ON PROBLEMS RELATED TO AN INEQUALITY OF DE LELLIS AND TOPPING

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Abstract. In this paper we study various problems related to an inequality proved recently by De Lellis and Topping.

1. Introduction

In this paper we consider various problems related to a recent result of De Lellis and Topping about the Schur Lemma

Theorem A (Almost Schur Lemma [5]). *For $n \geq 3$, if (M^n, g) is a closed Riemannian manifold with non-negative Ricci tensor, then*

$$\int_M |Ric - \frac{\overline{R}}{n} g|^2 dv(g) \leq \frac{n^2}{(n-2)^2} \int_M |Ric - \frac{R}{n} g|^2 dv(g), \tag{1}$$

where $\overline{R} = vol(g)^{-1} \int_M R dv(g)$ is the average of the scalar curvature R of g.

This result could be viewed as a quantitative version or a stability result of the Schur Lemma. Moreover, this result is optimal in the following sense: the constant in inequality (1) is the best and the non-negativity of the Ricci tensor can not be removed in general.

We observed in [8] that inequality (1) can be rewritten in terms of σ_k-scalar curvatures. Namely, it is equivalent to

$$\left(\int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} vol(g) \int_M \sigma_2(g) dv(g), \tag{2}$$

where

$$\sigma_1(g) = \frac{R_g}{2(n-1)} \quad \text{and} \quad \sigma_2(g) = \frac{1}{2(n-2)^2} \left\{ -|Ric|^2 + \frac{n}{4(n-1)} R^2 \right\}.$$

For the definition of $\sigma_k(g)$ scalar curvature for general k see below. With this observation and a nice argument of Gursky [15] we improved Theorem A in $n = 4$.

Theorem B [8] *If $n = 4$, and if (M^4, g) is a closed Riemannian manifold with non-negative scalar tensor, then (1) holds. Moreover, equality holds if and only if (M^4, g) is an Einstein manifold.*

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From Theorem B, one may naturally ask whether equality in (1) holds if and only if (M, g) is Einstein. The first result of this paper gives a positive answer.

Theorem 1. *Equality in Theorem A holds if and only if (M^n, g) is an Einstein manifold.*

With the observation mentioned above, it is natural to consider the following Yamabe type functional

$$\mathcal{E}(g) = \frac{vol(g)\displaystyle\int_M \sigma_2(g)dv(g)}{(\displaystyle\int_M \sigma_1(g)dv(g))^2}, \tag{3}$$

at least for metrics with $\displaystyle\int_M \sigma_1(g)dv(g) \neq 0$. For a metric g with nonnegative Ricci tensor Theorem A implies that

$$\mathcal{E}(g) \leq \frac{n-1}{2n}. \tag{4}$$

Theorem B implies that (4) holds for metrics with nonnegative scalar curvature, when $n = 4$. We conjectured in [8] that (4) for metrics with nonnegative scalar curvature if $n = 3$. However (4) is not true in general for metrics nonnegative scalar curvature if $n > 4$. In fact we have

Theorem 2. *If $n > 4$, for any metric g_0 with positive Yamabe constant, which is equivalent to the condition that there is a metric in $[g_0]$ with positive scalar curvature, we have*

$$\sup_{g\in[g_0]\cap\mathcal{C}_1} \mathcal{E}(g) = \infty \tag{5}$$

and

$$Y([g_0]) := \inf_{g\in[g_0]\cap\mathcal{C}_1} \mathcal{E}(g) < \frac{n-1}{2n}, \tag{6}$$

where $[g_0]$ is the conformal class of g_0 and $\mathcal{C}_k = \{g\,|\sigma_j(g) > 0\,\forall j \leq k\}$. Moreover, we have

$$Y([g_0]) > -\infty. \tag{7}$$

Furthermore, we have

$$Y([g_0]) > 0 \tag{8}$$

if and only if

$$\mathcal{C}_2([g_0]) := [g_0] \cap \mathcal{C}_2 \neq \emptyset.$$

In view of Theorem 2 it is natural to ask

Problem. *Is there a conformal metric $g \in [g_0] \cap \mathcal{C}_1$ achieving the infimum $Y([g_0])$ in $[g_0] \cap \mathcal{C}_1$, namely*

$$\mathcal{E}(g) = \inf_{g\in[g_0]\cap\mathcal{C}_1} \mathcal{E}(g)?$$

Or is there a metric $g \in [g_0] \cap \mathcal{C}_1$ which is a critical point of $\mathcal{E}$ in $[g_0]$?

Note that for the standard sphere $(\mathbb{S}^n, g_{\mathbb{S}^n})$ we have $\mathcal{E}(g_{\mathbb{S}^n}) = \frac{n-1}{2n}$, but

$$0 < Y([g_{\mathbb{S}^n}]) = \inf_{g\in[g_{\mathbb{S}^n}]\cap\mathcal{C}_1} \mathcal{E}(g) < \frac{n-1}{2n}.$$

It is easy to see that the standard round metric $g_{\mathbb{S}^n}$ is a critical point of $\mathcal{E}$. It would be interesting to know the value of $Y([g_{\mathbb{S}^n}])$.

We are also interested in the generalization of (2) to large k. Let us use the convention $\sigma_0 = 1$. Hence we can rewrite (2) as

$$\left(\int_M \sigma_1(g)dv(g)\right)^2 \geq \frac{2n}{n-1}\int_M \sigma_0(g)dv(g)\int_M \sigma_2(g)dv(g). \tag{9}$$

Note that the elementary symmetric functions σ_1 and σ_2 satisfy the Newton inequality $\sigma_1^2 \geq \frac{2n}{n-1}\sigma_0\sigma_2$. In general we have the Newton–MacLaurin formula for general k

$$\sigma_k^2(\Lambda) \geq c(n,k)\sigma_{k-1}(\Lambda)\cdot\sigma_{k+1}(\Lambda), \tag{10}$$

for $\Lambda \in \Gamma_k^+ := \{\Lambda \in \mathbb{R}^n \,|\, \sigma_j(\Lambda) > 0\,\forall\, j \leq k\}$. Here $c(n,k) = (k+1)(n-k+1)/(n-k)k$ and we used the convention that $\sigma_k = 0$ if $k < 0$ or $k > n$. Inspired by Theorem A and Theorem B we would like to ask under which conditions there holds

$$\left(\int_M \sigma_k(g)dv(g)\right)^2 \geq c(n,k)\int_M \sigma_{k-1}(g)dv(g)\int_M \sigma_{k+1}(g)dv(g). \tag{11}$$

At least, if the underlying manifold M is locally conformally flat, we have a generalization of Theorem B.

