GENERALIZED GINZBURG-LANDAU EQUATIONS IN HIGH DIMENSIONS

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ABSTRACT. In this work, we study critical points of the generalized Ginzburg-Landau equations in dimensions $n \geq 3$ which satisfy a suitable energy bound, but are not necessarily energy-minimizers. When the parameter in the equations tend to zero, such solutions are shown to converge to singular n-harmonic maps into spheres, and the convergence is strong away from a finite set consisting 1) of the infinite energy singularities of the limiting map, and 2) of points where bubbling off of finite energy n-harmonic maps could take place. The latter case is specific to dimensions greater than 2. We also exhibit a criticality condition satisfied by the limiting n-harmonic maps which constrains the location of the infinite energy singularities. Finally we construct an example of non-minimizing solutions to the generalized Ginzburg-Landau equations satisfying our assumptions.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ be a bounded smooth domain. Given $g : \partial\Omega \to \mathbb{S}^{n-1}$ a smooth prescribed map with the degree $d = \deg(g, \partial\Omega, \mathbb{S}^{n-1})$, we consider the functional

$$\mathbf{E}_\varepsilon(u,\Omega) = \int_\Omega \left[\frac{|\nabla u|^n}{n} + \frac{1}{4\varepsilon^n}\left(1-|u|^2\right)^2 \right] \mathrm{d}x \tag{1.1}$$

for $u \in W^{1,n}_g(\Omega, \mathbb{R}^n) = \left\{ w \in W^{1,n}(\Omega,\mathbb{R}^n) : w|_{\partial\Omega} = g \right\}$.

In the case of $n = 2$, the minimizers and critical points of this functional were studied by F.Bethuel, H.Brezis and F.Hélein [3] and many authors after them. In this case the critical points satisfy the so called Ginzburg-Landau system

$$\begin{cases} -\Delta u_\varepsilon &= \frac{1}{\varepsilon^2}\left(1-|u_\varepsilon|^2\right) u_\varepsilon & \text{in } \Omega \\ \quad u_\varepsilon &= g & \text{on } \partial\Omega\ . \end{cases} \tag{1.2}$$

A theorem from [3] is

Theorem (BBH1). *Assume that Ω is star-shaped, and that $d \neq 0$, then there exists a subsequence of $\varepsilon_k \to 0$, exactly $|d|$ distinct points $a_1, a_2, \cdots, a_{|d|}$, and a harmonic map $u_* \in \mathbf{C}^\infty(\Omega \setminus \{a_1, a_2, \cdots, a_{|d|}\})$ with boundary value g such that*

$$u_{\varepsilon_n} \to u_* \quad \textit{in} \quad \mathbf{C}^k_{\mathrm{loc}}(\Omega \setminus \cup_i \{a_i\}) \quad \textit{for} \quad \forall k \quad \textit{and in} \quad \mathbf{C}^{1,\alpha}_{\mathrm{loc}}(\bar\Omega \setminus \cup_i\{a_i\}) \quad \textit{for} \quad \forall \alpha < 1.$$

In addition, each singularity has degree sign(d).

The infinite energy singularities $a_1, a_2, \cdots, a_{|d|}$ of the $\mathbb{S}^1$-valued harmonic map u_* are not arbitrarily located. Given any configuration $b = (b_1, b_2, \cdots, b_{|d|})$ of distinct points in Ω, its renormalized energy is defined in [3] as

$$W(b,d,g) := -\pi \sum_{i\neq j} \ln|b_i - b_j| + \frac{1}{2}\int_{\partial\Omega} \Phi(g \times g_\tau) - \pi \sum_{i=1}^{|d|} R(b_i)$$

where Φ is the solution of the linear Neumann problem

$$\begin{cases} \Delta\Phi = 2\pi \sum_{i=1}^{|d|} \delta_{b_i} & \text{in } \Omega, \\ \dfrac{\partial\Phi}{\partial\nu} = g \times g_\tau & \text{on } \partial\Omega \end{cases} \tag{1.3}$$

where ν is the unit outward normal to $\partial\Omega$, τ is a unit tangent vector to $\partial\Omega$ and

$$R(x) = \Phi(x) - \sum_{i=1}^{|d|} \ln|x - b_i|.$$

Then the following holds.

Theorem (BBH2). *With the assumptions and notations of Theorem (BBH1), the following holds.*

1. *The configuration $\{a_1, a_2, \cdots, a_{|d|}\}$ minimizes $b \to W(b, d, g)$.*
2. *(Vanishing gradient property)*
   Near each singularity a_j,

$$u_*(z) = \frac{z - a_j}{|z - a_j|} e^{iH_j(z)}, \tag{1.4}$$

where H_j is a real harmonic function such that

$$H_j(z) = H_j(a_j) + O(|z - a_j|^2), \text{ as } z \to a_j. \tag{1.5}$$

In other words,

$$\nabla H_j(a_j) = 0. \tag{1.6}$$

In the case $n \geq 3$, the minimizers of $\mathbf{E}_\varepsilon(u, \Omega)$, and more generally critical points, satisfy

$$\begin{cases} -div\left(|\nabla u_\varepsilon|^{n-2} \nabla u_\varepsilon\right) &= \frac{1}{\varepsilon^n}\left(1 - |u_\varepsilon|^2\right) u_\varepsilon & \text{in } \Omega \\ u_\varepsilon &= g & \text{on } \partial\Omega\ . \end{cases} \tag{1.7}$$

Several authors have studied the sequences of minimizers of $\mathbf{E}_\varepsilon$ in the case $n \geq 3$, namely P.Strzelecki [30], M-C.Hong [18] and Z-C.Han and Y-Y.Li [11]. Let us recall the main results in [11]. For convenience, we define a constant

$$\kappa_n = \frac{1}{n}(n-1)^{\frac{n}{2}}\omega_n \tag{1.8}$$

where $\omega_n = |\mathbb{S}^{n-1}|$.

Theorem (HL). *Assume $d \neq 0$, $n \geq 3$. For any sequence $\varepsilon_k \to 0$, let $\{u_k\} \subset W_g^{1,n}(\Omega, \mathbb{R}^n)$ be the corresponding sequence of minimizer for $\mathbf{E}_{\varepsilon_k}$. Then there exists a subsequence $\{u_{k'}\}$, a collection of $|d|$ distinct points $\{a_1, a_2, \cdots, a_{|d|}\} \subset \Omega$, and an n-harmonic map $u_* : \Omega \setminus \cup_i \{a_i\} \to \mathbb{S}^{n-1}$ such that*

$$u_{k'} \to u_* \quad \text{strongly in} \quad \mathbf{W}^{1,n}_{\text{loc}}(\bar{\Omega} \setminus \cup_i \{a_i\}; \mathbb{R}^n), \tag{1.9}$$

$$u_{k'} \to u_* \quad \text{in} \quad \mathbf{C}^0_{\text{loc}}(\bar{\Omega} \setminus \cup_i \{a_i\}; \mathbb{R}^n), \tag{1.10}$$

$$u_{k'} \to u_* \quad \text{strongly in} \quad \mathbf{W}^{1,p}(\Omega; \mathbb{R}^n) \text{ for all } 1 \leq p < n. \tag{1.11}$$

Furthermore, $\deg(u_*, \partial B_\sigma, \mathbb{S}^{n-1}) = sign(d)$ *for all* $1 \leq j \leq |d|$ *and* $\sigma > 0$ *small enough. When $d = 0$, $u_{k'}$ converges to u_* strongly in $\mathbf{W}^{1,n} \cap \mathbf{C}^0$.*

From now on, we assume without loss of generality that the degree $d > 0$ is positive and the dimension $n \geq 3$.

Our first result is an analogue of Theorem(BBH2), i.e. the proof that the singularities of u_* minimize a renormalized energy as well. This renormalized energy was actually introduced by R.Hardt, F-H.Lin and C-Y.Wang [16] as follows.

Given d distinct points in Ω denoted $a = \{a_1, a_2, \cdots, a_d\}$, and for $\delta > 0$, let

$$\Omega_{a,\delta} = \Omega \setminus \cup_{i=1}^{d} B_\delta(a_i).$$

Then define for any δ small enough

$$\mathcal{W}_{a,\delta} = \left\{ w \in W^{1,n}(\Omega_{a,\delta}; \mathbb{S}^{n-1}) : w|\partial\Omega = g, \deg(w, \partial B_\delta(a_i)) = 1 \,\text{for all}\, i \right\}.$$

The renormalized energy of $a = \{a_1, a_2, \cdots, a_d\}$ is defined to be

$$W_g(a) := \lim_{\delta \to 0} \left(\min_{w \in \mathcal{W}_{a,\delta}} E_n(w, \Omega_{a,\delta}) - d\kappa_n |\ln \delta| \right), \tag{1.12}$$

where

$$E_n(w, \Omega_{a,\delta}) = \int_{\Omega_{a,\delta}} \frac{|\nabla w|^n}{n} \, \mathrm{d}x.$$

In particular, it is proved in [16] that the limit defining W_g exists, and is even increasing as $\delta \to 0$.

We have the following result.

Theorem 1.1. *Let $a = \{a_i\}_{i=1}^d$ be the limiting singular points of Theorem (HL), then*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) = d\kappa_n |\ln \varepsilon| + W_g(a) + d\gamma + o(1) \; as \, \varepsilon \to 0,$$

where γ is a constant defined in Section 2.1 below by (2.4). Moreover, the configuration $\{a_i\}_{i=1}^d$ minimizes W_g.

The results above deal only with sequences of energy-minimizers. The ones below deal with limits of solutions to the system (1.7).

Theorem 1.2. *Assume that for each $\varepsilon > 0$ the map u_ε, is a critical point of $\mathbf{E}_\varepsilon$ and that for some $M > 0$ independent of ε it holds that*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n \, |\ln \varepsilon| + M. \tag{1.13}$$

Then there exists a subsequence $\{\varepsilon\}$ tending to zero, a collection of d distinct points $\{a_1, a_2, \cdots, a_d\}$ $\subset \Omega$, a finite subset S_1 of $\bar{\Omega}$, and a stationary n-harmonic map $u_0 : \Omega_0 := \Omega \setminus \{a_1, a_2, \cdots, a_d\} \to \mathbb{S}^{n-1}$, such that

$$u_\varepsilon \to u_0 \quad \textit{strongly in} \quad \mathbf{W}^{1,n}_{\mathrm{loc}}(\Omega_0 \setminus S_1, \mathbb{R}^n)$$

and for any $1 \leq p < n$

$$u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in} \quad \mathbf{W}^{1,p}(\Omega, \mathbb{R}^n).$$

Furthermore, $\deg(u_0, \partial B_\sigma(a_j), \mathbb{S}^{n-1}) = 1$*, for $1 \leq j \leq d$ and any small enough $\sigma > 0$.*

It was proved by R. Jerrard in [19] that the upper bound condition (1.13) is sufficient to guarantee the local weak convergence in Ω_0 of a subsequence. Here we improve this to strong convergence for solutions of the system (1.7). However, contrary to the case $n = 2$ we need to remove a finite set S corresponding to the bubbling-off of nontrivial finite energy n-harmonic maps from $\mathbb{R}^n$ to $\mathbb{S}^{n-1}$ which do not exist when $n = 2$.

In the case $n = 3$ an example of such a map is the Hopf fibration, and recently T.Rivi¿$\frac{1}{2}$re in [24] showed that there exists in fact many of them. This multiplicity arises in particular from a richer topology, due to the non-trivial fundamental group $\pi_3(\mathbb{S}^2)$, for which the Hopf map is a generator. This hints at the fact that the moduli space of critical points of the generalized Ginzburg-Landau equations for small parameter ε could be quite rich too. For $n > 3$ the same situation is expected because of homotopy groups of the spheres, for example, $\pi_7(\mathbb{S}^4), \pi_{15}(\mathbb{S}^8)$, or other topological invariants.

Theorem 1.2 contains a criticality condition satisfied by the points $\{a_1, a_2, \cdots, a_d\}$ hidden in the word "stationary n-harmonic map" that we now define.

Definition 1.3. *Let $u : \Omega_0 \to \mathbb{S}^{n-1}$ be an n-harmonic map, where $\Omega_0 = \Omega \setminus \{a_1, a_2, \cdots, a_d\}$. We say u is a* stationary n-harmonic map *if its stress-energy tensor*

$$T_{i,j} := |\nabla u|^{n-2} \langle \partial_i u, \partial_j u \rangle - \frac{1}{n} |\nabla u|^n \delta_{i,j}$$

satisfies

$$\sum_i \partial_i T_{i,j} = 0$$

in Ω_0, and if for any $1 \leq k \leq d$ and $\rho > 0$ such that $\partial B_\rho(a_k) \subset \Omega_0$ it holds that

$$\int_{\partial B_\rho(a_k)} \sum_i T_{i,j} \nu_i = 0, \tag{1.14}$$

where $\nu = (\nu_1, \cdots, \nu_n)$ is the outward-pointing normal to $\partial B_\rho(a_k)$.

When both conditions are satisfied we say that T_{ij} is divergence free *in Ω_0.*

The following proposition links the property of being a stationary n-harmonic map with the vanishing gradient property (1.6). Unfortunately it is not clear yet whether its assumptions are satisfied for the stationary n-harmonic maps arising as limits of critical points of the Ginzburg-Landau functional in dimension n.

Proposition 1.4. *Assume $u : \Omega_0 \subset \mathbb{R}^n \to \mathbb{S}^{n-1}$ is a stationary $n-$harmonic map in the above sense, where $\Omega_0 = \Omega \setminus (\{a_1, \cdots, a_d\}$, and that $\deg(u, a_k) = 1$. Assume that around each singular point a_k, one has the asymptotic expansion*

$$u(x) = \mathrm{e}^{B_k(x)} \frac{x - a_k}{|x - a_k|}$$

where $B_k(x) \in so(n)$ is an antisymmetric matrix satisfying $B_k(a_k) = 0$ and such that $x \to B_k(x)$ is C^1 in a neighborhood of a_k. Then

$$\sum_{i=1}^n \partial_i B_k(a_k) e_i = 0, \tag{1.15}$$

where $(e_1, \cdots, e_n)$ is the canonical basis in $\mathbb{R}^n$. Equivalently, we have the expansion

$$u(x) = \frac{x - a_k}{|x - a_k|} + \frac{Q_k(x - a_k)}{|x - a_k|} + O(|x - a_k|^2), \tag{1.16}$$

where $Q_k(x)$ is a harmonic polynomial of degree 2. In particular, when $n = 2$, we have $B_k(x) = O(|x - a_k|^2)$.

Finally we will construct an example of a sequence of non-minimizing critical points satisfying the hypothesis of Theorem 1.2.

Theorem 1.5. *Let $n = 3$. There exists a domain $\Omega \subset \mathbb{R}^3$, a boundary map $g : \partial\Omega \to \mathbb{S}^{n-1}$, and for every small enough $\varepsilon > 0$ a non minimizing critical point u_ε of the functional $\mathbf{E}_\varepsilon(u, \Omega)$ such that the energy bound (1.13) holds.*

The paper is organized as follows. In Section 2, we prove Theorem 1.1. In Section 3 we prove the Pohozaev inequality. Section 4 is devoted to the proof of Theorem 1.2 and of Proposition 1.4. Theorem 1.5 is proved in the last section.

2. Renormalized Energy

In this section, we study the renormalized energy for minimizers of n-dimensional Ginzburg-Landau type functional. We show that it coincides with the renormalized energy for n-harmonic maps. The proof of Theorem 1.1 mimics the strategy in [3]. It can be divided into the following two lemmas.

Lemma 2.1. *Let $\bar{a} = \{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_d\}$ be any configuration of d distinct points in Ω. Then there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n |\ln \varepsilon| + W_g(\bar{a}) + d\gamma + o(1).$$

where γ is the constant defined in (2.4).

Lemma 2.2. *With the notations of Theorem (HL), assume $\{\varepsilon\}$ converges to zero and that u_ε converges to the $\mathbb{S}^{n-1}$-valued n-harmonic map u_* strongly in $\mathbf{W}^{1,n}_{\text{loc}}(\bar{\Omega} \setminus \cup_i \{a_i\}; \mathbb{R}^n)$. Then*

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \geq d\kappa_n |\ln \varepsilon| + W_g(a) + d\gamma - o(1),$$

where $a = \{a_1, a_2, \cdots, a_d\}$.

2.1. **Estimates when $\Omega = B_R$ and $g(x) = g_0 = \frac{x}{|x|}$.** We begin by introducing quantities wich are the counterparts to those introduced in[3] for the case $n = 2$. Let

$$\mathbf{I}(\varepsilon, R) = \min_{u(x) = x/|x| \text{ on } \partial B_R} \mathbf{E}_\varepsilon(u, B_R). \tag{2.1}$$

A scaling argument shows that

$$\mathbf{I}(\varepsilon, R) = \mathbf{I}(1, R/\varepsilon) = \mathbf{I}(\varepsilon/R, 1).$$

Lemma 2.3. *Let $\mathbf{I}(t) = \mathbf{I}(t, 1)$. Then the function $t \to \mathbf{I}(t) + \kappa_n \ln(t)$ is increasing on $(0, 1)$, where κ_n is defined by in (1.8), and has a limit as $t \searrow 0$.*

Proof. Assume $0 < t_1 < t_2 < 1$ and let u_t be the minimizer of $\mathbf{I}(1, \frac{1}{t})$. Let

$$v(x) = \begin{cases} u_{t_2} & \text{if } |x| < \frac{1}{t_2}, \\ \frac{x}{|x|} & \text{if } \frac{1}{t_2} \leq |x| \leq \frac{1}{t_1}. \end{cases} \tag{2.2}$$

Then by the definition of $\mathbf{I}(t)$, we have

$$\begin{aligned} &\mathbf{I}(t_1) = \mathbf{I}(1, t_1^{-1}) \leq \mathbf{E}_1(v, B_{1/t_1}) \\ &= \mathbf{I}(t_2) + \int_{B_{t_1^{-1}} \setminus B_{t_2^{-1}}} \frac{1}{n} \left| \nabla \left(\frac{x}{|x|} \right) \right|^n \mathrm{d}\, x = \mathbf{I}(t_2) + \int_{t_2^{-1}}^{t_1^{-1}} \frac{(n-1)^{\frac{n}{2}}}{n \cdot r} \cdot \omega_n \,\mathrm{d}\, r \\ &= \mathbf{I}(t_2) + \frac{(n-1)^{\frac{n}{2}}}{n} \omega_n \cdot \ln \frac{t_2}{t_1} \end{aligned} \tag{2.3}$$

In view of (1.8), this proves that $t \to \mathbf{I}(t) + \kappa_n \ln(t)$ is increasing on $(0, 1)$.

