

HYPERBOLIC ALEXANDROV-FENCHEL QUERMASINTEGRAL INEQUALITIES II

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ABSTRACT. In this paper we first establish an optimal Sobolev type inequality for hypersurfaces in \mathbb{H}^n (see Theorem 1.1). As an application we obtain hyperbolic Alexandrov-Fenchel inequalities for curvature integrals and quermassintegrals. Precisely, we prove a following geometric inequality in the hyperbolic space \mathbb{H}^n , which is a hyperbolic Alexandrov-Fenchel inequality,

$$\int_{\Sigma} \sigma_{2k} \geq C_{n-1}^{2k} \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k} \frac{n-1-2k}{n-1}} \right\}^k,$$

provided that Σ is a horospherical convex, where $2k \leq n-1$. Equality holds if and only if Σ is a geodesic sphere in \mathbb{H}^n . Here $\sigma_j = \sigma_j(\kappa)$ is the j -th mean curvature and $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$ is the set of the principal curvatures of Σ . Also, an optimal inequality for quermassintegrals in \mathbb{H}^n is as following:

$$W_{2k+1}(\Omega) \geq \frac{\omega_{n-1}}{n} \sum_{i=0}^k \frac{n-1-2k}{n-1-2k+2i} C_k^i \left(\frac{nW_1(\Omega)}{\omega_{n-1}} \right)^{\frac{n-1-2k+2i}{n-1}},$$

provided that $\Omega \subset \mathbb{H}^n$ is a domain with $\Sigma = \partial\Omega$ horospherical convex, where $2k \leq n-1$. Equality holds if and only if Σ is a geodesic sphere in \mathbb{H}^n . Here $W_r(\Omega)$ is quermassintegrals in integral geometry.

1. INTRODUCTION

In this paper we first establish Sobolev type inequalities for hypersurfaces in the hyperbolic space \mathbb{H}^n . Let g be a Riemannian metric on a Riemannian manifold. Its k th Gauss-Bonnet curvature (or Lovelock curvature) L_k is defined by (see [13] for instance)

$$(1.1) \quad L_k := \frac{1}{2^k} \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}}.$$

Here R_{ij}^{kl} is the Riemannian curvature with respect to g , and the generalized Kronecker delta is defined by

$$\delta_{i_1 i_2 \dots i_r}^{j_1 j_2 \dots j_r} = \det \begin{pmatrix} \delta_{i_1}^{j_1} & \delta_{i_1}^{j_2} & \dots & \delta_{i_1}^{j_r} \\ \delta_{i_2}^{j_1} & \delta_{i_2}^{j_2} & \dots & \delta_{i_2}^{j_r} \\ \vdots & \vdots & \ddots & \vdots \\ \delta_{i_r}^{j_1} & \delta_{i_r}^{j_2} & \dots & \delta_{i_r}^{j_r} \end{pmatrix}.$$

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When $k = 1$, L_1 is just the scalar curvature R . When $k = 2$, it is the so-called (second) Gauss-Bonnet curvature

$$L_2 = \|Rm\|^2 - 4\|Ric\|^2 + R^2,$$

where Rm , Ric are the Riemannian curvature tensor, and the Ricci tensor with respect to g respectively. The Gauss-Bonnet curvature L_k is a very natural generalization of the scalar curvature. When the underlying manifold is local conformally flat, L_k equals to the σ_k -scalar curvature up to a constant multiple, precisely (cf. [14])

$$(1.2) \quad L_k = 2^k k! (n-1-k)(n-2-k) \cdots (n-2k) \sigma_k(g).$$

Here the σ_k -scalar curvature was introduced in Viaclovsky [30] by

$$(1.3) \quad \sigma_k(g) := \sigma_k(\Lambda_g),$$

and Λ_g is the set of the eigenvalues of the Schouten tensor A_g defined by

$$(1.4) \quad A_g = \frac{1}{n-3} \left(Ric_g - \frac{R_g}{2(n-2)} g \right).$$

Here we consider the $(n-1)$ -dimensional manifold M with metric g . The σ_k -scalar curvature is also a very natural generalization of the scalar curvature R (in fact, $\sigma_1(g) = \frac{1}{2(n-2)}R$) and has been intensively studied in the fully nonlinear Yamabe problem. The fully nonlinear Yamabe problem for σ_k is a generalization of ordinary Yamabe problem for the scalar curvature R . In the ordinary Yamabe problem, the following functional, the so-called Yamabe functional, plays a crucial role

$$(1.5) \quad \mathcal{F}_1(g) = (\text{vol}(g))^{-\frac{n-3}{n-1}} \int R_g d\mu(g).$$

For a given conformal class $[g] = \{e^{-2u}g \mid u \in C^\infty(M)\}$, the Yamabe constant is defined by

$$Y_1([g]) = \inf_{\tilde{g} \in [g]} \mathcal{F}_1(\tilde{g}).$$

By the resolution of the Yamabe problem, Aubin and Schoen [2, 25] proved that for any metric g on M

$$(1.6) \quad Y_1([g]) \leq Y_1([g_{\mathbb{S}^{n-1}}]) \quad \text{and} \quad Y_1([g]) < Y_1([g_{\mathbb{S}^{n-1}}]) \quad \text{for any } (M, [g]) \text{ other than } [g_{\mathbb{S}^{n-1}}],$$

where $[g_{\mathbb{S}^{n-1}}]$ is the conformal class of the standard round metric on the sphere \mathbb{S}^{n-1} . From this, one can see the importance of the constant $Y_1([g_{\mathbb{S}^{n-1}}])$. In fact, one can prove that

$$(1.7) \quad Y_1([g_{\mathbb{S}^{n-1}}]) = (n-1)(n-2)\omega_{n-1}^{\frac{2}{n-1}},$$

where ω_{n-1} is the volume of $g_{\mathbb{S}^{n-1}}$. It is trivial to see that (1.7) is equivalent to

$$(1.8) \quad \int_M L_1 d\mu(g) = \int_M R_g d\mu(g) \geq (n-1)(n-2)\omega_{n-1}^{\frac{2}{n-1}} \text{vol}(g)^{\frac{n-3}{n-1}},$$

for any $g \in [g_{\mathbb{S}^{n-1}}]$, which is in fact an optimal Sobolev inequality. See [20]. As a natural generalization, we proved in [19] a generalized Sobolev inequality for σ_k -scalar curvature $\sigma_k(g)$, which states

$$(1.9) \quad \int_M \sigma_k(g) d\mu(g) \geq \frac{C_{n-1}^k}{2^k} \omega_{n-1}^{\frac{2k}{n-1}} \text{vol}(g)^{\frac{n-1-2k}{n-1}},$$

for any $g \in \mathcal{C}_{k-1}([g_{\mathbb{S}^{n-1}}])$, where $\mathcal{C}_{k-1}([g_{\mathbb{S}^{n-1}}]) = [g_{\mathbb{S}^{n-1}}] \cap \Gamma_{k-1}^+$ and $\Gamma_k^+ = \{g \mid \sigma_j(g) > 0, \forall j \leq k\}$. In this paper, we denote $C_{n-1}^k = \frac{(n-1)!}{k!(n-1-k)!}$. By (1.2) inequality (1.9) can be written in the following form

$$(1.10) \quad \int_{\Sigma} L_k d\mu(g) \geq C_{n-1}^{2k} (2k)! \omega_{n-1}^{\frac{2k}{n-1}} (\text{vol}(g))^{\frac{n-1-2k}{n-1}},$$

for any $g \in \mathcal{C}_{k-1}([g_{\mathbb{S}^{n-1}}])$. We call both inequalities (1.8), (1.10) *optimal Sobolev inequalities* and would like to investigate which classes of metrics satisfy the optimal Sobolev inequalities. (1.8) and (1.10) mean that a suitable subclass of the conformal class of the standard round metric satisfies the optimal Sobolev inequalities. From (1.6) we know in any conformal class other than the conformal class of the standard round metric, there exist many metrics which do not satisfy the optimal Sobolev inequality. Hence it is natural to ask if there are other interesting classes of metrics satisfy the optimal Sobolev inequality? Observe that for a closed hypersurface Σ in \mathbb{R}^n ,

$$(1.11) \quad L_k = (2k)! \sigma_{2k},$$

where σ_{2k} is the $2k$ -mean curvature of Σ , which is defined by

$$\sigma_j = \sigma_j(\kappa),$$

where $\kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1})$, κ_j ($1 \leq j \leq n-1$) is the principal curvature of B , and B is the 2nd fundamental form of Σ induced by the standard Euclidean metric. The classical Alexandrov-Fenchel inequality (see [27] for instance) implies for convex hypersurfaces in \mathbb{R}^n that

$$(1.12) \quad \int_{\Sigma} L_k(g) d\mu(g) = (2k)! \int_{\Sigma} \sigma_{2k} d\mu(g) \geq C_{n-1}^{2k} (2k)! \omega_{n-1}^{\frac{2k}{n-1}} |\Sigma|^{\frac{n-1-2k}{n-1}}.$$

