 0$ such that for any $x \in \mathbb{R}$ and $R > 1$ we have $|\sharp(\Lambda_1 \cap [x-R, x+R] - 2R| \le C$ then $\widetilde{\mathbb{W}}_1(\Lambda) < +\infty$.

7. A Penrose lattice

We now describe the construction of a Penrose-type lattice Λ such that $\tilde{W}(\Lambda) < +\infty$. Of course it would be better to show that Λ satisfies the hypothesis of Theorem 2, but this is to our knowledge an open problem.

For simplicity, we consider the Robinson triangle decompositions in Penrose's second tilling (P2)–kite and dart tiling, or in Penrose's third tilling (P3)–rhombus tiling, (see [29]). The construction is as follows: Ω_1 and Ω_2 are two Robinson triangles, namely, Ω_1 is an obtuse Robinson triangle having side lengths $(1, 1, \varphi)$, while Ω_2 is an acute triangle with sidelengths $(\varphi, \varphi, 1)$, where $\varphi = (1+\sqrt{5})/2$; the scaled-up domain $\varphi\Omega_1$ decomposes as the union of a copy of Ω_1 and a copy of Ω_2, where the interiors are disjoint — and such that $\varphi\Omega_2$ decomposes as the union of one copy of Ω_1 and two copies of Ω_2 with disjoint interiors (see figure).

For $i = 1, 2$ we choose a point p_i in the interior of Ω_i.

Then we proceed by induction, starting with Ω_1 choosing p_1 as the origin, then scaling up by φ, then decomposing, then scaling up again, then decomposing each piece, etc... After n steps we have a (large domain) $\varphi^n\Omega_1$ tiled by a number of copies of either Ω_1 or Ω_2. In each tile we have a distinguished point, the union of these points is denoted Λ_n. As $n \to +\infty$ and modulo a subsequence, Λ_n converges to a discrete set Λ, which is uniformly separated since the distance between two points is no less than $\min\left(d(p_1, \partial\Omega_1), d(p_2, \partial\Omega_2)\right)$.

Theorem 5. *We have $\widetilde{\mathbb{W}}_m(\Lambda) < +\infty$, for some $m > 0$.*

Proof. As usual, we choose some $\eta > 0$ and prove that there exists a solution E_η of $-\operatorname{div} E_\eta = 2\pi(\delta_p^{(\eta)} - m)$ for some $m > 0$ such that $\mathcal{W}(E_\eta) < +\infty$.

For each n we define a current E_n as follows. On each copy of Ω_i we let E_n be equal to (a copy of) ∇U_i, where

$$\begin{cases} -\Delta U_i &= 2\pi\left(\delta_{p_i}^{(\eta)} - \frac{1}{|\Omega_i|}\right) & \text{in } \Omega_i \\ \frac{\partial U_i}{\partial \nu} &= 0 & \text{on } \partial\Omega_i. \end{cases}$$

Then E_n converges as $n \to +\infty$ to a current $\tilde{E}_\eta$ such that the following holds in $\mathbb{R}^2$

$$-\operatorname{div} \tilde{E}_\eta = 2\pi\left(\sum_{p\in\Lambda} \delta_p^{(\eta)} - \alpha\right),$$

where $\alpha = 1/|\Omega_i|$ on each copy of Ω_i. It is not difficult to check that $\mathcal{W}(\tilde{E}_\eta) < +\infty$, but the background density α is not constant. We need to add a correction to $\tilde{E}_\eta$, which is the object of the following

Lemma 3. *There exist $m > 0$ and a solution of the following equation in $\mathbb{R}^2$*

$$-\operatorname{div} E' = \alpha - m \tag{40}$$

such that $\|E'\|_\infty < +\infty$.

