ON PROBLEMS RELATED TO AN INEQUALITY OF DE LELLIS AND TOPPING

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Abstract. In this paper we study various problems related to an inequality proved recently by De Lellis and Topping.

1. Introduction

In this paper we consider various problems related to a recent result of De Lellis and Topping about the Schur Lemma

**Theorem A** (Almost Schur Lemma [5]). For \( n \geq 3 \), if \((M^n, g)\) is a closed Riemannian manifold with non-negative Ricci tensor, then

\[
\int_M |\text{Ric} - \frac{R}{n} g|^2 dv(g) \leq \frac{n^2}{(n-2)^2} \int_M |\text{Ric} - \frac{R}{n} g|^2 dv(g),
\]

where \( R = \text{vol}(g)^{-1} \int_M R dv(g) \) is the average of the scalar curvature \( R \) of \( g \).

This result could be viewed as a quantitative version or a stability result of the Schur Lemma. Moreover, this result is optimal in the following sense: the constant in inequality (1) is the best and the non-negativity of the Ricci tensor can not be removed in general.

We observed in [8] that inequality (1) can be rewritten in terms of \( \sigma_k \)-scalar curvatures. Namely, it is equivalent to

\[
\left( \int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} \text{vol}(g) \int_M \sigma_2(g) dv(g),
\]

where

\[
\sigma_1(g) = \frac{R_g}{2(n-1)} \quad \text{and} \quad \sigma_2(g) = \frac{1}{2(n-2)^2} \left\{ -|\text{Ric}|^2 + \frac{n}{4(n-1)} R^2 \right\}.
\]

For the definition of \( \sigma_k(g) \) scalar curvature for general \( k \) see below. With this observation and a nice argument of Gursky [15] we improved Theorem A in \( n = 4 \).

**Theorem B** [8] If \( n = 4 \), and if \((M^4, g)\) is a closed Riemannian manifold with non-negative scalar tensor, then (1) holds. Moreover, equality holds if and only if \((M^4, g)\) is an Einstein manifold.

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From Theorem B, one may naturally ask whether equality in (1) holds if and only if \((M, g)\) is Einstein. The first result of this paper gives a positive answer.

**Theorem 1.** Equality in Theorem A holds if and only if \((M^n, g)\) is an Einstein manifold.

With the observation mentioned above, it is natural to consider the following Yamabe type functional

\[
E(g) = \frac{\text{vol}(g) \int_M \sigma_2(g) dv(g)}{\left( \int_M \sigma_1(g) dv(g) \right)^2},
\]

at least for metrics with \(\int_M \sigma_1(g) dv(g) \neq 0\). For a metric \(g\) with nonnegative Ricci tensor

Theorem A implies that

\[
E(g) \leq \frac{n-1}{2n}.
\]

Theorem B implies that (4) holds for metrics with nonnegative scalar curvature, when \(n = 4\). We conjectured in [8] that (4) for metrics with nonnegative scalar curvature if \(n = 3\). However (4) is not true in general for metrics nonnegative scalar curvature if \(n > 4\). In fact we have

**Theorem 2.** If \(n > 4\), for any metric \(g_0\) with positive Yamabe constant, which is equivalent to the condition that there is a metric in \([g_0]\) with positive scalar curvature, we have

\[
\sup_{g \in [g_0] \cap C_1} E(g) = \infty
\]

and

\[
Y([g_0]) := \inf_{g \in [g_0] \cap C_1} E(g) < \frac{n-1}{2n},
\]

where \([g_0]\) is the conformal class of \(g_0\) and \(C_k = \{g | \sigma_j(g) > 0 \forall j \leq k\}\). Moreover, we have

\[
Y([g_0]) > -\infty.
\]

Furthermore, we have

\[
Y([g_0]) > 0
\]

if and only if

\[
C_2([g_0]) := [g_0] \cap C_2 \neq \emptyset.
\]

In view of Theorem 2 it is natural to ask

**Problem.** Is there a conformal metric \(g \in [g_0] \cap C_1\) achieving the infimum \(Y([g_0])\) in \([g_0] \cap C_1\), namely

\[
E(g) = \inf_{g \in [g_0] \cap C_1} E(g)?
\]

Or is there a metric \(g \in [g_0] \cap C_1\) which is a critical point of \(E\) in \([g_0]\)?
Note that for the standard sphere \((\mathbb{S}^n, g_{\mathbb{S}^n})\) we have \(E(g_{\mathbb{S}^n}) = \frac{n-1}{2n}\), but
\[
0 < Y([g_{\mathbb{S}^n}]) = \inf_{\mathcal{C}_1} E(g) < \frac{n-1}{2n}.
\]

It is easy to see that the standard round metric \(g_{\mathbb{S}^n}\) is a critical point of \(E\). It would be interesting to know the value of \(Y([g_{\mathbb{S}^n}])\).

We are also interested in the generalization of (2) to large \(k\). Let us use the convention \(\sigma_0 = 1\). Hence we can rewrite (2) as
\[
\left( \int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} \int_M \sigma_0(g) dv(g) \int_M \sigma_2(g) dv(g).
\]

Note that the elementary symmetric functions \(\sigma_1\) and \(\sigma_2\) satisfy the Newton inequality \(\sigma_2 \geq \frac{2n}{n-1} \sigma_0 \sigma_2\). In general we have the Newton–MacLaurin formula for general \(k\)
\[
\sigma_k^2(\Lambda) \geq c(n,k) \sigma_{k-1}(\Lambda) \cdot \sigma_{k+1}(\Lambda),
\]
for \(\Lambda \in \Gamma_k^+ := \{\Lambda \in \mathbb{R}^n | \sigma_j(\Lambda) > 0 \forall j \leq k\}\). Here \(c(n,k) = (k+1)(n-k+1)/(n-k)k\)
and we used the convention that \(\sigma_k = 0\) if \(k < 0\) or \(k > n\). Inspired by Theorem A and Theorem B we would like to ask under which conditions there holds
\[
\left( \int_M \sigma_k(g) dv(g) \right)^2 \geq c(n,k) \int_M \sigma_{k-1}(g) dv(g) \int_M \sigma_{k+1}(g) dv(g).
\]

At least, if the underlying manifold \(M\) is locally conformally flat, we have a generalization of Theorem B.

