ON A CONFORMAL QUOTIENT EQUATION. II

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ABSTRACT. In this paper we show that two conformal invariants $Y_{2,1}$ and $\tilde{Y}_{2,1}$ defined in (1) and (2) resp. coincide and are achieved by a conformal metric $g \in \Gamma_2^+$ $(n \geq 4)$, which satisfies a conformal quotient equation. The paper is a continuation of our paper [13]

1. INTRODUCTION

Let (M, g_0) be a compact Riemannian manifold with metric g_0 and $[g_0]$ the conformal class of g_0. Let S_g be the Schouten tensor of the metric g defined by

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right).$$

Here Ric_g and R_g are the Ricci tensor and scalar curvature of a metric g respectively. The importance of the Schouten tensor in conformal geometry can be viewed in the following decomposition of the Riemann curvature tensor

$$Riem_g = W_g + S_g \owedge g,$$

where $\owedge$ is the Kulkani-Nomizu product. Note that $g^{-1} \cdot W_g$ is invariant in a given conformal class.

Define $\sigma_k(g)$ be the *σ_k-scalar curvature* or *k-scalar curvature* by

$$\sigma_k(g) := \sigma_k(g^{-1} \cdot S_g),$$

where $g^{-1} \cdot S_g$ is locally defined by $(g^{-1} \cdot S_g)^i_j = \sum_k g^{ik}(S_g)_{kj}$ and σ_k is the kth elementary symmetric function. Here for an $n \times n$ symmetric matrix A we define $\sigma_k(A) = \sigma_k(\Lambda)$, where $\Lambda = (\lambda_1, \cdots, \lambda_n)$ is the set of eigenvalues of A. It is clear that $\sigma_1(g)$ is a constant multiple of the scalar curvature R_g. The k-scalar curvature $\sigma_k(g)$, which was first considered by Viaclovsky [33], is a natural generalization of the scalar curvature.

Let

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \cdots, \lambda_n) \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0, \forall j \leq k\}$$

be Garding's cone. A metric g is said to be k-positive or simply $g \in \Gamma_k^+$ if $g^{-1} \cdot S_g \in \Gamma_k^+$ for every point $x \in M$. We call u is k-admissible if $e^{-2u} g_0 \in \Gamma_k^+$. Set $\mathcal{C}_k([g_0]) = \Gamma_k^+ \cap [g_0]$.

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As in [11], we define a Yamabe type constant

$$Y_{2,1}([g_0]) := \begin{cases} \inf\limits_{g\in\mathcal{C}_1([g_0])} \dfrac{\int \sigma_2(g)dvol(g)}{(\int \sigma_1(g)dvol(g))^{\frac{n-4}{n-2}}}, & \text{if } n>4, \\ \int \sigma_2(g)dvol(g), & \text{if } n=4, \\ \sup\limits_{g\in\mathcal{C}_1([g_0])} \int \sigma_2(g)dvol(g)\times\int \sigma_1(g)dvol(g), & \text{if } n=3. \end{cases} \tag{1}$$

In [11], we prove the following proposition.

Proposition 1. *Let (M^n, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_1^+$ and $n \geq 3$. Then the conformal invariant $Y_{2,1}([g_0])$ is positive if and only if there is a conformal metric $g \in [g_0] \cap \Gamma_2^+$.*

As in [21], we also define another Yamabe invariant in the other cone $\mathcal{C}_2([g_0])$ when it is not empty, that is

$$\tilde{Y}_{2,1}([g_0]) := \begin{cases} \inf\limits_{g\in\mathcal{C}_2([g_0])} \dfrac{\int \sigma_2(g)dvol(g)}{(\int \sigma_1(g)dvol(g))^{\frac{n-4}{n-2}}}, & \text{if } n>4, \\ \int \sigma_2(g)dvol(g), & \text{if } n=4, \\ \sup\limits_{g\in\mathcal{C}_2([g_0])} \int \sigma_2(g)dvol(g)\times\int \sigma_1(g)dvol(g), & \text{if } n=3. \end{cases} \tag{2}$$

By the definition, when the dimension $n = 4$, we have $Y_{2,1}([g_0]) = \tilde{Y}_{2,1}([g_0])$. In this paper, we consider $n \neq 4$. Since $\mathcal{C}_2([g_0]) \subset \mathcal{C}_1([g_0])$, it is clear that $Y_{2,1}([g_0]) \leq \tilde{Y}_{2,1}([g_0])$ when $n > 4$ and $Y_{2,1}([g_0]) \geq \tilde{Y}_{2,1}([g_0])$ when $n = 3$. Hence, a natural question is to know if these two invariants are same. Here we will give an affirmative answer under the suitable assumptions. One of our main results in this paper is

Theorem 1. *Let (M^n, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_2^+$ and $n \geq 3$. Assume that $0 < Y_{2,1}([g_0]) < +\infty$. Then we have*

$$Y_{2,1}([g_0]) = \tilde{Y}_{2,1}([g_0]) \tag{3}$$

Moreover, if $n > 4$, then $Y_{2,1}([g_0])$ can be achieved by some conformal metric $g \in \Gamma_2^+ \cap [g_0]$.

In the case $n \geq 4$, the invariant $Y_{2,1}([g_0])$ is finite real number. Moreover, in the case $n \geq 5$, we have always $\tilde{Y}_{2,1}(M, [g_0]) \leq \tilde{Y}_{2,1}(\mathbb{S}^n)$,where $\tilde{Y}_{2,1}(\mathbb{S}^n)$ is defined for the conformal class of the standard sphere. And the equality holds if and only if $M = \mathbb{S}^n$ is the standard sphere. Hence the assumption $Y_{2,1}([g_0]) < +\infty$ is need just for the case $n = 3$.Till now we do not know if it is bounded, though we believe it is true. This is a Sobolev type inequality. Recently we obtained in [15] another (optimal) Sobolev type inequality for 3-dimensional manifolds. This is related to a geometric inequality, which was recently obtained by B Andrews [9] and De Lellis-Topping [10]. See also [14] and [16].

Following the definition of the sigma invariant of Schoen [30] (see also [26]) one can define a differential invariant by using $Y_{2,1}$ ($n > 4$)

$$\tau_2(M) = \sup_{\mathcal{C}_1([g])\neq\emptyset} Y_{2,1}([g]).$$

Previously we wanted to use $\tilde{Y}_{2,1}$ to define it. The advantage to use $Y_{2,1}$ is that it might be easier to study. With Theorem 1 we know that both are the same, provided $\tau_2(M) > 0$. One can show that

$$\tau_2(M) \leq \tau_2(\mathbb{S}^n) = \tau_2(\mathbb{S}^{n-1} \times \mathbb{S}).$$

We hope to use it to study the classification of 5-dimensional manifolds, as Bray-Neves [3] and Akutagawa-Neves [2] did for 3-dimensional manifolds by using the sigma invariant.

2. Yamabe type flows

Set

$$\mathcal{F}_2(g) = \int_M \sigma_2(g) dvol(g).$$

For any small $\varepsilon \in (0, 1)$, consider the following perturbed functional

$$\mathcal{F}_{1,\varepsilon}(g) = \int_M e^{2\varepsilon u}\sigma_1(g) dvol(g),$$

for $g = e^{-2u}g_0$. The variation of $\mathcal{F}_{1,\varepsilon}$ is given

$$\begin{aligned}\frac{d}{dt}\mathcal{F}_{1,\varepsilon}(g) &= \frac{n-2-2\varepsilon}{2}\int e^{2\varepsilon u}\sigma_1(g)g^{-1}\cdot\frac{d}{dt}g dvol(g) - \frac{1}{2}\int e^{2\varepsilon u}\Delta_g(g^{-1}\cdot\frac{d}{dt}g) dvol(g)\\ &= \frac{n-2-2\varepsilon}{2}\int e^{2\varepsilon u}\sigma_1(g)g^{-1}\cdot\frac{d}{dt}g dvol(g) - \frac{1}{2}\int \Delta_g(e^{2\varepsilon u})g^{-1}\cdot\frac{d}{dt}g dvol(g),\end{aligned}$$

where Δ_g is the Laplacian operator with respect to $g = e^{-2u}g_0$. It is easy to see that

$$\Delta_g(e^{2\varepsilon u}) = 2\varepsilon e^{2(1+\varepsilon)u}\left(\Delta u + (2\varepsilon - (n-2))|\nabla u|^2\right).$$

Set for $g = e^{-2u}g_0$

$$\begin{aligned}\sigma_{1,\varepsilon}(g) &= \frac{n-2-2\varepsilon}{n-2-4\varepsilon}\left\{\sigma_1(g) - \frac{2\varepsilon}{n-2-2\varepsilon}e^{2u}\left(\Delta u + (2\varepsilon-(n-2))|\nabla u|^2\right)\right\}\\ &= \sigma_1(g) + \varepsilon e^{2u}\left\{|\nabla u|^2 + \frac{2}{n-2-4\varepsilon}\sigma_1(g_0)\right\}.\end{aligned} \tag{4}$$

From the computation given above we have

Lemma 1. *We have*

$$\frac{d}{dt}\mathcal{F}_{1,\varepsilon}(g) = \frac{n-2-4\varepsilon}{2}\int e^{2\varepsilon u}\sigma_{1,\varepsilon}(g)g^{-1}\cdot\frac{d}{dt}(g) dvol(g).$$

Now we introduce a flow, which non-increases (resp. non-decreases) $\mathcal{F}_2$ when $n \geq 4$ (resp. $n = 3$) and preserves $\mathcal{F}_{1,\varepsilon}$.

$$\frac{du}{dt} = -\frac{1}{2}g^{-1}\frac{d}{dt}g = h_\varepsilon\left(e^{-2u}\frac{\sigma_2(g)}{\sigma_1(g)}\right) - h_\varepsilon\left(r_\varepsilon(g)\left(1+\frac{e^{2u}\nu}{\sigma_1(g)}\right)e^{(2\varepsilon-2)u}\right) + s_\varepsilon(g), \tag{5}$$

where $r_\varepsilon(g)$ and $s_\varepsilon(g)$ are space constants, given by

$$r_\varepsilon(g) = \frac{\int_M \sigma_2(g)dvol(g)}{\int_M e^{2\varepsilon u}\sigma_{1,\varepsilon}(g)dvol(g)} = \frac{n-2-4\varepsilon}{n-2-2\varepsilon}\frac{\int_M \sigma_2(g)dvol(g)}{\int_M e^{2\varepsilon u}\sigma_1(g)dvol(g)} \tag{6}$$

and

$$\int_M e^{2\varepsilon u}\sigma_{1,\varepsilon}(g)\left\{h_\varepsilon\left(e^{-2u}\frac{\sigma_2(g)}{\sigma_1(g)}\right) - h_\varepsilon\left(r_\varepsilon(g)\left(1+\frac{e^{2u}\nu}{\sigma_1(g)}\right)e^{(2\varepsilon-2)u}\right) + s_\varepsilon(g)\right\}dvol(g) = 0, \tag{7}$$

and

$$\nu = \varepsilon\left(|\nabla u|^2 + \frac{2}{n-2-4\varepsilon}\sigma_1(g_0)\right). \tag{8}$$