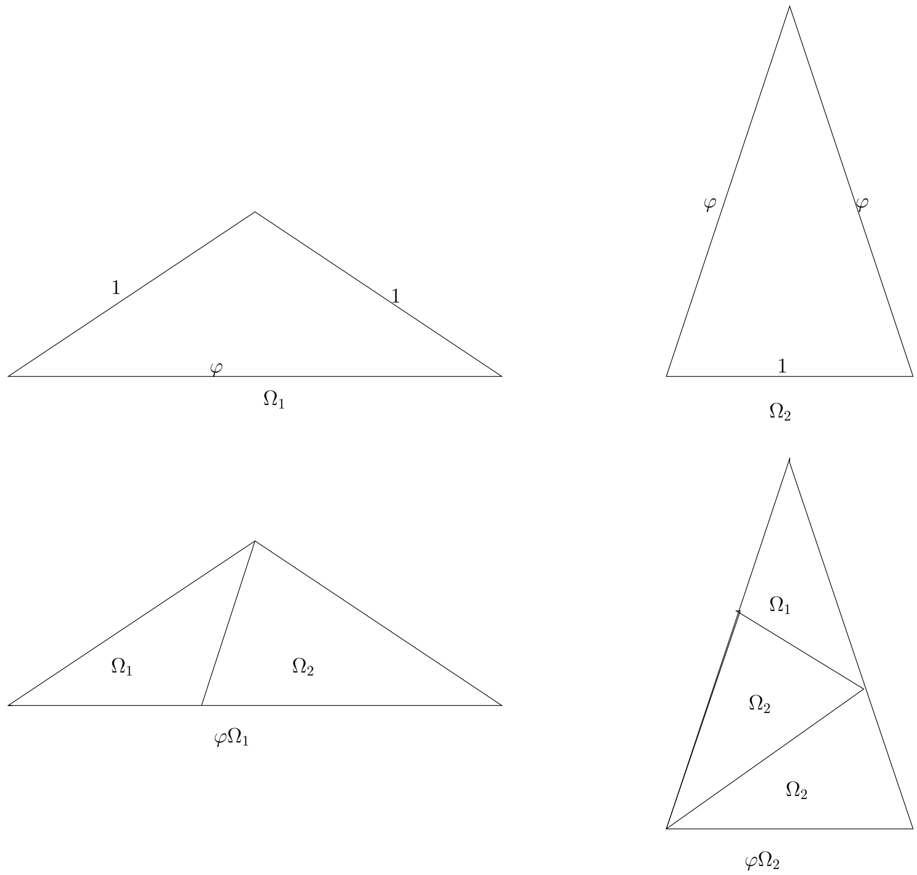

FIGURE 1

Assuming the lemma is true we let $E_\eta = \tilde{E_\eta} + E'$. Then $-\operatorname{div} E_\eta = 2\pi\left(\sum_{p\in\Lambda}\delta_p^{(\eta)} - m\right)$ thus the corresponding $E \in \mathcal{F}_{\Lambda,m}$ for the background m, and the fact that $\mathcal{W}(\tilde{E_\eta}) < +\infty$ and $E' \in L^\infty$ implies that $W(E_\eta) < +\infty$ and the Theorem. $\square$

Proof of Lemma 3. The current E' is obtained as the limit of E_n, where E_n solves

$$\left\{\begin{array}{lcll} -\operatorname{div} E_n & = & 2\pi(\alpha_n - m_n) & \text{in } \varphi^n\Omega_1 \\ E_n\cdot\nu & = & 0 & \text{on } \partial(\varphi^n\Omega_1), \end{array}\right. \tag{41}$$

where $\alpha_n : \varphi^n\Omega_1 \to \mathbb{R}$ is the function equal to $1/|\Omega_i|$ on each of the copies of Ω_i, $i = 1, 2$ which tile $\varphi^n\Omega_1$, and where m_n is equal to the average of α_n on $\varphi^n\Omega_1$.

The current E_n is defined recursively. First we define the equivalent of α_n for Ω_2-type domains: For any integer n we tile $\varphi^n\Omega_2$ by one copy of $\varphi^{n-1}\Omega_1$ and two copies of $\varphi^{n-1}\Omega_2$, then we tile each of the three pieces, etc... until we have tiled $\varphi^n\Omega_2$ by copies of either Ω_1 or Ω_2. then we let $\beta_n : \varphi^n\Omega_2 \to \mathbb{R}$ be the function equal to $1/|\Omega_i|$ on each of the copies of Ω_i, $i = 1, 2$. We also define q_n to be the equivalent of m_n, i.e. the average of β_n on $\varphi^n\Omega_2$. Finally we define $\bar{E}_n$ to be the equivalent of E_n for type 2 domains, i.e. the solution of (41) with α_n replaced by β_n, m_n replaced by q_n and Ω_1 replaced by Ω_2.