Theorem 3. *Let $n \geq 3$ and $k \in [n/2-1, n/2)$. When (M^n, g) is locally conformally flat with $g \in \mathcal{C}_k$, then (11) holds. Moreover, equality holds if and only if (M, g) is a space form.*

Now one may ask if there is a De Lellis-Topping type result for a suitable "Ricci curvature" such that the corresponding Theorem B type result is Theorem 3. There are really such curvatures, the Lovelock curvatures, which were introduced by Lovelock, but at least went back to Lanczos [17] in 1938. For the definition, see [19] and Section 5 below. We remark here that the Lovelock curvatures are natural generalizations of the Einstein tensor, other than the Ricci tensor.

Theorem 4. *Let (M^n, g) be a closed Riemannnian manifold with non-positive Ricci tensor and $1 \leq k < n/2$. We have*

$$\int_M |R^{(k)} - \overline{R}^{(k)}|^2 dv(g) \leq \frac{4n(n-1)}{(n-2k)^2}\int_M |E^{(k)} + \frac{n-2k}{2n}R^{(k)}g|^2 dv(g),$$

where $\overline{R}^{(k)}$ is the average of $R^{(k)}$. Here $trE^{(k)} = -\frac{n-2k}{2}R^{(k)}$ is defined below.

When $k = 1$, Theorem 4 is just Theorem A. For a given $k > 1$, if $n = 2(k+1)$ and (M^n, g) is local conformally flat, then Theorem 3 is just Theorem 4 with a slightly different condition $g \in \Gamma_k^+$. Note that the condition $g \in \Gamma_k^+$ with $k \geq n/2$ implies the condition that g has non-negative Ricci curvature. See [12]. The condition $g \in \Gamma_k^+$ with $k < n/2$ is not stronger than the condition that g has non-negative Ricci curvature.

The paper is organized as follows. In Section 2 we prove the rigidity result, Theorem 1. In Section 3 we first recall the definition of σ_k-scalar curvature and then prove Theorem 2 by choosing the suitable test metrics. In the construction of such metrics we need to pay extra attention to assure that all test metrics have positive scalar curvature. In Section 4 we generalize Theorem B to large k. In Section 5 we recall the definition of generalized Einstein tensors and then prove a De Lellis-Topping type result for these Einstein tensors.

2. Rigidity

Proof of Theorem 1. Assume that the equality holds, or equivalently,

$$\int_M |R - \overline{R}|^2 dv(g) = \frac{4n(n-1)}{(n-2)^2} \int_M |Ric - \frac{R}{n} g|^2 dv(g). \tag{12}$$

Then from the proof of Theorem A in [5] we know there exists some $\lambda \in \mathbb{R} \cup \{\infty\}$

$$Ric - \frac{R}{n} g = \lambda(\nabla^2 f - \frac{\Delta f}{n} g), \tag{13}$$

and

$$Ric(\nabla f, \nabla f) = 0, \tag{14}$$

where $\Delta f = R - \overline{R}$ with $\int_M f = 0$. Here by $\lambda = \infty$ we mean $\nabla^2 f - \frac{\Delta f}{n} g = 0$. In this case, integrating this equality we obtain

$$\int_M |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n} dv(g) = 0.$$

From (14) we have

$$\int_M |\nabla^2 f|^2 dv(g) = \int_M |\Delta f|^2 dv(g) = \int_M (R - \overline{R})^2 dv(g). \tag{15}$$

Therefore,

$$0 = \int_M |\Delta f|^2 dv(g) = \int_M (R - \overline{R})^2 dv(g)$$

which, together with (12), means that g is an Einstein metric.

Now we consider $\lambda \in \mathbb{R}$. In the following, we use the normal coordinates to calculate. Recall the fact the Ricci tensor is non-negative. From the Cauchy-Schwarz inequality and (14), for all $x \in M$ and all tangent vector $Y \in T_x M$,

$$|Ric(\nabla f(x), Y)|^2 \leq Ric(\nabla f(x), \nabla f(x)) Ric(Y, Y) = 0, \tag{16}$$

that is, $Ric(\nabla f, \cdot) = 0$, or equivalently

$$R^i_j f_i = 0. \tag{17}$$

Here we use Einstein summation convention. From (13), (17) and $R_{ij,j} = \frac{1}{2}R_i$, we have

$$\begin{aligned}
\frac{1}{2}R_i &= \lambda(f_{ij} - \frac{\Delta f g_{ij}}{n})_j + (\frac{R g_{ij}}{n})_j \quad = \quad \lambda f_{ijj} - \lambda\frac{(\Delta f)_i}{n} + \frac{R_i}{n} \\
&= \lambda f_{jji} - \lambda\frac{(\Delta f)_i}{n} + \frac{R_i}{n} \\
&= (\lambda + \frac{1-\lambda}{n})R_i,
\end{aligned}$$

which implies that either $\lambda = \frac{n-2}{2(n-1)}$ or $\lambda \neq \frac{n-2}{2(n-1)}$ and $R_i = 0$. In the latter case, R is constant and it follows from (12) that g is an Einstein metric. Now we consider the former case, ie. $\lambda = \frac{n-2}{2(n-1)}$. (13) will be read as

$$Ric = \frac{n-2}{2(n-1)}\nabla^2 f + \frac{R}{2(n-1)}g + \frac{(n-2)\overline{R}}{2n(n-1)}g. \tag{18}$$

By differentiating (17) and using $R_{ij,j} = \frac{1}{2}R_i$ we have

$$\frac{1}{2}R^i f_i + R^{ij} f_{ij} = 0. \tag{19}$$

Combining (18) and (19) gives

$$\frac{1}{2}R^i f_i + \frac{n-2}{2(n-1)}|\nabla^2 f|^2 + \frac{R(R-\overline{R})}{2(n-1)} + \frac{(n-2)\overline{R}(R-\overline{R})}{2n(n-1)} = 0. \tag{20}$$

Since M is compact, there exists some point $x_0 \in M$ such that $R(x_0) = \max R$. At this point, we have $R - \overline{R} \geq 0$, $R \geq 0$, $\overline{R} \geq 0$ and $R^i = 0$. From (20) we have $\max R = \overline{R}$, and hence $R \equiv \overline{R}$. g is also an Einstein metric in this case. ■

Theorem A and Theorem 1 give a characterization of Einstein metrics. We remark that a metric g satisfying (13) is called an Ricci almost soliton in [21], which is a generalization of the Ricci soliton.

