A refined result on sign changing solutions for a critical elliptic problem

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Abstract : In this work, we consider sign changing solutions to the critical elliptic problem $\Delta u + |u|^{\frac{4}{N-2}} u = 0$ in Ω_ε and $u = 0$ on $\partial\Omega_\varepsilon$, where $\Omega_\varepsilon := \Omega - (\bigcup_{i=1}^m (a_i + \varepsilon\Omega_i))$ for small parameter $\varepsilon > 0$ is a perforated domain, Ω and Ω_i with $0 \in \Omega_i$ ($\forall i = 1, \cdots, m$) are bounded regular general domains without symmetry in $\mathbb{R}^N$ and a_i are points in Ω for all $i = 1, \cdots, m$. As ε goes to zero, we construct by gluing method solutions with multiple blow up at each point a_i for all $i = 1, \cdots, m$.

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1 Introduction

In this paper we consider the semilinear critical elliptic problem

$$\begin{cases} \Delta u + |u|^{\frac{4}{N-2}} u = 0 & \text{in } \Omega \\ u = 0 & \text{on } \partial\Omega \end{cases} \tag{1}$$

where Ω is a bounded regular domain in $\mathbb{R}^N$, $N \geq 3$.

It is well know that the Sobolev embedding $H_0^1(\Omega) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega)$ is not compact and for this reason solvability of (1) is a quite delicate issue. Pohozaev's identity [29] shows that problem (1) has only the trivial solution if the domain Ω is assumed to be strictly starshaped. On the other hand, if Ω is an annulus then (1) has a (unique) positive solution in the class of functions with radial symmetry [19]. In the nonsymmetric case, Coron [9] found via variational methods that (1) is solvable under the assumption that Ω is a domain exhibiting a small hole. Substantial improvement of this result was obtained by Bahri and Coron [1], showing that if some homology group of Ω with coefficients in $\mathbf{Z}_2$ is not trivial, then (1) has at least one positive solution (see also [2, 5, 6, 20, 25, 31] for related results). If the domain Ω has several round holes, then a multiplicity result for positive solutions to (1) is obtained in [30]. Existence and qualitative behavior of sign changing solutions for elliptic problems with critical nonlinearity have been investigated by several authors in the last years (see [3, 4, 7, 18, 22]).

Tower of bubbles type solutions for the slightly supercritical problems are obtained [10, 11, 12, 28, 15, 16, 26]. In critical case, the same phenomenon is discovered for sign

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change solutions in [17, 23]. In these works, the authors treat the case in which the removed domains are small balls by Lyapunov reduction method (in particular for the domains perforated with one or two small balls). In this paper, by an alternative approach–gluing method, we study this problem for general pierced domains with arbitrary small holes without any symmetric condition neither on the perforated domains nor on the removed holes. We will not care of the location of the holes because we have a N-parameter family of conformal transformations for the annular domains, which corresponds to the translations at infinity. Moreover, the exact asymptotic profiles for the theses solutions are described. The proof here uses a gluing technique inspired from [21, 15]. However, the analysis is more delicate because of the presence of the translations and ones at infinity. They give the different asymptotic profiles near the boundary. To overcome the difficulties caused by the conformal invariance, we introduce some new weighted functional spaces on which we could get precise blow-up information of such solutions related to the two types of translations.

We briefly describe the plan of the paper. In section 2 we state the main results (Theorem 1). Some examples and some comments are illustrated in Section 3. Section 4 is devoted to radial solutions. We study the linearized operator in Section 5 and the main theorem will be proved in the last section.

2 Statement of the result

From now on we assume that Ω is a bounded regular domain in $\mathbb{R}^N$. Let Ω_i for all $i = 1, \cdots, m$ be a bounded regular domain in $\mathbb{R}^N$ such that $0 \in \Omega_i$. Given m points $a_i \in \Omega$ for all $i = 1, \cdots, m$, we consider a perforated domain $\Omega_\varepsilon := \Omega - (\bigcup_{i=1}^m (a_i + \varepsilon\Omega_i))$ for small parameter $\varepsilon > 0$. Given a positive integer $k \in \mathbb{N}$ and m integers $l_i \in \{0, 1\}$ for all $i = 1, \cdots, m$, we want to construct sign changed bubble tree solutions u_ε of (1) in Ω_ε which looks like a k-bubble around each point a_i. In other words, the sequence u_ε converges to 0 in any $\mathcal{C}^k$ topology away from the points a_i, as the parameter ε tends to 0.

Let G denote Green's function for the Laplace operator with Dirichlet boundary condition on Ω and let H denote Robin's function, i.e. the regular part of Green's function. Namely

$$G(y, z) := |y - z|^{2-N} - H(y, z),$$

for $(y, z) \in \Omega \times \Omega$. Observe that $\Delta_y H = 0$ in $\Omega \times \Omega$ and $G = 0$ on $\partial(\Omega \times \Omega)$.

We define the $m \times m$ matrix

$$M := (m_{ij})_{1 \leq i,j \leq m},$$

whose entries are given by

$$m_{ii} := H(a_i, a_i) > 0 \qquad \text{and} \qquad m_{ij} := -(-1)^{l_j - l_i} G(a_i, a_j), \tag{2}$$

if $i \neq j$.

Now we consider the Green type function for the Laplace operator with Dirichlet boundary condition in the exterior domain $\mathbb{R}^N - \Omega_i$ for any $1 \leq i \leq m$, that is,

$$\begin{array}{rlcl} G_i : & (\mathbb{R}^N - \Omega_i) \times \Omega_i & \to & \mathbb{R} \\ & (y, z) & \mapsto & G_i(y, z) \end{array}$$

satisfying

$$\begin{cases} \triangle_y G_i(\cdot, z) &= 0 \quad \text{in } \mathbb{R}^N - \Omega_i \\ G_i(\cdot, z) &= 0 \quad \text{on } \partial\Omega_i \end{cases}$$

and

$$-\frac{G_i(I_z(\cdot), z)}{|\cdot - z|^{N-2}} + \frac{1}{|\cdot - z|^{N-2}} \text{ is a regular function in } I_z(\mathbb{R}^N - \Omega_i)$$

Here for any $z \in \mathbb{R}^N$, let us denote I_z the inversion map around z in $\mathbb{R}^N - \{z\}$, namely,

$$\begin{array}{rlll} I_z : & \mathbb{R}^N - \{z\} & \rightarrow & \mathbb{R}^N - \{z\} \\ & y & \mapsto & \dfrac{y - z}{|y - z|^2} + z \end{array}$$

Let us define $\forall (y, z) \in I_z(\mathbb{R}^N - \Omega_i) \times \Omega_i$

$$H_i(y, z) = -\frac{G_i(I_z(y), z)}{|y - z|^{N-2}} + \frac{1}{|y - z|^{N-2}}$$

We make some hypotheses about the domains Ω_i and the matrix M :
(A1) For all $i = 1, \cdots, m$, there exists some point $a_i^* \in \Omega_i$ such that $\nabla_y H_i(a_i^*, a_i^*) = 0$ and the $N \times N$ matrix $\nabla(\nabla_y H_i(z, z))$ is not degenerated at the point $z = a_i^*$;
(A2) The functional $\mathcal{F} : (\mathbb{R}_+^*)^m \longrightarrow \mathbb{R}$ defined by

$$\mathcal{F}(\Lambda) := \Lambda\, M\, {}^t\Lambda + (2k - 1) \sum_{i=1}^{k} \left(e^{\frac{(N-2)kC_N}{2}} \sqrt{H_i(a_i^*, a_i^*)} \Lambda_i^{-1} \right)^{\frac{2}{2k-1}},$$

has a non-degenerate critical point in $(\mathbb{R}_+^*)^m$, where $\Lambda = (\Lambda_1, \ldots, \Lambda_m)$.

Here ∇_y (or ∇_1) designates the derivative with respect to the first variable. Granted the above definitions, our result reads :

Theorem 1 *Given $k \in \mathbb{N}$, assume that $N \geq 3$ and (**A1**) and (**A2**) are verified. Then, there exists $\varepsilon_0 > 0$ and for all $\varepsilon \in (0, \varepsilon_0)$ there exists u_ε a solution of (1) in Ω_ε, such that*

$$|\nabla u_\varepsilon|^2\, dx \quad \rightharpoonup \quad C_N^{(1)} \sum_{i=1}^{m} k\, \delta_{a_i},$$

in the sense of measures, where the constant $C_N^{(1)}$ is given by

$$C_N^{(1)} := (N(N-2))^{\frac{N+2}{4}} \int_{\mathbb{R}^N} \left(\frac{1}{1 + |x|^2} \right)^{\frac{N+2}{2}} dx.$$

More precisely, let us denote

$$r_{\varepsilon,1} = \varepsilon^{\frac{1}{k(N+2)}},\ r_{\varepsilon,2} = \varepsilon^{1 - \frac{1}{k(N+2)}},\ B_{\varepsilon,1} = -\log r_{\varepsilon,1},\ B_{\varepsilon,2} = -\log r_{\varepsilon,2}$$

Near each a_i the solutions u_ε has k-multiple blow up in the sense that there exists $c > 0$ (independent of ε) and parameters $D_{i,\varepsilon} > 0$, $A_{i,\varepsilon} \in \mathbb{R}$, $a'_{i,\varepsilon} \in \Omega_i$ and $b_{i,\varepsilon} \in \mathbb{R}^N$ such that

$$\frac{1}{c} < D_{i,\varepsilon} < c$$

$$|A_{i,\varepsilon}| < c$$

$$|a'_{i,\varepsilon}| < c$$

$$|b_{i,\varepsilon}| < c$$

$$\eta_i = D_{i,\varepsilon}\varepsilon^{\frac{N-2}{4k}}$$

and

$$\lim_{\varepsilon\to 0} \Bigg(\left\| u_\varepsilon(\cdot + a_i + \varepsilon a'_{i,\varepsilon}) - (-1)^{l_i} T_{b_{i,\varepsilon}}(u_{\eta_i, A_{i,\varepsilon}}(\cdot)) \right\|_{L^\infty(B(a_i, r_{\varepsilon,1}) - B(a_i,\sqrt{\varepsilon}))} + \varepsilon^{-\frac{N-2}{2}} \left\| u_\varepsilon(\cdot + a_i + \varepsilon a'_{i,\varepsilon}) - (-1)^{l_i} T_{b_{i,\varepsilon}}(u_{\eta_i, A_{i,\varepsilon}}(\cdot)) \right\|_{\mathcal{C}^0_{2-N}(B(a_i,\sqrt{\varepsilon}) - B(a_i, r_{\varepsilon,2}))} \Bigg) = 0$$

Here for any $b \in \mathbb{R}^N$ T_b is a translation at the infinity in the function spaces, that is, for any given the real function ψ on the $\mathbb{R}^N$, one has

$$T_b(\psi)(z) = \left| \frac{z}{|z|} - b|z| \right|^{2-N} \psi\left(\frac{\frac{z}{|z|^2} - b}{\left|\frac{z}{|z|^2} - b\right|^2} \right)$$

$u_{\eta_i, A_{i,\varepsilon}}(\cdot)$ is a radial solution in Section 4 and the norm $\|\cdot\|_{\mathcal{C}^0_{2-N}}$ will be defined in the section 5.

Alternatively, the sequence u_ε converges to 0 (in any $\mathcal{C}^k$ topology) away from the points a_i, as the parameter ε tends to 0. Near each a_i the solution u_ε has multiple blow up in the sense that there exists $c > 0$ (independent of ε), $a_{i,\varepsilon} \in \Omega$ and parameters $d_{i,j,\varepsilon} > 0$ such that

$$\frac{1}{c} < d_{i,j,\varepsilon} < c,$$

$$|b_{i,\varepsilon}| < c$$

$$a_{i,\varepsilon} \to a_i,$$

and

$$\lim_{\varepsilon\to 0} \left\| u_\varepsilon(\cdot) - (N(N-2))^{\frac{N-2}{4}} \sum_{j=1}^{k} (-1)^{j+l_i} \left(\frac{\bar{\varepsilon}_{i,j}}{1 + \bar{\varepsilon}_{i,j}^2 |\cdot - a_{i,\varepsilon}|^2} \right)^{\frac{N-2}{2}} \right\|_{H^1(B_{a_i, r_{\varepsilon,1}} - \varepsilon\Omega_i)} = 0.$$

Here

$$\bar{\varepsilon}_{i,j} := d_{i,j,\varepsilon} \left(\varepsilon^{\frac{1}{2} - j}\right)^{\frac{1}{k}}.$$

3 Applications and comments

Comment 1 We consider the case where $\Omega_i = B$ is the unit ball. We can check the condition (**A1**) for the point $a_i^* = 0$. Indeed, the direct calculations lead to

$$H_i(y,z) = \left(|z|^2 \left|y - z - \frac{z}{1-|z|^2}\right|^2 + \frac{1}{(1-|z|^2)^2} + \frac{2}{1-|z|^2}\left(\langle y, z\rangle - |z|^2 - \frac{|z|^2}{1-|z|^2}\right)\right)^{\frac{2-N}{2}}$$

so that

$$\nabla_y H_i(z,z) = (2-N)z.$$

Therefore, $\nabla_y H_i(0,0) = 0$ and $\nabla(\nabla_y H_i(0,0))|_{z=0} = (2-N)I$, where the $N \times N$ matrix I is the identity matrix. By a perturbation arguments, if for any $i = 1, \cdots, m$, Ω_i is close to the balls, the condition (**A1**) is also satisfied.