Then, by using Theorem 1.1 of [19], there exists a constant $C > 0$ such that if $u \in W^{1,n}(B_1, \mathbb{R}^n)$ and $u(x) = x/|x|$ on the boundary, then

$$\mathbf{E}_\varepsilon(u, B_1) \geq \kappa_n |\ln(\varepsilon)| - C.$$

This implies that $\mathbf{I}(t) + \kappa_n \ln(t)$ is bounded below on $(0,1)$ and then, using the monotonicity, that the limit exists as $t \searrow 0$. □

We may now define the constant γ:

$$\gamma := \lim_{t\to 0}\{\mathbf{I}(t) + \kappa_n \ln(t)\}. \tag{2.4}$$

Note that — due to the invariance of $\mathbf{E}_\varepsilon$ under isometries of the target — if we replace $x/|x|$ in the definition of $I(t)$ by $Rx/|x|$, where $R \in O(n)$, the function I and thus the constant γ are unchanged.

2.2. **Proof of Lemma 2.1.** We construct a comparison map which is in $W^{1,n}_g(\Omega, \mathbb{R}^n)$ to obtain the upper bound.

Let $\bar{a} = \{\bar{a}_1, \bar{a}_2, \cdots, \bar{a}_d\}$ be any configuration of d distinct points in Ω. For any $\frac{1}{2} > \delta > 0$ such that the balls $B(\bar{a}_i, 4\delta)$ are disjoint and included in Ω, let $w_{\bar{a},\delta}$ denote a minimizer for E_n, that is, $E_n(w_{\bar{a},\delta}, \Omega_{\bar{a},\delta}) = \min_{w\in\mathcal{W}_{a,\delta}} E_n(w, \Omega_{a,\delta})$. Then, from Lemma 9.1 in [16], for any $\mu > 0$ there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$ there exists rotations $\{R_i\}_{1\le i\le d}$ such that for any $1 \le i \le d$

$$\|w_{\bar{a},\delta}(\bar{a}_i + 4\delta\cdot) - R_i\|_{C^1(B_1\setminus B_{1/2})} < \mu/3d. \tag{2.5}$$

Now, from (1.12), for any $\mu > 0$ and any $\delta > 0$ small enough depending on μ, we have

$$E_{\bar{a},\delta}(w_{\bar{a},\delta}) \le W_g(\bar{a}) + d\kappa_n |\ln\delta| + \mu/3. \tag{2.6}$$

We choose such a δ so that (2.5) holds, and we define the comparison map $u \in W^{1,n}_g(\Omega, \mathbb{R}^n)$ by letting

$$u(x) = \begin{cases} w_{\bar{a},\delta} & \text{if } x \in \Omega_{\bar{a},4\delta}, \\ v_i(x) & \text{if } x \in B_{4\delta}(\bar{a}_i) \setminus B_{2\delta}(\bar{a}_i) \ \ 1 \le i \le d \\ R_i u_{2\delta}(x - a_i) & \text{if } x \in B_{2\delta}(\bar{a}_i) \ \ 1 \le i \le d, \end{cases} \tag{2.7}$$

where $u_{2\delta}$ is the minimizer for $\mathbf{I}(\varepsilon, 2\delta)$ (see (2.1)) and for $1 \le i \le d$, $v_i(x)$ is the interpolation map

$$v_i = \frac{v_i^*}{|v_i^*|}, \quad v_i^*(a_i + y) = \left(2 - \frac{|y|}{2\delta}\right) R_i \frac{y}{|y|} + \left(\frac{|y|}{2\delta} - 1\right) w_{\bar{a},\delta}(a_i + y).$$

From (2.5) it is not difficult to show that for $1 \le i \le d$

$$\int_{B_{4\delta}(\bar{a}_i)\setminus B_{2\delta}(\bar{a}_i)} \frac{|\nabla v_i|^n}{n} \,\mathrm{d}\,x \le \kappa_n \ln 2 + C\mu/d, \tag{2.8}$$

where $C > 0$ is some positive constant independent of μ.

To compare the energy of u with that of $w_{\bar{a},\delta}$ on $B_{4\delta}(a_i) \setminus B_\delta(a_i)$ we need an energy lower bound for the latter, provided by the following well known Lemma (see [16] or [19] or [11])

Lemma 2.4 (Annulus estimate)**.** *If* $0 < r < s < \infty$, $v \in W^{1,n}(B_s\setminus B_r, \mathbb{S}^{n-1})$, *and* $\deg(v, \partial B_\rho) = D \ne 0$ *for almost all* $\rho \in (r, s)$, *then*

$$\int_{B_s\setminus B_r} \frac{1}{n} |\nabla u|^n \,\mathrm{d}\,x \ge |D|^{\frac{n}{n-1}} \kappa_n \ln\frac{s}{r}. \tag{2.9}$$

From (2.9) we obtain

$$\mathbf{E}_\varepsilon\left(w_{\bar{a},\delta}, B_{4\delta}(a_i) \setminus B_\delta(a_i)\right) \ge \kappa_n \ln 4.$$

Together with (2.6) this yields

$$\begin{aligned}\mathbf{E}_\varepsilon(u,\Omega_{\bar a,4\delta}) &= \int_{\Omega_{\bar a,\delta}} \frac{1}{n}\left|\nabla w_{\bar a,\delta}\right|^n \mathrm{d}\,x - \int_{\cup_i(B_{4\delta}(\bar a_i)\setminus B_\delta(\bar a_i))} \frac{1}{n}\left|\nabla w_{\bar a,\delta}\right|^n \mathrm{d}\,x \\ &\le W_g(\bar a) - d\kappa_n\log 4 + \mu/3 + d\kappa_n|\log\delta|.\end{aligned} \tag{2.10}$$

In the balls $B_{2\delta}(\bar a_i)$, there exists a constant $\varepsilon_0>0$ such that for any $\varepsilon<\varepsilon_0$

$$\mathbf{E}_\varepsilon(u_{2\delta},B_{2\delta}) = \mathbf{I}(\varepsilon,2\delta) = \mathbf{I}(\varepsilon/(2\delta),1) \le \gamma + \kappa_n\left|\ln(\varepsilon/(2\delta))\right| + \mu/3d. \tag{2.11}$$

Combining (2.8),(2.10) and (2.11) we have the desired upper bound

$$\begin{aligned}\mathbf{E}_\varepsilon(u_\varepsilon,\Omega) &\le \mathbf{E}_\varepsilon(u,\Omega) \\ &= \mathbf{E}_\varepsilon(u,\Omega_{\bar a,4\delta}) + \sum_{i=1}^d \int_{B_{4\delta}(\bar a_i)\setminus B_{2\delta}(\bar a_i)} \frac{1}{n}\left|\nabla v_i\right|^n \mathrm{d}\,x + \sum_{i=1}^d \mathbf{E}_\varepsilon(u_{2\delta},B_{2\delta}) \\ &\le W_g(\bar a) + d\kappa_n\left|\ln\varepsilon\right| + d\gamma + \mu(C+2/3).\end{aligned} \tag{2.12}$$

Since this bound is true for any $\mu>0$, this concludes the proof of Lemma 2.1.

2.3. **Proof of Lemma 2.2.** Let $a=\{a_1,a_2,\cdots,a_d\}$ be the singularities of u_*, which are distinct and belong to Ω. From the convergence $u_\varepsilon\to u_*$, we have a lower bound for $\mathbf{E}_\varepsilon(u_\varepsilon)$ away from the singularities. Then we need to prove that for $\rho>0$ small enough, and for any $1\le i\le d$, as $\varepsilon\to 0$,

$$\mathbf{E}_\varepsilon(u_\varepsilon,B_\rho(a_i)) \ge \mathbf{I}(\varepsilon,\rho) + o(1). \tag{2.13}$$

In order to prove (2.13), we need the following equivalent of (2.5) for u_*.

Lemma 2.5. *The limiting map u_* is in $C^{1,\alpha}(\Omega\setminus\{a_1,a_2,\cdots,a_d\})$. The restriction of u_* on any small sphere around a_i has degree equal to 1.*
Moreover, letting $u_{i,r}(x):=u_*(a_i+rx)$, we have the following result: For any $i=1,\cdots,d$, there exists a decreasing sequence $\sigma_k\to 0$, and rotations $R_{i,k}$ such that

$$\lim_k \|u_{i,\sigma_k}(\cdot) - R_{i,k}\|_{W^{1,n}(\mathbb{S}^{n-1},\mathbb{R}^n)} = 0. \tag{2.14}$$

Proof. Since u_* is an n-harmonic map into the sphere which is locally minimizing in $\Omega\setminus\{a_1,a_2,\cdots,a_d\}$, the regularity theory of [13] insures that it is in $C^{1,\alpha}$ in this set.

To prove the remaining statements, we begin by proving a basic fact. Let $\{f_k\}$ be a sequence of maps in $\subset W^{1,n}(\mathbb{S}^{n-1},\mathbb{S}^{n-1})$ with degree greater than or equal to 1. Then

$$\lim_k \int_{\mathbb{S}^{n-1}} \left|\nabla_{\tan} f_k\right|^n \mathrm{d}\mathcal{H}^{n-1} = n\kappa_n$$

if and only if there exists a sequence of rotations R_k such

$$\lim_k \|f_k - R_k\|_{W^{1,n}(\mathbb{S}^{n-1},\mathbb{R}^n)} = 0.$$

Here $\nabla_{\tan} f$ is the gradient of f on the sphere.

The reverse implication is clear. For the direct implication note that there is a compact embedding of $W^{1,n}(\mathbb{S}^{n-1},\mathbb{R}^n)$ into $C^0(\mathbb{S}^{n-1},\mathbb{R}^n)$. Thus the degree is conserved under the weak convergence in $W^{1,n}$. Then any weak limit of f_k has degree at least one and n-energy no greater than $n\kappa_n$, thus its energy is exactly $n\kappa_n$, its degree is one, and it is a rotation. Moreover the convergence is strong since we have weak convergence and convergence of the energies, using Brezis-Lieb's Lemma, which proves the statement.

We now argue by contradiction. Assume the lemma were false, then there would exist an index i, and positive numbers $\eta, \bar\sigma$ such that, for any $\rho < \bar\sigma$,

$$\rho \int_{\partial B_\rho(a_i)} |\nabla_{\tan} u_*|^n \, d\mathcal{H}^{n-1} \geq n(\kappa_n + \eta).$$

This would imply that for any $\sigma < \bar\sigma$

$$\int_{B_{\bar\sigma(a_i)} \setminus B_\sigma(a_i)} \frac{1}{n} |\nabla u_*|^n \, d\,x \geq (\kappa_n + \eta) \ln \frac{\bar\sigma}{\sigma}. \tag{2.15}$$

But, from Theorem 1.2 of [19] or Proposition 3.8 in [11]), we have

$$\mathbf{E}_\varepsilon(u_\varepsilon, B_\sigma(a_i)) \geq \kappa_n \ln \frac{\sigma}{\varepsilon} - C(n, \Omega, g) \tag{2.16}$$

where $C(n, \Omega, g)$ is a constant independent of ε.

Then for ε small enough, say $\varepsilon < \varepsilon_1$, (2.15) and (2.16) would imply

$$\mathbf{E}_\varepsilon(u_\varepsilon, \cup B_{\bar\sigma}(a_i)) \geq d\kappa_n \ln \frac{\bar\sigma}{\varepsilon} + \eta \ln \frac{\bar\sigma}{\sigma} - C(n, \Omega, g), \tag{2.17}$$

since

$$\liminf_{\varepsilon \to 0} \int_{\cup(B_{\bar\sigma}(a_i) \setminus B_\sigma(a_i))} |\nabla u_\varepsilon|^n \geq \int_{\cup(B_{\bar\sigma}(a_i) \setminus B_\sigma(a_i))} |\nabla u_*|^n.$$

This contradicts the upper bound (2.12) if $\bar\sigma/\sigma$ is chosen large enough. This completes the proof of the lemma. □

Remark 2.6. *We could actually prove a stronger result modeled after* (2.5) *by the method of* [16]*: Given i, for any $\mu > 0$, there exists a positive δ_0 such that if $\delta < \delta_0$, then*

$$\|u_{i,\delta} - R_{i,\delta}\|_{C^1(B_1 \setminus B_{1/2})} < \mu$$

for some rotation $R_{i,\delta}$.

We now complete the proof of Lemma 2.2. For any $\mu > 0$, from the definition of W_g, there exists $\delta_0 > 0$ such that for any $\delta < \delta_0$, we have

$$\mathbf{E}_{a,\delta}(w_{a,\delta}, \Omega_{a,\delta}) \geq W_g(a) + d\kappa_n |\ln \delta| - \mu/6. \tag{2.18}$$

Then, from Lemma 2.5, for any i there exists a sequence (σ_k) converging to 0 and a sequence of rotations $R_{i,k}$ such that

$$\|u_{i,\sigma_k}(\cdot) - R_{i,k}\|_{W^{1,n}(\mathbb{S}^{n-1}, \mathbb{R}^n)} < \mu, \tag{2.19}$$

where $u_{i,\sigma_k} = u_*(a_i + \sigma_k x)$. We may choose for each i some k such that $\sigma_k < \delta_0/2$. Let us fix some i and let $\rho = \sigma_k$ and $R = R_{i,k}$. We define $\widetilde{u}_\varepsilon$ on $B_{2\rho}(a_i)$ as follows.

On $B_\rho(a_i)$, we let $\widetilde{u}_\varepsilon = u_\varepsilon$, while on $B_{2\rho(a_i)} \setminus B_\rho(a_i)$ we interpolate between u_ε and the rotation R by letting, for any $\sigma \in \mathbb{S}^{n-1}$ and any $r \in [\rho, 2\rho]$

$$|\widetilde{u}_\varepsilon|(a_i + r\sigma) = \frac{r - \rho}{\rho} + \frac{2\rho - r}{\rho} |u_\varepsilon(a_i + \rho\sigma)|,$$

$$\frac{\widetilde{u}_\varepsilon}{|\widetilde{u}_\varepsilon|}(a_i + r\sigma) = \frac{v}{|v|}(r\sigma),$$

where

$$v(r\sigma) = \frac{r - \rho}{\rho} R\sigma + \frac{2\rho - r}{\rho} u_\varepsilon(a_i + \rho\sigma).$$

It is not difficult to check, using on the one hand (2.19) and on the other hand the uniform convergence of $|u_\varepsilon|$ to 1 on $B_{2\rho(a_i)} \setminus B_\rho(a_i)$, that

$$\frac{\widetilde{u}_\varepsilon}{|\widetilde{u}_\varepsilon|} - R\left(\frac{\cdot - a_i}{|\cdot - a_i|}\right) \to 0$$

in $W^{1,n}(B_{2\rho} \setminus B_\rho)$. Also, again using the uniform convergence of $|u_\varepsilon|$, we find that pointwise

$$|\nabla |u_\varepsilon|| - |\nabla |\widetilde{u}_\varepsilon|| \geq -c_\varepsilon,$$

where $c_\varepsilon \to 0$ as $\varepsilon \to 0$.

Finally we note that $|\widetilde{u}_\varepsilon|$ is closer to 1 pointwise than $|u_\varepsilon|$, which together with the two previous bounds and the fact that $\widetilde{u}_\varepsilon = u_\varepsilon$ on B_ρ yields

$$\limsup_{\varepsilon\to 0} \left(\mathbf{E}_\varepsilon(\widetilde{u}_\varepsilon, B_{2\rho(a_i)}) - \mathbf{E}_\varepsilon(u_\varepsilon, B_{2\rho(a_i)})\right) \leq 0. \tag{2.20}$$

On the other hand, since $\widetilde{u}_\varepsilon$ is a rotation on $\partial B_{2\rho}(a_i)$, we have

$$\mathbf{E}_\varepsilon\left(\widetilde{u}_\varepsilon, B_{2\rho_i(a_i)}\right) \geq \mathbf{I}(\varepsilon, 2\rho_i). \tag{2.21}$$

Then, from the strong convergence of u_ε to u_* on $\Omega_{2\rho} := \Omega \setminus \cup_i B_{2\rho_i}(a_i)$ we have

$$\liminf_{\varepsilon\to 0} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega_{2\rho}) \geq \frac{1}{n}\int_{\Omega_{2\rho}} |\nabla u_*|^n. \tag{2.22}$$

To estimate the right-hand side we may use Lemma 2.5 again to construct for any small enough $\eta > 0$ a comparison map $v : \Omega \setminus \cup_i B_\eta(a_i) \to \mathbb{S}^{n-1}$ such that v is a rotation on $\partial B_\eta(a_i)$ for each i, such that $v = u_*$ on $\Omega_{2\rho}$ and such that on each annulus $B_{2\rho_i}(a_i) \setminus B_\eta(a_i)$ the n-energy of v is μ/d-close to $\kappa_n \log(2\rho_i/\eta)$.

Then the n-energy of v on Ω_η is bounded below by the n-energy of $w_{a,\eta}$, which itself is μ/d-close to $W_g(a) + d\kappa_n|\log\eta|$ if η is small enough. It follows that

$$\frac{1}{n}\int_{\Omega_{2\rho}} |\nabla u_*|^n \geq W_g(a) + \kappa_n \sum_i \log\frac{1}{2\rho_i} - 2\mu. \tag{2.23}$$

Putting Together (2.20), (2.21), (2.22) and (2.23) we deduce that

$$\mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \geq W_g(a) + \sum_i \left(\kappa_n \log\frac{1}{2\rho_i} + I(\varepsilon, 2\rho_i)\right) - C\mu - o(1).$$

In view of (2.4), this completes the proof of Lemma 2.2.

2.4. **Proof of Theorem 1.1.** Applying Lemma 2.1, we obtain

$$\limsup_{\varepsilon\to 0} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n|\ln\varepsilon| + W_g(\bar{a}) + d\gamma$$

which implies

$$\limsup_{\varepsilon\to 0} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \leq d\kappa_n|\ln\varepsilon| + \min W_g(\bar{a}) + d\gamma \tag{2.24}$$

Let $\{\varepsilon\}$ be sequence tending to 0 such that u_ε converges to the $\mathbb{S}^{n-1}$-valued n-harmonic map u_* in $\Omega \setminus \{a_1, \ldots, a_d\}$. Then by Lemma 2.2, we get

$$\liminf_{\varepsilon\to 0} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \geq d\kappa_n|\ln\varepsilon| + W_g(a) + d\gamma$$

which implies

$$\liminf_{\varepsilon\to 0} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega) \geq d\kappa_n|\ln\varepsilon| + \min W_g(\bar{a}) + d\gamma \tag{2.25}$$

Gathering (2.24) and (2.25), Theorem 1.1 is proved.