Here $h_\varepsilon : \mathbb{R}_+ \to \mathbb{R}$ is smooth concave function satisfying

$$h_\varepsilon(t) = \begin{cases} t & \text{if } t \leq 1 \\ \alpha_\varepsilon t^{1-\frac{\varepsilon}{2}} + \beta_\varepsilon & \text{if } t \geq 2; \end{cases} \tag{9}$$

$$h_\varepsilon'(t) + h_\varepsilon''(t)t \geq 0 \qquad \forall t \geq 0; \tag{10}$$

where the constants $\alpha_\varepsilon > 0$, β_ε are bounded as $\varepsilon \to 0$ and $\alpha_\varepsilon \to 1$ as $\varepsilon \to 0$. From the definition, we infer

$$h_\varepsilon'(t) = \alpha_\varepsilon(1-\frac{\varepsilon}{2})t^{-\frac{\varepsilon}{2}} \text{ if } t \geq 2, \tag{11}$$

Lemma 2. *Flow (5) preserves $\mathcal{F}_{1,\varepsilon}$ and non-increases (resp. non-decreases) $\mathcal{F}_2$ when $n > 4$ (resp. $n = 3$). And hence r_ε is non-increasing (resp. non-decreasing) along the flow when $n > 4$ (resp. $n = 3$).*

Proof. By the definition of $s_\varepsilon(g)$ and Lemma 1, flow (5) preserves $\mathcal{F}_{1,\varepsilon}$. By the definition of s_ε and r_ε, we can compute as follows

$$\begin{aligned}
&-\frac{2}{n-4}\frac{d}{dt}\mathcal{F}_2(g)\\
=\ &-\int_M \sigma_2(g)g^{-1}\cdot\frac{d}{dt}gdvol(g)\\
=\ &-\int(\sigma_2(g)-r_\varepsilon(g)e^{2\varepsilon u}\sigma_{1,\varepsilon}(g))g^{-1}\cdot\frac{d}{dt}gdvol(g)\\
=\ &2\int(\sigma_2(g)-r_\varepsilon(g)e^{2\varepsilon u}\sigma_{1,\varepsilon}(g))\\
&\times\left(h_\varepsilon\left(e^{-2u}\frac{\sigma_2(g)}{\sigma_1(g)}\right)-h_\varepsilon\left(r_\varepsilon(g)\left(1+\frac{e^{2u}\nu}{\sigma_1(g)}\right)e^{(2\varepsilon-2)u}\right)\right)dvol(g)\\
=\ &2\int e^{2u}\sigma_1(g)\left(e^{-2u}\frac{\sigma_2(g)}{\sigma_1(g)}-r_\varepsilon(g)\left(1+\frac{e^{2u}\nu}{\sigma_1(g)}\right)e^{(2\varepsilon-2)u}\right)\\
&\times\left(h_\varepsilon\left(e^{-2u}\frac{\sigma_2(g)}{\sigma_1(g)}\right)-h_\varepsilon\left(r_\varepsilon(g)\left(1+\frac{e^{2u}\nu}{\sigma_1(g)}\right)e^{(2\varepsilon-2)u}\right)\right)dvol(g).
\end{aligned}\tag{12}$$

■

Lemma 3. *(see [11]) For $1<k\le n$ set $F=\frac{\sigma_k}{\sigma_{k-1}}$. We have*

1) *the matrix $(F^{ij})(W)$ is semi-positive definite for $W\in\Gamma^+_{k-1}$ and is positive definite for $W\in\Gamma^+_{k-1}\backslash\mathcal{R}_1$, where $\mathcal{R}_1$ is the set of symmetric matrices of rank 1.*
2) *The function F is concave in the cone Γ^+_{k-1}. When $k=2$, for all $W\in\Gamma^+_1$ and for all $R=(r_{ij})\in\mathcal{S}_n$, we have*

$$\sum_{ijkl}\frac{\partial^2}{\partial w_{ij}\partial w_{kl}}\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)r_{ij}r_{kl}\ =\ -\frac{\sum_{ij}(\sigma_1(W)r_{ij}-\sigma_1(R)w_{ij})^2}{\sigma_1^3(W)}.\tag{13}$$

Set

$$\tilde{\mathcal{F}}_{2,\varepsilon}(g)=(\mathcal{F}_{1,\varepsilon})^{-\frac{n-4}{n-2\varepsilon-2}}\int_M\sigma_2(g)\,dvol(g)\tag{14}$$

and

$$\begin{aligned}
Y_\varepsilon(M,[g_0])&=\inf_{g\in\mathcal{C}_1([g_0])}\tilde{\mathcal{F}}_{2,\varepsilon}(g),\\
\tilde{Y}_\varepsilon(M,[g_0])&=\inf_{g\in\mathcal{C}_2([g_0])}\tilde{\mathcal{F}}_{2,\varepsilon}(g),
\end{aligned}$$

In the case $n>4$ and under the assumptions as in Theorem 1, the above discussion shows that (5) decreases the functional $\tilde{\mathcal{F}}_{2,\varepsilon}(g)$. If g is a stationary point of the flow, then the metric g satisfies the following perturbed equation

$$\frac{\sigma_2(g)}{\sigma_1(g)}-ce^{2\varepsilon u}\frac{e^{2u}\nu}{\sigma_1(g)}=ce^{2\varepsilon u},\tag{15}$$

or equivalently

$$\frac{\sigma_2(g)}{\sigma_{1,\varepsilon}(g)} = ce^{2\varepsilon u}, \tag{16}$$

where $c > 0$ is some positive constant. This is the perturbed equation that we use to approximate the following equation

$$\frac{\sigma_2(g)}{\sigma_1(g)} = 1. \tag{17}$$

We will show that Y_ε is achieved at $u_\varepsilon \in \mathcal{C}_2([g_0])$ for any small $\varepsilon > 0$, which is clearly a solution of (15). Hence, we can conclude

$$Y_{2,1}([g_0]) = \tilde{Y}_{2,1}([g_0]),$$

since $e^{-2u_\varepsilon} g_0$ converges to the extremal metric when $\varepsilon \to 0$. Similarly, we have the same result in the case $n = 3$.

3. Local estimates

In this section, we will study local estimates for flow (5) and equation (15). In this paper, C and C' denote positive constants, which in general are independent of ε. They vary from line to line. Recall $\nu = \varepsilon \left(|\nabla u|^2 + \frac{2\sigma_1(g_0)}{n-2-4\varepsilon} \right)$. Note that $\nu \geq 0$ and $\sigma_{1,\varepsilon}(g) = \sigma_1(g) + e^{2u}\nu$. By the standard implicit function theorem we have the following short-time existence result. Let $T^* \in (0, \infty]$ be the maximum of the existence of the flow.

Theorem 2. *Assume $n \geq 3$. Let u be a solution of (5) in a geodesic ball $B_r \times [0, T]$ for $T < T^*$ and $r < r_0$, the injectivity radius of M. We suppose that the function $r_\varepsilon(g(t))$ is positive and bounded on $[0, T^*)$. There are constant $\varepsilon_0 > 0$ depending only on (B_r, g_0), and constant C depending only on (B_r, g_0) (independent of ε) and the upper bound of $r_\varepsilon(g(t))$ on $[0, T]$ such that for any $\varepsilon \in (0, \varepsilon_0)$ and $(x, t) \in B_{r/2} \times [0, T]$*

$$|\nabla u|^2 + |\nabla^2 u| \leq C(1 + e^{-\frac{2-2\varepsilon}{1-\varepsilon/2} \inf_{(x,t)\in B_r \times [0,T]} u(x,t)}). \tag{18}$$

Proof. Let $W = (w_{ij})$ be an $n \times n$ matrix with $w_{ij} = \nabla^2_{ij} u + u_i u_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}$. Here u_i and u_{ij} are the first and second derivatives of u with respect to the background metric g_0. Set $K = r_\varepsilon(g(t)) e^{(2\varepsilon-2)u}$ and $K_1 = K \left(1 + \frac{\nu}{\sigma_1(W)} \right)$. Let $F_\varepsilon : \Gamma^1_+ \times \mathbb{R}^+ \times \mathbb{R} \times \mathbb{R}^+ \to \mathbb{R}$ be regular function defined by

$$\begin{aligned} F_\varepsilon(W, \nu, u, t) &:= h_\varepsilon \left(\frac{\sigma_2(W)}{\sigma_1(W)} \right) - h_\varepsilon \left(r_\varepsilon(g(t)) \left(1 + \frac{\nu}{\sigma_1(W)} \right) e^{(2\varepsilon-2)u} \right) \\ &= h_\varepsilon \left(\frac{\sigma_2(W)}{\sigma_1(W)} \right) - h_\varepsilon(K_1) \end{aligned} \tag{19}$$

Set

$$
\begin{aligned}
F_\varepsilon^{ij}(W,\nu,u,t) &:= \frac{\partial F_\varepsilon}{\partial w_{ij}}(W,\nu,u,t) \\
&= h_\varepsilon'\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)\frac{\sigma_1(W)T^{ij}-\sigma_2(W)\delta^{ij}}{\sigma_1^2(W)} + h_\varepsilon'(K_1)\frac{K\nu\delta^{ij}}{\sigma_1^2(W)}
\end{aligned} \tag{20}
$$

where $(T^{ij}) = (\sigma_1(W)\delta^{ij} - w^{ij})$ is the first Newton transformation associated with W, and δ^{ij} is the Kronecker symbol. As $W \in \Gamma_+^1$ and h_ε' is positive on $(0,+\infty)$, we see that (F_ε^{ij}) is positive definite.