In this paper we use $|\Sigma|$ to denote the area of Σ with respect to the induced metric. Inequality (1.12) means that the induced metric of any convex hypersurfaces in \mathbb{R}^n satisfy the optimal Sobolev inequalities. The convexity can be weakened. See the work of Guan-Li [18], Huisken [21] and Chang-Wang [6].

In this paper we prove that the induced metric of horospherical convex hypersurfaces in \mathbb{H}^n also satisfy the optimal Sobolev inequalities.

Theorem 1.1. *Let $2k < n - 1$. Any horospherical convex hypersurfaces Σ in \mathbb{H}^n satisfies*

$$(1.13) \quad \int_{\Sigma} L_k d\mu(g) \geq C_{n-1}^{2k} (2k)! \omega_{n-1}^{\frac{2k}{n-1}} |\Sigma|^{\frac{n-1-2k}{n-1}},$$

equality holds if and only if Σ is a geodesic sphere.

A hypersurface in \mathbb{H}^n is *horospherical convex* if all principal curvatures are larger than or equal to 1. The horospherical convexity is a natural geometric concept, which is equivalent to the geometric convexity in Riemannian manifolds. For any hypersurface in \mathbb{H}^n , the Gauss-Bonnet curvature L_k of the induced metric of the hypersurface can be expressed in terms of the curvature integrals by (see also Lemma 3.1 below)

$$(1.14) \quad L_k = C_{n-1}^{2k} (2k)! \sum_{j=0}^k (-1)^j \frac{C_k^j}{C_{n-1}^{2k-2j}} \sigma_{2k-2j}.$$

Comparing (1.12) for \mathbb{R}^n with (1.13) for \mathbb{H}^n and (1.11) with (1.14), we obtain the same inequality for L_k , while L_k has different expression in terms of the curvature integrals. We remark that when $2k = n - 1$, (1.13) is an equality for any hypersurface diffeomorphic to a sphere, i.e.,

$$\int_{\Sigma} L_{\frac{n-1}{2}} dv(g) = (n-1)! \omega_{n-1}.$$

This follows that the Gauss-Bonnet-Chern theorem.

As a first direct application, we establish Alexandrov-Fenchel type inequalities for curvature integrals.

Theorem 1.2. *Let $2k \leq n - 1$. Any horospherical convex hypersurface $\Sigma \subset \mathbb{H}^n$ satisfies*

$$(1.15) \quad \int_{\Sigma} \sigma_{2k} \geq C_{n-1}^{2k} \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k} \frac{n-1-2k}{n-1}} \right\}^k,$$

equality holds if and only if Σ is a geodesic sphere.

When $k = 1$ Theorem 1.1, and hence Theorem 1.2, is true even for any star-shaped and two-convex hypersurfaces in \mathbb{H}^n , i.e., $\sigma_1 \geq 0$ and $\sigma_2 \geq 0$, which was proved by Li-Wei-Xiong in a recent work [22]. When $k = 2$, Theorem 1.1 was proved in our recent paper [15]. Due to the complication of the variational structure of $\int \sigma_k$ in the hyperbolic space, the case $k \geq 2$ is quite different from the case $k = 1$. For case $k \geq 2$ the horospherical convexity of the hypersurface Σ plays an essential role.

At the end of this paper we show that a similar inequality holds for σ_1 and propose a conjecture for general odd σ_{2k+1} .

Another application is an optimal inequality for quermassintegrals in \mathbb{H}^n . For a (geodesically) convex domain $\Omega \subset \mathbb{H}^n$ with $\Sigma = \partial\Omega$, quermassintegrals are defined by

$$(1.16) \quad W_r(\Omega) := \frac{(n-r)\omega_{r-1} \cdots \omega_0}{n\omega_{n-2} \cdots \omega_{n-r-1}} \int_{\mathcal{L}_r} \chi(L \cap \Omega) dL,$$

where \mathcal{L}_r is the space of r -dimensional totally geodesic subspaces L in \mathbb{H}^n , ω_r is the area of the r -dimensional standard round sphere and dL is the natural (invariant) measure on \mathcal{L}_r (cf. [24], [28]). As in the Euclidean case we take $W_0(\Omega) = Vol(\Omega)$. With these definitions, unlike the euclidean case, the quermassintegral in \mathbb{H}^n do not coincide with the mean curvature integrals, but they are closely related (cf. [28])

$$(1.17) \quad \frac{1}{C_{n-1}^r} \int_{\Sigma} \sigma_r = n \left(W_{r+1}(\Omega) + \frac{r}{n-r+1} W_{r-1}(\Omega) \right), \quad W_0(\Omega) = Vol(\Omega), \quad W_1(\Omega) = \frac{1}{n} |\Sigma|.$$

The relationship between W_0 and W_1 , the hyperbolic isoperimetric inequality, was established by Schmidt [26] 70 years ago. When $n = 2$, the hyperbolic isoperimetric inequality is

$$L^2 \geq 4\pi A + A^2,$$

where L is the length of a curve γ in \mathbb{H}^2 and A is the area of the enclosed domain by γ . In general, this hyperbolic isoperimetric inequality has no explicit form. There are many attempts to establish relationship between $W_k(\Omega)$ in the hyperbolic space \mathbb{H}^n . See, for example, [24]

and [29]. In [11], Gallego-Solanes proved by using integral geometry the following interesting inequality for convex domains in \mathbb{H}^n , precisely, there holds,

$$(1.18) \quad W_r(\Omega) > \frac{n-r}{n-s} W_s(\Omega), \quad r > s,$$

which implies

$$(1.19) \quad \int_{\Sigma} \sigma_k d\mu > c C_{n-1}^k |\Sigma|,$$

where $c = 1$ if $k > 1$ and $c = (n-2)/(n-1)$ if $k = 1$ and $|\Sigma|$ is the area of Σ . Here $d\mu$ is the area element of the induced metric. The constants in (1.18) and (1.19) are optimal in the sense that one can not replace them by bigger constants. However, they are far away being optimal.

As another application of Theorem 1.1, we have the following optimal inequalities of $W_k(\Omega)$ for general odd k in terms of $W_1 = \frac{1}{n} |\Sigma|$.

Theorem 1.3. *Let $2k \leq n-1$. If $\Omega \subset \mathbb{H}^n$ be a domain with $\Sigma = \partial\Omega$ horospherical convex, then*

$$(1.20) \quad W_{2k+1}(\Omega) \geq \frac{\omega_{n-1}}{n} \sum_{i=0}^k \frac{n-1-2k}{n-1-2k+2i} C_k^i \left(\frac{nW_1(\Omega)}{\omega_{n-1}} \right)^{\frac{n-1-2k+2i}{n-1}},$$

where ω_{n-1} is the area of the unit sphere \mathbb{S}^{n-1} . Equality holds if and only if Σ is a geodesic sphere.

As a direct corollary we solve an isoperimetric problem for horospherical convex surfaces with fixed W_1 .

Corollary 1.4. *Let $2k \leq n-1$. In a class of horospherical convex hypersurfaces in \mathbb{H}^n with fixed W_1 , the minimum of W_{2k+1} is achieved by and only by the geodesic spheres.*

Corollary 1.4 answers a question asked in the paper of Gao, Hug and Schneider [12] in this case.