3. An equivalent inequality in terms of σ_k scalar

Let us first recall the definition of the k-scalar curvature, which was first introduced by Viaclovsky [24] and has been intensively studied by many mathematicians, see for example [10], [25] and the references in [6]. There are many geometric applications of analysis developed in the study of the k-scalar curvature. For example, a 4-dimensional sphere Theorem was proved in [4] (see also [3], a 3-dimensional sphere Theorem in [6] and [1], an eigenvalue estimates for the Dirac operator in [26] and various geometric inequalities in [14]). The results of this section and next section are also applications of this analysis.

Let

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)} \cdot g\right)$$

be the Schouten tensor of g. For an integer k with $1 \le k \le n$ let σ_k be the k-th elementary symmetric function in $\mathbb{R}^n$. The k-scalar curvature is

$$\sigma_k(g) := \sigma_k(\Lambda_g),$$

where Λ_g is the set of eigenvalue of the matrix $g^{-1} \cdot S_g$. In particular, $\sigma_1(g) = \operatorname{tr} S$ and $\sigma_2 = \frac{1}{2}((\operatorname{tr} S)^2 - |S|^2)$. It is trivial to see that

$$\begin{aligned} \sigma_1(g) &= \frac{R}{2(n-1)}, \\ \sigma_2(g) &= \frac{1}{2(n-2)^2}\left\{-|Ric|^2 + \frac{n}{4(n-1)}R^2\right\}, \\ \left|Ric - \frac{R}{n}g\right|^2 &= |Ric|^2 - \frac{R^2}{n}. \end{aligned}$$

From above it is easy to have the following observation.

Observation. ([8]) *Inequality (1) is equivalent to (2).*

In [8] we proved Theorem B, namely there is an inequality

$$\mathcal{E}(g) \le \frac{n-1}{2n}, \tag{21}$$

provided that g is a metric of non-negative scalar curvature and $n = 4$. We conjectured that this statement is true for $n = 3$. In this Section we show Theorem 2, namely this statement is not true for $n > 4$.

We first prove one part of Theorem 2 in

Proposition 1. *Let $n > 4$ and $g_0 \in \mathcal{C}_1$. Then we have* (7)*. Moreover,* (8) *holds if and only if*

$$\mathcal{C}_2([g_0]) := [g_0] \cap \mathcal{C}_2 \neq \emptyset.$$

Proof. Recall the ordinary Yamabe constant Y_1 and another Yamabe type constant $Y_{2,1}$ studied in [6]

$$Y_1([g_0]) := \inf_{g \in \mathcal{C}_1([g_0])} \frac{\displaystyle\int_M \sigma_1(g) dv(g)}{(vol(g))^{\frac{n-2}{n}}} \quad \text{and} \quad Y_{2,1}([g_0]) := \inf_{g \in \mathcal{C}_1([g_0])} \frac{\displaystyle\int_M \sigma_2(g) dv(g)}{(\displaystyle\int_M \sigma_1(g) dv(g)^{\frac{n-4}{n-2}}}.$$

By a direct computation we have in [6]

$$
\begin{aligned}
2\int \sigma_2(g)dv(g) &= \frac{n-4}{2}\int \sigma_1(g)|\nabla u|^2_{g_0}e^{2u}dv(g) + \frac{n-4}{4}\int |\nabla u|^4_{g_0}e^{4u}dv(g) \\
&\quad + \int e^{2u}\sigma_1(g)\sigma_1(g_0)dv(g) - \int e^{4u}|S(g_0)|^2_{g_0}dv(g) \\
&\quad +(4-n)\int \sum_{i,j} S(g_0)^{ij}u_iu_jdv(g) + \int \sigma_1(g_0)|\nabla u|^2_{g_0}e^{4u}dv(g) \\
&\quad + \int e^{4u}\langle \nabla u, \nabla \sigma_1(g_0)\rangle_{g_0}dv(g).
\end{aligned} \tag{22}
$$

It follows that

$$
\int \sigma_2(g)dv(g) \;\; \geq \;\; \frac{n-4}{16}\int |\nabla u|^4_{g_0}e^{4u}dv(g) - c\int e^{4u}dv(g), \tag{23}
$$

provided that $g \in \Gamma_1^+$. Moreover we have

$$
\int_M \sigma_1(g)dv(g) = \int \left(\frac{n-2}{2}|\nabla u|^2 + \sigma_1(g_0)\right)e^{2u}dv(g) \geq Y_1([g_0])(vol(g))^{\frac{n-2}{n}}. \tag{24}
$$

From (23), (24) and Hölder's inequality Hölder's inequality, we have

$$
\begin{aligned}
\int \sigma_2(g)dv(g) - c'\int e^{4u}dv(g) &\geq c(\int |\nabla u|^4_{g_0}e^{4u}dv(g) + \int e^{4u}dv(g)) \\
&\geq c(\int_M \sigma_1(g)dv(g))^2(vol(g))^{-1}.
\end{aligned} \tag{25}
$$

so that

$$
\int \sigma_2(g)dv(g) \geq (c_1 - c_2Y_1([g_0])^{-2})(\int_M \sigma_1(g)dv(g))^2(vol(g))^{-1}.
$$

that is, $Y([g_0]) \geq c_1 - c_2Y_1([g_0])^{-2} > -\infty$. This proves (7).

Now we assume that (8) hold. Since $g_0 \in \mathcal{C}_1$ we have $Y_1([g_0]) > 0$. It is clear that for $g \in \mathcal{C}_1([g_0])$

$$
\frac{\int_M \sigma_2(g)dv(g)}{(\int_M \sigma_1(g)dv(g))^{\frac{n-4}{n-2}}} = \mathcal{E}(g)\left(\frac{\int_M \sigma_1(g)dv(g)}{(vol(g))^{\frac{n-2}{n}}}\right)^{\frac{n}{n-2}} \geq Y([g_0])(Y_1([g_0]))^{\frac{n}{n-2}}.
$$

It follows that $Y_{2,1}([g_0]) > 0$, which is equivalent to the non-emptiness of $\mathcal{C}_2([g_0])$ by a result in [6]. See also [23].

