A critical elliptic problem for polyharmonic operators

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Abstract

In this paper, we study the existence of solutions for a critical elliptic problem for polyharmonic operators. We prove the existence result in some general domain by minimizing on some infinite-dimensional Finsler manifold for some suitable perturbation of the critical nonlinearity when the dimension of domain is larger than critical one. For the critical dimensions, we prove also the existence of solutions in domains perforated with the small holes. Some unstable solutions are obtained at higher level sets by Coron's topological method, provided that the minimizing solution does not exist.

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1 Introduction

This paper is a sequel to [21] on some semilinear critical problems for polyharmonic operators. Let $K \in \mathbb{N}$ and $\Omega \subset \mathbb{R}^N (N \geq 2K+1)$ be a smooth bounded domain in $\mathbb{R}^N$. We consider the semilinear polyharmonic problem with homogeneous Dirichlet boundary condition

$$\begin{cases} (-\Delta)^K u = |u|^{s-2}u + f(x,u) & \text{in } \Omega \\ u = Du = \cdots = D^{K-1}u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where $s := \frac{2N}{N-2K}$ denotes the critical Sobolev exponent for $(-\Delta)^K$ and $f(x,u)$ is a lower-order perturbation of $|u|^{s-2}u$ (see the assumption (H2) below). The equation (1) is of variational type: Solutions of (1) correspond to critical points of the energy functional

$$\mathcal{E}(u) = \frac{1}{2}\|u\|^2_{K,2,\Omega} - \frac{1}{s}\int_\Omega |u|^s - \int_\Omega F(x,u), \tag{2}$$

defined on the Hilbert space

$$H_0^K(\Omega) = \left\{ v \in H^K(\Omega) \mid \quad D^i v = 0 \text{ on } \partial\Omega \quad \forall \quad 0 \leq i < K \right\}$$

which is endowed with the scalar product

$$(u,v)_\Omega = \begin{cases} \displaystyle\int_\Omega ((-\triangle)^M u)((-\triangle)^M v) & \text{if } K = 2M \text{ is even} \\ \displaystyle\int_\Omega (\nabla(-\triangle)^M u)(\nabla(-\triangle)^M v) & \text{if } K = 2M+1 \text{ is odd} \end{cases} \tag{3}$$

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and $\|\cdot\|_{K,2,\Omega}$ is the corresponding norm, $F(x,u) := \displaystyle\int_0^u f(x,t)dt$ is the primitive of f.

We assume that

(H1) $f(x,u) : \Omega \times \mathbb{R} \to \mathbb{R}$ is continuous and $\displaystyle\sup_{x\in\Omega,|u|\le M} |f(x,u)| < \infty$ for every $M > 0$;

(H2) $f(x,u) = a(x)u + g(x,u)$ with $a(x) \in L^\infty(\Omega) \cap C^\infty(\Omega)$, $g(x,u) = o(u)$ as $u \to 0$ uniformly in x and $g(x,u) = o(|u|^{s-1})$ as $u \to \infty$ uniformly in x.

From (H1) to (H2), it follows $f(x,0) = 0$ and that f is a lower-order perturbation of $|u|^{s-2}u$ at infinite in the sense that $\displaystyle\lim_{u\to\infty} \frac{f(x,u)}{|u|^{s-1}} = 0$ uniformly in $x \in \Omega$. Moreover, we assume that $f(x,u)$ satisfies

(H3) $\frac{\partial f}{\partial u}(x,u)$ is continuous on $\Omega \times \mathbb{R}$;

(H4) $|\frac{\partial f}{\partial u}(x,u)| \le C(1 + |u|^{s-2})$, $\forall u \in \mathbb{R}$ uniformly in $x \in \Omega$;

(H5) $f_1(x,u) := \frac{f(x,u)}{u}$ is non-decreasing in $u > 0$ and non-increasing in $u < 0$ for a.e. $x \in \Omega$.

For $K = 1$, $f(x,u) = \lambda u$ and $\lambda \in (0, \lambda_1)$ where λ_1 is the first eigenvalue of $-\Delta$ for Dirichlet boundary condition, the problem has a strong background from some variational problems in geometry and physics, such as the Yamabe's problem with lack of compactness. This was considered by Brezis and Nirenberg for positive solutions in their pioneer work in [5]. Then it has been studied extensively in the last three decades. We recall briefly some results about the existence and multiplicity of sign-changing solutions to the problem (1) for $K = 1$ and $f(x,u) = \lambda u$. For any fixed $\lambda > 0$, the first multiplicity result was due to Cerami, Fortunato and Struwe [8]. They obtained the number of the solutions of (1) is bounded below by the number of the eigenvalues of $-\Delta$ lying in the open interval $(\lambda, \lambda + S|\Omega|^{-2/N})$, where S is the best constant for the Sobolev embedding $D^{1,2}(\mathbb{R}^N) \hookrightarrow L^{2^*}(\mathbb{R}^N)$ (see the definition below) and $|\Omega|$ denotes the Lebesgue measure of Ω. Capozzi, Fortunato and Palmieri in [7] established the existence of a nontrivial solution for $\lambda > 0$ which is not an eigenvalue of $-\Delta$ when $N \ge 4$ and for any $\lambda > 0$ when $N \ge 5$ (see also [45]). In [11], Devillanova and Solimini proved that, if $N \ge 7$, then (1) has infinitely many solutions for every $\lambda > 0$. They proved also in [12] that, if $N \ge 4$ and $\lambda \in (0, \lambda_1)$, then there exist at least $\frac{N}{2} + 1$ pairs of nontrivial solutions. Clapp and Weth [9] has extended this last result to all $\lambda > 0$ with $N \ge 4$. In the same paper they also obtained some extensions to critical biharmonic problems for $N \ge 8$. When the domain Ω is a ball and $N \ge 4$, Fortunato and Jannelli [15] proved there are infinitely many sign-changing solutions which are built using the symmetry of the domain Ω. Schechter and Zou in [37] showed the same result for any domain Ω when $N \ge 7$. In particular, if $\lambda \ge \lambda_1$, it has and only has infinitely many sign-changing solutions except zero. Their work is based on the estimates of Morse indices of nodal solutions.

Concerning the polyharmonic case, Pucci and Serrin in [35] has studied the problem (1) for $K = 2$ and $\lambda > 0$ when Ω is a ball. They proved that it admits nontrivial radial symmetric solutions for all $\lambda \in (0, \lambda_1)$ if and only if $N \ge 8$. If $N = 5, 6, 7$, then there exists $\lambda_* \in (0, \lambda_1)$ such that the problem admits no nontrivial radial symmetric solutions whenever $\lambda \in (0, \lambda_*]$. Here λ_1 is understood as the first eigenvalue of Δ^2 for Dirichlet boundary conditions. This is the counterpart of the well known result of [5] on the nonexistence for radial symmetric solutions for small λ in dimension $N = 3$ and $K = 1$ (where $\lambda_* = \lambda_1/4$). They called these dimensions as critical dimensions. They conjectured that for general $K \geq 1$, the critical dimensions are $2K + 1, \cdots, 4K - 1$. The conjecture is not completely solved for all $K \geq 1$. Grunau [23] defined later the notion of weakly critical dimensions as the space dimensions for which a necessary condition for the existence of a positive radial solution of (1) in B_1 is $\lambda \in (\lambda^*, \lambda_1)$ for some $\lambda^* > 0$. He proved that the conjecture is true in the weak sense. Gazzola, Grunau and Squassina [18] proved nonexistence of positive radial symmetric solutions for Navier boundary condition for small $\lambda > 0$. They established also some existence results for $\lambda = 0$. Their result strongly depends on the geometry of domains. For biharmonic operators, Bartsch, Weth and Willem in [3] and Ebobisse and Ahmedou in [13] have studied the problem (1) on domains with nontrivial topology under Dirichlet boundary condition and Navier boundary condition respectively. For related problems, we infer to [4], [14], [16], [17], [22], [24], [32] and the references therein.

For general case $K \geq 1$, Ge has studied in [21] the same type of equation (1) for Navier boundary condition when $f(x, u) = \lambda u$ with $0 \leq \lambda < \lambda_1$ and λ_1 the first eigenvalues of $(-\Delta)^K$. He established the existence of positive solutions in some general domain under the suitable assumptions. In particular unstable solutions in higher level set are obtained by Coron's topological method in domains perforated with the small holes.

The purpose of this paper is to continue the study of the semilinear polyharmonic problem (1) to general $K \geq 1$ with Dirichlet boundary condition for general domains. Let us denote the polyharmonic operator

$$\mathcal{L} := (-\Delta)^K - a(x)$$

and $\lambda_1(\Omega) \leq \lambda_2(\Omega) \leq \cdots \leq \lambda_n(\Omega) \leq \cdots$ the eigenvalues of $\mathcal{L}$ under the homogeneous Dirichlet boundary condition. It is well known that each eigenvalue $\lambda_k(\Omega)$, $k \geq 1$, can be described as the minimax value

$$\lambda_k(\Omega) = \min_{V \subset H_0^K(\Omega), \dim V = k} \max_{v \in V} \frac{\int_\Omega v\mathcal{L}v}{\int_\Omega v^2}.$$

It follows that $\lambda_k(\Omega)$ is a non-increasing functional on the domains, that is, if $\Omega_2 \subset \Omega_1$, then $\lambda_k(\Omega_2) \geq \lambda_k(\Omega_1)$. Moreover, from the unique continuation principle, we have $\lambda_k(\Omega_2) > \lambda_k(\Omega_1)$ for any $k \geq 1$, provided Ω_1 is connected (see [27, 34]). For the perforated domain $\Omega := \Omega_1 \setminus \Omega_2$ with the smooth bounded domains $\Omega_2 \subset \Omega_1$, with the help of the above description, we have

$$\lim \lambda_k(\Omega_1 \setminus \Omega_2) = \lambda_k(\Omega_1),$$

where the limit is taken as the diameter of Ω_2 goes to 0. To this aim, it suffices to consider $\Omega_2 = B(x, \epsilon)$ balls with small radius $\epsilon > 0$ in the sequel.

Assume now $\lambda_n(\Omega) \le 0$ and $\lambda_{n+1}(\Omega) > 0$ for some $n \ge 1$. Let $e_i(x)$ be an eigenfunction associated to $\lambda_k(\Omega)$ with $\|e_i\|_{K,2,\Omega} = 1$ for any $1 \le i \le n$. Define

$$\mathcal{M} := \{v \in H_0^K(\Omega) \setminus \{0\} | \quad d\mathcal{E}(v)(w) = 0, \quad \forall w \in \mathrm{Span}(v, e_1, \cdots, e_n)\}.$$

We prove in Section 2 that under the hypothesis (H1) to (H5), $\mathcal{M}$ is then a complete C^1 Finsler manifold and it will be a $C^{1,1}$ Finsler manifold with additional assumptions (H6) to (H7) (see section 2). This permits to consider the following minimization problem

$$\kappa := \inf_{v \in \mathcal{M}} \mathcal{E}(v).$$

We can prove then $\kappa \le \frac{K}{N}(S_K(\Omega))^{\frac{N}{2K}}$ for any f satisfying (H1) to (H5), where we denote

$$S_K(\Omega) := \inf_{v \in H_0^K(\Omega)\setminus\{0\}} \frac{\|v\|^2_{K,2,\Omega}}{\|v\|^2_{L^s(\Omega)}}$$

the best constant for the embedding $H_0^K(\Omega) \hookrightarrow L^s(\Omega)$. Here, as for $K = 1$, it is well known that $S_K(\Omega)$ is independent of Ω and $S_K(\Omega) = S_K(\mathbb{R}^N) := \inf_{v \in H^K(\mathbb{R}^N)\setminus\{0\}} \frac{\|v\|^2_{K,2,\mathbb{R}^N}}{\|v\|^2_{L^s(\mathbb{R}^N)}}$. Therefore we denote it by S_K in the sequel (see [17, 20]). We prove then that for non critical dimension case $N \ge 4K$ and $f(x, u) = \mu u$, the infimum above is achieved by some $u \in \mathcal{M}$ which is a solution of (1). This method is a generalization of the Nehari manifold method ([31]) and can be seen as an alternative approach to the linking method (see [39]). For the critical dimension $2K < N < 4K$, the existence of solutions to (1) is a delicate issue. To our knowledge, there are few results on it. The reason is that the minimizing method fails, for example, for $K = 1$, when $\Omega \subset \mathbb{R}^3$ is a ball and when $f(x, u) = \lambda u$ with $0 < \lambda < \frac{\lambda_1}{4}$. It is well known that there are no positive solutions. In Section 3, we study the existence of solutions for some perforated domains in such critical dimensions. We analyze the concentration phenomenon when κ equals to $\frac{K}{N}(S_K)^{\frac{N}{2K}}$. Then following Coron's strategy of topological argument, we obtain the existence of unstable critical points in higher level sets for domains perforated with small holes.