**Theorem 3.** Let \(n \geq 3\) and \(k \in [n/2 - 1, n/2)\). When \((M^n, g)\) is locally conformally flat with \(g \in C_k\), then (11) holds. Moreover, equality holds if and only if \((M, g)\) is a space form.

Now one may ask if there is a De Lellis-Topping type result for a suitable “Ricci curvature” such that the corresponding Theorem B type result is Theorem 3. There are really such curvatures, the Lovelock curvatures, which were introduced by Lovelock, but at least went back to Lanczos [17] in 1938. For the definition, see [19] and Section 5 below. We remark here that the Lovelock curvatures are natural generalizations of the Einstein tensor, other than the Ricci tensor.

**Theorem 4.** Let \((M^n, g)\) be a closed Riemannian manifold with non-positive Ricci tensor and \(1 \leq k < n/2\). We have
\[
\int_M |R^{(k)} - \overline{R}^{(k)}|^2 dv(g) \leq \frac{4n(n-1)}{(n-2k)^2} \int_M |E^{(k)} + \frac{n-2k}{2n} R^{(k)} g|^2 dv(g),
\]
where \(\overline{R}^{(k)}\) is the average of \(R^{(k)}\). Here \(\text{tr}E^{(k)} = -\frac{n-2k}{2} \overline{R}^{(k)}\) is defined below.
When $k = 1$, Theorem 4 is just Theorem A. For a given $k > 1$, if $n = 2(k+1)$ and $(M^n, g)$ is local conformally flat, then Theorem 3 is just Theorem 4 with a slightly different condition $g \in \Gamma_k^+$. Note that the condition $g \in \Gamma_k^+$ with $k \geq n/2$ implies the condition that $g$ has non-negative Ricci curvature. See [12]. The condition $g \in \Gamma_k^+$ with $k < n/2$ is not stronger than the condition that $g$ has non-negative Ricci curvature.

The paper is organized as follows. In Section 2 we prove the rigidity result, Theorem 1. In Section 3 we first recall the definition of $\sigma_k$-scalar curvature and then prove Theorem 2 by choosing the suitable test metrics. In the construction of such metrics we need to pay extra attention to assure that all test metrics have positive scalar curvature. In Section 4 we generalize Theorem B to large $k$. In Section 5 we recall the definition of generalized Einstein tensors and then prove a De Lellis-Topping type result for these Einstein tensors.

2. Rigidity

**Proof of Theorem 1.** Assume that the equality holds, or equivalently,

$$
\int_M |R - \bar{R}|^2 dv(g) = \frac{4n(n-1)}{(n-2)^2} \int_M |Ric - \frac{R}{n} g|^2 dv(g).
$$

Then from the proof of Theorem A in [5] we know there exists some $\lambda \in \mathbb{R} \cup \{\infty\}$

$$
Ric - \frac{R}{n} g = \lambda (\nabla^2 f - \frac{\Delta f}{n} g),
$$

and

$$
Ric(\nabla f, \nabla f) = 0,
$$

where $\Delta f = R - \bar{R}$ with $\int_M f = 0$. Here by $\lambda = \infty$ we mean $\nabla^2 f - \frac{\Delta f}{n} g = 0$. In this case, integrating this equality we obtain

$$
\int_M |\nabla^2 f|^2 - \frac{(\Delta f)^2}{n} dv(g) = 0.
$$

From (14) we have

$$
\int_M |\nabla^2 f|^2 dv(g) = \int_M |\Delta f|^2 dv(g) = \int_M (R - \bar{R})^2 dv(g).
$$

Therefore,

$$
0 = \int_M |\Delta f|^2 dv(g) = \int_M (R - \bar{R})^2 dv(g)
$$

which, together with (12), means that $g$ is an Einstein metric.

Now we consider $\lambda \in \mathbb{R}$. In the following, we use the normal coordinates to calculate. Recall the fact the Ricci tensor is non-negative. From the Cauchy-Schwarz inequality and (14), for all $x \in M$ and all tangent vector $Y \in T_x M$,

$$
|Ric(\nabla f(x), Y)|^2 \leq Ric(\nabla f(x), \nabla f(x)) Ric(Y, Y) = 0,
$$

$$
Ric(\nabla f, \nabla f) = 0,
$$

and

$$
Ric(\nabla f, \nabla f) = 0.
$$

Therefore,

$$
0 = \int_M |\Delta f|^2 dv(g) = \int_M (R - \bar{R})^2 dv(g)
$$

which, together with (12), means that $g$ is an Einstein metric.
that is, $\text{Ric}(\nabla f, \cdot) = 0$, or equivalently

(17) $R^i_j f_i = 0$.

Here we use Einstein summation convention. From (13), (17) and $R_{ij,j} = \frac{1}{2} R_i$, we have

$$
\frac{1}{2} R_i = \lambda (f_{ij} - \frac{\Delta f g_{ij}}{n})_j + (\frac{R g_{ij}}{n})_j = \lambda f_{ij} - \lambda (\frac{\Delta f}{n})_i + \frac{R}{n} \\
= \lambda f_{ij} - \lambda (\frac{\Delta f}{n})_i + \frac{R}{n} \\
= (\lambda + \frac{1 - \lambda}{n}) R_i,
$$

which implies that either $\lambda = \frac{n-2}{2(n-1)}$ or $\lambda \neq \frac{n-2}{2(n-1)}$ and $R_i = 0$. In the latter case, $R$ is constant and it follows from (12) that $g$ is an Einstein metric. Now we consider the former case, i.e. $\lambda = \frac{n-2}{2(n-1)}$. (13) will be read as

(18) $\text{Ric} = \frac{n-2}{2(n-1)} \nabla^2 f + R \frac{2}{2(n-1)} g + (\frac{n-2}{2n}) \frac{R}{g}$.

By differentiating (17) and using $R_{ij,j} = \frac{1}{2} R_i$ we have

(19) $\frac{1}{2} R^i_i + R^j_j f_i = 0$.

Combining (18) and (19) gives

(20) $\frac{1}{2} R^i_i + \frac{n-2}{2(n-1)} |\nabla^2 f|^2 + \frac{R (R - \frac{R}{2})}{2(n-1)} g + \frac{(n-2)(R - \frac{R}{2})}{2n(n-1)} = 0$.

