A NEW MASS FOR ASYMPTOTICALLY FLAT MANIFOLDS

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1. Introduction

One of the important results in differential geometry is the Riemannian positive mass theorem (PMT): *Any asymptotically flat Riemannian manifold $\mathcal{M}^n$ with a suitable decay order and with nonnegative scalar curvature has nonnegative ADM mass.* This theorem was proved by Schoen and Yau [50] for manifolds of dimension $n \le 7$ using a minimal hypersurface argument and by Witten [57] for any spin manifold. See also [44]. For locally conformally flat manifolds the proof was given in [51] using the developing map. Very recently, the PMT was proved for all asymptotically flat Riemannian manifold $\mathcal{M}^n$ which are represented by a graph in $\mathbb{R}^{n+1}$ by Lam [35]. For general higher dimensional manifolds, the proof of the positive mass theorem was announced by Lohkamp [40] by an argument extending the minimal hypersurface argument of Schoen and Yau and by Schoen in [49]. There are many generalizations of the positive mass theorem. For example, a refinement of the PMT, the Riemannian Penrose inequality is proved by Huisken-Ilmanen [33] and Bray [4] for $n = 3$ and by Bray and Lee [3] for $n \le 7$. See the excellent surveys on the Riemannian Penrose inequality of Bray [5] and Mars [42].

The ADM mass was introduced by Arnowitt, Deser, and Misner [1] for asymptotically flat Riemannian manifolds. A complete manifold $(\mathcal{M}^n, g)$ is said to be an asymptotically flat(AF) of order τ (with one end) if there is a compact K such that $\mathcal{M} \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$ for some $R > 0$ and in the standard coordinates in $\mathbb{R}^n$, the metric g has the following expansion

$$g_{ij} = \delta_{ij} + \sigma_{ij}$$

with

$$|\sigma_{ij}| + r|\partial \sigma_{ij}| + r^2 |\partial^2 \sigma_{ij}| = O(r^{-\tau}),$$

where r and ∂ denote the Euclidean distance and the standard derivative operator on $\mathbb{R}^n$ respectively. The ADM mass is defined by

$$m_1(g) := m_{ADM} := \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j}) n_j dS, \tag{1.1}$$

where ω_{n-1} is the volume of $(n-1)$-dimensional standard unit sphere and S_r is the Euclidean coordinate sphere, dS is the volume element on S_r induced by the Euclidean metric, n is the outward unit normal to S_r in $\mathbb{R}^n$ and the derivative is the ordinary partial derivative.

In a seminal paper Bartnik [2] proved that the ADM mass is well-defined for asymptotically flat Riemannian manifolds with a suitable decay order τ and it is a geometric invariant. Precisely,

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it does not depend on the choice of the coordinates, provided

$$\tau > \frac{n-2}{2}. \tag{1.2}$$

With this restriction, the ADM mass cannot be defined for many other interesting asymptotically flat Riemannian manifolds. For example, the following metric

$$g^{(2)}_{\mathrm{Sch}} = \left(1 - \frac{2m}{r^{\frac{n}{2}-2}}\right)^{-1} dr^2 + r^2 d\Theta^2 = \left(1 + \frac{m}{2r^{\frac{n-4}{2}}}\right)^{\frac{8}{n-4}} g_{\mathbb{R}^n}, \tag{1.3}$$

plays an important role as the Schwarzschild metric in the (pure) Gauss-Bonnet gravity [12]. Its decay order is $\frac{n-4}{2}$, which is smaller than $\frac{n-2}{2}$. Here $d\Theta^2$ is the standard metric on $\mathbb{S}^{n-1}$. For the discussion of this metric and more general Schwarzschild type metrics, see Section 6 below.

It is well-known that the ADM mass is very closely related to the scalar curvature. In fact, on an asymptotically flat manifold with decay order τ, the scalar curvature has the following expression [48]

$$R(g) = \partial_j(g_{ij,i} - g_{ii,j}) + O(r^{-2\tau-2}).$$

From this expression one can check that

$$\lim_{r\to\infty} \int_{S_r} (g_{ij,i} - g_{ii,j}) n_j dS$$

is well defined, provide that $\tau > \frac{n-2}{2}$ and R is integrable. This term gives the ADM mass after a normalization. From this interpretation one can easily see the mathematical meaning of the ADM mass. This also motivates us to introduce a new mass by using the following second Gauss-Bonnet curvature[1]

$$L_2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 = |W|^2 + 8(n-2)(n-3)\sigma_2,$$

where W is the Weyl tensor and σ_2 is the so-called σ_2-scalar curvature. More discussion about the Gauss-Bonnet curvature and the σ_2-scalar curvature will be given in the next section. In this paper we use the Einstein summation convention.

Definition 1.1 (Gauss-Bonnet-Chern Mass). *Let $n \geq 5$. Suppose $(\mathcal{M}^n, g)$ is an asymptotically flat manifold of decay order*

$$\tau > \frac{n-4}{3}, \tag{1.4}$$

and the second Gauss-Bonnet curvature given by $L_2 = R_{ijkl}R^{ijkl} - 4R_{ij}R_{ij} + R^2$ is integrable in $(\mathcal{M}^n, g)$. We define the Gauss-Bonnet-Chern mass-energy by

$$m_2(g) := m_{GBC}(g) = c_2(n) \lim_{r\to\infty} \int_{S_r} P^{ijkl}\partial_l g_{jk} n_i dS, \tag{1.5}$$

where $c_2(n) = \frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}}$, n is the outward unit normal to S_r, dS is the area element of S_r and the tensor P is defined

$$P^{ijkl} = R^{ijkl} + R^{jk}g^{il} - R^{jl}g^{ik} - R^{ik}g^{jl} + R^{il}g^{jk} + \frac{1}{2}R(g^{ik}g^{jl} - g^{il}g^{jk}).$$

[1]The second named author would like to thank Professor Schoen for his suggestion to use this way to define a mass by using the σ_k-curvature ($k \geq 2$) in Toronto in 2005.

We remark that when $n = 4$ one can also define the m_2 mass, but in this case (i.e., $n = 4$) m_2 always vanishes. See also the discussion in Section 7. In fact one easily check that m_2 vanishes for asymptotically flat manifolds of decay decay order larger than $\frac{n-4}{2}$. Hence the ordinary Schwarzschild metric

$$g^{(1)}_{\mathrm{Sch}} = \left(1 - \frac{2m}{r^{n-2}}\right)^{-1} dr^2 + r^2 d\Theta^2 = \left(1 + \frac{m}{2r^{n-2}}\right)^{\frac{4}{n-4}} g_{\mathbb{R}^n}$$

considered in the Einstein gravity have a vanishing GBC mass whenever it can be defined, for it has a decay order $\tau = n - 2 > \frac{n-4}{2}$. For the metric given in (1.3) one can check that the Gauss-Bonnet mass $m_2(g) = m^2$, which is positive. See Section 6 below.

Our work is partly motivated by the study of the σ_2-curvature and partly by the study of Einstein-Gauss-Bonnet gravity, in which there is a similar mass defined for the Gauss-Bonnet

$$R + \Lambda + \alpha L_2, \tag{1.6}$$

where Λ is the cosmological constant and α is a parameter. In contrast, if one considers only the term L_2, it is called the pure Gauss-Bonnet, or pure Lovelock gravity in physics. The study of Einstein-Gauss-Bonnet gravity was initiated by the work of Boulware, Deser, Wheeler [6], [56]. A mass for (1.6) was introduced by Deser-Tekin [23] and [24]. See also [21],[43],[11] and especially [12] and references therein.

Similar to the work of Bartink for the ADM mass we first show that the GBC mass m_2 is a geometric invariant in the following

Theorem 1.2. *Suppose $(\mathcal{M}^n, g)(n \geq 5)$ is an asymptotically flat manifold of decay order $\tau > \frac{n-4}{3}$ and the second Gauss-Bonnet curvature L_2 is integrable in $(\mathcal{M}^n, g)$, then the Gauss-Bonnet-Chern mass m_2 is well-defined and does not depend the choice of the asymptotical coordinates used in the definition.*

Now it is a natural question to ask:

Is the GBC mass m_2 nonegative when the Gauss-Bonnet curvature L_2 is nonnegative?

Due to the lack of methods, we can not yet attack this question for a general asymptotically flat manifold. Instead, we leave this question as a conjecture and provide a strong support in the following result. Precisely, the problem has an affirmative answer, if the asymptotically flat manifold $\mathcal{M}^n$ can be embedded in $\mathbb{R}^{n+1}$ as a graph over $\mathbb{R}^n$.

Definition 1.3. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth function and let f_i, f_{ij} and f_{ijk} denote the first, the second and the third derivative of f. f is called* asymptotically flat of order τ *if*

$$f_i(x) = O(|x|^{-\tau/2}),$$
$$|x||f_{ij}(x)| + |x|^2|f_{ijk}(x)| = O(|x|^{-\tau/2})$$

at infinity for some $\tau > (n-4)/3$.

Theorem 1.4 (Positve Mass Theorem). *Let $(\mathcal{M}^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$. Let $L_2 \triangleq R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2$ be the Gauss-Bonnet curvature with respect to g and m_2 be the corresponding mass defined in the definition 1.1.Then*

$$m_2 = \frac{1}{4(n-1)(n-2)(n-3)\omega_{n-1}} \int_{\mathcal{M}^n} L_2 \cdot \frac{1}{\sqrt{1+|\nabla_\delta f|^2}} dV_g.$$

Particularly, $L_2 \geq 0$ yields $m_2 \geq 0$.

For the more details see Section 4 below. This result is motivated by the recent work of Lam [35] mentioned above. See also the work in [32], [18] and [19].

Remark 1.5. *In this paper, for simplicity we focus on the mass defined by the second Gauss-Bonnet curvature. From the proof one can easily generalize the results to the mass defined by the generalized Gauss-Bonnet curvature*

$$\begin{aligned} L_k &= \frac{1}{2^k} \delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}} \\ &= P_k^{ijlm} R_{ijlm} \end{aligned}$$

by (2.1) and (7.1) with $k < n/2$. See the discussion in Section 7.

More interesting is that we have a Penrose type inequality, at least for the graphs.