Comment 2 Now assume $l_1 = \cdots = l_m$. Thus, the condition (**A2**) is verified if and only if the matrix M is positive definite. To see this, we assume first M is positive definite. The minimization of the functional $\mathcal{F}$ guarantees the existence of the minimum point $\Lambda = (\Lambda_1, \cdots, \Lambda_m) \in (\mathbb{R}_+^*)^m$. Let us denote the $m \times m$ diagonal matrix by $Q_1 = \text{diag}\left(\left(e^{\frac{(N-2)kC_N}{2}}\sqrt{H_1(a_1^*,a_1^*)}\right)^{\frac{2}{2k-1}}\Lambda_1^{-\frac{4k}{2k-1}}, \cdots, \left(e^{\frac{(N-2)kC_N}{2}}\sqrt{H_m(a_m^*,a_m^*)}\right)^{\frac{2}{2k-1}}\Lambda_m^{-\frac{4k}{2k-1}}\right)$. At such point, the second differential $d^2\mathcal{F}(\Lambda) = 2M + \frac{4k+2}{2k-1}Q_1$ is positive definite, which yields the desired result. Conversely, we suppose the functional $\mathcal{F}$ has a critical point $\Lambda \in (\mathbb{R}_+^*)^m$. Let us denote by $Q = \text{diag}\,(\Lambda_1, \cdots, \Lambda_m)$ the $m \times m$ diagonal matrix. We consider the matrix $Q^t M Q$. The sum of k-th line is just the real number

$$m_{ii}\Lambda_i^2 + \sum_{j \neq i} m_{ij}\Lambda_i\Lambda_j = \left(e^{\frac{(N-2)kC_N}{2}}\sqrt{H_i(a_i^*,a_i^*)}\right)^{\frac{2}{2k-1}}\Lambda_i^{-\frac{2}{2k-1}}$$

since $m_{ii} > 0$ and $m_{ij} < 0$ for all $i \neq j$. Therefore, the matrix $Q^t M Q$ is dominated by the elements at the diagonal, which yields it is positive definite. Consequently, the matrix M is also positive definite.

Comment 3 Now assume all entries m_{ij} of the matrix M are positive. The functional $\mathcal{F}$ admits always a minimum point $\Lambda \in (\mathbb{R}_+^*)^m$. At this point, we have $Q^t d^2\mathcal{F}(\Lambda) Q = 2(Q^t M Q + Q_2)$, where the matrix Q_2 is diagonal, namely $Q_2 = \frac{2k+1}{2k-1}\text{diag}\left(\sum_{j=1}^m m_{ij}\Lambda_i\Lambda_j\right)$. Therefore, the matrix $Q^t d^2\mathcal{F}(\Lambda) Q = 2(Q^t M Q + Q_2)$ is dominated by the elements at the diagonal so that it is positive definite. Finally, the second differential $d^2\mathcal{F}(\Lambda)$ is positive definite.

Comment 4 The non-degeneracy condition in (**A1**) could be weakened.

Application 1 We consider the cases $m = 1$ and $\Omega_1 = B(0,1)$ or $m = 2$, $l_1 = 0$, $l_2 = 1$ and $\Omega_1 = \Omega_2 = B(0,1)$. The conditions (**A1**) and (**A2**) are satisfied. As a consequence of Theorem 1, there exists u_ε a solution of (1) in Ω_ε. Such a result has been obtained in [17].

Application 2 When $l_1 = \cdots = l_m$, $\Omega_1 = \cdots = \Omega_m = B(0,1)$ and the matrix M is positive definite. Both conditions (**A1**) and (**A2**) are satisfied. We find solutions u_ε of (1) in Ω_ε. When $k = 1$, this result has been obtained by Rey [30]. When $m = 2$, it is just a result in [17].

4 Positive radial solutions of $\Delta u + u^{\frac{N+2}{N-2}} = 0$

We recall some well known facts about positive radial solutions of

$$\Delta u + u^{\frac{N+2}{N-2}} = 0, \tag{3}$$

It is standard to look for radial positive solutions of (3) of the form

$$u(x) = |x|^{-\frac{N-2}{2}} v(-\log |x|). \tag{4}$$

If we set $t = -\log |x|$, then v is a solution of an autonomous second order nonlinear ordinary differential equation (see [8]) :

$$\partial_t^2 v - \left(\frac{N-2}{2}\right)^2 v + v^{\frac{N+2}{N-2}} = 0, \tag{5}$$

We introduce the function

$$H(x,y) := \frac{1}{2}\, y^2 - \frac{(N-2)^2}{8}\, x^2 + \frac{(N-2)}{2N} x^{\frac{2N}{N-2}}. \tag{6}$$

If v is a solution of (5), then

$$\partial_t H\left(v, \partial_t v\right) = 0.$$

In particular, this implies that $H\left(v, \partial_t v\right) \equiv c$ along the solution. When $c = 0$, there exists an unique solution up to translation in t of (5), which is defined on $\mathbb{R}$ and explicitly given by

$$w_0(t) := \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}} (\cosh t)^{\frac{2-N}{2}}\,. \tag{7}$$

The related solution of (3) is regular and positive on $\mathbb{R}^N$. When $c < 0$, the unique solution up to translation in t of (5) is well defined on $\mathbb{R}$, but the related one of (3) has a singularity point 0 on $\mathbb{R}^N$. When $c > 0$, the corresponding positive solution of (3) vanishes on the boundary of some annular domain. In this section, we are interested in case $c > 0$ and in particular the symptotic behavior of such solution when $c \to 0$.

Given $\eta > 0$, let v_η be the unique solution of

$$\left\{\begin{array}{rcl} \frac{d^2}{dt^2} v - \left(\frac{N-2}{2}\right)^2 v + v^{\frac{N+2}{N-2}} & = & 0 \\ v(0) & = & 0 \\ \frac{d}{dt} v(0) & = & \eta \end{array}\right. \tag{8}$$

As the hamiltonian quantity H is conserved, such solution changes the sign. Suppose $T_\eta > 0$ such that $v_\eta > 0$ on $(0, T_\eta)$ and $v_\eta(T_\eta) = 0$, that is, T_η is the first zero of v_η on the half line $\mathbb{R}^+$. Clearly, one has $\frac{d}{dt} v_\eta(T_\eta) = -\eta$ since H is constant. We first give some technic results about T_η.

Proposition 1 *Under the above assumptions, we have*
1) T_η is decreasing in $\eta \in (0,+\infty)$.
2) when $\eta \to 0^+$

$$T_\eta = \frac{4}{N-2}\log\frac{1}{\eta} + c_N + O(\eta^{\frac{N-2}{N}} + \eta^{\frac{4}{N}}\log\frac{1}{\eta}), \tag{9}$$

where

$$c_N = \frac{4}{N-2}\log(N-2) + \log N(N-2) \tag{10}$$

Proof. As H is conserved, the orbit $\{(v,\partial_t v)\}$ is symmetric with respect to the axis Oy so that we have $\frac{d}{dt}v_\eta(\frac{T_\eta}{2}) = 0$. Moreover, $\alpha_\eta := v_\eta(\frac{T_\eta}{2})$ satisfies

$$-\frac{(N-2)^2}{4}(\alpha_\eta)^2 + \frac{(N-2)}{N}(\alpha_\eta)^{\frac{2N}{N-2}} = \eta^2 \tag{11}$$

Clearly, $\alpha_\eta = \max_{[0,T_\eta]} v_\eta$ and $\alpha_\eta \to \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}}$ as $\eta \to 0^+$. Moreover, it follows from (11) that

$$\alpha_\eta = \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}} + O(\eta^2) \text{ as } \eta \to 0^+. \tag{12}$$

We claim α_η increases in $\eta \in (0,+\infty)$.
To see this, observe that

$$-\frac{(N-2)^2}{4}(\alpha_\eta)^2 + \frac{(N-2)}{N}(\alpha_\eta)^{\frac{2N}{N-2}} > 0.$$

Therefore,

$$\alpha_\eta \geq \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}}$$

Now we consider the function $f(x) = -\frac{(N-2)^2}{4}x^2 + \frac{(N-2)}{N}x^{\frac{2N}{N-2}}$. It is clear $f'(x) = -\frac{(N-2)^2}{2}x + 2x^{\frac{N+2}{N-2}} \geq 0$ provided $x \geq \left(\frac{(N-2)^2}{4}\right)^{\frac{N-2}{4}}$, that is, f is increasing on the interval $(\left(\frac{(N-2)^2}{4}\right)^{\frac{N-2}{4}}, +\infty)$. Thus, the desired claim yields.
We prove by contradiction the first part of the result. Suppose there exists some $0 < \eta_1 < \eta_2$ such that $T_{\eta_1} \leq T_{\eta_2}$. Set $w_\eta(t) := v_\eta(t + \frac{T_\eta}{2})$. Clearly, $w_{\eta_1}(0) = \alpha_{\eta_1} < \alpha_{\eta_2} = w_{\eta_2}(0)$. We distinguish two cases:

Case 1. $\forall t \in (0, T_{\eta_1}/2)$, $w_{\eta_1}(t) < w_{\eta_2}(t)$.

We consider the first eigenvalue problem

$$\lambda_\eta := \min_{\beta \in H_0^1((-T_\eta/2, T_\eta/2))} E_\eta(\beta)$$

where

$$E_\eta(\beta) := \frac{\displaystyle\int_{-T_\eta/2}^{T_\eta/2} (\beta')^2 - w_\eta^{\frac{4}{N-2}}(t)\beta^2}{\displaystyle\int_{-T_\eta/2}^{T_\eta/2} \beta^2}$$

and $H_0^1((-T_\eta/2, T_\eta/2)) := \{\beta \in L^2 | \quad \beta' \in L^2, \beta(-T_\eta/2) = \beta(T_\eta/2) = 0\}$ is the classic Sobolev space. As w_η is a positive eigenfunction associated to the eigenvalue $-\frac{(N-2)^2}{4}$, we obtain

$$\lambda_\eta = -\frac{(N-2)^2}{4}. \tag{13}$$

Set the extension of w_{η_1} by

$$\bar{w}_{\eta_1}(t) := \begin{cases} w_{\eta_1}(t) & \text{if } |t| \le T_{\eta_1}/2 \\ 0 & \text{if } |t| \ge T_{\eta_1}/2 \end{cases}$$

The simple calculation leads to

$$E_{\eta_2}(\bar{w}_{\eta_1}) < E_{\eta_1}(w_{\eta_1}) = -\frac{(N-2)^2}{4}$$

which contradicts (13).

Case 2. $\exists t_0 \in (0, T_{\eta_1}/2)$ such that $w_{\eta_1}(t_0) = w_{\eta_2}(t_0) = \bar{x}$.

We write for $i = 1, 2$

$$T_{\eta_i}/2 - t_0 = \int_{t_0}^{T_{\eta_i}/2} dt = \int_0^{\bar{x}} \frac{dx}{|\partial_t x|} = \int_0^{\bar{x}} \frac{dx}{\sqrt{\eta_i^2 + \frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \tag{14}$$

We state $\forall x \in (0, \bar{x})$

$$\frac{1}{\sqrt{\eta_2^2 + \frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} < \frac{1}{\sqrt{\eta_1^2 + \frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}}$$

which implies

$$T_{\eta_2}/2 - t_0 < T_{\eta_1}/2 - t_0,$$

that is, $T_{\eta_2} < T_{\eta_1}$. This gives the desired contradiction. Therefore, we prove the first part of Proposition 1.

To handle the second part, we write

$$\frac{T_\eta}{2} = \int_0^{\alpha_\eta} \frac{dx}{\sqrt{\eta^2 + \frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \tag{15}$$

Now we divide the interval $(0, \alpha_\eta)$ into three parts provided $\eta < 1$ and $\frac{2}{N-2}\eta^{(N-2)/N} < \alpha_\eta - \eta^{2(N-2)/N}$

$$\begin{aligned} & (0, \alpha_\eta) \\ := \; & (0, \tfrac{2}{N-2}\eta^{(N-2)/N}) \cup [\tfrac{2}{N-2}\eta^{(N-2)/N}, \alpha_\eta - \eta^{2(N-2)/N}) \cup [\alpha_\eta - \eta^{2(N-2)/N}, \alpha_\eta) \\ = \; & I_1 \cup I_2 \cup I_3 \end{aligned}$$

We estimate successively these integrals.
Integral on I_1

$$
\begin{aligned}
&\int_{I_1} \frac{dx}{\sqrt{\eta^2 + \frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \\
= \;& (1 + O(\eta^{\frac{4}{N}})) \int_{I_1} \frac{dx}{\sqrt{\eta^2 + \frac{(N-2)^2}{4}x^2}} \\
= \;& (1 + O(\eta^{\frac{4}{N}})) \frac{2}{N-2} \mathrm{Arcsh}(\eta^{-\frac{2}{N}}) \\
= \;& (1 + O(\eta^{\frac{4}{N}})) \frac{2}{N-2} \ln(\eta^{-\frac{2}{N}} + \sqrt{1 + \eta^{-\frac{4}{N}}}) \\
= \;& \frac{4}{N(N-2)} \ln\frac{1}{\eta} + \frac{2}{N-2} \ln 2 + O(\eta^{\frac{4}{N}} \ln\frac{1}{\eta} + \eta^{\frac{4}{N}}) \\
= \;& \frac{4}{N(N-2)} \ln\frac{1}{\eta} + \frac{2}{N-2} \ln 2 + O(\eta^{\frac{4}{N}} \ln\frac{1}{\eta})
\end{aligned}
\tag{16}
$$