In all this paper, C, C' and c denote generic positive constant independent of u, even their value could be changed from one line to another one. We give also some notations here. The space $\mathcal{D}^{K,2}(\mathbb{R}^N)$ (resp. $\mathcal{D}^{K,2}(\mathbb{R}^N_+)$) is the completion of $C_0^\infty(\mathbb{R}^N)$ (resp. $C_0^\infty(\mathbb{R}^N_+)$) for the norm $\|\cdot\|_{K,2,\mathbb{R}^N}$ (resp. $\|\cdot\|_{K,2,\mathbb{R}^N_+}$).

2 Study of the energy functional $\mathcal{E}$ on $\mathcal{M}$

We begin this section by studying some properties of the set $\mathcal{M}$. Observe that $v \in \mathcal{M}$ is equivalent to say $v \ne 0$ and satisfying

$$\begin{aligned} l_0(v) &:= \|v\|^2_{K,2,\Omega} - \|v\|^s_{L^s(\Omega)} - \int_\Omega f(x, v)v = 0 \\ l_i(v) &:= (v, e_i)_\Omega - \int_\Omega |v|^{s-2} v e_i - \int_\Omega f(x, v)e_i = 0, \quad \forall 1 \le i \le n. \end{aligned} \tag{4}$$

Let us denote $V_0 := \mathrm{Span}(e_1, \cdots, e_n)$ the $n-$dimensional vector space spanned by $e_1, \cdots, e_n$. We prove now the following proposition.

Proposition 1 *Suppose (H1) to (H5) are satisfied. Then $\mathcal{M}$ is a complete C^1 Finsler manifold. Furthermore, suppose that*

(H6) $\frac{\partial^2 f}{\partial u^2}(x,u)$ *is continuous on* $\Omega \times \mathbb{R}$ *and* $u \mapsto |u|^{s-2}u$ *is* C^2 *on* $\mathbb{R}$*;*

(H7) $|\frac{\partial^2}{\partial u^2} f(x,u)| \leq C(|u|+1)^{s-3}$, $\forall u \in \mathbb{R}$ *uniformly in* $x \in \Omega$.

Then $\mathcal{M}$ is a complete $C^{1,1}$ Finsler manifold.

Proof. The proof is divided into several steps.

Step 1. *$\mathcal{M}$ is not empty.*

By the assumptions (H1)-(H2), $\mathcal{E}$ is a continuous functional on $H_0^K(\Omega)$. Fixing $v \notin V_0$ and let $V := \mathrm{Span}(v, e_1, \cdots, e_n)$. Clearly, for all $w \in V$, we have

$$\mathcal{E}(w) \leq \frac{1}{2}\|w\|_{K,2,\Omega}^2 - \frac{1}{2}\int_\Omega a(x)w^2 - \frac{1}{s}\int_\Omega |w|^s, \tag{5}$$

since it follows from (H2) and (H5) that $\frac{g(x,u)}{u} \geq 0$ and $F(x,u) \geq \frac{1}{2}a(x)u^2$ for all $u \in \mathbb{R}\backslash\{0\}$ and for a.e. $x \in \Omega$. As V is a finite dimensional vector space, all the norms on it are equivalent. In particular, the norms $\|\cdot\|_{K,2,\Omega}$ and $\|\cdot\|_{L^s(\Omega)}$ are equivalent on V. This implies

$$\lim_{w \in V, w \to \infty} \mathcal{E}(w) = -\infty. \tag{6}$$

On the other hand, again from (H2), we infer for any given $\varepsilon > 0$, there exists $C > 0$ such that for all $u \in \mathbb{R}$ and for a.e. $x \in \Omega$

$$g(x,u) \leq \varepsilon|u| + C|u|^{s-1}, \qquad F(x,u) \leq \frac{1}{2}(a(x)+\varepsilon)u^2 + \frac{C}{s}|u|^s, \tag{7}$$

so that for all $w \in V$

$$\mathcal{E}(w) \geq \frac{1}{2}\|w\|_{K,2,\Omega}^2 - \frac{1}{2}\int_\Omega (a(x)+\varepsilon)w^2 - \frac{1+C}{s}\int_\Omega |w|^s.$$

Since $v \notin V_0$, we can choose $v' \in V \cap (V_0)^\perp$ such that $\frac{1}{2}\|v'\|_{K,2,\Omega}^2 - \frac{1}{2}\int_\Omega a(x)(v')^2 > 0$. By taking a sufficiently small $\varepsilon > 0$, we have

$$\frac{1}{2}\|v'\|_{K,2,\Omega}^2 - \int_\Omega \frac{1}{2}(a(x)+\varepsilon)(v')^2 \geq \varepsilon\|v'\|_{K,2,\Omega}^2. \tag{8}$$

As a consequence, we obtain

$$\sup_{w \in V} \mathcal{E}(w) > 0. \tag{9}$$

Together with (6), there exists $\tilde{v} \in V$ such that $\mathcal{E}(\tilde{v}) = \max_{w \in V} \mathcal{E}(w)$ since V is a finite dimensional vector space. Clearly, $\tilde{v} \in \mathcal{M}$.

Step 2. *$\mathcal{M}$ is closed.*

We define the map

$$\begin{array}{rlcl} L: & H_0^K(\Omega) & \to & \mathbb{R}^{n+1} \\ & v & \mapsto & (l_0(v),\cdots,l_n(v)). \end{array}$$

In view of the assumptions (H1)-(H2), L is continuous on $H_0^K(\Omega)$. Let $(v_k)\subset\mathcal{M}$ be a sequence in $\mathcal{M}$ such that $v_k\to v$ in $H_0^K(\Omega)$. Then we get $L(v)=0$. Now it suffices to show $v\neq 0$. First, we note $v_k\notin V_0$ for all $k\in\mathbb{N}$. Indeed, we have

$$\|v_k\|_{K,2,\Omega}^2-\int_\Omega a(x)v_k^2=\|v_k\|_{L^s(\Omega)}^s+\int_\Omega g(x,v_k)v_k. \tag{10}$$

If we have $v_k\in V_0$ for some $k\geq 1$, the term on the left hand is non-positive. But that one on the right hand is non-negative. Thus, $\|v_k\|_{L^s(\Omega)}^s=0$ and the desired contradiction $v_k\neq 0$ gives the result. Now, we claim there exists some positive number $c>0$ such that $\|v_k\|_{K,2,\Omega}>c$. We denote the orthogonal projection of v_k on V_0 by

$$v_k^{\|}:=\sum_{i=1}^n (v_k,e_i)_\Omega e_i$$

and $v_k^\perp$ its orthogonal complementary

$$v_k^\perp:=v_k-v_k^{\|}.$$

As $v_k\in\mathcal{M}$, we obtain

$$(v_k,v_k^{\|})_\Omega-\int_\Omega a(x)v_kv_k^{\|}=\int_\Omega(|v_k|^{s-2}+\frac{g(x,v_k)}{v_k})v_kv_k^{\|}.$$

Together with (10), we have

$$\begin{aligned} &\|v_k^\perp\|_{K,2,\Omega}^2-\int_\Omega a(x)(v_k^\perp)^2-(\|v_k^{\|}\|^2-\int_\Omega a(x)(v_k^{\|})^2)\\ = & \int_\Omega(|v_k|^{s-2}+\frac{g(x,v_k)}{v_k})((v_k^\perp)^2-(v_k^{\|})^2) \end{aligned}$$

which implies

$$\|v_k^\perp\|_{K,2,\Omega}^2-\int_\Omega a(x)(v_k^\perp)^2\leq\int_\Omega(|v_k|^{s-2}+\frac{g(x,v_k)}{v_k})(v_k^\perp)^2, \tag{11}$$

since

$$\|v_k^{\|}\|_{K,2,\Omega}^2-\int_\Omega a(x)(v_k^{\|})^2\leq 0.$$

Gathering (5), (8) and (11), we get

$$\begin{aligned} 2\varepsilon\|v_k^\perp\|_{K,2,\Omega}^2 &\leq \|v_k^\perp\|^2-\int_\Omega(a(x)+\varepsilon)(v_k^\perp)^2\\ &\leq(1+C)\int_\Omega|v_k|^{s-2}(v_k^\perp)^2\\ &\leq(1+C)\|v_k\|_{L^s(\Omega)}^{s-2}\|v_k^\perp\|_{L^s(\Omega)}^2\leq C'(1+C)\|v_k\|_{L^s(\Omega)}^{s-2}\|v_k^\perp\|_{K,2,\Omega}^2 \end{aligned}$$

Finally, $\|v_k\|_{L^s(\Omega)} \geq c > 0$ and the desired claim follows.

Step 3. *$dL(v)$ is surjective and its kernel splits for all $v \in \mathcal{M}$.*

By (H3) and (H4), $f(x,u)u$ and $f(x,u)$ are C^1 on $\Omega \times \mathbb{R}$ and

$$\left|\frac{\partial(f(x,u)u)}{\partial u}(x,u)\right| \leq C(1+|u|^{s-1}), \quad \text{uniformly in } x \in \Omega \text{ and } \forall u \in \mathbb{R}. \tag{12}$$

Therefore, L is C^1 on $H_0^K(\Omega)$ provided the assumptions (H1)-(H4) hold. A direct calculation leads to

$$\begin{aligned} dl_0(v)(w) &= 2(v,w)_\Omega - s\int_\Omega |v|^{s-2}vw - \int_\Omega (f(x,v) + v\frac{\partial f(x,v)}{\partial v})w, \\ dl_i(v)(w) &= (w,e_i)_\Omega - (s-1)\int_\Omega |v|^{s-2}we_i - \int_\Omega \frac{\partial f(x,v)}{\partial v}we_i, \quad \forall 1 \leq i \leq n. \end{aligned} \tag{13}$$