Theorem 1.6 (Penrose Inequality). *Let Ω be a bounded open set in $\mathbb{R}^n$ and $\Sigma = \partial\Omega$. If $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ is a smooth asymptotically flat function such that each connected component of $f(\Sigma)$ is in a level set of f and $|\nabla f(x)| \to \infty$ as $x \to \Sigma$. Let Ω_i be connected components of $\Omega, i = 1, \cdots, k$ and let $\Sigma_i = \partial\Omega_i$ and assume that each Ω_i is convex, then*

$$\begin{aligned} m_2 &\geq \frac{c_2(n)}{2} \int_{\mathcal{M}^n} \frac{L_2}{\sqrt{1+|\nabla f|^2}} dV_g + \sum_{i=1}^k \frac{1}{4} \left(\frac{\int_{\Sigma_i} R}{(n-1)(n-2)\omega_{n-1}} \right)^{\frac{n-4}{n-3}} \\ &\geq \frac{c_2(n)}{2} \int_{\mathcal{M}^n} \frac{L_2}{\sqrt{1+|\nabla f|^2}} dV_g + \sum_{i=1}^k \frac{1}{4} \left(\frac{\int_{\Sigma_i} H}{(n-1)\omega_{n-1}} \right)^{\frac{n-4}{n-2}} \\ &\geq \frac{c_2(n)}{2} \int_{\mathcal{M}^n} \frac{L_2}{\sqrt{1+|\nabla f|^2}} dV_g + \sum_{i=1}^k \frac{1}{4} \left(\frac{|\Sigma_i|}{\omega_{n-1}} \right)^{\frac{n-4}{n-1}} \\ &\geq \frac{c_2(n)}{2} \int_{\mathcal{M}^n} \frac{L_2}{\sqrt{1+|\nabla f|^2}} dV_g + \frac{1}{4} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-4}{n-1}}. \end{aligned}$$

In particular, $L_2 \geq 0$ yields

$$m_2 \geq \frac{1}{4} \left(\frac{\int_\Sigma R}{(n-1)(n-2)\omega_{n-1}} \right)^{\frac{n-4}{n-3}} \geq \frac{1}{4} \left(\frac{|\Sigma|}{\omega_{n-1}} \right)^{\frac{n-4}{n-1}}.$$

Moreover, the equalities are achieved by the metric (1.3.)

Our Penrose inequalities are optimal since one can check that equality in the Penrose inequality is achieved by the metrics (1.3). Also as illustrated in the Remark 6.5, the metric (1.3) can be realized as the induced metric of a graph.

Our results open many interesting questions and establish a natural relationship between many interesting functionals of intrinsic curvatures and extrinsic curvatures, which we will discuss at the end of this paper.

The rest of the paper is organized as follows. In Section 2 we recall the definitions of the Gauss-Bonnet curvature and the Lovelock curvature. The new mass is defined and proved being a geometric invariant in Section 3. In Section 4 we show that the new mass is nonnegative if the Gauss-Bonnet curvature is nonnegative for graphs. The Penrose type inequality will be proved in Section 5. In Section 6, we will discuss the metric (1.3), which is an important example and compute its Gauss-Bonnet mass explicitly. Further generalizations, problems and conjectures are discussed in Section 7.

2. Lovelock curvatures

In this section, let us recall the work of Lovelock [37] on generalized Einstein tensors. Let

$$E = Ric - \frac{1}{2}Rg$$

be the Einstein tensor. The Einstein tensor is very important in physics, and certainly also in mathematics. It admits an important property, namely it is a conversed quantity, i.e.,

$$E^i_{j,i} = 0.$$

In this paper we use the summation convention. In [37] Lovelock studied the classification of tensors A satisfying

(i) $A^{ij} = A^{ji}$, i.e., A is symmetric;
(ii) $A^{ij} = A^{ij}(g_{AB}, g_{AB,C}, g_{AB,CD})$;
(iii) $A^{ij}{}_{;j} = 0$, ie. A is divergence-free;
(iv) A^{ij} is linear in the second derivatives of g_{AB}.

It is clear that the Einstein tensor satisfies all the conditions. In fact, the Einstein tensor is the unique tensor satisfying all four conditions, up to a multiple constant. Lovelock classified all 2-tensors satisfying (i)–(iii). He proved that any 2-tensor satisfying (i)–(iii) has the form

$$\sum_j \alpha_j E^{(j)}$$

with certain constants α_j, $j \geq 0$, where the 2-tensor $E^{(k)}$ is defined by

$$E^{(k)}{}_{ij} := -\frac{1}{2^{k+1}} g_{\alpha i} \delta^{\alpha i_1 i_2 \cdots j_{2k-1} i_{2k}}_{j j_1 j_2 \cdots j_{2k-1} i_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}.$$

Here the generalized Kronecker delta is defined by

$$\delta^{j_1 j_2 \cdots j_r}_{i_1 i_2, \dots i_r} = \det \begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_r}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_r}_{i_2} \\ \vdots & \vdots & \vdots & \vdots \\ \delta^{j_1}_{i_r} & \delta^{j_2}_{i_r} & \cdots & \delta^{j_r}_{i_r} \end{pmatrix}.$$

As a convention we set $E^{(0)} = 1$. It is clear to see that $E^{(1)}$ is the Einstein tensor. The tensor ${E^{(k)}}_{ij}$ is a very natural generalization of the Einstein tensor. We call $E^{(k)}$ the *$k-th$ Lovelock curvature* and its trace

$$L_k := \frac{1}{2^k}\delta^{i_1 i_2\cdots i_{2k-1}i_{2k}}_{j_1 j_2\cdots j_{2k-1}j_{2k}} {R_{i_1 i_2}}^{j_1 j_2}\cdots {R_{i_{2k-1}i_{2k}}}^{j_{2k-1}j_{2k}} \tag{2.1}$$

the *k-th Gauss-Bonnet curvature*, or simply the *Gauss-Bonnet curvature*. Both have been intensively studied in the Gauss-Bonnet gravity, which is a generalization of the Einstein gravity. One could check that $L_k = 0$ if $2k > n$. When $2k = n$, L_k is in fact the Euler density, which was studied by Chern [9, 10] in his proof of the Gauss-Bonnet-Chern theorem. See also a nice survey [58] on the proof of the Gauss-Bonnet-Chern theorem. For $k < n/2$, L_k is therefore called the dimensional continued Euler density in physics. The above curvatures have been studied by many mathematicians and physicists, see for instance Pattersen [45] and Labbi [34].

In this paper we focus on the case $k = 2$. The results can be generalized to $k < n/2$. For the discussion see Section 7. One can also check that

$$E^{(2)}_{\mu\nu} = 2RR_{\mu\nu} - 4R_{\mu\alpha}{R^\alpha}_\nu - 4R_{\alpha\beta}{{{R^\alpha}_\mu}^\beta}_\nu + 2R_{\mu\alpha\beta\gamma}{R_\nu}^{\alpha\beta\gamma} - \frac{1}{2}g_{\mu\nu}L_2,$$

and

$$L_2 = \frac{1}{4}\delta^{i_1 i_2 i_3 i_4}_{j_1 j_2 j_3 j_4}{R^{j_1 j_2}}_{i_1 i_2}{R^{j_3 j_4}}_{i_3 i_4} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$

L_2 is called the Gauss-Bonnet term in physics. A direct computation gives the relation of L_2 with σ_2-scalar curvature and the Weyl tensor

$$\begin{aligned} L_2 &= |W|^2 - 4\frac{n-3}{n-2}|Ric|^2 + \frac{n(n-3)}{(n-1)(n-2)}R^2 \\ &= |W|^2 + \frac{n-3}{n-2}\left(\frac{n}{n-1}R^2 - 4|Ric|^2\right) \\ &= |W|^2 + 8(n-2)(n-3)\sigma_2. \end{aligned} \tag{2.2}$$

Here the σ_k-scalar curvature σ_2 has been intensively studied in the σ_k-Yamabe problem, which is first studied by Viaclovsky and Chang-Gursky-Yang. For the study of the σ_k- Yamabe problem, see for example the survey of Guan [26] and Viaclovsky [54].

As a generalization of the Einstein metric, the solution of the following equation is called *(string-inspired) Einstein-Gauss-Bonnet* metric

$$E^{(2)}_{\mu\nu} = \lambda g_{\mu\nu}.$$

$E^{(2)}$ was already given by Lanczos [36] in 1938 and is called the Lanczos tensor. If g is such a metric, it is clear that

$$\lambda = \frac{1}{n}g^{\mu\nu}E^{(2)}_{\mu\nu} = \frac{4-n}{2n}L_2 = \frac{4-n}{2n}\left(8(n-2)(n-3)\sigma_2(g) + |W|^2\right).$$

Since $E^{(2)}$ is divergence free, λ must be constant in this case. This is a Schur type result. An almost Schur lemma for $E^{(k)}$ was proved in [30], which generalizes a result of Andrews, De Lellis-Topping [22].

3. The Gauss-Bonnet-Chern Mass

In this section, we will introduce a new mass by using the Gauss-Bonnet curvature for asymptotically flat manifolds. This would be compared with the ADM mass which can be defined from the scalar curvature as told in the introduction. Moreover, following the approach from [2],[41], we are able to show that this new defined mass is geometrical invariant, i.e. it does not depend on the choice of asymptotic coordinates at infinity. Recall for a complete Riemannian manifold $(\mathcal{M}, g)$, the Gauss-Bonnet curvature is given by

$$L_2 = R_{ijkl}R^{ijkl} - 4R_{ij}R^{ij} + R^2.$$

One ingredient to the main theorems in this section is the observation that the Gauss-Bonnet curvature has the following decomposition

$$L_2 = R_{ijkl}P^{ijkl},$$

where

$$P^{ijkl} = R^{ijkl} + R^{jk}g^{il} - R^{jl}g^{ik} - R^{ik}g^{jl} + R^{il}g^{jk} + \frac{1}{2}R(g^{ik}g^{jl} - g^{il}g^{jk}). \tag{3.1}$$

This decomposition of L_2 will play a crucial role in the following discussion. It is very easy to see that this $(0, 4)$ tensor P has the same symmetric property as the Riemannian curvature tensor, precisely,

$$P^{ijkl} = -P^{jikl} = -P^{ijlk} = P^{klij}. \tag{3.2}$$

Also one can easily check that P satisfies the first Bianchi identity. Another key ingredient is to note that P is divergence-free. Before we discuss further, let us clarify the convention for Riemannian curvature first:

$$R_{ijkl} = R^m_{ijl}g_{mk}, \quad R_{ik} = g^{jl}R_{ijkl} = R^j_{jik},$$
$$R^{ijk}{}_m = R^{ijks}g_{ms}, \quad R^{ij}{}_k{}^m = R^{ijsm}g_{ks}.$$

Lemma 3.1.

$$\nabla_i P^{ijkl} = \nabla_j P^{ijkl} = \nabla_k P^{ijkl} = \nabla_l P^{ijkl} = 0.$$

Proof. This lemma follows directly from the differential Bianchi identity.