Now assume the non-emptiness of $\mathcal{C}_2([g_0])$. Let $g \in \mathcal{C}_1([g_0])$. First define a nonlinear eigenvalue of $\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0}$ by

$$
\lambda(g_0, \sigma_2) := \inf_{g=e^{-2u}g_0 \in \mathcal{C}_1([g_0])} \frac{\int \sigma_2(g)dv(g)}{\int e^{4u}dv(g)}.
$$

We have proved in [6] that $\lambda(g_0, \sigma_2) > 0$, i.e.,

$$\int_M \sigma_2(g) \geq \lambda(g_0, \sigma_2) \int_M e^{4u} dv(g) \tag{26}$$

for any $g = e^{-2u} g_0 \in \Gamma_1^+$.

From (23), (24), (26) and Hölder's inequality, we deduce

$$\begin{aligned} \int \sigma_2(g) dv(g) &\geq c\left(\int |\nabla u|^4_{g_0} e^{4u} dv(g) + \int e^{4u} dv(g)\right) \\ &\geq c\left(\int_M \sigma_1(g) dv(g)\right)^2 (vol(g))^{-1}. \end{aligned} \tag{27}$$

This is what we want to show. ■

We in fact proved that the following four statements are equivalent for a conformal class $[g_0]$ with $\mathcal{C}_1 \neq \emptyset$ $(n > 4)$.

(i) $\mathcal{C}_2([g_0]) \neq \emptyset$,
(ii) $Y_{2,1}([g_0]) > 0$,
(iii) $Y([g_0]) > 0$,
(iv) $\lambda(g_0, \sigma_2) > 0$.

Proposition 2. *Let $n \geq 3$ and $\mathcal{C}_1([g_0]) \neq \emptyset$. Then there exist a metric $g \in \mathcal{C}_1([g_0])$ with*

$$\mathcal{E}(g) < \frac{n-2}{2n}.$$

Proof. Let $\tilde{g} \in \mathcal{C}_1([g_0])$ be a Yamabe solution, ie. $\sigma_1(\tilde{g}) = const$. From the Newton inequality $\sigma_2(\Lambda) \leq \frac{n-2}{2n} \sigma_1^2(\Lambda)$ for any $\Lambda \in \mathbb{R}^n$ and equality holds if and only if $\Lambda = c(1, 1, \cdots 1)$ for some $c \in \mathbb{R}$, we have

$$\begin{aligned} \int_M \sigma_2(\tilde{g}) dv(\tilde{g}) &\leq \frac{n-2}{2n} \int_M \sigma_1(\tilde{g})^2 dv(\tilde{g}) \\ &= \frac{n-2}{2n} \frac{\left(\int_M \sigma_1(\tilde{g}) dv(\tilde{g})\right)^2}{vol(\tilde{g})}. \end{aligned}$$

Hence $\mathcal{E}(\tilde{g}) \leq \frac{n-2}{2n}$. From above it is easy to see that $\mathcal{E}(\tilde{g}) = \frac{n-2}{2n}$ if and only if $\tilde{g}$ is an Einstein metric. In this case, it is clear that $Ric(\tilde{g})$ is positive definite. Then we choose a nearby, not Einstein metric $\tilde{g}_1$ with positive Ricci tensor. Then by Theorem 1, we have $\mathcal{E}(\tilde{g}_1) < \frac{n-2}{2n}$. ■

The proof is motivated by an argument of Gursky in [15].

Now we remain to prove

Proposition 3. *Let $n > 4$ and $\mathcal{C}_1([g_0]) \neq \emptyset$. Then (5) holds. Namely*

$$\bar{Y}([g_0]) := \sup_{g \in [g_0] \cap \mathcal{C}_1} \mathcal{E}(g) = \infty.$$

To prove the Proposition we use the gluing method developed by Gromov-Lawson [9] (see also [22]).

Improving slightly the construction given in [7], which is motivated by [9] and [22], we have

Lemma 1. *Assume $n > 4$. Let g_0 be in Theorem 2. For any small constant $\delta, \lambda \in (0,1)$ such that $\lambda^{\frac{3}{8}} >> \delta >> \lambda^{\frac{1}{2}}$, there exists a constant $\delta_1 > 0$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfying:*

(i) $\delta_1 = \lambda^{-1}\delta^3$, $\delta << \delta_1 << \delta^{\frac{1}{3}}$,
(ii) *The metric $g = e^{-2u}g_0$ has positive scalar curvature in B_{δ_1},*
(iii) $u = \log(\lambda + |x|^2) + b_0$ *for* $|x| \leq \delta$,
(iv) $u = \log|x|$ *for* $|x| \geq \delta_1$,
(v) $vol(B_{\delta_1}\backslash B_\delta, g) = O(\delta_1^n\delta^{-n})$, $\int_{B_{\delta_1}\backslash B_\delta}\sigma_1(g)dv(g) = O(\delta_1^{n-2}\delta^{2-n})$ *and* $\int_{B_{\delta_1}\backslash B_\delta}\sigma_2(g)dv(g) = O(\delta_1^{n-4}\delta^{4-n})$, *where* $b_0 = -\log\delta_1 + O(1)$.

Lemma 2. *Assume $n > 4$. Let g_0 be in Theorem 2. For any small constant $\delta, \lambda \in (0,1)$ such that $\lambda^{\frac{3}{8}} >> \delta >> \lambda^{\frac{1}{2}}$, define a conformal metric $g = e^{-2u}g_0$ with $u(x) = \log(\lambda + |x|^2) + b_0$ in B_δ, where b_0 is some constant. Then g has positive scalar curvature in B_δ and we have the following*

$$vol(B_\delta, g) = e^{-nb_0}\lambda^{-\frac{n}{2}}[B + O(\lambda) + O((\lambda\delta^{-2})^{\frac{n}{2}})], \tag{28}$$

$$\int_{B_\delta}\sigma_1(g)dv(g) = e^{(2-n)b_0}\lambda^{1-\frac{n}{2}}[2nB + O(\lambda) + O((\lambda\delta^{-2})^{\frac{n}{2}})], \tag{29}$$

$$\int_{B_\delta}\sigma_2(g)dv(g) = e^{(4-n)b_0}\lambda^{2-\frac{n}{2}}[2n(n-1)B + O(\lambda) + O((\lambda\delta^{-2})^{\frac{n}{2}})]. \tag{30}$$

Here $B = \int_{\mathbb{R}^n}\frac{1}{(1+|x|^2)^n}dx$.