In order to prove Theorem 1.1, motivated by [15] and [22] (see also [4] and [9]), we consider the following functional

$$(1.21) \quad Q(\Sigma) := |\Sigma|^{-\frac{n-1-2k}{n-1}} \int_{\Sigma} L_k.$$

Here L_k is the Gauss-Bonnet curvature with respect to the induced metric g on Σ . This is a Yamabe type functional. One of crucial points of this paper is to show that functional Q is non-increasing under the following inverse curvature flow

$$(1.22) \quad \frac{\partial \Sigma_t}{\partial t} = \frac{n-2k}{2k} \frac{\sigma_{2k-1}}{\sigma_{2k}} \nu,$$

where ν is the outer normal of Σ_t , provided that the initial hypersurface is horospherical convex. One can show that horospherical convexity is preserved by flow (1.22). By the convergence results of Gerhardt [16] on the inverse curvature flow (1.22), we show that the flow approaches

to surfaces whose induced metrics belong to the conformal class of the standard round sphere metric. Therefore, we can use the result (1.10) to

$$Q(\Sigma) \geq \lim_{t \rightarrow \infty} Q(\Sigma_t) \geq C_{n-1}^{2k} (2k)! \omega_{n-1}^{\frac{2k}{n-1}}.$$

The rest of this paper is organized as follows. In Section 2 we present some basic facts about the elementary functions σ_k and recall the generalized Sobolev inequality (1.10) from [19]. In Section 3, We present the relationship between various geometric quantities including the intrinsic geometric quantities $\int_{\Sigma} L_k$, the curvature integrals $\int_{\Sigma} \sigma_k$ and the quermassintegrals $W_r(\Omega)$. In Section 4 we prove the crucial monotonicity of Q and analyze its asymptotic behavior under flow (1.22). The proof of our main theorems are given in Section 5. In Section 6, we show that a similar inequality holds for σ_1 and propose a conjecture for integral integrals σ_{2k+1} .

2. PRELIMINARIES

Let σ_k be the k -th elementary symmetry function $\sigma_k : \mathbb{R}^{n-1} \rightarrow \mathbb{R}$ defined by

$$\sigma_k(\Lambda) = \sum_{i_1 < \dots < i_k} \lambda_{i_1} \cdots \lambda_{i_k} \quad \text{for } \Lambda = (\lambda_1, \dots, \lambda_{n-1}) \in \mathbb{R}^{n-1}.$$

For a symmetric matrix B , denote $\lambda(B) = (\lambda_1(B), \dots, \lambda_n(B))$ be the eigenvalues of B . We set

$$\sigma_k(B) := \sigma_k(\lambda(B)).$$

The Garding cone Γ_k^+ is defined as

$$\Gamma_k^+ = \{\Lambda \in \mathbb{R}^{n-1} \mid \sigma_j(\Lambda) > 0, \quad \forall j \leq k\}.$$

A symmetric matrix B is called belong to Γ_k^+ if $\lambda(B) \in \Gamma_k^+$. We collect the basic facts about σ_k , which will be directly used in this paper. For other related facts, see a survey of Guan [17] or [22].

$$(2.1) \quad \sigma_k(B) = \frac{1}{k!} \delta_{j_1 \dots j_k}^{i_1 \dots i_k} b_{i_1}^{j_1} \cdots b_{i_k}^{j_k},$$

where $B = (b_j^i)$. In the following, for simplicity of notation we denote

$$p_k = \frac{\sigma_k}{C_{n-1}^k}.$$

Lemma 2.1. *For $\Lambda \in \Gamma_k^+$, we have the following Newton-MacLaurin inequalities*

$$(2.2) \quad \frac{p_{k-1} p_{k+1}}{p_k^2} \leq 1,$$

$$(2.3) \quad \frac{p_1 p_{k-1}}{p_k} \geq 1.$$

Moreover, equality holds in (2.2) or (2.3) at Λ if and only if $\Lambda = c(1, 1, \dots, 1)$.

The Newton-MacLaurin inequalities play a very important role in proving geometric inequalities mentioned above. However, we will see that these inequalities are not precise enough to show our inequality (1.13).

Let $\mathbb{H}^n = \mathbb{R}^+ \times \mathbb{S}^{n-1}$ with the hyperbolic metric

$$\bar{g} = dr^2 + \sinh^2 r g_{\mathbb{S}^{n-1}},$$

where $g_{\mathbb{S}^{n-1}}$ is the standard round metric on the unit sphere \mathbb{S}^{n-1} and $\Sigma \subset \mathbb{H}^n$ a smooth closed hypersurface in \mathbb{H}^n with a unit outward normal ν . Let h be the second fundamental form of Σ and $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ the set of principal curvatures of Σ in \mathbb{H}^n with respect to ν . The k -th mean curvature of Σ is defined by

$$\sigma_k = \sigma_k(\kappa).$$

We now consider the following curvature evolution equation

$$(2.4) \quad \frac{d}{dt} X = F\nu,$$

where $\Sigma_t = X(t, \cdot)$ is a family of hypersurfaces in \mathbb{H}^n , ν is the unit outward normal to $\Sigma_t = X(t, \cdot)$ and F is a speed function which may depend on the position vector X and principal curvatures of Σ_t . One can check that [23] along the flow

$$(2.5) \quad \frac{d}{dt} \int_{\Sigma} \sigma_k d\mu = (k+1) \int_{\Sigma} F \sigma_{k+1} d\mu + (n-k) \int_{\Sigma} F \sigma_{k-1} d\mu,$$

and thus

$$(2.6) \quad \frac{d}{dt} \int_{\Sigma} p_k d\mu = \int_{\Sigma} ((n-k-1)p_{k+1} + kp_{k-1}) F d\mu.$$

If one compares flow (2.4) in \mathbb{H}^n with a similar flow of hypersurfaces in \mathbb{R}^n , the last term in (2.5) is an extra term. This extra term comes from -1 , the sectional curvature of \mathbb{H}^n and makes the phenomenon of \mathbb{H}^n quite different from the one of \mathbb{R}^n .

As mentioned above we use the following inverse flow

$$(2.7) \quad \frac{d}{dt} X = \frac{p_{2k-1}}{p_{2k}} \nu.$$

By using the result of Gerhardt [16] we have

Proposition 2.2. *If the initial hypersurface Σ is horospherical convex, then the solution for the flow (2.7) exists for all time $t > 0$ and preserves the condition of horospherical convexity. Moreover, the hypersurfaces Σ_t become more and more umbilical in the sense of*

$$|h_i^j - \delta_i^j| \leq C e^{-\frac{t}{n-1}}, \quad t > 0,$$

i.e., the principal curvatures are uniformly bounded and converge exponentially fast to one. Here $h_i^j = g^{ik} h_{kj}$, where g is the induced metric and h is the second fundamental form.

Proof. For the long time existence of the inverse curvature flow, see the work of Gerhardt [16]. The preservation of the horospherical convexity along flow (2.7) was proved in [15] with the help of a maximum principle for tensors of Andrews [1]. \square

Let g be a Riemannian metric on M^{n-1} . Denote Ric_g and R_g the Ricci tensor and the scalar curvature of g respectively. The Schouten tensor A_g is defined by (1.4). The σ_k -scalar curvature, which is introduced by Viaclovsky [30], is defined by

$$\sigma_k(g) := \sigma_k(A_g).$$

This is a natural generalization of the scalar curvature R . In fact, $\sigma_1(g) = \frac{1}{2(n-2)}R$. Recall that M is of dimension $n - 1$. We now consider the conformal class $[g_{\mathbb{S}^{n-1}}]$ of the standard sphere \mathbb{S}^{n-1} and the following functionals defined by

$$(2.8) \quad \mathcal{F}_k(g) = vol(g)^{-\frac{n-1-2k}{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_k(g) d\mu, \quad k = 0, 1, \dots, n-1.$$

If a metric g satisfies $\sigma_j(g) > 0$ for any $j \leq k$, we call it k -positive and denote $g \in \Gamma_k^+$. From Theorem 1.A in [19] we have

Proposition 2.3. *Let $0 < k < \frac{n-1}{2}$ and $g \in [g_{\mathbb{S}^{n-1}}]$ k -positive. We have*

$$(2.9) \quad \mathcal{F}_k(g) \geq \mathcal{F}_k(g_{\mathbb{S}^{n-1}}) = \frac{C_{n-1}^k}{2^k} \omega_{n-1}^{\frac{2k}{n-1}}.$$

Inequality (2.9) is a generalized Sobolev inequality, since when $k = 1$ inequality (2.9) is just the optimal Sobolev inequality. See for example [20]. For another Sobolev inequalities, see also [3] and [7].