$$\begin{aligned}
\nabla_i P^{ijkl} &= -\nabla^k R^{ijl}{}_i - \nabla^l R^{ij}{}_i{}^k + \nabla^l R^{jk} - \nabla^k R^{jl} \\
&\quad -\frac{1}{2}\nabla^k R g^{jl} + \frac{1}{2}\nabla^l R g^{jk} + \frac{1}{2}\nabla_i R(g^{ik}g^{jl} - g^{il}g^{jk}) \\
&= \nabla^k R^{jl} - \nabla^l R^{jk} + \nabla^l R^{jk} - \nabla^k R^{jl} - \frac{1}{2}\nabla^k R g^{jl} + \frac{1}{2}\nabla^l R g^{jk} \\
&\quad +\frac{1}{2}\nabla^k R g^{jl} - \frac{1}{2}\nabla^l R g^{jk} \\
&= 0.
\end{aligned}$$

The rest follows from the symmetry property (3.2) of P. □

This divergence-free property of P was observed already in physic's literature, see for instance [17]. In view of Lemma 3.1, we are able to derive the corresponding expression of the mass-energy in the Einstein Gauss-Bonnet gravity. We observe that for the asymptotically flat manifolds, the Gauss-Bonnet curvature can be expressed as a divergence term beside some terms with faster decay. First, in the local coordinates, the curvature tensor is expressed as

$$R_{ijk}^m = \partial_i \Gamma_{jk}^m - \partial_j \Gamma_{ik}^m + \Gamma_{is}^m \Gamma_{jk}^s - \Gamma_{js}^m \Gamma_{ik}^s.$$

Then by the divergence-free property of P and the fact that the quadratic terms of Christoffel symbols has faster decay, we compute

$$\begin{aligned}
L_2 &= R_{ijkl}P^{ijkl} = g_{km}R_{ijl}^m P^{ijkl} \\
&= g_{km}(\partial_i \Gamma_{jl}^m - \partial_j \Gamma_{il}^m)P^{ijkl} + O(r^{-4-3\tau}) \\
&= g_{km}\left[\nabla_i(\Gamma_{jl}^m P^{ijkl}) - \nabla_j(\Gamma_{il}^m P^{ijkl})\right] + O(r^{-4-3\tau}) \\
&= \frac{1}{2}\nabla_i\left[(g_{jk,l}+g_{kl,j}-g_{jl,k})P^{ijkl}\right] - \frac{1}{2}\nabla_j\left[(g_{ik,l}+g_{kl,i}-g_{il,k})P^{ijkl}\right] + O(r^{-4-3\tau}) \\
&= 2\nabla_i\left(g_{jk,l}P^{ijkl}\right) + O(r^{-4-3\tau}) \\
&= 2\partial_i\left(g_{jk,l}P^{ijkl}\right) + O(r^{-4-3\tau}), \qquad (3.3)
\end{aligned}$$

where the fifth equality follows from (3.2).

With this divergence expression of L_2, one can check that the limit

$$\lim_{r\to\infty}\int_{S_r} P^{ijkl}\partial_l g_{jk} n_i dS_r$$

exists and is finite provided that L_2 is integrable and the decay order $\tau > \frac{n-4}{3}$, and hence we have:

Theorem 3.2. *Suppose $(\mathcal{M}^n, g)(n \geq 5)$ is an asymptotically flat manifold with decay order $\tau > \frac{n-4}{3}$ and the Gauss-Bonnet curvature $L_2 = R_{ijkl}R^{ijkl} - 4R_{ij}R_{ij} + R^2$ is integrable in $(\mathcal{M}^n, g)$, then the mass $m_2(g)$ defined in Definition 1.1 is well-defined.*

We call m_2 the Gauss-Bonnet-Chern mass, or just the GBC mass. The definition of the Gauss-Bonnet mass involves the choice of asymptotic coordinates. So it is natural to ask if it is a geometric invariant which does not depend on the choices of asymptotic coordinates as the ADM mass. We have an affirmative answer.

Theorem 3.3. *If $(\mathcal{M}^n, g)$ is asymptotically flat of order $\tau > \frac{n-4}{3}$, then $m_2(g)$ depends only on the metric g.*

Proof. The argument follows closely the one given by Bartink in the proof of ADM mass [2]. See also [41]. The key is to realize that when changing the asymptotic coordinates, some extra terms which do not decay fast enough to have vanishing integral can be gathered in a divergence, thus its integral over any closed hypersurface vanishes. The first step is the same as observed in

[2],[41]. For the convenience of readers, we sketch it.

Step 1. Suppose $\{x^i\}$ and $\{\hat{x}^i\}$ are the two choices of asymptotic coordinates on $\mathcal{M} \setminus K$. In view of [2], after composing with an Euclidean motion, we may assume

$$\hat{x}^i = x^i + \varphi^i, \quad \text{ where } \varphi^i \in C^{2,\alpha}_{1-\tau}$$

for some $0 < \alpha < 1$. For the definition of these weighted spaces, please refer to [2, 41] for the details. Then the radial distance functions $r = |x|$ and $\hat{r} = |\hat{x}|$ are related by

$$C^{-1} r \le \hat{r} \le Cr, \quad \text{with some constant} \quad C > 0.$$

Let $\{S_R : r = R\}$ and $\{\hat{S}_R : \hat{r} = R\}$ be two spheres and $A_R : C^{-1}R \le \hat{r} \le CR$ an annulus. The divergence theorem yields

$$\left| \int_{\hat{S}_R} P^{ijkl} \partial_l g_{jk} \hat{n}_i d\hat{S} - \int_{S_R} P^{ijkl} \partial_l g_{jk} n_i dS \right| \ \le \ \int_{A_R} \left| \partial_i (P^{ijkl} \partial_l g_{jk}) \right| dx.$$

Due to (3.3) together with the assumption that L_2 is integrable and $\tau > \frac{n-4}{3}$, the integral

$$\int_{A_R} \left| \partial_i (P^{ijkl} \partial_l g_{jk}) \right| dx \to 0 \quad \text{as} \quad R \to \infty.$$

Therefore we can replace S_R by $\hat{S}_R$ in the definition of $m_2(g)$ without changing the mass.

Step 2. Denote $\partial_i = \frac{\partial}{\partial x^i}$, $\hat{\partial}_i = \frac{\partial}{\partial \hat{x}^i}$, $g_{ij} = g(\partial_i, \partial_j)$ and $\hat{g}_{ij} = g(\hat{\partial}_i, \hat{\partial}_j)$, then we have

$$\begin{aligned} \hat{\partial}_i &= \partial_i - \partial_i \varphi^s \partial_s + O(r^{-\tau}), \\ \hat{g}_{ij} &= g_{ij} - \partial_i \varphi^j - \partial_j \varphi^i + O(r^{-2\tau}), \\ \hat{\partial}_k \hat{g}_{ij} &= \partial_k g_{ij} - \partial_k \partial_i \varphi^j - \partial_k \partial_j \varphi^i + O(r^{-1-2\tau}). \end{aligned} \tag{3.4}$$

We compute

$$\begin{aligned} & \int_{S_r} \left(P^{ijkl} \partial_l g_{jk} - \hat{P}^{ijkl} \hat{\partial}_l \hat{g}_{jk} \right) n_i dS \\ = & \int_{S_r} P^{ijkl} (\partial_l g_{jk} - \hat{\partial}_l \hat{g}_{jk}) n_i dS + \int_{S_r} (P^{ijkl} - \hat{P}^{ijkl}) \hat{\partial}_l \hat{g}_{jk} n_i dS \\ = & \ I + II. \end{aligned}$$

In view of Lemma 3.1 together with (3.2) and (3.4), we compute

$$
\begin{aligned}
I &= \int_{S_r} P^{ijkl}(\partial_l\partial_j\varphi^k + \partial_l\partial_k\varphi^j)n^i dS + O(r^{-3-3\tau})\\
&= \int_{S_r} P^{ijkl}(\partial_l\partial_j\varphi^k)n_i dS + O(r^{-3-3\tau})\\
&= \int_{S_r} P^{ijkl}(\partial_j\partial_l\varphi^k)n_i dS + O(r^{-3-3\tau})\\
&= \int_{S_r} P^{ijkl}\partial_j\big((\partial_l\varphi^k)n_i\big) dS + O(r^{-3-3\tau})\\
&= \int_{S_r} \partial_j\big(P^{ijkl}(\partial_l\varphi^k)n_i\big) dS + O(r^{-3-3\tau})\\
&= \int_{S_r} [\partial_j - \langle n, \partial_j\rangle n]\big(P^{ijkl}(\partial_l\varphi^k)n_i\big) dS + \int_{S_r} \langle n, \partial_j\rangle n\big(P^{ijkl}(\partial_l\varphi^k)n_i\big) dS + O(r^{-3-3\tau}),
\end{aligned}
$$

where the fourth equality follows from (3.2) and $n(f) = \frac{\partial}{\partial n}f$. The first integral in I vanishes from the divergence theorem. We will show that the second integral decays fast and vanishes as r approaching infinity.