Lemma 3. *Let g_0 be as in Theorem 2, $n > 4$ and B_{r_0} be a geodesic ball with respect to g_0 for some r_0. Then there exists a conformal metric $g = e^{-2u}g_0$ in $B_{r_0}\setminus\{0\}$ satisfying:*

(i) *The metric $g = e^{-2u}g_0$ has positive scalar curvature in $B_{r_0}\setminus\{0\}$,*
(ii) $u = \log|x|$ *for* $|x| \leq r_2$,
(iii) $u = b_1$ *for* $|x| \geq r_1$,

where $r_2 < r_1 < r_0$ and b_1 is a constant.

Proof of Proposition 3. Let $\{x_k\}_{k=1}^K$ be K points in M and $B_{r_{0K}}(x_k)$ be disjoint geodesic balls centered as x_k with radius r_{0K}, where $r_{0K} \to 0$ as $K \to \infty$. For any $K \in \mathbb{N}$, we choose some $\delta = o(K^{-\gamma})$ such that $\delta_K := K^\gamma\delta \to 0$ as $K \to \infty$ for some γ chosen later. For simplicity, set $\delta_k = \lambda_k^{\frac{3}{7}}$, which satisfies the assumption on δ, λ in Lemma 1 and define $\delta_{1k} = \lambda_k^{-1}\delta_k^3 = \delta_k^{\frac{2}{3}}$ and b_{0k} as in Lemma 1. Also define r_{1K}, $r_{2K} \leq r_{0K}$ and b_{1K} as in Lemma 3 (independent of k). We point out that r_{1K} and r_{2K} can be chosen as small as we want. For sufficient small δ_k with $\delta_{1k} \leq r_{2K}$, define a sequence of metrics $g_K = e^{-2u_K}g_0$ as follows. In $M\setminus B_{r_{0K}}(x_k)$, $g = e^{-2b_{1K}}g_0$, where b_{1K} (independent of k) is given in Lemma

3. We define

$$u_K = \begin{cases} \log(\lambda_k + |x-x_k|^2) + b_{0k}, & x \in B_{\delta_k}(x_k) \\ \log|x-x_k|, & x \in B_{r_{2K}}(x_k) \setminus B_{\delta_{1k}}(x_k) \\ b_{1K}, & x \in M \setminus \bigcup_{k=1}^K B_{r_{1K}}(x_k) \end{cases} \tag{31}$$

and in $B_{\delta_{1k}}(x_k) \setminus B_{\delta_k}(x_k)$, we define u_K as in Lemma 1, while in $B_{r_{1K}}(x_k) \setminus B_{r_{2K}}(x_k)$ we define u_K as in Lemma 3. From the construction in Lemma 1 and Lemma 3, we see that g_K is smooth and has positive scalar curvature. It follows directly from Lemma 1 and Lemma 2 that

$$\begin{aligned}
vol(B_{\delta_k}(x_k), g_K) &= \delta_k^{-\frac{1}{2}n}[B + O(\delta_k^{\frac{7}{3}}) + O(\delta_k^{\frac{n}{6}})], \\
\int_{B_{\delta_k}(x_k)} \sigma_1(g_K) dv(g_K) &= \delta_k^{1-\frac{1}{2}n}[2nB + O(\delta_k^{\frac{7}{3}}) + O(\delta_k^{\frac{n}{6}})], \\
\int_{B_{\delta_k}(x_k)} \sigma_2(g_K) dv(g_K) &= \delta_k^{2-\frac{1}{2}n}[2n(n-1)B + O(\delta_k^{\frac{7}{3}}) + O(\delta_k^{\frac{n}{6}})], \\
vol(B_{\delta_{1k}}(x_k) \setminus B_{\delta_k}(x_k), g_K) = O(\delta_k^{\frac{-n}{3}}) &= \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n}{6}}), \\
\int_{B_{\delta_{1k}}(x_k)\setminus B_{\delta_k}(x_k)} \sigma_1(g_K) dv(g_K) = O(\delta_k^{\frac{2-n}{3}}) &= \delta_k^{1-\frac{1}{2}n} O(\delta_k^{\frac{n-2}{6}}), \\
\int_{B_{\delta_{1k}}(x_k)\setminus B_{\delta_k}(x_k)} \sigma_2(g_K) dv(g_K) = O(\delta_k^{\frac{4-n}{3}}) &= \delta_k^{2-\frac{1}{2}n} O(\delta_k^{\frac{n-4}{6}}).
\end{aligned}$$

One can also choose r_{1K} and then r_{2K} sufficiently far away from δ_k for any $k = 1, \cdots, K$ such that

$$\begin{aligned}
vol(B_{r_{2K}}(x_k) \setminus B_{\delta_{1k}}(x_k), g_K) &= \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n}{6}}), \\
\int_{B_{r_{2K}}(x_k)\setminus B_{\delta_{1k}}(x_k)} \sigma_1(g_K) dv(g_K) &= \delta_k^{1-\frac{1}{2}n} O(\delta_k^{\frac{n-2}{6}}), \\
\int_{B_{r_{2K}}(x_k)\setminus B_{\delta_{1k}}(x_k)} \sigma_2(g_K) dv(g_K) &= \delta_k^{2-\frac{1}{2}n} O(\delta_k^{\frac{n-4}{6}}), \\
vol(M \setminus \bigcup_{k=1}^K B_{r_{2K}}(x_k), g_K) &= f_0(r_{2K}) = \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n}{6}}), \\
\int_{M\setminus \bigcup_{k=1}^K B_{r_{2K}}(x_k)} \sigma_1(g_K) dv(g_K) &= f_1(r_{2K}) = \delta_k^{1-\frac{1}{2}n} O(\delta_k^{\frac{n-2}{6}}), \\
\int_{M\setminus \bigcup_{k=1}^K B_{r_{2K}}(x_k)} \sigma_2(g_K) dv(g_K) &= f_2(r_{2K}) = \delta_k^{2-\frac{1}{2}n} O(\delta_k^{\frac{n-4}{6}}).
\end{aligned}$$

for some functions f_i, $i = 0, 1, 2$. Combining all the above estimates and using $\delta_k = k^\gamma \delta$, we obtain

$$(32)\quad \mathcal{E}(g_K)=\frac{vol(g_K)\int_M\sigma_2(g_K)dv(g_K)}{(\int_M\sigma_1(g_K)dv(g_K))^2}=\frac{\sum_{k=1}^K k^{-\frac12 n\gamma}\sum_{k=1}^K k^{(2-\frac12 n)\gamma}}{\left(\sum_{k=1}^K k^{(1-\frac12 n)\gamma}\right)^2}\left[\frac{n-1}{2n}+o(1)\right].$$