Since $M$ is compact, there exists some point $x_0 \in M$ such that $R(x_0) = \max R$. At this point, we have $R - \frac{R}{2} \geq 0$, $R \geq 0$, $\frac{R}{2} \geq 0$ and $R' = 0$. From (20) we have $\max R = \frac{R}{2}$, and hence $R \equiv \frac{R}{2}$. $g$ is also an Einstein metric in this case.

Theorem A and Theorem 1 give a characterization of Einstein metrics. We remark that a metric $g$ satisfying (13) is called an Ricci almost soliton in [21], which is a generalization of the Ricci soliton.

3. An equivalent inequality in terms of $\sigma_k$ scalar

Let us first recall the definition of the $k$-scalar curvature, which was first introduced by Viaclovsky [24] and has been intensively studied by many mathematicians, see for example [10], [25] and the references in [6]. There are many geometric applications of analysis developed in the study of the $k$-scalar curvature. For example, a 4-dimensional sphere Theorem was proved in [4] (see also [3], a 3-dimensional sphere Theorem in [6] and [1], an eigenvalue estimates for the Dirac operator in [26] and various geometric inequalities in [14]). The results of this section and next section are also applications of this analysis. Let

$$
S_g = \frac{1}{n-2} \left( \text{Ric}_g - \frac{R_g}{2(n-1)} \cdot g \right)
$$
be the Schouten tensor of $g$. For an integer $k$ with $1 \leq k \leq n$ let $\sigma_k$ be the $k$-th elementary symmetric function in $\mathbb{R}^n$. The $k$-scalar curvature is

$$\sigma_k(g) := \sigma_k(\Lambda_g),$$

where $\Lambda_g$ is the set of eigenvalue of the matrix $g^{-1} \cdot S_g$. In particular, $\sigma_1(g) = \text{tr} S$ and $\sigma_2 = \frac{1}{2}((\text{tr} S)^2 - |S|^2)$. It is trivial to see that

$$\sigma_1 = \frac{R}{2(n-1)},$$

$$\sigma_2 = \frac{1}{2(n-2)^2} \left\{ -|\text{Ric}|^2 + \frac{n}{4(n-1)}R^2 \right\},$$

$$\left| \text{Ric} - \frac{R}{n} g \right|^2 = |\text{Ric}|^2 - \frac{R^2}{n}.$$

From above it is easy to have the following observation.

**Observation.** ([8]) Inequality (1) is equivalent to (2).

In [8] we proved Theorem B, namely there is an inequality

(21)

$$\mathcal{E}(g) \leq \frac{n-1}{2n},$$

provided that $g$ is a metric of non-negative scalar curvature and $n = 4$. We conjectured that this statement is true for $n = 3$. In this Section we show Theorem 2, namely this statement is not true for $n > 4$.

We first prove one part of Theorem 2 in

**Proposition 1.** Let $n > 4$ and $g_0 \in C_1$. Then we have (7). Moreover, (8) holds if and only if

$$\mathcal{C}_2([g_0]) := [g_0] \cap \mathcal{C}_2 \neq \emptyset.$$

**Proof.** Recall the ordinary Yamabe constant $Y_1$ and another Yamabe type constant $Y_{2,1}$ studied in [6]

$$Y_1([g_0]) := \inf_{g \in \mathcal{C}_1([g_0])} \frac{\int_M \sigma_1(g) dv(g)}{(vol(g))^{\frac{n-2}{n}}} \quad \text{and} \quad Y_{2,1}( [g_0]) := \inf_{g \in \mathcal{C}_1([g_0])} \frac{\int_M \sigma_2(g) dv(g)}{(\int_M \sigma_1(g)dv(g))^{\frac{n-4}{n-2}}}.$$
By a direct computation we have in [6]
\[
2 \int \sigma_2(g) dv(g) = \frac{n-4}{2} \int \sigma_1(g) |\nabla u|^2 e^{2u} dv(g) + \frac{n-4}{4} \int |\nabla u|^{4} e^{4u} dv(g) \\
+ \int e^{2u} \sigma_1(g) \sigma_1(g) dv(g) - \int e^{4u} |S(g_0)|^2 dv(g) \\
+ (4-n) \int \sum_{i,j} S(g_0)^{ij} u_i u_j dv(g) + \int \sigma_1(g_0) |\nabla u|^2 e^{4u} dv(g) \\
+ \int e^{4u} (\nabla u, \nabla \sigma_1(g_0))_{g_0} dv(g).
\]
(22)

It follows that
\[
\int \sigma_2(g) dv(g) \geq \frac{n-4}{16} \int |\nabla u|^4_{g_0} e^{4u} dv(g) - c \int e^{4u} dv(g),
\]
provided that \(g \in \Gamma_1^+\). Moreover we have
\[
\int_M \sigma_1(g) dv(g) = \int \left( \frac{n-2}{2} |\nabla u|^2 + \sigma_1(g_0) \right) e^{2u} dv(g) \geq Y_1([g_0]) (\text{vol}(g))^{\frac{n-2}{n}}.
\]
(23)

From (23), (24) and Hölder’s inequality, we have
\[
\int \sigma_2(g) dv(g) - c' \int e^{4u} dv(g) \geq c(\int |\nabla u|^4_{g_0} e^{4u} dv(g) + \int e^{4u} dv(g)) \\
\geq c(\int_M \sigma_1(g) dv(g))^{2}(\text{vol}(g))^{-1}.
\]
(25)

so that
\[
\int \sigma_2(g) dv(g) \geq (c_1 - c_2 Y_1([g_0])^{-2}) (\int_M \sigma_1(g) dv(g))^{2}(\text{vol}(g))^{-1}.
\]

that is, \(Y([g_0]) \geq c_1 - c_2 Y_1([g_0])^{-2} > -\infty\). This proves (7).

Now we assume that (8) hold. Since \(g_0 \in C_1\) we have \(Y_1([g_0]) > 0\). It is clear that for \(g \in C_1([g_0])\)
\[
\left( \int_M \sigma_2(g) dv(g) \right)^{\frac{n-4}{n-2}} = \mathcal{E}(g) \left( \int_M \sigma_1(g) dv(g) \right)^{\frac{n}{n-2}} \geq Y([g_0]) (Y_1([g_0]))^{\frac{n}{n-2}}.
\]

It follows that \(Y_{2,1}([g_0]) > 0\), which is equivalent to the non-emptiness of \(C_2([g_0])\) by a
result in [6]. See also [23].