We claim $dL(v)|_V$, the restriction on V of $dL(v)$, is a bijective endomorphism from V on $\mathbb{R}^{n+1}$. As V and $\mathbb{R}^{n+1}$ have the same dimension, it suffices to prove $\mathrm{Ker}(dL(v)|_V) = \{0\}$. Let $w \in \mathrm{Ker}(dL(v)|_V)$ and write $w = \mu v + \sum_{i=1}^n \mu_i e_i$ where $\mu, \mu_i \in \mathbb{R}$ for each i. Combining (4) and (13), we get

$$\begin{aligned} dl_0(v)(w) &= -(s-2)\int_\Omega |v|^{s-2}vw - \int_\Omega (-f(x,v) + v\frac{\partial f(x,v)}{\partial v})w = 0, \\ dl_i(v)(w) &= -\int_\Omega (-\frac{f(x,v)}{v} + \frac{\partial f(x,v)}{\partial v})\mu ve_i - (s-2)\int_\Omega |v|^{s-2}\mu ve_i \\ &\quad + (\textstyle\sum_{j=1}^n \mu_j e_j, e_i)_\Omega - (s-1)\displaystyle\int_\Omega |v|^{s-2}e_i\sum_{j=1}^n \mu_j e_j - \int_\Omega \frac{\partial f(x,v)}{\partial v}e_i\sum_{j=1}^n \mu_j e_j = 0, \end{aligned} \tag{14}$$

for all $1 \leq i \leq n$. On the other hand, we have

$$\mu dl_0(v)(w) + \sum_{i=1}^n \mu_i dl_i(v)(w) = 0.$$

Together with (14), we infer

$$\begin{aligned} &(s-2)\int_\Omega |v|^{s-2}w^2 + \int_\Omega |v|^{s-2}(\sum_{j=1}^n \mu_j e_j)^2 + \int_\Omega (-\frac{f(x,v)}{v} + \frac{\partial f(x,v)}{\partial v})w^2 \\ &+ \int_\Omega \frac{g(x,v)}{v}(\sum_{j=1}^n \mu_j e_j)^2 - (\sum_{j=1}^n \mu_j e_j, \sum_{i=1}^n \mu_i e_i)_\Omega + \int_\Omega a(x)(\sum_{j=1}^n \mu_j e_j)^2 = 0. \end{aligned}$$

We know from (H2) and (H5) that $-\frac{f(x,v)}{v} + \frac{\partial f(x,v)}{\partial v} \geq 0$, $\frac{g(x,v)}{v} \geq 0$ and

$$(\sum_{j=1}^n \mu_j e_j, \sum_{i=1}^n \mu_i e_i)_\Omega - \int_\Omega a(x)(\sum_{j=1}^n \mu_j e_j)^2 \leq 0.$$

Finally, we deduce

$$vw(x) = 0, \quad v\sum_{j=1}^n \mu_j e_j(x) = 0 \quad \text{for a.e. } x \in \Omega \tag{15}$$

and

$$(\sum_{j=1}^{n}\mu_j e_j,\sum_{i=1}^{n}\mu_i e_i)_\Omega-\int_\Omega a(x)(\sum_{j=1}^{n}\mu_j e_j)^2=0. \tag{16}$$

Thus we have

$$\mu v^2=vw-v\sum_{j=1}^{n}\mu_j e_j=0$$

which yields $\mu=0$. Moreover, it follows from (16) that

$$\mathcal{L}w=0.$$

By the unique continuation principle, we have either $w\equiv 0$ or $w(x)\neq 0$ for a.e. $x\in\Omega$. Indeed, we state first w is regular. All the derivatives of w vanish a.e. on the set $\{x\in\Omega; w(x)=0\}$ provided this set is not a negligible measurable set. Thus, w vanishes of infinite order at such points. By the strong unique continuation principle [27], w vanishes. Going back to (15), we have $w\equiv 0$ and the desired claim follows. As a consequence, for all $v\in\mathcal{M}$, $dL(v)$ is surjective and $H_0^K(\Omega)=\ker(dL(v))\oplus V$. $\mathcal{M}$ is thus a complete C^1 Finsler manifold (see [26]). Furthermore, $\mathcal{M}$ is a complete $C^{1,1}$ Finsler manifold provided (H6) and (H7) are satisfied. ■

For any $v\in H_0^K(\Omega)\setminus V_0$, we denote by

$$V^+:=\{tv+\sum_{i=1}^{n}\mu_i e_i|\quad \text{for all } t>0,\ \mu_i\in\mathbb{R}\},$$

the $(n+1)-$dimensional half space spanned by v and $\{e_i\}$ for all $1\leq i\leq n$. We have the following

Lemma 1 *Under the assumptions (H1) to (H5), then there exists an unique $v_0\in\mathcal{M}$ such that*

$$\mathcal{M}\cap V^+=\{v_0\}. \tag{17}$$

Moreover we have

$$\mathcal{E}(v_0)=\max_{w\in V^+}\mathcal{E}(w). \tag{18}$$

Proof. Given $v\in H_0^K(\Omega)\setminus V_0$, we define for any $t>0$ the $n-$dimensional affine vector space

$$V_t:=tv+V_0.$$

We divide the proof into several steps.

Step 1. *For any $t>0$ there exists an unique $v(t)\in V_t$ such that $\mathcal{E}(v(t))=\max\limits_{V_t}\mathcal{E}$. Moreover, $\{v(t),t>0\}$ is a C^1 curve in V^+.*

From (H1) to (H4), it is known that $\mathcal{E}$ is C^2 on V^+. Thanks to (6), we have

$$\lim_{w\in V_t,w\to\infty}\mathcal{E}(w)=-\infty$$

Thus there exists some $v(t) \in V_t$ such that $\mathcal{E}(v(t)) = \max_{w\in V_t} \mathcal{E}(w)$. A direct calculation leads to

$$d^2\mathcal{E}(v)(w,w) = \|w\|^2_{K,2,\Omega} - \int_\Omega a(x)w^2 - \int_\Omega ((s-1)|v|^{s-2} + \frac{\partial g(x,v)}{\partial v})w^2$$

By (H5), we infer

$$\frac{g(x,v)}{v} \geq 0 \text{ and } \frac{\partial g(x,v)}{\partial v} \geq \frac{g(x,v)}{v} \geq 0.$$

Hence, $d^2\mathcal{E}(v) < 0$ on V_t, that is, the functional $\mathcal{E}$ is strictly concave on V_t. This yields the uniqueness. We note $\{v(t), t > 0\} = \{w \in V^+ |\quad d\mathcal{E}(w)|_{V_0} = 0\}$. As the second variation $d^2\mathcal{E}$ of $\mathcal{E}$ is negative define on V_0, it follows from the Implicit Function Theorem that $\{v(t), t > 0\}$ is a C^1 curve in V^+ which finishes the proof of step 1.

Step 2. *For all $w \in \mathcal{M} \cap V^+$, the restriction of $\mathcal{E}$ on V^+ has a strictly local maximum at w.*

Recall $V := \text{Span}(v, e_1, \cdots, e_n)$. Let $v \neq 0$ satisfying $d\mathcal{E}(v)|_V = 0$ and $w = \mu v + \sum_{i=1}^n \mu_i e_i \in V$. As in the proof of Proposition 1, we have by (H2),

$$\begin{aligned} d^2\mathcal{E}(v)(w,w) = \quad & -(s-2)\int_\Omega |v|^{s-2}w^2 - \int_\Omega |v|^{s-2}(\sum_{j=1}^n \mu_j e_j)^2 \\ & - \int_\Omega (-\frac{f(x,v)}{v} + \frac{\partial f(x,v)}{\partial v})w^2 - \int_\Omega \frac{g(x,v)}{v}(\sum_{j=1}^n \mu_j e_j)^2 \\ & + (\sum_{j=1}^n \mu_j e_j, \sum_{i=1}^n \mu_i e_i)_\Omega - \int_\Omega a(x)(\sum_{j=1}^n \mu_j e_j)^2 \end{aligned}$$

which implies from (H1) to (H5)

$$d^2\mathcal{E}(v)(w,w) < 0 \quad \text{ provided } w \neq 0.$$

Therefore, the desired claim follows.

Step 3. *There exists an unique $t_0 > 0$ such that $v(t_0) \in \mathcal{M}$. Moreover, $d\mathcal{E}(v(t))(v(t)) > 0$ for any $0 < t < t_0$ and $d\mathcal{E}(v(t))(v(t)) < 0$ for any $t > t_0$.*

With the same arguments as in the proof of Proposition 1, we have

$$\sup_{w\in V^+} \mathcal{E}(w) > 0. \tag{19}$$

On the other hand, it follows from (5) that $\forall w \in V_0$

$$\mathcal{E}(w) \leq 0. \tag{20}$$

In particular, we obtain

$$\sup_{w\in V^+} \mathcal{E}(w) = \sup_{w\in \overline{V^+}} \mathcal{E}(w) = \sup_{t>0} \mathcal{E}(v(t)),$$

where $\overline{V^+}$ is the closure of V^+. Combining (6), (19) and (20) and using the continuity of $\mathcal{E}$ on $\overline{V^+}$, there exists some $v_0 \in \mathcal{M} \cap V^+$ such that

$$\mathcal{E}(v_0) = \sup_{w \in V^+} \mathcal{E}(w).$$

We know

$$\mathcal{M} \cap V^+ \subset \{w \in V^+ | \quad d\mathcal{E}(w)|_{V_0} = 0\} = \{v(t)| \quad t > 0\} \tag{21}$$

so that there exists $t_0 > 0$ such that $v(t_0) = v_0$. Set $\alpha(t) := \mathcal{E}(v(t))$ then $\alpha'(t) = d\mathcal{E}(v(t))(v'(t)) = \frac{l_0(v(t))}{t}$ since $v'(t) - v \in V_0$ and $d\mathcal{E}(v(t))|_{V_0} = 0$. We claim $\mathcal{M} \cap V^+ = \{v(t)| \quad \alpha'(t) = 0\}$. Obviously, $\mathcal{M} \cap V^+ \subset \{v(t)| \quad \alpha'(t) = 0\}$. Conversely, for any $v(t)$ with $\alpha'(t) = 0$, by the method of Lagrange multipliers, there exists $\mu_1, \cdots, \mu_n \in \mathbb{R}$ such that $d\mathcal{E}(v(t))|_V + \sum_{i=1}^{n} \mu_i dl_i(v(t))|_V = 0$. Hence, we have on V_0,

$$\sum_{i=1}^{n} \mu_i d^2\mathcal{E}(v(t))(\cdot, e_i) = 0.$$

By virtue of the fact $d^2\mathcal{E}(v)|_{V_0} < 0$ for all $v \in V_t$, we infer $\mu_1 = \cdots \mu_n = 0$ which proves the claim. Applying (6), we infer

$$\lim_{t \to +\infty} \alpha(t) = -\infty,$$

since

$$\lim_{t \to +\infty} \inf_{w \in V_t} \|w\|_{K,2,\Omega} = +\infty.$$

It follows from Step 2 that there exists only strictly local maximum points for $\alpha(t)$. Hence, t_0 is the only critical point of $\alpha(t)$. Moreover, $\alpha'(t) > 0$ for any $0 < t < t_0$ and $\alpha'(t) < 0$ for any $t > t_0$. The lemma is proved. ∎

Now let us consider the minimization problem

$$\kappa := \inf_{v \in \mathcal{M}} \mathcal{E}(v). \tag{22}$$

We have then

Lemma 2 *Under assumptions (H1) to (H5), there holds*

$$\kappa \leq \frac{K}{N} (S_K)^{\frac{N}{2K}}. \tag{23}$$

Proof. Let $B(x_0, R) \subset \Omega$ for some $x_0 \in \Omega$ and $R > 0$. We consider for some small number $\nu > 0$ and for all $\epsilon \in (0, \nu)$, the function

$$u_\epsilon(x) := C_{N,K} \frac{\epsilon^{(N-2K)/2}}{(\epsilon^2 + |x - x_0|^2)^{(N-2K)/2}},$$

where the constant $C_{N,K}$ independent of ϵ is chosen such that $\|u_\epsilon\|^s_{L^s(\mathbb{R}^N)} = \|u_\epsilon\|^2_{K,2,\mathbb{R}^N} = (S_K)^{\frac{N}{2K}}$. Let $\xi \in C_0^\infty(B(x_0,R))$ be a fixed cut-off function satisfying $0 \le \xi \le 1$ and $\xi \equiv 1$ on $B(x_0,R/2)$. Putting $w_\epsilon := \xi u_\epsilon \in C_0^\infty(\Omega)$ as in [5] and [24], we obtain as $\epsilon \to 0$

$$\|w_\epsilon\|^s_{L^s} = (S_K)^{\frac{N}{2K}} + O(\epsilon^N) \quad \text{and} \quad \|w_\epsilon\|^2_{K,2,\Omega} = (S_K)^{\frac{N}{2K}} + O(\epsilon^{N-2K}). \tag{24}$$