Choose γ such that $(1-\frac12 n)\gamma=-1-\beta$ with $\beta\in(0,\frac{2}{n-4})$. Then we have

$$-\frac12 n\gamma=\frac{n}{n-2}(-1-\beta)<-1,\quad (2-\frac12 n)\gamma=\frac{n-4}{n-2}(-1-\beta)>-1.$$

Therefore, $\sum_{k=1}^\infty k^{-\frac12 n\gamma}$ and $\sum_{k=1}^\infty k^{(1-\frac12 n)\gamma}$ converge, meanwhile $\sum_{k=1}^\infty k^{(2-\frac12 n)\gamma}$ diverges. In view of (32), we see that $\mathcal{E}(g_K)$ can be made to be arbitrary large when K goes to infinity. Hence we finished the proof of (5). ∎

Remark 1. *Using Lemmas given above and an argument from Aubin, we can show a weaker form of* (6)*.*

$$(33)\qquad Y([g_0]):=\inf_{g\in[g_0]\cap\mathcal{C}_1}\mathcal{E}(g)\le\frac{n-1}{2n}.$$

This is an Aubin type inequality. Using the same gluing argument we can show the metrics constructed in Lemma 1 and Lemma 3 are in the class Γ_k^+*, provided* $g_0\in\Gamma_k^+$ *and* $k<n/2$*, and hence*

$$(34)\qquad \inf_{g\in[g_0]\cap\mathcal{C}_k}\mathcal{E}(g)\le\frac{n-1}{2n},$$

for any $k<n/2$*, provided that* $[g_0]\cap\mathcal{C}_k\neq\emptyset$*. We do not know if the inequality in* (34) *is strict, though we believe this. Similarly, one can show a slightly stronger form of* (5)

$$(35)\qquad \sup_{g\in[g_0]\cap\mathcal{C}_k}\mathcal{E}(g)=\infty,$$

for any $k<n/2$*, provided that* $[g_0]\cap\mathcal{C}_k\neq\emptyset$*. Comparing with the inequality of De Lellis-Topping* (1)*, i.e.,*

$$\mathcal{E}(g)\le\frac{n-1}{2n},\qquad \textit{for any } g \textit{ with } Ric\ge 0,$$

it indicates that the condition $Ric\ge 0$ *is "stronger" than the condition* $g\in\mathcal{C}_k$ *with* $k<n/2$*. Remark that a metric* $g\in\mathcal{C}_k$ *with* $k\ge n/2$ *have positive Ricci tensor* [12].

4. A geometric inequality for large k

In this Section, we will prove Theorem 3, namely

$$(36)\qquad \left(\int_M\sigma_k(g)dv(g)\right)^2\ge c(n,k)\int_M\sigma_{k-1}(g)dv(g)\int_M\sigma_{k+1}(g)dv(g)$$

holds if (M,g) is locally conformally flat and $g \in \Gamma_k^+$ with $k \in [n/2-1, n/2)$. The constraint $k \in [n/2-1, n/2)$ equals to

$$k = \begin{cases} \dfrac{n-1}{2}, & \text{if } n \text{ is odd,} \\ \dfrac{n}{2}-1, & \text{if } n \text{ is even.} \end{cases}$$

Proof of Theorem 3. First all, we may assume that $\displaystyle\int \sigma_{k+1}(g)dv(g) > 0$.

We first consider the case n is even and $k = \frac{n}{2}-1$. In this case we use the argument of Gursky [15] as in [8] and a following Yamabe problem

$$Y_k([g]) := \inf_{\tilde g \in \mathcal{C}_k([g])} \frac{\displaystyle\int_M \sigma_k(\tilde g)dv(\tilde g)}{(\displaystyle\int_M \sigma_{k-1}(\tilde g)dv(\tilde g))^{\frac{n-2k}{n-2(k-1)}}},$$

where $\mathcal{C}_k([g]) := [g] \cap \Gamma_k^+$, which was studied in [14]. Since (M,g) is locally conformally flat, it was proved in [14] that Y_k is achieved by a conformal metric $g_k \in \mathcal{C}_k$ satisfying

$$\frac{\sigma_k(g_k)}{\sigma_{k-1}(g_k)} = a_k, \tag{37}$$

for some constant $a_k > 0$, which implies that $\displaystyle\int_M \sigma_k(g_k)dv(g_k) = a_k \int_M \sigma_{k-1}(g_k)dv(g_k)$. Now by (10) we have

$$\begin{aligned} \int_M \sigma_{k+1}(g_k)dv(g_k) &\le c(n,k) \int_M \frac{\sigma_k(g_k)^2}{\sigma_{k-1}(g_k)} dv(g_k) \\ &= c(n,k)a_k \int_M \sigma_k(g_k)dv(g_k) \\ &= c(n,k) \left(\frac{\displaystyle\int_M \sigma_k(g_k)dv(g_k)}{(\displaystyle\int_M \sigma_{k-1}(g_k)dv(g_k))^{1/2}} \right)^2 = c(n,k)Y_k([g_k])^2, \end{aligned}$$

where we have used that $k = n/2-1$. Since $k+1 = n/2$ and the manifold is locally conformally flat, we know that $\int \sigma_{k+1}(g)dv(g)$ is constant in a given conformal class [24]. Hence we have

$$\begin{aligned} \int_M \sigma_{k+1}(g)dv(g) &= \int_M \sigma_{k+1}(g_k)dv(g_k) \le c(n,k)Y_k([g_k])^2 \\ &\le c(n,k) \left(\frac{\displaystyle\int_M \sigma_k(g)dv(g)}{(\displaystyle\int_M \sigma_{k-1}(g)dv(g))^{1/2}} \right)^2. \end{aligned}$$

In the last inequality we have used that g_k achieves the minimum Y_k. From the proof it is clear that equality holds if and only if

$$\sigma_{k+1}(g)\sigma_{k-1}(g) = c(n,k)\sigma_k^2(g),$$

that is, g is an Einstein metric.