Now assume the non-emptiness of \(C_2([g_0])\). Let \(g \in C_1([g_0])\). First define a nonlinear
eigenvalue of \(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}\) by
\[
\lambda(g_0, \sigma_2) := \inf_{g = e^{-2u} \in C_1([g_0])} \frac{\int \sigma_2(g) dv(g)}{\int e^{4u} dv(g)}.
\]
We have proved in [6] that \( \lambda(g_0, \sigma_2) > 0 \), i.e.,

\[
(26) \quad \int_M \sigma_2(g) \geq \lambda(g_0, \sigma_2) \int_M e^{4u} dv(g)
\]

for any \( g = e^{-2u}g_0 \in \Gamma_1^+ \).

From (23), (24), (26) and Hölder’s inequality, we deduce

\[
(27) \quad \int_M \sigma_2(g) dv(g) \geq c \left( \int_M |\nabla u|_{g_0}^4 e^{4u} dv(g) + \int_M e^{4u} dv(g) \right) \geq c \left( \int_M \sigma_1(g) dv(g) \right)^2 (\text{vol}(g))^{-1}.
\]

This is what we want to show.

Proposition 2. Let \( n \geq 3 \) and \( C_1([g_0]) \neq \emptyset \). Then there exist a metric \( g \in C_1([g_0]) \) with \( E(g) < \frac{n-2}{2n} \).

Proof. Let \( \tilde{g} \in C_1([g_0]) \) be a Yamabe solution, i.e., \( \sigma_1(\tilde{g}) = \text{const} \). From the Newton inequality \( \sigma_2(\Lambda) \leq \frac{n-2}{2n} \sigma_1^2(\Lambda) \) for any \( \Lambda \in \mathbb{R}^n \) and equality holds if and only if \( \Lambda = c(1,1,\cdots,1) \) for some \( c \in \mathbb{R} \), we have

\[
\int_M \sigma_2(\tilde{g}) dv(\tilde{g}) \leq \frac{n-2}{2n} \int_M \sigma_1(\tilde{g})^2 dv(\tilde{g}) = \frac{n-2}{2n} \left( \int_M \sigma_1(\tilde{g}) dv(\tilde{g}) \right)^2 \text{vol}(\tilde{g})^{-1}.
\]

Hence \( E(\tilde{g}) \leq \frac{n-2}{2n} \). From above it is easy to see that \( E(\tilde{g}) = \frac{n-2}{2n} \) if and only if \( \tilde{g} \) is an Einstein metric. In this case, it is clear that \( \text{Ric}(\tilde{g}) \) is positive definite. Then we choose a nearby, not Einstein metric \( \tilde{g}_1 \) with positive Ricci tensor. Then by Theorem 1, we have \( E(\tilde{g}_1) < \frac{n-2}{2n} \).

The proof is motivated by an argument of Gursky in [15].

Now we remain to prove

Proposition 3. Let \( n > 4 \) and \( C_1([g_0]) \neq \emptyset \). Then (5) holds. Namely

\[
\bar{Y}([g_0]) := \sup_{g \in [g_0] \cap C_1} E(g) = \infty.
\]
To prove the Proposition we use the gluing method developed by Gromov-Lawson [9] (see also [22]).

Improving slightly the construction given in [7], which is motivated by [9] and [22], we have

**Lemma 1.** Assume $n > 4$. Let $g_0$ be in Theorem 2. For any small constant $\delta, \lambda \in (0,1)$ such that $\frac{\lambda^2}{\delta} \gg \delta \gg \frac{1}{\delta}$, there exists a constant $\delta_1 > 0$ and a function $u : \mathbb{R}^n \to \mathbb{R}$ satisfying:

1. $\delta_1 = \lambda^{-1} \delta^3$, $\delta << \delta_1 << \frac{1}{\delta}$,
2. The metric $g = e^{-2u} g_0$ has positive scalar curvature in $B_{\delta_1}$,
3. $u = \log(\lambda + |x|^2) + b_0$ for $|x| \leq \delta$,
4. $u = \log |x|$ for $|x| \geq \delta_1$,
5. $\text{vol}(B_{\delta_1} \cup B_{\delta}, g) = O(\delta_1^n \delta^{-n})$, $\int_{B_{\delta_1} \setminus B_{\delta}} \sigma_1(g) dv(g) = O(\delta_1^{n-\frac{2}{3} \delta^2 - n})$ and $\int_{B_{\delta_1} \setminus B_{\delta}} \sigma_2(g) dv(g) = O(\delta_1^{n-4 \delta^2 - n})$, where $b_0 = -\log \delta_1 + O(1)$.

**Lemma 2.** Assume $n > 4$. Let $g_0$ be in Theorem 2. For any small constant $\delta, \lambda \in (0,1)$ such that $\frac{\lambda^2}{\delta} \gg \delta \gg \frac{1}{\delta}$, define a conformal metric $g = e^{-2u} g_0$ with $u(x) = \log(\lambda + |x|^2) + b_0$ in $B_{\delta}$, where $b_0$ is some constant. Then $g$ has positive scalar curvature in $B_{\delta}$ and we have the following

\begin{align*}
\text{vol}(B_{\delta}, g) &= e^{-nb_0} \lambda^{-\frac{n}{2}} [B + O(\lambda) + O((\lambda\delta^{-2}) \frac{n}{2})], \\
\int_{B_{\delta}} \sigma_1(g) dv(g) &= e^{(2-n)b_0} \lambda^{1-\frac{n}{2}} [2nB + O(\lambda) + O((\lambda\delta^{-2}) \frac{n}{2})], \\
\int_{B_{\delta}} \sigma_2(g) dv(g) &= e^{(4-n)b_0} \lambda^{2-\frac{n}{2}} [2n(n-1)B + O(\lambda) + O((\lambda\delta^{-2}) \frac{n}{2})].
\end{align*}

Here $B = \int_{\mathbb{R}^n} \frac{1}{(1+|x|^2)^n} dx$.