Since on the coordinate sphere S_r, the outward unit normal induced by the Euclidean metric $n \triangleq n^i \frac{\partial}{\partial x^i} = \nabla r$, we thus have

$$
n_i \triangleq \delta_{ij} n^j = n^i = \frac{x^i}{r}.
$$

By Lemma 3.1 and (3.3), we derive

$$
\begin{aligned}
&\int_{S_r} \langle n, \partial_j\rangle n\big(P^{ijkl}(\partial_l\varphi^k)n_i\big) dS\\
= &\int_{S_r} n_j n^t \frac{\partial}{\partial x^t}\big(P^{ijkl}(\partial_l\varphi^k)n_i\big) dS\\
= &\int_{S_r} n_j n^t P^{ijkl}\big(\partial_t\partial_l\varphi^k\big)n_i dS + \int_{S_r} n_j n^t P^{ijkl}(\partial_l\varphi^k)\frac{\partial}{\partial x^t}(n_i) dS + \int_{S_r} O(r^{-3-3\tau}) dS\\
= &\int_{S_r} \frac{x^i x^j x^t}{r^3} P^{ijkl}(\partial_t\partial_l\varphi^k) dS + \int_{S_r} \frac{x^j x^t}{r^2} P^{ijkl}(\partial_l\varphi^k)\Big(\frac{\delta_{it}}{r} - \frac{x^i x^t}{r^3}\Big) dS + \int_{S_r} O(r^{-3-3\tau}) dS\\
= &\int_{S_r} O(r^{-3-3\tau}) dS.
\end{aligned}
$$

Hence we obtain

$$
I = \int_{S_r} O(r^{-3-3\tau}) dS.
$$

For the second term II, we calculate

$$
II = \int_{S_r} \Big[(P^{ijkl} - \hat{P}^{ijkl})\partial_l g_{jk} n_i + O(r^{-3-3\tau})\Big] dS,
$$

and

$$\begin{aligned}
P^{ijkl} - \hat{P}^{ijkl} &= P_{ijkl} - \hat{P}_{ijkl} + O(r^{-2-2\tau}) \\
&= (R^k_{ijl} - \hat{R}^k_{ijl}) + (R_{jk} - \hat{R}_{jk})g_{il} - (R_{jl} - \hat{R}_{jl})g_{ik} - (R_{ik} - \hat{R}_{ik})g_{jl} \\
&\quad +(R_{il} - \hat{R}_{il})g_{jk} + \frac{1}{2}(R - \hat{R})(g_{ik}g_{jl} - g_{il}g_{jk}) + O(r^{-2-2\tau}).
\end{aligned}$$

From the expression of the curvature tensor and the Ricci tensor in local coordinates

$$\begin{aligned}
R^k_{ijl} &= -\frac{1}{2}(\partial_i\partial_k g_{jl} - \partial_i\partial_l g_{jk} - \partial_j\partial_k g_{il} + \partial_j\partial_l g_{ik}) + O(r^{-2-2\tau}), \\
R_{jk} &= \frac{1}{2}(\partial_i\partial_k g_{ji} - \partial_i\partial_i g_{jk} - \partial_j\partial_k g_{ii} + \partial_j\partial_i g_{ik}) + O(r^{-2-2\tau}),
\end{aligned}$$

and the difference

$$\hat{\partial}_k\hat{g}_{ij} = \partial_k g_{ij} - \partial_k\partial_i\varphi^j - \partial_k\partial_j\varphi^i + O(r^{-1-2\tau}),$$

we have

$$\begin{aligned}
R^k_{ijl} - \hat{R}^k_{ijl} &= -\frac{1}{2}\big[\partial_i\partial_k\partial_j\varphi^l + \partial_i\partial_k\partial_l\varphi^j - \partial_i\partial_l\partial_j\varphi^k - \partial_i\partial_l\partial_k\varphi^j - \partial_j\partial_k\partial_i\varphi^l \\
&\quad -\partial_j\partial_k\partial_l\varphi^i + \partial_j\partial_l\partial_i\varphi^k + \partial_j\partial_l\partial_k\varphi^i\big] + O(r^{-2-2\tau}) \\
&= O(r^{-2-2\tau}).
\end{aligned}$$

Similarly we have

$$R_{jk} - \hat{R}_{jk} = O(r^{-2-2\tau}).$$

Thus we obtain

$$II = \int_{S_r} O(r^{-3-3\tau})dS.$$

Combining the two things together, we obtain

$$\int_{S_r} (P^{ijkl}\partial_l g_{jk} - \hat{P}^{ijkl}\hat{\partial}_l\hat{g}_{jk})n_i dS = I + II = \int_{S_r} O(r^{-3-3\tau})dS,$$

which implies that

$$\lim_{r\to\infty}\int_{S_r} (P^{ijkl}\partial_l g_{jk} - \hat{P}^{ijkl}\hat{\partial}_l\hat{g}_{jk})n_i dS = 0,$$

when $\tau > \frac{n-4}{3}$. Therefore we conclude $m(g) = m(\hat{g})$ and finish the proof. $\square$

For the Euclidean metric, the GBC mass m_2 is trivially zero. Examples with non-vanishing GBC mass will be given in Section 6 later.

4. Positive mass theorem in the graph case

In this section, we investigate the special case that asymptotically flat manifolds are given as graphs of asymptotically constant functions over Euclidean space $\mathbb{R}^n$. As in the Riemannian positive mass theorem studied by Lam[35], for the new defined Gauss-Bonnet mass, we can show that the corresponding Riemannian positive mass holds for graphs when the Gauss-Bonnet curvature replaces the scalar curvature in all dimensions $n \geq 5$.

Following the notation in [35], suppose $(\mathcal{M}^n, g) = (\mathbb{R}^n, \delta + df \otimes df)$ be the graph of a smooth asymptotically flat function $f : \mathbb{R}^n \to \mathbb{R}$ which is as defined in the definition 1.3. Then

$$g_{ij} = \delta_{ij} + f_i f_j,$$

and the inverse of g_{ij} is

$$g^{ij} = \delta_{ij} - \frac{f_i f_j}{1 + |\nabla f|^2},$$

where the norm and the derivative ∇f are taken with respect to the flat metric δ. It is clear that such a graph is an asymptotically flat manifold of order τ in the sense of Definition 1.1. The Christoffel symbols Γ^k_{ij} with respect to the metric g and its derivatives can be computed as follows:

$$\begin{aligned}
\Gamma^k_{ij} &= \frac{f_{ij} f_k}{1 + |\nabla f|^2}, \\
\Gamma^k_{ij,l} &= \frac{f_{ijl} f_k + f_{ij} f_{kl}}{1 + |\nabla f|^2} - \frac{2 f_{ij} f_k f_s f_{ls}}{(1 + |\nabla f|^2)^2}.
\end{aligned} \tag{4.1}$$

The expression for the curvature tensor follows directly. For the convenience of the reader, we compute in the following lemma.

Lemma 4.1.

$$R_{ijkl} = \frac{f_{ik} f_{jl} - f_{il} f_{jk}}{1 + |\nabla f|^2}.$$

Proof. We begin with the $(1,3)$-type curvature tensor:

$$\begin{aligned}
R^l_{ijk} &= \Gamma^l_{jk,i} - \Gamma^l_{ik,j} + \Gamma^l_{is}\Gamma^s_{jk} - \Gamma^l_{js}\Gamma^s_{ik} \\
&= \frac{f_{jki} f_l + f_{jk} f_{li}}{1 + |\nabla f|^2} - \frac{2 f_{jk} f_l f_{is} f_s}{(1 + |\nabla f|^2)^2} - \frac{f_{ijk} f_l + f_{ik} f_{lj}}{1 + |\nabla f|^2} \\
&\quad + \frac{2 f_{ik} f_l f_{js} f_s}{(1 + |\nabla f|^2)^2} + \frac{f_{is} f_{jk} f_l f_s - f_{js} f_{ik} f_l f_s}{(1 + |\nabla f|^2)^2} \\
&= \frac{f_{il} f_{jk} - f_{ik} f_{jl}}{1 + |\nabla f|^2} + \frac{(f_{ik} f_{js} - f_{jk} f_{is}) f_s f_l}{(1 + |\nabla f|^2)^2}.
\end{aligned}$$

Then we have

$$
\begin{aligned}
R_{ijkl} &= R^m_{ijl} g_{km} \\
&= \Big(\frac{f_{im}f_{jl} - f_{il}f_{jm}}{1+|\nabla f|^2} + \frac{(f_{il}f_{js} - f_{jl}f_{is})f_s f_m}{(1+|\nabla f|^2)^2}\Big)(\delta_{mk} + f_m f_k) \\
&= \frac{f_{ik}f_{jl} - f_{il}f_{jk}}{1+|\nabla f|^2} + \frac{(f_{im}f_{jl} - f_{il}f_{jm})f_m f_k}{1+|\nabla f|^2} + \frac{(f_{il}f_{js} - f_{jl}f_{is})f_s f_k}{1+|\nabla f|^2} \\
&= \frac{f_{ik}f_{jl} - f_{il}f_{jk}}{1+|\nabla f|^2}.
\end{aligned}
$$

□

Remark 4.2. *This proof use the intrinsic definition of the curvature tensor. One can also calculate it from the Gauss formula via the extrinsic geometry.*

The divergence-free property of P enables us to express the Gauss-Bonnet curvature L_2 as a divergence term. This is a key ingredient to show the corresponding positive mass theorem for the GBC mass in the graph case.

Lemma 4.3.

$$\partial_i (P^{ijkl} \partial_l g_{jk}) = \frac{1}{2} L_2.$$

Proof.

$$\partial_i (P^{ijkl} \partial_l g_{jk}) = \partial_i P^{ijkl} \partial_l g_{jk} + P^{ijkl} \partial_i \partial_l g_{jk}.$$

We begin with the first term. Here it is important to use Lemma 3.1 to eliminate the terms of derivative of f of order three. In view of (3.2), we compute

$$
\begin{aligned}
& (\partial_i P^{ijkl})\partial_l g_{jk} \\
= & (\nabla_i P^{ijkl} - P^{sjkl}\Gamma^i_{is} - P^{iskl}\Gamma^j_{is} - P^{ijsl}\Gamma^k_{is} - P^{ijks}\Gamma^l_{is})\partial_l g_{jk} \\
= & -\big(P^{sjkl}\frac{f_{is}f_i}{1+|\nabla f|^2} + P^{iskl}\frac{f_{is}f_j}{1+|\nabla f|^2} + P^{ijsl}\frac{f_{is}f_k}{1+|\nabla f|^2} + P^{ijks}\frac{f_{is}f_l}{1+|\nabla f|^2}\big)(f_{jl}f_k + f_{kl}f_j) \\
= & -\big[P^{sjkl}\frac{f_{is}f_{jl}f_i f_k}{1+|\nabla f|^2} + P^{ijsl}\frac{f_{is}f_{jl}|\nabla f|^2 + f_{is}f_{kl}f_k f_j}{1+|\nabla f|^2} + P^{ijks}\frac{f_{is}f_{jl}f_k f_l + f_{is}f_{kl}f_j f_l}{1+|\nabla f|^2}\big] \\
= & -\frac{P^{ijkl}}{1+|\nabla f|^2}\big[(f_{is}f_{jl} + f_{il}f_{js})f_k f_s + (f_{ik}f_{sl} + f_{il}f_{ks})f_j f_s + |\nabla f|^2 f_{ik}f_{jl}\big] \\
= & -\frac{|\nabla f|^2}{1+|\nabla f|^2} f_{ik}f_{jl}P^{ijkl},
\end{aligned}
$$

where we have relabeled the indices in the fourth equality and used the property (3.2) in the third and fifth equality.