Integral on I_3
Gathering with (11) and (12), there holds for any $x \in I_3$

$$
\begin{aligned}
&\eta^2 + \frac{(N-2)^2}{4} x^2 - \frac{(N-2)}{N} x^{\frac{2N}{N-2}} \\
= \;& \frac{(N-2)^2}{4} x^2 - \frac{(N-2)}{N} x^{\frac{2N}{N-2}} - \frac{(N-2)^2}{4} (\alpha_\eta)^2 + \frac{(N-2)}{N} (\alpha_\eta)^{\frac{2N}{N-2}} \\
= \;& \left((N-2) \left(\frac{N(N-2)}{4} \right)^{\frac{N-2}{4}} + o(1) \right) (\alpha_\eta - x)
\end{aligned}
\tag{17}
$$

Therefore, we can estimate

$$
\begin{aligned}
&\int_{I_3} \frac{dx}{\sqrt{\eta^2 + \frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \\
= \;& O(\eta^{\frac{N-2}{N}})
\end{aligned}
\tag{18}
$$

Similarly, we have

$$
\begin{aligned}
&\int_{\alpha_\eta - \eta^{2(N-2)/N}}^{\left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}}} \frac{dx}{\sqrt{\frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \\
= \;& O(\eta^{\frac{N-2}{N}})
\end{aligned}
\tag{19}
$$

since from (12), it follows $\alpha_\eta - \eta^{2(N-2)/N} < \left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}}$ provided η is sufficiently small.
Integral on I_2
For any $x \in I_2$, we estimate

$$
\eta^2 + \frac{(N-2)^2}{4} x^2 - \frac{(N-2)}{N} x^{\frac{2N}{N-2}} = (1 + O(\eta^{\frac{4}{N}}))(\frac{(N-2)^2}{4} x^2 - \frac{(N-2)}{N} x^{\frac{2N}{N-2}}) \tag{20}
$$

Together with (19), we infer

$$\begin{aligned}
&\int_{I_2} \frac{dx}{\sqrt{\eta^2 + \frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \\
&= \quad (1 + O(\eta^{\frac{4}{N}})) \int_{I_2} \frac{dx}{\sqrt{\frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \\
&= \quad (1 + O(\eta^{\frac{4}{N}})) \int_{\frac{2}{N-2}\eta^{(N-2)/N}}^{\left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}}} \frac{dx}{\sqrt{\frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} + O(\eta^{\frac{N-2}{N}})
\end{aligned} \tag{21}$$

On the other hand, we recall when $c = 0$ the exact solution of (5) is $w_0(t)$ given by (7) so that

$$\begin{aligned}
&\int_{\frac{2}{N-2}\eta^{(N-2)/N}}^{\left(\frac{N(N-2)}{4}\right)^{\frac{N-2}{4}}} \frac{dx}{\sqrt{\frac{(N-2)^2}{4}x^2 - \frac{(N-2)}{N}x^{\frac{2N}{N-2}}}} \\
&= \quad \frac{1}{2}\ln N(N-2) + \frac{2}{N-2}\log\frac{N-2}{2} + \frac{2}{N}\ln\frac{1}{\eta} + O(\eta^{\frac{4}{N}})
\end{aligned} \tag{22}$$

Gathering (16), (18)-(19) and (21)-(22), the desired result (9) yields.

■

In the next result, we give the asymptotic expansion of the solutions to (8).

Proposition 2 *For all $k \in \mathbb{N}$, there exists a positive constant $c_k > 0$ such that for all $\eta < 1$ and for all $t \in (-\frac{T_\eta}{2}, \frac{T_\eta}{2})$*

$$|\partial_t^k (v_\eta(t) - \eta \sinh(\frac{N-2}{2}t))| \le c_k\, \eta^{\frac{N+2}{N-2}}\, e^{\frac{N+2}{2}t} \tag{23}$$

where $\sinh(t) = (e^t - e^{-t})/2$.

Proof. We drop the indices η to keep the notations simple and we consider the case $t \ge 0$ as v is a odd function. We view v as a solution of a non homogeneous linear second order ordinary differential equation in $(0, \frac{T_\eta}{2})$

$$\partial_t^2 v - \left(\frac{N-2}{2}\right)^2 v = -v^{\frac{N+2}{N-2}}$$

By variation of the parameters formula,

$$v(t) = \eta\, \sinh(\frac{N-2}{2}t) - e^{\frac{N-2}{2}t} \int_0^t e^{(2-N)s} \int_0^s e^{\frac{N-2}{2}\zeta}\, v(\zeta)^{\frac{N+2}{N-2}}\, d\zeta\, ds, \tag{24}$$

This in particular implies that $v(t) \le \eta \sinh(\frac{N-2}{2}t)$ for all $t \in [0, \frac{T_\eta}{2})$.

We can therefore use the bounds

$$v(t) \le \eta\, \sinh(\frac{N-2}{2}t) \le \eta\, e^{\frac{N-2}{2}t} \tag{25}$$

in (23) to conclude that

$$|v(t) - \eta\, \sinh(\frac{N-2}{2}t)| \leq e^{\frac{N-2}{2}t} \int_0^t e^{(2-N)s} \int_0^s e^{\frac{N-2}{2}\zeta}\, v(\zeta)^{\frac{N+2}{N-2}}\, d\zeta\, ds \leq c\, \eta^{\frac{N+2}{N-2}}\, e^{\frac{N+2}{2}t}. \quad (26)$$

This completes the proof of the estimate of v. The estimates for the derivatives follow similarly. ■

5 The linear analysis.

In this section we analyze the linearized operator around the radial solutions of (1). In doing so our aim is to derive precise estimates for these solutions which will be needed in the forthcoming construction. We begin with the definition of weighted spaces in cylindrical coordinates.

Definition 1 *Given $\delta \in \mathbb{R}$ and $-\infty \leq t_1 < t_2 \leq +\infty$, the space $\mathcal{C}^0_\delta((t_1,t_2) \times S^{N-1})$ is defined to be the set of continuous functions $w \in \mathcal{C}^0_{loc}((t_1,t_2) \times S^{N-1})$ for which the following norm is finite :*

$$\|w\|_{\mathcal{C}^0_\delta((t_1,t_2)\times S^{N-1})} := \|e^{-\delta s}\, w\|_{L^\infty((t_1,t_2)\times S^{N-1})}. \quad (27)$$

Now assume that Ω is a regular bounded open subset of $\mathbb{R}^N$ and $\Sigma := \{a_1, \dots, a_m\}$ is a finite set of points of Ω. We choose $r_0 > 0$ in such a way that the closed balls $B(a_i, 2r_0)$, for $i = 1, \dots, m$ are disjoint and included in Ω. For all $r \in (0, r_0)$, we define

$$\Omega_{int,r} := \bigcup_{i=1}^m B(a_i, r) \qquad \text{and} \qquad \Omega_{ext,r} := \Omega - \overline{\Omega}_{int,r}.$$

We define the weighted spaces :

Definition 2 *Given $\nu \in \mathbb{R}$, the space $\mathcal{C}^0_\nu(\overline{\Omega} - \Sigma)$ is defined to be the set of continuous functions $w \in C^0_{loc}(\overline{\Omega} - \Sigma)$ for which the following norm is finite :*

$$\|w\|_{C^0_\nu(\overline{\Omega}-\Sigma)} := \|w\|_{L^\infty(\overline{\Omega_{ext,r_0}})} + \sum_{j=1}^m \|r^{-\nu}\, w(a_j + \cdot)\|_{L^\infty(B(0,2r_0)-\{0\})}. \quad (28)$$

Given a subset $\Omega_1 \subset \Omega - \Sigma$ we define the space $C^0_\nu(\Omega_1)$ to be the space of restrictions of functions of $C^0_\nu(\overline{\Omega} - \Sigma)$ to $\overline{\Omega}_1$. This space is endowed with the induced norm.

Recall given $\eta > 0$, the solution v_η of (8) can be extended to a sign change regular periodic function on $\mathbb{R}$ with the periode equal to $2T_\eta$, which solves the following ODE

$$\left\{ \begin{array}{rcl} \frac{d^2}{dt^2}v - \left(\frac{N-2}{2}\right)^2 v + |v|^{\frac{4}{N-2}}v &=& 0 \\ v(0) &=& 0 \\ \frac{d}{dt}v(0) &=& \eta \end{array} \right. \quad (29)$$

Given a bounded real number $A \in \mathbb{R}$, set $v_{\eta,A}(\cdot) := v_\eta(\cdot + A)$. Thus, the related the radial function

$$u_{\eta,A}(x) := |x|^{-\frac{N-2}{2}}\, v_{\eta,A}(-\log|x|) \quad (30)$$

solves the equation

$$\Delta u + |u|^{\frac{4}{N-2}} u = 0 \text{ in } \mathbb{R}^N - \{0\} \tag{31}$$

In this section, we study the linearization of the above nonlinear equation about the radial function $u_{\eta,A}$. This operator is defined by

$$L_{\eta,A} := \Delta + \frac{N+2}{N-2} |u_{\eta,A}|^{\frac{4}{N-2}}.$$

Given a positive interger $k \in \mathbb{N}$, we define $r_{\varepsilon,1} := \varepsilon^{\frac{1}{k(N+2)}}$, $r_{\varepsilon,2} := \varepsilon^{1-\frac{1}{k(N+2)}}$, $B_{\varepsilon,1} := -\log r_{\varepsilon,1}$ and $B_{\varepsilon,2} := -\log r_{\varepsilon,2}$. We write $\eta := D\varepsilon^{\frac{N-2}{4k}}$ with $D \in \mathbb{R}$ a positive number. Given a sufficiently large number $c > 0$, we assume

$$|A| < c \text{ and } \frac{1}{c} < D < c$$

For the sake of simplicity in the notations, we drop the η, A, ε, k indices. We can write any function ψ defined in the annular domain $B(0, r_{\varepsilon,1}) - B(0, r_{\varepsilon,2})$ as

$$\psi(x) = |x|^{-\frac{N-2}{2}} w(-\log |x|, \theta),$$

so that the study of L reduces to the study of the linear operator

$$\mathcal{L} := \partial_t^2 - \left(\frac{N-2}{2}\right)^2 + \Delta_{S^{N-1}} + \frac{N+2}{N-2} |v|^{\frac{4}{N-2}} \tag{32}$$

on the cylinder $[B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1}$, where $\Delta_{S^{N-1}}$ denotes the Laplace-Beltrami operator on the sphere S^{N-1}.

We denote by (e_j, λ_j) the set of eigendata of $\Delta_{S^{N-1}}$

$$\Delta_{S^{N-1}} e_j = -\lambda_j \, e_j.$$

We also assume that the eigenvalues are counted with multiplicity, that $\lambda_j \leq \lambda_{j+1}$ and that the e_j are normalized by

$$\int_{S^{N-1}} e_j^2 \, d\omega = 1.$$

We now prove some uniform estimates for a right inverse for the operator $\mathcal{L}$.

Proposition 3 *Assume that $\delta \in (-\frac{N+2}{2}, -\frac{N}{2})$ is fixed. Then, there exists $\varepsilon(k,c) \in (0, +\infty)$ such that, if $\varepsilon \in (0, \varepsilon(k,c))$, then, for all $f \in \mathcal{C}^0_\delta([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1})$, there exists a unique solution $w \in \mathcal{C}^0_\delta([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1})$ of*

$$\mathcal{L}\, w = f \tag{33}$$

in $[B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1}$ which satisfies

$$\begin{cases} w(B_{\varepsilon,2}, \theta) = 0 \\ w(B_{\varepsilon,1}, \theta) \in Span\{e_j \,:\, j = 0, \ldots, N\}. \end{cases} \tag{34}$$

Furthermore,

$$\|w\|_{\mathcal{C}^0_\delta} \leq c \, \|f\|_{\mathcal{C}^0_\delta} \tag{35}$$

for some constant c which does not depend on ε, A, D.

Proof. The proof is divided in three parts. In the first part we explain how to solve the equation (33) when the function f does not have any component on e_j for $j = 0, \dots, N$ in its eigenfunction decomposition. Next, in the second part, we obtain a uniform estimate for the solution already obtained. Finally, in the last part, we explain how to solve (33) when the eigenfunction decomposition of f has components on $e_0, \dots, e_N$.

Step 1 For the time being, we assume that the eigenfunction decomposition of the function f is given by

$$f(t, \theta) = \sum_{j \geq N+1} f^j(t)\, e_j(\theta). \tag{36}$$

Observe that, as η tends to 0 we have

$$\lim_{\eta \to 0} \sup \frac{N+2}{N-2} |v_{\eta,A}|^{\frac{4}{N-2}} = \frac{N(N+2)}{4}.$$

Now the eigenfunction decomposition of the Laplace-Betrami operator on S^{N-1} induces a decomposition of the operator $\mathcal{L}$ into the sequence of operators

$$\mathcal{L}_j := \partial_t^2 - \left(\frac{N-2}{2}\right)^2 + \frac{N+2}{N-2} |v|^{\frac{4}{N-2}} - \lambda_j$$

Using these above limits together with the fact that $\lambda_j \geq 2N$ for $j \geq N+1$, we conclude that, for $j \geq N+1$ the potential is negative provided ε is close enough to 0. In particular, this implies that it is possible to solve

$$\mathcal{L}\, w = f$$

on any $[B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1}$, with $w = 0$ as boundary data (recall that the operator $\mathcal{L}$ is self adjoint. When restricted to the set of functions spanned by e_j, for $j \geq N+1$, we can use its variational structure to solve it.)