**Lemma 3.** Let $g_0$ be as in Theorem 2, $n > 4$ and $B_{r_0}$ be a geodesic ball with respect to $g_0$ for some $r_0$. Then there exists a conformal metric $g = e^{-2u} g_0$ in $B_{r_0} \setminus \{0\}$ satisfying:

1. The metric $g = e^{-2u} g_0$ has positive scalar curvature in $B_{r_0} \setminus \{0\}$,
2. $u = \log |x|$ for $|x| \leq r_2$,
3. $u = b_1$ for $|x| \geq r_1$,

where $r_2 < r_1 < r_0$ and $b_1$ is a constant.

**Proof of Proposition 3.** Let $\{x_k\}_{k=1}^K$ be $K$ points in $M$ and $B_{r_0}(x_k)$ be disjoint geodesic balls centered as $x_k$ with radius $r_{0K}$, where $r_{0K} \to 0$ as $K \to \infty$. For any $K \in \mathbb{N}$, we choose some $\delta = o(K^{-\gamma})$ such that $\delta_K := K^\gamma \delta \to 0$ as $K \to \infty$ for some $\gamma$ chosen later.

For simplicity, set $\delta_k = \lambda_k^2$, which satisfies the assumption on $\delta, \lambda$ in Lemma 1 and define $\delta_{1k} = \lambda_k^{1-\delta} = \delta_1^2$ and $b_{0k}$ as in Lemma 1. Also define $r_{1K}, r_{2K} \leq r_{0K}$ and $b_{1K}$ as in Lemma 3 (independent of $k$). We point out that $r_{1K}$ and $r_{2K}$ can be chosen as small as we want. For sufficient small $\delta_k$ with $\delta_{1k} \leq r_{2K}$, define a sequence of metrics $g_K = e^{-2u_K} g_0$ as follows. In $M \setminus B_{r_{0K}}(x_k)$, $g = e^{-2b_{1K}} g_0$, where $b_{1K}$ (independent of $k$) is given in Lemma
3. We define

\[
    u_K = \begin{cases} 
    \log(\lambda_k + |x - x_k|^2) + b_0k, & x \in B_{\delta_k}(x_k) \\
    \log |x - x_k|, & x \in B_{r_{2K}}(x_k) \setminus B_{\delta_k}(x_k) \\
    b_1K, & x \in M \setminus \bigcup_{k=1}^K B_{r_{1K}}(x_k) 
    \end{cases}
\]  

(31)

and in $B_{\delta_k}(x_k) \setminus B_{\delta_k}(x_k)$, we define $u_K$ as in Lemma 1, while in $B_{r_{1K}}(x_k) \setminus B_{r_{2K}}(x_k)$ we define $u_K$ as in Lemma 3. From the construction in Lemma 1 and Lemma 3, we see that $g_K$ is smooth and has positive scalar curvature. It follows directly from Lemma 1 and Lemma 2 that

\[
\begin{align*}
    \text{vol}(B_{\delta_k}(x_k), g_K) &= \delta_k^{-\frac{1}{2}n} [B + O(\delta_k^2) + O(\delta_k^n)], \\
    \int_{B_{\delta_k}(x_k)} \sigma_1(g_K) dv(g_K) &= \delta_k^{1 - \frac{1}{2}n} [2nB + O(\delta_k^2) + O(\delta_k^n)], \\
    \int_{B_{\delta_k}(x_k)} \sigma_2(g_K) dv(g_K) &= \delta_k^{-\frac{1}{2}n} [2n(n-1)B + O(\delta_k^2) + O(\delta_k^n)], \\
    \int_{B_{\delta_k}(x_k)} \sigma_1(g_K) dv(g_K) &= O(\delta_k^{-\frac{2}{n}}) = \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n}{2}}), \\
    \int_{B_{\delta_k}(x_k)} \sigma_2(g_K) dv(g_K) &= O(\delta_k^{-\frac{3}{n}}) = \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n-2}{2}}).
\end{align*}
\]

One can also choose $r_{1K}$ and then $r_{2K}$ sufficiently far away from $\delta_k$ for any $k = 1, \ldots, K$ such that

\[
\begin{align*}
    \text{vol}(B_{r_{2K}}(x_k) \setminus B_{\delta_k}(x_k), g_K) &= \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n}{2}}), \\
    \int_{B_{r_{2K}}(x_k) \setminus B_{\delta_k}(x_k)} \sigma_1(g_K) dv(g_K) &= \delta_k^{1 - \frac{1}{2}n} O(\delta_k^{\frac{n-2}{2}}), \\
    \int_{B_{r_{2K}}(x_k) \setminus B_{\delta_k}(x_k)} \sigma_2(g_K) dv(g_K) &= \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n-4}{2}}), \\
    \text{vol}(M \setminus \bigcup_{k=1}^K B_{r_{2K}}(x_k), g_K) &= f_0(r_{2K}) = \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n}{2}}), \\
    \int_{M \setminus \bigcup_{k=1}^K B_{r_{2K}}(x_k)} \sigma_1(g_K) dv(g_K) &= f_1(r_{2K}) = \delta_k^{1 - \frac{1}{2}n} O(\delta_k^{\frac{n-2}{2}}), \\
    \int_{M \setminus \bigcup_{k=1}^K B_{r_{2K}}(x_k)} \sigma_2(g_K) dv(g_K) &= f_2(r_{2K}) = \delta_k^{-\frac{1}{2}n} O(\delta_k^{\frac{n-4}{2}}).
\end{align*}
\]

for some functions $f_i$, $i = 0, 1, 2$. Combining all the above estimates and using $\delta_k = k^2 \delta$, we obtain
\[ E(g_K) = \frac{\text{vol}(g_K) \int_M \sigma_2(g_K) dv(g_K)}{\left( \int_M \sigma_1(g_K) dv(g_K) \right)^2} = \sum_{k=1}^K k^{-\frac{1}{2}n\gamma} \sum_{k=1}^K k^{(2-\frac{1}{2}n)\gamma} \left[ \frac{n-1}{2n} + o(1) \right]. \]

Choose \( \gamma \) such that \( (1 - \frac{1}{2}n)\gamma = -1 - \beta \) with \( \beta \in (0, \frac{2}{n-2}) \). Then we have

\[ -\frac{1}{2} n\gamma = \frac{n}{n-2}(-1 - \beta) < -1, \quad (2 - \frac{1}{2} n)\gamma = \frac{n-4}{n-2}(-1 - \beta) > -1. \]

Therefore, \( \sum_{k=1}^\infty k^{-\frac{1}{2}n\gamma} \) and \( \sum_{k=1}^\infty k^{(1-\frac{1}{2}n)\gamma} \) converge, meanwhile \( \sum_{k=1}^\infty k^{(2-\frac{1}{2}n)\gamma} \) diverges.