3. Divergence Free Stress-Energy Tensor and Pohozaev Inequality

In this section we introduce the stress-energy tensor for critical points of $\mathbf{E}_\varepsilon$ and derive the corresponding Pohozaev identity.

3.1. **Stress-Energy Tensor.** The derivation of the fact that the stress-energy tensor of a solution of (1.7) is divergence free is not trivial because of the a priori insufficient regularity of solutions. We prove it through a regularization procedure. Note that from [11], a solution of (1.7) which is in $W^{1,n}$ is in $C^{1,\alpha}$ for some $\alpha \in (0,1)$.

We start with the following well-known fact.

Lemma 3.1. *Let $u_\varepsilon \in W^{1,n}_g(\Omega, \mathbb{R}^n)$ be a solution of equations (1.7), where $g : \partial\Omega \to \mathbf{S}^{n-1}$. Then we have $|u_\varepsilon| \le 1$ in Ω.*

Proof. Assuming u_ε is a solution of (1.7), we have

$$\begin{aligned}\frac{1}{2}\mathrm{div}\left(|\nabla u_\varepsilon|^{n-2}\nabla|u_\varepsilon|^2\right) &= \left\langle \mathrm{div}\left(|\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon\right), u_\varepsilon\right\rangle + |\nabla u_\varepsilon|^n \\ &= \left\langle \frac{1}{\varepsilon^n}\left(|u_\varepsilon|^2-1\right)u_\varepsilon, u_\varepsilon\right\rangle + |\nabla u_\varepsilon|^n \\ &= \frac{1}{\varepsilon^n}\left(|u_\varepsilon|^2-1\right)|u_\varepsilon|^2 + |\nabla u_\varepsilon|^n.\end{aligned} \tag{3.1}$$

Multiplying by $(|u_\varepsilon|^2-1)_+$ and integrating by parts we find, using the fact that $|u_\varepsilon| = 1$ on the boundary, that

$$-\frac{1}{2}\int_{\Omega_+}|\nabla u_\varepsilon|^{n-2}\left|\nabla|u_\varepsilon|^2\right|^2 = \int_{\Omega_+}\frac{1}{\varepsilon^n}\left(|u_\varepsilon|^2-1\right)^2|u_\varepsilon|^2 + |\nabla u_\varepsilon|^n\left(|u_\varepsilon|^2-1\right), \tag{3.2}$$

where $\Omega_+ = \{|u_\varepsilon| > 1\}$.

It follows that the right-hand side is equal to zero, and therefore that $\Omega_+ = \varnothing$. □

The stress-energy tensor for equation (1.7) is defined, for any $u \in W^{1,n}$ by

$$T_{i,j}(u) = |\nabla u|^{n-2}\langle\partial_i u, \partial_j u\rangle - \left(\frac{1}{n}|\nabla u|^n + \frac{1}{4\varepsilon^n}(1-|u|^2)^2\right)\delta_{i,j}. \tag{3.3}$$

We have

Lemma 3.2. *For any solution u_ε of (1.7) and any $j = 1, \cdots, n$*

$$\sum_{i=1}^n \partial_i T_{i,j}(u_\varepsilon) = 0. \tag{3.4}$$

In particular, for any C^1 vector field $Y = (Y_1, \cdots, Y_n) \in C^1(\bar\Omega, \mathbb{R}^n)$, there holds

$$\sum_{i,j}\int_{\partial\Omega} T_{i,j}Y_j\nu_i = \sum_{i,j}\int_\Omega T_{i,j}\partial_i Y_j. \tag{3.5}$$

Proof. We consider, given a ball $B \subset \Omega$ and $\delta \ge 0$ the following functional

$$\mathbf{F}_\delta(w) := \frac{1}{n}\int_B (|\nabla w|^2 + \delta^2)^{n/2}\,\mathrm{d}x - \int_B f(u_\varepsilon)w\,\mathrm{d}x, \quad \text{where} \quad f(u_\varepsilon) = \frac{u_\varepsilon}{\varepsilon^n}(1-|u_\varepsilon|^2),$$

defined on the space

$$W = \left\{w \in W^{1,n}(B, \mathbb{R}^n) \mid w = u_\varepsilon \text{ on } \partial B\right\}.$$

Note that for any fixed ε we have $f(u_\varepsilon) \in W^{1,n} \cap L^\infty$. This functional is coercive and strictly convex, thus it has a unique minimizer w_δ. Then, for any smooth compactly supported vector field $X : B \to \mathbb{R}^n$ we may define a family of smooth diffeomorphisms of B by letting $\varphi_t(x) = x + tX(x)$, and the minimality of w_δ implies, if the derivative exists, that

$$\frac{\mathrm{d}}{\mathrm{d}t}_{|t=0} \mathbf{F}_\delta(w_\delta \circ \varphi_t) = 0.$$

A standard computation shows that indeed the derivative exists — this is where the regularization by δ is important — and that its vanishing is equivalent to

$$\int_B T^\delta_{i,j}(w_\delta)\partial_i X^j = -\int_B (X \cdot \nabla f(u_\varepsilon)) \cdot w_\delta, \tag{3.6}$$

where

$$T^\delta_{i,j}(w_\delta) = (|\nabla w_\delta|^2 + \delta^2)^{\frac{n-2}{2}} \partial_i w_\delta \partial_j w_\delta - \delta_{ij} f_\delta(w_\delta)$$

and

$$f_\delta(w) := \frac{1}{n}(|\nabla w|^2 + \delta^2)^{n/2} - f(u_\varepsilon) \cdot w.$$

Now we wish to pass to the limit as $\delta \to 0$ in (3.6). Let us assume for a moment that the following holds

$$w_\delta \to u_\varepsilon \text{ strongly in } W^{1,n} \text{ as } \delta \to 0. \tag{3.7}$$

Then passing to the limit in (3.6) yields

$$\int_B T_{i,j}(u_\varepsilon)\partial_i X^j = -\int_B \left(X \cdot \nabla \frac{(1 - |u_\varepsilon|^2)u_\varepsilon}{\varepsilon^n} \right) \cdot u_\varepsilon, \tag{3.8}$$

where

$$T_{i,j}(u_\varepsilon) = |\nabla u_\varepsilon|^{n-2} \partial_i u_\varepsilon \partial_j u_\varepsilon - \left(\frac{1}{n} |\nabla u_\varepsilon|^n - \frac{1}{\varepsilon^n}(1 - |u_\varepsilon|^2)|u_\varepsilon|^2 \right) \delta_{ij}.$$

Then we note that

$$\begin{aligned} - \partial_j \left(\frac{1}{\varepsilon^n}(1 - |u_\varepsilon|^2)|u_\varepsilon|^2 \right) + \partial_j \left(\frac{1}{\varepsilon^n}(1 - |u_\varepsilon|^2)u_\varepsilon \right) \cdot u_\varepsilon = \\ = -\frac{1}{\varepsilon^n}(1 - |u_\varepsilon|^2)u_\varepsilon \cdot \partial_j u_\varepsilon = \frac{1}{4\varepsilon^n}\partial_j(1 - |u_\varepsilon|^2)^2. \end{aligned}$$

Inserting in (3.8) then proves (3.4) and the lemma.

It remains to prove (3.7). First we note that $\{w_\delta\}_{\delta>0}$ is a bounded family in $W^{1,n}$. Indeed any fixed test-function in $W^{1,n}$ provides an upper bound for $\mathbf{F}_\delta(w_\delta)$ independent of δ, using the embedding of $W^{1,n}$ in L^1 and the fact that $f(u_\varepsilon) \in L^\infty$. Using again the pointwise bound of $f(u_\varepsilon)$ and Poincarï¿½'s inequality this upper bound implies easily that $\{w_\delta\}_{\delta>0}$ is a bounded family in $W^{1,n}$. We consider a weakly converging subsequence, which is strongly convergent in any L^p, and denote by w_0 its weak limit.

Then $w_0 = u_\varepsilon$, because w_0 is a minimizer of $\mathbf{F}_0$, but $\mathbf{F}_0$ is convex and therefore has a unique minimizer, which must be u_ε because it satisfies the corresponding Euler-Lagrange equation. Now we clearly have by lower semicontinuity

$$\int_B |\nabla u_\varepsilon|^n \leq \liminf_{\delta \to 0} \int_B |\nabla w_\delta|^n. \tag{3.9}$$

On the other hand, from the minimality of w_δ we have $\mathbf{F}_\delta(w_\delta) \le \mathbf{F}_\delta(u_\varepsilon)$. Passing to the limit we find

$$\limsup_{\delta\to 0} \int_B |\nabla w_\delta|^n \le \int_B |\nabla u_\varepsilon|^n , \tag{3.10}$$

where we have used the strong convergence in L^p to pass to the limit in $f(u_\varepsilon)w_\delta$. Comparing (3.9) and (3.10) we deduce the strong convergence of w_δ to u_ε. ☐

From Lemma 3.2 we derive the following Pohozaev Inequality.

Proposition 3.3. *Let $D \subset \mathbb{R}^n$ be a bounded strictly star-shaped domain with respect to $x_0 \in D$, and $\alpha > 0$ be such that $(x - x_0)\cdot\nu \ge \alpha \operatorname{diam}(D)$ for all $x \in \partial D$.*

Then there exists a constant C depending only on n, α such that, for any solution u_ε of (1.7),

$$\begin{aligned}&\int_D \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2 + \alpha \operatorname{diam}(D)\int_{\partial D} |\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2\\ &\le C(n,\alpha)\operatorname{diam}(D)\int_{\partial D} \frac{1}{n}|\nabla u_\varepsilon|^{n-2}\,|\nabla_\tau u_\varepsilon|^2 + \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2,\end{aligned} \tag{3.11}$$

where $|\nabla_\tau u_\varepsilon|^2 = |\nabla u_\varepsilon|^2 - |\partial_\nu u_\varepsilon|^2$ and $C(n,\alpha) = 2 + \frac{n^2(n-1)}{2(n-2)\alpha}$.

Proof. Let $Y(x) = x - x_0$, then $\partial_i(Y_j) = \delta_{ij}$. From (3.5) we obtain by choosing as our basis an orthonormal frame $\nu, \tau_1, \dots, \tau_{n-1}$, where ν is the outward pointing normal to D and $\tau_1, \dots, \tau_{n-1}$ is an orthonormal basis of tangent vectors to ∂D,

$$\begin{aligned}&\int_{\partial D}\langle Y,\nu\rangle\left(|\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2 - \frac{1}{n}|\nabla u_\varepsilon|^n - \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2\right) +\\ &\qquad + \sum_{k=1}^{n-1}\int_{\partial D}\langle Y,\tau_k\rangle\,|\nabla u_\varepsilon|^{n-2}\,\langle \partial_\nu u_\varepsilon, \partial_{\tau_k} u_\varepsilon\rangle = -\frac{n}{4\varepsilon^n}\int_D\left(1-|u_\varepsilon|^2\right)^2.\end{aligned} \tag{3.12}$$

For each $k = 1, \dots, n-1$ we have

$$|\langle \partial_\nu u_\varepsilon, \partial_{\tau_k} u_\varepsilon\rangle| \le \frac{1}{2}\left(\frac{\alpha(n-2)}{n(n-1)}|\partial_\nu u_\varepsilon|^2 + \frac{n(n-1)}{(n-2)\alpha}|\nabla_\tau u_\varepsilon|^2\right).$$

Together with (3.12) this implies, using the bounds

$$\langle Y,\tau_k\rangle \le \operatorname{diam}(D), \quad \alpha\operatorname{diam}(D) \le \langle Y,\nu\rangle \le \operatorname{diam}(D),$$

that

$$\begin{aligned}&\int_D \frac{n}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2 + \alpha\operatorname{diam}(D)\int_{\partial D}\left(\frac{n-1}{n} - \frac{n-2}{2n}\right)|\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2 \le\\ &\quad \le \operatorname{diam}(D)\int_{\partial D}\frac{1}{n}|\nabla u_\varepsilon|^{n-2}\,|\nabla_\tau u_\varepsilon|^2 + \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2 + \frac{1}{2}\frac{n(n-1)}{(n-2)\alpha}|\nabla u_\varepsilon|^{n-2}\,|\nabla_\tau u_\varepsilon|^2,\end{aligned}$$

where we have used $|\nabla_\tau u_\varepsilon|^2 = \sum_k |\partial_{\tau_k} u_\varepsilon|^2$. We deduce

$$\begin{aligned}&\frac{n}{2\varepsilon^n}\int_D\left(1-|u_\varepsilon|^2\right)^2 + \alpha\operatorname{diam}(D)\int_{\partial D}|\nabla u_\varepsilon|^{n-2}\,|\partial_\nu u_\varepsilon|^2 \le\\ &\qquad \operatorname{diam}(D)\int_{\partial D}\frac{1}{2}\left(\frac{1}{n} + \frac{n(n-1)}{2\alpha(n-2)}\right)|\nabla u_\varepsilon|^{n-2}\,|\nabla_\tau u_\varepsilon|^2 + \frac{1}{2\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2,\end{aligned}$$

which proves (3.11) with a suitable constant $C(n, \alpha)$. □

4. Proof of Theorem 1.2 and of Proposition 1.4

In this section, we analyze the behavior of critical points u_ε to the equations (1.7) which satisfy the energy upper bound (1.13). For this we combine the ball construction of R.Jerrard (see [19]) with the use of the Pohozaev identity above. We recall the main result of [19], expressed in a form suitable for us. Below we use the notation $r(B)$ for the radius of a ball and $\deg(B, u_\varepsilon)$ or simply $\deg(B)$ for the degree of the map $u_\varepsilon/|u_\varepsilon| : \partial B \to \mathbb{S}^{n-1}$.

Proposition 4.1. *Assume that $u_\varepsilon \in W^{1,n}_g(\Omega, \mathbb{R}^n)$ is such that for some $\beta \in (0, 1)$*

$$[u_\varepsilon]_{C^\beta(\bar\Omega)} \le C\varepsilon^{-\beta}, \tag{4.1}$$

and let

$$S_\varepsilon := \left\{ x \in \Omega : |u_\varepsilon| < \frac{1}{2} \right\}. \tag{4.2}$$

Then, there exists a constant C' depending only on C and the dimension n such that for any $t \ge C'\varepsilon\,|\log\varepsilon|$ there exists a finite collection of disjoint balls $\{B^t_i\}_{i\in I_t}$ such that $\cup_{i\in I_t} B^t_i$ is increasing with respect to t and such that for any $i \in I_t$

$$B^t_i \cap S_\varepsilon \neq \varnothing, \quad \mathbf{E}_\varepsilon(u_\varepsilon, B^t_i) \ge r(B^t_i)\frac{\Lambda_\varepsilon(t)}{t}, \quad r(B^t_i) \ge |d_i|t,$$

where $d_i = \deg(B^t_i, u_\varepsilon)$ if $B^t_i \subset \Omega$ and $d_i = 0$ otherwise, and where Λ_ε is a function defined on $\mathbb{R}_+$ such that

$$\frac{\Lambda_\varepsilon(t)}{t} \text{ is decreasing on } \mathbb{R}_+ \text{ and } \forall t \ge \varepsilon,\ \Lambda_\varepsilon(t) \ge \kappa_n \log\frac{t}{\varepsilon} - C'.$$

Note that in this proposition, the map u_ε is not assumed to solve (1.7) or to satisfy an energy bound. The above statement differs from the equivalent in [19] in that the latter does not require (4.1) but then requires to distinguish among essential and non-essential connected components of S_ε. We do not give details but it is well-known in the case $n = 2$ (see for instance [27]) that this distinction can be removed by assuming (4.1), and the case of general n is identical.

4.1. Weak convergence.

We now begin to prove Theorem 1.2. It is well known, this is proved in [19], that under the hypothesis of the theorem $\{u_\varepsilon\}_\varepsilon$ converges weakly in $W^{1,n}_{\mathrm{loc}}(\Omega \setminus S)$, where S is a finite set. It is to be noted that the energy bound is sufficient for this to hold (see [19]). This begins with the following consequence of Proposition 4.1.

Proposition 4.2. *There exists constants C, ε_0 depending only on Ω, g, M such that the following holds.*

Assume that $0 < \varepsilon < \varepsilon_0$, that $u_\varepsilon \in W^{1,n}_g(\Omega, \mathbb{R}^n)$ solves (1.7) and that $\mathbf{E}_\varepsilon(u_\varepsilon) \le \kappa_n d\,|\log\varepsilon| + M$, where $d = \deg(g)$. Then for any $C\varepsilon^{\frac{1}{4}} < \sigma < 1/C$ there exists a collection of points $\{x_i\}_{i\in I}$ in Ω, where the cardinal of I is bounded by a number depending only on Ω, g, M, such that

$$\mathbf{E}_\varepsilon\left(u_\varepsilon, \Omega \setminus \cup_{i\in I} B(x_i, \sigma)\right) \le \kappa_n d \log\frac{1}{\sigma} + C. \tag{4.3}$$

Proof. First we extend u_ε to a δ-neighbourhood Ω_δ of Ω, with $\delta > 0$ by letting on $u_\varepsilon(x) = g(\pi x)$ for any $x \in \Omega_\delta \setminus \Omega$, where π is the nearest point projection to $\partial\Omega$. If δ is small enough, this extension is well defined and such that

$$\mathbf{E}_\varepsilon\left(u_\varepsilon, \Omega_\delta \setminus \Omega\right) \le C, \quad |u_\varepsilon| = 1 \text{ on } \Omega_\delta \setminus \Omega. \tag{4.4}$$

Since u_ε solves (1.7) the bound (4.1) holds (see [11]) and therefore we may apply Proposition 4.1 to find that for any $t > C\varepsilon\,|\log\varepsilon|$, there exists a finite collection of disjoint balls $\{B_i^t\}_{i\in I_t}$ covering S_ε and such that for any $i \in I_t$

$$B_i^t \cap S_\varepsilon \neq \varnothing, \quad \mathbf{E}_\varepsilon(u_\varepsilon, B_i^t) \geq r(B_i^t)\frac{\Lambda_\varepsilon(t)}{t}, \quad r(B_i^t) \geq |d_i| t, \tag{4.5}$$

where $d_i = \deg(B_i^t, u_\varepsilon)$ if $B_i^t \subset \Omega$ and $d_i = 0$ otherwise.