Lemma 4. *F_ε is concave in Γ_1^+.*

Proof. To show this, we compute

$$
\begin{aligned}
&\frac{\partial^2 F_\varepsilon}{\partial w_{ij}w_{kl}} \\
=\ & h_\varepsilon''\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)\frac{\partial}{\partial w_{ij}}\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)\frac{\partial}{\partial w_{kl}}\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right) + h_\varepsilon'\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)\frac{\partial^2}{\partial w_{ij}\partial w_{kl}}\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right) \\
& -h_\varepsilon''(K_1)\left(\frac{K\nu}{\sigma_1^2(W)}\right)^2\delta^{ij}\delta^{kl} - 2h_\varepsilon'(K_1)\frac{K\nu}{\sigma_1^3(W)}\delta^{ij}\delta^{kl} \\
:=\ & I + II.
\end{aligned}
$$

where

$$
I := h_\varepsilon''\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)\frac{\partial}{\partial w_{ij}}\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)\frac{\partial}{\partial w_{kl}}\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right) + h_\varepsilon'\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)\frac{\partial^2}{\partial w_{ij}\partial w_{kl}}\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right)
$$

and

$$
II := -h_\varepsilon''(K_1)\left(\frac{K\nu}{\sigma_1^2(W)}\right)^2\delta^{ij}\delta^{kl} - 2h_\varepsilon'(K_1)\frac{K\nu}{\sigma_1^3(W)}\delta^{ij}\delta^{kl}.
$$

Recall that h_ε is concave and by lemma 3, $\dfrac{\sigma_2(W)}{\sigma_1(W)}$ is concave in W in the cone Γ_+^1. Thus (I), as a matrix, is non-positive definite. On the other hand, it follows from (10) we have

$$
\left(\left[-h_\varepsilon''(K_1)\frac{K\nu}{\sigma_1(W)} - h_\varepsilon'(K_1)\right]\frac{K\nu}{\sigma_1^3(W)}\delta^{ij}\delta^{kl}\right) \le 0
$$

since $K>0$, $\nu>0$, $h_\varepsilon'(t) + h_\varepsilon''(t)t \ge 0$ for all $t \ge 0$ and $\sigma_1(W) > 0$. Therefore, we prove the Lemma. ∎

From the proof of Lemma 4, we in fact have

$$
\sum_{ijkl}\frac{\partial^2 F_\varepsilon}{\partial w_{ij}\partial w_{kl}}r^{ij}r^{kl} \le -h_\varepsilon'(K_1)\frac{K\nu}{\sigma_1^3(W)}\left(\sum_i r^{ii}\right)^2. \tag{21}
$$

For the simplicity of notations, we now drop the index ε, if there is no confusion. We try to show the local estimates for first and second order derivatives together. Let $S(TM)$ denote the unit tangent bundle of M with respect to the background metric g_0. We define a function $\tilde{G}:\ S(TM)\times[0,T]\ \to\mathbb{R}$

$$\tilde{G}(e,t)=(\nabla^2u+|\nabla u|^2g_0)(e,e) \tag{22}$$

Without loss of generality, we assume $r=1$. Let $\rho\in C_0^\infty(B_1)$ be a cut-off function defined as in [19] such that

$$\begin{aligned}\rho &\geq 0, && \text{in } B_1,\\ \rho &= 1, && \text{in } B_{1/2},\\ |\nabla\rho(x)| &\leq 2b_0\rho^{1/2}(x), && \text{in } B_1,\\ |\nabla^2\rho| &\leq b_0, && \text{in } B_1.\end{aligned} \tag{23}$$

Here $b_0>1$ is a constant. Since $e^{-2u}g_0\in\Gamma_1^+$, to bound $|\nabla u|$ and $|\nabla^2u|$ we only need to bound $(\nabla^2u+|\nabla u|^2g_0)(e,e)$ from above for all $e\in S(TM)$ and for all $t\in[0,T]$. To see this, denote $G(e,t)=\rho(x)\tilde{G}(e,t)$. Assume $x_0\in M$ and $(e_1,t_0)\in S(T_{x_0}M)\times[0,T]$ such that

$$G(e_1,t_0)=\max_{S(TM)\times[0,T]}G(e,t), \tag{24}$$

$$t_0>0, \tag{25}$$

$$G(e_1,t_0)>n\max_{B_1}\sigma_1(g_0). \tag{26}$$

Let $(e_1,\cdots,e_n)$ be a orthonormal basis at point (x_0,t_0). It follows from the fact $W\in\Gamma_1^+$

$$\begin{aligned}nG(e_1,t_0) &\geq \rho(\Delta u+n|\nabla u|^2)\geq\rho(n|\nabla u|^2+\frac{n-2}{2}|\nabla u|^2-\sigma_1(g_0)),\\ &\geq \frac{3n-2}{2}\rho|\nabla u|^2-\frac{1}{n}G(e_1,t_0),\end{aligned}$$

so that

$$G(e_1,t_0)\geq\frac{\frac{3n-2}{2}}{n+\frac{1}{n}}\rho|\nabla u|^2\geq\frac{21}{20}\rho|\nabla u|^2.$$

Consequently, we obtain

$$\nabla^2_{11}u(x_0,t_0)\geq\frac{1}{20}|\nabla u|^2(x_0,t_0) \tag{27}$$

Set for any $i\neq j\in\{1,\cdots,n\}$

$$e'=\frac{1}{\sqrt{2}}(e_i\pm e_j).$$

We have

$$G(e',t_0)=\frac{1}{2}(G(e_i,t_0)+G(e_j,t_0))\pm\rho\nabla^2_{ij}u(x_0,t_0). \tag{28}$$

Thus, there holds

$$\rho|\nabla^2_{ij}u(x_0,t_0)| \leq G(e_1,t_0) - \frac{1}{2}(G(e_i,t_0)+G(e_j,t_0)). \tag{29}$$

On the other hand, we have $\forall i = 1,\cdots,n$

$$(n-1)G(e_1,t_0) + G(e_i,t_0) \geq \rho(\Delta u + n|\nabla u|^2) \geq \rho(\frac{3n-2}{2}|\nabla u|^2 - \sigma_1(g_0)), \tag{30}$$

which implies

$$G(e_i,t_0) \geq \rho(\frac{3n-2}{2}|\nabla u|^2 - \sigma_1(g_0)) - (n-1)G(e_1,t_0). \tag{31}$$

Together with (29), we deduce

$$\rho|\nabla^2_{ij}u(x_0,t_0)| \leq nG(e_1,t_0) - \frac{3n-2}{2}\rho|\nabla u|^2 + \rho\sigma_1(g_0) \leq (n+1)G(e_1,t_0). \tag{32}$$

(Indeed, at all point (x,t), the estimate $\rho|\nabla^2_{ij}u| \leq (n+1)G(e_1,t_0)$ holds). Now choose the normal coordinates around x_0 such that at point x_0

$$\frac{\partial}{\partial x_1} = e_1$$

and consider the function on $M\times[0,T]$

$$G(x,t) = \rho(x)(u_{11} + |\nabla u|^2)(x,t).$$

(without the confusion, we denotes also this function by G). Clearly, (x_0,t_0) is a maximum point of $G(x,t)$ on $M\times[0,T]$. At (x_0,t_0), we have

$$0 \leq G_t = \rho(u_{11t} + 2\sum_l u_l u_{lt}), \tag{33}$$

$$0 = G_j = \frac{\rho_j}{\rho}G + \rho(u_{11j} + 2\sum_{l\geq 1} u_l u_{lj}), \quad \text{for any } j, \tag{34}$$

$$0 \geq (G_{ij}) = \left(\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G + \rho(u_{11ij} + \sum_{l\geq 1}(2u_{li}u_{lj} + 2u_l u_{lij}))\right). \tag{35}$$

Recall that (F^{ij}) is definite positive. Hence, we have

$$\begin{aligned} 0 &\geq \sum_{i,j\geq 1} F^{ij}G_{ij} - G_t \\ &\geq \sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G + \rho\sum_{i,j\geq 1}F^{ij}(u_{11ij} + \sum_{l\geq 1}(2u_{li}u_{lj} + 2u_l u_{lij})) \\ &\quad -\rho(u_{11t} + 2\sum_{l\geq 1} u_l u_{lt}). \end{aligned} \tag{36}$$

First, from the definition of ρ, we have

$$\sum_{i,j\geq 1} F^{ij}\frac{\rho\rho_{ij} - 2\rho_i\rho_j}{\rho^2}G \geq -C\sum_{i,j\geq 1}|F^{ij}|\frac{1}{\rho}G, \tag{37}$$

and

$$\sum_{i,j\geq 1}|F^{ij}| \quad \geq \quad \sum_i F^{ii} \geq C\sum_{i,j\geq 1}|F^{ij}| \tag{38}$$

since F is positive definite . Using the facts that

$$u_{kij} = u_{ijk} + \sum_m R_{mikj}u_m, \tag{39}$$

$$u_{kkij} = u_{ijkk} + \sum_m (2R_{mikj}u_{mk} - Ric_{mj}u_{mi} - Ric_{mi}u_{mj} - Ric_{mi,j}u_m + R_{mikj,k}u_m) \tag{40}$$

and

$$(\sum_l u_l^2)_{11} = 2\sum_l (u_{11l}u_l + u_{1l}^2) + O(|\nabla u|^2), \tag{41}$$

we have

$$\begin{aligned} &\sum_{i,j\geq 1} F^{ij}u_{11ij} \\ \geq\ &\sum_{i,j\geq 1} F^{ij}\left(w_{ij11} - (u_{11})_i u_j - u_i(u_{11})_j + \sum_{l\geq 1}(u_{1l}^2 + u_{11l}u_l)(g_0)_{ij}\right) \\ &-2\sum_{i,j\geq 1} F^{ij}u_{i1}u_{j1} - C(1+|\nabla^2 u| + |\nabla u|^2)\sum_{i,j\geq 1}|F^{ij}| \end{aligned} \tag{42}$$

and

$$\begin{aligned} \sum_{i,j,l} F^{ij}u_l u_{lij} \quad \geq \quad &\sum_{i,j,l} F^{ij}u_l w_{ijl} - \sum_{i,j,l} F^{ij}(u_l u_{il}u_j + u_l u_i u_{jl}) \\ &+\frac{1}{2}\sum_{i,j} F^{ij}\langle \nabla u, \nabla(|\nabla u|^2)\rangle (g_0)_{ij} - C(1+|\nabla u|^2)\sum_{i,j\geq 1}|F^{ij}|. \end{aligned} \tag{43}$$

Combining (42) and (43), we deduce

(44)
$$\begin{aligned}
&\sum_{i,j\geq 1} F^{ij}(u_{11ij}+2\sum_{l\geq 1}(u_{li}u_{lj}+u_l u_{lij}))\\
\geq\ &\sum_{i,j\geq 1} F^{ij}(w_{ij11}+2\sum_{l\geq 1} w_{ijl}u_l)+2\sum_{i,j\geq 1}F^{ij}\sum_{l\geq 2}u_{li}u_{lj}+\sum_{i,j,l\geq 1}u_{1l}^2F^{ij}(g_0)_{ij}\\
&-\sum_{i,j}F^{ij}\left[(u_{11}+|\nabla u|^2)_i u_j+u_i(u_{11}+|\nabla u|^2)_j-\langle\nabla u,\nabla(u_{11}+|\nabla u|^2)\rangle(g_0)_{ij}\right]\\
&-C(1+|\nabla^2 u|+|\nabla u|^2)\sum_{i,j\geq 1}|F^{ij}|\\
\geq\ &\sum_{i,j}F^{ij}(w_{ij11}+2\sum_l w_{ijl}u_l)+u_{11}^2\sum_{i,j}F^{ij}(g_0)_{ij}\\
&+\sum_{i,j}F^{ij}\left(\rho_i u_j+\rho_j u_i-\langle\nabla\rho,\nabla u\rangle(g_0)_{ij}\right)\frac{G}{\rho^2}-C(1+|\nabla^2 u|+|\nabla u|^2)\sum_{i,j\geq 1}|F^{ij}|.
\end{aligned}$$

Now, we want to estimate $\sum_{i,j,l}F^{ij}w_{ijl}u_l$ and $\sum_{i,j}F^{ij}w_{ij11}$ respectively. For the first term $\sum_{i,j,l}F^{ij}w_{ijl}u_l$, we have

$$\sum_{i,j,l}F^{ij}w_{ijl}u_l = \sum_l F_l u_l+h'(K_1)\frac{K}{\sigma_1(W)}\sum_l \nu_l u_l+h'(K_1)K_1(2\varepsilon-2)\sum_l u_l^2. \tag{45}$$