In view of (32), we see that \( E(g_K) \) can be made to be arbitrary large when \( K \) goes to infinity. Hence we finished the proof of (5).

**Remark 1.** Using Lemmas given above and an argument from Aubin, we can show a weaker form of (6).

\[ Y([g_0]) := \inf_{g \in [g_0] \cap C_k} E(g) \leq \frac{n-1}{2n}. \]

This is an Aubin type inequality. Using the same gluing argument we can show the metrics constructed in Lemma 1 and Lemma 3 are in the class \( \Gamma_k^+ \), provided \( g_0 \in \Gamma_k^+ \) and \( k < n/2 \), and hence

\[ \inf_{g \in [g_0] \cap C_k} E(g) \leq \frac{n-1}{2n}, \]

for any \( k < n/2 \), provided that \( [g_0] \cap C_k \neq \emptyset \). We do not know if the inequality in (34) is strict, though we believe this. Similarly, one can show a slightly stronger form of (5)

\[ \sup_{g \in [g_0] \cap C_k} E(g) = \infty, \]

for any \( k < n/2 \), provided that \( [g_0] \cap C_k \neq \emptyset \). Comparing with the inequality of De Lellis-Topping (1), i.e.,

\[ E(g) \leq \frac{n-1}{2n}, \quad \text{for any } g \text{ with } \text{Ric} \geq 0, \]

it indicates that the condition \( \text{Ric} \geq 0 \) is “stronger” than the condition \( g \in C_k \) with \( k < n/2 \). Remark that a metric \( g \in C_k \) with \( k \geq n/2 \) have positive Ricci tensor [12].

### 4. A geometric inequality for large \( k \)

In this Section, we will prove Theorem 3, namely

\[ \left( \int_M \sigma_k(g) dv(g) \right)^2 \geq c(n, k) \int_M \sigma_{k-1}(g) dv(g) \int_M \sigma_{k+1}(g) dv(g) \]

\[ \quad \left( \int_M \sigma_k(g) dv(g) \right)^2 \geq c(n, k) \int_M \sigma_{k-1}(g) dv(g) \int_M \sigma_{k+1}(g) dv(g) \]
holds if \((M, g)\) is locally conformally flat and \(g \in \Gamma_k^+\) with \(k \in [n/2 - 1, n/2)\). The constraint \(k \in [n/2 - 1, n/2)\) equals to

\[
k = \begin{cases} 
  \frac{n-1}{2}, & \text{if } n \text{ is odd}, \\
  \frac{n}{2} - 1, & \text{if } n \text{ is even}.
\end{cases}
\]

**Proof of Theorem 3.** First all, we may assume that \(\int_M \sigma_{k+1}(g) dv(g) > 0\).

We first consider the case \(n\) is even and \(k = \frac{n}{2} - 1\). In this case we use the argument of Gursky [15] as in [8] and a following Yamabe problem

\[
Y_k([g]) := \inf_{\hat{g} \in C_k([g])} \frac{\int_M \sigma_k(\hat{g}) dv(\hat{g})}{\left( \int_M \sigma_{k-1}(\hat{g}) dv(\hat{g}) \right)^{n/(4(k-1))}},
\]

where \(C_k([g]) := [g] \cap \Gamma_k^+\), which was studied in [14]. Since \((M, g)\) is locally conformally flat, it was proved in [14] that \(Y_k\) is achieved by a conformal metric \(g_k \in C_k\) satisfying

\[
(37) \quad \frac{\sigma_k(g_k)}{\sigma_{k-1}(g_k)} = a_k,
\]

for some constant \(a_k > 0\), which implies that \(\int_M \sigma_k(g_k) dv(g_k) = a_k \int_M \sigma_{k-1}(g_k) dv(g_k)\).

Now by (10) we have

\[
\int_M \sigma_{k+1}(g_k) dv(g_k) \leq c(n, k) \int_M \frac{\sigma_k(g_k)^2}{\sigma_{k-1}(g_k)} dv(g_k) = c(n, k) a_k \int_M \sigma_k(g_k) dv(g_k) = c(n, k) \left( \int_M \frac{\sigma_k(g_k) dv(g_k)}{\left( \int_M \sigma_{k-1}(g_k) dv(g_k) \right)^{1/2}} \right)^2 = c(n, k) Y_k([g_k])^2,
\]

where we have used that \(k = n/2 - 1\). Since \(k + 1 = n/2\) and the manifold is locally conformally flat, we know that \(\int \sigma_{k+1}(g) dv(g)\) is constant in a given conformal class [24]. Hence we have

\[
\int_M \sigma_{k+1}(g) dv(g) = \int_M \sigma_{k+1}(g_k) dv(g_k) \leq c(n, k) Y_k([g_k])^2 \leq c(n, k) \left( \int_M \frac{\sigma_k(g) dv(g)}{\left( \int_M \sigma_{k-1}(g) dv(g) \right)^{1/2}} \right)^2.
\]
In the last inequality we have used that $g_k$ achieves the minimum $Y_k$. From the proof it is clear that equality holds if and only if

$$\sigma_{k+1}(g)\sigma_{k-1}(g) = c(n, k)\sigma_k^2(g),$$

that is, $g$ is an Einstein metric.

Now we consider the case that $n$ is odd and $k = \frac{n-1}{2}$. In this case we consider the following Yamabe type problem.