3. RELATIONSHIP BETWEEN VARIOUS GEOMETRIC QUANTITIES

The Gauss-Bonnet curvatures L_k , and hence $\int_{\Sigma} L_k$ are intrinsic geometric quantities, which depend only on the induced metric g on Σ and do not depend on the embeddings of (Σ, g) . Lemma 3.2 and Lemma 3.3 below imply that σ_{2k} , $\int \sigma_{2k}$ and W_{2k+1} are also intrinsic. σ_{2k+1} , $\int \sigma_{2k+1}$ and W_{2k} are extrinsic. The functionals $\int_{\Sigma} L_k$ are new geometric quantities for the study of the integral geometry in \mathbb{H}^n . In this section we present the relationship between these geometric quantities.

We first have a relation between L_k and σ_k .

Lemma 3.1. *For a hypersurface (Σ, g) in \mathbb{H}^n , its Gauss-Bonnet curvature L_k can be expressed by higher order mean curvatures*

$$(3.1) \quad L_k = C_{n-1}^{2k} (2k)! \sum_{i=0}^k C_k^i (-1)^i p_{2k-2i}.$$

Hence we have

$$(3.2) \quad \int_{\Sigma} L_k = C_{n-1}^{2k} (2k)! \sum_{i=0}^k C_k^i (-1)^i \int_{\Sigma} p_{2k-2i} = C_{n-1}^{2k} (2k)! \sum_{i=0}^k (-1)^i \frac{C_k^i}{C_{n-1}^{2k-2i}} \int_{\Sigma} \sigma_{2k-2i}.$$

Proof. First recall the Gauss formula

$$R_{ij}{}^{kl} = (h_i^k h_j^l - h_i^l h_j^k) - (\delta_i^k \delta_j^l - \delta_i^l \delta_j^k),$$

where $h_i^j := g^{ik}h_{kj}$ and h is the second fundamental form. Then substituting the Gauss formula above into (1.1) and recalling (2.1), we have by a straightforward calculation,

$$\begin{aligned}
 L_k &= \frac{1}{2^k} \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}} \\
 &= \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} (h_{i_1}^{j_1} h_{i_2}^{j_2} - \delta_{i_1}^{j_1} \delta_{i_2}^{j_2}) \dots (h_{i_{2k-1}}^{j_{2k-1}} h_{i_{2k}}^{j_{2k}} - \delta_{i_{2k-1}}^{j_{2k-1}} \delta_{i_{2k}}^{j_{2k}}) \\
 &= \sum_{i=0}^k C_k^i (-1)^i (n-2k)(n-2k+1) \dots (n-1-2k+2i) ((2k-2i)! \sigma_{2k-2i}) \\
 &= C_{n-1}^{2k} (2k)! \sum_{i=0}^k C_k^i (-1)^i p_{2k-2i}.
 \end{aligned}$$

Here in the second equality we use the symmetry of generalized Kronecker delta and in the third equality we use (2.1) and the basic property of generalized Kronecker delta

$$(3.3) \quad \delta_{j_1 j_2 \dots j_{p-1} j_p}^{i_1 i_2 \dots i_{p-1} i_p} \delta_{i_1}^{j_1} = (n-p) \delta_{j_2 j_3 \dots j_p}^{i_2 i_3 \dots i_p},$$

which follows from the Laplace expansion of determinant. □

Motivated by the expression (3.1), we introduce the following notations,

$$(3.4) \quad \tilde{L}_k = \sum_{i=0}^k C_k^i (-1)^i p_{2k-2i}, \quad \tilde{N}_k = \sum_{i=0}^k C_k^i (-1)^i p_{2k-2i+1}.$$

It is clear that

$$L_k = (2k)! C_{n-1}^{2k} \tilde{L}_k, \quad N_k = (2k)! C_{n-1}^{2k} \tilde{N}_k.$$

Lemma 3.2. *We have*

$$(3.5) \quad \sigma_{2k} = C_{n-1}^{2k} p_{2k} = C_{n-1}^{2k} \left(\sum_{i=0}^k C_k^i \tilde{L}_i \right),$$

and hence

$$\int_{\Sigma} \sigma_{2k} = C_{n-1}^{2k} \sum_{i=0}^k C_k^i \int_{\Sigma} \tilde{L}_i = \frac{1}{(2k)!} \sum_{i=0}^k C_k^i \int_{\Sigma} L_i.$$

To show Theorem 1.3 below, we need

Lemma 3.3. *The quermassintegral W_{2k+1} can be expressed in terms of integral of \tilde{L}_i*

$$(3.6) \quad W_{2k+1}(\Omega) = \frac{1}{n} \sum_{i=0}^k C_k^i \frac{n-1-2k}{n-1-2k+2i} \int_{\Sigma} \tilde{L}_{k-i}.$$

Proof. We use the induction argument to show (3.6). When $k = 0$, we have by (1.17) that $W_1(Q) = \frac{1}{n}|\Sigma|$. We then assume that (3.6) holds for $k - 1$, that is

$$\begin{aligned} W_{2k-1}(\Omega) &= \frac{1}{n} \sum_{j=0}^{k-1} C_{k-1}^j \frac{n+1-2k}{n+1-2k+2j} \tilde{L}_{k-1-j} \\ (3.7) \qquad &= \frac{1}{n} \int_{\Sigma} \sum_{i=1}^k C_{k-1}^{i-1} \frac{n+1-2k}{n-1-2k+2i} \tilde{L}_{k-i}. \end{aligned}$$

By (1.17) and (3.5), we have

$$\begin{aligned} W_{2k+1}(\Omega) &= \frac{1}{n} \int_{\Sigma} p_{2k} - \frac{2k}{n-2k+1} W_{2k-1}(\Omega) \\ &= \frac{1}{n} \int_{\Sigma} \sum_{i=0}^k C_k^i \tilde{L}_i - \frac{2k}{n-2k+1} W_{2k-1}(\Omega). \end{aligned}$$

Substituting (3.7) into above, one immediately obtains (3.6) for k . Thus we complete the proof. \square

One can also show the following relation between the quermassintegrals and the curvature integrals.

Lemma 3.4.

$$(3.8) \quad W_{2k+1}(\Omega) = \frac{1}{n} \sum_{j=0}^k (-1)^j \frac{(2k)!!(n-2k-1)!!}{(2k-2j)!!(n-2k-1+2j)!!} \frac{1}{C_{n-1}^{2k-2j}} \int_{\Sigma} \sigma_{2k-2j},$$

where

$$(2k-1)!! := (2k-1)(2k-3)\cdots 1 \quad \text{and} \quad (2k)!! := (2k)(2k-2)\cdots 2.$$

Proof. One can show this relation by a direct computation. See also [24] or [29]. \square

4. MONOTONICITY

In this section we prove the monotonicity of functional Q under inverse curvature flow. First, we have the variational formula for $\int \tilde{L}_k$.