For the second term $\sum_{i,j,l}F^{ij}w_{ij11}$, we have

(46)
$$\begin{aligned}
&\sum_{i,j}F^{ij}w_{ij11}\\
=\ &F_{11}-\sum_{i,j,k,m}\frac{\partial^2 F}{\partial w_{ij}\partial w_{km}}w_{ij1}w_{km1}-2\sum_{i,j}\frac{\partial^2 F}{\partial w_{ij}\partial u}w_{ij1}u_1-2\sum_{i,j}\frac{\partial^2 F}{\partial w_{ij}\partial\nu}w_{ij1}\nu_1\\
&-\frac{\partial^2 F}{\partial\nu^2}\nu_1^2-2\frac{\partial^2 F}{\partial\nu\partial u}\nu_1u_1-\frac{\partial^2 F}{\partial u^2}u_1^2-\frac{\partial F}{\partial\nu}\nu_{11}-\frac{\partial F}{\partial u}u_{11}
\end{aligned}$$

It follows from (21) that

$$-\sum_{i,j,k,m}\frac{\partial^2 F}{\partial w_{ij}\partial w_{km}}w_{ij1}w_{km1}\geq h^{'}(K_1)\frac{K\nu}{\sigma_1^3(W)}\left(\sum_i w_{ii1}\right)^2 \tag{47}$$

Using the facts (9) to (11), we can estimate successively

$$\begin{aligned} & -2\sum_{i,j}\frac{\partial^2 F}{\partial w_{ij}\partial u}w_{ij1}u_1 \\ \geq\ & -2(2-2\varepsilon)h^{'}(K_1)\frac{K\nu}{\sigma_1^2(W)}\left[2\sigma_1(W)u_1^2+\frac{1}{8\sigma_1(W)}\left(\sum_i w_{ii1}\right)^2\right], \end{aligned} \tag{48}$$

$$\begin{aligned} & -2\sum_{i,j}\frac{\partial^2 F}{\partial w_{ij}\partial \nu}w_{ij1}\nu_1 \\ \geq\ & -2h^{'}(K_1)\frac{K}{\sigma_1^2(W)}\left[\frac{\sigma_1(W)}{\nu}\nu_1^2+\frac{\nu}{4\sigma_1(W)}\left(\sum_i w_{ii1}\right)^2\right], \end{aligned} \tag{49}$$

$$\begin{aligned} & -2\frac{\partial^2 F}{\partial \nu\partial u}\nu_1 u_1 \\ \geq\ & -2h^{'}(K_1)\frac{K\nu}{\sigma_1(W)}u_1^2-2h^{'}(K_1)\frac{K}{\nu\sigma_1(W)}\nu_1^2, \end{aligned} \tag{50}$$

$$-\frac{\partial^2 F}{\partial \nu^2}\nu_1^2 = h^{''}(K_1)\frac{K^2}{\sigma_1^2(W)}\nu_1^2 \geq -h^{'}(K_1)\frac{K}{\sigma_1^2(W)}\nu_1^2, \tag{51}$$

$$-\frac{\partial^2 F}{\partial u^2}u_1^2 \geq 0 \tag{52}$$

and

$$2h'(K_1)\frac{K}{\sigma_1(W)}\sum_l \nu_l u_l \geq -h'(K_1)\frac{K}{\sigma_1(W)}\left[\frac{|\nabla\nu|^2}{\nu}+\nu|\nabla u|^2\right]. \tag{53}$$

These estimates, together with (45) and (46), imply

$$
\begin{aligned}
&\sum_{i,j} F^{ij}(w_{ij11}+2\sum_l w_{ijl}u_l)\\
\geq\ & (F_{11}+\sum_l 2F_l u_l)-11h^{'}(K_1)\frac{K\nu}{\sigma_1(W)}|\nabla u|^2\\
&-5h^{'}(K_1)\frac{K}{\sigma_1(W)\nu}|\nabla\nu|^2-h^{'}(K_1)\frac{K}{\sigma_1^2(W)}|\nabla\nu|^2\\
&-4h^{'}(K_1)K_1|\nabla u|^2+h^{'}(K_1)\frac{K}{\sigma_1(W)}\nu_{11}-(2-2\varepsilon)h^{'}(K_1)K_1u_{11}\\
\geq\ & F_{11}+\sum_l 2F_l u_l-15h^{'}(K_1)\frac{K\nu}{\sigma_1(W)}|\nabla u|^2\\
&-5h^{'}(K_1)\frac{K}{\sigma_1(W)\nu}|\nabla\nu|^2-h^{'}(K_1)\frac{K}{\sigma_1^2(W)}|\nabla\nu|^2\\
&-C(1+e^{(2\varepsilon-2)u})\frac{G}{\rho}+h^{'}(K_1)\frac{K}{\sigma_1(W)}\nu_{11}-2h^{'}(K_1)\frac{K\nu}{\sigma_1(W)}u_{11}
\end{aligned}
\tag{54}
$$

Here we the fact $u_{11}\geq 0$ at the point (x_0,t_0). Remark

$$
\begin{aligned}
\left|\sum_l u_{il}u_l\right| &= \left|\sum_l\left(w_{il}-u_iu_l+\frac{|\nabla u|^2}{2}(g_0)_{il}-(S_{g_0})_{il}\right)u_l\right|\\
&\leq \sqrt{\sigma_1^2(W)-2\sigma_2(W)}|\nabla u|+\frac{|\nabla u|^3}{2}+|S_{g_0}||\nabla u|,
\end{aligned}
\tag{55}
$$

so that together with (34) and (41), there holds

$$
\begin{aligned}
&h^{'}(K_1)\frac{K}{\sigma_1(W)}\nu_{11}\\
=\ & \varepsilon h^{'}(K_1)\frac{K}{\sigma_1(W)}(|\nabla u|^2+\frac{2}{n-2-4\varepsilon}\sigma_1(g_0))_{11}\\
=\ & \varepsilon h^{'}(K_1)\frac{K}{\sigma_1(W)}\left[2\sum_l\left(\frac{-G\rho_l u_l}{\rho^2}-2\sum_j u_ju_{jl}u_l+u_{1l}^2\right)+O(|\nabla u|^2+1)\right]\\
\geq\ & \varepsilon h^{'}(K_1)\frac{K}{\sigma_1(W)}\left[\frac{-2G\langle\nabla\rho,\nabla u\rangle}{\rho^2}-C|\nabla u|^2|\nabla^2u|-C(|\nabla u|^4+1)\right]
\end{aligned}
\tag{56}
$$

Finally, we deduce that

$$\begin{aligned}
&\sum_{i,j} F^{ij}(w_{ij11} + 2\sum_l w_{ijl}u_l) \\
\geq\ & F_{11} + 2\sum_l F_l u_l - 15h^{'}(K_1)\frac{K\nu}{\sigma_1(W)}|\nabla u|^2 - 5h^{'}(K_1)\frac{K}{\sigma_1(W)\nu}|\nabla\nu|^2 \\
& -2h^{'}(K_1)\frac{K\nu}{\sigma_1(W)}u_{11} - h^{'}(K_1)\frac{K}{\sigma_1^2(W)}|\nabla\nu|^2 \\
& -C(1+e^{(2\varepsilon-2)u})\frac{G}{\rho} - C\varepsilon h^{'}(K_1)\frac{K}{\sigma_1(W)}\left[\frac{G|\nabla u|}{\rho^{3/2}} + \frac{G}{\rho}(1+|\nabla u|^2)\right].
\end{aligned} \tag{57}$$

Now we claim that there is a constant $C > 0$ independent of ε such that

$$G \leq C\left[1 + \sqrt{\rho}\left(\sum_i F^{ii}\right)^{-1}\left(1 + e^{(2\varepsilon-2)u}\right) + \varepsilon\rho\left(1 + e^{(2\varepsilon-2)u}\right)\right]. \tag{58}$$

We divide the proof of the claim into two cases.

Case 1. $\dfrac{\nu}{\sigma_1(W)} \geq 1$.

It is clear that

$$C'\varepsilon(1+|\nabla u|^2) \leq \nu \leq C\varepsilon(1+|\nabla u|^2) \tag{59}$$

and

$$|\nabla\nu|^2 \leq C\varepsilon\nu\frac{G^2}{\rho^2}, \tag{60}$$

for some positive constants C' and C with $C' < C$. Recall

$$\sum_i F^{ii} = h'(\frac{\sigma_2(W)}{\sigma_1(W)})\left(n - 1 - \frac{n\sigma_2(W)}{\sigma_1^2(W)}\right) + h'(K_1)\frac{nK\nu}{\sigma_1^2(W)} \tag{61}$$

Thus, we can get that

$$\begin{aligned}
h'(K_1)\frac{K\nu}{\sigma_1(W)}|\nabla u|^2 &\leq \left(\sum_i F^{ii}\right)\frac{\sigma_1(W)}{n}|\nabla u|^2 \leq \left(\sum_i F^{ii}\right)\frac{\nu}{n}|\nabla u|^2 \\
&\leq C\varepsilon\left(\sum_i F^{ii}\right)\frac{G^2}{\rho^2},
\end{aligned} \tag{62}$$

$$h'(K_1)\frac{K\nu}{\sigma_1(W)}u_{11} \leq C\varepsilon\left(\sum_i F^{ii}\right)\frac{G^2}{\rho^2}, \tag{63}$$

$$h'(K_1)\frac{K}{\nu\sigma_1(W)}|\nabla\nu|^2 \le h'(K_1)\frac{K}{\sigma_1^2(W)}|\nabla\nu|^2 \le C\varepsilon\left(\sum_i F^{ii}\right)\frac{G^2}{\rho^2}, \tag{64}$$

and

$$C\varepsilon h^{'}(K_1)\frac{K}{\sigma_1(W)}\left[\frac{G|\nabla u|}{\rho^{3/2}}+\frac{G(1+|\nabla u|^2)}{\rho}\right] \le C\varepsilon\left(\sum_i F^{ii}\right)\left[\frac{G^{3/2}}{\rho^2}+\frac{G^2}{\rho^2}\right]. \tag{65}$$

Combining (57) and (62) to (65), we obtain

$$\begin{aligned}&\sum_{i,j}F^{ij}(w_{ij11}+2\sum_l w_{ijl}u_l)\\ \ge\ & F_{11}+2\sum_l F_l u_l - C\varepsilon\left(\sum_i F^{ii}\right)\left(\frac{G}{\rho}\right)^2 - C(1+e^{(2\varepsilon-2)u})\frac{G}{\rho},\end{aligned} \tag{66}$$

so that it follows from (36) to (38) and (44)

$$\begin{aligned}0\ \ge\ & -C\left(\sum_i F^{ii}\right)\frac{G}{\rho}+\rho u_{11}^2\left(\sum_i F^{ii}\right)-C\left(\sum_i F^{ii}\right)\frac{G\sqrt{G}}{\rho}\\ & -C\left(\sum_i F^{ii}\right)G - C\varepsilon\left(\sum_i F^{ii}\right)\frac{G^2}{\rho}-C(1+e^{(2\varepsilon-2)u})G.\end{aligned} \tag{67}$$

Therefore, we prove (58), provided ε is sufficiently small.