It is clear that as $\epsilon \to 0$, we have

$$\begin{array}{ll} w_\epsilon \rightharpoonup 0 & \text{weakly in } H_0^K(\Omega), \\ w_\epsilon \rightharpoonup 0 & \text{weakly in } L^s(\Omega), \text{ strongly in } L^q(\Omega) \quad (\forall q < s) \text{ and a.e. in } \Omega. \end{array}$$

Therefore, there holds

$$f(x, w_\epsilon) \to 0 \qquad \text{strongly in } L^{\frac{s}{s-1}}(\Omega). \tag{25}$$

Indeed, for any $M > 0$, let

$$f_M(x,u) := \begin{cases} f(x,u), & \text{if} \quad |u| \le M \\ 0, & \text{if} \quad |u| > M. \end{cases}$$

From (H1) to (H2), it follows that $\forall \delta > 0$, there exists $M > 0$ such that

$$|f_M(x,u) - f(x,u)| \le \delta |u|^{s-1} \qquad \text{for a.e. } x \in \Omega \text{ and } \forall u \in \mathbb{R}.$$

Therefore, we have

$$\begin{aligned} \|f(x,w_\epsilon)\|_{L^{\frac{s}{s-1}}} &\le \|f(x,w_\epsilon) - f_M(x,w_\epsilon)\|_{L^{\frac{s}{s-1}}} + \|f_M(x,w_\epsilon)\|_{L^{\frac{s}{s-1}}} \\ &\le \delta \|w_\epsilon\|^{s-1}_{L^s} + \|f_M(x,w_\epsilon)\|_{L^{\frac{s}{s-1}}}. \end{aligned} \tag{26}$$

Using Lesbegue's theorem, we infer that $\forall \beta > 0$

$$\|f_M(x,w_\epsilon)\|_{L^\beta} \to 0.$$

Letting $\epsilon \to 0$ in (26), we obtain

$$\limsup_{\epsilon \to 0} \|f(x,w_\epsilon)\|_{L^{\frac{s}{s-1}}} \le 2\delta C.$$

Thus (25) is proved. Similarly, we have

$$\lim_{\epsilon \to 0} \int_\Omega F(x, w_\epsilon) = 0.$$

Set $e_0 = w_\epsilon$. Clearly, $e_0, e_1, \cdots, e_n$ are linearly independent. Denote V^ϵ the $n+1$ dimensional vector space spanned by $e_0, \cdots, e_n$ and let $\tilde{w}_\epsilon \in V^\epsilon \cap \mathcal{M}$. We claim

$$\lim_{\epsilon \to 0} \|w_\epsilon - \tilde{w}_\epsilon\|_{K,2,\Omega} = 0.$$

For this purpose, fix some small number $r > 0$. For all $(\gamma_0, \cdots, \gamma_n) \in \mathbb{R}^{n+1}$ with $\sum_{i=0}^n \gamma_i^2 = r^2$, with the same arguments as above, we have the following expansions:

$$\begin{aligned}
\int_\Omega F(x, w_\epsilon + \sum_{i=0}^n \gamma_i e_i) &= \int_\Omega F(x, \sum_{i=1}^n \gamma_i e_i) + o(1), \\
\|w_\epsilon + \sum_{i=0}^n \gamma_i e_i\|_{K,2,\Omega}^2 &= (1+\gamma_0)^2 \|w_\epsilon\|_{K,2,\Omega}^2 + \|\sum_{i=1}^n \gamma_i e_i\|_{K,2,\Omega}^2 + o(1), \\
\int_\Omega |w_\epsilon + \sum_{i=0}^n \gamma_i e_i|^s &= |1+\gamma_0|^s \int_\Omega |w_\epsilon|^s + \int_\Omega |\sum_{i=1}^n \gamma_i e_i|^s + o(1)
\end{aligned}$$

where $o(1)$ tends to 0 uniformly with respect to $(\gamma_0, \cdots, \gamma_n)$. As a consequence, we infer

$$\begin{aligned}
\mathcal{E}(w_\epsilon + \textstyle\sum_{i=0}^n \gamma_i e_i) &\le \frac{1}{2}(1+\gamma_0)^2 \|w_\epsilon\|_{K,2,\Omega}^2 - \frac{1}{s}|1+\gamma_0|^s \|w_\epsilon\|_{L^s(\Omega)}^s \\
&\quad + \frac{1}{2}\|\sum_{i=1}^n \gamma_i e_i\|_{K,2,\Omega}^2 - \frac{1}{s}\|\sum_{i=1}^n \gamma_i e_i\|_{L^s(\Omega)}^s \\
&\quad - \frac{1}{2}\int_\Omega a(x)(\sum_{i=1}^n \gamma_i e_i)^2 + o(1) \qquad (27) \\
&\le \frac{1}{2}(1+\gamma_0)^2 \|w_\epsilon\|_{K,2,\Omega}^2 - \frac{1}{s}|1+\gamma_0|^s \|w_\epsilon\|_{L^s(\Omega)}^s \\
&\quad + \frac{1}{2}\sum_{i=1}^n \gamma_i^2 \lambda_i(\Omega) - \frac{1}{s}\|\sum_{i=1}^n \gamma_i e_i\|_{L^s(\Omega)}^s + o(1),
\end{aligned}$$

since $F(x,u) \ge \frac{1}{2}a(x)u^2$ for a.e. $x \in \Omega$. Gathering (24) and (27), we deduce

$$\mathcal{E}(w_\epsilon + \sum_{i=0}^n \gamma_i e_i) < \mathcal{E}(w_\epsilon)$$

provided ϵ is sufficiently small. On the other hand, $\mathcal{E}(\tilde{w}_\epsilon) = \sup_{v \in V^\epsilon} \mathcal{E}(v)$. Hence, we have $w_\epsilon - \tilde{w}_\epsilon = \sum_{i=0}^n \gamma_i e_i$ with $\Gamma = (\gamma_0, \cdots, \gamma_n) \in \mathbb{R}^{n+1}$ satisfying $|\Gamma|^2 = \sum_{i=0}^n \gamma_i^2 < r^2$, that is, the claim is proved. Now, applying (24) and (27), we infer

$$\lim_{\epsilon \to 0} \mathcal{E}(\tilde{w}_\epsilon) = \lim_{\epsilon \to 0} \mathcal{E}(w_\epsilon) = \frac{K}{N}(S_K)^{\frac{N}{2K}}.$$

This yields the desired result. ■

The following lemma concerns the linear perturbation problem for the non critical dimensions case.

Lemma 3 *We suppose $N \ge 4K$ and $f(x,u) = \mu u$ for some $\mu > 0$. Then we have*

$$\kappa < \frac{K}{N}(S_K)^{\frac{N}{2K}}. \qquad (28)$$

Proof. We keep the same notations as in the proof of Lemma 2. Direct calculations lead to

$$\|w_\epsilon\|_{L^{s-1}}^{s-1} = O(\epsilon^{(N-2K)/2}), \quad \|w_\epsilon\|_{L^1} = O(\epsilon^{(N-2K)/2}) \quad \text{and} \quad \|w_\epsilon\|_{L^2}^2 \ge c_1 \epsilon^{2K} \qquad (29)$$

for some positive constant $c_1 > 0$. When $N = 4K$, c_1 could be any large constant as wanted. We have also for any $i = 1, \cdots, n$

$$(w_\epsilon, e_i)_\Omega = \int_\Omega w_\epsilon (-\Delta)^K e_i = O(\epsilon^{(N-2K)/2}). \tag{30}$$

Together with (24), we obtain, since $\lim_{\epsilon \to 0} \Gamma = 0$,

$$\begin{aligned} \|\tilde{w}_\epsilon\|^2_{K,2,\Omega} &= (1+\gamma_0)^2 (S_K)^{\frac{N}{2K}} + \|\textstyle\sum_{i=1}^n \gamma_i e_i\|^2_{K,2,\Omega} + o(\epsilon^{(N-2K)/2}) \\ \|\tilde{w}_\epsilon\|^2_{L^2(\Omega)} &\geq c_1 (1+\gamma_0)^2 \epsilon^{2K} + \|\textstyle\sum_{i=1}^n \gamma_i e_i\|^2_{L^2(\Omega)} + o(\epsilon^{(N-2K)/2}) \end{aligned} \tag{31}$$

On the other hand, using the fact that function $|\cdot|^s$ is convex on $\mathbb{R}$, we have

$$\begin{aligned} \|\tilde{w}_\epsilon\|^s_{L^s(\Omega)} &\geq \int_\Omega (1+\gamma_0)^s |e_0|^s + \int_\Omega s(1+\gamma_0)^{s-1} |e_0|^{s-2} e_0 \sum_{i=1}^n \gamma_i e_i \\ &\geq (1+\gamma_0)^s (S_K)^{\frac{N}{2K}} + o(\epsilon^{(N-2K)/2}). \end{aligned} \tag{32}$$

Gathering (31) and (32), there holds

$$\begin{aligned} \mathcal{E}(\tilde{w}_\epsilon) &\leq \frac{1}{2}(1+\gamma_0)^2 (S_K)^{\frac{N}{2K}} - \frac{1}{s}(1+\gamma_0)^s (S_K)^{\frac{N}{2K}} + \frac{1}{2}\|\sum_{i=1}^n \gamma_i e_i\|^2_{K,2,\Omega} \\ &\quad -\tfrac{\mu}{2}\|\textstyle\sum_{i=1}^n \gamma_i e_i\|^2_{L^2(\Omega)} - \frac{c_1 \mu}{2}(1+\gamma_0)^2 \epsilon^{2K} + o(\epsilon^{(N-2K)/2}) \\ &\leq \frac{K}{N}(S_K)^{\frac{N}{2K}} - \frac{c_1 \mu}{2}(1+\gamma_0)^2 \epsilon^{2K} + o(\epsilon^{(N-2K)/2}), \end{aligned}$$

since

$$\frac{1}{2}\|\sum_{i=1}^n \gamma_i e_i\|^2_{K,2,\Omega} - \frac{\mu}{2}\|\sum_{i=1}^n \gamma_i e_i\|^2_{L^2(\Omega)} \leq 0$$

and

$$\sup_{t>0}\left(\frac{1}{2}t^2 (S_K)^{\frac{N}{2K}} - \frac{1}{s}t^s (S_K)^{\frac{N}{2K}}\right) = \frac{K}{N}(S_K)^{\frac{N}{2K}}.$$

Finally,

$$\mathcal{E}(\tilde{w}_\epsilon) < \frac{K}{N}(S_K)^{\frac{N}{2K}}$$

provided ϵ is sufficiently small, which yields the desired result. ■

Now we state our main result of this section.

Theorem 1 *Suppose (H1) to (H5) and (28) are satisfied. Then there exists $u \in \mathcal{M}$ such that $\mathcal{E}(u) = \kappa$ and u is a solution to (1).*

Proof. The strategy of the proof is standard. Let $(u_k) \subset \mathcal{M}$ be a minimizing sequence for $\mathcal{E}$. We prove first that (u_k) is bounded and then we can extract a subsequence, if necessary, which converges to some limit u. We prove then $u \neq 0$, $u \in \mathcal{M}$ and u is a minimizer for κ.