Lemma 4.1. *Along the inverse flow (2.7), we have*

$$(4.1) \quad \frac{d}{dt} \int_{\Sigma} \tilde{L}_k = (n-1-2k) \int \tilde{L}_k + (n-1-2k) \int_{\Sigma} \left(\tilde{N}_k \frac{p_{2k-1}}{p_{2k}} - \tilde{L}_k \right).$$

Proof. It follows from (2.6) that along the inverse flow (2.4), we have

$$\begin{aligned}
 \frac{d}{dt} \int_{\Sigma} \tilde{L}_k &= \int_{\Sigma} \sum_{i=0}^k C_k^i (-1)^i \left((n-1-2k+2i)p_{2k-2i+1} + 2(k-i)p_{2k-2i-1} \right) F \\
 &= \int_{\Sigma} \sum_{i=0}^k C_k^i (-1)^i (n-1-2k+2i)p_{2k-2i+1} F + \int_{\Sigma} \sum_{j=1}^k C_k^{j-1} (-1)^{j-1} 2(k-j+1)p_{2k-2j+1} F \\
 &= \int_{\Sigma} \sum_{i=0}^k C_k^i (-1)^i (n-1-2k)p_{2k-2i+1} F + \int_{\Sigma} \sum_{j=1}^k 2(-1)^j \left(C_k^j j - C_k^{j-1} (k-j+1) \right) p_{2k-2j+1} F \\
 &= (n-1-2k) \int_{\Sigma} \sum_{i=0}^k C_k^i (-1)^i p_{2k-2i+1} \\
 &= (n-1-2k) \int_{\Sigma} \tilde{N}_k F \\
 &= (n-1-2k) \int_{\Sigma} \tilde{L}_k + (n-1-2k) \int_{\Sigma} \left(\tilde{N}_k F - \tilde{L}_k \right).
 \end{aligned}$$

Substituting $F = \frac{p_{2k-1}}{p_{2k}}$ into above, we get the desired result. \square

In order to show the monotonicity of the functional Q defined in (1.21) under the inverse flow (2.7), we need to show the non-positivity of the last term in (4.1). That is

$$(4.2) \quad \frac{p_{2k-1}}{p_{2k}} \tilde{N}_k - \tilde{L}_k \leq 0.$$

When $k = 1$, (4.2) is just

$$\frac{p_1}{p_2} (p_3 - p_1) - (p_2 - 1) \leq 0,$$

which follows from the Newton-Maclaurin inequalities in Lemma 2.1. In fact, it is clear that

$$\frac{p_1}{p_2} (p_3 - p_1) - (p_2 - 1) = \left(\frac{p_1 p_3}{p_2} - p_2 \right) + \left(1 - \frac{p_1^2}{p_2} \right).$$

Hence the non-positivity follows, for both terms are non-positive, by Lemma 2.1. This was used in [22]. When $k \geq 2$, the proof of (4.2) becomes more complicated. When $k = 2$, one needs to show the non-positivity of

$$(4.3) \quad \frac{p_3}{p_4} (p_5 - 2p_3 + p_1) - (p_4 - 2p_2 + 1) = \left(\frac{p_3}{p_4} p_5 - p_4 \right) + 2 \left(p_2 - \frac{p_3^2}{p_4} \right) + \left(\frac{p_3}{p_4} p_1 - 1 \right).$$

By Lemma 2.1, the first two terms are non-positive, but the last term is non-negative. It was showed in [15] that (4.3) is non-positive if $\kappa \in \mathbb{R}^{n-1}$ satisfying

$$(4.4) \quad \kappa \in \{ \kappa = (\kappa_1, \kappa_2, \dots, \kappa_{n-1}) \in \mathbb{R}^{n-1} \mid \kappa_i \geq 1 \}.$$

We want to show that (4.2) is true for general $k \leq \frac{1}{2}(n-1)$. This is one of key points of this paper. Now the case is more complicated than the case $k = 2$.

Proposition 4.2. *For any κ satisfying (4.4), we have*

$$(4.5) \quad \frac{p_{2k-1}}{p_{2k}} \tilde{N}_k - \tilde{L}_k \leq 0.$$

Equality holds if and only if one of the following two cases holds

$$(4.6) \quad \text{either (i) } \kappa_i = \kappa_j \forall i, j, \quad \text{or (ii) } \exists i \text{ with } \kappa_i < 1 \text{ \& } \kappa_j = 1 \forall j \neq i.$$

We sketch the proof into several steps. Before the proof, we introduce the notation of \sum_{cyc} to simplify notations. Precisely, given $n-1$ numbers $(\kappa_1, \kappa_2, \dots, \kappa_{n-1})$, we denote $\sum_{cyc} f(\kappa_1, \dots, \kappa_{n-1})$ the cyclic summation which takes over all *different* terms of the type $f(\kappa_1, \dots, \kappa_{n-1})$. For instance,

$$\begin{aligned} \sum_{cyc} \kappa_1 &= \kappa_1 + \kappa_2 + \dots + \kappa_{n-1}, & \sum_{cyc} \kappa_1^2 \kappa_2 &= \sum_{i=1}^{n-1} \left(\kappa_i^2 \sum_{j \neq i} \kappa_j \right), \\ \sum_{cyc} \kappa_1 (\kappa_2 - \kappa_3)^2 &= \sum_{i=1}^{n-1} \left(\kappa_i \sum_{\substack{1 \leq j < k \leq n-1 \\ j, k \neq i}} (\kappa_j - \kappa_k)^2 \right), \\ &= (n-3) \sum_{cyc} \kappa_1 \kappa_2^2 - 6 \sum_{cyc} \kappa_1 \kappa_2 \kappa_3. \end{aligned}$$

Lemma 4.3. *For any κ satisfying (4.4), we have*

$$(4.7) \quad \tilde{N}_k - p_1 \tilde{L}_k \leq 0.$$

Equality holds if and only if one of the following two cases holds

$$\text{either (i) } \kappa_i = \kappa_j \forall i, j, \quad \text{or (ii) } \exists i \text{ with } \kappa_i > 1 \text{ \& } \kappa_j = 1 \forall j \neq i.$$

Proof. It is crucial to observe that (4.7) is indeed equivalent to the following inequality:

$$(4.8) \quad \sum_{1 \leq i_m \leq n-1, i_j \neq i_l (j \neq l)} \kappa_{i_1} (\kappa_{i_2} \kappa_{i_3} - 1) (\kappa_{i_4} \kappa_{i_5} - 1) \cdots (\kappa_{i_{2k-2}} \kappa_{i_{2k-1}} - 1) (\kappa_{i_{2k}} - \kappa_{i_{2k+1}})^2 \geq 0,$$

where the summation takes over all the $(2k+1)$ -elements permutation of $\{1, 2, \dots, n-1\}$. For the convenience of the reader, we sketch the proof of (4.8) briefly. First, note that from (3.4) that

$$\begin{aligned} (p_1 \tilde{L}_k - \tilde{N}_k) &= p_1 \sum_{i=0}^k C_k^{k-i} (-1)^{k-i} p_{2i} - \sum_{i=0}^k C_k^{k-i} (-1)^{k-i} p_{2i+1} \\ &= \sum_{i=0}^k (-1)^{k-i} C_k^i (p_1 p_{2i} - p_{2i+1}). \end{aligned}$$

Next we calculate each term $p_1 p_{2i} - p_{2i+1}$ carefully. By using

$$(n-1)C_{n-1}^j = (j+1)C_{n-1}^{j+1} + (n-1)C_{n-1}^{j-1},$$

we have

$$\begin{aligned}\sigma_j \sigma_1 &= \left(\sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_j} \right) \left(\sum_{cyc} \kappa_{i_{j+1}} \right) \\ &= (j+1) \left(\sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_j} \kappa_{i_{j+1}} \right) + \sum_{cyc} \kappa_{i_1}^2 \kappa_{i_2} \cdots \kappa_{i_j},\end{aligned}$$

and

$$\begin{aligned}& p_1 p_{2j} - p_{2j+1} \\ &= \frac{1}{(n-1)C_{n-1}^{2j}} \left(\sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_{2j}} \right) \left(\sum_{cyc} \kappa_{i_{2j+1}} \right) - \frac{1}{C_{n-1}^{2j+1}} \sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_{2j+1}} \\ &= \frac{1}{(n-1)C_{n-1}^{2j} C_{n-1}^{2j+1}} \left(C_{n-1}^{2j+1} (2j+1) \sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_{2j}} \kappa_{i_{2j+1}} + C_{n-1}^{2j+1} \sum_{cyc} \kappa_{i_1}^2 \kappa_{i_2} \cdots \kappa_{i_{2j}} \right. \\ &\quad \left. - (n-1) C_{n-1}^{2j} \sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_{2j+1}} \right) \\ &= \frac{1}{(n-1)C_{n-1}^{2j} C_{n-1}^{2j+1}} \cdot \frac{C_{n-1}^{2j+1}}{n-2j} \sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_{2j-1}} (\kappa_{i_{2j}} - \kappa_{i_{2j+1}})^2 \\ &= \frac{(2j)!(n-2j-2)!}{(n-1) \cdot (n-1)!} \sum_{cyc} \kappa_{i_1} \kappa_{i_2} \cdots \kappa_{i_{2j-1}} (\kappa_{i_{2j}} - \kappa_{i_{2j+1}})^2.\end{aligned}$$