Define

$$E_k(g) := \int_M \sigma_{k-1}(g)dv(g) \int_M \sigma_{k+1}(g)dv(g)\left(\int_M \sigma_k(g)dv(g)\right)^2$$

and

$$\tilde{Y}_k([g_0]) := \sup_{g \in \mathcal{C}_k([g_0])} E_k(g).$$

The Euler-Lagrange equation of (38) is a Yamabe type equation

$$\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g) = -2s_k(g),$$

where $r_k(g)$ and $s_k(g)$ are two positive constants defined by

$$r_k(g) = \int_M \sigma_{k+1}(g)dv(g) \int_M \sigma_{k-1}(g)dv(g)$$

and

$$s_k(g) = \int_M \sigma_{k+1}(g)dv(g) \int_M \sigma_{k-1}(g)dv(g).$$

By the key Lemma in [6] we have: For $g_0 \in \Gamma_k^+$ Equation (39) is an elliptic and concave equation. We want to find the maximum of $E_k$, $\tilde{Y}_k([g_0])$. In order to do so, we consider a Yamabe type flow

$$-g^{-1} \cdot \frac{d}{dt} g = \frac{\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g)}{\sigma_k(g)} + 2s_k(g).$$

**Proposition 4.** Flow (40) preserves $\int_M \sigma_k(g)dv(g)$, while it increases

$$\int_M \sigma_{k-1}(g)dv(g) \int_M \sigma_{k+1}(g)dv(g).$$

**Proof.** It is clear that the flow preserves $\int_M \sigma_k(g)dv(g)$. By a direct computation we have

$$\frac{d}{dt} \left( \int_M \sigma_{k-1}(g)dv(g) \int_M \sigma_{k+1}(g)dv(g) \right)$$

$$= -\frac{1}{2} \int_M \sigma_{k-1}(g)dv(g) \int_M (\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g))g^{-1} \cdot \frac{d}{dt} g$$

$$= \frac{1}{2} \int_M \sigma_{k-1}(g)dv(g) \int_M \sigma_k(g) \left(\frac{\sigma_{k+1}(g) - 3r_k(g)\sigma_{k-1}(g)}{\sigma_k(g)} + 2s_k(g)\right)^2 \geq 0.$$
Proposition 5. Flow (40) is a parabolic equation.

Proof. See [6].

Since \((M, g)\) is locally conformally flat, we can use the argument in [13] to show that the flow converges to a solution of (39). This argument used a crucial argument in [28] for the ordinary Yamabe flow, to show that there is a uniform estimate for gradients. Here we will not repeat it. Hence for any \(g \in [g_0] \cap C_k\) by using flow (40) we find a \(\tilde{g} \in [g_0] \cap C_k\) satisfying (39). Since the flow increases \(E_k\) we have \(E_k(g) \leq E(\tilde{g})\). Now one can show that \(\tilde{g}\) is in fact a metric with constant sectional curvature.

Theorem 5. Let \(n\) be odd and \(k = (n - 1)/2\). If \((M, g)\) is a locally conformally flat with \(g \in \Gamma_k^+\) and \(\int_M \sigma_{k+1}(g)dv(g) > 0\), then there is a conformal metric \(g_1 \in [g]\) with constant sectional curvature.

Proof. The proof follows from the proof given in [6] directly. In fact the argument would imply the cone \(\Gamma^+_{k+1}\) is not empty. Then it follows from [12] \((M, g)\) has positive Ricci curvature. By Theorem of Myers, \(\pi_1(M)\) is finite. Hence the universal cover of \(M\) is compact and locally conformally flat and thus conformal to the standard \(n\)-sphere. The argument in [1] would also work. See also closely related results in [11] and [2].

By this Theorem 5, without loss of generality we may assume that \((M, g_0)\) is the standard round metric. Since \(\tilde{g}\) satisfies a conformal equation (39), the classification result in [18] implies that \(\tilde{g}\) is also a metric with constant sectional curvature, and hence \(\mathcal{E}(\tilde{g}) = c(n, k)\). Therefore we have proved

\[
\mathcal{E}_k(g) \leq \mathcal{E}(\tilde{g}) = c(n, k).
\]

Equality holds if and only if \(\mathcal{E}_k(g) = c(n, k)\), which means that \(g\) is a maximum of \(\mathcal{E}_k\) and hence satisfies (39). By Theorem 5 again, \((M, g)\) is a space form. Now we complete the proof of Theorem 3.

5. LOVELOCK

In this section, let us first recall the work of Lovelock [19] on generalized Einstein tensors. See also [20], [27] and [16].

Let

\[
E_{AB} = R_{AB} - \frac{1}{2}Rg_{AB}
\]

be the Einstein tensor. It is clear that \(g\) is an Einstein metric if and only if

\[
E_{AB} = \lambda g_{AB}.
\]

The Einstein tensor is very important in theoretical physics. It is a conversed quantity, i.e.,

\[
E^B_{A,B} = 0.
\]

It would be an interesting to generalize the Einstein tensor. In [19] Lovelock studied the classification of tensors \(A\) satisfying
(i) $A^i = A^j$, i.e., $A$ is symmetric.
(ii) $A^i = A^j (g_{AB}, g_{AB,C}, g_{AB,CD})$.
(iii) $A^i j = 0$, i.e., $A$ is divergence-free.
(iv) $A^i j$ is linear in the second derivatives of $g_{AB}$.

It is clear that the Einstein tensor satisfies all conditions. Lovelock classified all 2-tensors satisfying (i)–(iii). Let us first define

$$L_k = R^{(k)} := \frac{1}{2k} \delta^{i_1 j_1 \cdots i_2 j_2 \cdots i_{2k-1} j_{2k}} R_{i_1 i_2 \cdots j_1 j_2 \cdots j_{2k-1} j_{2k}}.$$  

Here the generalized Kronecker delta is defined by

$$\delta_{i_1 j_1 \cdots i_r j_r} = \det\begin{pmatrix}
\delta_{j_1 i_1} & \delta_{j_2 i_1} & \cdots & \delta_{j_r i_1} \\
\delta_{j_1 i_2} & \delta_{j_2 i_2} & \cdots & \delta_{j_r i_2} \\
\vdots & \vdots & \ddots & \vdots \\
\delta_{j_1 i_r} & \delta_{j_2 i_r} & \cdots & \delta_{j_r i_r}
\end{pmatrix}.$$  