Step 1. *(u_k) is a bounded sequence in $H_0^K(\Omega)$.*

Recall that (u_k) satisfies (4) and

$$\frac{1}{2}\|u_k\|^2_{K,2,\Omega} - \frac{1}{s}\|u_k\|^s_{L^s(\Omega)} - \int_\Omega F(x,u_k) = \kappa + o(1) \tag{33}$$

so that

$$\frac{K}{N}\|u_k\|^s_{L^s(\Omega)} + \int_\Omega \left(\frac{f(x,u_k)u_k}{2} - F(x,u_k)\right) = \kappa + o(1). \tag{34}$$

From (H5), for a.e $x \in \Omega$ and $\forall u \in \mathbb{R}$, we have

$$F(x,u) \leq \frac{1}{2} f(x,u)u,$$

which in turn (34) implies

$$\|u_k\|^s_{L^s(\Omega)} \leq \frac{N}{K}\kappa + o(1). \tag{35}$$

We infer from (H2) that for a.e $x \in \Omega$ and $\forall u \in \mathbb{R}$, we have also

$$F(x,u) \leq \frac{1}{2}a(x)u^2 + \frac{2\epsilon|u|^s}{s} + C, \tag{36}$$

thus

$$\int_\Omega F(x,u_k) \leq \frac{2\epsilon}{s}\|u_k\|^s_{L^s(\Omega)} + \frac{1}{2}\int_\Omega a(x)u_k^2 + C.$$

Together with (33) and (35),

$$\begin{aligned} \|u_k\|^2_{K,2,\Omega} &= \frac{2}{s}\|u_k\|^s_{L^s(\Omega)} + 2\int_\Omega F(x,u_k) + 2\kappa + o(1) \\ &\leq C(\|u_k\|^s_{L^s(\Omega)} + \|u_k\|^2_{L^s(\Omega)}) + C + 2\kappa + o(1) \\ &\leq C. \end{aligned}$$

Hence Step 1 is proved.

Extracting a subsequence, there exists some $u \in H^K_0(\Omega)$ such that

$$\begin{aligned} &u_k \rightharpoonup u \quad \text{weakly in } H^K_0(\Omega), \\ &u_k \rightharpoonup u \quad \text{weakly in } L^s(\Omega), \text{ strongly in } L^q(\Omega) \quad (\forall q < s) \text{ and a.e. on } \Omega, \end{aligned}$$

so that

$$l_i(u) = 0 \qquad \forall 1 \leq i \leq n. \tag{37}$$

Setting $v_k = u_k - u$, we have

$$\begin{aligned} \|u_k\|^2_{K,2,\Omega} &= \|v_k\|^2_{K,2,\Omega} + \|u\|^2_{K,2,\Omega} + o(1) \\ \|u_k\|^s_{L^s(\Omega)} &= \|u\|^s_{L^s(\Omega)} + \|v_k\|^s_{L^s(\Omega)} + o(1). \end{aligned} \tag{38}$$

Step 2. *We have $u \neq 0$.*

Suppose by contradiction that $u = 0$. As in the proof of Lemma 2, we have

$$f(x,u_k) \to 0 \qquad \text{in } L^{\frac{s}{s-1}}(\Omega) \tag{39}$$

and

$$F(x, u_k) \to 0 \qquad \text{in } L^1(\Omega). \tag{40}$$

Combining (4), (33), (39) and (40), we deduce

$$\|u_k\|_{L^s(\Omega)}^s = \frac{N}{K}\kappa + o(1), \qquad \|u_k\|_{K,2,\Omega}^2 = \frac{N}{K}\kappa + o(1)$$

which yields

$$\frac{\|u_k\|_{K,2,\Omega}^2}{\|u_k\|_{L^s(\Omega)}^2} = \left(\frac{N}{K}\kappa\right)^{\frac{s-2}{s}} + o(1) < S_K \quad \text{for sufficiently large } k.$$

This contradiction gives $u \neq 0$. Consequently, we have $u \notin V_0$ because of (37).

Step 3. *We have $u \in \mathcal{M}$ and $\mathcal{E}(u) = \kappa$.*

We need to prove $l_0(u) = 0$ to conclude that $u \in \mathcal{M}$ and $\mathcal{E}(u) = \kappa$. So we should exclude two cases: (i) $l_0(u) < 0$ and (ii) $l_0(u) > 0$. First we suppose that the case (i) occurs. In this case there exists $t \in (0, 1)$ such that $u(t) \in \mathcal{M}$ because of the Step 3 of Lemma1. Set $v_k := u_k - u$ as before and $\tilde{u}_k := tu_k + u(t) - tu = tv_k + u(t)$. We define for all $w \in H_0^K(\Omega)$,

$$\mathcal{E}_\infty(w) := \frac{1}{2}\|w\|_{K,2,\Omega}^2 - \frac{1}{s}\int_\Omega |w|^s.$$

As $v_k \rightharpoonup 0$ weakly in $H_0^K(\Omega)$, we obtain

$$\mathcal{E}(\tilde{u}_k) = \mathcal{E}_\infty(tv_k) + \mathcal{E}(u(t)) + o(1).$$

Suppose $\mathcal{E}(u(t)) > \kappa$, otherwise $\mathcal{E}(u(t)) = \kappa$ and then we finish the proof. By Lemma 1 and the fact $\tilde{u}_k - tu_k \in V_0$, we have

$$\mathcal{E}(\tilde{u}_k) \le \mathcal{E}(u_k) = \kappa + o(1)$$

which implies $\mathcal{E}_\infty(tv_k) < 0$ for sufficiently large k. In particular, $v_k \neq 0$. Consequently, for sufficiently large k,

$$\|tv_k\|_{L^s(\Omega)}^s > \frac{s}{2}\|tv_k\|_{K,2,\Omega}^2 \ge \frac{s}{2}S_K\|tv_k\|_{L^s(\Omega)}^2 > S_K\|tv_k\|_{L^s(\Omega)}^2 \tag{41}$$

so that

$$\|v_k\|_{L^s(\Omega)}^s > (S_K)^{\frac{N}{2K}}. \tag{42}$$

On the other hand, we have

$$\|v_k\|_{L^s(\Omega)}^s = \|u_k\|_{L^s(\Omega)}^s - \|u\|_{L^s(\Omega)}^s + o(1) \le \frac{N}{K}\kappa - \|u\|_{L^s(\Omega)}^s + o(1), \tag{43}$$

which contradicts (42) by using Lemma 2. Thus case (i) is impossible.

Now we treat the case (ii). By the same arguments in the Step 2, we have

$$\int_\Omega f(x, u_k)u_k = \int_\Omega f(x, u)u + o(1), \tag{44}$$

$$\int_\Omega F(x,u_k) = \int_\Omega F(x,u) + o(1). \tag{45}$$

Thus, according to (34), (44) and (45), we have

$$\|v_k\|_{L^s(\Omega)}^s = \frac{N}{K}\kappa + \frac{N}{K}\int_\Omega (F(x,u) - \frac{1}{2}f(x,u)u) - \|u\|_{L^s(\Omega)}^s + o(1). \tag{46}$$

Similarly, we have

$$\|v_k\|_{K,2,\Omega}^2 = \frac{N}{K}\kappa + \frac{N}{K}\int_\Omega F(x,u) + (1 - \frac{N}{2K})\int_\Omega f(x,u)u - \|u\|_{K,2,\Omega}^2 + o(1). \tag{47}$$

Combining (45), (47), we see that $l_0(u) > 0$ implies for sufficiently large k

$$\|v_k\|_{L^s(\Omega)}^s > \|v_k\|_{K,2,\Omega}^2.$$

Consequently, by the definition of S_K, we obtain $\|v_k\|_{L^s(\Omega)}^s > (S_K)^{\frac{N}{2K}}$ for sufficiently large k. This is (42) and as before, we conclude that (ii) does not occur and thus $u \in \mathcal{M}$. Moreover

$$\mathcal{E}(u_k) = \mathcal{E}(u) + \mathcal{E}_\infty(v_k) + o(1) \qquad \text{and} \qquad \|v_k\|_{L^s(\Omega)}^s = \|v_k\|_{K,2,\Omega}^2 + o(1).$$

Thus

$$\mathcal{E}(u) = \mathcal{E}(u_k) - \frac{K}{N}\|v_k\|_{K,2,\Omega}^2 + o(1).$$

Finally, we deduce $\|v_k\|_{K,2,\Omega}^2 = o(1)$ and therefore $\mathcal{E}(u) = \kappa$.

Step 4. *u is a solution to (1).*

In fact u is a critical point of $\mathcal{E}$ on $\mathcal{M}$. By the method of Lagrange multipliers, there exists $\mu, \mu_1, \cdots, \mu_n \in \mathbb{R}$ such that

$$d\mathcal{E}(u) + \mu dl_0(u) + \sum_{i=1}^n \mu_i dl_i(u) = 0.$$

We consider its restriction on V, this means

$$\left(\mu dl_0(u) + \sum_{i=1}^n \mu_i dl_i(u)\right)|_V = 0$$

since $d\mathcal{E}(u)|_V = 0$. On the other hand, we have seen from Proposition 1 that $dL(u)|_V$ is an isomorphism from V on $\mathbb{R}^{n+1}$. Consequently, $\mu = \mu_1 = \cdots = \mu_n = 0$, that is, $d\mathcal{E}(u) = 0$. Finally, u solves the problem (1) which finishes the proof. ∎

3 Existence of solutions for some perforated domains

In this section, we analyze first the concentration phenomenon for the problem (1). For this purpose, set

$$F_K(v) := \begin{cases} ((-\Delta)^M v)^2 & \text{if } K = 2M \\ |\nabla(-\Delta)^M v|^2 & \text{if } K = 2M+1. \end{cases}$$

Similarly to Theorem 6 of [21], we have the following theorem and here we just give a sketch of the proof.

Theorem 2 *Suppose the assumptions (H1) to (H5) are satisfied. Moreover, suppose that*

$$\kappa = \frac{K}{N}(S_K)^{\frac{N}{2K}} \tag{48}$$

and

$$\mathcal{E}(v) > \kappa, \qquad \forall v \in \mathcal{M}. \tag{49}$$

Let $(u_k) \subset \mathcal{M}$ be a minimizing sequence for κ, that is, $\lim_{n\to\infty} \mathcal{E}(u_k) = \kappa$. Then there exists $x_0 \in \overline{\Omega}$ such that

$$\mu_k := \zeta_\Omega \, F_K(u_k) \, dx \rightharpoonup S_K \delta_{x_0} \text{ weakly in } \mathcal{R}(\mathbb{R}^N)$$

and

$$\nu_k := \zeta_\Omega \, |u_k|^s \, dx \rightharpoonup S_K \delta_{x_0} \text{ weakly in } \mathcal{R}(\mathbb{R}^N),$$

where $\mathcal{R}(\mathbb{R}^N)$ denotes the space of non-negative Radon measures on $\mathbb{R}^N$ with finite mass, δ_{x_0} denotes the Dirac measure concentrated at x_0 with mass equal to 1 and ζ_Ω designates the characteristic function of Ω.