In (4.8), the coefficient of $\kappa_1 \kappa_2 \cdots \kappa_{2j-1} (\kappa_{2j} - \kappa_{2j+1})^2$ is

$$\begin{aligned}2(-1)^{k-j} C_{k-1}^{j-1} (2j-1)! C_{n-2j-2}^{2k-2j} [2(k-j)]! &= \frac{(-1)^{k-j}}{k} C_k^j (2j)!(n-2j-2)! \\ &= (-1)^{k-j} C_k^j \frac{(2j)!(n-2j-2)!}{(n-1) \cdot (n-1)!} \cdot \frac{(n-1) \cdot (n-1)!}{k}.\end{aligned}$$

Therefore we have

$$\begin{aligned}0 &\leq \sum_{1 \leq i_m \leq n-1, i_j \neq i_l (j \neq l)} \kappa_{i_1} (\kappa_{i_2} \kappa_{i_3} - 1) (\kappa_{i_4} \kappa_{i_5} - 1) \cdots (\kappa_{i_{2k-2}} \kappa_{i_{2k-1}} - 1) (\kappa_{i_{2k}} - \kappa_{i_{2k+1}})^2 \\ &= \frac{(n-1) \cdot (n-1)!}{k} \sum_{j=0}^k (-1)^{k-j} C_k^j (p_1 p_{2j} - p_{2j+1}) \\ &= \frac{(n-1) \cdot (n-1)!}{k} (p_1 \tilde{L}_k - \tilde{N}_k).\end{aligned}$$

This finishes the proof. \square

In view of (4.8), we have the following remark which will be used later.

Remark 4.4. For any $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ satisfying $0 < \kappa_i \leq 1$, ($i = 1, \dots, n-1$), then $(-1)^{k-1} (\tilde{N}_k - p_1 \tilde{L}_k) \leq 0$.

Lemma 4.5. *For any κ satisfying (4.4), we have*

$$\tilde{N}_k \geq 0, \quad \tilde{L}_k \geq 0.$$

Proof. They are equivalent to the following inequalities respectively:

$$(4.9) \quad \sum_{1 \leq i_m \leq n-1, i_j \neq i_l (j \neq l)} \kappa_{i_1} (\kappa_{i_2} \kappa_{i_3} - 1) (\kappa_{i_4} \kappa_{i_5} - 1) \cdots (\kappa_{i_{2k-2}} \kappa_{i_{2k-1}} - 1) (\kappa_{i_{2k}} \kappa_{i_{2k+1}} - 1) \geq 0,$$

$$(4.10) \quad \sum_{1 \leq i_m \leq n-1, i_j \neq i_l (j \neq l)} (\kappa_{i_2} \kappa_{i_3} - 1) (\kappa_{i_4} \kappa_{i_5} - 1) \cdots (\kappa_{i_{2k-2}} \kappa_{i_{2k-1}} - 1) (\kappa_{i_{2k}} \kappa_{i_{2k+1}} - 1) \geq 0.$$

where the summation takes over all the $(2k+1)$ -elements permutation of $\{1, 2, \dots, n-1\}$. The proof to show the equivalence of (4.9), (4.10) is exactly the same as the one of (4.8). Hence we omit it here. \square

Remark 4.6. *For any $\kappa = (\kappa_1, \dots, \kappa_{n-1})$ satisfying $0 < \kappa_i \leq 1$, ($i = 1, \dots, n-1$), then $(-1)^k \tilde{N}_k \geq 0$, $(-1)^k \tilde{L}_k \geq 0$.*

Making use of Lemma 4.3 and Remark 4.4, we can show the following result which is stronger than Proposition 4.2.

Lemma 4.7. *For any κ satisfying (4.4), we have*

$$p_{2k} \tilde{N}_k - p_{2k+1} \tilde{L}_k \leq 0.$$

Proof. According to the induction argument proved in [15] (see p.8), we only need to prove it for $n-1 = 2k+1$. Let $z_i = \frac{1}{\kappa_i} \leq 1$, and

$$\hat{p}_i = p_i(z_1, z_2, \dots, z_{2k+1}).$$

It is clear that

$$(4.11) \quad \hat{p}_j = \frac{p_{2k+1-j}}{p_{2k+1}}.$$

By Remark 4.4, we have

$$(4.12) \quad (-1)^{k-1} \sum_{i=0}^k C_k^i (-1)^i \hat{p}_{2k-2i+1} - (-1)^{k-1} \hat{p}_1 \sum_{i=0}^k C_k^i (-1)^i \hat{p}_{2k-2i} \leq 0,$$

which is equivalent to

$$(4.13) \quad (-1)^{k-1} \sum_{i=0}^k C_k^i (-1)^i \frac{p_{2i}}{p_{2k+1}} - (-1)^{k-1} \frac{p_{2k}}{p_{2k+1}} \sum_{i=0}^k C_k^i (-1)^i \frac{p_{2i+1}}{p_{2k+1}} \leq 0.$$

Thus we have

$$(4.14) \quad \sum_{i=0}^k C_k^i (-1)^{k-i} p_{2i} - \frac{p_{2k}}{p_{2k+1}} \sum_{i=0}^k C_k^i (-1)^{k-i} p_{2i+1} \geq 0,$$

which implies $\frac{p_{2k}}{p_{2k+1}} \tilde{N}_k - \tilde{L}_k \leq 0$. \square

Proof of Proposition 4.2. Then by the Newton-MacLaurin inequality $p_{2k-1}p_{2k+1} \leq p_{2k}^2$, we obtain

$$\frac{p_{2k-1}}{p_{2k}} \tilde{N}_k - \tilde{L}_k \leq \frac{p_{2k}}{p_{2k+1}} \tilde{N}_k - \tilde{L}_k \leq 0,$$

which is exactly (4.5). Here we have used Lemma 4.5. \square

Remark 4.8. Proposition 4.2 holds for $\kappa \in \mathbb{R}^{n-1}$ with $\kappa_i \kappa_j \geq 1$ for any i, j . This is equivalent to the condition that the sectional curvature of Σ is non-negative.

Remark 4.9. From the proof of Proposition 4.2, it is easy to see that (4.5) has an inverse inequality for $\kappa \in \mathbb{R}^{n-1}$ with $0 \leq \kappa_i \leq 1$.

Now we have a monotonicity of $Q(\Sigma_t)$ defined by (1.21) under the flow (2.7).

Theorem 4.10. Functional Q is non-increasing under the flow (2.7), provided that the initial surface is horospherical convex.

Proof. It follows from (3.1), (3.4) and Proposition 4.2 that

$$(4.15) \quad \frac{d}{dt} \int_{\Sigma} L_k \leq (n-1-2k) \int_{\Sigma} L_k.$$

On the other hand, by (2.6) and (2.3), we also have

$$(4.16) \quad \frac{d}{dt} |\Sigma_t| = \int_{\Sigma_t} \frac{p_{2k-1}}{p_{2k}} (n-1) p_1 d\mu \geq (n-1) |\Sigma_t|.$$

Combining (4.15) and (4.16) together, we complete the proof. \square

Remark 4.11. From the above proof, one can check that to obtain a monotonicity of Q it is enough to choose $F = \frac{1}{p_1}$. Then from (4.1) and (4.7), it holds for all k

$$\begin{aligned} \frac{d}{dt} \int \tilde{L}_k &= (n-2k-1) \int \tilde{L}_k + (n-2k-1) \left(\frac{1}{p_1} \tilde{N}_k - \tilde{L}_k \right) \\ &\leq (n-2k-1) \int \tilde{L}_k. \end{aligned}$$

5. PROOF OF MAIN THEOREMS

Now we are ready to show our main theorems.