It remains to prove that there exists a constant $c > 0$ which does not depend on ε, A and D such that

$$\sup |e^{-\delta t}\, w| \leq c \sup |e^{-\delta t}\, f|. \tag{37}$$

Step 2 The proof of (37) is by contradiction. If it were false for all choice of ε, A and D without loss of generality, there would exist a sequence ε_n tending to 0, a sequence of reals A_n tending to A_∞, a sequence of reals D_n tending to D_∞, a sequence of functions (f_n) and a sequence $(w_n)_n$ of solutions of (33) and (34) such that

$$\|f_n\|_{\mathcal{C}^0_\delta} \equiv 1 \qquad \text{and} \qquad \lim_{n \to +\infty} A_n := \sup e^{-\delta t} |w_n| = +\infty. \tag{38}$$

We denote $B_{n,1} = B_{\varepsilon_n,1}$, $B_{n,2} = B_{\varepsilon_n,2}$ and $v_n = v_{\eta_n, A_n}$. Obviously, there exists a point $(t_n, \theta_n) \in (B_{n,1}, B_{n,2}) \times S^{N-1}$ where the above supremum is achieved, namely $A_n = e^{-\delta t_n}\, |w_n(t_n, \theta_n)|$. Observe that elliptic estimates imply that

$$\sup e^{-\delta t} |\nabla w_n| \leq c\,(1 + A_n) \tag{39}$$

and this in turn implies that the sequences $(t_n - B_{n,1})_n$ and $(B_{n,2} - t_n)_n$ remain bounded away from 0.

We define $\tilde{t}_n > B_{n,1}$ to be the nearest local maximal point of the function $|v_n(t)|$ to the point t_n. We distinguish several cases according to the behavior of the sequence $(t_n)_n$.

Case 1. Assume that the sequence $(t_n - \tilde{t}_n)_n$ is bounded. In this case, we define the function $\tilde{w}_n$ by

$$\tilde{w}_n(t,\theta) = \frac{1}{A_n}\, e^{-\delta \tilde{t}_n}\, w_n(t + \tilde{t}_n, \theta).$$

Observe that the sequence of functions $(|v_n(\cdot + \tilde{t}_n)|)_n$ converges on compact to $t \to (N(N-2))^{\frac{N-2}{4}}(\cosh t)^{\frac{2-N}{2}}$. Up to a subsequence, we can assume that the sequence $(t_n - \tilde{t}_n)_n$ converges to t_∞. Moreover, we can assume that the sequence $(\tilde{w}_n)_n$ converges on compacts to $\tilde{w}_\infty$ a nontrivial solution of

$$\partial_t^2 \tilde{w}_\infty + \Delta_{S^{N-1}}\, \tilde{w}_\infty - \frac{(N-2)^2}{4}\, \tilde{w}_\infty + \frac{N(N+2)}{4}\, (\cosh t)^{-2}\, \tilde{w}_\infty = 0. \tag{40}$$

Moreover, $\tilde{w}_\infty$ is bounded by a constant times $e^{\delta t}$. The fact that $\tilde{w}_\infty$ is not identically equal to 0 follows from the fact that $|\tilde{w}_n(t_n - \tilde{t}_n, \theta_n)| = e^{\delta(t_n - \tilde{t}_n)}$ and hence remains bounded away from 0.

We consider the eigenfunction decomposition of $\tilde{w}_\infty$

$$\tilde{w}_\infty = \sum_{j=N+1}^{\infty} a_j\, e_j.$$

At $-\infty$ the function a_j is either blowing up like $t \longrightarrow e^{-\gamma_j t}$ or decaying like $t \longrightarrow e^{\gamma_j t}$, where

$$\gamma_j := \sqrt{\lambda_j + \frac{(N-2)^2}{4}}.$$

The choice of $\delta \in (-\frac{N+2}{2}, -\frac{N}{2})$ implies that $-\delta < \gamma_j$ for all $j \geq N+1$. Hence a_j decays exponentially at $-\infty$. Multiplying the equation (40) by $a^j\, e_j$ and integrating by parts over $\mathbb{R}$ (all integrations are justified because a_j decays exponentially at both $\pm\infty$), we get

$$\begin{aligned}\int_{-\infty}^{+\infty} |\partial_t a^j|^2 + (\lambda_j + \frac{(N-2)^2}{4})(a^j)^2 \quad &= \quad \frac{N(N+2)}{4} \int_{-\infty}^{+\infty} (\cosh s)^{-2}\, (a^j)^2 \\ &\leq \quad \frac{N(N+2)}{4} \int_{-\infty}^{+\infty} (a^j)^2.\end{aligned}$$

Since $j \geq N+1$, we have $\lambda_j \geq 2N$, and hence we conclude that $a_j \equiv 0$. Hence, $\tilde{w}_\infty \equiv 0$, a contradiction.

Case 2. Assume that the sequence $(t_n - \tilde{t}_n)$, the sequence $(t_n - B_{n,1})_n$ and the sequence $(B_{n,2} - t_n)_n$ are all unbounded. In this case, we define the function $\tilde{w}_n$ by

$$\tilde{w}_n(t,\theta) = \frac{1}{A_n}\, e^{-\delta t_n}\, w_n(t + t_n, \theta).$$

Observe that this time the sequence of functions $(v_n(\cdot + t_n))_n$ converge to 0 on compacts. Up to a subsequence, we can assume that the sequence $(\tilde{w}_n)_n$ converges on compacts to $\tilde{w}_\infty$ a nontrivial solution of

$$\partial_t^2 \tilde{w}_\infty + \Delta_{S^{N-1}}\, \tilde{w}_\infty - \frac{(N-2)^2}{4}\, \tilde{w}_\infty = 0.$$

Moreover, $\tilde{w}_\infty$ is bounded by a constant times $e^{\delta t}$.

Again, we consider the eigenfunction decomposition of $\tilde{w}_\infty$

$$\tilde{w}_\infty = \sum_{j=N+1}^{\infty} a_j \, e_j$$

and we see that a_j is a linear combination of $t \longrightarrow e^{-\gamma_j t}$ and $t \longrightarrow e^{\gamma_j t}$. The choice of $\delta \in (-\frac{N+2}{2}, -\frac{N}{2})$ implies that $\delta > -\gamma_j$ for all $j \geq N+1$. Hence a_j cannot be bounded by $e^{\delta t}$ unless it is identically 0. We conclude that $a_j \equiv 0$. Hence, $\tilde{w}_\infty \equiv 0$, a contradiction.

Case 3. Assume that the sequence $(t_n - B_{n,1})_n$ is bounded (resp. that the sequence $(B_{n,2} - t_n)_n$ is bounded) and that the sequence $(t_n - \tilde{t}_n)$ is unbounded. This case can be treated as in case 2. The only difference is that this time $\tilde{w}_\infty$ is defined on $[\underline{t}_\infty, +\infty) \times S^{N-1}$ (resp. on $(-\infty, \bar{t}_\infty] \times S^{N-1}$) and is equal to 0 on $\{\underline{t}_\infty\} \times S^{N-1}$ (resp. on $\{\bar{t}_\infty\} \times S^{N-1}$). We omit the details.

Since we have reached a contradiction in each case, the proof of the claim is complete the proof of the result in the case where the eigenfunction decomposition of f does not involve any e_j for $j = 0, \ldots, N$.

Step 3. Now we consider the case where the function f is collinear to e_j, namely

$$f(t, \theta) = f^j(t) \, e_j(\theta)$$

for some $0 \leq j \leq N$. We consider the equation

$$\begin{cases} \partial_t^2 a^j - (\lambda_j + \left(\frac{N-2}{2}\right)^2) \, a^j + \frac{N+2}{N-2} \, |v|^{\frac{4}{N-2}} \, a^j = f^j & \text{in } [B_{\varepsilon,1}, B_{\varepsilon,2}] \\ a^j(B_{\varepsilon,2}) = \partial_t a^j(B_{\varepsilon,2}) = 0 \end{cases} \tag{41}$$

Observe that

$$|f^j(t)| \leq \|f\|_{\mathcal{C}^0_\delta} \, e^{\delta t}.$$

We consider the bounded neighborhood around $B_{\varepsilon,2}$. For ε sufficiently small (or equivalently, η sufficiently small), δ is not an indicial root of the operator $\mathcal{L}$ and it follows from Cauchy's theorem that there exists a unique solution of (41) such that for any given intervalle $[E, 0]$, the function $a^j(\cdot + B_{\varepsilon,2})$ is uniformly bounded on $[E, 0]$ with respect to ε, A and D.

We claim that there exists a constant $c > 0$ such that

$$\sup_{[B_{\varepsilon,1}, B_{\varepsilon,2}]} e^{-\delta t} \, |a^j| \leq c \sup_{[B_{\varepsilon,1}, B_{\varepsilon,2}]} e^{-\delta t} \, |f^j|$$

provided ε is close enough 0. As before, we argue by contradiction. Assume that the claim is not true. Then there would exist a sequence $(\varepsilon_n)_n$ tending to 0, a sequence of reals A_n tending to A_∞, a sequence of reals D_n tending to D_∞, a sequence of functions $(f^j_n)_n$ and a sequence of solutions $(a^j_n)_n$ of (41) such that

$$\sup_{[B_{n,1}, B_{n,2}]} e^{-\delta t} \, |f^j_n| = 1 \qquad \text{and} \qquad A_n := \sup_{[B_{n,1}, B_{n,2}]} e^{-\delta t} \, |a^j_n|$$

tends to $+\infty$. We can define t_n such that $A_n = e^{-\delta t_n}\, |a_n^j(t_n, \theta_n)|$.

As in Step 2, we define $\tilde{t}_n > B_{n,1}$ to be the nearest local maximal point of the function $|v_n(t)|$ to the point t_n. We distinguish several cases according to the behavior of the sequence $(t_n)_n$. When the sequence $(t_n - \tilde{t}_n)$ is bounded, we define the function $\tilde{a}_n^j$ by

$$\tilde{a}_n^j(t) = \frac{1}{A_n}\, e^{-\delta \tilde{t}_n}\, a_n^j(t + \tilde{t}_n).$$

We can assume that, up to a subsequence, the sequence $(\tilde{a}_n^j)_n$ converges on compacts to $\tilde{a}_\infty$ a nontrivial solution of

$$\partial_t^2 \tilde{a}_\infty - \lambda_j\, \tilde{a}_\infty - \frac{(N-2)^2}{4}\, \tilde{a}_\infty + \frac{N(N+2)}{4}\, (\cosh t)^{-2}\, \tilde{a}_\infty = 0.$$

When the sequence $(t_n - \tilde{t}_n)$ is unbounded, we define the function $\tilde{a}_n^j$ by

$$\tilde{a}_n^j(t) = \frac{1}{A_n}\, e^{-\delta t_n}\, a_n^j(t + t_n).$$

Again, up to a subsequence, the sequence $(\tilde{a}_n^j)_n$ converges on compacts to $\tilde{a}_\infty$ a nontrivial solution of

$$\partial_t^2 \tilde{a}_\infty - \lambda_j\, \tilde{a}_\infty - \frac{(N-2)^2}{4}\, \tilde{a}_\infty = 0.$$

Moreover, up to a subsequence $B_{n,2} - t_n \to +\infty$ because of the remark at the beginning of the step. Thus, the above solution is defined on $\mathbb{R}$ or on $(t_\infty, +\infty)$. Furthermore, $\tilde{a}_\infty$ is bounded by a constant times $e^{\delta t}$. However, the choice of $\delta \in (-\frac{N+2}{2}, -\frac{N}{2})$ implies that $\delta < -\gamma_j$ for all $j = 0, \ldots, N$ and there are non nontrivial solutions of the above homogeneous problems which are bounded by $e^{\delta t}$ at $+\infty$. Hence, $\tilde{a}_\infty \equiv 0$, a contradiction. This completes the proof of the result. ■

With the same arguments, we have also the following result.

Proposition 4 *Assume that $\delta \in (\frac{N}{2}, \frac{N+2}{2})$ is fixed. Then, there exists $\varepsilon'(k, c) \in (0, +\infty)$ such that, if $\varepsilon \in (0, \varepsilon'(k, c))$, then, for all $f \in \mathcal{C}^0_\delta([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1})$, there exists a unique solution $w \in \mathcal{C}^0_\delta([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1})$ of (33) satisfying*

$$\left\{ \begin{array}{l} w(B_{\varepsilon,1}, \theta) = 0 \\ w(B_{\varepsilon,2}, \theta) \in Span\{e_j \, : \, j = 0, \ldots, N\}. \end{array} \right. \tag{42}$$

Furthermore, estimate (35) holds true.

Definition 3 *Given $\delta_1, \delta_2, \delta_3 \in \mathbb{R}$ and $-\infty \le t_1 < t_2 \le +\infty$, the space $\mathcal{C}^0_{\delta_1}((t_1, t_2) \times S^{N-1}) + \mathcal{C}^0_{\delta_2}((t_1, t_2) \times S^{N-1})$ is define to be the set of continuous functions $w = w_1 + w_2$ where $w_1 \in \mathcal{C}^0_{\delta_1}((t_1, t_2) \times S^{N-1})$ and $w_2 \in \mathcal{C}^0_{\delta_2}((t_1, t_2) \times S^{N-1})$. On $\mathcal{C}^0_{\delta_1}((t_1, t_2) \times S^{N-1}) + \mathcal{C}^0_{\delta_2}((t_1, t_2) \times S^{N-1})$ we define the following norm :*

$$\|w\|_{\delta_1, \delta_2, \delta_3} := \inf_{\{(w_1, w_2), w_1 + w_2 = w\}} \|w_1\|_{\mathcal{C}^0_{\delta_1}((t_1, t_2) \times S^{N-1})} + \varepsilon^{\delta_3} \|w_2\|_{\mathcal{C}^0_{\delta_2}((t_1, t_2) \times S^{N-1})}. \tag{43}$$

Gathering the propositions 3 and 4, we have the following result.