Proof. As in the proof of Theorem 1, we see that (u_k) is bounded in $H_0^K(\Omega)$. Extracting a subsequence, there exists some $u \in H_0^K(\Omega)$ such that

$$\begin{array}{ll} u_k \rightharpoonup u & \text{weakly in } H_0^K(\Omega), \\ u_k \rightharpoonup u & \text{weakly in } L^s(\Omega) \text{ and a.e. on } \Omega. \end{array}$$

Moreover, for all $1 \le j \le n$, we have $l_j(u) = 0$. Furthermore, we have $u = 0$. Otherwise, with the same arguments as in Theorem 1, we infer $u \in \mathcal{M}$ and $\mathcal{E}(u) = \kappa$ which contradicts (49). Now the rest of proof is just a consequence of concentration compactness principle (for details cf [28, 19, 21]). ■

In the following, we give some classification result. First we recall a basic fact for non-existence result on the half space $\mathbb{R}^N_+$. It can be stated as follows:

Lemma 4 *Let $u \in \mathcal{D}^{K,2}(\mathbb{R}^N_+)$ be a weak positive solution of the problem*

$$\begin{cases} (-\Delta)^K u = |u|^{s-2} u & \text{in } \mathbb{R}^N_+ \\ u = Du = \cdots = D^{K-1} u = 0 & \text{on } \partial\mathbb{R}^N_+. \end{cases} \tag{50}$$

Then $u \equiv 0$.

A stronger result have been obtained by Reichel and Weth in[36] very recently. Here we give a proof based on the Pohozaev formula (see [29]).

Proof. It follows from the Pohozaev formula $D^K u = 0$ on $\partial\mathbb{R}^N_+$ (see the details cf [21] for the Navier boundary conditions). Now, $(-\Delta)^{K-1}((-\Delta)u) = u^s > 0$ in $\mathbb{R}^N_+$ verifying Dirichlet boundary condition $(-\Delta)u = \cdots = D^{K-2}(-\Delta)u = 0$ on $\partial\mathbb{R}^N_+$. Thanks to the Boggio's result, we know the Green function for the operator $(-\Delta)^{K-1}$ on the half space with Dirichlet boundary condition is positive. Thus, $(-\Delta)u > 0$ in $\mathbb{R}^N_+$. From Hopf's Maximum principle, $\frac{\partial u}{\partial n} > 0$ on $\partial\mathbb{R}^N_+$. This contradiction finishes the proof of Lemma. ■

A similar problem in the whole space can be stated as follows:

Lemma 5 *Let $u \in \mathcal{D}^{K,2}(\mathbb{R}^N)$ be a weak positive solution of the problem*

$$(-\Delta)^K u = |u|^{s-2}u \quad \text{in } \mathbb{R}^N. \tag{51}$$

Then there exists a constant $\lambda \geq 0$ and a point $x_0 \in \mathbb{R}^N$ such that

$$u(x) = \left(\frac{2\lambda}{1+\lambda^2|x-x_0|^2}\right)^{\frac{N-2K}{2}}. \tag{52}$$

This result has been proved by Wei-Xu (Theorem 1.3 in [44]).

Lemma 6 *Let $u \in \mathcal{D}^{K,2}(\mathbb{R}^N_+)$ (resp. $u \in \mathcal{D}^{K,2}(\mathbb{R}^N)$) be a weak sign changing solution of the problem (50) (resp. (51)). Then*

$$E_\infty(u) \geq \frac{2K}{N}(S_K)^{\frac{N}{2K}}. \tag{53}$$

Proof. Our proof is an adaptation of Gazzola-Grunau-Squassina's approach [18]. We consider the closed convex cone

$$\mathcal{C}_1 = \{v \in \mathcal{D}^{K,2}(\mathbb{R}^N_+) \mid \quad v \geq 0 \text{ a.e. in } \mathbb{R}^N_+\}$$

and its dual cone

$$\mathcal{C}_2 = \{w \in \mathcal{D}^{K,2}(\mathbb{R}^N_+) \mid \quad (w,v)_{\mathbb{R}^N_+} \leq 0 \quad \forall v \in \mathcal{C}_1\}.$$

We claim that $\mathcal{C}_2 \subset -\mathcal{C}_1$. Given $h \in C_0^\infty(\mathbb{R}^N_+) \cap \mathcal{C}_1$, let v be the solution to the problem

$$(-\Delta)^K v = h \qquad \text{in} \quad \mathbb{R}^N_+.$$

Again from the Boggio's result, we have $v \geq 0$ since the Green function for the operator $(-\Delta)^K$ on the half space with Dirichlet boundary condition is positive. Consequently, for all $w \in \mathcal{C}_2$, we have

$$\int_{\mathbb{R}^N_+} hw = \int_{\mathbb{R}^N_+} (-\Delta)^K v w = (v,w)_{\mathbb{R}^N_+} \leq 0.$$

This implies $w \leq 0$ a.e. in $\mathbb{R}^N_+$. Hence the claim is proved. Using a result of Moreau [30], for any $u \in \mathcal{D}^{K,2}(\mathbb{R}^N_+)$, there exists an unique pair $(u_1, u_2) \in \mathcal{C}_1 \times \mathcal{C}_2$ such that

$$u = u_1 + u_2 \quad \text{with} \quad (u_1, u_2)_{\mathbb{R}^N_+} = 0.$$

Now let u be a sign-changing solution of the problem (50). Then $u_i \neq 0$ for all $\;= 1, 2$. From the above claim, we see $u_1 \geq 0$ and $u_2 \leq 0$ so that $|u(x)|^{s-2}u(x)u_i(x) \leq |u_i(x)|^s$ for $i = 1, 2$. Applying the Sobolev inequality for u_i $(i = 1, 2)$, we obtain

$$S_K\|u_i\|^2_{L^s} \leq \|u_i\|^2_{K,2,\mathbb{R}^N_+} = (u, u_i)_{\mathbb{R}^N_+} = \int_{\mathbb{R}^N_+}(-\Delta)^K uu_i \leq \int_{\mathbb{R}^N_+}|u_i(x)|^s = \|u_i\|^s_{L^s}$$

so that

$$\|u_i\|^s_{L^s(\Omega)} \geq (S_K)^{\frac{N}{2K}}.$$

Consequently, using the fact $\|u\|^2_{K,2,\mathbb{R}^N_+} = \|u\|^s_{L^s}$, we infer

$$\begin{aligned}\mathcal{E}_\infty(u) &= \frac{K}{N}\|u\|^2_{K,2,\mathbb{R}^N_+} = \frac{K}{N}(\|u_1\|^2_{K,2,\mathbb{R}^N_+} + \|u_2\|^2_{K,2,\mathbb{R}^N_+}) \\ &\geq \frac{K}{N}S_K(\|u_1\|^2_{L^s} + \|u_2\|^2_{L^s}) \geq \frac{2K}{N}(S_K)^{\frac{N}{2K}}.\end{aligned}$$

Similarly, we have the same result for $u \in \mathcal{D}^{K,2}(\mathbb{R}^N)$. ∎

Theorem 3 *Assume (H1), (H2), (H5), (48) and (49) are satisfied. Let $(u_k) \subset H^K_0(\Omega)$ be a (P.S.)$_\beta$ sequence such that*

$$\mathcal{E}(u_k) \to \beta \in (\frac{K}{N}(S_K)^{\frac{N}{2K}}, \frac{2K}{N}(S_K)^{\frac{N}{2K}}) \tag{54}$$

$$d\mathcal{E}(u_k) \to 0 \quad in\ (H^K_0(\Omega))^*. \tag{55}$$

Then (u_k) is precompact in $H^K_0(\Omega)$.

Proof. The blow up analysis for (P.S.)$_\beta$ sequences is more or less standard. Its proof follows from the P. Lions' concentration compactness principle and it is close to one in [21]. The only difference is that we need Lemma 6 to rule out sign changing bubbles. We leave this part to interested readers. ∎

As a consequence, we have

Corollary 1 *Under the assumptions (H1) to (H5), (48) and (49), assume moreover*

(H8) $e_n(\Omega) < 0$.

Let $(u_k) \subset \mathcal{M}$ be a (P.S.)$_\beta$ sequence for $\mathcal{E}$ on $\mathcal{M}$ such that

$$\mathcal{E}(u_k) \to \beta \in (\frac{K}{N}(S_K)^{\frac{N}{2K}}, \frac{2K}{N}(S_K)^{\frac{N}{2K}}), \tag{56}$$

$$\|d\mathcal{E}(u_k)\|_{(T_{u_k}\mathcal{M})^*} \to 0. \tag{57}$$

Then (u_k) is precompact in $\mathcal{M}$.

Proof. As in the proof of Theorem 1, (u_k) is a bounded sequence in $H_0^K(\Omega)$. On the other hand, using (33), (34) and (H2), we infer that(u_k) is bounded from below by some positive constant in $H_0^K(\Omega)$ and also in $L^s(\Omega)$. Set V^k the $n+1$ dimensional vector space spanned by $u_k, e_1, \cdots, e_n$. If there is no confusion, we drop the index k. We claim there exists some positive constant $c > 0$ independent of k such that $\forall k \in \mathbb{N}$, $\forall w \in H_0^K(\Omega)$, we can decompose

$$w = w_1 + w_2 \tag{58}$$

where $w_1 \in V^k$ and $w_2 \in T_{u_k}\mathcal{M}$ satisfying

$$\|w_1\|_{K,2,\Omega} \leq c\|w\|_{K,2,\Omega}, \qquad \|w_2\|_{K,2,\Omega} \leq c\|w\|_{K,2,\Omega}.$$

Set $e_0 = u_k$ and $\theta_i = dl_i(u_k)(w) \in \mathbb{R}$ for all $i = 0, \cdots, n$. Using (13) and the fact that (u_k) is a bounded sequence in $H_0^K(\Omega)$, the vector $\Theta = (\theta_0, \cdots, \theta_n)^T$ is bounded in $\mathbb{R}^{n+1}$ with respect to k. Moreover, we can estimate

$$|\Theta| \leq c\|w\|_{K,2,\Omega}.$$

Define $(n+1) \times (n+1)$ symmetric matrix $M(k) = (m_{ij})_{0 \leq i,j \leq n}$ by

$$m_{ij} = d^2\mathcal{E}(u_k)(e_i, e_j).$$

We write

$$w_1 = \sum_{i=0}^{n} \psi_i e_i$$

where $\psi_i \in \mathbb{R}$. Denote the vector $\Psi = (\psi_0, \cdots, \psi_n)^T \in \mathbb{R}^{n+1}$. Again from (13), the decomposition (58) is equivalent to solve

$$d^2\mathcal{E}(u_k)(w_1, e_i) = dl_i(u_k)(w) \qquad \forall 0 \leq i \leq n,$$

that is, $M(k)\Psi = \Theta$. As in the proof of Lemma 1, the matrix is negative definite. Clearly, the matrix $M(k)$ is uniformly bounded. We show there exists $c > 0$ independent of k such that

$$M(k) \leq -cI$$

where I is the identity matrix. For this purpose, for any vector $\Gamma^T = (\gamma_0, \cdots, \gamma_n) \in \mathbb{R}^{n+1}$, denote $\xi = \sum_{i=0}^n \gamma_i e_i$ we have

$$\begin{aligned}
\Gamma^T M(k)\Gamma &= d^2\mathcal{E}(u_k)(\xi, \xi) \\
&\leq -(s-2)\int_\Omega |u_k|^{s-2}\xi^2 - \int_\Omega |u_k|^{s-2}(\sum_{j=1}^n \gamma_j e_j)^2 \\
&\quad +(\sum_{j=1}^n \gamma_j e_j, \sum_{i=1}^n \gamma_i e_i)_\Omega - \int_\Omega a(x)(\sum_{j=1}^n \gamma_j e_j)^2 \\
&\leq -\frac{s-2}{s-1}\int_\Omega |u_k|^s \gamma_0^2 + \sum_{j=1}^n \gamma_j^2 \lambda_j(\Omega).
\end{aligned}$$

Thus, the desired result follows. As a consequence, $(\Psi = (M(k))^{-1}\Theta)_k$ is a bounded sequence. More precisely, we infer

$$\|w_1\|_{K,2,\Omega} \leq c\|w\|_{K,2,\Omega}.$$

Therefore,

$$\|w_2\|_{K,2,\Omega} \leq (\|w\|_{K,2,\Omega} + \|w_1\|_{K,2,\Omega}) \leq c\|w\|_{K,2,\Omega},$$

that is, the claim is proved. Hence,

$$|d\mathcal{E}(u_k)(w)| \quad = |d\mathcal{E}(u_k)(w_2)| \leq c\|d\mathcal{E}(u_k)\|_{(T_{u_k}\mathcal{M})^*}\|w\|_{K,2,\Omega}.$$

Thus, there holds

$$\|d\mathcal{E}(u_k)\|_{(H_0^K(\Omega))^*} \leq c\|d\mathcal{E}(u_k)\|_{(T_{u_k}\mathcal{M})^*}$$

so that

$$\lim_{n\to\infty} \|d\mathcal{E}(u_k)\|_{(H_0^K(\Omega))^*} = 0.$$

Finally, applying Theorem 3, we finish the proof. ∎

Now, we can prove the main result for domains with the small holes. Recall that $\Omega = \Omega_1 \setminus \Omega_2$ is a bounded domain satisfying $\Omega_2 \subset B(0,\epsilon)$ and Ω_1 is fixed. To search solutions of (1) in such Ω, we minimize the energy functional $\mathcal{E}$ on the Finsler manifold $\mathcal{M}$. We see that the concentration phenomenon occurs if $\mathcal{E}$ can not reach the minimum. In this case, we will employ Coron's strategy to search unstable critical points in higher level sets.