If $t > C\varepsilon^{\frac14}$ (which is greater than $C\varepsilon\,|\log\varepsilon|$ if ε is small enough) we have $\Lambda_\varepsilon(t) \geq \frac{1}{C}\,|\log\varepsilon|$ if ε is small enough, therefore the energy bound for u_ε implies that for every $i \in I_t$

$$r(B_i^t) \leq Ct. \tag{4.6}$$

In turn, since $B_i^t \cap \Omega \neq \varnothing$, this implies that if $t < 1/C$, then $B_i^t \subset \Omega_\delta$ and therefore

$$\mathbf{E}_\varepsilon(u_\varepsilon, B_i^t) \geq \left|\deg(B_i^t, u_\varepsilon)\right| \Lambda_\varepsilon(t) \geq \kappa_n \left|\deg(B_i^t, u_\varepsilon)\right| \left(\log\frac{t}{\varepsilon} - C\right). \tag{4.7}$$

We change variables by letting $\sigma = Ct$, and denoting $\{x_j\}_{j\in J}$ the centers of the balls for which $\deg(B_i^t, u_\varepsilon) \neq 0$ we have by comparing (4.7) and the energy upper bound for u_ε that the cardinal of J is bounded above independently of ε. Moreover

$$\mathbf{E}_\varepsilon\left(u_\varepsilon, \cup_{i\in j} B(x_j,\sigma)\right) \geq \kappa_n \left(\sum_i \left|\deg(B_i^t, u_\varepsilon)\right|\right)\left(\log\frac{t}{\varepsilon} - C\right).$$

Since the balls B_i^t are included in Ω_δ, the sum of the degrees must be d so that

$$\mathbf{E}_\varepsilon\left(u_\varepsilon, \cup_{i\in j} B(x_j,\sigma)\right) \geq \kappa_n d \log\frac{\sigma}{\varepsilon} - C.$$

Using (4.4) and the energy bound for u_ε we deduce (4.3). □

We are ready to prove

Proposition 4.3. *Under the assumptions of Theorem 1.2, there exists a subsequence $\{u_\varepsilon\}_\varepsilon$, d distinct points $\{a_1, a_2, \cdots, a_d\} \subset \Omega$, and an $\mathbb{S}^{n-1}$-valued map $u_0 : \Omega \setminus \{a_1, a_2, \cdots, a_d\} \to \mathbb{S}^{n-1}$ such that, as $\varepsilon \to 0$,*

$$u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in} \quad \mathbf{W}^{1,n}_{\mathrm{loc}}(\Omega \setminus \{a_1, a_2, \cdots, a_d\}, \mathbb{R}^n) \tag{4.8}$$

and for any $1 \leq p < n$

$$u_\varepsilon \rightharpoonup u_0 \quad \textit{weakly in} \quad \mathbf{W}^{1,p}(\Omega, \mathbb{R}^n). \tag{4.9}$$

Moreover, $\deg(u_0, \partial B_\sigma(a_j), \mathbb{S}^{n-1}) = 1$, *for $1 \leq j \leq d$ and for any small enough $\sigma > 0$.*

Step 1. Assuming the hypothesis of Theorem 1.2 are satisfied, then using Proposition 4.2 and extracting subsequences as in [19], Proposition 5.1, there exists a finite subset $S = \{a_1, \ldots, a_\ell\}$ of $\overline{\Omega}$ such that u_ε converges to u_0 weakly in $W^{1,n}_{\mathrm{loc}}(\Omega \setminus \{a_1, \ldots, a_\ell\})$.

Another consequence of Proposition 4.2 is that for any $p < n$, $\{u_\varepsilon\}_\varepsilon$ is bounded in $W^{1,p}$. Indeed, proceeding as in [29], we decompose Ω as follows. For any integer k we let $\omega_k = \cup_{i\in I_k} B(x_i^k, 2^k C\varepsilon^{\frac14})$, where the x_i^k are provided by Proposition 4.2 applied with $\sigma_k = 2^k C\varepsilon^{\frac14}$. Choosing K such that $2^K C\varepsilon^{\frac14} \in [1/2, 1]$, we have $\Omega = \cup_k \Omega_k$, where $k = 0, \ldots, K+1$ and

$$\Omega_0 = \omega_0, \quad \Omega_{K+1} = \Omega \setminus \omega_K, \quad \text{and } \Omega_k = \omega_k \setminus \omega_{k-1} \text{ if } 1 \leq k \leq K.$$

Then, from (4.3) and Hölder's inequality

$$\begin{aligned}\int_\Omega |\nabla u_\varepsilon|^p \,\mathrm{d}x &\leq \sum_{k=0}^{K+1} \int_{\Omega_k} |\nabla u_\varepsilon|^p \,\mathrm{d}x \\ &\leq \sum_{k=0}^{K+1} |\Omega_k|^{1-p/n} \left(\int_{\Omega_k} |\nabla u_\varepsilon|^n \,\mathrm{d}x\right)^{p/n} \\ &\leq |\Omega_0|^{1-p/n}\mathbf{E}_\varepsilon(u_\varepsilon) + C\sum_{k=1}^{K+1} \sigma_k^{2(1-p/n)} |\log \sigma_k| \,.\end{aligned}$$

Since σ_k is a geometric progression the sum is bounded independently of ε. The first term is bounded as well since $|\Omega_0| \leq C\varepsilon^{\frac{1}{2}}$. It follows that $\{u_\varepsilon\}_\varepsilon$ is bounded in $W^{1,p}$.

Step 2. We now prove that the points $\{a_1, \dots, a_\ell\}$ actually belong to Ω, and that the degree of u_0 around each of them is equal to one.

As before we extend u_ε to a δ-neighbourhood $\hat\Omega_\delta$ of Ω by letting on $u_\varepsilon(x) = g(\pi x)$ for any $x \in \hat\Omega_\delta \setminus \Omega$, where π is the nearest point projection to $\partial\Omega$ so that (4.4) holds.

To prove that the degrees are equal to 1 we argue by contradiction. Using the annulus estimate (2.9), we find that

$$\frac{1}{n}\int_{\Omega^\sigma} |\nabla u_*|^n \geq \kappa_n \sum_{i=1}^{\ell} |d_i|^{\frac{n}{n-1}} \log\frac{\eta}{\sigma}, \tag{4.10}$$

for any $0 < \sigma < \eta$ such that the annuli $B_\eta(a_i) \setminus B_\sigma(a_i)$ are disjoint and included in $\hat\Omega_\delta$, where Ω^σ is the complement in $\hat\Omega_\delta$ of the union $\cup_i B_\sigma(a_i)$.

From Proposition 4.2, and given a small enough $\delta > 0$ there are points $\{x_{i,\varepsilon}\}_i$ such that for each $\varepsilon > 0$ small enough (4.3) holds. Passing to a subsequence, we may assume these points converge to some points $\{x_j^*\}$, and clearly we must have for ε small enough that

$$\frac{1}{n}\int_{\hat\Omega_\delta \setminus \cup_j B_{2\sigma}(x_j^*)} |\nabla u_*|^n \leq \kappa_n d \log\frac{1}{\sigma} + C. \tag{4.11}$$

this implies in particular that for every $1 \leq i \leq \ell$ we have $a_i \in \cup_j B_{2\sigma}(x_j^*)$. But then

$$\Omega^{3\sigma} \subset \hat\Omega_\delta \setminus \cup_j B_{2\sigma}(x_j^*),$$

and thus, if

$$\sum_{i=1}^{\ell} |d_i|^{\frac{n}{n-1}} > d,$$

then (4.10) and (4.11) contradict each by fixing $\eta > 0$ small enough and letting σ tend to 0.

The proof that the points lie in the interior is similar, except that it relies on the following boundary version of Lemma 2.4.

Lemma 4.4 (Boundary annulus estimate)**.** *Let $a \in \partial\Omega$. Assume there are $\mu_0 \in (\frac{1}{2}, 1)$, $\mu_1 > 1$ and $\sigma_1 > 0$ such that for any $r < \sigma_1$, there holds*

$$|\partial\Omega \cap B_r(a)| \leq \mu_1 \omega_n r^{n-1} \quad \text{and} \quad |\Omega \cap \partial B_r(a)| \leq \omega_n \mu_0 r^{n-1}.$$

Given $0 < r < s < \sigma_1$, assume $u \in W^{1,n}(B_s(a) \setminus B_r(a), \mathbb{S}^{n-1})$, and letting g be the restriction of u to $\partial\Omega \cap (B_s(a) \setminus B_r(a))$, that g is Lipschitz and $\deg u|_{\partial(\Omega \cap B_\rho(a))} = j \neq 0$ for almost all

$\rho \in (r,s)$. *Then*

$$\int_{\Omega\cap(B_s(a)\setminus B_r(a))} \frac{|\nabla u|^n}{n}\,\mathrm{d}\,x \geq \mu_0^{-\frac{1}{n-1}}|j|^{\frac{n}{n-1}}\kappa_n \ln\frac{s}{r} - 4|j|\mu_1\|g\|_{\mathrm{Lip}}^{n-1}\frac{s^n-r^n}{n}. \tag{4.12}$$

Proof. Using Lemma 1.4 in [16], we have for any $\rho \in (r,s)$

$$\int_{\Omega\cap\partial B_\rho(a)} |\nabla_{\tan} u|^n \,\mathrm{d}\mathcal{H}^{n-1} \geq \mu_0^{-\frac{1}{n-1}}|j|^{\frac{n}{n-1}}\kappa_n n\rho^{-1} - 4|j|\mu_1\kappa_n n\|g\|_{\mathrm{Lip}}^{n-1}\rho^{n-1}$$

Integrating it over (r,s), we deduce the desired inequality. □

Then we argue by contradiction. Using (4.12), we find as above that

$$\frac{1}{n}\int_{\Omega_\sigma} |\nabla u_*|^n \geq \kappa_n D\log\frac{\eta}{\sigma} - C, \tag{4.13}$$

where, denoting by k the number of points belonging to $\partial\Omega$,

$$D = k\mu_0^{-\frac{1}{n-1}} + (d-k),$$

for some $\mu_0 \in (1/2,1)$ which depends only on Ω, g. The important point is that if indeed some point is on the boundary, i.e. $k \neq 0$, then $D > d$, which we now assume.

But now (4.13) contradicts (4.11) again by letting σ tend to 0, which proves that the points $a_1,\ldots,a_d$ belong to Ω. □

4.2. **Improved convergence, u_0 is n-harmonic.** We now wish to prove that the limit u_0 in Proposition 4.3 is n-harmonic. This requires some improved convergence estimates, which in turn require to use the Pohozaev identity in a suitable way. We use the method in [4, 29, 11]. For any $x_0 \in \Omega$, $\rho > 0$, we define

$$f(x_0,\rho) = \rho\int_{\partial B_\rho(x_0)\bigcap\Omega} \frac{|\nabla u_\varepsilon|^n}{n} + \frac{1}{4\varepsilon^n}\left(1-|u_\varepsilon|^2\right)^2.$$

Then we have a "Courant-Lebesgue lemma" type result (see Lemma 2.3 in [29] and Lemma 3.5 in [11]).

Lemma 4.5. *Assume u_ε is a solution of* (1.7). *Then the following holds*

(i) If the upper bound (1.13) holds, then for any point $x_0 \in \Omega$, and $0 < \varepsilon \leq \mathrm{e}^{-1}$, we have

$$\inf_{\varepsilon^{1/2}\leq\rho\leq\varepsilon^{1/4}} f(x_0,\rho) \leq \frac{4\mathbf{E}_\varepsilon(u_\varepsilon,\Omega\cap B_{\varepsilon^{1/4}}(x_0))}{|\ln\varepsilon|} \leq C_1 := 4d\kappa_n + 4M$$

and

$$\inf_{5\varepsilon^{1/4}\leq\rho\leq 5\varepsilon^{1/8}} f(x_0,\rho) \leq \frac{8\mathbf{E}_\varepsilon(u_\varepsilon,\Omega\cap B_{5\varepsilon^{1/8}}(x_0))}{|\ln\varepsilon|} \leq 2C_1.$$

(ii). There exists γ and ε_0 depending on Ω, g, such that for $0 < \varepsilon < \varepsilon_0$ and $\varepsilon^{1/2} \leq \rho \leq 5\varepsilon_0^{1/8}$

$$f(x_0,\rho) \leq \gamma \quad \Longrightarrow \quad \inf_{B_\rho\bigcap\Omega} |u_\varepsilon| \geq 1/2.$$

Proof of the lemma. (i). For $0 < \varepsilon \leq \mathrm{e}^{-1}$, we have

$$\frac{1}{4}|\ln\varepsilon| \inf_{\varepsilon^{1/2}\leq\rho\leq\varepsilon^{1/4}} f(x_0,\rho) \leq \int_{\varepsilon^{1/2}}^{\varepsilon^{1/4}} f(x_0,\rho)\frac{1}{\rho}\,\mathrm{d}\rho \leq \mathbf{E}_\varepsilon(u_\varepsilon,\Omega\cap B_{\varepsilon^{1/4}}(x_0)) \leq \mathbf{E}_\varepsilon(u_\varepsilon,\Omega),$$

proving the first estimate in (i). The second inequality in (i) is similar.

(ii). From Proposition 3.3 in [11] we know that u_ε is Hölder continuous, and more precisely for any $\beta \in (0,1)$

$$[u_\varepsilon]_{C^\beta(\bar{\Omega})} \leq C_2 \varepsilon^{-\beta},$$

where C_2 is a positive constant independent of ε.

Fix $\beta = \frac{1}{2}$ and assume without loss of generality that $C_2 > \sqrt{5}$. Since Ω is a smooth bounded domain, there exists $\rho_0(\Omega) > 0$ such that $\forall \rho \in (0, \rho_0(\Omega))$ and for all $x_0 \in \Omega$, $D = B_\rho(x_0) \bigcap \Omega$ is strongly star-shaped w.r.t. some $y_0 \in D$ and $(x - y_0) \cdot \nu \geq \dfrac{1}{4}\rho$ for $\forall x \in \partial D$. We assume $\varepsilon < \rho$.

Now assume by contradiction that $y \in D$ is such that $|u_\varepsilon(y)| \leq \dfrac{1}{2}$, then

$$|x - y| \leq \frac{\varepsilon}{(4C_2)^{\frac{1}{\beta}}}, \quad \Longrightarrow \quad |u_\varepsilon(x)| \leq \frac{3}{4}$$

and it easily follows that there exists a constant $C_3 > 0$ independent of ε such that

$$\int_D \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 \geq C_3. \tag{4.14}$$

Using the Pohozaev inequality (3.11) which holds in D with $\alpha = \frac{1}{4}$, we find

(4.15)

$$\int_D \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 + \frac{\rho}{4} \int_{\partial D} |\nabla u_\varepsilon|^{n-2} |\partial_\nu u_\varepsilon|^2 \leq C \left(f(x_0, \rho) + \rho \int_{\partial D \cap \partial \Omega} \frac{|\nabla u_\varepsilon|^{n-2} |\nabla_\tau g|^2}{n} \right).$$

To conclude we need to absorb the integral on the right-hand side with the left-hand side. For this we note that $|\nabla u_\varepsilon|^2 = |\partial_\nu u_\varepsilon|^2 + |\nabla_\tau g|^2$, hence it follows from Young's inequality that

$$|\nabla u_\varepsilon|^{n-2} |\nabla_\tau g|^2 \leq 2^{\frac{n-2}{2}} (|\partial_\nu u_\varepsilon|^{n-2} |\nabla_\tau g|^2 + |\nabla_\tau g|^n) \leq \frac{1}{4C} |\partial_\nu u_\varepsilon|^n + C' |\nabla_\tau g|^n, \tag{4.16}$$

where C' is some positive constant depending on n. On the other hand, recall that g is smooth. Combining (4.15) and (4.16), we infer

$$\begin{aligned}
&\int_D \frac{1}{4\varepsilon^n} \left(1 - |u_\varepsilon|^2\right)^2 + \frac{\rho}{4} \int_{\partial D} |\nabla u_\varepsilon|^{n-2} |\partial_\nu u_\varepsilon|^2 \\
&\leq C \left[f(x_0, \rho) + \rho C' \int_{B_\rho(x_0) \cap \partial \Omega} \frac{|\nabla_\tau g|^n}{n} \right] + \frac{\rho}{4n} \int_{B_\rho(x_0) \cap \partial \Omega} |\partial_\nu u_\varepsilon|^n \\
&\leq C f(x_0, \rho) + C_4(n, \Omega, g, \alpha) \rho^n + \frac{\rho}{4n} \int_{\partial D} |\partial_\nu u_\varepsilon|^n \\
&\leq C f(x_0, \rho) + C_4(n, \Omega, g, \alpha) \rho^n + \frac{\rho}{4} \int_{\partial D} |\nabla u_\varepsilon|^{n-2} |\partial_\nu u_\varepsilon|^2.
\end{aligned} \tag{4.17}$$

Now we choose $\varepsilon_0 < \min(e^{-1}, (\rho_0(\Omega)/5)^8)$ satisfying $C_4(n, \Omega, g, \alpha) 5^n \varepsilon_0^{n/8} < \frac{C_3}{2}$ and γ such that $C\gamma < \frac{C_3}{2}$. Then (4.17) contradicts (4.14), which proves (ii). □

From this lemma we deduce improved convergence estimates.

Lemma 4.6. *For any $K \subset\subset \bar{\Omega} \setminus \{a_1, \cdots, a_d\}$, we have*

(a) $|u_\varepsilon| \longrightarrow 1$ uniformly in K, as $\varepsilon \longrightarrow 0$,

(b) $\dfrac{1}{\varepsilon^n} \displaystyle\int_K \left(1 - |u_\varepsilon|^2\right)^2 + \int_K |\nabla u_\varepsilon|^{n-2} |\nabla |u_\varepsilon||^2 \longrightarrow 0$, as $\varepsilon \longrightarrow 0$,

(c) $\dfrac{(1-|u_\varepsilon|^2)|u_\varepsilon|^2}{\varepsilon^n}$ *is bounded in* $L^1(K)$ *independently of* ε.

Proof of (a). We argue by contradiction. If the result were false there would exist $\delta > 0$, and for $\varepsilon > 0$ arbitrarily small there would exist $y_\varepsilon \in K$ such that $|u_\varepsilon(y_\varepsilon)| \le 1-\delta$. From Proposition 4.2, we know that if $\eta > 0$ is chosen small enough then $\mathbf{E}(u_\varepsilon, B_\eta(y_\varepsilon))$ is bounded independently of ε.