Proposition 5 *Assume that* $\delta_1 \in (-\frac{N+2}{2}, -\frac{N}{2})$, $\delta_2 \in (\frac{N}{2}, \frac{N+2}{2})$ *and* $\delta_3 \in \mathbb{R}$ *are fixed. Then, there exists* $\varepsilon'(k,c) \in (0, +\infty)$ *such that, if* $\varepsilon \in (0, \varepsilon'(k,c))$*, then, for all* $f \in \mathcal{C}^0_{\delta_1}([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1}) + \mathcal{C}^0_{\delta_2}([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1})$ *, there exists a unique solution* $w \in \mathcal{C}^0_{\delta_1}([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1}) + \mathcal{C}^0_{\delta_2}([B_{\varepsilon,1}, B_{\varepsilon,2}] \times S^{N-1})$ *of (33) satisfying*

$$\begin{cases} w(B_{\varepsilon,1}, \theta) \in Span\{e_j \,:\, j = 0, \ldots, N\} \\ w(B_{\varepsilon,2}, \theta) \in Span\{e_j \,:\, j = 0, \ldots, N\}. \end{cases} \tag{44}$$

Furthermore, the following estimate holds

$$\|w\|_{\delta_1,\delta_2,\delta_3} \le c\, \|f\|_{\delta_1,\delta_2,\delta_3} \tag{45}$$

6 Bubble tree solutions in general domains

We recall

$$r_{\varepsilon,1} = \varepsilon^{\frac{1}{k(N+2)}},\ r_{\varepsilon,2} = \varepsilon^{1-\frac{1}{k(N+2)}},\ B_{\varepsilon,1} = -\log r_{\varepsilon,1},\ B_{\varepsilon,2} = -\log r_{\varepsilon,2}$$

We define the translations at the infinity. Given $b, a \in \mathbb{R}^N$, set

$$\begin{array}{rccc} T_{b,a}: & \mathbb{R}^N & \to & \mathbb{R}^N \\ & x & \mapsto & \dfrac{\frac{x-a}{|x-a|^2} - b}{\left|\frac{x-a}{|x-a|^2} - b\right|^2} + a \end{array}$$

These translations induce the conformal transformations on the space of the real functions on $\mathbb{R}^N$. More precisely, let $\psi : \mathbb{R}^N \to \mathbb{R}$ be a real function. We set

$$T_{b,a}(\psi)(x) := \left| \frac{x-a}{|x-a|} - b|x-a| \right|^{2-N} \psi(T_{b,a}(x))$$

Given $\mathbf{a}' = (a'_1, \cdots, a'_m) \in \Omega_1 \times \cdots \times \Omega_m$ and $\mathbf{b} = (b_1, \cdots, b_m) \in (\mathbb{R}^N)^m$ (depending on ε, but uniformly bounded as $\varepsilon \to 0$), we divide

$$\Omega_\varepsilon := \Omega_{int,\varepsilon} \cup \Omega_{ext,\varepsilon}$$

where

$$\begin{aligned} \Omega_{int,\varepsilon} &= \bigcup_{i=1}^m \Omega_{int,i,\varepsilon} \\ &:= \bigcup_{i=1}^m T_{-b_i, a_i+\varepsilon a'_i}(B(a_i + \varepsilon a'_i, r_{\varepsilon,1})) - T_{-b_i, a_i+\varepsilon a'_i}(B(a_i + \varepsilon a'_i, r_{\varepsilon,2})) \\ &= \bigcup_{i=1}^m \Bigg(B(a_i + \varepsilon a'_i - \frac{b_i r_{\varepsilon,1}^2}{1 - |b_i|^2 r_{\varepsilon,1}^2}, \frac{r_{\varepsilon,1}}{1 - |b_i|^2 r_{\varepsilon,1}^2}) \\ &\qquad - B(a_i + \varepsilon a'_i - \frac{b_i r_{\varepsilon,2}^2}{1 - |b_i|^2 r_{\varepsilon,2}^2}, \frac{r_{\varepsilon,2}}{1 - |b_i|^2 r_{\varepsilon,2}^2}) \Bigg) \end{aligned}$$

and

$$\begin{aligned}
\Omega_{ext,\varepsilon} &= \bigcup_{i=0}^{m} \Omega_{ext,i,\varepsilon} \\
&:= (\Omega - \bigcup_{i=1}^{m} T_{-b_i,a_i+\varepsilon a_i'}(B(a_i+\varepsilon a_i', r_{\varepsilon,1}))) \\
&\quad \cup \bigcup_{i=1}^{m} \Big(T_{-b_i,a_i+\varepsilon a_i'}(B(a_i+\varepsilon a_i', r_{\varepsilon,2})) - (a_i+\varepsilon\Omega_i)\Big) \\
&= (\Omega - \bigcup_{i=1}^{m} B(a_i+\varepsilon a_i' - \frac{b_i r_{\varepsilon,1}^2}{1-|b_i|^2 r_{\varepsilon,1}^2}, \frac{r_{\varepsilon,1}}{1-|b_i|^2 r_{\varepsilon,1}^2})) \\
&\quad \cup \bigcup_{i=1}^{m} \left(B(a_i+\varepsilon a_i' - \frac{b_i r_{\varepsilon,2}^2}{1-|b_i|^2 r_{\varepsilon,2}^2}, \frac{r_{\varepsilon,2}}{1-|b_i|^2 r_{\varepsilon,2}^2}) - (a_i+\varepsilon\Omega_i)\right)
\end{aligned}$$

Given $\alpha \in (0,1)$, we define two functional spaces

$$\begin{aligned}
\mathcal{E}_1 : &= \Big\{\varphi \in C^{2,\alpha}(S^{N-1}) \quad : \quad \int_{S^{N-1}} \varphi\, e_j\, d\omega = 0, \quad j = 0,\dots,N \\
&\qquad \text{and} \quad \|\varphi\|_{\mathcal{C}^{2,\alpha}} \le \varepsilon^{\frac{1}{k(N+2)} + \frac{N-2}{4k}} \Big\}
\end{aligned}$$

and

$$\begin{aligned}
\mathcal{E}_2 : &= \Big\{\phi \in C^{2,\alpha}(S^{N-1}) \quad : \quad \int_{S^{N-1}} \varphi\, e_j\, d\omega = 0, \quad j = 0,\dots,N \\
&\qquad \text{and} \quad \|\phi\|_{\mathcal{C}^{2,\alpha}} \le \varepsilon^{\frac{N-1}{k(N+2)} - \frac{N-2}{2} + \frac{N-2}{4k}} \Big\}.
\end{aligned}$$

In this section, we only give the details for the dimension $N \geq 6$. In the other cases, the analysis is similar.

6.1 Some basic properties about harmonic functions

We recall some well known result concerning harmonic extension of functions which are defined on S^{N-1} (see [27]).

Lemma 1 *Given $\varphi \in C^{2,\alpha}(S^{N-1})$, we define V_φ to be the unique harmonic extension of φ in $B(0,1)$, namely*

$$\begin{cases} \Delta V_\varphi = 0 & \text{in} \quad B(0,1) \\ V_\varphi = \varphi & \text{on} \quad \partial B(0,1) \end{cases} \tag{46}$$

Assume that φ is $L^2(S^{N-1})$ orthogonal to $e_0,\dots,e_N$, then

$$\|V_\varphi\|_{\mathcal{C}^0_2(B(0,1)-\{0\})} \le c\,\|\varphi\|_{\mathcal{C}^0(S^{N-1})}$$

for some constant $c > 0$ which does not depend on φ.

Using the fact that Kelvin's transform of an harmonic function V

$$W(x) = |x|^{2-N}\, V(\frac{x}{|x|})$$

is harmonic, the above result translates into:

Lemma 2 *Given $\varphi \in C^{2,\alpha}(S^{N-1})$, we define W_φ to be the unique harmonic extension of φ in $\mathbb{R}^N - B(0,1)$ which decays at ∞. Namely*

$$\left\{ \begin{array}{rclll} \Delta W_\phi & = & 0 & in & \mathbb{R}^N - B(0,1) \\ W_\phi & = & \phi & on & \partial B(0,1) \end{array} \right. \tag{47}$$

and W_φ tends to 0 at ∞. Assume that φ is $L^2(S^{N-1})$ orthogonal to $e_0, \cdots, e_N$ then

$$\|W_\phi\|_{\mathcal{C}^0_{-N}(\mathbb{R}^N - B(0,1))} \leq c \, \|\phi\|_{\mathcal{C}^0(S^{N-1})}$$

for some constant $c > 0$ which does not depend on ϕ.

6.2 Solution of the nonlinear problem in $\Omega_{int,\varepsilon}$.

Let $\delta_1 = \delta \in (-\frac{N+2}{2} + \frac{2}{N}, -\frac{N}{2})$ be fixed. We choose $\delta_2 = -\delta$ and $\delta_3 = \delta$. Given a m functions $\Psi_1 := (\varphi_{1,1}, \ldots, \varphi_{m,1}) \in (\mathcal{E}_1)^m$, a m functions $\Phi_2 := (\phi_{1,2}, \ldots, \phi_{m,2}) \in (\mathcal{E}_2)^m$, m points $\mathbf{a}' := (a'_1, \ldots, a'_m) \in \Omega_1 \times \cdots \Omega_m$ and m points $\mathbf{b} := (b_1, \ldots, b_m) \in (\mathbb{R}^N)^m$, we construct a solution of problem (1) in $\Omega_{int,\varepsilon}$ whose boundary is, in some sense, parameterized by $(T_{b_1, a_1+\varepsilon a'_1}(\varphi_{1,1}), \cdots, T_{b_m, a_m+\varepsilon a'_m}(\varphi_{m,1}))$ and $(T_{b_1, a_1+\varepsilon a'_1}(\phi_{1,2}), \cdots, T_{b_m, a_m+\varepsilon a'_m}(\phi_{m,2}))$. Namely we would like to solve

$$\left\{ \begin{array}{lll} \Delta u_{int,i} + |u_{int,i}|^{\frac{4}{N-2}} u_{int,i} = 0 & \text{in} & \Omega_{int,i,\varepsilon} \\ T_{-b_i, a_i+\varepsilon a'_i}(u_{int,i}) - \varphi_{i,1} \in \text{Span}\{e_0, \ldots, e_N\} & \text{on} & \partial B(a_i + \varepsilon a'_i, r_{\varepsilon,1}) \\ T_{-b_i, a_i+\varepsilon a'_i}(u_{int,i}) - \phi_{i,2} \in \text{Span}\{e_0, \ldots, e_N\} & \text{on} & \partial B(a_i + \varepsilon a'_i, r_{\varepsilon,2}) \end{array} \right. \tag{48}$$

Or equivalently, we will solve

$$\left\{ \begin{array}{lll} \Delta u_{int,i,b_i,a'_i} + |u_{int,i,b_i,a'_i}|^{\frac{4}{N-2}} u_{int,i,b_i,a'_i} = 0 & \text{in} & B(x_i, r_{\varepsilon,1}) - B(a_i + \varepsilon a'_i, r_{\varepsilon,2}) \\ u_{int,i,b_i,a'_i} - \varphi_{i,1} \in \text{Span}\{e_0, \ldots, e_N\} & \text{on} & \partial B(a_i + \varepsilon a'_i, r_{\varepsilon,1}) \\ u_{int,i,b_i,a'_i} - \phi_{i,2} \in \text{Span}\{e_0, \ldots, e_N\} & \text{on} & \partial B(a_i + \varepsilon a'_i, r_{\varepsilon,2}) \end{array} \right. \tag{49}$$

Here we denote by $u_{int,i,b_i,a'_i} := T_{-b_i, a_i+\varepsilon a'_i}(u_{int,i})$ (for the simplicity, we denote by u_i). For each $i = 1, \ldots, m$, we denote by $V_{\varphi_{i,1}}$ (resp. $W_{\phi_{i,2}}$) the unique harmonic extension of $\varphi_{i,1}$ (resp. $\phi_{i,2}$) in $B(a_i + \varepsilon a'_i, r_{\varepsilon,1})$ (resp. $\mathbb{R}^N - B(a_i + \varepsilon a'_i, r_{\varepsilon,2})$), namely

$$\left\{ \begin{array}{rclll} \Delta V_{\varphi_{i,1}} & = & 0 & \text{in} & B(a_i + \varepsilon a'_i, r_{\varepsilon,1}) \\ V_{\varphi_{i,1}} & = & \varphi_{i,1} & \text{on} & \partial B(a_i + \varepsilon a'_i, r_{\varepsilon,1}) \end{array} \right. \tag{50}$$

and

$$\left\{ \begin{array}{rclll} \Delta W_{\phi_{i,2}} & = & 0 & \text{in} & B(a_i + \varepsilon a'_i, r_{\varepsilon,2}) \\ W_{\phi_{i,2}} & = & \phi_{i,2} & \text{on} & \partial B(a_i + \varepsilon a'_i, r_{\varepsilon,2}) \end{array} \right. \tag{51}$$

Thus, $T_{b_i,a_i+\varepsilon a'_i}(V_{\varphi_{i,1}})$ (resp. $T_{b_i,a_i+\varepsilon a'_i}(W_{\phi_{i,2}})$) the unique harmonic extension of $T_{b_i,a_i+\varepsilon a'_i}(\varphi_{i,1})$ (resp. $T_{b_i,a_i+\varepsilon a'_i}(\phi_{i,2})$) in $T_{-b_i,a_i+\varepsilon a'_i}(B(a_i+\varepsilon a'_i, r_{\varepsilon,1}))$ (resp. $\mathbb{R}^N - T_{-b_i,a_i+\varepsilon a'_i}(B(a_i+\varepsilon a'_i, r_{\varepsilon,2}))$). It follows from Lemmas 1 and 2, together with a scaling argument, that

$$\|V_{\varphi_{i,1}}\|_{\mathcal{C}^0_2(B(a_i+\varepsilon a'_i,r_{\varepsilon,1})-\{a_i+\varepsilon a'_i\})} \leq c\, r_{\varepsilon,1}^{-2}\, \|\varphi_{i,1}\|_{\mathcal{C}^0(S^{N-1})} \tag{52}$$

$$\|W_{\phi_{i,2}}\|_{\mathcal{C}^0_{-N}(\mathbb{R}^N-B(a_i+\varepsilon a'_i,r_{\varepsilon,2}))} \leq c\, r_{\varepsilon,2}^{N}\, \|\phi_{i,2}\|_{\mathcal{C}^0(S^{N-1})}. \tag{53}$$

We keep the notations in the previous sections, and we look for a solution of problem (49) in $B(a_i+\varepsilon a'_i, r_{\varepsilon,1}) - B(a_i+\varepsilon a'_i, r_{\varepsilon,2})$ of the form

$$u_i = (-1)^{l_i} u_{\eta_i,A_i}(\cdot - a_i - \varepsilon a'_i) + V_{\varphi_{i,1}} + W_{\phi_{i,2}} + w_i \tag{54}$$

where the function u_{η_i,A_i} is the radial solution of problem (49) which has been obtained in Section 5 (see (30)) and where the functions w_i is small.