Theorem 4 *Let Ω be a bounded domain satisfying the above assumption. Assume (H1) to (H7) hold. Then there exists $\eta > 0$ such that for all $\epsilon < \eta$, the problem (1) admits a non trivial solution in Ω.*

Proof. Thanks to Lemma 2, we have $\kappa \leq \frac{K}{N}(S_K)^{\frac{N}{2K}}$. In the case $\kappa < \frac{K}{N}(S_K)^{\frac{N}{2K}}$, the desired result follows from Theorem 1. So we suppose $\kappa = \frac{K}{N}(S_K)^{\frac{N}{2K}}$. If there exists $u \in \mathcal{M}$ such that $\mathcal{E}(u) = \kappa$, we finish the proof by Step 4 in the proof of Theorem 1. Hence, we assume $\forall v \in \mathcal{M}$ there holds $\mathcal{E}(v) > \kappa$. From the properties of eigenvalues $\lambda_i(\Omega)$ described in the previous sections, (H8) is always satisfied for the perforated domain Ω, provided ϵ is sufficiently small. In fact, in case $\lambda_i(\Omega_1) \neq 0$ for all $i \in \mathbb{N}$, it follows from the continuity of $\lambda_i(\Omega)$. In the case $\lambda_n(\Omega_1) = \cdots = \lambda_{n+k}(\Omega_1) = 0$, we have $\lambda_n(\Omega) > 0$.

We devide the proof into several steps.

Step 1. We choose a radially symmetric function $\varphi \in C_0^\infty(\mathbb{R}^N)$ such that $0 \leq \varphi \leq 1$, $\varphi \equiv 1$ on the annulus $\{x \in \mathbb{R}^N | \quad 1/2 < |x| < 1\}$ and $\varphi \equiv 0$ outside the annulus $\{x \in \mathbb{R}^N | \quad 1/4 < |x| < 2\}$. For any $R \geq 1$, define

$$\varphi_R(x) = \begin{cases} \varphi(Rx) & \text{if} \quad 0 \leq |x| < 1/R \\ 1 & \text{if} \quad 1/R \leq |x| < R \\ \varphi(x/R) & \text{if} \quad |x| \geq R. \end{cases}$$

Denote the unit sphere $S^{N-1} = \{x \in \mathbb{R}^N | \quad |x| = 1\}$. For $\sigma \in S^{N-1}$, $0 \leq t < 1$, we set

$$u_t^\sigma(x) = C_{N,K}\left[\frac{1-t}{(1-t)^2 + |x - t\sigma|^2}\right]^{\frac{N-2K}{2}} \in H^K(\mathbb{R}^N),$$

where the choice of $C_{N,K}$ is such that $\|u_t^\sigma\|^2_{K,2,\mathbb{R}^N} = \|u_t^\sigma\|^s_{L^s(\mathbb{R}^N)} = (S_K)^{\frac{N}{2K}}$. Let $\tilde{w}^\sigma_{t,R}(x) = u_t^\sigma(x)\varphi_R(x)$ and $w^\sigma_{t,R}(x) = (4R)^{\frac{N-2K}{2}}\tilde{w}^\sigma_{t,R}(4Rx)$. Hence $w^\sigma_{t,R} \in H_0^K(B(0,1/2)\backslash B(0,1/16R^2))$, $\forall \sigma \in S^{N-1}$ and $\forall t \in [0,1)$. Clearly,

$$\|\tilde{w}^\sigma_{t,R}\|_{L^s(\mathbb{R}^N)} = \|w^\sigma_{t,R}\|_{L^s(\mathbb{R}^N)} \tag{59}$$

$$\|\tilde{w}^\sigma_{t,R}\|_{K,2,\mathbb{R}^N} = \|w^\sigma_{t,R}\|_{K,2,\mathbb{R}^N}. \tag{60}$$

A direct computation leads to $\forall R > 1$

$$\|\tilde{w}^\sigma_{t,R} - u_t^\sigma\|^2_{K,2,\mathbb{R}^N} \leq C(1-t)^{N-2K}R^{2K-N} \tag{61}$$

and

$$\|\tilde{w}^\sigma_{t,R} - u_t^\sigma\|^s_{L^s(\mathbb{R}^N)} \leq CR^{-N}(1-t)^N. \tag{62}$$

Consequently

$$\lim_{R\to\infty}\|\tilde{w}^\sigma_{t,R}\|^2_{K,2,\mathbb{R}^N} = \lim_{R\to\infty}\|\tilde{w}^\sigma_{t,R}\|^s_{L^s(\mathbb{R}^N)} = (S_K)^{\frac{N}{2K}}$$

uniformly for $t \in [0,1)$ and $\sigma \in S^{N-1}$. Set $\bar{w}^\sigma_{t,R} \in \mathcal{M} \cap \mathrm{Vect}\{e_1(\Omega), \cdots, e_n(\Omega), w^\sigma_{t,R}\}$ where $\Omega = \Omega_1 \setminus \Omega_2$, $B(0,1/2) \subset \Omega_1$ and $\Omega_2 \subset B(0,1/16R^2)$. Thanks to the Implicit Function Theorem, the continuous map

$$\begin{array}{rlcl} w_R: & S^{N-1}\times[0,1) & \to & H_0^K(\Omega) \\ & (\sigma,t) & \mapsto & w^\sigma_{t,R} \end{array}$$

yields a continuous map

$$\begin{array}{rlcl} \bar{w}_R: & S^{N-1}\times[0,1) & \to & \mathcal{M} \\ & (\sigma,t) & \mapsto & \bar{w}^\sigma_{t,R}. \end{array}$$

Recall Ω_1 is fixed. A basic observation is that $e_i(\Omega) \to e_i(\Omega_1)$ for all $i = 1, \cdots, n$ in $C^\infty_{loc}(\Omega_1 \setminus \{0\})$ away from 0 and strongly in $H_0^K(\Omega_1)$ as $R \to +\infty$. We remark that

$$\mathcal{E}(u) \leq \frac{1}{2}\|u\|^2_{K,2,\Omega} - \frac{1}{s}\|u\|^s_{L^s(\Omega)} - \frac{1}{2}\int_\Omega a(x)u^2.$$

In the following, we consider the simple case $F(x,u) = \frac{1}{2}a(x)u^2$ (we can treat the general case with the same argumens). Fix some small number $r > 0$. As in the proof of Lemma 2, for all $\Gamma = (\gamma_0, \cdots, \gamma_n) \in \mathbb{R}^{n+1}$ with $\sum_{i=0}^n \gamma_i^2 \leq r^2$, we infer

$$\begin{aligned} \sup_{t,\sigma,\Omega_2} \mathcal{E}(w^\sigma_{t,R} + \sum_{i=0}^n \gamma_i e_i) \quad \leq \quad & \frac{1}{2}(1+\gamma_0)^2(S_K)^{\frac{N}{2K}} - \frac{1}{s}|1+\gamma_0|^s(S_K)^{\frac{N}{2K}} \\ & + \frac{1}{2}\sum_{i=1}^n \gamma_i^2\lambda_i(\Omega_1) - \frac{1}{s}\|\sum_{i=1}^n \gamma_i e_i(\Omega_1)\|^s_{L^s(\Omega_1)} + o(1), \end{aligned} \tag{63}$$

where $o(1)$ is uniformly with respect to Γ as $R \to \infty$. Consequently, we deduce

$$\sup_{t,\sigma,\Omega_2} \mathcal{E}(w^\sigma_{t,R} + \sum_{i=0}^n \gamma_i e_i) < \mathcal{E}(w^\sigma_{t,R}) \quad \text{for } \sum_{i=0}^n \gamma_i^2 = r^2$$

provided R is sufficiently large. This implies

$$\bar{w}_{t,R}^{\sigma} - w_{t,R}^{\sigma} = \sum_{i=0}^{n} \gamma_i e_i(\Omega) \quad \text{for some } |\Gamma| < r,$$

so that

$$\lim_{R\to\infty} \sup_{t,\sigma,\Omega_2} \mathcal{E}(\bar{w}_{t,R}^{\sigma}) = \frac{K}{N}(S_K)^{\frac{N}{2K}}.$$

Hence, we can choose $R_0 > 0$ such that for any $R \geq R_0$

$$\sup_{t\in[0,1),\ \sigma\in S^{N-1},\ \Omega_2\subset B(0,1/16R^2)} \mathcal{E}(\bar{w}_{t,R}^{\sigma}) < \frac{2K}{N}(S_K)^{\frac{N}{2K}}. \tag{64}$$

Thus we can define a map

$$\begin{array}{rccc} \alpha: & B(0,1) & \to & \mathcal{M} \\ & (t,\sigma) & \mapsto & \bar{w}_{t,R_0}^{\sigma}. \end{array}$$

Step 2. Set $\eta := 1/16R_0^2$ and fix $\Omega_2 \subset B(0,\eta)$. From (59) to (62), we infer that

$$\lim_{t\to 1} \|\bar{w}_{t,R_0}^{\sigma}\|_{K,2,\Omega}^2 = \lim_{t\to 1} \|\bar{w}_{t,R_0}^{\sigma}\|_{L^s(\Omega)}^s = (S_K)^{\frac{N}{2K}} \quad \text{uniformly for } \sigma \in S^{N-1}$$

which implies for any $\sigma \in S^{N-1}$

$$\lim_{t\to 1} \mathcal{E}(\alpha(t,\sigma)) = \frac{K}{N}(S_K)^{\frac{N}{2K}}.$$