Case 2. $\dfrac{\nu}{\sigma_1(W)} < 1$.
We distingush two cases.

a) $\sigma_2(W) \ge 0$.
Then from (55), we have

$$|\nabla\nu| \le C\varepsilon(\sigma_1(W)|\nabla u| + |\nabla u|^3 + |\nabla u| + 1). \tag{68}$$

In view of (59), we have

$$\begin{aligned}&h'(K_1)\frac{K\nu}{\sigma_1(W)}|\nabla u|^2 + h'(K_1)\frac{K}{\sigma_1(W)\nu}|\nabla\nu|^2 + h'(K_1)\frac{K}{\sigma_1^2(W)}|\nabla\nu|^2 + h'(K_1)\frac{K\nu}{\sigma_1(W)}|u_{11}|\\ \le\ & C\left(1+e^{(2\varepsilon-2)u}\right)\frac{G}{\rho},\end{aligned} \tag{69}$$

and

$$C\varepsilon h^{'}(K_1)\frac{K}{\sigma_1(W)}\left[\frac{G|\nabla u|}{\rho^{3/2}}+\frac{G}{\rho}(1+|\nabla u|^2)\right] \le C\left(1+e^{(2\varepsilon-2)u}\right)\frac{G}{\rho^{3/2}}. \tag{70}$$

Hence, we infer also (58), provided ε is sufficiently small.

b) $\sigma_2(W) < 0$.
We have also (70) and

$$\begin{aligned} & h'(K_1)\frac{K\nu}{\sigma_1(W)}|\nabla u|^2 + h'(K_1)\frac{K\nu}{\sigma_1(W)}|u_{11}| \\ \leq\ & C\left(1+e^{(2\varepsilon-2)u}\right)\frac{G}{\rho}, \end{aligned} \tag{71}$$

Thanks of (55), we have

$$|\nabla\nu| \leq C\varepsilon(\sqrt{\sigma_1^2(W)-2\sigma_2(W)}|\nabla u| + |\nabla u|^3 + |\nabla u| + 1) \tag{72}$$

so that together with (59) and (61) we obtain

$$\begin{aligned} & h'(K_1)\frac{K}{\sigma_1^2(W)}|\nabla\nu|^2 \\ \leq\ & h'(K_1)\frac{K}{\sigma_1(W)\nu}|\nabla\nu|^2 \\ \leq\ & Ch'(K_1)K\varepsilon^2\left(\frac{\sigma_1^2(W)-2\sigma_2(W)}{\sigma_1(W)\nu}|\nabla u|^2 + \frac{|\nabla u|^6+1}{\nu^2}\right) \\ \leq\ & C(1+e^{(2\varepsilon-2)u})\frac{G}{\rho} + C\varepsilon K\sigma_1(W)\left(\sum_i F^{ii}\right) \\ \leq\ & C(1+e^{(2\varepsilon-2)u})\frac{G}{\rho}\left(1+\varepsilon\sum_i F^{ii}\right), \end{aligned} \tag{73}$$

Finally we imply that the claim (58) holds in this case, provided ε is sufficiently small. It is easy to see from (9) that $h_\varepsilon^{'}(2)$ is uniformly bounded from below by a positive constant for all $\varepsilon \in [0, 1/2]$. Hence, we have

$$\left(\sum_i F^{ii}\right)^{-1} \leq \begin{cases} C, & \text{if } \dfrac{\sigma_2(W)}{\sigma_1(W)} \leq 2, \\ C\left(\dfrac{G}{\rho}\right)^{\varepsilon/2}, & \text{if } \dfrac{\sigma_2(W)}{\sigma_1(W)} \geq 2, \end{cases} \tag{74}$$

which, together with (58), implies that

$$G \leq \begin{cases} C\left(1+e^{(2\varepsilon-2)u}\right), & \text{if } \dfrac{\sigma_2(W)}{\sigma_1(W)} \leq 2; \\ C\left(1+e^{\frac{(4\varepsilon-4)u}{2-\varepsilon}}\right), & \text{if } \dfrac{\sigma_2(W)}{\sigma_1(W)} \geq 2. \end{cases} \tag{75}$$

Therefore, we have finished the proof of the Theorem. ∎

The same proof gives the local estimates for the elliptic equation (16).

Theorem 3. *Assume $n \geq 3$ and $\varepsilon \in [0, \varepsilon_0)$. Let u be a solution of (16) in a geodesic ball B_r for $r < r_0$, the injectivity radius of M. There is a constant C depending only on (B_r, g_0) (independent of ε) such that for any $\varepsilon \in [0, \varepsilon_0)$ and $x \in B_{r/2}$*

$$|\nabla u|^2 + |\nabla^2 u| \leq C(1 + e^{-(2-2\varepsilon) inf_{x \in B_r} u(x)}). \tag{76}$$

4. A Sobolev inequality

The Sobolev inequality is a very important analytic tool in many problems arising from analysis and geometry. It plays a crucial role in the resolution of the Yamabe problem, which was solved completely by Yamabe [36], Trudinger [32], Aubin [1] and Schoen [29]. See various optimal Sobolev inequalities in [25]. In this section we are interested in a similar type inequality for the class of a fully nonlinear conformal quotient operators. In [20], [21], [12] and [17], the Sobolev inequality was generalized to the various fully nonlinear operators.

In this section, we establish the Sobolev inequality relating $\int_M \sigma_2(g) dvol(g)$ and $\int_M \sigma_{1,\varepsilon}(g) dvol(g)$ for a general manifold, which will be used in the next section.

Theorem 4. *Let (M, g_0) be a compact Riemannian manifold with $g_0 \in \Gamma_2^+$ and the dimension $n > 4$. Assume $\varepsilon \in [0, 1/2]$. Then there exists a positive constant $C > 0$ depending only on (M, g_0) (and independent of ε) such that for any C^2 function u with $e^{-2u} g_0 \in \mathcal{C}_1([g_0])$ we have*

$$\int_M \sigma_2(e^{-2u} g_0) dvol(e^{-2u} g_0) \geq C \left(\int_M e^{2\varepsilon u} \sigma_1(e^{-2u} g_0) dvol(e^{-2u} g_0) \right)^{\frac{n-4}{n-2-2\varepsilon}}. \tag{77}$$

Equivalently, for such a function u we have

$$\begin{aligned} &\int_M e^{(4-n)u} \sigma_2(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) dvol(g_0) \\ \geq\ & C \left(\int_M e^{(2+2\varepsilon-n)u} \sigma_1(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}) dvol(g_0) \right)^{\frac{n-4}{n-2-2\varepsilon}}. \end{aligned} \tag{78}$$

Proof. Let $g = e^{-2u} g_0$. We have shown in [11] the invariant $Y_{2,1}([g_0]) > 0$ and for any $e^{-2u} g_0 \in \mathcal{C}_1([g_0])$

$$\int_M \sigma_2(e^{-2u} g_0) dvol(e^{-2u} g_0) \geq C_1 \int_M |\nabla u|^4 e^{(4-n)u} dvol(g_0) - C \int_M e^{(4-n)u} dvol(g_0), \tag{79}$$

$$\int_M \sigma_2(e^{-2u} g_0) dvol(e^{-2u} g_0) \geq C_1 \int_M e^{(4-n)u} dvol(g_0), \tag{80}$$

for some positive constants $C_1 > 0$ and $C > 0$. Hence, we deduce

$$\int_M \sigma_2(e^{-2u} g_0) dvol(e^{-2u} g_0) \geq C \int_M |\nabla u|^4 e^{(4-n)u} dvol(g_0). \tag{81}$$

It is easy to say that

$$\begin{aligned}
&\int_M e^{2\varepsilon u}\sigma_1(e^{-2u}g_0)dvol(e^{-2u}g_0)\\
=\ &\int\left(\Delta u-\frac{n-2}{2}|\nabla u|^2+\sigma_1(g_0)\right)e^{(2+2\varepsilon-n)u}dvol(g_0)\\
=\ &\int\left(\frac{n-2-4\varepsilon}{2}|\nabla u|^2+\sigma_1(g_0)\right)e^{(2+2\varepsilon-n)u}dvol(g_0) \qquad (82)\\
\le\ &\frac{n-2-4\varepsilon}{2}\left(\int|\nabla u|^4e^{(4-n)u}dvol(g_0)\int e^{(4\varepsilon-n)u}dvol(g_0)\right)^{1/2}\\
&+(\sup\sigma_1(g_0))\int e^{(2+2\varepsilon-n)u}dvol(g_0).
\end{aligned}$$

Recall the definition of the conformal invariants [11]

$$Y_1([g_0])=\inf_{g\in\mathcal{C}_1([g_0])}\frac{\int\sigma_1(g)dvol(g)}{(vol(g))^{\frac{n-2}{n}}}. \qquad (83)$$

and

$$\left(\int_M dvol(e^{-2u}g_0)\right)^{\frac{n-4}{n}}\le(Y_{2,1}([g_0])Y_1([g_0])^{\frac{n-4}{n-2}})^{-1}\int_M\sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0). \qquad (84)$$

By the Hölder inequality, we get for any $\alpha\in[0,n/2]$

$$\left(\int_M e^{\alpha u}dvol(e^{-2u}g_0)\right)^{\frac{n-4}{n-\alpha}}\le C\int_M\sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0), \qquad (85)$$

where C is a positive constant independent of α. (81), (82) and (85) imply

$$\begin{aligned}
&\int_M e^{2\varepsilon u}\sigma_1(e^{-2u}g_0)dvol(e^{-2u}g_0)\\
\le\ &C\left[\left(\int_M\sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0))\right)^{1/2}\left(\int_M\sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0))\right)^{\frac{n-4\varepsilon}{2(n-4)}}\right.\\
&\left.+\left(\int_M\sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0))\right)^{\frac{n-2-2\varepsilon}{n-4}}\right] \qquad (86)\\
\le\ &C\left(\int_M\sigma_2(e^{-2u}g_0)dvol(e^{-2u}g_0))\right)^{\frac{n-2-2\varepsilon}{n-4}}.
\end{aligned}$$

We finish the proof of Theorem. ∎

Remark 1. *In* [13]*, we proved the Sobolev inequality (77) in the cone* $\mathcal{C}_2([g_0])$ *when the conformal invariant* $Y_{2,1}([g_0])>0$*.*

5. Proof of Theorem 1 in the case $n \geq 5$

Now we can prove that Y_ε is achived for any small $\varepsilon > 0$.

Proposition 2. *For $\varepsilon_0 > \varepsilon > 0$, flow (5) globally converges to a solution of (15). As a direct appliaction, Y_ε is achieved by a function u_ε satisfying (15).*

Proof. We divide the proof into 3 steps.