The second term is also simplified by (3.2).

$$
\begin{aligned}
P^{ijkl}\partial_i\partial_l g_{jk} &= P^{ijkl}\partial_i\partial_l(f_j f_k) \\
&= P^{ijkl}\cdot(f_{ijl}f_k + f_{ikl}f_j + f_{ik}f_{jl} + f_{ij}f_{kl}) \\
&= P^{ijkl}f_{ik}f_{jl}.
\end{aligned}
$$

Combining these two terms together, we arrive at

$$\begin{aligned}\partial_i(P^{ijkl}\partial_l g_{jk}) &= P^{ijkl}\cdot\frac{f_{ik}f_{jl}}{1+|\nabla f|^2}\\ &= \frac{1}{2}P^{ijkl}\cdot(\frac{f_{ik}f_{jl}-f_{il}f_{jk}}{1+|\nabla f|^2}).\end{aligned}\tag{4.2}$$

Recall that

$$L_2 = P^{ijkl}R_{ijkl},$$

and invoke the expression of the curvature tensor in Lemma 4.1, we complete the proof of the lemma. □

Lam showed a similar result for the scalar curvature, which is the crucial observation in [35]. See also the first paragraph of Section 7 below.

Now we are ready to prove our main Theorem 1.4.

Proof of Theorem 1.4. In view of Lemma 4.3 and the divergence theorem in (R^n,δ), we derive

$$\begin{aligned}m_2 &= \lim_{r\to\infty} c_2(n)\int_{S_r} P^{ijkl}\partial_l g_{jk}n^i dS_r\\ &= c_2(n)\int_{R^n}\partial_i(P^{ijkl}\partial_l g_{jk})dV_\delta\\ &= \frac{c_2(n)}{2}\int_{R^n} L_2 dV_\delta\\ &= \frac{c_2(n)}{2}\int_{\mathcal{M}^n} L_2\cdot\frac{1}{\sqrt{1+|\nabla f|^2}}dV_g,\end{aligned}$$

where $c_2(n)=\frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}}$ and the last equality holds due to the fact

$$dV_g=\sqrt{\det g}dV_\delta=\sqrt{1+|\nabla f|^2}dV_\delta.$$

□

5. Penrose inequality for graphs on $\mathbb{R}^n$

In this section we investigate the Penrose inequality related to the GBC mass for the manifolds which can be realized as graphs over $\mathbb{R}^n$. Let Ω be a bounded open set in $\mathbb{R}^n$ and $\Sigma=\partial\Omega$. If $f:\mathbb{R}^n\setminus\Omega\to\mathbb{R}$ is a smooth asymptotically flat function such that each connected component of $\{(x,f(x))\,|\,x\in\Sigma\}$ is in a level set of f and

$$|\nabla f(x)|\to\infty \quad \text{as } x\to\Sigma,\tag{5.1}$$

then the graph of f, $(\mathcal{M}^n,g)=(\mathbb{R}^n\setminus\Omega,\delta+df\otimes df)$, is an asymptotically flat manifold with an area horizon Σ. See Remark 5.1 below. Without loss of generality we may assume that $\{(x,f(x))\,|\,x\in\Sigma\}$ is included in $f^{-1}(0)$. In this case we can identify $\{(x,f(x))\,|\,x\in\Sigma\}$ with Σ.

On Σ, the outer unit normal vector induced by δ is

$$n \triangleq n^i \frac{\partial}{\partial x^i} = -\frac{\nabla f}{|\nabla f|}.$$

Then

$$n^i = -\frac{f_i}{|\nabla f|} \quad \text{and} \quad n_i \triangleq \delta_{ij} n^j = n^i.$$

As in the proof of Lemma 4.3, integrating by parts now gives

$$\begin{aligned}
m_2 &= \lim_{r\to\infty} c_2(n) \int_{S_r} P^{ijkl} \partial_l g_{jk} n_i dS \\
&= \frac{c_2(n)}{2} \int_{\mathbb{R}^n \setminus \Omega} L_2 \cdot \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g - c_2(n) \int_\Sigma P^{ijkl} \partial_l g_{jk} n_i dS_r \\
&= \frac{c_2(n)}{2} \int_{\mathcal{M}^n} L_2 \cdot \frac{1}{\sqrt{1+|\nabla f|^2}} dV_g - c_2(n) \int_\Sigma P^{ijkl} (f_{lj} f_k + f_{kl} f_j) n_i dS_r \\
&= \frac{c_2(n)}{2} \int_{\mathcal{M}^n} \frac{L_2}{\sqrt{1+|\nabla f|^2}} dV_g - c_2(n) \int_\Sigma P^{ijkl} f_{lj} f_k n_i dS_r,
\end{aligned}$$

where the last term in the third equality vanishes because of symmetry. From the definition of P^{ijkl} together with the expression $g^{ij} = \delta_{ij} - \frac{f_i f_j}{1+|\nabla f|^2}$ we have

$$\begin{aligned}
& P^{ijkl} f_{jl} f_k n_i \\
= \; & R^{ijkl} f_{jl} f_k n_i + R^{jk} (f_{ij} f_k - \frac{f_{lj} f_l f_k}{1+|\nabla f|^2} f_i) n_i - R^{ik} (f_{jj} f_k - \frac{f_{lj} f_l f_j}{1+|\nabla f|^2} f_k) n_i \\
& + R^{il} f_j \frac{f_{jl}}{1+|\nabla f|^2} n_i - R^{jl} f_{jl} \cdot \frac{f_i}{1+|\nabla f|^2} n_i + \frac{1}{2} R \cdot \frac{f_{jj} f_i - f_{ij} f_j}{1+|\nabla f|^2} n_i \\
= \; & I + II + III + IV + V + VI.
\end{aligned}$$

Recall that we have assumed that Σ is in a level set of f. At any given point $p \in \Sigma$, we choose the coordinates such that $\{\frac{\partial}{\partial x^2}, \cdots, \frac{\partial}{\partial x^n}\}$ denotes the tangential space of Σ and $\frac{\partial}{\partial x^1}$ denotes the normal direction of Σ. To clarify the notations, in the following we will use the convention that the Latin letters stand for the index: $1, 2, \cdots, n$ and the Greek letters stand for the index: $2, \cdots, n$. Now the computations are all done at the given point p. It is easy to see that we have

$$f_\alpha = 0 \quad \text{and} \quad f_{\alpha\beta} = A_{\alpha\beta} |\nabla f| = A_{\alpha\beta} |f_1|,$$

where $A_{\alpha\beta}$ is the second fundamental form of the isometric embedding (Σ, h) into the Euclidean space $\mathbb{R}^n$. In other words, h is the induced metric. Note that (Σ, h) is also an isometric embedding from (Σ, h) into the graph.

Next, we calculate each term in $P^{ijkl}f_{jl}f_kn_i$. The point in the computation is to distinguish the tangential direction and the normal direction carefully.

$$
\begin{aligned}
I &= R^{1\alpha1\beta}f_{\alpha\beta}f_1n^1 = R^{1\alpha1\beta}A_{\alpha\beta}|f_1|f_1\cdot(-\frac{f_1}{|f_1|})\\
&= -R^{1\alpha1\beta}A_{\alpha\beta}f_1^2,\\
II &= R^{j1}(f_{1j}f_1-\frac{f_{1j}f_1^3}{1+f_1^2})\cdot n^1 = (R^{1j}f_{1j})\frac{f_1}{1+f_1^2}\cdot(-\frac{f_1}{|f_1|})\\
&= -(R^{1j}f_{1j})\frac{f_1^2}{(1+f_1^2)|f_1|},\\
III &= -R^{11}\big[(H_0|f_1|+f_{11})f_1-\frac{f_{11}f_1^3}{1+f_1^2})\big]\cdot n^1\\
&= -R^{11}\big[(H_0|f_1|f_1+\frac{f_{11}f_1}{1+f_1^2})\big]\cdot(-\frac{f_1}{|f_1|})\\
&= R^{11}\big[H_0f_1^2+\frac{f_{11}f_1^2}{(1+f_1^2)|f_1|})\big],\\
IV &= R^{1l}f_{1l}\frac{f_1}{1+f_1^2}\cdot n^1 = -R^{1l}f_{1l}\frac{f_1^2}{(1+f_1^2)|f_1|},\\
V &= -(R^{jl}f_{jl})\frac{f_1}{1+f_1^2}\cdot(-\frac{f_1}{|f_1|}) = (R^{jl}f_{jl})\frac{f_1^2}{(1+f_1^2)|f_1|}\\
&= (2R^{1l}f_{1l}+R^{\alpha\beta}f_{\alpha\beta}-R^{11}f_{11})\frac{f_1^2}{(1+f_1^2)|f_1|}\\
&= (2R^{1l}f_{1l}+R^{\alpha\beta}A_{\alpha\beta}|f_1|-R^{11}f_{11})\frac{f_1^2}{(1+f_1^2)|f_1|},\\
VI &= \frac{1}{2}R\frac{(H_0|f_1|+f_{11})f_1-f_{11}f_1}{1+f_1^2}\cdot(-\frac{f_1}{|f_1|})\\
&= -\frac{1}{2}RH_0\frac{f_1^2}{1+f_1^2}.
\end{aligned}
$$

Noting that $II+IV$ cancels the first term of V and the second term in III cancels the third term in V we get

$$
\begin{aligned}
P^{ijkl}f_{jl}f_kn_i &= I+II+III+IV+V+VI\\
&= -R^{1\alpha1\beta}A_{\alpha\beta}f_1^2+R^{11}H_0f_1^2+R^{\alpha\beta}A_{\alpha\beta}\cdot\frac{f_1^2}{(1+f_1^2)}-\frac{1}{2}RH_0\frac{f_1^2}{1+f_1^2}. \qquad (5.2)
\end{aligned}
$$

Similarly, for the embedding $(\Sigma^{n-1},h)\hookrightarrow(\mathcal{M}^n,g)$ we denote the outer unit normal vector induced by g by $\tilde{n}$, and the corresponding second fundamental form by $\tilde{A}_{\alpha\beta}$. Then a direct calculation gives

$$\tilde{A}_{\alpha\beta}=\frac{1}{\sqrt{1+f_1^2}}A_{\alpha\beta}.$$

Remark 5.1. *From the above formula we have the following equivalent statements, provided $\Sigma \subset \mathbb{R}^n$ is strictly mean convex:*

- $|\nabla f| = \infty$ *on* Σ*;*
- Σ *is minimal, i.e.,* $\mathrm{tr}\tilde{A} = 0$*;*
- Σ *is totally geodesic, i.e.,* $\tilde{A} = 0$.

Therefore Σ is an area-minimizing horizon if and only if $|\nabla f| = \infty$ on Σ and if and only if Σ is totally geodesic. Hence $|\nabla f| = \infty$ is a natural assumption.