Step 3. For any $v \in \mathcal{M}$, let

$$\gamma(v) = \int_{\Omega} x|v(x)|^s dx \in \mathbb{R}^N$$

denotes its center mass. We claim there exists $\tilde{\delta} > 0$ such that for any $v \in \mathcal{M}$ satisfying $\mathcal{E}(v) \leq \frac{K}{N}(S_K)^{\frac{N}{2K}} + \tilde{\delta}$, we have

$$\gamma(v) \in \mathbb{R}^N \setminus B(0, \epsilon_2 (S_K)^{\frac{N}{2K}}/2) \tag{65}$$

where $B(0,\epsilon_2) \subset \Omega_2$. Otherwise, we can find a sequence $(v_n) \subset \mathcal{M}$ satisfying

$$\lim_{n\to\infty} \mathcal{E}(v_n) = \frac{K}{N}(S_K)^{\frac{N}{2K}}, \tag{66}$$

$$\gamma(v_n) \in B(0, \epsilon_2 (S_K)^{\frac{N}{2K}}/2). \tag{67}$$

Applying Theorem 2, there exists $x_0 \in \bar{\Omega}$ such that

$$\zeta_\Omega |v_n(x)|^s dx \rightharpoonup (S_K)^{\frac{N}{2K}} \delta_{x_0}.$$

Consequently,

$$\gamma(v_n) \to (S_K)^{\frac{N}{2K}} x_0 \notin B(0, \epsilon_2 (S_K)^{\frac{N}{2K}})$$

which contradicts (67). Thus, the desired claim yields. Choosing $t_0 \in [0,1)$ such that $\forall \sigma \in S^{N-1}$ and $\forall t \in [t_0, 1)$, we have $\mathcal{E}(\alpha(t,\sigma)) < \frac{K}{N}(S_K)^{\frac{N}{2K}} + \tilde{\delta}$, we set

$$\beta := \min_{f \in H} \max_{(t,\sigma)\in(0,t_0]\times S^{N-1}} \mathcal{E}(f(t,\sigma)),$$

where H is the set of any function homotopic to α on $B(0,t_0)$ with the fixed boundary data, that is,

$$H = \{f| \quad f : \overline{B(0,t_0)} \to \mathcal{M} \text{ is continuous}, f|_{\partial B(0,t_0)} = \alpha|_{\partial B(0,t_0)} \text{ and } f \text{ is homotopic to } \alpha\}.$$

We see that $\forall f \in H$, $\gamma \circ f : \overline{B(0,t_0)} \to \mathbb{R}^N$ is a contraction of the loop $\gamma \circ \alpha|_{\partial B(0,t_0)} \subset \mathbb{R}^N \setminus B(0, \epsilon_2 (S_K)^{\frac{N}{2K}}/2)$. On the other hand, it follows from Steps 1 and 2

$$\lim_{t\to 1} \gamma \circ \alpha(t,\sigma) = (S_K)^{\frac{N}{2K}} \frac{\sigma}{4R_0} \qquad \text{uniformly in } \sigma \in S^{N-1}.$$

Thus, $\gamma \circ \alpha|_{\partial B(0,t_0)}$ is a non-trivial loop in $\mathbb{R}^N \setminus B(0, \epsilon_2 (S_K)^{\frac{N}{2K}}/2)$. Using (67), we obtain

$$\sup_{(t,\sigma)\in\overline{B(0,t_0)}} \mathcal{E}(f(t,\sigma)) \geq \frac{K}{N}(S_K)^{\frac{N}{2K}} + \tilde{\delta},$$

which implies

$$\beta \geq \frac{K}{N}(S_K)^{\frac{N}{2K}} + \tilde{\delta} > \frac{K}{N}(S_K)^{\frac{N}{2K}}.$$

On the other hand, it follows from Step 1

$$\beta \leq \sup_{(t,\sigma)\in\overline{B(0,t_0)}} \mathcal{E}(\alpha(t,\sigma)) < \frac{2K}{N}(S_K)^{\frac{N}{2K}}.$$

Recalling Theorem 1 and Corollary 1 and using the deformation lemma, we infer β is a critical value. Finally, the problem (1) admits a non trivial critical point u such that $E(u) = \beta$. ■

Remark 0 *The condition $a \in L^\infty(\Omega) \cap C^\infty(\Omega)$ could be weakened.*

Remark 1 *We can use the above strategy to treat also the Navier boundary conditions.*

References

[1] A. Ambrosetti and P. Rabinowitz, *Dual variational methods in critical point theory and applications*, J. Funct. Anal. **14**, 349-381 (1973).

[2] T. Aubin, *Equations differentielles non lineaires et probleme de Yamabe concernant la courbure scalaire*, J. Math. Pure Appl. **55**, 269-296 (1976).

[3] T. Bartsch, T. Weth and M. Willem, *A Sobolev inequality with remainder term and critical equations on domanins with topology for the polyharmonic operator*, Calc. Var. and PDE. **18**, 253-268 (2003).

[4] F. Bernis, J. Garcia-Azorero and I. Peral, *Existence and multiplicity of nontrivial solutions in semilinear critical problems of fourth order*, Adv. Diff. Eq. **1**, 219-240 (1996).

[5] H. Brezis and L. Nirenberg, *Positive solutions of nonlinear elliptic equations involving critical Sobolev exponents*, Comm. Pure Appl. Math. **36**, 437-477 (1983).

[6] S-Y A. Chang and P.C. Yang, *Extremal metrics of zeta function determinants on 4-manifolds*, Ann. Math. (2) **142**, No.1, 171-212 (1995).

[7] A. Capozzi, D. Fortunato and G. Palmieri, *An existence results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré, Anal. Non Linéaire. **2**, No.6, 463-470 (1985).

[8] G. Cerami, D. Fortunato and M. Struwe, *Bifurcation and multiplicity results for nonlinear elliptic problems involving critical Sobolev exponents*, Ann. Inst. H. Poincaré, Anal. Non Linéaire **1**, No.5, 341-350 (1984).

[9] M. Clapp and T. Weth, *Multiple solutions for the Brezis-Nirenberg problem*, Adv. Diff. Eq. **10**, 463-480 (2005).

[10] J. M. Coron, *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sc. Paris, **299**, Ser. I, 209-212 (1984).

[11] G. Devillanova and S. Solimini, *Concentration estimates and multiple solutions to elliptic problem at critical growth*, Adv. Diff. Eq. **7**, 1257-1280 (2002).

[12] G. Devillanova and S. Solimini, *A multiplicity result for elliptic equations at critical growth in low dimension*, Comm. Contemp. Math. **5**, 171-177 (2003).

[13] F.Ebobisse and M. Ould Ahmedou, *On a nonlinear fourth order elliptic equation involving the critical Sobolev exponent*, Nonlinear Anal., **52**, No.5, 1535-1552 (2003).

[14] D.E. Edmunds, D. Fortunato and E. Jannelli, *Critical exponents, critical dimensions and the biharmonic operator* Arch. Rat. Mech. Anal. **112**, No.3, 269-289 (1990).

[15] D. Fortunato and E. Jannelli, *Infinitely many solutions for some nonlinear elliptic problems in symmetrical domains*, Proc. Roy. Soc. Edinburgh, **105A**, 205-213 (1987).

[16] F. Gazzola, *Critical growth problems for polyharmonic operators*, Proc. Roy. Soc. Edinb., **128A**, 251-263 (1998).

[17] F. Gazzola and H.C. Grunau, *Critical dimensions and higher order Sobolev inequalities with remainder terms*, NoDEA, **8**, 35-44 (2001).

[18] F. Gazzola, H.C. Grunau and M. Squassina, *Existence and nonexistence results for critical growth biharmonic elliptic equations*, Calc. Var. and PDE., **18**, 117–143 (2003).

[19] Y. Ge, *Estimations of the best constant involving the L^2 norm in Wente's inequality and compact H-surfaces in Euclidean space*, Control, Optimisation and Calculus of Variations, **3**, 263-300 (1998).

[20] Y. Ge, *Sharp Sobolev inequalities in critical dimensions*, Michigan Math. J., **51**, 27-45 (2003).

[21] Y. Ge, *Positive solutions in semilinear critical problems for polyharmonic operators*, J. Math. Pures Appl., **84**, 199-245 (2005).

[22] H.C. Grunau, *Critical exponents and multiple critical dimensions for polyharmonic operators. II*, Boll. Un. Mat. Ital. (7) **9-B**, 815-847 (1995).

[23] H.C. Grunau, *On a conjecture of P.Pucci-J.Serrin*, Analysis, **16**, 399-403 (1996).

[24] H.C. Grunau, *Positive solutions to semilinear polyharmonic Dirichlet problems involving critical Sobolev exponents*, Calc. Var. and PDE., **3**, 243-252 (1995).

[25] E. Hebey and F. Robert, *Coercivity and Struwe's compactness for Paneitz type operators with constant coefficients*, Calc Var. and PDE., **13**, 491-517 (2001).

[26] S. Lang, *Introduction to differentiable manifolds*, Interscience, New York, (1962).

[27] C. Lin, *Strong unique continuation for m-th powers of a Laplacian operator with singular coefficients*, Proc. Amer. Math. Soc., **135**, 569–578 (2007).

[28] P. L. Lions, *The concentration-compactness principle in the calculus of variations: The limit case. Part I and Part II*, Rev. Mat. Ibero. **1**(1) 145-201 (1985) and **1**(2) 45-121 (1985).

[29] E. Mitidieri, *A Rellich type identity and applications*, Comm. Partial Differential Equations **18**, 125–151 (1993).

[30] J.J. Moreau, *Décomposition orthogonale d'un espace hilbertien selon deux cônes mutuellement polaires*, C. R. Acad. Sci. Paris **255**, 238–240 (1962).

[31] Z. Nehari, On a class of nonlinear second-order differential equations, *Trans. Amer. Math. Soc.* 95(1960), 101-123.

[32] E.S. Noussair, C.A. Swanson and J. Yang, *Critical semilinear biharmonic equations in R^N*, Proc. Roy. Soc. Edinburgh, **121A**, 139-148 (1992).

[33] R. S. Palais, *Lusternik-Schnirelman theory on Banach manifolds*, Topology **5**, 115-132 (1966).

[34] M.H. Protter, *Unique continuation for elliptic equation*, Trans. Amer. Math. Soc. **95**, 81-91 (1960).

[35] P. Pucci and J. Serrin, *Critical exponents and critical dimensions for polyharmonic operators*, J. Math. Pures Appl. **69**, 55-83 (1990).

[36] Reichel and T. Weth, *A priori bounds and a Liouville theorem on a half-space for higher-order elliptic Dirichlet problems*, Math. Z., **261**, 805-827 (2009).

[37] M. Schechter and W. M. Zou, *On the Brézis-Nirenberg problem*, to appear in Arch. Rat. Mech. Anal., 2010.

[38] M. Struwe, *A global compactness result for elliptic boundary value problem involving limiting nonlinearities*, Math. Z., **26**, 511-517 (1984).

[39] M. Struwe, *Variational Methods*, Springer, Berlin-Heidelberg-New York-Tokyo (1990).

[40] C.A. Swanson, *The best Sobolev constant*, Appl. Anal. **47**, No.4, 227-239 (1992).

[41] G. Talenti, *Best constant in Sobolev inequality*, Ann. Mat. Pura Appl., IV. Ser. **110**, 353-372 (1976).

[42] G. Talenti, *Elliptic equations and rearrangements*, Ann. Sci. Norm. Sup. Pisa Cl. Sci., Ser. IV **3**, 697-718 (1976).

[43] R.C.A.M. Van der Vorst, *Best constant for the embedding of the space* $H^2 \cap H_0^1(\Omega)$ *into* $L^{2N/(N-4)}\Omega$, Diff. Int. Equ., **6**, No.2, 259-276 (1993).

[44] J. Wei and X. Xu, *Classification of solutions of higher order conformally invariant equations*, Math. Ann. **313**, No.2, 207-228 (1999).

[45] D. Zhang, *On multiple solutions of* $\Delta u + \lambda u + |u|^{\frac{4}{N-2}}u = 0$, Nonlin. Anal. TMA, **13**, 353-372 (1989).

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