As usual, we introduce the polar coordinates $(t,\theta) \in (-\log r_{\varepsilon,1}, -\log r_{\varepsilon,2}) \times S^{N-1}$ in each $B(a_i+\varepsilon a'_i, r_{\varepsilon,1}) - B(a_i+\varepsilon a'_i, r_{\varepsilon,2})$. Given a function β, defined on $B(a_i+\varepsilon a'_i, r_{\varepsilon,1}) - B(a_i+\varepsilon a'_i, r_{\varepsilon,2})$, we agree that the function $\tilde\beta$ is the function defined on $(-\log r_{\varepsilon,1}, -\log r_{\varepsilon,2}) \times S^{N-1}$ which is determined by the relation

$$\beta(x) = |x - a_i - \varepsilon a'_i|^{-\frac{N-2}{2}}\, \tilde\beta(-\log|x - a_i - \varepsilon a'_i|, \theta). \tag{55}$$

With these notations, we need to find a function $\tilde u_i$ and $d_{i,0,1}, \ldots, d_{i,N,1}, d_{i,0,2}, \ldots, d_{i,N,2} \in \mathbb{R}$ such that

$$\partial_t^2 \tilde u_i - \frac{(N-2)^2}{4}\, \tilde u_i + \Delta_{S^{N-1}} \tilde u_i = -|\tilde u_i|^{\frac{N-2}{4}}\, \tilde u_i \tag{56}$$

in $[-\log r_{\varepsilon,1}, -\log r_{\varepsilon,2}] \times S^{N-1}$ and

$$\tilde u_i(-\log r_{\varepsilon,1}, \theta) = r_{\varepsilon,1}^{\frac{N-2}{2}}\, \varphi_{i,1}(\theta) + \sum_{j=0}^{N} d_{i,j,1}\, e_j$$

$$\tilde u_i(-\log r_{\varepsilon,2}, \theta) = r_{\varepsilon,2}^{\frac{N-2}{2}}\, \phi_{i,2}(\theta) + \sum_{j=0}^{N} d_{i,j,2}\, e_j$$

on S^{N-1}.

We will obtain a solution of this equation as a fixed point for some contraction mapping. We define

$$\mathcal{E}_{int,\varepsilon} := \left\{ \tilde w \in \mathcal{C}^0([-\log r_{\varepsilon,1}, -\log r_{\varepsilon,2}] \times S^{N-1}) : \quad \|\tilde w\|_{\mathcal{C}^0_{\delta,-\delta,\delta}} \leq \kappa\, \varepsilon^{-\frac{1}{k(N+2)}+\frac{\delta+N}{2k}} \right\} \tag{57}$$

where the parameter $\kappa > 0$ will be fixed later on.

We define

$$\begin{array}{cccc} h: & \mathbb{R} & \to & \mathbb{R} \\ & s & \mapsto & |s|^{\frac{N-2}{4}} s \end{array}$$

We write (56) as

$$\mathcal{L}\, \tilde w_i = -Q_{\varphi_{i,1},\phi_{i,2}}(\tilde w_i) \tag{58}$$

where the linear operator $\mathcal{L}$ is given by

$$\mathcal{L} := \partial_t^2 + \Delta_{S^{N-1}} - \frac{(N-2)^2}{4} + \frac{N+2}{N-2}|v_{\eta_i,A_i}|^{\frac{N-2}{4}}$$

and where $Q_{\varphi_{i,1},\phi_{i,2}}$ collects the nonlinear terms

$$Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{w}_i) := h((-1)^{l_i} v_{\eta_i,A_i} + \tilde{V}_{\varphi_{i,1}} + \tilde{W}_{\phi_{i,2}} + \tilde{w}_i) - h((-1)^{l_i} v_{\eta_i,A_i}) - h'((-1)^{l_i} v_{\eta_i,A_i})\, \tilde{w}_i.$$

We know $\mathbf{a}'$ is bounded. As before, we write $\eta_i = D_i \varepsilon^{\frac{N-2}{4k}}$. Suppose D_i is bounded from above and from below, A_i is bounded, and $\mathbf{b}$ is bounded, that is,

$$\frac{1}{\Theta} < D_i < \Theta \tag{59}$$

$$|A_i| < \Theta \tag{60}$$

$$|b_i| < \Theta, \tag{61}$$

where Θ ia a sufficiently large number to be fixed later. We state if $N \geq 6$, then $\forall (s,t) \in \mathbb{R}^2$

$$|h(s+t) - h(s) - h'(s)t| \leq c_1 |t|^{\frac{N+2}{N-2}} \tag{62}$$

On the other hand, it follows from Lemmas 1 and 2

$$|\tilde{V}_{\varphi_{i,1}}| \leq e^{-\frac{(N+2)t}{2}} \varepsilon^{\frac{N-2}{4k} - \frac{1}{k(N+2)}}$$

$$|\tilde{W}_{\phi_{i,2}}| \leq e^{\frac{(N+2)t}{2}} \varepsilon^{\frac{N-2}{4k} + \frac{N+2}{2} - \frac{1}{k(N+2)}}$$

Together with (62), we deduce if ε is sufficiently small, then

$$|Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{w}_i)| \leq c_2 \left[|v_{\eta_i,A_i}|^{\frac{4}{N-2}} (|\tilde{V}_{\varphi_{i,1}}| + |\tilde{W}_{\phi_{i,2}}|) + |\tilde{V}_{\varphi_{i,1}}|^{\frac{N+2}{N-2}} + |\tilde{W}_{\phi_{i,2}}|^{\frac{N+2}{N-2}} + |\tilde{w}_i|^{\frac{N+2}{N-2}} \right] \tag{63}$$

which implies

$$\|Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{w}_i)\|_{\delta,-\delta,\delta} \leq c_3\, \varepsilon^{\frac{N+\delta}{2k} - \frac{1}{k(N+2)}} (1 + c_4\, \kappa^{\frac{N+2}{N-2}} \varepsilon^{\frac{4}{N-2}(\frac{N+\delta}{2k} - \frac{1}{k(N+2)})}). \tag{64}$$

Here the constants c_2, c_3 and c_4 are independent of κ and ε.

Given $\tilde{\psi} \in \mathcal{E}_{int,\varepsilon}$ we use the result of Proposition 5 to solve

$$\mathcal{L}\,\tilde{\xi} = -Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{\psi})$$

It follows from Proprosition 5 and the above estimate that, given κ, there exists $\varepsilon_0 > 0$ (depending on κ) such that the mapping defined by $T_i(\tilde{\psi}) = \tilde{\xi}$ is well defined, provided $\varepsilon \in (0, \varepsilon_0)$. To see this, we have

$$\|T_i(\tilde{\psi})\|_{\delta,-\delta,\delta} \leq c\, c_3\, \varepsilon^{\frac{N+\delta}{2k} - \frac{1}{k(N+2)}} (1 + c_4\, \kappa^{\frac{N+2}{N-2}} \varepsilon^{\frac{4}{N-2}(\frac{N+\delta}{2k} - \frac{1}{k(N+2)})})$$

Thus, if we choose $\kappa = c\, c_3 + 1$, the desired result yields.

Moreover, for all $\tilde{\psi}_1, \tilde{\psi}_2 \in \mathcal{E}_{int,\varepsilon}$, one can check that

$$\begin{aligned} \|T_i(\tilde{\psi}_1) - T_i(\tilde{\psi}_2)\|_{\delta,-\delta,\delta} &\leq c\,\|Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{\psi}_1) - Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{\psi}_2)\|_{\delta,-\delta,\delta} \\ &\leq c\,((1+\kappa)\varepsilon^{\frac{N-2}{4k}+\frac{N}{2k(N+2)}})^{\frac{4}{N-2}}\|\tilde{\psi}_1 - \tilde{\psi}_2\|_{\delta,-\delta,\delta}, \end{aligned} \tag{65}$$

since

$$|Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{\psi}_1) - Q_{\varphi_{i,1},\phi_{i,2}}(\tilde{\psi}_2)| \leq c\,(|\tilde{V}_{\varphi_{i,1}}| + |\tilde{W}_{\phi_{i,2}}| + |\tilde{\psi}_1| + |\tilde{\psi}_2|)^{\frac{4}{N-2}}|\tilde{\psi}_1 - \tilde{\psi}_2|$$

Consequently, for ε sufficiently small, the mapping T_i is a contraction from $\mathcal{E}_{int,\varepsilon}$ into itself and hence admits a unique fixed point in this set. This yields a solution u_i of (49).

Keeping the notations in Section 4, we set $C_{N,\eta} := T_\eta - \frac{4}{N-2}\ln\frac{1}{\eta} = C_N + O(\eta^{\frac{N-2}{N}} + \eta^{\frac{4}{N}}\log\frac{1}{\eta})$. If we define the function u_{int} to be equal to $T_{b_i,a_i+\varepsilon a_i'}(u_i)$ on $\Omega_{int,i,\varepsilon}$, we have proven the :

Proposition 6 *Given $\mathbf{a}' \in \Omega_1 \times \cdots \times \Omega_m$, $\mathbf{b} \in (\mathbb{R}^N)^m$, $\mathbf{A} := (A_1, \cdots, A_m) \in \mathbb{R}^m$, $\mathbf{D} := (D_1, \cdots, D_m) \in (\mathbb{R}^+)^m$, $\Psi_1 \in (\mathcal{E}_1)^m$ and $\Phi_2 \in (\mathcal{E}_2)^m$, there exists a solution u_{int} of (48) in $\Omega_{int,\varepsilon}$ satisfying boundary conditions*

$$T_{-b_i,a_i+\varepsilon a_i'}(u_{int})|_{\partial B(a_i+\varepsilon a_i', r_{\varepsilon,1})} - \varphi_{i,1} \in Span\{e_j \quad : \quad j = 0,\ldots,N\}$$

$$T_{-b_i,a_i+\varepsilon a_i'}(u_{int})|_{\partial B(a_i+\varepsilon a_i', r_{\varepsilon,2})} - \phi_{i,2} \in Span\{e_j \quad : \quad j = 0,\ldots,N\}$$

for all $1 \leq i \leq m$. Moreover, the sequence of solutions u_{int} blows up at each a_i as ε tends to 0 in such a way that

$$|\nabla u_{int}|^2\,dx \rightharpoonup C_N^{(1)}\sum_{i=1}^{m} k\,\delta_{a_i}$$

in the sense of measures. Here $C_N^{(1)}$ is the constant defined in Theorem 1. Finally, this solution can be expanded as

$$\begin{aligned} T_{-b_i,a_i+\varepsilon a_i'}(u_{int}) = \ & (-1)^{l_i}D_i\varepsilon^{\frac{N-2}{4k}}\left[-e^{\frac{(2-N)A_i}{2}} + e^{\frac{(N-2)A_i}{2}}|x-a_i-\varepsilon a_i'|^{2-N}\right] \\ & + V_{\varphi_{i,1}} + \mathcal{O}(\varepsilon^{\frac{N-2}{4k}+\frac{2}{k(N+2)}}(1+\varepsilon^{\frac{N+2}{2}-\frac{N+3}{k(N+2)}} + \kappa\varepsilon^{\frac{N^2+2N+2N\delta-4}{4k(N+2)}})) \end{aligned} \tag{66}$$

in $B(a_i+\varepsilon a_i', r_{\varepsilon,1}) - B(a_i+\varepsilon a_i', r_{\varepsilon,1}/2)$ and

$$\begin{aligned} T_{-b_i,a_i+\varepsilon a_i'}(u_{int}) = \ & (-1)^{l_i+k}D_i\varepsilon^{\frac{N-2}{4k}}\Big[-e^{\frac{(N-2)(kC_{N,\eta}-A_i)}{2}}D_i^{-2k}\varepsilon^{\frac{2-N}{2}} \\ & + e^{\frac{(2-N)(kC_{N,\eta}-A_i)}{2}}D_i^{2k}\varepsilon^{\frac{N-2}{2}}|x-a_i-\varepsilon a_i'|^{2-N}\Big] \\ & + W_{\phi_{i,2}} + \mathcal{O}(\varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}+\frac{N}{k(N+2)}}(1+\varepsilon^{\frac{N+2}{2}-\frac{N+3}{k(N+2)}} + \kappa\varepsilon^{\frac{N^2+2N+2N\delta-4}{4k(N+2)}})) \end{aligned} \tag{67}$$

in $B(a_i+\varepsilon a_i', 2r_{\varepsilon,2}) - B(a_i+\varepsilon a_i', r_{\varepsilon,2})$