$L_k$ is called the Lovelock curvature. When $2k = n$, $R^{(k)}$ is the Euler density. We could check that $R^{(k)} = 0$ if $2k > n$. For $k < n/2$, $R^{(k)}$ is called the dimensional continued Euler density in Physics. Let us define a 2-tensor $E^{(k)}$ by

$$E^{(k)}_{i j} := \frac{1}{2k+1} g_{AB} \delta^{i_1 j_1 \cdots i_2 j_2 \cdots i_{2k-1} j_{2k}} R_{i_1 i_2 \cdots j_1 j_2 \cdots j_{2k-1} j_{2k}}$$  

locally. It is clear that

$$\text{tr} E^{(k)} = -\frac{n - 2k}{2} R^{(k)}.$$  

One can check that

$$E^{(k)}_{i j} j_i = 0,$$

i.e., $E^{(k)}$ satisfies (i)–(iii). Lovelock proved that any 2-tensor satisfying (i)–(iii) has the form

$$\sum_j \alpha_j E^{(j)}$$  

with certain constants $\alpha_j$, $j \geq 0$. Here we set $E^{(0)} = 0$. It is clear to see that $E^{(1)}$ is the Einstein tensor and

$$R^{(1)} = R,$$

which is the scalar curvature.

One can also check that

$$E^{(2)}_{\mu \nu} = 2 R_{\mu \nu} - 4 R_{\mu \alpha} R^\alpha \nu - 4 R_{\mu \alpha \beta \gamma} R^\alpha \beta \nu + 2 R_{\mu \alpha \beta} R_{\nu \beta \gamma} - \frac{1}{2} g_{\mu \nu} L_2$$  

and

$$L_2 = \frac{1}{4} \delta^{i_1 j_1 i_2 j_2 i_3 j_3 i_4} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} R_{i_1 i_2 j_1 j_2} R_{i_3 i_4 j_3 j_4} = R_{\mu \nu \rho \sigma} R^{\mu \nu \rho \sigma} - 4 R_{\mu \nu} R^{\mu \nu} + R^2.$$
$L_2$ is called the Gauss-Bonnet term in Physics. A direct computation gives

\begin{align}
L_2 &= |W|^2 - 4 \frac{n-3}{n-2} |\text{Ric}|^2 + \frac{n(n-3)}{(n-1)(n-2)} R^2 \\
&= |W|^2 + \frac{n-3}{n-2} \left( \frac{n}{n-1} R^2 - 4 |\text{Ric}|^2 \right) \\
&= |W|^2 + 8(n-2)(n-3) \sigma_2.
\end{align}

When $n = 4$, $L_2$ is the Euler density and its integration is the Euler characteristic. It is clear that by definition $L_k = c \sigma_k(g)$ if $(M, g)$ is locally conformally flat.

As a generalization of the Einstein metric, the solution of the following equation is called (string-inspired) Einstein-Gauss-Bonnet metric

$$E^{(2)}_{\mu\nu} = \lambda g_{\mu\nu}.$$  

$E^{(2)}$ was already given by Lanczos [17] in 1938 and is called Lanczos tensor. If $g$ is such a metric, it is clear that

$$\lambda = \frac{1}{n} g^{\mu\nu} E^{(2)}_{\mu\nu} = \frac{4-n}{2n} L_2 = \frac{4-n}{2n} \left( 8(n-2)(n-3) \sigma_2(g) + |W|^2 \right).$$

Since $E^{(2)}$ is divergence free, namely

$$E^{(2)}_{\alpha \beta ; \beta} = 0,$$

it follows that $\lambda$ must be constant.

It is naturally to consider the generalization of Einstein metrics for all $k < n/2$. We call a metric $g$ is $k$-Einstein if

$$E^{(k)} = \lambda g,$$

with $\lambda$ constant. Such metrics have been studied intensively in physical literatures and also by mathematicians. See for instance [20], [27] and [16]. One can show that if a metric $g$ satisfies the property that its $k$-Einstein tensor proportional to itself pointwisely, ie.

$$E^{(k)} = \lambda g$$

for a function $\lambda$, then the $\lambda$ is constant, which follows from the fact that $E^{(k)}$ is divergence free. This is a generalization of the Schur Lemma.

It is interesting to see if the almost Schur Lemma of De Lellis-Topping could be generalized. Theorem 4 gives an affirmative answer.

**Proof of theorem 4.** Let $R^{(k)} = L_k$. The proof is almost the same in [5]. Let $f$ be the unique solution of

$$\Delta f = R^{(k)} - \overline{R}^{(k)},$$

with $\int f = 0$. Since $E^{(k)}$ is divergence-free, we have

$$dR^{(k)} = \frac{2n}{n-2k} \delta (E^{(k)}) + \frac{n-2k}{2n} R^{(k)} g.$$
Their argument shows that
\[
\int |R^{(k)} - \bar{R}^{(k)}|^2 \leq \frac{2n}{n-2k} \|E^{(k)}\| + \frac{n-2k}{2n} R^{(k)} g \|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2}.
\]
A Bochner formula gives
\[
\|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2}^2 = \frac{n-1}{n} \int |R^{(k)} - \bar{R}^{(k)}|^2 - \int \text{Ric}(\nabla f, \nabla f)
\]
Thus we have
\[
\int |R^{(k)} - \bar{R}^{(k)}|^2 \leq \frac{4n(n-1)}{(n-2k)^2} \int_M |E^{(k)}| + \frac{n-2k}{2n} R^{(k)} g \|\nabla^2 f - \frac{\Delta f}{n} g\|_{L^2}^2 dv(g).
\]

When $k = 1$ the inequality is equivalent to the almost Schur Lemma, Theorem A. If $(M, g)$ is locally conformally flat, Theorems 3 and 4 are the same under slightly different conditions.

It is natural to ask the following Yamabe type problem.

**Problem.** Given a metric $g_0$ and an integer $k \in [2, n/2)$, is there a conformal metric $g \in [g_0]$ with
\[ R^{(k)} = \text{const}. \]
Especially, when $k = 2$ and $n > 4$, is there a conformal metric $g \in [g_0]$ with
\[ R^{(2)} = 8(n-2)(n-3)\sigma_2(g) + |W|^2 = \text{const}. \]

When $(M, g_0)$ is locally conformally flat, $R^{(k)} = \sigma_k$. Thus, this problem is just the $\sigma_k$-Yamabe problem on a locally conformally flat manifold, which was solved already.

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