Step 1. For a fixed small number $\varepsilon > 0$, the solution u of flow (5) has a uniform C^0 bound, which is independent of t.

The proof use the optimality of the local estimate (18). First all, since flow (5) does not increase $\mathcal{F}_2$, $\mathcal{F}_2(g)$ is bounded from above along the flow. By (84), we konw that $\int_M dvol(g)$ is bounded from above. From the Hölder inequality, $V_\varepsilon(g) = \int_M e^{2\varepsilon u} dvol(g)$ is also bounded.

Let $T^* \in (0, \infty]$ be the maximum of the existence of the flow. For any $T \in [0, T^*)$, set

$$m(T) = \min_{(x,s)\in M\times[0,T]} u(x, s).$$

We show that there is a constant $C_0 > 0$ independent of T (depending on ε) such that

$$\inf_{T\in[0,T^*)} m(T) > -C_0. \tag{87}$$

We assume by contradiction that $\inf_{t\in[0,T^*)} m(t) = -\infty$. Let T_i be a sequence tending to T^* with $m(T_i) \to -\infty$ as $i \to \infty$. Let $(x_i, t_i) \in M \times [0, T_i]$ with $u(x_i, t_i) = m(T_i)$. Fix $\delta \in (\frac{2}{5}, \frac{1}{2})$, we consider $r_i = \dfrac{\varepsilon}{2}|m(T_i)|e^{(1-\delta\varepsilon)m(T_i)}$. Clearly, we have $r_i \to 0$. It follows from Theorem 2 that for sufficiently large i and for any $x \in B_{r_i}(x_i)$

$$\begin{aligned} u(x, t_i) &\leq m(T_i) + (\sup_{B_{r_i}(x_i)} |\nabla u|) r_i \\ &\leq m(T_i) + Ce^{(\frac{\varepsilon}{2-\varepsilon}-1)m(T_i)} \frac{\varepsilon}{2}|m(T_i)|e^{(1-\delta\varepsilon)m(T_i)} \\ &= m(T_i) + C\frac{\varepsilon}{2}|m(T_i)|e^{\varepsilon(\frac{1}{2-\varepsilon}-\delta)m(T_i)} \\ &\leq (1-\kappa)m(T_i), \end{aligned}$$

for some $\kappa \in (0, (\delta - \frac{2}{n})\varepsilon)$. Note that $\delta - \frac{2}{n} > 0$, for $n \geq 5$. Therefore, we obtain

$$\begin{aligned} \int_{B(x_i,r_i)} e^{2\varepsilon u} dvol(g) &\geq \int_{B(x_i,r_i)} e^{(2\varepsilon-n)m(T_i)(1-\kappa)} dvol(g_0) \geq Ce^{(2\varepsilon-n)m(T_i)(1-\kappa)} r_i^n \\ &\geq C\left(\frac{|m(T_i)|\varepsilon}{2}\right)^n \to \infty. \end{aligned}$$

where we have used $n \geq 5$. Hence, this fact contradicts the boundedness of V_ε. This proves the claim. This claim, together with the local estimates and the fact $\mathcal{F}_{1,\varepsilon}$ is preserved along the flow, implies that $\|u(t)\|_{C^2}$ has a unform bound.

Step 2 There is a constant $C_0 > 0$, independent of $T \in [0, T^*)$ such that $\sigma_1(g(x,t)) > C_0$ for any $t \in [0,T]$ and $x \in M$.

¿From the Sobolev inequality and Lemma 2, the function $r_\varepsilon(g(t))$ is bounded from below and from above by the positive constants. Recall

$$\nu = \varepsilon(|\nabla u|^2 + 2\sigma_1(g_0)/(n-2-4\varepsilon)),$$

$$W = (w_{ij}) = (\nabla^2_{ij} u + u_i u_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}),$$

$$K = r_\varepsilon(g(t)) e^{(2\varepsilon-2)u},\ K_1 = K\left(1 + \frac{\nu}{\sigma_1(W)}\right)$$

and

$$F_\varepsilon(W, \nu, u, t) = h_\varepsilon\left(\frac{\sigma_2(W)}{\sigma_1(W)}\right) - h_\varepsilon\left(K_1\right).$$

Let us consider a function $H_\varepsilon : M \times [0,T]$ defined by

$$\begin{aligned} H_\varepsilon &:= F_\varepsilon - e^{-u} \\ &= u_t - s_\varepsilon(g) - e^{-u}, \end{aligned}$$

where u_t denotes the derivative of u with respect to t. Without loss of generality, we assume that the minimum of H_ε is achieved at $(x_0, t_0) \in M \times (0,T]$ and at (x_0, t_0)

$$\frac{\sigma_2(W)}{\sigma_1(W)} < 1.$$

Recall that $h_\varepsilon(t) = t$ for $t < 1$. Hence, in a small neighborhood of (x_0, t_0)

$$H_\varepsilon = \frac{\sigma_2(W)}{\sigma_1(W)} - h_\varepsilon(K_1) - e^{-u}.$$

Let us use $O(1)$ denote terms with a uniform bound (perhaps depending on ε). Using $\frac{dr_\varepsilon(g)}{dt} \le 0$, we have near (x_0, t_0)

$$\begin{aligned} \frac{d}{dt} H_\varepsilon &\geq \operatorname{tr}\left(A\nabla^2_g(H_\varepsilon + e^{-u})\right) - h'_\varepsilon(K_1) K \frac{2\varepsilon\langle \nabla_{g_0} u, \nabla_{g_0}(H_\varepsilon + e^{-u})\rangle_{g_0}}{\sigma_1(W)} \\ &\quad + k_\varepsilon(x,t) u_t, \end{aligned} \tag{88}$$

where

$$A := \frac{(\sigma_1^2(W) - \sigma_2(W))I - \sigma_1(W)W}{\sigma_1^2(W)} + h'_\varepsilon(K_1)\frac{K\nu I}{\sigma_1^2(W)}$$

is positive definite and

$$k_\varepsilon(x,t) := e^{-u} + (2-2\varepsilon)h'_\varepsilon(K_1)K_1$$

is a positive function and I denotes the identity matrix. To simplify the notations, we drop the index ε as before. We prove first There is a constant $C_2 > 0$, independent of

$T \in [0, T^*)$ such that $\sigma_1(g(x_0, t_0)) > C_2$. Since (x_0, t_0) is the minimum of H in $M \times [0, T]$, at this point, we have $\frac{dH}{dt} \leq 0$, $H_l = 0 \quad \forall l$ and (H_{ij}) is non-negative definite. Note that

$$(\nabla_g^2)_{ij} H = H_{ij} + u_i H_j + u_j H_i - \sum_l u_l H_l \delta_{ij} = H_{ij},$$

at (x_0, t_0), where H_j and H_{ij} are the first and second derivatives with respect to the back-ground metric g_0. From the positivity of A and (88), we have

$$\begin{aligned} 0 &\geq H_t - \sum_{i,j} A^{ij} H_{ij} \\ &\geq \sum_{i,j} A^{ij} \{(e^{-u})_{ij} + u_i (e^{-u})_j + u_j (e^{-u})_i - \sum_l u_l (e^{-u})_l \delta_{ij}\} + k(x,t) u_t \\ &= e^{-u} \sum_{i,j} A^{ij} \{-w_{ij} + S(g_0)_{ij} + \frac{1}{2} |\nabla u|^2 \delta_{ij}\} + k(x,t) u_t \\ &\geq e^{-u} \sum_{i,j} A^{ij} (-w_{ij} + S(g_0)_{ij}) + k(x,t) u_t. \end{aligned} \tag{89}$$

Here we have

$$\sum_{i,j} A^{ij} w_{ij} = \frac{\sigma_2(W)}{\sigma_1(W)} + h'(K_1) \frac{K\nu}{\sigma_1(W)}.$$

On the other hand, we have

$$\begin{aligned} \sum_{i,j} A^{ij} S(g_0)_{ij} &= \frac{\sigma_1(g_0)(\sigma_1^2(W) - \sigma_2(W))}{\sigma_1^2(W)} - \frac{1}{\sigma_1(W)} \sum_{i,j} w^{ij} S(g_0)_{ij} \\ &\quad + h'(K_1) \frac{K\nu \sigma_1(g_0)}{\sigma_1^2(W)} \\ &= -\frac{\sigma_2(W)\sigma_1(g_0)}{\sigma_1^2(W)} + h'(K_1) \frac{K\nu\sigma_1(g_0)}{\sigma_1^2(W)} \\ &\quad - \frac{1}{\sigma_1(W)} \sum_{i,j} w^{ij} S(g_0)_{ij} + \sigma_1(g_0). \end{aligned} \tag{90}$$

As W is bounded, we deduce that

$$\begin{aligned} \sum_{i,j} A^{ij} (-w_{ij} + S(g_0)_{ij}) &= \Big(h'(K_1) \frac{K\nu\sigma_1(g_0)}{\sigma_1^2(W)} \\ &\quad - \frac{\sigma_2(W)\sigma_1(g_0)}{\sigma_1^2(W)} + O(\frac{1}{\sigma_1(W)}) + O(1) \Big). \end{aligned} \tag{91}$$

$$k(x_0, t_0) = (2 - 2\varepsilon) h'(K_1) \frac{K\nu}{\sigma_1(W)} + O(1). \tag{92}$$

$$\begin{aligned} u_t(x_0,t_0) &= H(x_0,t_0)+s_\varepsilon(g(t_0))+e^{-u(x_0,t_0)} \geq H(x_0,t_0)+O(1) \\ &= \frac{\sigma_2(W)}{\sigma_1(W)}-h(K_1)+O(1) \end{aligned} \tag{93}$$

We divide the proof into two cases.

Case 1. $\sigma_2(W)\geq 0$.

It is clear

$$0\leq \sigma_2(W)\leq \frac{1}{2}\sigma_1^2(W)$$

Thus

$$0 \geq e^{-u}h'(K_1)\frac{K\nu\sigma_1(g_0)}{\sigma_1^2(W)}-(2-2\varepsilon)h(K_1)h'(K_1)\frac{K\nu}{\sigma_1(W)}+O(1)+O(\frac{1}{\sigma_1(W)}) \tag{94}$$

Assume that $\frac{1}{\sigma_1(W)}$ is sufficiently large. Then

$$h'(K_1)\frac{K\nu\sigma_1(g_0)}{\sigma_1^2(W)}\geq \frac{C_3}{\sigma_1^{2-\varepsilon/2}(W)} \tag{95}$$

and

$$(2-2\varepsilon)h(K_1)h'(K_1)\frac{K\nu}{\sigma_1(W)}\leq \frac{C_4}{\sigma_1^{2-\varepsilon}(W)} \tag{96}$$

for some positive constants C_3 and C_4 independent of T. This implies boundness of $\sigma_1(W)$ at the point (x_0,t_0) from below by some positive constant independent of T.

Case 2. $\sigma_2(W)<0$.

In this case, (94) holds also since $-\frac{\sigma_1(g_0)\sigma_2(W)}{\sigma_1^2(W)}+k(x_0,t_0)\frac{\sigma_2(W)}{\sigma_1(W)}>0$, provided $\frac{1}{\sigma_1(W)}$ is sufficiently large. Hence, the desired result yields.