By the Gauss-Codazzi equation,

$$\hat{R}^{\alpha\beta\gamma\delta} = A^{\alpha\gamma}A^{\beta\delta} - A^{\alpha\delta}A^{\beta\gamma},$$

where $\hat{R}_m$ is the corresponding curvature tensor with respect to the induced metric h on Σ. On the other hand, we have

$$\hat{R}^{\alpha\beta\gamma\delta} = R^{\alpha\beta\gamma\delta} + \tilde{A}^{\alpha\gamma}\tilde{A}^{\beta\delta} - \tilde{A}^{\alpha\delta}\tilde{A}^{\beta\gamma} = R^{\alpha\beta\gamma\delta} + \frac{A^{\alpha\gamma}A^{\beta\delta} - A^{\alpha\delta}A^{\beta\gamma}}{1+f_1^2},$$

which yields

$$\begin{aligned} R^{\alpha\beta\gamma\delta} &= \frac{f_1^2}{1+f_1^2}(A^{\alpha\gamma}A^{\beta\delta} - A^{\alpha\delta}A^{\beta\gamma}) \\ &= \frac{f_1^2}{1+f_1^2}\hat{R}^{\alpha\beta\gamma\delta}. \end{aligned} \tag{5.3}$$

Similarly, we have

$$\begin{aligned} \tilde{F}^{\alpha\beta} &\triangleq R^{\alpha\gamma\beta\delta}g_{\gamma\delta} \\ &= \frac{|\nabla f|^2}{1+|\nabla f|^2}(H_0 A_{\alpha\beta} - A_{\alpha\gamma}A_{\gamma\beta}) = \frac{|\nabla f|^2}{1+|\nabla f|^2}\hat{R}^{\alpha\beta}, \end{aligned} \tag{5.4}$$

and

$$\tilde{F} \triangleq R^{\alpha\beta}g_{\alpha\beta} = \frac{|\nabla f|^2}{1+|\nabla f|^2}(H_0^2 - A_{\alpha\beta}A_{\alpha\beta}) = \frac{|\nabla f|^2}{1+|\nabla f|^2}\hat{R}. \tag{5.5}$$

We then go back to the equality (5.2). Noting the facts

$$R^{\alpha\beta} = \tilde{F}^{\alpha\beta} + R^{1\alpha1\beta}\cdot g_{11} = \tilde{F}^{\alpha\beta} + R^{1\alpha1\beta}(1+f_1^2),$$

and

$$R = \tilde{F} + 2R^{11}g_{11} = \tilde{F} + 2R^{11}(1+f_1^2).$$

We compute

$$
\begin{aligned}
P^{ijkl}f_{jl}f_kn_i &= -R^{1\alpha 1\beta}A_{\alpha\beta}f_1^2 + R^{11}H_0f_1^2 + R^{\alpha\beta}A_{\alpha\beta}\cdot\frac{f_1^2}{(1+f_1^2)} - \frac{1}{2}RH_0\frac{f_1^2}{1+f_1^2} \\
&= -R^{1\alpha 1\beta}A_{\alpha\beta}f_1^2 + R^{11}H_0f_1^2 + \left[\tilde{F}^{\alpha\beta} + R^{1\alpha 1\beta}(1+f_1^2)\right]A_{\alpha\beta}\cdot\frac{f_1^2}{(1+f_1^2)} \\
&\quad -\frac{1}{2}\left[\tilde{F} + 2R^{11}(1+f_1^2)\right]\cdot H_0\frac{f_1^2}{1+f_1^2} \\
&= (\tilde{F}^{\alpha\beta}A_{\alpha\beta} - \frac{1}{2}\tilde{F}H_0)\cdot\frac{f_1^2}{1+f_1^2} \\
&= (\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{R}h^{\alpha\beta})A_{\alpha\beta}\left(\frac{|\nabla f|^2}{1+|\nabla f|^2}\right)^2,
\end{aligned}
$$

where we have used (5.4) and (5.5). Therefore we conclude

$$
\int_\Sigma P^{ijkl}f_{jl}f_kn^idS_r = \int_\Sigma (\frac{|\nabla f|^2}{1+|\nabla f|^2})^2(\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{R}h^{\alpha\beta})A_{\alpha\beta}dS.
$$

One can check that

$$
-(\hat{R}^{\alpha\beta} - \frac{1}{2}\hat{R}h^{\alpha\beta})A_{\alpha\beta} = 3H_3,
$$

where H_k means the k-th mean curvature, which is defined by the k-th elementary symmetric function on the principal curvatures of the second fundamental form A.

Making use of the assumption that $|\nabla f(x)| \to \infty$ as $x \to \Sigma$, thus

$$
\begin{aligned}
m_2 &= \frac{c_2(n)}{2}\int_{\mathcal{M}^n}\frac{L_2}{\sqrt{1+|\nabla f|^2}}dV_g - c_2(n)\int_\Sigma P^{ijkl}f_{lj}f_kn^idS \\
&= \frac{c_2(n)}{2}\int_{\mathcal{M}^n}\frac{L_2}{\sqrt{1+|\nabla f|^2}}dV_g + c_2(n)\int_\Sigma(\frac{|\nabla f|^2}{1+|\nabla f|^2})^2\cdot 3H_3dS \\
&= \frac{c_2(n)}{2}\int_{\mathcal{M}^n}\frac{L_2}{\sqrt{1+|\nabla f|^2}}dV_g + c_2(n)\int_\Sigma 3H_3dS,
\end{aligned}
$$

where $c_2(n) = \frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}}$.

To summarize, we have showed that

Proposition 5.2. *Let Ω be a bounded open set in $\mathbb{R}^n$ and $\Sigma = \partial\Omega$. If $f : \mathbb{R}^n \setminus \Omega \to \mathbb{R}$ is a smooth asymptotically flat function such that each connected component of $f(\Sigma)$ is in a level set of f and $|\nabla f(x)| \to \infty$ as $x \to \Sigma$. Let H_3 denotes the 3-th mean curvature of Σ induced by Euclidean metric. Then*

$$
m_2 = \frac{c_2(n)}{2}\int_{\mathcal{M}^n}\frac{L_2}{\sqrt{1+|\nabla f|^2}}dV_g + c_2(n)\int_\Sigma 3H_3dS.
$$

Let Ω_i be connected components of $\Omega, i = 1, \cdots, k$ and let $\Sigma_i = \partial\Omega_i$ and assume that each Ω_i is convex. The rest to show the Penrose inequality in the graph case is the same as the one in [35], that to use the Aleksandrov-Fenchel inequality [46] .

Lemma 5.3. *Assume Σ is a convex hypersurface in $\mathbb{R}^n$. Let H_k denote k-th elementary symmetric function corresponding to the second fundamental form A with respect to the induced metric by δ and area $|\Sigma|$, then*

$$\begin{aligned}\frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}}\int_\Sigma 3H_3 dS &\geq \frac{1}{4}\left(\frac{\int_\Sigma R dS}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-4}{n-3}}\\ &\geq \frac{1}{4}\left(\frac{\int_\Sigma H dS}{(n-1)\omega_{n-1}}\right)^{\frac{n-4}{n-2}}\\ &\geq \frac{1}{4}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-4}{n-1}}.\end{aligned}$$

Proof. By the Aleksandrov-Fenchel inequality, we infer

$$\begin{aligned}\frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}}\int_\Sigma 3H_3 dS &\geq \frac{1}{4}\left(\frac{\int_\Sigma 2H_2 dS}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-4}{n-3}}\\ &\geq \frac{1}{4}\left(\frac{\int_\Sigma H_1 dS}{(n-1)\omega_{n-1}}\right)^{\frac{n-4}{n-2}}\\ &\geq \frac{1}{4}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-4}{n-1}}.\end{aligned}$$

On the other hand, it follows from the Gauss equation $2H_2 = R$. Hence the desired results yields. $\square$

Now we are ready to finish the proof of Penrose inequality.

Proof of Theorem 1.6. In view of the Proposition 5.2 together with the Lemma 5.3, we have showed the first part of Theorem 1.6. To check that the metric (1.3) in the example 6.1 attains the equality in the Penrose-type inequality. First, it follows from the calculation of appendix that the Gauss-Bonnet curvature L_2 with respect to the metric (1.3) equals to 0. Then the horizon is $\{S_{\rho_0} : \rho_0^{\frac{n}{2}-2} = 2m\}$ which implies the right hand side of Penrose-type inequality is

$$\begin{aligned} RHS &= \frac{1}{4}\left(\frac{\omega_{n-1}{\rho_0}^{n-1}}{\omega_{n-1}}\right)^{\frac{n-4}{n-1}}\\ &= \frac{1}{4}{\rho_0}^{n-4} = \frac{1}{4}(2m)^2\\ &= m^2 = m_2.\end{aligned}$$

$\square$

Remark 5.4. *By the work of Guan-Li* [27] *one can reduce the assumption of convexity of Σ to the assumption that Σ is of star-shape, $H > 0$, $R > 0$ and H_3 is non-negative. See also the related work of* [15].

6. Schwardschild metrics

Example 6.1. *($\mathcal{M}^n = I \times \mathbb{S}^{n-1}, g$) with coordinates (ρ, θ), a general Schwardschild metrics are given*

$$g^k_{\rm Sch} = (1 - \frac{2m}{\rho^{\frac{n}{k}-2}})^{-1} d\rho^2 + \rho^2 d\Theta^2,$$

where $d\Theta^2$ is the round metric in $\mathbb{S}^{n-1}$, $m \in \mathbb{R}$ is the 'total mass' of corresponding black hole solutions in the Lovelock gravity [37]. *When $k = 1$ we recover the Schwarzschild solutions of the Einstein gravity.*

Motivated by the Schwarzschild solutions, the above metrics also have a form of conformally flat which is more convenient for computation. One can check that the corresponding coordinate transformation is as follows:

$$(1 - \frac{2m}{\rho^{\frac{n}{k}-2}})^{-1} \rho^2 + \rho^2 d\Theta^2 = (1 + \frac{m}{2r^{\frac{n}{k}-2}})^{\frac{4k}{n-2k}} (dr^2 + r^2 d\Theta^2).$$

For our purpose, in this paper we focus on the case $k = 2$, namely,

$$g^2_{\rm Sch} = (1 - \frac{2m}{\rho^{\frac{n}{2}-2}})^{-1} d\rho^2 + r^2 d\Theta^2 = (1 + \frac{m}{2r^{\frac{n}{2}-2}})^{\frac{8}{n-4}} \delta,$$

where δ is the standard Euclidean metric, which was given in the Introduction.