Below it will be convenient to abuse notation by writing $\varphi^n\Omega_i$ for a copy of $\varphi^n\Omega_i$. Then we have $\varphi^n\Omega_1 = \varphi^{n-1}\Omega_1 \cup \varphi^{n-1}\Omega_2$. We let

$$E_n = E_{n-1}\mathbf{1}_{\varphi^{n-1}\Omega_1} + \bar{E}_{n-1}\mathbf{1}_{\varphi^{n-1}\Omega_2} + \nabla U_n \mathbf{1}_{\varphi^n\Omega_1}, \tag{42}$$

where

$$\begin{cases} -\Delta U_n & = & 2\pi(m_n - m_{n-1})\mathbf{1}_{\varphi^{n-1}\Omega_1} + 2\pi(m_n - q_{n-1})\mathbf{1}_{\varphi^{n-1}\Omega_2} & \text{in } \varphi^n\Omega_1 \\ \frac{\partial U_n}{\partial \nu} & = & 0 & \text{on } \partial(\varphi^n\Omega_1). \end{cases} \tag{43}$$

It is straightforward to check that E_n satisfies (41) assuming E_{n-1} and $\bar{E}_{n-1}$ do.

The relation (42) is the recursion relation which repeated n times allows to write E_n as equal to a sum of on the one hand error terms ∇U_k (or their type 2 equivalent that we denote V_k), for k between 1 and n, and on the other hand of a vector field which on each elementary tile of type Ω_1 of $\varphi^n\Omega_1$ is equal to E_0 and on a tile of type Ω_2 is equal to $\bar{E}_0$. However from (41) we may take $E_0 = 0$ and $\bar{E}_0 = 0$, thus we are left with evaluating the error terms.

Claim: There exists $C > 0$ such that for any integer $k > 0$ we have

$$\|\nabla U_k\|_\infty, \|\nabla V_k\|_\infty \le C\varphi^{-3k}.$$

This clearly proves that the sum of errors for $k = 1 \dots n$ is bounded in L^∞ independently of n and therefore that $\{E_n\}$ is bounded in L^∞. Then the limit E' is in L^∞.

To prove the lemma, it remains to prove the claim, and to show that E' satisfies (40) for some $m > 0$, which in view of (41) amounts to showing that $\{m_n\}_n$ converges to such an m. For this we define u_{2n} (resp. u_{2n+1}) be the number of elementary tiles of type Ω_1 (resp. Ω_2) in $\varphi^n\Omega_1$. We define similarly v_{2n} and v_{2n+1} by replacing Ω_1 by Ω_2. Therefore $u_0 = 1$, $u_1 = 0$, $v_0 = 0$, $v_1 = 1$. We have the following recurrence relations

$$u_{2n+2} = u_{2n} + u_{2n+1}, \quad u_{2n+3} = u_{2n} + 2u_{2n+1},$$

which we can summarize as the single relation $u_{n+2} = u_{n+1} + u_n$. Similarly $v_{n+2} = v_{n+1} + v_n$. It follows that

$$u_n = \varphi^n \frac{1}{\varphi+2} + (-\varphi)^{-n}\frac{\varphi+1}{\varphi+2}, \quad v_n = \varphi^n\frac{\varphi}{\varphi+2} + (-\varphi)^{-n}\frac{-\varphi}{\varphi+2}.$$

We have $u_n = a\varphi^n + O(\varphi^{-n})$ and $v_n = b\varphi^n + O(\varphi^{-n})$ with $a = \frac{1}{\varphi+2}$ and $b = \frac{\varphi}{\varphi+2}$ strictly positive. Then we easily deduce that

$$m_n = \frac{u_{2n} + u_{2n+1}}{u_{2n}|\Omega_1| + u_{2n+1}|\Omega_2|} = m + O(\varphi^{-4n}),$$

where

$$m = \frac{1+\varphi}{|\Omega_1| + \varphi|\Omega_2|},$$

and similarly that $q_n = m + O(\varphi^{-4n})$. This proves in particular the convergence of $\{m_n\}_n$. Moreover it shows that the right-hand side of (43) is bounded by $C\varphi^{-4n}$. By elliptic regularity (lemma 1 and lemma 2) we deduce that

$$\|\nabla U_n\|_\infty \le C|\varphi^n\Omega_1|^{\frac{1}{2}}\varphi^{-4n} = C|\Omega_1|^{\frac{1}{2}}\varphi^{-3n},$$

and a similar bound for V_n. This proves the claim, and the lemma □

Remark 6. *The above construction could easily be generalized to similar recursive constructions.*

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