Now for any $(x,t)\in M\times[0,T]$ we have

$$H(x,t)\geq H(x_0,t_0)=O(1) \tag{97}$$

so that

$$h(\frac{\sigma_2(W)}{\sigma_1(W)})(x,t)-h(K_1)(x,t)\geq O(1) \tag{98}$$

Therefore, we infer

$$-h\left(r_\varepsilon(g(t))e^{(2\varepsilon-2)u}\left(1+\frac{\nu}{\sigma_1(W)}\right)\right)(x,t)\geq O(1) \tag{99}$$

since we have always

$$\begin{cases} h(\frac{\sigma_2(W)}{\sigma_1(W)})(x,t) = \frac{\sigma_2(W)}{\sigma_1(W)}(x,t) < 0 & \text{if } \sigma_2(W)(x,t) < 0 \\ \frac{\sigma_2(W)}{\sigma_1(W)}(x,t) \le \frac{1}{2}\sigma_1(W)(x,t) \le O(1) & \text{if } \sigma_2(W)(x,t) \ge 0 \end{cases} \tag{100}$$

Finally, $K_1(x,t)$ is bounded from above and yields that there exists $C_0 > 0$ independent of T such that $\sigma_1(W)(x,t) > C_0$.

Step 3. Now we can finish the proof of Proposition 2. From Step 2, we know that the flow is uniformly parabolic. In view of Step 1, Krylov's theory implies $u(t)$ has a uniform $C^{2,\alpha}$ bound. Hence, $T^* = \infty$. One can also show that $u(t)$ globally converges to $u(\infty)$, which clearly is a solution of (15) for $\varepsilon_0 > \varepsilon > 0$. From the local estimates, the set of solutions of (15) for $c = 1$ with the uniform bounded energy functional $\mathcal{F}_2$ is bounded in C^2 norm. Since (16) is concave in W, from the Evans-Krylov theory, this set is compact in $C^{2,\alpha}$ norm. Now it is easy to show that Y_ε is achieved. ∎

Proof of Theorem 1 in the case $n \ge 5$. By Proposition 2, for small $\varepsilon > 0$ we have a solution u_ε of (15) which has $\tilde{\mathcal{F}}_{2,\varepsilon}(u_\varepsilon) = Y_\varepsilon$. It is easy to show that

$$\lim_{\varepsilon\to 0} Y_\varepsilon = \lim_{\varepsilon\to 0} \tilde{Y}_\varepsilon = Y_{2,1}.$$

If $\lim_{\varepsilon\to 0} \min_{x\in M} u_\varepsilon(x) > -\infty$, then local estimates imply that u_ε (taking a subsequence) converges in $C^{2,\alpha}$ to u, which is a solution of (3). We are done.

If $\lim_{\varepsilon\to 0} \min_{x\in M} u_\varepsilon(x) = -\infty$, we can use the local estimates and the classification of solutions of (17) in the standard sphere to get a contradiction to the facts $\tilde{Y}_{2,1}([g_0]) \le \tilde{Y}_{2,1}(S^n)$ and equality holds if only if $(M,[g_0])$ is the standard sphere. This so-called the blow-up analysis for this class of fully nonlinear conformal equations becomes more or less standard. Here we leave the proof to the interested reader.

Now $Y_{2,1}([g_0])$ is achieved by some $g \in \mathcal{C}_1([g_0])$ which solves (17). Hence $g \in \mathcal{C}_2([g_0])$ and yields that $Y_{2,1}([g_0]) = \tilde{Y}_{2,1}([g_0])$. ∎

6. Proof of Theorem 1 in the case $n = 3$

Now we want to consider the existence of the following equation

$$F_\varepsilon(g) = \frac{\sigma_2(g) - \varepsilon e^{4u}}{\sigma_1(g)} = constant, \tag{101}$$

with $g = e^{-2u} g_0$ for $\varepsilon > 0$ a positive number. In this paper we will choose ν as a small positive constant. Following [20], [12] and [13] we will introduce a suitable Yamabe type flow to study equation (101).

For any $\varepsilon \in (0, +\infty)$ and for $g = e^{-2u} g_0$, consider the following perturbed functional

$$\mathcal{E}_\varepsilon(g) := \begin{cases} \dfrac{2}{n-4} \displaystyle\int_M (\sigma_2(g) - \varepsilon e^{4u}) dvol(g), & \text{if } n \neq 4, \\ -\displaystyle\int_0^1 \int_M (\sigma_2(g_t) - 2\varepsilon e^{4tu}) u dvol(g_t) dt, & \text{if } n = 4, \end{cases}$$

where $g_t = e^{-2tu} g_0$. When $\varepsilon = 0$, the functional was considered in [33], [5] and [4]. Recall

$$\mathcal{F}_1(g) = \int_M \sigma_1(g) dvol(g) \quad \text{and} \quad \mathcal{F}_2(g) = \int_M \sigma_2(g) dvol(g)$$

From the variational formula given in [33], [5] and [4], we have

$$\frac{d}{dt}\mathcal{E}_\varepsilon(g) = \int (\sigma_2(g) - \varepsilon e^{4u}) g^{-1} \cdot \frac{d}{dt} g dvol(g) \tag{102}$$

and

$$\frac{d}{dt}\mathcal{F}_1(g) = \frac{n-2}{2} \int \sigma_1(g) g^{-1} \cdot \frac{d}{dt} g dvol(g). \tag{103}$$

Now we introduce a Yamabe type flow, which non-increases $\mathcal{E}_\varepsilon$ and preserves $\mathcal{F}_1$.

$$\frac{du}{dt} = -\frac{1}{2} g^{-1} \frac{d}{dt} g = e^{-2u} \frac{\sigma_2(g) - \varepsilon e^{4u}}{\sigma_1(g)} - r_\varepsilon(g) e^{-2u} + s_\varepsilon(g), \tag{104}$$

where $r_\varepsilon(g)$ and $s_\varepsilon(g)$ are space constants, given by

$$r_\varepsilon(g) := \frac{\mathcal{F}_2(g) - \int_M \varepsilon e^{4u} dvol_g}{\mathcal{F}_1(g)} \tag{105}$$

and

$$\int_M \sigma_1(g) \left\{ e^{-2u} \frac{\sigma_2(g) - \varepsilon e^{4u}}{\sigma_1(g)} - r_\varepsilon(g) e^{-2u} + s_\varepsilon(g) \right\} dvol(g) = 0. \tag{106}$$

We collect some basic facts proved in [11].

Lemma 5. *Flow (104) preserves $\mathcal{F}_1$ and non-increases $\mathcal{E}_\varepsilon$. Hence when $n \geq 4$, then r_ε is non-increasing along the flow, and when $n = 3$, then r_ε is non-decreasing along the flow.*

Given $\varepsilon > 0$, assume $g_0 \in \mathcal{C}_1([g_0])$. By Lemma 3, (104) is parabolic. By the standard implicit function theorem we have the short-time existence result. Let $T^* \in (0, \infty]$ so that $[0, T^*)$ is the maximum interval for the existence of the flow $g(t) \in \Gamma_1^+$.

Proposition 3. *Assume that $n \geq 3$, $\varepsilon > 0$ and $g_0 \in \Gamma_1^+$. Let u be a solution of (104) in a geodesic ball $B_R \times [0, T]$ for $T < T^*$ and $R < \tau_0$, the injectivity radius of M.*

(1) *Assume that* $\forall t \in [0,T]$, *there holds*

$$r_\varepsilon(t) \le 0.$$

Then there is a constant C *depending only on* (B_R, g_0) *(independent of* ε *and* T*) such that for any* $(x,t) \in B_{R/2} \times [0,T]$

$$|\nabla u|^2 + |\nabla^2 u| \le C. \tag{107}$$

(2) *Assume that* $\forall t \in [0,T]$, *there holds*

$$r_\varepsilon(t) > 0.$$

Then there is a constant C *depending only on* (B_R, g_0) *(independent of* ε *and* T*) such that for any* $(x,t) \in B_{R/2} \times [0,T]$

$$|\nabla u|^2 + |\nabla^2 u| \le C(1 + \sup_{t\in[0,T]} r_\varepsilon(t) \times e^{-2\inf_{(x,t)\in B_R\times[0,T]} u(x,t)}). \tag{108}$$

Now we define

$$a_\varepsilon := \inf_{g\in\mathcal{C}_1([g_0])} \frac{\mathcal{E}_\varepsilon(g)}{(\int_M \sigma_1(g) dvol(g))^{\frac{n-4}{n-2}}} \quad \text{if } n \neq 4; \tag{109}$$

If a_ε is achieved by a metric $g = e^{-2u} g_0$, the g satisfies

$$\frac{\sigma_2(g) - \varepsilon e^{4u}}{\sigma_1(g)} = \kappa, \tag{110}$$

for some constant κ. Equivalently, we will consider the energy functional $\mathcal{E}_\varepsilon$ on the normalized cone $\tilde{\mathcal{C}}_1([g_0])$

$$\tilde{\mathcal{C}}_1([g_0]) := \left\{ g \in \mathcal{C}_1([g_0]) \mid \int_M \sigma_1(g) dvol(g) = 1 \right\}. \tag{111}$$

Using the same arguments as in Proposition 3, we have the following local estimate.

Proposition 4. *Assume that* $n \ge 3$, $\varepsilon > 0$ *and* $g_0 \in \Gamma_1^+$. *Let* u *be a solution of (110) in a geodesic ball* $B_R \times [0,T]$ *for* $T < T^*$ *and* $R < \tau_0$, *the injectivity radius of* M.

(1) *Assume*

$$\kappa \le 0.$$

Then there is a constant C *depending only on* (B_R, g_0) *(independent of* ν *and* T*) such that for any* $(x,t) \in B_{R/2} \times [0,T]$

$$|\nabla u|^2 + |\nabla^2 u| \le C. \tag{112}$$

(2) *Assume*

$$\kappa > 0.$$

Then there is a constant C depending only on (B_R, g_0) (independent of ε and T) such that for any $(x,t) \in B_{R/2} \times [0,T]$

$$|\nabla u|^2 + |\nabla^2 u| \leq C(1 + \kappa \times e^{-2\inf_{(x,t)\in B_R\times[0,T]} u(x,t)}). \tag{113}$$

Now we consider $n = 3$ and can prove that a_ε is achieved for any small $\varepsilon > 0$.

Proposition 5. *Assume $g_0 \in \Gamma_2^+$ and $n = 3$. For $\frac{1}{2} > \varepsilon > 0$, flow (104) globally converges to a solution of (110). As a direct appliaction, a_ε is achieved by a function u_ε satisfying (110) for $\kappa > 0$, provided ε is suficiently small.*

Proof. We divide the proof into 3 steps.