On the other hand, using $|u_\varepsilon(y_\varepsilon)| \le 1-\delta$ and Lemma 4.5, (ii), we have for every $\varepsilon^{\frac{1}{2}} < \rho < \varepsilon^{\frac{1}{4}}$

$$f(y_\varepsilon, \rho) \ge \gamma_\delta > 0,$$

where γ_δ is independent of ε. Integrating $f(y_\varepsilon,\rho)/\rho$ with respect to $\varepsilon^{\frac{1}{2}} < \rho < \varepsilon^{\frac{1}{4}}$ would then imply that $\mathbf{E}(u_\varepsilon, B_\eta(y_\varepsilon)) \ge \frac{1}{4}|\log\varepsilon|\gamma_\delta$, contradicting the boundedness of $\mathbf{E}(u_\varepsilon, B_\eta(y_\varepsilon))$. □

Proof of (b). As K is compact, we can cover it with finitely many balls $\{B_r(x_i)\}_{i\in I}$ of radius r such that the concentric balls of radius $2r$ that we denote $B_{2r}(x_i)$ do not touch S. Then $\mathbf{E}_\varepsilon(u_\varepsilon, \Omega\cap\cup_i A_i)$ is bounded independently of ε, where A_i is the annulus $B_{2r}(x_i)\setminus B_r(x_i)$. Using Fubini's theorem, there must therefore exist for each ε a radius $r_\varepsilon \in (r, 2r)$ such that

$$\sum_i \int_{\Omega\cap\partial B_{r_\varepsilon}(x_i)} \frac{1}{n}|\nabla u_\varepsilon|^n + \frac{1}{4\varepsilon^n}(1-|u_\varepsilon|^2)^2 \le C. \tag{4.18}$$

Now taking the scalar product of (1.7) with $u_\varepsilon \dfrac{1-|u_\varepsilon|^2}{4|u_\varepsilon|^2}$ — recall that now we know that $|u_\varepsilon| \to 1$ locally uniformly outside the points $\{a_i\}_{1\le i\le d}$ — and integrating on $D_i := \Omega\cap B_{r_\varepsilon}(x_i)$ we get

$$\int_{D_i} \frac{1}{4\varepsilon^n}(1-|u_\varepsilon|^2)^2 = -\int_{\partial D_i} |\nabla u_\varepsilon|^{n-2}\frac{1-|u_\varepsilon|^2}{4|u_\varepsilon|}\partial_\nu|u_\varepsilon| + \\ + \int_{D_i} |\nabla u_\varepsilon|^{n-2}\nabla u_\varepsilon\cdot\nabla\left(u_\varepsilon\frac{1-|u_\varepsilon|^2}{4|u_\varepsilon|^2}\right),$$

and then

$$\int_{D_i} \frac{1}{4\varepsilon^n}(1-|u_\varepsilon|^2)^2 + \int_{D_i} |\nabla u_\varepsilon|^{n-2}\frac{|\nabla|u_\varepsilon||^2}{2|u_\varepsilon|^2} = \\ = \int_{D_i} |\nabla u_\varepsilon|^n\frac{1-|u_\varepsilon|^2}{4|u_\varepsilon|^2} - \int_{\partial D_i} |\nabla u_\varepsilon|^{n-2}\frac{1-|u_\varepsilon|^2}{4|u_\varepsilon|}\partial_\nu|u_\varepsilon|. \tag{4.19}$$

From (4.18) and using Hölder's inequality, we have

$$\int_{\Omega\cap\partial D_i} |\nabla u_\varepsilon|^{n-2}\frac{1-|u_\varepsilon|^2}{4|u_\varepsilon|}\partial_\nu|u_\varepsilon| \le C\varepsilon, \tag{4.20}$$

while the integrand vanishes on $\partial\Omega\cap\partial D_i$ since $|u_\varepsilon| = 1$ there. On the other hand from the uniform convergence of $|u_\varepsilon|$ to 1 on $\cup_i D_i$ and the boundedness of u_ε in $W^{1,n}(\cup_i D_i)$ we have

$$\lim_{\varepsilon\to 0}\int_{D_i} |\nabla u_\varepsilon|^n\frac{1-|u_\varepsilon|^2}{4|u_\varepsilon|^2} = 0.$$

Together with (4.20) and (4.19), this implies that

$$\lim_{\varepsilon\to 0}\left(\int_{D_i} \frac{1}{4\varepsilon^n}(1-|u_\varepsilon|^2)^2 + \int_{D_i} |\nabla u_\varepsilon|^{n-2}\frac{|\nabla|u_\varepsilon||^2}{2|u_\varepsilon|^2}\right) = 0,$$

which finishes the proof since $K\subset\cup_{i\in I}D_i$, and I is independent of ε. □

Proof of (c). The proof is similar to that of (b). Again we cover K with balls $\{B_r(x_i)\}_{i\in I}$ such that $B_{2r}(x_i)$ doesn't touch S. Then $\{u_\varepsilon\}_\varepsilon$ is bounded in $W^{1,n}$ hence also $W^{1,n-1}$ on $\cup_i A_i$, where $A_i = B_{2r}(x_i) \setminus B_r(x_i)$, and thus there exists for each $\varepsilon > 0$ a radius $r_\varepsilon \in (r, 2r)$ such that

$$\sum_i \int_{\Omega\cap\partial B_{r_\varepsilon}(x_i)} \frac{1}{n} |\nabla u_\varepsilon|^{n-1} \leq C. \tag{4.21}$$

Now taking the scalar product of (1.7) with u_ε and integrating on $D_i := \Omega \cap B_{r_\varepsilon}(x_i)$ we find

$$\int_{D_i\cap\Omega} \frac{(1-|u_\varepsilon|^2)\,|u_\varepsilon|^2}{\varepsilon^n} = -\int_{\partial D_i} |\nabla u_\varepsilon|^{n-2}\partial_\nu u_\varepsilon \cdot u_\varepsilon + \int_{D_i} |\nabla u_\varepsilon|^n . \tag{4.22}$$

Because $|u_\varepsilon| \leq 1$ in Ω and $|u_\varepsilon| = 1$ on the boundary we have

$$|\nabla u_\varepsilon|^{n-2}\partial_\nu u_\varepsilon \cdot u_\varepsilon \geq 0, \quad \text{on } \partial D_i \cap \partial\Omega,$$

and from (4.21) we deduce

$$\int_{\Omega\cap\partial D_i} |\nabla u_\varepsilon|^{n-2}\partial_\nu u_\varepsilon \cdot u_\varepsilon \leq C.$$

Inserting in (4.22) we find that

$$\int_{D_i\cap\Omega} \frac{(1-|u_\varepsilon|^2)\,|u_\varepsilon|^2}{\varepsilon^n} \leq C + \int_{D_i} |\nabla u_\varepsilon|^n ,$$

which finishes the proof since $\{u_\varepsilon\}_\varepsilon$ is bounded in $W^{1,n}(\cup_i D_i)$ and $K \subset \cup_i D_i$. □

We now recall the following result from [15].

Lemma 4.7 (Theorem [15]). *Assume $1 < p < \infty$ and for each $i = 1, 2, \cdots$, let $u_i \in W^{1,p}(\Omega, \mathbb{R}^n)$ be a weak solution of the following equation*

$$div\left(|\nabla u_i|^{p-2}\,\nabla u_i\right) + f_i = 0$$

with $K := \sup_i \|u_i\|_{W^{1,p}} + \sup_i \|f_i\|_{L^1} < \infty$. If $u_i \rightharpoonup u$ weakly in $W^{1,p}$, then $u_i \to u$ strongly in $W^{1,q}$ whenever $1 < q < p$.

We may now state

Proposition 4.8. *Assuming the hypothesis of Theorem 1.2 are satisfied, the map u_0 is an $\mathbb{S}^{n-1}$-valued n-harmonic map in $\Omega \setminus \{a_1, \ldots, a_d\}$.*

Proof. It follows from Lemma 4.6, (c) and Lemma4.7 that for any $1 < q < n$

$$u_\varepsilon \to u_0 \text{ strongly in } W^{1,q}_{\text{loc}}(\overline{\Omega} \setminus \{a_1, \cdots a_d\}, \mathbb{R}^n).$$

From Equation (1.7) , we have

$$\text{div}(|\nabla u_\varepsilon|^{n-2}\,\nabla u_\varepsilon \wedge u_\varepsilon) = 0 \text{ in } \mathcal{D}'(\Omega).$$

By the strong convergence in $W^{1,q}_{\text{loc}}(\Omega \setminus \{a_1, \cdots a_d\}, \mathbb{R}^n)$ for any $q < n$, we may pass to the limit above to find

$$\text{div}(|\nabla u_0|^{n-2}\,\nabla u_0 \wedge u_0) = 0 \text{ in } \mathcal{D}'(\Omega \setminus \{a_1, \cdots a_d\}).$$

It is well-known (see [5, 25, 28]) that a map $u \in W^{1,n}(\Omega \setminus \{a_1, \cdots a_d\}), \mathbb{S}^{n-1})$ is $n-$harmonic if and only if it satisfies the above equation. Therefore u_0 is $n-$harmonic map and the proposition is proved. □

4.3. $\eta-$**Regularity.** We now wish to prove the strong convergence of $\{u_\varepsilon\}_\varepsilon$ outside a finite set, which is needed to pass to the limit in the stationarity condition (3.4). However, as already mentionned in the introduction, strong convergence should not be expected to hold outside of S in general, because of bubbling. What we will prove though is that there may only be a finite number of bubbles, just as it is the case for Palais-Smale sequences of n-harmonic maps (see [22]).

The goal is thus to prove compactness under a small energy hypothesis. This is done by proving that a Campanato-space type estimate

$$\operatorname{osc}(u_\varepsilon, x, \rho) := \int_{\Omega\cap B(x,\rho)} |u_\varepsilon - (u_\varepsilon)_{x,\rho}|^n \le C\rho^{n+\delta} \tag{4.23}$$

holds for any $x \in \overline{\Omega} \setminus \{a_1, \ldots, a_d\}$, where $\delta > 0$ and C do not depend on x, ρ or ε, and where $(u_\varepsilon)_{x,\rho}$ denotes the average of u_ε on $B(x,\rho)$.

We distinguish the case where where ρ is much smaller than ε, which is easy, from the case where ρ is much larger than ε.

Lemma 4.9. *Assume $u_\varepsilon \in W^{1,n}_g$ is a solution of* (1.7). *Then, given $\beta > 1$, there exists $\delta, C > 0$ depending only on β, Ω, g such that for any $x \in \overline{\Omega}$ and any $\rho \in (0, \varepsilon^\beta)$ the estimate* (4.23) *holds.*

Proof. From [11], Proposition 3.3 we have that the $C^{0,\frac{1}{2}}$ norm of u_ε is bounded by $C\varepsilon^{-\frac{1}{2}}$, therefore on $B(x,\rho)$ we have

$$|u_\varepsilon - (u_\varepsilon)_{x,\rho}| \le C\left(\frac{\rho}{\varepsilon}\right)^{\frac{1}{2}},$$

and then

$$\operatorname{osc}(u_\varepsilon, x, \rho) \le \rho^n \rho^{(1-\frac{1}{\beta})\frac{n}{2}},$$

which finishes the proof. □

To prove the estimate for larger balls, we begin by rewriting (1.7) in terms of the modulus and "phase" of u_ε. Let

$$\rho_\varepsilon = |u_\varepsilon|, \quad \theta_\varepsilon = \frac{u_\varepsilon}{|u_\varepsilon|}. \tag{4.24}$$

Then $u_\varepsilon = \rho_\varepsilon\theta_\varepsilon$ and thus, from (1.7), we have

$$-\operatorname{div}(|\nabla u_\varepsilon|^{n-2} (\rho_\varepsilon \nabla\theta_\varepsilon + \theta_\varepsilon\nabla\rho_\varepsilon)) = \frac{1}{\varepsilon^n}(1 - |u_\varepsilon|^2)u_\varepsilon,$$

and then

$$-\operatorname{div}(|\nabla u_\varepsilon|^{n-2} \nabla\rho_\varepsilon)\,\theta_\varepsilon - 2\,|\nabla u_\varepsilon|^{n-2} \nabla\theta_\varepsilon \cdot \nabla\rho_\varepsilon - \operatorname{div}(|\nabla u_\varepsilon|^{n-2} \nabla\theta_\varepsilon)\rho_\varepsilon = \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)u_\varepsilon.$$

Taking the scalar product with θ_ε, we infer

$$-\operatorname{div}(|\nabla u_\varepsilon|^{n-2} \nabla\rho_\varepsilon) - 0 - \operatorname{div}(|\nabla u_\varepsilon|^{n-2} \nabla\theta_\varepsilon)\rho_\varepsilon \cdot \theta_\varepsilon = \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon,$$

where we have used the fact that $\theta_\varepsilon \cdot \nabla\theta_\varepsilon = 0$. This same fact implies that

$$-\operatorname{div}(|\nabla u_\varepsilon|^{n-2} \nabla\theta_\varepsilon) \cdot \theta_\varepsilon = |\nabla u_\varepsilon|^{n-2} |\nabla\theta_\varepsilon|^2,$$

and therefore

$$-\operatorname{div}(|\nabla u_\varepsilon|^{n-2} \nabla\rho_\varepsilon) + |\nabla u_\varepsilon|^{n-2} |\nabla\theta_\varepsilon|^2 \rho_\varepsilon = \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon. \tag{4.25}$$

Now recall that $\theta_\varepsilon = \dfrac{u_\varepsilon}{|u_\varepsilon|}$, thus $\nabla\theta_\varepsilon = -\nabla\rho_\varepsilon \dfrac{u_\varepsilon}{|u_\varepsilon|^2} + \dfrac{\nabla u_\varepsilon}{\rho_\varepsilon}$. Hence, from (4.25), straightforward calculations yield

$$
\begin{aligned}
-\mathrm{div}(\, |\nabla u_\varepsilon|^{n-2}\, \rho_\varepsilon^2 \nabla\theta_\varepsilon) &= \mathrm{div}[|\nabla u_\varepsilon|^{n-2}\, (u_\varepsilon \nabla\rho_\varepsilon - \rho_\varepsilon \nabla u_\varepsilon)] \\
&= \mathrm{div}(|\nabla u_\varepsilon|^{n-2}\, \nabla\rho_\varepsilon) u_\varepsilon + |\nabla u_\varepsilon|^{n-2}\, \nabla u_\varepsilon \cdot \nabla\rho_\varepsilon - \mathrm{div}(|\nabla u_\varepsilon|^{n-2}\, \nabla u_\varepsilon)\rho_\varepsilon - |\nabla u_\varepsilon|^{n-2}\, \nabla u_\varepsilon \cdot \nabla\rho_\varepsilon \\
&= |\, \nabla u_\varepsilon\, |^{n-2}\, \rho_\varepsilon^2\, |\, \nabla\theta_\varepsilon\, |^2\, \theta_\varepsilon - \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon^2\theta_\varepsilon + \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon^2\theta_\varepsilon \\
&= |\nabla u_\varepsilon|^{n-2}\, \rho_\varepsilon^2\, |\nabla\theta_\varepsilon|^2\, \theta_\varepsilon.
\end{aligned}
$$

Finally, we get the system

$$
-\,\mathrm{div}\,(|\nabla u_\varepsilon|^{n-2}\nabla\rho_\varepsilon) + |\nabla u_\varepsilon|^{n-2}|\nabla\theta_\varepsilon|^2\rho_\varepsilon = \frac{1}{\varepsilon^n}(1-\rho_\varepsilon^2)\rho_\varepsilon, \tag{4.26}
$$

$$
-\,\mathrm{div}\,(|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon) - |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon = 0. \tag{4.27}
$$

Lemma 4.10. *Let $x \in \bar\Omega$ and $r > \varepsilon$ such that $B_r(x) \cap \Omega \subset \{x \in \Omega|\ |u_\varepsilon(x)| \geq \frac{1}{2}\}$. Then there exists some positive constant $C > 0$ depending only on Ω and n but independent of ε such that*

$$
\int_{B_{r/2}(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}|\nabla\rho_\varepsilon|^2 + \frac{(1-\rho_\varepsilon)^2}{\varepsilon^n} \leq C\|1-\rho_\varepsilon\|_\infty^{\frac{n-3}{n-1}} \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n,
$$

provided that $\|1-\rho_\varepsilon\|_\infty$ is sufficiently small.

Proof. Let ξ be a smooth function compactly supported in $B_r(x)$ and such that $0 \leq \xi \leq 1$, $\xi|_{B_{r/2}(x)} \equiv 1$, and $|\nabla\xi| \leq \frac{2}{r}$. Taking $(1-\rho_\varepsilon)\xi^2$ as a test function in (4.26), we find

$$
\begin{aligned}
\int_{B_r(x)\cap\Omega} \xi^2|\nabla u_\varepsilon|^{n-2}|\nabla\rho_\varepsilon|^2 + \rho_\varepsilon\frac{(1-\rho_\varepsilon^2)(1-\rho_\varepsilon)\xi^2}{\varepsilon^n} &= \\
\int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}|\nabla\theta_\varepsilon|^2\rho_\varepsilon(1-\rho_\varepsilon)\xi^2 &+ 2|\nabla u_\varepsilon|^{n-2}(1-\rho_\varepsilon)\xi\nabla\rho_\varepsilon\cdot\nabla\xi,
\end{aligned} \tag{4.28}
$$

where we have used the fact that $\xi(1-\rho_\varepsilon) = 0$ on $\partial(B_r(x)\cap\Omega)$.

To estimate the right-hand side we first note that

$$
\int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}|\nabla\theta_\varepsilon|^2\rho_\varepsilon(1-\rho_\varepsilon)\xi^2 \leq C\|1-\rho_\varepsilon\|_\infty \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n. \tag{4.29}
$$

To deal with the second term we bound the integrand using Young's inequality, after noting that $|\nabla\xi| < 2/r$, to find

$$
|\nabla u_\varepsilon|^{n-2}|\nabla\rho_\varepsilon|\frac{\xi(1-\rho_\varepsilon)}{r} \leq C|\nabla u_\varepsilon|^n(1-\rho_\varepsilon)^{\frac{n-3}{n-1}} + C\frac{\xi^n(1-\rho_\varepsilon)^3}{r^n}.
$$

The integral of the second term on the right-hand side may be absorbed by the left-hand side of (4.28) if $\|1-\rho_\varepsilon\|_\infty$ is small enough, since $r > \varepsilon$, and we deduce that

$$
\int_{B_{r/2}(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}|\nabla\rho_\varepsilon|^2 + \frac{(1-\rho_\varepsilon)^2}{\varepsilon^n} \leq C\|1-\rho_\varepsilon\|_\infty^{\frac{n-3}{n-1}} \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n,
$$

proving the lemma. □

4.3.1. *Large interior balls.* Here we adapt the proof of [23] to obtain (4.23). Letting

$$e(x,r,u_\varepsilon) := \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n, \tag{4.30}$$

we wish to prove that for some $\theta \in (0,1)$, the inequality $e_\varepsilon(x,r/2,u_\varepsilon) \le \theta e_\varepsilon(x,r,u_\varepsilon)$ holds uniformly with respect to x, r, ε. This is well known to imply (4.23).