Proof of Theorem 1.1. First recall the definition (1.21) of the functional Q , (1.13) is equivalent to

$$(5.1) \quad Q(\Sigma) \geq C_{n-1}^{2k} (2k)! \omega_{n-1}^{\frac{2k}{n-1}}.$$

Let $\Sigma(t)$ be a solution of flow (2.7) obtained by the work of Gerhardt [16]. This flow preserves the horospherical convexity and non-increases for the functional Q . Hence, to show (5.1) we only need to show

$$(5.2) \quad \lim_{t \rightarrow \infty} Q(\Sigma_t) \geq C_{n-1}^{2k} (2k)! \omega_{n-1}^{\frac{2k}{n-1}}.$$

Since Σ is a horospherical convex hypersurface in (\mathbb{H}^n, \bar{g}) , it is written as graph of function $r(\theta)$, $\theta \in \mathbb{S}^{n-1}$. We denote $X(t)$ as graphs $r(t, \theta)$ on \mathbb{S}^{n-1} with the standard metric \hat{g} . We set $\lambda(r) = \sinh(r)$ and we have $\lambda'(r) = \cosh(r)$. It is clear that

$$(\lambda')^2 = (\lambda)^2 + 1.$$

We define $\varphi(\theta) = \Phi(r(\theta))$. Here Φ verifies

$$\Phi' = \frac{1}{\lambda}.$$

We define another function

$$v = \sqrt{1 + |\nabla\varphi|_{\hat{g}}^2}.$$

By [16], we have the following results.

Lemma 5.1.

$$\lambda = O(e^{-\frac{t}{n-1}}), \quad |\nabla\varphi| + |\nabla^2\varphi| = O(e^{-\frac{t}{n-1}}).$$

From Lemma (5.1), we have the following expansions:

$$(5.3) \quad \lambda' = \lambda(1 + \frac{1}{2}\lambda^{-2}) + O(e^{-\frac{4t}{n-1}}),$$

and

$$(5.4) \quad \frac{1}{v} = 1 - \frac{1}{2}|\nabla\varphi|_{\hat{g}}^2 + O(e^{-\frac{4t}{n-1}}).$$

We have also

$$(5.5) \quad \nabla\lambda = \lambda\lambda'\nabla\varphi.$$

The second fundamental form of Σ is written in an orthogonal basis (see [10] for example)

$$\begin{aligned} h_i^j &= \frac{\lambda'}{v\lambda} \left(\delta_i^j - \frac{\varphi_i^j}{\lambda'} + \frac{\varphi_i\varphi_l\varphi^{jl}}{v^2\lambda'} \right) \\ &= \delta_i^j + \left(\frac{1}{2\lambda^2} - \frac{1}{2}|\nabla\varphi|^2 \right) \delta_i^j - \frac{\varphi_i^j}{\lambda} + O(e^{-\frac{4t}{n-1}}), \end{aligned}$$

where the second equality follows from (5.3) and (5.4). We set

$$(5.6) \quad T_i^j = \left(\frac{1}{2\lambda^2} - \frac{1}{2}|\nabla\varphi|^2 \right) \delta_i^j - \frac{\varphi_i^j}{\lambda},$$

then from the Gauss equations, we obtain

$$\begin{aligned} R_{ij}^{kl} &= -(\delta_i^k\delta_j^l - \delta_i^l\delta_j^k) + (h_i^k h_j^l - h_i^l h_j^k) \\ &= \delta_i^k T_j^l + T_i^k \delta_j^l - T_i^l \delta_j^k - \delta_i^l T_j^k + O(e^{-\frac{4t}{n-1}}). \end{aligned}$$

It follows from (1.1) that

$$\begin{aligned}
 L_k &= \frac{1}{2^k} \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} R_{i_1 i_2}^{j_1 j_2} \dots R_{i_{2k-1} i_{2k}}^{j_{2k-1} j_{2k}} \\
 &= 2^k \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} T_{i_1}^{j_1} \delta_{i_2}^{j_2} \dots T_{i_{2k-1}}^{j_{2k-1}} \delta_{i_{2k}}^{j_{2k}} + O(e^{-\frac{(2k+2)t}{n-1}}) \\
 &= 2^k (n-1-k) \dots (n-2k) \delta_{j_1 j_3 \dots j_{2k-1}}^{i_1 i_3 \dots i_{2k-1}} T_{i_1}^{j_1} T_{i_3}^{j_3} \dots T_{i_{2k-1}}^{j_{2k-1}} + O(e^{-\frac{(2k+2)t}{n-1}}) \\
 &= 2^k k! (n-1-k) \dots (n-2k) \sigma_k(T) + O(e^{-\frac{(2k+2)t}{n-1}}).
 \end{aligned}$$

Here in the second equality we use the fact

$$\begin{aligned}
 \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} T_{i_1}^{j_1} \delta_{i_2}^{j_2} &= \delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} \delta_{i_1}^{j_1} T_{i_2}^{j_2} \\
 &= -\delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} T_{i_1}^{j_2} \delta_{i_2}^{j_1} = -\delta_{j_1 j_2 \dots j_{2k-1} j_{2k}}^{i_1 i_2 \dots i_{2k-1} i_{2k}} \delta_{i_1}^{j_2} T_{i_2}^{j_1},
 \end{aligned}$$

and in the third equality we use (2.1) and (3.3).

Recall $\varphi_i = \lambda_i / \lambda \lambda'$, then by (5.3) we have

$$(5.7) \quad \varphi_{ij} = \frac{\lambda_{ij}}{\lambda^2} - \frac{2\lambda_i \lambda_j}{\lambda^3} + O(e^{-\frac{3t}{n-1}}).$$

By the definition of the Schouten tensor,

$$A_{\hat{g}} = \frac{1}{n-3} \left(Ric_{\hat{g}} - \frac{R_{\hat{g}}}{2(n-2)} \hat{g} \right) = \frac{1}{2} \hat{g}.$$

Its conformal transformation formula is well-known (see for example [30])

$$(5.8) \quad A_{\lambda^2 \hat{g}} = -\frac{\nabla^2 \lambda}{\lambda} + \frac{2\nabla \lambda \otimes \nabla \lambda}{\lambda^2} - \frac{1}{2} \frac{|\nabla \lambda|^2}{\lambda^2} \hat{g} + A_{\hat{g}} = -\frac{\nabla^2 \lambda}{\lambda} + \frac{2\nabla \lambda \otimes \nabla \lambda}{\lambda^2} - \frac{1}{2} \frac{|\nabla \lambda|^2}{\lambda^2} \hat{g} + \frac{1}{2} \hat{g}.$$

Substituting (5.5) and (5.7) into (5.6), together with (5.8), we have

$$T_i^j = ((\lambda^2 \hat{g})^{-1} A_{\lambda^2 \hat{g}})_i^j + O(e^{-\frac{4t}{n-1}}),$$

which implies

$$(5.9) \quad L_k = 2^k k! (n-1-k) \dots (n-2k) \sigma_k(A_{\lambda^2 \hat{g}}) + O(e^{-\frac{(2k+2)t}{n-1}}).$$