Now we consider the case that n is odd and $k = \frac{n-1}{2}$. In this case we consider the following Yamabe type problem.

Define

$$\mathcal{E}_k(g) := \frac{\int_M \sigma_{k-1}(g)dv(g) \int_M \sigma_{k+1}(g)dv(g)}{(\int_M \sigma_k(g)dv(g))^2} \tag{38}$$

and

$$\widetilde{Y}_k([g_0]) := \sup_{g\in\mathcal{C}_k([g_0])} \mathcal{E}_k(g).$$

The Euler-Lagrange equation of (38) is a Yamabe type equation

$$\frac{\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g)}{\sigma_k(g)} = -2s_k(g), \tag{39}$$

where $r_k(g)$ and $s_k(g)$ are two positive constants defined by

$$r_k(g) = \frac{\int_M \sigma_{k+1}(g)dv(g)}{\int_M \sigma_{k-1}(g)dv(g)} \quad \text{and} \quad s_k(g) = \frac{\int_M \sigma_{k+1}(g)dv(g)}{\int_M \sigma_k(g)dv(g)}.$$

By the key Lemma in [6] we have: For $g_0 \in \Gamma_k^+$ Equation (39) is an elliptic and concave equation. We want to find the maximum of $\mathcal{E}_k$, $Y_k([g_0])$. In order to do so, we consider a Yamabe type flow

$$-g^{-1}\cdot \frac{d}{dt}g = \frac{\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g)}{\sigma_k(g)} + 2s_k(g). \tag{40}$$

Proposition 4. *Flow (40) preserves* $\int_M \sigma_k(g)dv(g)$, *while it increases*

$$\int_M \sigma_{k-1}(g)dv(g) \int_M \sigma_{k+1}(g)dv(g).$$

Proof. It is clear that the flow preserves $\int_M \sigma_k(g)dv(g)$. By a direct computation we have

$$\begin{aligned}
&\frac{d}{dt}\left(\int_M \sigma_{k-1}(g)dv(g) \int_M \sigma_{k+1}(g)dv(g)\right)\\
= \; & -\frac{1}{2}\int_M \sigma_{k-1}(g)dv(g) \int_M (\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g))g^{-1}\cdot\frac{d}{dt}g\\
= \; & \frac{1}{2}\int_M \sigma_{k-1}(g)dv(g)\int_M \sigma_k(g)\left(\frac{\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g)}{\sigma_k(g)} + 2s_k(g)\right)^2 \geq 0.
\end{aligned}$$

∎

Proposition 5. *Flow (40) is a parabolic equation.*

Proof. See [6]. ∎

Since (M, g) is locally conformally flat, we can use the argument in [13] to show that the flow converges to a solution of (39). This argument used a crucial argument in [28] for the ordinary Yamabe flow, to show that there is a uniform estimate for gradients. Here we will not repeat it. Hence for any $g \in [g_0] \cap \mathcal{C}_k$ by using flow (40) we find a $\tilde{g} \in [g_0] \cap \mathcal{C}_k$ satisfying (39). Since the flow increases $\mathcal{E}_k$ we have $\mathcal{E}_k(g) \le \mathcal{E}(\tilde{g})$. Now one can show that $\tilde{g}$ is in fact a metric with constant sectional curvature.

Theorem 5. *Let n be odd and $k = (n-1)/2$. If (M, g) is a locally conformally flat with $g \in \Gamma_k^+$ and $\int_M \sigma_{k+1}(g) dv(g) > 0$, then there is a conformal metric $g_1 \in [g]$ with constant sectional curvature.*

Proof. The proof follows from the proof given in [6] directly. In fact the argument would imply the cone Γ_{k+1}^+ is not empty. Then it follows from [12] (M, g) has positive Ricci curvature. By Theorem of Myers, $\pi_1(M)$ is finite. Hence the universal cover of M is compact and locally conformally flat and thus conformal to the standard n-sphere. The argument in [1] would also work. See also closely related results in [11] and [2]. ∎

By this Theorem 5, without loss of generality we may assume that (M, g_0) is the standard round metric. Since $\tilde{g}$ satisfies a conformal equation (39), the classification result in [18] implies that $\tilde{g}$ is also a metric with constant sectional curvature, and hence $\mathcal{E}(\tilde{g}) = c(n, k)$. Therefore we have proved

$$\mathcal{E}_k(g) \le \mathcal{E}(\tilde{g}) = c(n, k).$$

Equality holds if and only if $\mathcal{E}_k(g) = c(n, k)$, which means that g is a maximum of $\mathcal{E}_k$ and hence satisfies (39). By Theorem 5 again, (M, g) is a space form. Now we complete the proof of Thereom 3. ∎

5. Lovelock

In this section, let us first recall the work of Lovelock [19] on generalized Einstein tensors. See also [20], [27] and [16].

Let

$$E_{AB} = R_{AB} - \frac{1}{2} R g_{AB}$$

be the Einstein tensor. It is clear that g is an Einstein metric if and only if

$$E_{AB} = \lambda g_{AB}. \tag{41}$$

The Einstein tensor is very important in theoretical physics. It is a conversed quantity, i.e.,

$$E^B_{A,B} = 0.$$

It would be an interesting to generalize the Einstein tensor. In [19] Lovelock studied the classification of tensors A satisfying

(i) $A^{ij} = A^{ji}$, ie, A is symmetric.
(ii) $A^{ij} = A^{ij}(g_{AB}, g_{AB,C}, g_{AB,CD})$.
(iii) ${A^{ij}}_j = 0$, ie. A is divergence-free.
(iv) A^{ij} is linear in the second derivatives of g_{AB}.