First we recall some definitions of the Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ and the $BMO(\mathbb{R}^n)$ and their basic properties. Let $\Psi \in \mathcal{C}_0^\infty(\mathbb{R}^n)$ be a function satisfying $\int_{\mathbb{R}^n}\Psi = 1$. For each $t > 0$, set $\Psi_t(x) = t^{-n}\Psi(\frac{x}{t})$. The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is the set of all functions $g \in L^1(\mathbb{R}^n)$ such that the maximal function $f^*(x) := \sup_{t>0}|\Psi_t * f(x)| \in L^1(\mathbb{R}^n)$. The Hardy space $\mathcal{H}^1(\mathbb{R}^n)$ is equipped with the norm $\|f\|_{\mathcal{H}^1(\mathbb{R}^n)} := \|f\|_{L^1(\mathbb{R}^n)} + \|f^*\|_{L^1(\mathbb{R}^n)}$. The Space $BMO(\mathbb{R}^n)$ is the subset of functions $f \in L^1_{\text{loc}}(\mathbb{R}^n)$ with bounded mean oscillations in the sense that $\|f\|_{BMO(\mathbb{R}^n)} := \sup_{x\in\mathbb{R}^n,t>0} ⨍_{B_t(x)}|f - f_{x,t}| < \infty$, where $f_{x,t} := ⨍_{B_t(x)} f$ is the average of f on $B_t(x)$.

The famous theorem of Fefferman and Stein in [9] states.

Theorem. *$\mathcal{H}^1(\mathbb{R}^n)^* = BMO(\mathbb{R}^n)$. That is, there is a constant $C = C(n)$ such that for all $f \in \mathcal{H}^1(\mathbb{R}^n)\bigcap C^\infty$ and $g \in BMO(\mathbb{R}^n)$ there holds*

$$\left|\int_{\mathbb{R}^n} f\cdot g\right| \le C\,||f||_{\mathcal{H}^1(\mathbb{R}^n)}\,||g||_{BMO(\mathbb{R}^n)}\,.$$

In our paper, functions are defined on Ω. When we say a function $f \in \mathcal{H}^1_{\text{loc}}(\Omega)$, we mean that in each relatively compact domain $U \subset\subset \Omega$, f agrees with a function in $\mathcal{H}^1(\mathbb{R}^n)$. And we define

$$||f||_{\mathcal{H}^1(U)} = \inf\{||g||_{\mathcal{H}^1(\mathbb{R}^n)} : f|_U = g|_U\}.$$

We recall a result in [6] (see also [23]).

Lemma 4.11. *Assume $u \in W_0^{1,p}(B_r(0))$ and $E \in L^{p'}(B_r(0),\mathbb{R}^n)$ with $\frac{1}{p} + \frac{1}{p'} = 1$, $p > 1$ and $p' > 1$. If E is divergence free in $B_r(0)$, then $\nabla u\cdot E \in \mathcal{H}^1_{\text{loc}}(B_r(0))$ and there holds*

$$\|\nabla u\cdot E\|_{\mathcal{H}^1(B_{r/2}(0))} \le C\|\nabla u\|_{L^p(B_r(0))}\|E\|_{L^{p'}(B_r(0))}$$

where $C > 0$ is some constant independent of u, E and r. In particular, when $x \in \Omega$ and $u \in W_0^{1,p}(\Omega)$, we have

$$\|\nabla u\cdot E\|_{\mathcal{H}^1(B_{r/2}(x)\cap\Omega)} \le C\|\nabla u\|_{L^p(B_r(x)\cap\Omega)}\|E\|_{L^{p'}(B_r(x)\cap\Omega)}$$

Proof. The first part is in [6]. For the second part, if $B_{3r/4}(x) \subset \Omega$, it is done in [23]. Otherwise, we fix a cut-off function $\xi \in C_0^1(B_r(x))$ such that $\xi|_{B_{r/2}(x)} \equiv 1$ and $\|\nabla\xi\|_\infty \le \frac{3}{r}$. By Poincaré's inequality, we have

$$\|\nabla(\xi u)\|_{L^p(B_r(x)\cap\Omega)} \le C\|\nabla u\|_{L^p(B_r(x)\cap\Omega)}.$$

Therefore, the desired result follows from the first part. □

The main step in proving (4.23) for larger balls is the following

Proposition 4.12. *Assume $\{u_\varepsilon\}_\varepsilon$ satisfy the hypothesis of Theorem 1.2, and that $u_\varepsilon \rightharpoonup u_0$ in $W^{1,n}_{\text{loc}}(\Omega\setminus\cup_{1\le i\le d}\{a_i\})$. Then there exist $\eta > 0, \bar\theta \in (0,1)$ such that the following holds.*

For any compact subset K of $\Omega\setminus\{a_i\}_{1\le i\le d}$ and any $\varepsilon > 0$ small enough depending on K, if $e(x,r,u_\varepsilon) \le \eta$ and $B_r(x) \subset K$, then

$$\int_{B_{r/4}(x)} \rho_\varepsilon^2|\nabla u_\varepsilon|^{n-2}|\nabla\theta_\varepsilon|^2 \le \bar\theta\|\nabla u_\varepsilon\|^n_{L^n(B_r(x))}.$$

Proof. We first estimate the norm of $|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon$ in the Hardy space $\mathcal{H}^1(B_{r/2}(x))$. Using the fact $|\theta_\varepsilon|=1$, we have

$$\partial_i\theta_\varepsilon\cdot\theta_\varepsilon=0$$

so that we can write

$$\begin{aligned}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon^k &= |u_\varepsilon|^{n-2}\rho_\varepsilon^2(\sum_{i,j}\partial_i\theta_\varepsilon^j\partial_i\theta_\varepsilon^j\theta_\varepsilon^k)\\ &= |u_\varepsilon|^{n-2}\rho_\varepsilon^2[\sum_{i,j}\partial_i\theta_\varepsilon^j(\partial_i\theta_\varepsilon^j\theta_\varepsilon^k-\partial_i\theta_\varepsilon^k\theta_\varepsilon^j)].\end{aligned}\tag{4.31}$$

Let

$$B_j=\nabla\theta_\varepsilon^j,\quad E_{jk}=|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2(\nabla\theta_\varepsilon^j\theta_\varepsilon^k-\nabla\theta_\varepsilon^k\theta_\varepsilon^j).$$

Then clearly $E_{jk}\in L^{\frac{n}{n-1}}(B_r(x))$ and $B_j\in L^n(B_r(x))$. Moreover B_j is a gradient hence $\mathrm{curl}B_j=0$. It also holds that $\mathrm{div}E_{jk}=0$. Indeed, using (4.27) and (4.31), for any $\Omega'\subset\Omega\setminus\{a_1,\cdots,a_d\}$, and $\phi\in W_0^{1,n}(\Omega',\mathbb{R})$, we have

$$\begin{aligned}\int_{\Omega'}\mathrm{div}E_{jk}\cdot\phi &= -\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2(\nabla\theta_\varepsilon^j\theta_\varepsilon^k)\cdot\nabla\phi+\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2(\nabla\theta_\varepsilon^k\theta_\varepsilon^j)\cdot\nabla\phi\\ &= -\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon^j\cdot\nabla(\phi\theta_\varepsilon^k)+\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon^k\cdot\nabla(\phi\theta_\varepsilon^j)\\ &= -\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\phi\theta_\varepsilon^j\theta_\varepsilon^k+\int_{\Omega'}|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\phi\theta_\varepsilon^k\theta_\varepsilon^j\\ &= 0.\end{aligned}\tag{4.32}$$

It follows from Lemma (4.11) that $E_{jk}\cdot B_j\in\mathcal{H}^1(B_{r/2}(x))$ and

$$||E_{jk}\cdot B_j||_{\mathcal{H}^1(B_{r/2}(x))}\le C\,||E_{jk}||_{L^{\frac{n}{n-1}}(B_r(x))}\,||B_j||_{L^n(B_r(x))}\,.$$

Together with (4.31), we deduce

$$\||\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon\|_{\mathcal{H}^1(B_{r/2}(x))}\le C\|\nabla u_\varepsilon\|_{L^n(B_r(x))}^{n-2}\|\nabla\theta_\varepsilon\|_{L^n(B_r(x))}^2,\tag{4.33}$$

since $0\le\rho_\varepsilon<1$. Let $\xi\in C_0^1(B_r(x))$ be a non-negative cut-off function satisfying $\xi_{|B_{r/4}(x)}\equiv1$, $\xi_{|\mathbb{R}^n\setminus B_{r/2}(x)}\equiv0$ and $|\nabla\xi|\le\frac{5}{r}$. We let

$$\bar\theta_{\varepsilon,x,r}=\fint_{B_{r/2}(x)\setminus B_{r/4}(x)}\theta_\varepsilon(x)\,\mathrm{d}x$$

be the average of θ_ε on the annulus $B_{r/2}(x)\setminus B_{r/4}(x)$. Set $\psi=\xi(\theta_\varepsilon-\bar\theta_{\varepsilon,x,r})$. Using again Poincaré's inequality, we have

$$\begin{aligned}\int_{B_r(x)}|\nabla\psi|^n &= \int_{B_{r/4}(x)}|\nabla\theta_\varepsilon|^n+\int_{B_{r/2}(x)\setminus B_{r/4}(x)}|\nabla\psi|^n\\ &\le \int_{B_{r/4}(x)}|\nabla\theta_\varepsilon|^n+2^n\int_{B_{r/2}(x)\setminus B_{r/4}(x)}|\nabla\theta_\varepsilon|^n+\frac{5^n}{r^n}|\theta_\varepsilon-\bar\theta_{\varepsilon,x,r}|^n\\ &\le \int_{B_{r/4}(x)}|\nabla\theta_\varepsilon|^n+C\int_{B_{r/2}(x)\setminus B_{r/4}(x)}|\nabla\theta_\varepsilon|^n\\ &\le C\int_{B_r(x)}|\nabla\theta_\varepsilon|^n\end{aligned}$$

Again from Poincaré's inequality and Hölder's inequality, we have $\forall z \in \mathbb{R}^n$ and $\forall s > 0$

$$\fint_{B_s(z)} |\psi - \bar{\psi}_{z,s}| \leq Cs^{1-n} \int_{B_s(z)} |\nabla\psi| \leq C \left(\int_{B_s(z)} |\nabla\psi|^n \right)^{1/n} \leq C\|\nabla\theta_\varepsilon\|_{L^n(B_r(x))},$$

which implies

$$\|\psi\|_{BMO(\mathbb{R}^n)} \leq C\|\nabla\theta_\varepsilon\|_{L^n(B_r(x))}, \tag{4.34}$$

Here $\bar{\psi}_{z,s}$ is the average of function ψ on the ball $B_s(z)$. Taking ψ as a test function in (4.27) and using (4.33) and (4.34) , we obtain

$$\begin{aligned} \int_{B_r(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2 \nabla\theta_\varepsilon \cdot \nabla\psi &= \int_{B_r(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2 |\nabla\theta_\varepsilon|^2\theta_\varepsilon \cdot \psi \\ &\leq C\||\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon\|_{\mathcal{H}^1(B_{r/2}(x))}\|\psi\|_{BMO(\mathbb{R}^n)} \\ &\leq C_1\|\nabla u_\varepsilon\|^{n-2}_{L^n(B_r(x))}\|\nabla\theta_\varepsilon\|^3_{L^n(B_r(x))} \end{aligned} \tag{4.35}$$

On the other hand, it follows from Poincaré's inequality and Hölder's inequality that

$$\begin{aligned} &\int_{B_{r/4}(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2 \\ &\leq \int_{B_r(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon\cdot\nabla\psi + \left| \int_{B_{\frac{r}{2}}(x)\setminus B_{\frac{r}{4}}(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon||\nabla\xi||\theta_\varepsilon - \bar{\theta}_{\varepsilon,x,r}| \right| \\ &\leq \int_{B_r(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon\cdot\nabla\psi + C\|\nabla u_\varepsilon\|^{n-2}_{L^n(B_{\frac{r}{2}}(x)\setminus B_{\frac{r}{4}}(x))}\|\nabla\theta_\varepsilon\|^2_{L^n(B_{\frac{r}{2}}(x)\setminus B_{\frac{r}{4}}(x))} \\ &\leq \int_{B_r(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon\cdot\nabla\psi + C_2\|\nabla u_\varepsilon\|^n_{L^n(B_{\frac{r}{2}}(x)\setminus B_{\frac{r}{4}}(x))}, \end{aligned} \tag{4.36}$$

Adding $C_2\|\nabla u_\varepsilon\|^n_{L^n(B_{\frac{r}{4}}(x)}$ to both sides we find

$$(1+C_2)\int_{B_{r/4}(x)} \rho_\varepsilon^2|\nabla u_\varepsilon|^{n-2}|\nabla\theta_\varepsilon|^2 \leq \int_{B_r(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon\cdot\nabla\psi + C_2\|\nabla u_\varepsilon\|^n_{L^n(B_{r/4}(x))}.$$

Then, using (4.35) and Hölder's inequality to bound the right-hand side,

$$\int_{B_{r/4}(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2 \leq \left(\frac{C_1}{1+C_2}\|\nabla u_\varepsilon\|_{L^n(B_r(x))} + \frac{C_2}{1+C_2} \right) \|\nabla u_\varepsilon\|^n_{L^n(B_r(x))}. \tag{4.37}$$

Now we can choose $\eta > 0$ such that

$$\alpha := \frac{C_1}{1+C_2}\eta^{\frac{1}{n}} + \frac{C_2}{1+C_2} < 1,$$

and from the uniform convergence of ρ_ε to 1 on K we know that, if ε is small enough depending on K, then $\alpha/{\rho_\varepsilon}^2$ is smaller on $B_r(x)$ than $\bar{\theta} := (1+\alpha)/2$, which is less than 1. For this choice of η the hypothesis of the proposition then imply, for small enough ε and in view of (4.37) that

$$\int_{B_{r/4}(x)} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2 \leq \bar{\theta}\|\nabla u_\varepsilon\|^n_{L^n(B_r(x))},$$

where $\bar{\theta} < 1$. □

Using Lemmas 4.10 and 4.6, Proposition 4.12 implies the following (Note that we need to assume that $r > \varepsilon$ so that the hypothesis of Lemma 4.10 are satisfied.)

Corollary 4.13. *Under the hypothesis of Proposition 4.12, there exist* $\eta > 0, \tau \in (0,1)$ *such that the following holds.*

For any compact subset K *of* $\Omega \setminus \{a_i\}_{1\le i\le d}$ *there exist* $\varepsilon_0 > 0, r_0 > 0$ *depending on* K *such that*

$$\varepsilon < \varepsilon_0,\ x \in K,\ r < r_0,\ \text{and } e(x,r,u_\varepsilon) \le \eta, \qquad \Longrightarrow\ e(x,r/4,u_\varepsilon) \le \tau e(x,r,u_\varepsilon).$$

Now we are ready to prove interior η-regularity, putting together the estimate on small balls and large balls.

Theorem 4.14. *Assume* $\{u_\varepsilon\}_\varepsilon$ *satisfy the hypothesis of Theorem 1.2, and that* $u_\varepsilon \rightharpoonup u_0$ *in* $W^{1,n}_{\text{loc}}(\Omega \setminus \cup_{1\le i\le d}\{a_i\})$*. Then there exist* $\eta > 0$*,* $\alpha > 0$ *such that the following holds.*

For any compact subset K *of* $\Omega \setminus \{a_i\}_{1\le i\le d}$ *there exist* $\varepsilon_0 > 0, r_0 > 0$ *depending on* K *such that if* $x \in K$*,* $\varepsilon \in (0,\varepsilon_0)$*,* $r \in (0,r_0)$ *and* $e(x,r,u_\varepsilon) \le \eta$ *then we have*

$$\|u_\varepsilon\|_{C^\alpha(B_{r/2}(x))} \le C. \tag{4.38}$$

where C *is some positive constant independent of* ε*.*

Proof. First we recall the property of Campanato spaces that given an open set U, the C^α norm on U is equivalent to

$$\|f\|_C = \sup_{\substack{x\in U\\ \rho<\operatorname{diam}(U)}} \frac{1}{\rho^\alpha}\left(\fint_U |f - \bar f_{U\cap B_\rho(x)}|^n\right)^{\frac1n},$$

where $\bar f_A$ denotes the average of f on A.

Therefore, proving (4.38) amounts to proving that for any $y \in B_{r/2}(x)$ and any $\rho < r$ it holds that

$$\operatorname{osc}(u_\varepsilon, y, \rho) \le C\rho^n\rho^{n\alpha}, \tag{4.39}$$

In view of Poincarï¿½'s inequality we may alternatively show that

$$e(y,\rho,u_\varepsilon) \le C\rho^{n\alpha}. \tag{4.40}$$

We choose some $\beta > 1$. In the case where $y \in B_{r/2}(x)$ and $\rho < \varepsilon^\beta$, Lemma 4.9 provides the desired estimate, with exponent $\alpha_1 = \frac12(1-\frac1\beta)$.

In the case where $\rho \ge \varepsilon$, we use Corollary 4.13 to deduce that

$$e(y,\rho,u_\varepsilon) \le \bar\theta e(y,2\rho,u_\varepsilon) \le \cdots \le (\bar\theta)^n e(y,r,u_\varepsilon) \le \eta(\bar\theta)^n,$$

where n is the integer part of $\log_2(r/\rho)$. It follows by straightforwardly that

$$e(y,\rho,u_\varepsilon) \le C\rho^{n\delta}, \quad \delta = -\frac1n\frac{\log\bar\theta}{\log 2},$$

note that $\delta > 0$ since $\bar\theta < 1$.

Finally, in the case where $\varepsilon^\beta \le \rho \le \varepsilon$ we simply use the above bound, noting that $\varepsilon \le \rho^{1/\beta}$ to find

$$e(y,\rho,u_\varepsilon) \le e(y,\varepsilon,u_\varepsilon) \le C\varepsilon^{n\delta} \le C\rho^{\frac{n\delta}{\beta}}.$$

Therefore, choosing $\alpha = \min(\alpha_1, \delta/\beta)$, either estimate (4.39) or (4.40) is satisfied for any $y \in B_{r/2}(x)$ and any $\rho < r$. □

4.3.2. *Boundary η-regularity.* The boundary version of Proposition 4.12 is

Proposition 4.15. *Assume $\{u_\varepsilon\}_\varepsilon$ satisfy the hypothesis of Theorem 1.2, and that $u_\varepsilon \rightharpoonup u_0$ in $W^{1,n}_{\rm loc}(\Omega \setminus \cup_{1\le i\le d}\{a_i\})$. Assume moreover that Ω has C^2 boundary and that the boundary data $g : \partial\Omega \to \mathbb{S}^{n-1}$ is C^1. Then there exist $C, \eta, \varepsilon_0, r_0 > 0$ and $\bar\theta \in (0,1)$ such that if $r < r_0$, if $\varepsilon < \varepsilon_0$ and if $x \in \partial\Omega$ then*

$$e(x,r,u_\varepsilon) < \eta \quad \Longrightarrow \quad \int_{B_{r/4}(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}|\nabla\theta_\varepsilon|^2 \le Cr^n + \bar\theta \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n. \tag{4.41}$$

If, moreover, we assume that $r > \varepsilon$ then,

$$e(x, r/4, u_\varepsilon) \le \frac{\bar\theta + 1}{2} e(x,r,u_\varepsilon) + Cr^n. \tag{4.42}$$

Proof. Denote by v a C^1 extension of the boundary map g, and let

$$B^1_j = \nabla(\theta^j_\varepsilon - v^j), \quad B^2_j = \nabla v^j, \quad E_{jk} = |\nabla u_\varepsilon|^{n-2}\, \rho_\varepsilon^2 (\nabla\theta^j_\varepsilon \theta^k_\varepsilon - \nabla\theta^k_\varepsilon\theta^j_\varepsilon).$$

Then

$$|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta^k_\varepsilon = \sum_j E_{jk}(B^1_j + B^2_j) = F^k_1 + F^k_2,$$

where $F^k_1 = \sum_j E_{jk}\cdot B^1_j$ and similarly for F^k_2.