Since we have found the solution of (49) with the form (54), we have

$$-\Delta w_i = h((-1)^{l_i}u_{\eta_i,A_i}(\cdot - a_i - \varepsilon a_i') + V_{\varphi_{i,1}} + W_{\phi_{i,2}} + w_i) - h((-1)^{l_i}u_{\eta_i,A_i}(\cdot - a_i - \varepsilon a_i')) \tag{68}$$

so that in $B(a_i+\varepsilon a_i', r_{\varepsilon,1}) - B(a_i+\varepsilon a_i', r_{\varepsilon,1}/2)$

$|\Delta w_i| \le |u_{\eta_i,A_i}(\cdot - a_i - \varepsilon a_i')|^{\frac{4}{N-2}} |V_{\varphi_{i,1}} + W_{\phi_{i,2}} + w_i| \le c|V_{\varphi_{i,1}} + W_{\phi_{i,2}} + w_i| \le c\varepsilon^{\frac{N-2}{4k}+\frac{1}{k(N+2)}}$,

since $\delta > -\frac{N+2}{2}$. Using the standard elliptic theory, we have

$$\|r_{\varepsilon,1}\nabla w_i\|_{L^\infty(B(a_i+\varepsilon a_i', r_{\varepsilon,1})-B(a_i+\varepsilon a_i', 3r_{\varepsilon,1}/4))} \le c\varepsilon^{\frac{N-2}{4k}} (\varepsilon^{\frac{3}{k(N+2)}} + \kappa\varepsilon^{\frac{2}{k(N+2)}+\frac{N^2+2N+2N\delta-4}{4k(N+2}})$$

Recall

$$N^2 + 2N + 2N\delta - 4 > 0$$

Thus,

$$\|r_{\varepsilon,1}\nabla w_i\|_{L^\infty(B(a_i+\varepsilon a_i', r_{\varepsilon,1})-B(a_i+\varepsilon a_i', 3r_{\varepsilon,1}/4))} \le c\varepsilon^{\frac{N-2}{4k}+\frac{2}{k(N+2)}}$$

By the regularity theory, for all $\alpha \in (0,1)$, on the boundary $\partial B(a_i+\varepsilon a_i', r_{\varepsilon,1})$

$$\|r_{\varepsilon,1}\,\partial_n w_i\|_{C^{1,\alpha}(S^{N-1})} \le c\varepsilon^{\frac{N-2}{4k}+\frac{2}{k(N+2)}} \tag{69}$$

With the same arguments, we infer for all $\alpha \in (0,1)$, on the boundary $\partial B(a_i+\varepsilon a_i', r_{\varepsilon,2})$

$$\|r_{\varepsilon,2}\,\partial_n w_i\|_{C^{1,\alpha}(S^{N-1})} \le c\varepsilon^{\frac{N-2}{4k}+\frac{N}{k(N+2)}-\frac{N-2}{2}} \tag{70}$$

6.3 Solutions of the nonlinear problem in $\Omega_{ext,0,\varepsilon}$

Recall $\mathbf{a}' := (a_1', \ldots, a_m') \in \Omega_1 \times \cdots \Omega_m$ and $\mathbf{b} := (b_1, \ldots, b_m) \in (\mathbb{R}^N)^m$. Given a m functions $\Phi_1 = (\phi_{1,1}, \ldots, \phi_{m,1}) \in (\mathcal{E}_1)^m$, we now construct a family of solution of (48) in $\Omega_{ext,0,\varepsilon}$ which in some sense is parameterized by $T_{b_i,a_i+\varepsilon a_i'}(\Phi_1) = (T_{b_i,a_i+\varepsilon a_i'}(\phi_{1,1}), \ldots, T_{b_i,a_i+\varepsilon a_i'}(\phi_{m,1}))$. More precisely, denote by $W_{T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1})}$ the unique harmonic extension of $T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1})$ in $\mathbb{R}^N - T_{-b_i,a_i+\varepsilon a_i'}(B(a_i+\varepsilon a_i', r_{\varepsilon,1}))$ which decays at ∞. Let χ be a C^∞ cut-off function defined in $\mathbb{R}^N$, such that $\chi|_{B(0,1)} \equiv 1$ and $\chi \equiv 0$ on $\mathbb{R}^N - B(0,2)$ and $\chi \ge 0$. Given a m positive numbers $\Lambda := (\Lambda_1, \cdots, \Lambda_m) \in (\mathbb{R}^+)^m$ and $\mathbf{g} := (g_1, \cdots, g_m) \in (\mathbb{R}^N)^m$ a m vectors in $\mathbb{R}^N$, we assume for all $i \in \{1, \cdots, m\}$

$$\frac{1}{\Theta} < \Lambda_i < \Theta \tag{71}$$

where the real number Θ is defined in the previous subsection. Fix $\nu \in (-N+2, -N+3)$ and choose $\alpha' \in (0, \frac{\nu+N-2}{k(N+2)(\nu+N-1)})$. We look for a solution of the following equation in $\Omega_{ext,0,\varepsilon}$

$$\begin{cases} \Delta u_{ext,0} + |u_{ext,0}|^{\frac{4}{N-2}} u_{ext,0} = 0 \quad \text{in} \quad \Omega_{ext,0,\varepsilon} \\ T_{-b_i,a_i+\varepsilon a_i'}(u_{ext,0}) - \phi_{i,1} \in \text{Span}\{e_0, \ldots, e_N\} \quad \text{on} \quad \partial B(a_i+\varepsilon a_i', r_{\varepsilon,1}) \\ u_{ext,0} = 0 \quad \text{on} \quad \partial\Omega \end{cases} \tag{72}$$

We write $u_{ext,0}$ in the following form

$$\begin{aligned} u_{ext,0} \;=\; & \sum_{i=1}^m (-1)^{l_i} \Lambda_i \varepsilon^{\frac{N-2}{4k}} G(\cdot, a_i+\varepsilon a_i') \\ & + \sum_{i=1}^m \chi\Big(\frac{\cdot - a_i - \varepsilon a_i'}{\varepsilon^{\alpha'}}\Big) W_{T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1})} \\ & + \sum_{i=1}^m \chi\Big(\frac{\cdot - a_i - \varepsilon a_i'}{\varepsilon^{\alpha'}}\Big) \left\langle g_i, T_{b_i,a_i+\varepsilon a_i'}\Big(\frac{\cdot - a_i - \varepsilon a_i'}{|\cdot - a_i - \varepsilon a_i'|^N}\Big) \right\rangle + w_{ext,0} \end{aligned} \tag{73}$$

where $\mathbf{g}$ is small and the function $w_{ext,0}$ is assumed to be small and to satisfy $w_{ext,0}|_{\partial\Omega_{ext,\varepsilon}} = 0$.

We use the maximum principle to reduce (72) to

$$\left\{\begin{array}{rcll} -\Delta w_{ext,0} & = & q + Q_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(w_{ext,0}) & \text{in} \quad \Omega_{ext,0,\varepsilon} \\ w_{ext} & = & 0 & \text{on} \quad \partial\Omega_{ext,0,\varepsilon} \end{array}\right. \tag{74}$$

where

$$\begin{aligned} Q_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(w) \;:=\; & h\Bigg(\sum_{i=1}^m (-1)^{l_i}\Lambda_i \varepsilon^{\frac{N-2}{4k}}\, G(\cdot, a_i+\varepsilon a_i') \\ & + \sum_{i=1}^m \chi(\frac{\cdot - a_i - \varepsilon a_i'}{\varepsilon^{\alpha'}}) W_{T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1})} \\ & + \sum_{i=1}^m \chi(\frac{\cdot - a_i - \varepsilon a_i'}{\varepsilon^{\alpha'}}) \left\langle g_i, T_{b_i,a_i+\varepsilon a_i'}(\frac{\cdot - a_i - \varepsilon a_i'}{|\cdot - a_i - \varepsilon a_i'|^N})\right\rangle + w\Bigg) \end{aligned}$$

and where the function q is given by

$$\begin{aligned} q(z) \;=\; & \sum_{i=1}^m \frac{1}{\varepsilon^{2\alpha'}} \Delta\chi(\frac{z - a_i - \varepsilon a_i'}{\varepsilon^{\alpha'}}) \left(W_{T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1})}(z) + \left\langle g_i, T_{b_i,a_i+\varepsilon a_i'}(\frac{z - a_i - \varepsilon a_i'}{|z - a_i - \varepsilon a_i'|^N})\right\rangle \right) \\ & + \frac{2}{\varepsilon^{\alpha'}} \sum_{i=1}^m \nabla\chi(\frac{z - a_i - \varepsilon a_i'}{\varepsilon^{\alpha'}}) \cdot \nabla \left(W_{T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1})}(z) + \left\langle g_i, T_{b_i,a_i+\varepsilon a_i'}(\frac{z - a_i - \varepsilon a_i'}{|z - a_i - \varepsilon a_i'|^N})\right\rangle \right) \end{aligned}$$

We define

$$P_{\varepsilon,1} := \{\mathbf{g} \in (\mathbb{R}^N)^m \;:\; |\mathbf{g}| \leq \varepsilon^{\frac{N-2}{4k} + \frac{N}{k(N+2)}}\}$$

and consider

$$\mathcal{E}_{ext,0,\varepsilon} := \{w \in C^0_\nu(\Omega_{ext,0,\varepsilon}) \;:\; \|w\|_{C^0_\nu} \leq \kappa_1\, \varepsilon^{\frac{N-2}{4k} + \frac{2-\nu}{k(N+2)}} \quad \text{and} \quad w|_{\partial\Omega_{ext,0,\varepsilon}} = 0\},$$

where κ_1 is a constant to be fixed later. It is clear that

$$\left| T_{b_i,a_i+\varepsilon a_i'}(\frac{z - a_i - \varepsilon a_i'}{|z - a_i - \varepsilon a_i'|^N}) \right| \leq \frac{1}{|z - a_i - \varepsilon a_i'|^{N-1}}(1 + \Theta|z - a_i - \varepsilon a_i'|).$$

On the other hand, for all $\Phi_1 \in (\mathcal{E}_1)^m$, it follows from Lemma 2

$$|W_{\phi_{i,1}}(z)| \leq c\varepsilon^{\frac{N-2}{4k} + \frac{N+1}{k(N+2)}} |z - a_i - \varepsilon a_i'|^{-N}$$

which implies in the small $B(a_i + \varepsilon a_i', 2\varepsilon^{\alpha'})$

$$|W_{T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1})}(z)| \leq c\varepsilon^{\frac{N-2}{4k} + \frac{N+1}{k(N+2)}} |z - a_i - \varepsilon a_i'|^{-N}(1 + 2\Theta|z - a_i - \varepsilon a_i'| + \Theta^2|z - a_i - \varepsilon a_i'|^2),$$

Therefore, for all $\mathbf{g} \in P_{\varepsilon,1}$ and $\Phi_1 \in (\mathcal{E}_1)^m$, we estimate

$$\begin{aligned} \|q\|_{C^0_{\nu-2}(\Omega_{ext,0,\varepsilon})} & \leq c(1 + \Theta^2\varepsilon^{2\alpha'})\, \varepsilon^{\frac{N-2}{4k}} (\varepsilon^{\frac{N+1}{k(N+2)} + \alpha'(-\nu-N)} + \varepsilon^{\frac{N}{k(N+2)} + \alpha'(-\nu-N+1)}) \\ & \leq c(1 + \Theta^2\varepsilon^{2\alpha'})\, \varepsilon^{\frac{N-2}{4k} + \frac{2-\nu}{k(N+2)}} \end{aligned} \tag{75}$$

since $\alpha' \in (0, \frac{\nu+N-2}{k(N+2)(\nu+N-1)})$. Here c is a constant independent of Θ and ε. Given $w \in \mathcal{E}_{ext,0,\varepsilon}$, we obtain with little work

$$\begin{aligned}
&\|Q_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(w)\|_{C^0_{\nu-2}} \\
\leq\ & c\varepsilon^{\frac{N+2}{4k}}[\Theta^{\frac{N+2}{N-2}}\varepsilon^{\frac{-N-\nu}{k(N+2)}} + (1+\Theta^2\varepsilon^{2\alpha'})\varepsilon^{\frac{N+1}{k(N-2)}+\alpha'(2-\nu-\frac{N(N+2)}{N-2})}) \\
& +(1+\Theta^2\varepsilon^{2\alpha'})\varepsilon^{\frac{N}{k(N-2)}+\alpha'(2-\nu-\frac{(N-1)(N+2)}{N-2})} + \kappa_1^{\frac{N+2}{N-2}}\varepsilon^{(-\nu+\frac{4N}{N-2})\frac{1}{k(N+2)}}] \\
\leq\ & c\varepsilon^{\frac{N-2}{4k}+\frac{2-\nu}{k(N+2)}}[\Theta^{\frac{N+2}{N-2}} + (1+\Theta^2\varepsilon^{2\alpha'})\varepsilon^{\frac{1}{k(N-2)}+\frac{N+\nu}{k(N+2)}+\frac{(N-1)(N+2)}{N-2}(\frac{1}{k(N+2)}-\alpha')} \\
& +\kappa_1^{\frac{N+2}{N-2}}\varepsilon^{\frac{N-1}{k(N-2)}}]
\end{aligned} \tag{76}$$

and for all $\psi_1, \psi_2 \in \mathcal{E}_{ext,0,\varepsilon}$

$$\begin{aligned}
&\|Q_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(\psi_1) - Q_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(\psi_2)\|_{C^0_{\nu-2}} \\
\leq\ & c\,\varepsilon^{\frac{N}{k(N+2)}}[\Theta^{\frac{4}{N-2}} + \varepsilon^{\frac{2}{k(N+2)}}\kappa_1^{\frac{4}{N-2}} + \varepsilon^{\frac{2}{k(N+2)}}(1+\Theta^2\varepsilon^{2\alpha'})^{\frac{4}{N-2}}]\|\psi_1-\psi_2\|_{C^0_\nu}
\end{aligned} \tag{77}$$