Next, we will study the correspondence between m and the Gauss-Bonnet mass m_2. Recall

$$\begin{aligned} m_2 &= \lim_{r\to\infty} \frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}} \int_{S_r} P^{ijkl} \partial_l g_{jk} n^i dS_r \\ &= \lim_{r\to\infty} \frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}} \int_{S_r} P_{ijkl} \partial_l g_{jk} n^i dS_r, \end{aligned}$$

where

$$P_{ijkl} = R_{ijkl} + R_{jk} g_{il} - R_{jl} g_{ik} - R_{ik} g_{jl} + R_{il} g_{jk} + \frac{1}{2} R (g_{ik} g_{jl} - g_{il} g_{jk}).$$

For the simplicity of notation, we introduce the notation of the Kulkarmi-Nomizu product denoted by $\owedge$, then we have

$$\begin{aligned} & (A \owedge B)(X, Y, Z, W) \\ = \;& A(X, Z)B(Y, W) - A(Y, Z)B(X, W) - A(X, W)B(Y, Z) + A(Y, W)B(X, Z), \end{aligned}$$

and the compressed expression

$$P = Rm - Ric \owedge g + \frac{1}{4} R (g \owedge g).$$

Suppose $g = e^{-2u}\delta$, a direct computation gives

$$
\begin{aligned}
Rm &= e^{-2u}(\nabla_\delta^2 u + du \otimes du - \frac{1}{2}|\nabla_\delta u|^2\delta) ⃝\!\!\!\wedge\, \delta \\
Ric &= (n-2)(\nabla_\delta^2 u + \frac{1}{n-2}(\Delta_\delta u)\delta + du \otimes du - |\nabla_\delta u|^2) \\
R &= e^{2u}(2(n-1)\Delta_\delta u - (n-1)(n-2)|\nabla_\delta u|^2).
\end{aligned} \tag{6.1}
$$

Then we compute

$$
\begin{aligned}
P &= Rm - Ric ⃝\!\!\!\wedge\, g + \frac{1}{4}R(g ⃝\!\!\!\wedge\, g) \\
&= Rm - e^{-2u}Ric ⃝\!\!\!\wedge\, \delta + \frac{1}{4}e^{-4u}R(\delta ⃝\!\!\!\wedge\, \delta) \\
&= (n-3)e^{-2u}(-\nabla_\delta^2 u + \frac{1}{2}(\Delta_\delta u)\delta - du \otimes du - \frac{n-4}{4}|\nabla_\delta u|^2\delta) ⃝\!\!\!\wedge\, \delta.
\end{aligned} \tag{6.2}
$$

Namely,

$$
\begin{aligned}
P_{ijkl} = (n-3)e^{-2u}\big[&- u_{ik}\delta_{jl} - u_{jl}\delta_{ik} + u_{il}\delta_{jk} + u_{jk}\delta_{il} + (u_{ss} - \frac{n-4}{2}u_s^2)(\delta_{ik}\delta_{jl} - \delta_{il}\delta_{jk}) \\
&-u_i u_k \delta_{jl} - u_j u_l \delta_{ik} + u_i u_l \delta_{jk} + u_j u_k \delta_{il}\big].
\end{aligned}
$$

Since $g = e^{-2u}\delta$, we can get

$$
\int_{S_r} P^{ijkl}\partial_l g_{jk} n_i dS_r = \int_{S_r} P^{ijkl}(\partial_l e^{-2u})\delta_{jk} n_i dS_r = \int_{S_r} -2e^{-2u}P^{ijjl}u_l n_i dS_r.
$$

In the light of (6.3), we calculate the integral term:

$$
\begin{aligned}
& -2e^{-2u}P^{ijjl}u_l \\
= & -2(n-3)e^{4u}u_l[-u_{il} - u_{il} + nu_{il} + u_{ss}\delta_{il} + (u_{ss} - \frac{n-4}{2}u_s^2)(\delta_{il} - n\delta_{il}) \\
& \qquad -u_i u_l - u_i u_l + nu_i u_l + u_s^2\delta_{il}] \\
= & -2(n-2)(n-3)e^{4u}u_l[u_{il} - u_{ss}\delta_{il} + u_i u_l + \frac{n-3}{2}u_s^2\delta_{il}].
\end{aligned}
$$

The special case that $(\mathcal{M}^n, g)$ is asymptotically flat manifold and conformally flat with a smooth spherically symmetric function, namely, $g = e^{-2u(r)}\delta$.

Denote the radial derivative of u by $u_r \triangleq \frac{\partial u}{\partial r}$, and we have

$$
u_i = u_r \cdot \frac{x_i}{r} \tag{6.3}
$$

$$
u_{ij} = u_{rr} \cdot \frac{x_i x_j}{r^2} + u_r(\frac{\delta_{ij}}{r} - \frac{x_i x_j}{r^3}), \tag{6.4}
$$

which yields

$$
u_{ii} = u_{rr} + \frac{n-1}{r}u_r. \tag{6.5}
$$

Then it follows from (6.3),(6.4) and (6.5) that

$$\begin{aligned}
&\int_{S_r} -2e^{-2u}P^{ijjl}u_l n_i dS\\
= &\int_{S_r} 2(n-2)(n-3)e^{4u}u_l(-u_{il}+u_{ss}\delta_{il}-u_iu_l-\frac{n-3}{2}u_s^2\delta_{il})\cdot\frac{x_i}{r}dS\\
= &\int_{S_r} 2(n-2)(n-3)e^{4u}u_r\frac{x_lx_i}{r^2}\cdot\big[-u_{rr}\cdot\frac{x_ix_l}{r^2}-u_r(\frac{\delta_{il}}{r}-\frac{x_ix_l}{r^3})+(u_{rr}+\frac{n-1}{r}u_r)\delta_{il}\\
&\qquad -\frac{x_ix_l}{r^2}u_r^2-\frac{n-3}{2}u_r^2\cdot\delta_{il}\big]dS\\
= &\int_{S_r} 2(n-2)(n-3)u_re^{4u}\big[-u_{rr}-\frac{u_r}{r}+\frac{u_r}{r}+u_{rr}+\frac{n-1}{r}u_r-u_r^2-\frac{n-3}{2}u_r^2\big]dS\\
= &\int_{S_r} 2(n-1)(n-2)(n-3)e^{4u}\cdot\big[\frac{u_r^2}{r}-\frac{1}{2}u_r^3\big]dS.
\end{aligned}$$

In particular, for the metric (1.3) in Example 6.1,

$$e^{-2u}=(1+\frac{m}{2r^{\frac{n}{2}-2}})^{\frac{8}{n-4}},$$

namely,

$$u=-\frac{4}{n-4}\log(1+\frac{m}{2r^{\frac{n}{2}-2}}),$$

and then

$$\begin{aligned}
u_r &= -\frac{4}{n-4}\cdot\frac{1}{1+\frac{m}{2r^{\frac{n}{2}-2}}}\cdot\frac{m}{2}\cdot(2-\frac{n}{2})r^{1-\frac{n}{2}}\\
&= \frac{m}{1+\frac{m}{2r^{\frac{n}{2}-2}}}r^{1-\frac{n}{2}}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&\int_{S_r} 2(n-1)(n-2)(n-3)e^{4u}\cdot\big[\frac{u_r^2}{r}-\frac{1}{2}u_r^3\big]dS\\
= &\int_{S_r} 2(n-1)(n-2)(n-3)\cdot(1+\frac{m}{2r^{\frac{n}{2}-2}})^{\frac{-16}{n-4}}\cdot\left[\frac{m^2}{(1+\frac{m}{2r^{\frac{n}{2}-2}})^2}r^{1-n}-\frac{1}{2}\frac{m^3}{(1+\frac{m}{2r^{\frac{n}{2}-2}})^3}r^{3-\frac{3}{2}n}\right]dS\\
= &\int_{S_r} 2(n-1)(n-2)(n-3)(1+\frac{m}{2r^{\frac{n}{2}-2}})^{\frac{-2(n+4)}{n-4}}\cdot m^2r^{1-n}dS\\
&-\int_{S_r}(n-1)(n-2)(n-3)(1+\frac{m}{2r^{\frac{n}{2}-2}})^{\frac{-4-3n}{n-4}}\cdot m^3r^{3-\frac{3}{2}n}dS.
\end{aligned}$$

When $n \geq 5$, the last term approaches 0 as $r \to \infty$. Therefore

$$\begin{aligned} & \lim_{r\to\infty}\int_{S_r}\frac{1}{2(n-1)(n-2)(n-3)\omega_{n-1}}P^{ijkl}\partial_l g_{jk}n^i dS_r \\ = & \lim_{r\to\infty}\int_{S_r}\frac{1}{\omega_{n-1}}(1+\frac{m}{2r^{\frac{n}{2}-2}})^{\frac{-2(n+4)}{n-4}}\cdot m^2 r^{1-n}dS_r \\ = & m^2, \end{aligned}$$

where ω_{n-1} is the volume of S_1 with the induced metric of standard Euclidean metric. Therefore the GBC mass m_2 of the metric (1.3) is exactly m^2 as claimed.

One interesting byproduct of the above computation is the following

Proposition 6.2. *Suppose $(\mathcal{M}^n, g), (n \geq 5)$ is asymptotically flat with decay order $\tau > \frac{n-4}{3}$ and $(\mathcal{M}^n, g)$ is spherically symmetric i.e. $g = e^{-2u(r)}\delta$, then*

$$m_2 = \lim_{r\to\infty}\frac{1}{\omega_{n-1}}\int_{S_r}\frac{u_r^2}{r}dS_r \geq 0.$$

Proof. By the above calculation, we have

$$m_2 = \lim_{r\to\infty}\frac{1}{\omega_{n-1}}\int_{S_r}e^{4u}(\frac{u_r^2}{r}-\frac{1}{2}u_r^3)dS_r.$$

We claim that under the assumption of the decay order, the second term vanishes as $r \to \infty$. In fact, since $(\mathcal{M}^n, g = e^{-2u(r)}\delta)$ is asymptotically flat with decay order τ, we have $u_r = O(r^{-1-\tau})$ and $u = O(r^{-\tau})$. Combining with the condition of decay order $\tau > \frac{n-4}{3}$, we thus get

$$u_r = o(r^{-\frac{n-1}{3}}),$$

which yields the second integral vanishes as as $r \to \infty$. Moreover, $e^{4u} = 1 + o(1)$. Finally, the desired result yields. ☐

Remark 6.3. *By this lemma and the previous positive mass theorem for graphs, there are no spherically symmetric asymptotically flat smooth functions on $\mathbb{R}^n$ whose graphs have negative Gauss-Bonnet curvature L_2 everywhere.*

Remark 6.4. *Under the same condition of Proposition 6.2, a direct computation gives the ADM total mass*

$$\begin{aligned} m_{ADM} & \triangleq \lim_{r\to\infty}\frac{1}{2(n-1)\omega_{n-1}}\int_{S_r}(g_{ij,i}-g_{ii,j})\cdot n^j dS_r \\ & = \lim_{r\to\infty}\frac{1}{\omega_{n-1}}\int_{S_r}e^{-2u}u_r dS_r. \end{aligned}$$

Thus if the manifold is asymptotically flat with decay order $\tau > \frac{n-2}{2}$ and $(\mathcal{M}^n, g)$ is spherically symmetric that $g = e^{-2u(r)}\delta$, unlike our case the ADM mass is not always nonnegative.