We estimate the Hardy norm of F_1 as in Proposition 4.12, and the Lebesgue norm of F_2 using Hölder's inequality to find

$$\|F^1\|_{\mathcal{H}^1(B_{r/2}(x)\cap\Omega)} \le C\|\nabla u_\varepsilon\|^{n-1}_{L^n(B_r(x)\cap\Omega)}\|\nabla(\theta_\varepsilon - v)\|_{L^n(B_r(x)\cap\Omega)}, \tag{4.43}$$

$$\|F^2\|_{L^{\frac{n}{n-1}}(B_r(x)\cap\Omega)} \le C\|\nabla u_\varepsilon\|^{n-1}_{L^n(B_r(x)\cap\Omega)}. \tag{4.44}$$

Then let $\psi = \xi(\theta_\varepsilon - v)$, where ξ is cut-off function defined as in Proposition 4.12. Noting that $\theta_\varepsilon - v$ vanishes on $\partial\Omega$ we may use Poincaré's inequality as in Proposition 4.12 and find that

$$\|\psi\|_{BMO(\mathbb{R}^n)} \le C\|\nabla(\theta_\varepsilon - v)\|_{L^n(B_r(x)\cap\Omega)}, \tag{4.45}$$

Using ψ as a test function in (4.27) we have as before

$$\int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon\cdot\nabla\psi = \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\theta_\varepsilon\cdot\psi,$$

and we estimate the latter using (4.43) and (4.45), to get,

$$\int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon\cdot\nabla\psi \le C\left(\|\nabla u_\varepsilon\|^n_{L^n(B_r(x)\cap\Omega)}\|\nabla(\theta_\varepsilon - v)\|_{L^n(B_r(x)\cap\Omega)} + \|\nabla u_\varepsilon\|^{n-1}_{L^n(B_r(x)\cap\Omega)} r\right), \tag{4.46}$$

where we have used Hï¿½lder's inequality and the fact that

$$\|\nabla(\theta_\varepsilon - v)\|_{L^n(B_r(x)\cap\Omega)} \le Cr + \|\nabla u_\varepsilon\|_{L^n(B_r(x)\cap\Omega)}.$$

Let us denote by D_r the set $B_r(x)\cap\Omega$. We estimate

$$\int_{D_{\frac{r}{4}}} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2 \le \int_{D_r} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\xi|\nabla\theta_\varepsilon|^2,$$

and note that $|\nabla\theta_\varepsilon|^2 \le 2\nabla\theta_\varepsilon\cdot\nabla(\theta_\varepsilon - v) + |\nabla v|^2$ to deduce, using Hï¿½lder's and Poincarï¿½'s inequalities together with (4.46), that

(4.47)
$$\begin{aligned}
&\int_{D_{\frac{r}{4}}} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2\\
\le& 2\int_{D_r} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon\cdot\nabla\psi + \int_{D_{\frac{r}{2}}} |\nabla u_\varepsilon|^{n-2}|\nabla v|^2 + \frac{C}{r}\int_{D_{\frac{r}{2}}\setminus D_{\frac{r}{4}}} |\nabla u_\varepsilon|^{n-2}|\nabla\theta_\varepsilon||\theta_\varepsilon - v|\\
\le& C\left(r^2\|\nabla u_\varepsilon\|^{n-2}_{L^n(D_r)} + r\|\nabla u_\varepsilon\|^{n-1}_{L^n(D_r)} + \|\nabla u_\varepsilon\|^n_{L^n(D_r)}\|\nabla(\theta_\varepsilon - v)\|_{L^n(D_r)}\right)\\
&+ C\left(\int_{D_{\frac{r}{2}}\setminus D_{\frac{r}{4}}} |\nabla u_\varepsilon|^n\right)^{\frac{n-1}{n}}\left(\int_{D_{\frac{r}{2}}\setminus D_{\frac{r}{4}}} |\nabla(\theta_\varepsilon - v)|^n\right)^{\frac{1}{n}}\\
\le& C_1\left(r^2\|\nabla u_\varepsilon\|^{n-2}_{L^n(D_r)} + r\|\nabla u_\varepsilon\|^{n-1}_{L^n(D_r)} + \|\nabla u_\varepsilon\|^n_{L^n(D_r)}(\|\nabla u_\varepsilon\|_{L^n(D_r)} + r)\right) + \int_{D_{\frac{r}{2}}\setminus D_{\frac{r}{4}}} |\nabla u_\varepsilon|^n.
\end{aligned}$$

Adding $C_1\|\nabla u_\varepsilon\|^n_{L^n(D_{\frac{r}{4}})}$ on both side and dividing by $C_1 + 1$ we get

$$\begin{aligned}
\int_{D_{\frac{r}{4}}} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2 \le\ & \|\nabla u_\varepsilon\|^n_{L^n(D_r)}\left(\tfrac{C_1}{C_1+1}\right)\left(\|\nabla u_\varepsilon\|_{L^n(D_r)} + r + 1\right)\\
&+ C\left(r^2\|\nabla u_\varepsilon\|^{n-2}_{L^n(D_r)} + r\|\nabla u_\varepsilon\|^{n-1}_{L^n(D_r)}\right).
\end{aligned}$$

But, given any small $\gamma > 0$ there exists a constant C such that $r^2a^{n-2} + ra^{n-1} \le C(a^n + r^n)$ for any $a > 0$. Inserting above we deduce

$$\int_{D_{\frac{r}{4}}} |\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2|\nabla\theta_\varepsilon|^2 \le \|\nabla u_\varepsilon\|^n_{L^n(D_r)}\left(\frac{C_1}{C_1+1}\right)\left(\|\nabla u_\varepsilon\|_{L^n(D_r)} + r + 1 + C\gamma + Cr^n\right),$$

and we may choose $\eta > 0$ and $\gamma > 0$ small enough so that the right-hand side is bounded by $Cr^n + \bar\theta\|\nabla u_\varepsilon\|^n_{L^n(D_r)}$ for some $\bar\theta \in (0,1)$. Then (4.42) follows using Lemmas 4.6 and 4.10, possibly modifying the constants $\bar\theta$ and C. □

We can now state the boundary η-regularity result.

Theorem 4.16. *Assume $\{u_\varepsilon\}_\varepsilon$ satisfy the hypothesis of Theorem 1.2, and that $u_\varepsilon \rightharpoonup u_0$ in $W^{1,n}_{\mathrm{loc}}(\Omega \setminus \cup_{1\le i\le d}\{a_i\})$. Assume moreover that the domain Ω is C^2 and that the boundary data $g : \partial\Omega \to \mathbb{S}^{n-1}$ is C^1. Then there exist $C, \eta, \varepsilon_0, r_0 > 0$ and $\bar\theta \in (0,1)$ such that if $r < r_0$, if $\varepsilon < \varepsilon_0$ and if $x \in \partial\Omega$ then*

(4.48)
$$e(x, r, u_\varepsilon) \le \eta \quad\Longrightarrow\quad \|u_\varepsilon\|_{C^\alpha(B_r(x)\cap\Omega)} \le C,$$

where C is independent of ε.

Proof. It follows from Lemma 4.9 and Proposition 4.15. The proof is similar to the one of Theorem 4.14. We leave the details to the interested reader. □

4.4. **Strong convergence.** The only statements in Theorem 1.2 remaining to be proved are first that there exists a finite set S_1 such that for any compact subset K of $\overline{\Omega} \setminus (S_1 \cup \{a_i\}_{1\le i\le d})$ we have

(4.49)
$$\lim_{\varepsilon\to 0} \|\nabla(u_\varepsilon - u_0)\|_{L^n(K)} = 0,$$

and second that the limiting map u_0 is stationary. In this section we prove (4.49), the stationarity will follow easily.

We define S_1 as follows.

$$S_1 = \bigcap_{r>0} \left\{ x \in \bar{\Omega} \setminus \{a_1, a_2, \cdots, a_d\} \mid \liminf_{\varepsilon \longrightarrow 0} \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n > \frac{\eta}{2} \right\} \tag{4.50}$$

where η is the constant in Theorems 4.14 and 4.16.

The set S_1 is finite because of our assumption of the upper bound for the energy. Indeed fix $\sigma > 0$ and recall $\Omega_\sigma = \Omega \setminus \cup_{1\le i\le d} B_\sigma(a_i)$. Then if there are k distinct points in $S_1 \cap \Omega_{2\sigma}$ we must have

$$\liminf_{\varepsilon\to 0} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega_\sigma) \ge k\frac{\eta}{2} + \frac{1}{n}\int_{\Omega_\sigma} |\nabla u_0|^n \ge k\frac{\eta}{2} + \kappa_n d \log\frac{1}{\sigma} - C,$$

where C is independent of σ. Using Proposition 4.2 we have that

$$\limsup_{\varepsilon} \mathbf{E}_\varepsilon(u_\varepsilon, \Omega_\sigma) \le \kappa_n d \log\frac{1}{\sigma} + C,$$

and we deduce a bound on k, hence on the cardinal of S_1.

Now fix a compact subset $K \subset\subset \bar{\Omega} \setminus (\{a_1, a_2, \cdots, a_d\} \cup S_1)$. Then for any $x \in K$, by the definition of S_1, there exists r, such that

$$\liminf_{\varepsilon\to 0} \int_{B_r(x)\cap\Omega} |\nabla u_\varepsilon|^n \le \frac{\eta}{2}.$$

Applying Theorems 4.14 and 4.16, up to a subsequence , u_ε is a bounded family in $C^\alpha(B_r(x)\cap\bar{\Omega})$ for some $\alpha > 0$. Since K is compact, by a covering argument, up to a subsequence, u_ε is a bounded family in $C^\alpha(K_1)$ where K_1 is some relatively compact neighborhood of K. Therefore, it follows from Arzela-Ascoli theorem that there is a subsequence — still denoted $\{u_\varepsilon\}$ —such that $u_\varepsilon \to u_0$ in $\mathbf{C}^{\alpha'}(K_1)$ for any $\alpha' < \alpha$.

If B_r is any ball of radius r such that $\overline{B}_r \cap (\{a_1, a_2, \cdots, a_d\} \cup S_1) = \varnothing$ then Proposition 4.2, implies that $\{u_\varepsilon\}_\varepsilon$ is a bounded sequence in $W^{1,n}(B_r \cap \Omega)$ and from Lemma 4.6 we have

$$\lim_{\varepsilon\to 0} \|\rho_\varepsilon - 1\|_{L^\infty(B_r\cap\Omega)} = 0, \quad \lim_{\varepsilon\to 0} \int_{B_r\cap\Omega} |\nabla u_\varepsilon|^{n-2} |\nabla\rho_\varepsilon|^2 = 0. \tag{4.51}$$

which implies in particular that $\lim_\varepsilon \|\nabla\rho_\varepsilon\|_{L^n(B_r\cap\Omega)} = 0$ and, from the boundness of $\|\nabla u_\varepsilon\|_{L^n(B_r\cap\Omega)}$ and Hï¿½lder's inequality, that for any $\gamma \in (0, n)$

$$\lim_{\varepsilon\to 0} \int_{B_r\cap\Omega} |\nabla u_\varepsilon|^{n-\gamma} |\nabla\rho_\varepsilon|^\gamma = 0. \tag{4.52}$$

Let ξ a smooth positive function compactly supported in B_r and such that $\xi_{|B_{r/2}} \equiv 1$ and $|\nabla\eta| \le \frac{3}{r}$. Taking $\xi(u_\varepsilon - u_{\varepsilon'})$ as a test function in (4.27) for u_ε and $u_{\varepsilon'}$ respectively, we have

$$\begin{aligned} &\int_{B_r} [|\nabla u_\varepsilon|^{n-2} \rho_\varepsilon^2 \nabla\theta_\varepsilon - |\nabla u_{\varepsilon'}|^{n-2} \rho_{\varepsilon'}^2 \nabla\theta_{\varepsilon'}] \cdot \nabla[(u_\varepsilon - u_{\varepsilon'})\xi] \\ &= \int_{B_r} [|\nabla u_\varepsilon|^{n-2} \rho_\varepsilon^2 |\nabla\theta_\varepsilon|^2 \theta_\varepsilon - |\nabla u_{\varepsilon'}|^{n-2} \rho_{\varepsilon'}^2 |\nabla\theta_{\varepsilon'}|^2 \theta_{\varepsilon'}] \cdot [(u_\varepsilon - u_{\varepsilon'})\xi] \\ &\le \max_{B_r} |u_\varepsilon - u_{\varepsilon'}| \int_{B_r} (|\nabla u_\varepsilon|^n + |\nabla u_{\varepsilon'}|^n) = o(1), . \end{aligned} \tag{4.53}$$

Then we recall for any $a, b \in \mathbb{R}^n$ and any $p \geq 2$, we have

$$|a-b|^p \leq 2^{p-2}(|a|^{p-2}+|b|^{p-2})\,|a-b|^2 \leq 2^{p-1}(|a|^{p-2}\,a-|b|^{p-2}\,b)\cdot(a-b). \tag{4.54}$$

Since $\|u_\varepsilon - u_0\|_{L^\infty(B_r\cap\Omega)} = 0$. Using (4.54), (4.51) to (4.53) and Hölder's inequality, we have

$$\begin{aligned}
\int_{B_{r/2}\cap\Omega} |\nabla u_\varepsilon - \nabla u_{\varepsilon'}|^n \leq C \int_{B_r\cap\Omega} & [|\nabla u_\varepsilon|^{n-2}\,\nabla u_\varepsilon - |\nabla u_{\varepsilon'}|^{n-2}\,\nabla u_{\varepsilon'}]\cdot[\xi\nabla(u_\varepsilon - u_{\varepsilon'})] \\
\leq C \int_{B_r\cap\Omega} & [|\nabla u_\varepsilon|^{n-2}\,\rho_\varepsilon^2\nabla\theta_\varepsilon - |\nabla u_{\varepsilon'}|^{n-2}\,\rho_{\varepsilon'}^2\nabla\theta_{\varepsilon'}]\cdot[\nabla(\xi(u_\varepsilon-u_{\varepsilon'})) - (u_\varepsilon - u_{\varepsilon'})\nabla\xi] \\
& + C\int_{B_r\cap\Omega}(|1-\rho_\varepsilon||\nabla u_\varepsilon|^{n-1} + |1-\rho_{\varepsilon'}||\nabla u_{\varepsilon'}|^{n-1})|\nabla(u_\varepsilon - u_{\varepsilon'})| \\
& + C\int_{B_r\cap\Omega}(|\nabla\rho_\varepsilon||\nabla u_\varepsilon|^{n-2} + |\nabla\rho_{\varepsilon'}||\nabla u_{\varepsilon'}|^{n-2})|\nabla(u_\varepsilon - u_{\varepsilon'})| \\
\leq C \int_{B_r\cap\Omega} & \Big|[|\nabla u_\varepsilon|^{n-2}\rho_\varepsilon^2\nabla\theta_\varepsilon - |\nabla u_{\varepsilon'}|^{n-2}\,\rho_{\varepsilon'}^2\nabla\theta_{\varepsilon'}]\cdot[(u_\varepsilon-u_{\varepsilon'})\nabla\xi]\Big| + o(1) \\
\leq C (& \|\nabla u_\varepsilon\|^{n-1}_{L^n(B_r\cap\Omega)} + \|\nabla u_{\varepsilon'}\|^{n-1}_{L^n(B_r\cap\Omega)})\|u_\varepsilon - u_{\varepsilon'}\|_{L^n(B_r\cap\Omega)} + o(1) = o(1).
\end{aligned} \tag{4.55}$$

In the last inequality we used the compact Sobolev embedding to deduce that $\|u_\varepsilon - u_{\varepsilon'}\|_{L^n(B_r\cap\Omega)} \to 0$ as $\varepsilon, \varepsilon' \to 0$.

The strong convergence $u_\varepsilon \to u_0$ in $\mathbf{W}^{1,n}(B_r\cap\Omega)$ thus holds, hence (4.49) by a finite covering argument.

4.5. **stationarity of u_0.** In this section we conclude the proof of Theorem 1.2 by showing that the limiting map u_0 is a stationary n-harmonic map in the sense of Definition 1.3.

First we recall from [23] that if $u : \Omega_0 = \Omega \setminus \{a_1, a_2, \cdots, a_d\} \to \mathbb{S}^{n-1}$ is a n-harmonic map and if we know that $u \in \mathbf{W}^{1,n}_{\text{loc}}(\Omega_0)$, then $u \in \mathbf{C}^{1,\alpha}_{\text{loc}}(\Omega_0)$ for some $\alpha \in (0,1)$.

By Lemma 3.4, $T_{i,j}(u_\varepsilon)$ is divergence free. Thus, for any ball $B_r(y) \subset \Omega$, and $j = 1, \ldots, n$ we have

$$\int_{B_r(y)} \sum_i \partial_i T_{i,j}(u_\varepsilon) = \int_{\partial B_r(y)} \sum_i \nu_i T_{i,j}(u_\varepsilon) = 0.$$

As a consequence, on any annulus $B_R(y) \setminus B_r(y) \subset \Omega_0 \setminus S_1$, we have

$$\int_{B_R(y)\setminus B_r(y)} \sum_i \frac{x_i - y_i}{|x-y|} T_{i,j}(u_\varepsilon) = 0.$$

Letting $\varepsilon \to 0$ and applying Lemma 4.6 and Theorem 1.2 we find

$$\int_{B_R\setminus B_r} \sum_i \frac{x_i - y_i}{|x-y|}\cdot T_{i,j}(u_0) = 0.$$

Recall that $u_0 \in \mathbf{C}^{1,\alpha}(\Omega_0)$, therefore for every $y \in \Omega$ and $s > 0$ with $\partial B_s(y) \subset \Omega_0$ we get

$$\int_{\partial B_\sigma(y)} \sum_i T_{i,j}(u_0)\nu_i = 0. \tag{4.56}$$

On the other hand, we know from Lemma 3.4 that $\sum_i \partial_i T_{i,j}(u_\varepsilon) = 0$ in Ω. Again from Lemma 4.6 and Theorem 1.2, it follows that

$$\sum_i \partial_i T_{i,j}(u_0) = 0, \quad \text{in } \Omega_0 \setminus S_1.$$

By the regularity of u_0 in Ω_0, we deduce that the above identity is also true in Ω_0. Hence u_0 is a stationary n-harmonic map and Theorem 1.2 is proved.