Step 1 There is a constant $C_0 > 0$, independent of $T \in [0, T^*)$ such that

$$\|u(t)\|_{C^2} \leq C_0. \tag{114}$$

Claim. There is a constant $C > 0$, independent of $T \in [0, T^*)$ such that

$$\int_M e^{4u(t)} dvol(g(t)) \leq C. \tag{115}$$

Without loss of generality, we can suppose $\mathcal{F}_1(g(t)) \equiv 1$. Thus, we obtain

$$\varepsilon \int_M e^{4u(t)} dvol(g(t)) = \mathcal{F}_2(g(t)) - r_\varepsilon(g(t)) \leq \mathcal{F}_2(g(t)) \leq Y_{2,1}([g_0]). \tag{116}$$

Thus, we prove the claim. As in [11], we have for all $g \in \mathcal{C}_1([g_0])$

$$\int \sigma_2(g) dvol(g) \;\; \leq \;\; -\frac{1}{16} \int |\nabla u|_{g_0}^4 e^{4u} dvol(g) + c \int e^{4u} dvol(g), \tag{117}$$

for some positive constant $c > 0$. Recall that $Y_{2,1}([g_0])$ is finite and $\int \sigma_2(g(t)) dvol(g(t)) \geq 0$, provided $r_\varepsilon(g(0)) > 0$, since $r_\varepsilon(g(t))$ is non-decreasing. Thus, we infer $\forall t \in [0, T^*)$

(118)

$$256 \int |\nabla e^{\frac{u}{4}}|_{g_0}^4 dvol(g_0) = \int |\nabla u|_{g_0}^4 e^{4u} dvol(g) \leq c \int e^{4u} dvol(g) \leq c \int (e^{u/4})^4 dvol(g_0),$$

which implies by the Sobolev's embedding Theorem for all $x, y \in M$

$$|e^{u(x,t)/4} - e^{u(y,t)/4}| \leq c(\int e^{4u} dvol(g))^{1/4} (d_{g_0}(x,y))^{1/4} \leq c(d_{g_0}(x,y))^{1/4} \tag{119}$$

where $d_{g_0}(x,y)$ is the distance between x and y with respect to the metric g_0. Set

$$\beta(t) := e^{\min_M u(\cdot,t)} = e^{u(x_t,t)} \tag{120}$$

and

$$\tilde{\beta}(t) := e^{\max_M u(\cdot,t)} = e^{u(\tilde{x}_t,t)} \tag{121}$$

for some $x_t, \tilde{x}_t \in M$. It follows from (119) that for any $y \in M$

$$e^{-u(y,t)/4} \geq (\beta(t)^{1/4} + c(d_{g_0}(x,y))^{1/4})^{-1} \tag{122}$$

which implies

$$\begin{aligned} Vol(g(t)) &= \int e^{-3u(t)} dvol(g_0) \geq \int (\beta(t)^{1/4} + c(d_{g_0}(x,y))^{1/4})^{-12} dvol(g_0) \\ &\geq c\int_0^R (\beta(t)^{1/4} + cr^{1/4})^{-12} r^2 dr \geq -c\ln\beta(t), \end{aligned} \tag{123}$$

provided $\beta(t) < 1/2$. On the other hand, we have always $Vol(g(t)) \leq (Y_1([g_0])\mathcal{F}_1(g(t)))^3$. Thus the lower boundness of $u(x,t)$ yields. Together with the local estimates and the fact $\mathcal{F}_1$ is preserved along the flow, this implies that $\|u(t)\|_{C^2}$ has a uniform bound.

Step 2. The flow preserves the positivity of the scalar curvature.

Proposition 6. *There is a constant $C_0 > 0$, independent of $T \in [0, T^*)$ such that $\sigma_1(g(t)) > C_0$ for any $t \in [0,T]$.*

Proof. Recall

$$W = (w_{ij}) = (\nabla^2_{ij} u + u_i u_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}).$$

Set

$$F_\varepsilon(W) = \frac{\sigma_2(W) - \varepsilon}{\sigma_1(W)} - \alpha e^{-2u}.$$

where $\alpha = 2Y_{2,1}([g_0])$. Thus, $F_\varepsilon = u_t + (r_\varepsilon(g(t)) - \alpha)e^{-2u} - s_\varepsilon(g(t))$. Without loss of generality, we assume that the minimum of F_ε is achieved at $(x_0, t_0) \in M \times (0,T]$. Let us use $O(1)$ denote terms with a uniform bound with respect to t (perhaps depending on ε). It is clear that $s_\varepsilon(g(t))$ and $r_\varepsilon(g(t))$ are bounded. Near (x_0, t_0), we have

$$\frac{d}{dt}F_\varepsilon = \sum_{ij} A^{ij}\left[(\nabla^2_g(F_\varepsilon))_{ij} - (r_\varepsilon(g) - \alpha)(\nabla^2_g(e^{-2u}))_{ij}\right] + 2\alpha e^{-2u}u_t, \tag{124}$$

where

$$A^{ij} := \frac{(\sigma_1^2(W) - \sigma_2(W) + \varepsilon)\delta^{ij} - \sigma_1(W)W^{ij}}{\sigma_1^2(W)}$$

is positive definite. To simplify the notations, we drop the index ε as before. Since (x_0, t_0) is the minimum of F in $M \times [0,T]$, at this point, we have $\frac{dF}{dt} \leq 0$, $F_l = 0 \quad \forall l$ and (F_{ij}) is non-negative definite. Note that

$$(\nabla^2_g)_{ij}F = F_{ij} + u_i F_j + u_j F_i - \sum_l u_l F_l \delta_{ij} = F_{ij},$$

at (x_0, t_0), where F_j and F_{ij} are the first and second derivatives with respect to the background metric g_0. From the positivity of A and (124), we have

(125)
$$\begin{aligned}
& -2\alpha e^{-2u} u_t \\
= \; & -2\alpha e^{-2u} \frac{\sigma_2(W) - \varepsilon}{\sigma_1(W)} + O(1) \\
\geq \; & F_t - \sum_{i,j} A^{ij} F_{ij} - 2\alpha e^{-2u} u_t \\
\geq \; & (\alpha - r_\varepsilon(g)) \sum_{i,j} A^{ij} \{(e^{-2u})_{ij} + u_i (e^{-2u})_j + u_j (e^{-2u})_i - \sum_l u_l (e^{-2u})_l \delta_{ij}\} \\
= \; & (\alpha - r_\varepsilon(g)) e^{-2u} \sum_{i,j} A^{ij} \{-2w_{ij} + 2u_i u_j + 2S(g_0)_{ij} + |\nabla u|^2 \delta_{ij}\} \\
\geq \; & (\alpha - r_\varepsilon(g)) e^{-2u} \left[\left(\frac{-2\sigma_2(W) - 2\varepsilon}{\sigma_1(W)} \right) + \sum_{i,j} A^{ij} (2u_i u_j + 2S(g_0)_{ij} + |\nabla u|^2 \delta_{ij}) \right].
\end{aligned}$$

Here we have used $\sum_{i,j} A^{ij} w_{ij} = \dfrac{\sigma_2(W) + \varepsilon}{\sigma_1(W)}$. On the other hand, we have

(126)
$$\sum_{i,j} A^{ij} S(g_0)_{ij} = \frac{(\sigma_1^2(W) - \sigma_2(W))\sigma_1(g_0)}{\sigma_1^2(W)} - \frac{1}{\sigma_1(W)} \sum_{i,j} W^{ij} S(g_0)_{ij} + \frac{\varepsilon \sigma_1(g_0)}{\sigma_1^2(W)}.$$

Going back to (125), we have

(127)
$$\begin{aligned}
& -2\alpha e^{-2u} \frac{\sigma_2(W) - \varepsilon}{\sigma_1(W)} + O(1) \\
\geq \; & F_t - \sum_{i,j} A^{ij} F_{ij} - 2\alpha e^{-2u} u_t \\
\geq \; & (\alpha - r_\varepsilon(g)) e^{-2u} \left[\frac{-2\sigma_2(W) - 2\varepsilon}{\sigma_1(W)} + \frac{2(\sigma_1^2(W) - \sigma_2(W))\sigma_1(g_0)}{\sigma_1^2(W)} \right. \\
& \left. - \frac{2}{\sigma_1(W)} \sum_{i,j} W^{ij} S(g_0)_{ij} + \frac{2\varepsilon \sigma_1(g_0)}{\sigma_1^2(W)} \right],
\end{aligned}$$

since (A^{ij}) is positive definite and $\alpha - r_\varepsilon(g) > Y_{2,1}([g_0])$ is positive. One can check $\sigma_2(g) = O(1)$ for $\|u\|_{C^2}$ is uniformly bounded and $\sum_{i,j} W^{ij} S(g_0)_{ij} = O(1)$. Also the term $\sigma_1^2(W) - \sigma_2(W)$ is always non-negative. From (127), we conclude that there is a positive constant $C_2 > 0$ (independent of T) such that

$$\sigma_1(W)(x_0, t_0) > C_2 > 0. \tag{128}$$

Now for any $(x,t) \in M \times [0,T]$, we have

$$\frac{\sigma_2(W)(x,t) - \varepsilon}{\sigma_1(W)(x,t)} \geq O(1)$$

so that there is a positive constant $C > 0$, independent of T, such that

$$\sigma_1(W)(x,t) \geq C$$

since we have $\sigma_2(W) \leq \frac{1}{2}\sigma_1^2(W)$ provided $\sigma_2(W) \geq 0$. This finishes the proof of the Proposition. ∎

Step 3. Now we can prove equation (110) admits a solution. From Steps 1 and 2, we know that the flow is uniformly parabolic. And, Krylov's theory implies $u(t)$ has a uniform $C^{2,\alpha}$ bound. Hence, $T^* = \infty$. One can also show that $u(t)$ globally converges to $u(\infty)$, which clearly is a solution of (110) for $k = r_\varepsilon(g(\infty))$. (Note that $r_\varepsilon(g(t))$ is monotone and bounded so that $r_\varepsilon(g(\infty))$ exists) (see [20]). So $u_\varepsilon = u(\infty) - \frac{1}{2}\log|r_\varepsilon(g(\infty))|$ solves (110) for $\kappa = 1$ (resp. $0, 1$) if $r_\varepsilon(g(\infty)) > 0$ (resp. $= 0, < 0$). Now, if ε is sufficiently small, we have $r_\varepsilon(g(0)) > 0$. Thus, there exists a minimizing solution to (110), that is, a_ε is achieved.

Proof of Theorem 1 in the case $n = 3$. Now let u_ε be a minimizing solution to (110) for $\kappa = 1$. It is clear $e^{-2u_\varepsilon} g_0 \in \mathcal{C}_2([g_0])$ and $\tilde{Y}_{2,1}([g_0]) \geq a_\varepsilon$. As $a_\varepsilon \to Y_{2,1}([g_0])$, we infer $\tilde{Y}_{2,1}([g_0]) \geq Y_{2,1}([g_0])$. On the other hand, $\tilde{Y}_{2,1}([g_0]) \leq Y_{2,1}([g_0])$ since $\mathcal{C}_2([g_0]) \subset \mathcal{C}_1([g_0])$. Finally, $\tilde{Y}_{2,1}([g_0]) = Y_{2,1}([g_0])$. This finishes the proof of Theorem. ∎

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