It is clear that the Einstein tensor satisfies all conditions. Lovelock classified all 2-tensors satisfying (i)–(iii). Let us first define

$$L_k = R^{(k)} := \frac{1}{2^k}\delta^{i_1i_2\cdots i_{2k-1}i_{2k}}_{j_1j_2\cdots j_{2k-1}i_{2k}} {R_{i_1i_2}}^{j_1j_2}\cdots {R_{i_{2k-1}i_{2k}}}^{j_{2k-1}j_{2k}}.$$

Here the generalized Kronecker delta is denied by

$$\delta^{j_1j_2\cdots j_r}_{i_1i_2,\cdots i_r} = \det\begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_r}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_r}_{i_2} \\ \vdots & \vdots & \vdots & \vdots \\ \delta^{j_1}_{i_r} & \delta^{j_2}_{i_r} & \cdots & \delta^{j_r}_{i_r} \end{pmatrix}.$$

L_k is called the lovelock curvature. When $2k = n$, $R^{(k)}$ is the Euler density. We could check that $R^{(k)} = 0$ if $2k > n$. For $k < n/2$, $R^{(k)}$ is called the dimensional continued Euler density in Physics. Let us define a 2-tensor $E^{(k)}$ by

$${E^{(k)}}_{ij} := -\frac{1}{2^{k+1}} g_{\alpha i}\delta^{\alpha i_1i_2\cdots j_{2k-1}i_{2k}}_{jj_1j_2\cdots j_{2k-1}i_{2k}} {R_{i_1i_2}}^{j_1j_2}\cdots {R_{i_{2k-1}i_{2k}}}^{j_{2k-1}j_{2k}}$$

locally. It is clear that

$$\mathrm{tr}E^{(k)} = -\frac{n-2k}{2}R^{(k)}.$$

One can check that

$${E^{(k)}}^i_{\ j,i} = 0,$$

ie, $E^{(k)}$ satisfies (i)–(iii). Lovelock proved that any 2-tensor satisfying (i)–(iii) has the form

$$\sum_j \alpha_j E^{(j)}$$

with certain constants α_j, $j \geq 0$. Here we set $E^{(0)} = 0$. It is clear to see that $E^{(1)}$ is the Einstein tensor and

$$R^{(1)} = R,$$

which is the scalar curvature.

One can also check that

$$E^{(2)}_{\mu\nu} = 2RR_{\mu\nu} - 4R_{\mu\alpha}{R^\alpha}_\nu - 4R_{\alpha\beta}{{{R^\alpha}_\mu}^\beta}_\nu + 2R_{\mu\alpha\beta\gamma}{R_\nu}^{\alpha\beta\gamma} - \frac{1}{2}g_{\mu\nu}L_2$$

and

$$L_2 = \frac{1}{4}\delta^{i_1i_2i_3i_4}_{j_1j_2j_3j_4}{R^{j_1j_2}}_{i_1i_2}{R^{j_3j_4}}_{i_3i_4} = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2.$$

L_2 is called the Gauss-Bonnet term in Physics. A direct computation gives

$$
\begin{aligned}
L_2 &= |W|^2 - 4\frac{n-3}{n-2}|Ric|^2 + \frac{n(n-3)}{(n-1)(n-2)}R^2 \\
&= |W|^2 + \frac{n-3}{n-2}(\frac{n}{n-1}R^2 - 4|Ric|^2) \qquad (42)\\
&= |W|^2 + 8(n-2)(n-3)\sigma_2.
\end{aligned}
$$

When $n = 4$, L_2 is the Euler density and its integration is the Euler characteristic. It is clear that by definition $L_k = c\sigma_k(g)$ if (M, g) is locally conformally flat.

As a generalization of the Einstein metric, the solution of the following equation is called *(string-inspired) Einstein-Gauss-Bonnet* metric

$$E^{(2)}_{\mu\nu} = \lambda g_{\mu\nu}.$$

$E^{(2)}$ was already given by Lanczos [17] in 1938 and is called Lanczos tensor. If g is such a metric, it is clear that

$$\lambda = \frac{1}{n}g^{\mu\nu}E^{(2)}_{\mu\nu} = \frac{4-n}{2n}L_2 = \frac{4-n}{2n}(8(n-2)(n-3)\sigma_2(g) + |W|^2).$$

Since $E^{(2)}$ is divergence free, namely

$$E^{(2),\beta}_{\alpha\beta} = 0,$$

it follows that λ must be constant.

It is naturally to consider the generalization of Einstein metrics for all $k < n/2$. We call a metric g is *k-Einstein* if

$$E^{(k)} = \lambda g,$$

with λ constant. Such metrics have been studied intensively in physical literatures and also by mathematicians. See for instance [20], [27] and [16]. One can show that if a metric g satisfies the property that its k-Einstein tensor proportional to itself pointwisely, ie.

$$E^{(k)} = \lambda g$$

for a function λ, then the λ is constant, which follows from the fact that $E^{(k)}$ is divergence free. This is a generalization of the Schur Lemma.

It is interesting to see if the almost Schur Lemma of De Lellis-Topping could be generalized. Theorem 4 gives an affirmative answer.

Proof of theorem 4. Let $R^{(k)} = L_k$. The proof is almost the same in [5]. Let f be the unique solution of

$$\Delta f = R^{(k)} - \overline{R}^{(k)},$$

with $\int f = 0$. Since $E^{(k)}$ is divergence-free, we have

$$dR^{(k)} = \frac{2n}{n-2k}\delta(E^{(k)} + \frac{n-2k}{2n}R^{(k)}g).$$

Their argument shows that

$$\int |R^{(k)} - \overline{R}^{(k)}|^2 \le \frac{2n}{n-2k} \|E^{(k)} + \frac{n-2k}{2n} R^{(k)} g\|_{L^2} \|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2}.$$

A Bochner formula gives

$$\|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2}^2 = \frac{n-1}{n} \int |R^{(k)} - \overline{R}^{(k)}|^2 - \int Ric(\nabla f, \nabla f)$$

Thus we have

$$\int |R^{(k)} - \overline{R}^{(k)}|^2 \le \frac{4n(n-1)}{(n-2k)^2} \int_M |E^{(k)} + \frac{n-2k}{2n} R^{(k)} g|^2 dv(g).$$

■

When $k = 1$ the inequality is equivalent to the almost Schur Lemma, Theorem A. If (M, g) is locally conformally flat, Theorems 3 and 4 are the same under slightly different conditions.

It is natural to ask the following Yamabe type problem.

Problem. Given a metric g_0 and an integer $k \in [2, n/2)$, is there a conformal metric $g \in [g_0]$ with

$$R^{(k)} = const.?$$

Especially, when $k = 2$ and $n > 4$, is there a conformal metric $g \in [g_0]$ with

$$R^{(2)} = 8(n-2)(n-3)\sigma_2(g) + |W|^2 = const.?$$

When (M, g_0) is locally conformally flat, $R^{(k)} = \sigma_k$. Thus, this problem is just the σ_k-Yamabe problem on a locally conformally flat manifold, which was solved already.

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