4.6. **Proof of Proposition 1.4.** We now assume that $u:\Omega\setminus\{a_1,\dots,a_d\}\subset\mathbb{R}^n\to\mathbb{S}^{n-1}$ is a stationary $n-$harmonic map such that $\deg(u,a_i)=1$, and that in a neighbourhood of each singular point a_i we have

$$u(x)=\mathrm{e}^{B(x)}\frac{x-a_i}{|x-a_i|},$$

where $B(x)\in so(n)$ is antisymmetric matrix satisfying $B(a_i)=0$ which is C^1 w.r.t. x.

We start by proving (1.15). Without loss of generality, we assume $a_i=0$. Then, letting $r=|x|$ and $\nu=\frac{x}{|x|}$, we have

$$\partial_j u(x)=\mathrm{e}^{B(x)}\left(\partial_j B(x)\nu+\partial_j\nu\right),\quad \partial_j\nu=\frac{1}{r}\left(e_j-\langle e_j,\nu\rangle\nu\right). \tag{4.57}$$

Recall that $\mathrm{e}^{B(x)}$ is an orthogonal matrix, hence

$$\begin{aligned}\langle\partial_i u,\partial_j u\rangle&=\langle\partial_i B\ \nu,\partial_j B\ \nu\rangle+\langle\partial_i B\ \nu,\partial_j\nu\rangle+\langle\partial_j B\ \nu,\partial_i\nu\rangle+\langle\partial_i\nu,\partial_j\nu\rangle.\\ &=\langle\partial_i\nu,\partial_j\nu\rangle+\left(\langle\partial_i B(0)\nu,\partial_j\nu\rangle+\langle\partial_j B(0)\ \nu,\partial_i\nu\rangle\right)+o\left(\frac{1}{r}\right).\end{aligned} \tag{4.58}$$

In the above, the derivatives of ν are of order $1/r$, and the other terms are of order 1. We deduce easily the leading order term in the expansion w.r.t. r of $|\nabla u|^{n-2}$, $|\nabla u|^n$ and since these expansions contain only even powers, we find that

$$|\nabla u|^{n-2}=\frac{(n-1)^{\frac{n-2}{2}}}{r^{n-2}}+O\left(\frac{1}{r^{n-4}}\right),\quad |\nabla u|^n=\frac{(n-1)^{\frac{n}{2}}}{r^n}+O\left(\frac{1}{r^{n-2}}\right). \tag{4.59}$$

From (4.58) and (4.59) we deduce that for every j

$$\begin{aligned}\sum_i\nu^i T_{i,j}(u_0)&=\frac{(n-1)^{\frac{n-2}{2}}}{r^{n-2}}\left(\langle\partial_\nu\nu,\partial_j\nu\rangle+\langle\partial_\nu\nu,\partial_j B(0)\nu\rangle+\langle\sum_i\nu^i\partial_i B(0)\nu,\partial_j\nu\rangle\right)-\\ &\qquad-\frac{(n-1)^{\frac{n}{2}}}{nr^n}\nu^j+o\left(\frac{1}{r^{n-1}}\right)\\ &=-\frac{(n-1)^{\frac{n}{2}}}{nr^n}\nu^j+\frac{(n-1)^{\frac{n-2}{2}}}{r^{n-2}}\langle\sum_i\nu^i\partial_i B(0)\nu,\partial_j\nu\rangle+o\left(\frac{1}{r^{n-1}}\right),\end{aligned}$$

where we have used the fact that $\partial_\nu\nu=0$. Since the integral of ν^j on ∂B_r is equal to 0 we deduce that, as $r\to 0$,

$$\int_{\partial B(0,r)}\sum_i\nu^i T_{i,j}(u_0)=\frac{(n-1)^{\frac{n-2}{2}}}{r^{n-2}}\int_{\partial B(0,r)}\langle\sum_i\nu^i\partial_i B(0)\nu,\partial_j\nu\rangle+o(1). \tag{4.60}$$

From the antisymmetry of $\partial_i B(0)$ we have $\langle\partial_i B(0)\nu,\nu\rangle=0$ so that, in view of (4.57),

$$\langle\partial_i B(0)\nu,\partial_j\nu\rangle=\langle\partial_i B(0)\nu,\frac{1}{r}e_j\rangle.$$

Now we write $\nu(x)$, $\partial_j\nu(x)$ in terms of the coordinates $x^1,\dots,x^n$ and get

$$\sum_i\langle\nu^i\partial_i B(0)\nu,\partial_j\nu\rangle=\sum_{i,k}\frac{x^i x^k}{r^3}\langle\partial_i B(0)e_k,e_j\rangle.$$

Since the integral of $x^i x^k$ on ∂B_r is equal to $\alpha r^{n+1}\delta_{ik}$ for some strictly positive α and replacing in (4.60) we finally obtain from (1.14) that

$$\int_{\partial B(0,r)} \sum_i \nu^i T_{i,j}(u_0) = \alpha(n-1)^{\frac{n-2}{2}} \langle \sum_i \partial_i B(0) e_i, e_j \rangle + o(1) = 0,$$

which proves (1.15) since the equality holds for $j = 1, \ldots, n$.

To prove (1.16) we use Taylor's expansion $B(x) = \sum_i \partial_i B(0) x^i + O(|x|^2)$ to obtain

$$u(x) = \nu + |x|\, \partial_\nu B(0)\nu + O(|x|^2).$$

Therefore, it follows from $\sum_i \partial_i B(0) e_i = 0$ that

$$\Delta\, |x|^2\, \partial_\nu B(0)\nu = 2\sum_{i=1}^n \partial_i B(0) e_k = 0.$$

Hence, the quadratic $Q(x) = |x|^2\, \partial_\nu B(0)\nu$ is harmonic. When $n = 2$, we write

$$B(x) = \begin{pmatrix} 0 & \alpha(x) \\ -\alpha(x) & 0 \end{pmatrix} \tag{4.61}$$

The above condition (1.6) is equivalent to $\nabla\alpha(0) = 0$. Hence $B(x) = O(|x|^2)$. This concludes the proof of Proposition 1.4.

5. Construction of non-minimizing sequence of critical points

In this section, we prove Theorem 1.5.

Let $n = 3$ and $x = (x', x_3)$ with $x' \in \mathbb{R}^2$. We consider a domain

$$\Omega = C \cup D_+ \cup D_-$$

consisting of a long cylinder $C = \{x \in \mathbb{R}^3 |\, |x'| \le 1, |x_3| \le L\}$ of radius 1 and length $2L$ plus two spherical caps at each end $D_+ = B(P,1) \cap \{x_3 \ge L\}$ and $D_- = B(Q,1) \cap \{x_3 \le -L\}$, where $P = (0,0,L)$ and $Q = (0,0,-L)$.

We define a boundary map $g : \partial\Omega \to \mathbb{S}^2$ of degree one defined on the spherical caps by

$$g(x) = \frac{x-P}{|x-P|} \text{ on } \partial D_+ \cap \partial\Omega, \qquad g(x) = \frac{x-Q}{|x-Q|} \text{ on } \partial D_- \cap \partial\Omega.$$

On the cylindrical part of the boundary we let, choosing an arbitrary $h > 0$,

$$g(x) = \sqrt{\frac{1}{1+h^2}}(x', -h) \text{ if } 1 \le x_3 \le L-1, \quad g(x) = \sqrt{\frac{1}{1+h^2}}(x', h) \text{ if } -L+1 \le x_3 \le -1,$$

and the boundary map interpolates between these on the remaining part of the boundary, namely for $x_3 \in [L-1, L] \cup [-L, 1-L] \cup [-1, 1]$. We also require that the interpolation be such that

$$g \circ S = S \circ g, \quad \text{where} \quad S(x', x_3) = (x', -x_3)$$

and for any $\theta \in \mathbb{R}$, identifying $\mathbb{R}^2$ with $\mathbb{C}$,

$$g \circ R_\theta = R_\theta \circ g, \quad \text{where} \quad R_\theta(z, x_3) = (e^{i\theta} z, x_3).$$

We define the sobolev spaces of equivariant maps by

$$\bar{W}(\Omega, \mathbb{R}^3) = \{u \in W^{1,3}_g(\Omega, \mathbb{R}^3) \mid u \circ S = S \circ u, u \circ R_\theta = R_\theta \circ u, \forall\theta\},$$

and $\bar{W}(\Omega, \mathbb{S}^2) = \{u \in \bar{W}(\Omega, \mathbb{R}^3) \mid |u| = 1 \text{ a.e.}\}$.

Let E_ε be the Ginzburg-Landau functional. What we will do now is to show that there exists $C > 0$ such that if $\varepsilon > 0$ is small enough, then

$$\min_{u\in W_g^{1,3}(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon(u) < \min_{u\in \bar{W}(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon(u) \le \kappa_n |\log\varepsilon| + C \tag{5.1}$$

This will prove Theorem 1.5 since a minimizer u_ε for $\min \bar{W}(\Omega,\mathbb{R}^3)$ is a solution to (1.7) by the symmetric criticality principle, and if ε is small enough, using (5.1), it is nonminimizing and satisfies the bound (1.13). It remains thus to prove (5.1).

Upper bound for $\min_{u\in\bar{W}(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon(u)$**.** Let $B(0,\frac12) = \{x \in \mathbb{R}^3, |x| \le \frac12\}$ be the ball with center 0 and radius equal to $\frac12$ and $v \in \bar{W}(\Omega \setminus B(0,\frac12),\mathbb{R}^3)$ be some given equivariant map satisfying $v = g$ on $\partial\Omega$ and $v(x) = \frac{x}{|x|}$ for all $x \in \partial B(0,\frac12)$. Define for any $x \in B(0,\frac12)$

$$u_\varepsilon(x) = \rho_\varepsilon(|x|)\frac{x}{|x|}, \quad \rho_\varepsilon(r) = \min\left(\left(\frac{r-\varepsilon}{\varepsilon}\right)_+, 1\right)$$

and for any $x \in \Omega \setminus B(0,\frac12)$

$$u_\varepsilon(x) = v(x)$$

Then u_ε is clearly equivariant and thus we have

$$\min_{u\in\bar{W}(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon(u) \le \mathbf{E}_\varepsilon(u_\varepsilon) = \mathbf{E}_\varepsilon(v,\Omega\setminus B(0,\frac12)) + \mathbf{E}_\varepsilon(u_\varepsilon, B(0,\frac12)) \le \kappa_n |\log\varepsilon| + C, \tag{5.2}$$

since $\mathbf{E}_\varepsilon(u_\varepsilon, B(0,\frac12)) = \kappa_n |\log\varepsilon| + O(1)$ and $\mathbf{E}_\varepsilon(v,\Omega\setminus B(0,\frac12)) = O(1)$. Here C is some constant depending on Ω.

Upper bound for $\min_{u\in W_g^{1,3}(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon(u)$**.** This upper bound is more delicate. Let $\mathbf{D}$ denote the unit disc in $\mathbb{R}^2$. The large spherical cap is defined to be

$$A = \left\{(x', x_3) \in \mathbb{S}^2 \mid, x_3 \ge -\delta\right\}, \quad \text{where} \quad \delta = \frac{h}{\sqrt{1+h^2}}.$$

and the small spherical cap is $B = \mathbb{S}^2 \setminus A$.

Define $g_B : \partial\mathbf{D} \to \mathbb{S}^2$ by $g_B(x) = (\frac{x}{\sqrt{|x|^2+\delta^2}}, -\delta)$ and let

$$b := \min\{\frac13 \int_{\mathbf{D}} |\nabla u|^3 \mid u : \mathbf{D} \to \mathbb{S}^2, u|_{\partial\mathbf{D}} = g_B\}, \tag{5.3}$$

Then a minimizer exists, and we claim that it is unique and satisfies $u_B^3 \le -\delta$. Indeed if $u_B = (u_B^1, u_B^2, u_B^3)$ is a minimizer then $u'_B = (u_B^1, u_B^2, -|u_B^3|)$ is another minimizer, and since

$$-\mathrm{div}\,(|\nabla u'_B|\nabla {u'_B}^3) = |\nabla u'_B|^3 {u'_B}^3 \le 0,$$

the maximum principle implies that ${u'_B}^3 \le -\delta$ in $\mathbf{D}$. Since ${u'_B}^3 = -|u_B^3|$, it follows that $u_B^3 \le -\delta$, hence u_B takes values into B, and from [8, 10, 26], the minimizer u_B is in fact unique.

Now we may define a "large" 3-harmonic map $u_A : \mathbf{D} \to \mathbf{S}^2$ — following the strategy of [2], but here no bubbling can occur — by minimizing the three energy over the set of maps $u : \mathbf{D} \to \mathbf{S}^2$ agreeing with g_B on the boundary and such that

$$Q(u) = \int_{\mathbf{D}} u \cdot \partial_x u \wedge \partial_y u = |A|,$$

where $|A|$ denotes the surface area of the large cap A. The minimizer exists and belongs to the same class, because $Q(u)$ is well known to be continuous with respect to weak convergence in

$W^{1,3}$, moreover it is a 3-harmonic map because $Q(u)$ is a null-lagrangian, and is distinct from u_B because $Q(u_B) = -|B|$. Therefore

$$b < a := \frac{1}{3}\int_{\mathbf{D}} |\nabla u_A|^3 = \min\{\frac{1}{3}\int_{\mathbf{D}} |\nabla u|^3 \mid u : \mathbf{D} \to \mathbb{S}^2, u|_{\partial \mathbf{D}} = g_B, Q(u) = |A|\}. \tag{5.4}$$

We claim that

$$\min_{W^{1,3}_g(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon \le \kappa_3 |\ln \varepsilon| + L(a+b) + C, \tag{5.5}$$

where C does not depend on ε or L.

To prove this, we define a test map u_ε as follows. For any $x \in \mathbf{D}$ we let

$$u_\varepsilon(x', x_3) = g_B(x') \text{ if } x_3 \in [1, L-1], \quad u_\varepsilon(x', x_3) = -g_A(x') \text{ if } x_3 \in [-1, 1-L],$$

and we extend u_ε on $\mathbf{D} \times [-1, 1]$ so that it is a finite 3-energy $\mathbb{S}^2$-valued map, which is possible since the boundary map is of degree zero. Similarly we may define u_ε on $\Omega \cap \{x_3 \le 1 - L\}$ as a finite 3-energy $\mathbb{S}^2$-valued map. It remains to define u_ε on $U := \Omega \cap \{x_3 \ge L-1\}$. For this purpose, we do this first in the ball $B(P, 1)$ by letting for any $x \in B(P, 1)$

$$u_\varepsilon(x) = \rho_\varepsilon(|x-P|)\frac{x-P}{|x-P|}, \quad \rho_\varepsilon(r) = \min\left(\left(\frac{r-\varepsilon}{\varepsilon}\right)_+, 1\right),$$

and extend u_ε on $U \setminus B(P, 1)$ by taking some finite 3-energy $\mathbb{S}^2$-valued map as above since the boundary map is of degree zero.

Then it is straightforward to check that $\mathbf{E}_\varepsilon(u_\varepsilon, U) \le \kappa_3 |\log \varepsilon|\, C$, and we have

$$\mathbf{E}_\varepsilon(u_\varepsilon, \mathbf{D} \times [1, L-1]) = (L-2)b, \quad \mathbf{E}_\varepsilon(u_\varepsilon, \mathbf{D} \times [1-L, -1]) = (L-2)a.$$

On the rest of Ω, the energy of u_ε is clearly independent of ε, L thus (5.5) is proved.

Lower bound for $\min_{\bar W(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon$. Now we claim that

$$\min_{u \in \bar W(\Omega,\mathbb{R}^3)} \mathbf{E}_\varepsilon(u) \ge \kappa_3 |\ln \varepsilon| + 2La - C, \tag{5.6}$$

where C is independent of ε, L.

Here the equivariance of maps in $\bar W$ plays a role. Denote by u_ε the minimizer of the functional E_ε over $\bar W(\Omega, \mathbb{R}^3)$ and u_0 its limit given by Theorem 1.2, which applies because of (5.2). The map u_0 has a single singularity of degree 1, but it is also equivariant and therefore this singularity must be located at the origin. Since u_0 is 3-harmonic and minimizing away from the origin, it has $C^{1,\alpha}$ bounds away from the origin and the boundary which are independent of L (see [13]). It follows straightforwardly from this — we omit the lengthy details — that

$$\mathbf{E}_\varepsilon(u_\varepsilon, B(0, 1/2)) \ge \kappa_3 |\ln \varepsilon| - C. \tag{5.7}$$

Moreover, since u_0 is in $C^{1,\alpha}(\Omega \setminus \{0\}, \mathbb{S}^2)$, for any $t \in (1, L-1)$ the degree of u_0 on $(\partial\Omega \cap \{x_3 \ge t\}) \cup (\Omega \cap \{x_3 = t\})$ is equal to 0. Therefore we have for any $t \in (1, L-1)$ that

$$Q(u_0|_{\{x_3=t\}}) = |A|,$$

and therefore

$$\frac{1}{3}\int_{\{x_3=t\}\cap\Omega} |\nabla u_0|^3 \ge a,$$

which implies

$$\frac{1}{3}\int_{\{x_3\ge 1\}\cap\Omega} |\nabla u_0|^3 \ge La + O(1). \tag{5.8}$$

Similarly, we get

$$\frac{1}{3}\int_{\{x_3\leq -1\}\cap\Omega} |\nabla u_0|^3 \geq La + O(1). \tag{5.9}$$

Applying Theorem 1.2, $u_\varepsilon \to u_0$ weakly in $W^{1,n}$ outside the origin, therefore (5.7), (5.8) and (5.9) imply (5.6).

Conclusion. We have proved (5.2), (5.5) and (5.6), from which (5.1) follows easily in view of (5.4), if we choose $\varepsilon > 0$ small enough and $L > 0$ large enough. As already explained, this proves Theorem 1.5.

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