As before, $\Sigma(t)$ is a horospherical convex hypersurface. As a consequence, Σ has the nonnegative sectional curvature so that $T + O(e^{-\frac{4t}{n-1}})$ is positive definite. We consider $\tilde{\lambda} := \lambda^{1-e^{-\frac{t}{n-1}}}$ and the conformal metric $\tilde{\lambda}^2 \hat{g}$. We have

$$\begin{aligned}
 \tilde{\lambda}^2 (\tilde{\lambda}^2 \hat{g})^{-1} A_{\tilde{\lambda}^2 \hat{g}} &= \frac{1}{2} e^{-\frac{t}{n-1}} I + \frac{1}{2} e^{-\frac{t}{n-1}} (1 - e^{-\frac{t}{n-1}}) \frac{|\nabla \lambda|^2}{\lambda^2} I - e^{-\frac{t}{n-1}} (1 - e^{-\frac{t}{n-1}}) \hat{g}^{-1} \frac{\nabla \lambda \otimes \nabla \lambda}{\lambda^2} \\
 &\quad + \lambda^2 (1 - e^{-\frac{t}{n-1}}) (\lambda^2 \hat{g})^{-1} A_{\lambda^2 \hat{g}}.
 \end{aligned}$$

Recall $\frac{1}{2} e^{-\frac{t}{n-1}} I + \lambda^2 (1 - e^{-\frac{t}{n-1}}) (\lambda^2 \hat{g})^{-1} A_{\lambda^2 \hat{g}} \in \Gamma_{n-1}^+$ for the sufficiently large t and $\frac{1}{2} e^{-\frac{t}{n-1}} (1 - e^{-\frac{t}{n-1}}) \frac{|\nabla \lambda|^2}{\lambda^2} I - e^{-\frac{t}{n-1}} (1 - e^{-\frac{t}{n-1}}) \hat{g}^{-1} \frac{\nabla \lambda \otimes \nabla \lambda}{\lambda^2} \in \Gamma_k^+$ for any $k \leq \frac{n-1}{2}$. Therefore, we infer $\tilde{\lambda}^2 \hat{g} \in \Gamma_k^+$ for any $k \leq \frac{n-1}{2}$. The Sobolev inequality (2.9) for the σ_k operator gives

$$(5.10) \quad (vol(\tilde{\lambda}^2 \hat{g}))^{-\frac{n-1-2k}{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_k(A_{\tilde{\lambda}^2 \hat{g}}) dvol_{\tilde{\lambda}^2 \hat{g}} \geq \frac{(n-1) \dots (n-k)}{2^k k!} \omega_{n-1}^{\frac{2k}{n-1}}.$$

On the other hand, we have

$$(5.11) \quad \begin{aligned} & (\text{vol}(\tilde{\lambda}^2 \hat{g}))^{-\frac{n-1-2k}{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_k(A_{\tilde{\lambda}^2 \hat{g}}) d\text{vol}_{\tilde{\lambda}^2 \hat{g}} \\ &= (1 + o(1)) (\text{vol}(\lambda^2 \hat{g}))^{-\frac{n-1-2k}{n-1}} \int_{\mathbb{S}^{n-1}} \sigma_k(A_{\lambda^2 \hat{g}}) d\text{vol}_{\lambda^2 \hat{g}}, \end{aligned}$$

since

$$\lambda^{-e^{-\frac{t}{n-1}}} = 1 + o(1).$$

As a consequence of (5.9), (5.10) and (5.11), we deduce

$$\lim_{t \rightarrow +\infty} (\text{vol}(\Sigma(t)))^{-\frac{n-1-2k}{n-1}} \int_{\Sigma(t)} L_k \geq (n-1)(n-2) \cdots (n-2k) \omega_{n-1}^{\frac{2k}{n-1}}.$$

When (5.1) is an equality, then Q is constant along the flow. Then (4.16) is an equality, which implies that equality in the inequality

$$\frac{p_{2k-1}}{p_{2k}} p_1 \geq 1,$$

holds. Therefore, Σ is a geodesic sphere. \square

Proof of Theorem 1.2. It follows from (3.1), (3.4) and Theorem 1.1 that when $n-1 > 2k$

$$(5.12) \quad \int_{\Sigma} \tilde{L}_k \geq \omega_{n-1}^{\frac{2k}{n-1}} (|\Sigma|)^{\frac{n-1-2k}{n-1}}.$$

Using the expression (3.5) of $\int_{\Sigma} \sigma_k$ in terms of $\int_{\Sigma} \tilde{L}_j$ we get the desired result

$$\int_{\Sigma} \sigma_{2k} \geq C_{n-1}^{2k} \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{1}{k} \frac{n-1-2k}{n-1}} \right\}^k.$$

By Theorem 1.1, equality holds if and only if Σ is a geodesic sphere.

When $n-1 = 2k$, since the hypersurface Σ is convex, we know that (1.13) is an equality when $n-1 = 2k$ by the Gauss-Bonnet-Chern theorem, even for any hypersurface diffeomorphic to a sphere. Hence in this case, we also have all the above inequalities with equality which in turn implies by [22] or [15] that Σ is a geodesic sphere. \square

Proof of Theorem 1.3. When $n-1 > 2k$, the proof follows directly from (5.12) and Lemma 3.3. When $n-1 = 2k$, the proof follows by the same reason as in Theorem 1.2. \square

From (1.17), it is easy to see that Theorem 1.3 implies Theorem 1.2, meanwhile Theorem 1.2 may not directly imply Theorem 1.3, since there are negative coefficients in (3.8) above.

6. ALEXANDROV-FENCHEL INEQUALITY FOR ODD k

In this section, we show an Alexandrov-Fenchel inequality for σ_1 , which follows from the result of Cheng-Zhou [8] and Theorem 1.2 (or more precisely from [22]).

Theorem 6.1. *Let $n \geq 2$. Any horospherical convex hypersurface $\Sigma \subset \mathbb{H}^n$ satisfies*

$$(6.1) \quad \int_{\Sigma} \sigma_1 \geq (n-1)\omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^2 + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2(n-2)}{n-1}} \right\}^{\frac{1}{2}}.$$

where ω_{n-1} is the area of the unit sphere \mathbb{S}^{n-1} and $|\Sigma|$ is the area of Σ . Equality holds if and only if Σ is a geodesic sphere.

Proof. Notice that the horospherical convex condition implies that the Ricci curvature of Σ is non-negative. We observe first that by a direct computation (1.4) in [8]

$$\int_{\Sigma} |H - \bar{H}|^2 \leq \frac{n-1}{n-2} \int_{\Sigma} |B - \frac{H}{n-1}g|^2,$$

is equivalent to

$$(6.2) \quad \int_{\Sigma} \sigma_2 \int_{\Sigma} \sigma_0 \leq \frac{n-2}{2(n-1)} \left(\int_{\Sigma} \sigma_1 \right)^2.$$

Then we use the optimal inequality for σ_2 proved in [22] (see also Theorem 1.2),

$$(6.3) \quad \int_{\Sigma} \sigma_2 \geq \frac{(n-1)(n-2)}{2} \left(\omega_{n-1}^{\frac{2}{n-1}} |\Sigma|^{\frac{n-3}{n-1}} + |\Sigma| \right),$$

to obtain the desired inequality for σ_1 ,

$$\int_{\Sigma} \sigma_1 \geq (n-1)\omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^2 + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2(n-2)}{n-1}} \right\}^{\frac{1}{2}}.$$

When (6.1) is an equality, in turn, (6.3) is also a equality, then it follows from [22] that the hypersurface is a geodesic sphere. \square

Motivated by Theorem 1.2 and (6.2), we would like propose the following

Conjecture 6.2. *Let $n-1 \geq 2k+1$. Any horospherical convex hypersurface $\Sigma \subset \mathbb{H}^n$ satisfies*

$$\int_{\Sigma} \sigma_{2k+1} \geq C_{n-1}^{2k+1} \omega_{n-1} \left\{ \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2}{2k+1}} + \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{2}{2k+1} \frac{(n-2k-2)}{n-1}} \right\}^{\frac{2k+1}{2}}.$$

Equality holds if and only if Σ is a geodesic sphere.

The conjecture follows from Theorem 1.2 and the following conjecture

$$(6.4) \quad \frac{(C_{n-1}^{2k+1})^2}{C_{n-1}^{2k+2} C_{n-1}^{2k}} \int_{\Sigma} \sigma_{2k+2} \int_{\Sigma} \sigma_{2k} \leq \left(\int_{\Sigma} \sigma_{2k+1} \right)^2.$$

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