Remark 6.5. *It is well-known that the Schwarzschild metric has zero scalar curvature and in the view of analogy, the Gauss-Bonnet curvature L_2 with respect to the metric (1.3) is 0. One can check from the above calculation. Moreover, the metrics in Example 6.1 can be realized as* *a graph with the induced metric from the Euclidean space $\mathbb{R}^{n+1}$. For example, when $n = 5$, the metric 1.3*

$$(\mathcal{M}^5, g) = \big(\mathbb{R}^5 \setminus \{0\}, \big(1 + \frac{m}{2r^{\frac{1}{2}}}\big)^8 g_{\mathbb{R}^5}\big)$$

can be isometrically embedded as a rotating parabola in $\{(x_1, x_2, x_3, x_4, x_5, w) \subset \mathbb{R}^6\}$. The outer end of metric (1.3) containing the infinity is the graph of the spherically symmetric function $f :\subset \mathbb{R}^5 \setminus B_{4m^2}(0) \to \mathbb{R}$ given by $f(r) = 2r^{\frac{1}{2}}\sqrt{8m(r^{\frac{1}{2}} - 2m)} - \frac{1}{6m}[8m(r^{\frac{1}{2}} - 2m)]^{\frac{3}{2}}$, where $r = |(x_1, x_2, x_3, x_4, x_5)|$.

Lemma 6.6. *Assume $(\mathcal{M}^n, g), (n \geq 5)$ a n-dimensional submanifold in $\mathbb{R}^{n+1}$. Then*

$$L_2 = 24H_4 := 24 \sum_{i<j<k<l} \lambda_i \lambda_j \lambda_k \lambda_l,$$

where $(\lambda_1, \cdots, \lambda_n)$ is the set of eigenvalues of the second fundamental form A.

Proof. We recall the Gauss equation

$$R^{ijkl} = A^{ik}A^{jl} - A^{il}A^{jk}.$$

Thus the desired result yields from direct calculations. □

Proposition 6.7. *Let $f : \mathbb{R}^n \to \mathbb{R}$ be a smooth radial function in Definition 1.3. Then the second fundamental form A has $n-1$ eigenvalues $\frac{f_r}{r\sqrt{1+f_r^2}}$ and one eigenvalue $\frac{f_{rr}}{(\sqrt{1+f_r^2})^3}$. Hence*

$$L_2 = 24\left(\frac{(n-1)!}{(n-5)!4!}\frac{f_r^4}{r^4(1+f_r^2)^2} + \frac{(n-1)!}{(n-4)!3!}\frac{f_r^3 f_{rr}}{r^3(1+f_r^2)^3}\right)$$

so that

$$m_2 = \lim_{r\to+\infty} \frac{1}{w_{n-1}} \int_{S_r} \frac{r^{n-4}f_r^4(r)}{4} \geq 0.$$

7. Generalization, Problems and conjectures

First of all, we can generalize our results to $k < n/2$. In the definition of the GBC mass, the proof of its geometric invariance and in the proof of the positive mass theorem for graphs over $\mathbb{R}^n$ one can see that the crucial things are the divergence free and the symmetry (and also anti-symmetry) of the tensor P. Hence, with a completely same argument, we can define a mass for L_k-curvature for any $k < n/2$. For general L_k-curvature the corresponding P_k curvature is

$$P_k{}^{stlm} := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-3} i_{2k-2} st}_{j_1 j_2 \cdots j_{2k-3} j_{2k-2} j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}} g^{j_{2k-1} l} g^{j_{2k} m}. \tag{7.1}$$

We can define a mass for $1 \leq k < n/2$ by

$$m_k = \frac{1}{c_k \omega_{n-1}} \lim_{r\to\infty} \int_{S_r} P_k{}^{ijml} \partial_l g_{jm} n_i dS_r, \tag{7.2}$$

with a dimensional constant $c_k(n) > 0$. This constant can be decided by computing Example 5.1 such that the mass $m_k = m^k$. Remark that for even k, the Gauss-Bonnet-Chern mass m_k of the metric g^k_{Sch} is positive even for negative m. One can check that P_k has the same property of divergence free and the same symmetry (and also anti-symmetry) as the tensor P. It is clear that $P_2 = P$ and since $R = \frac{1}{2}(g^{il}g^{jm} - g^{im}g^{jl})R_{ijlm}$

$$P_1{}^{ijlm} = \frac{1}{2}(g^{il}g^{jm} - g^{im}g^{jl}).$$

If we use this tensor P_1 to define a mass, it is just the ADM mass, with a slightly different, and certainly equivalent form

$$\frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int (g^{ik}\partial^j g_{ik} - g^{jk}\partial^i g_{jk})n_i dS_r.$$

However, it is interesting to see that with this form one can directly compute to obtain for the AMD mass m_1 that

$$m_1 = \frac{1}{2(n-1)\omega_{n-1}} \lim_{r\to\infty} \int_{S_r} \frac{1}{1+|\nabla f|^2}(f_{ii}f_i - f_{ij}f_i)\nu_j dS_r$$

without using a trick in the proof of Theorem 5 in [35] by adding a factor $1/(1+|\nabla f|^2)$. This is the reason why we need not use this trick in our proof of Theorem 1.4.

With the same crucial property of P_k, we can show the positive mass theorem and the Penrose inequality for m_k in the case of graphs, provided that the decay order satisfies

$$\tau > \frac{n-2k}{k+1}.$$

Moreover, using the Gauss-Bonnet curvature L_2 (and also L_k $(k < n/2)$) we can also introduce a GBC mass m_2^H for asymptotic hyperbolic manifolds in [28]. The study of the ADM mass for asymptotic hyperbolic manifolds was initiated by X. Wang [55] and Chruściel-Herzlich [7]. See also [59]. There are many interesting generalizations. Here we just mention the recent work of Dahl-Gicquaud-Sakovich [16] and Lima-Girão [20] for asymptotic hyperbolic graphs. In [28] we obtained a positive mass theorem for m_2^H for asymptotic hyperbolic graphs if $L_2(g) \le L_2(g_{\mathbb{H}^n})$, where $g_{\mathbb{H}^n}$ is the standard hyperbolic metric. A Penrose type inequality was also obtained.

There are many interesting problems we would like to consider for the new mass.

First all, it would be an interesting problem to consider the relationship between the Gauss-Bonnet-Chern mass and the Gauss-Bonnet-Chern theorem. The mass defined in Defintion 1.1 can be also defined for $n = 4$. In this case, the decay order (1.4) needs

$$\tau > 0.$$

However, this decay condition forces the m_2 mass vanishing. Nevertheless, it is interesting to use it to consider asymptotically cones in dimension 4. There are interesting results about the Gauss-Bonnet-Chern theorem on higher dimensional, noncompact manifolds using Q-curvature initiated by Chang-Qing-Yang [14].

It would be interesting to ask if Theorem 1.4 is true for general asymptotically flat manifolds

Problem 1. *Is the GBC mass m_2 nonnegative for an asymptotically flat manifold with $\tau > \frac{n-4}{3}$ and $L_2(g) \ge 0$.*

We conjecture that this is true, at least under an additional condition that the scalar curvature R is nonnegative. The Schwarzschild metric (1.3) has $L_2 = 0$ and $R > 0$. It would be already interesting if one can show its nonnegativity for locally conformally flat manifolds. We can generalize Theorem 1.4 to show the nonnegativity of m_2 for a class of hypersurfaces in a manifold with a certain product structure. This is related to the recent work of Lima [18] and [32] for the AMD mass. This, together with a positivity result for conformal flat metrics in $\mathbb{R}^m$, will be presented in a forthcoming paper [29].

The GBC mass m_2 is closely related to the σ_k Yamabe problem. With a suitable definition of the Green function for the σ_k Yamabe problem one would like to ask the existence of the Green function and its expansion. The leading term of the regular part in the expansion of the Green function should closely related to the mass m_2. The metric (1.3) does provide such an example. For the relationship between the ADM mass, the expansion of the ordinary Green function and the resolution of the ordinary Yamabe problem, see [47] and [41].

Problem 2. *Is there the rigidity result?*

Namely, is it true if the $m_2 = 0$, then $\mathcal{M} = \mathbb{R}^n$? Proposition 6.2 and Proposition 6.7 show that the rigidity holds for two (very) specially classes of manifolds, one is the class of spherically symmetry and conformally flat manifolds and another spherically symmetry graph. For these two classes of manifolds the mass vanishes implies that the manifold is isometric to the Euclidean space. Therefore, It is natural to conjecture that the rigidity holds, at least under additional condition that its scalar curvature is nonnegative. This is a difficult problem, even in the case of the asymptotically graphs. In this case, it is in fact a Bernstein type problem. Namely, is there a non-constant function satisfying

$$2L_2 = P^{ijkl} \cdot \big(\frac{f_{ik}f_{jl} - f_{il}f_{jk}}{1 + |\nabla f|^2}\big) = 0, \tag{7.3}$$

under the decay conditions given in Definition 1.3? For the related results on the rigidity of the AMD mass for graphs, see [32] and [19].

Problem 3. *Does Penrose inequality for the GBC mass holds on general asymptotically flat manifolds?*

Theorem 1.6 provides 3 inequalities. Two of them involve only intrinsic invariants, which we consider as the generalized forms of ordinary Penrose inequality [33] and [5]. Comparing the ordinary Penrose inequality, we conjecture

$$m_2 \geq \frac{1}{4}\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-4}{n-1}},$$

if Σ is an area outer minimizing horizon and

$$m_2 \geq \frac{1}{4}\left(\frac{\int_\Sigma R}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-4}{n-3}}$$

if Σ is an outer minimizing horizon for the functional

$$\int_\Sigma R,$$

whose Euler-Lagrange equation is

$$E^1_{ij}B_{ij} = 0.$$

Here E^1 is the ordinary Einstein tensor. For m_k ($k < n/2$), one should have k inequalities relating m_k with

$$\int_\Sigma L_j(g)dv(g), \quad j = 0, 1, 2, \cdots, k-1.$$

These functionals were considered in [38] and [34]. Note that $L_1 = R$.

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