The following result is standard

Lemma 3 *Assume that* $\nu \in (2-N, 0)$ *then for all* $0 \leq i \leq m$ *and* $f \in C^0_{\nu-2}(\Omega_{ext,i,\varepsilon})$*, there exists* $w \in C^0_\nu(\Omega_{ext,i,\varepsilon})$ *unique solution of*

$$\left\{\begin{array}{rcll} \Delta w &=& f & \text{in} \quad \Omega_{ext,i,\varepsilon} \\ w &=& 0 & \text{on} \quad \partial\Omega_{ext,i,\varepsilon}\,. \end{array}\right. \tag{78}$$

Furthermore, there holds

$$\|w\|_{C^0_\nu} \leq c\|f\|_{C^0_{\nu-2}}.$$

Proof. The existence of w is straightforward and the estimate relies on the fact that $x \to |x-a_i-\varepsilon a_i'|^\nu$ can be used as a barrier in $\Omega - B(a_i+\varepsilon a_i', r_{\varepsilon,1})$ or in $B(a_i+\varepsilon a_i', r_{\varepsilon,2}) - (a_i+\varepsilon\Omega_i)$. ■

We define the map

$$T_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{f}} : \mathcal{E}_{ext,0,\varepsilon} \longrightarrow \mathcal{E}_{ext,0,\varepsilon}$$

by $T_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(w) := \psi$ where ψ is the solution of

$$-\Delta\psi = q + Q_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(w).$$

It follows from the estimates (75) to (77) we could choose $\kappa_1 > 0$ (depending on Θ) in such a way that the mapping $T_{\Lambda,\Phi_1,\mathbf{b},\mathbf{a}',\mathbf{g}}(w)$ is well defined and is a contraction, provided ε is chosen small enough, say $\varepsilon \in (0, \varepsilon_1)$. In particular, this mapping has a unique fixed point in $\mathcal{E}_{ext,0,\varepsilon}$ which yields a solution of (74). Therefore, we have proved the following :

Proposition 7 *Given* $\mathbf{a}' \in \Omega_1 \times \cdots \times \Omega_m$, $\mathbf{b} \in (\mathbb{R}^N)^m$, $\Phi_1 \in (\mathcal{E}_1)^m$ *and* $\mathbf{g} \in P_{\varepsilon,1}$*, there exists* $u_{ext,0}$ *solution of equation (72) of the form (73) in* $\Omega_{ext,0,\varepsilon}$*, satisfying*

$$u_{ext,0} = \sum_{i=1}^m (-1)^{l_i}\Lambda_i\varepsilon^{\frac{N-2}{4k}}\, G(\cdot, a_i+\varepsilon a_i') + T_{b_i,a_i+\varepsilon a_i'}(\phi_{i,1}) + \left\langle g_i, T_{b_i,a_i+\varepsilon a_i'}(\frac{\cdot - a_i - \varepsilon a_i'}{|\cdot - a_i - \varepsilon a_i'|^N})\right\rangle$$

on $\partial T_{-b_i,a_i+\varepsilon a'_i}(B(a_i+\varepsilon a'_i, r_{\varepsilon,1}))$ *for all* $1 \le i \le m$ *and* $u_{ext,0} = 0$ *on* $\partial\Omega$. *Furthermore, the function* $u_{ext,0}$ *can be expanded as*

$$\begin{aligned} u_{ext,0} &= \sum_{i=1}^{m} \Big[(-1)^{l_i} \Lambda_i \varepsilon^{\frac{N-2}{4k}} G(\cdot, a_i + \varepsilon a'_i) + W_{T_{b_i,a_i+\varepsilon a'_i}}(\phi_{i,1}) \\ &\quad + \Big\langle g_i, T_{b_i,a_i+\varepsilon a'_i}\big(\frac{\cdot - a_i - \varepsilon a'_i}{|\cdot - a_i - \varepsilon a'_i|^N}\big)\Big\rangle \Big] + \mathcal{O}(\varepsilon^{\frac{N-2}{4k}+\frac{2}{k(N+2)}}) \end{aligned} \tag{79}$$

in $T_{-b_i,a_i+\varepsilon a'_i}(B(a_i+\varepsilon a'_i, 2r_{\varepsilon,1}) - B(a_i+\varepsilon a'_i, r_{\varepsilon,1}))$.

Similarly,

$$\|r_{\varepsilon,1}\, \partial_n T_{-b_i,a_i+\varepsilon a'_i}(w_{ext,0})\|_{C^{1,\alpha}(S^{N-1})} \le c_5 \varepsilon^{\frac{N-2}{4k}+\frac{2}{k(N+2)}} \tag{80}$$

where n is the outside unit normal vector on the boundary of $B(a_i+\varepsilon a'_i, r_{\varepsilon,1})$ and c_5 is a constant independent of ε (but depending on Θ).

6.4 Solutions of the nonlinear problem in $\Omega_{ext,i,\varepsilon}$ with $1 \le i \le m$

Given a m functions $\Psi_2 = (\varphi_{1,2}, \dots, \varphi_{m,2}) \in (\mathcal{E}_2)^m$, let $V_{T_{b_i,a_i+\varepsilon a'_i}}(\varphi_{i,2})$ be the unique harmonic extension of $T_{b_i,a_i+\varepsilon a'_i}(\varphi_{i,2}))$ in $T_{-b_i,a_i+\varepsilon a'_i}(B(a_i+\varepsilon a'_i, r_{\varepsilon,2}))$. We assume $\Theta\, r_{\varepsilon,2}$ is small, for example,

$$\Theta\, r_{\varepsilon,2} \le \frac{1}{100} \tag{81}$$

Fix $\nu_1 \in (-1, 0)$ and choose some $\alpha_1 \in (1 - \frac{\nu_1}{(\nu_1-1)k(N+2)}, 1)$. Given a m positive numbers $\Gamma := (\Gamma_1, \cdots, \Gamma_m) \in (\mathbb{R}^+)^m$ and $\mathbf{s} := (s_1, \cdots, s_m) \in (\mathbb{R}^N)^m$ a m vectors in $\mathbb{R}^N$, we now construct a family of solution of (1) in $\Omega_{ext,i,\varepsilon}$ which in some sense is parameterized by $T_{b_i,a_i+\varepsilon a'_i}(\varphi_{i,2})$, namely

$$\begin{cases} \Delta u_{ext,i} + |u_{ext,i}|^{\frac{4}{N-2}} u_{ext,i} = 0 & \text{in} \quad \Omega_{ext,i,\varepsilon} \\ T_{-b_i,a_i+\varepsilon a'_i}(u_{ext,i}) - \varphi_{i,2} \in \text{Span}\{e_0, \dots, e_N\} & \text{on} \quad \partial B(a_i+\varepsilon a'_i, r_{\varepsilon,2}) \\ u_{ext,i} = 0 \quad \text{on} \quad \partial(a_i + \varepsilon\Omega_i) \end{cases} \tag{82}$$

We write $u_{ext,i}$ in the following form

$$\begin{aligned} u_{ext,i} &= -(-1)^{l_i+k} \Gamma_i \varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}} G_i\big(\frac{\cdot - a_i}{\varepsilon}, a'_i\big) \\ &\quad + (1-\chi)\big(\frac{\cdot - a_i - \varepsilon a'_i}{\varepsilon^{\alpha_1}}\big) V_{T_{b_i,a_i+\varepsilon a'_i}}(\phi_{i,2}) \\ &\quad + (1-\chi)\big(\frac{\cdot - a_i - \varepsilon a'_i}{\varepsilon^{\alpha_1}}\big) \Big\langle s_i, T_{b_i,a_i+\varepsilon a'_i}(\cdot - a_i - \varepsilon a'_i)\Big\rangle + w_{ext,i} \end{aligned} \tag{83}$$

where the Green type function G_i is defined in Section 2, $\mathbf{s}$ is small and the function $w_{ext,i}$ is assumed to be small and to satisfy $w_{ext,i}|_{\partial\Omega_{ext,i,\varepsilon}} = 0$.

We use the maximum principle to reduce (82) to

$$\begin{cases} -\Delta w_{ext,i} = q + Q_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}}(w_{ext,i}) & \text{in} \quad \Omega_{ext,i,\varepsilon} \\ w_{ext,i} = 0 & \text{on} \quad \partial\Omega_{ext,i,\varepsilon} \end{cases} \tag{84}$$

where

$$
\begin{aligned}
Q_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}}(w) \quad := \quad & h\Bigg((-1)^{l_i+k}\Gamma_i\varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}}\,G_i(\frac{\cdot - a_i}{\varepsilon},a_i') \\
& +(1-\chi)(\frac{\cdot - a_i - \varepsilon a_i'}{\varepsilon^{\alpha_1}})V_{T_{b_i,a_i+\varepsilon a_i'}(\varphi_{i,2})} \\
& +(1-\chi)(\frac{\cdot - a_i - \varepsilon a_i'}{\varepsilon^{\alpha_1}})\left\langle s_i, T_{b_i,a_i+\varepsilon a_i'}(\cdot - a_i - \varepsilon a_i')\right\rangle + w\Bigg)
\end{aligned}
$$

and where the function q is given by

$$
\begin{aligned}
q(z) \quad = \quad & -\frac{1}{\varepsilon^{2\alpha_1}}\Delta\chi(\frac{z - a_i - \varepsilon a_i'}{\varepsilon^{\alpha_1}})\left(V_{T_{b_i,a_i+\varepsilon a_i'}(\varphi_{i,2})}(z) + \left\langle s_i, T_{b_i,a_i+\varepsilon a_i'}(z - a_i - \varepsilon a_i')\right\rangle\right) \\
& -\frac{2}{\varepsilon^{\alpha_1}}\nabla\chi(\frac{z - a_i - \varepsilon a_i'}{\varepsilon^{\alpha_1}})\cdot\nabla\left(V_{T_{b_i,a_i+\varepsilon a_i'}(\varphi_{i,2})}(z) + \left\langle s_i, T_{b_i,a_i+\varepsilon a_i'}(z - a_i - \varepsilon a_i')\right\rangle\right)
\end{aligned}
$$

We define

$$
P_{\varepsilon,2} := \{\mathbf{s} \in (\mathbb{R}^N)^m \ : \ |\mathbf{s}| \le \varepsilon^{\frac{N-2}{4k}-\frac{N}{2}+\frac{N}{k(N+2)}}\}
$$

and consider

$$
\mathcal{E}_{ext,1,\varepsilon} := \{w \in C^0_{\nu_1}(\Omega_{ext,i,\varepsilon}) \ : \ \|w\|_{C^0_{\nu_1}} \le \kappa_2\,\varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}-\nu_1+\frac{N+\nu_1}{k(N+2)}} \quad \text{and} \quad w|_{\partial\Omega_{ext,i,\varepsilon}} = 0\},
$$

where κ_2 is a constant to be fixed later. Using (81), it is clear that

$$
|T_{b_i,a_i+\varepsilon a_i'}(z - a_i - \varepsilon a_i')| \le c|z - a_i - \varepsilon a_i'|.
$$

On the other hand, for all $\Psi_2 \in (\mathcal{E}_2)^m$, it follows from Lemma 1

$$
|V_{\varphi_{i,2}}(z)| \le c\varepsilon^{\frac{N-2}{4k}-\frac{N+2}{2}+\frac{N+1}{k(N+2)}}\,|z - a_i - \varepsilon a_i'|^2
$$

which yields again from (81) in the small $B(a_i + \varepsilon a_i', 2r_{\varepsilon,2})$

$$
|V_{T_{b_i,a_i+\varepsilon a_i'}(\varphi_{i,2})}(z)| \le c\varepsilon^{\frac{N-2}{4k}-\frac{N+2}{2}+\frac{N+1}{k(N+2)}}\,|z - a_i - \varepsilon a_i'|^2.
$$

Here, the constants c are independent of Θ abd ε. On the other hand, follows from Maximum principle that $0 < G_i(\cdot, z) < 1$. Therefore, for all $\mathbf{s} \in P_{\varepsilon,2}$ and $\Psi_2 \in (\mathcal{E}_2)^m$, we estimate with little work

$$
\|q\|_{C^0_{\nu_1-2}(\Omega_{ext,i,\varepsilon})} \le c\,\varepsilon^{\alpha_1(1-\nu_1)+\frac{N-2}{4k}-\frac{N}{2}+\frac{N}{k(N+2)}} \le c\,\varepsilon^{-\nu_1+\frac{N-2}{4k}-\frac{N-2}{2}+\frac{N+\nu_1}{k(N+2)}} \tag{85}
$$

since $\alpha_1 > 1 - \frac{\nu_1}{(\nu_1-1)k(N+2)}$. Given $w \in \mathcal{E}_{ext,1,\varepsilon}$, we obtain with little work

$$
\|Q_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}}(w)\|_{C^0_{\nu_1-2}} \le \ c\,\varepsilon^{-\nu_1+\frac{N-2}{4k}-\frac{N-2}{2}+\frac{N+\nu_1}{k(N+2)}}(\Theta^{\frac{N+2}{N-2}} + \varepsilon^{\frac{N-1}{k(N-2)}} + \kappa_2^{\frac{N+2}{N-2}}\varepsilon^{\frac{N}{k(N-2)}}) \tag{86}
$$

and for all $\psi_1, \psi_2 \in \mathcal{E}_{ext,1,\varepsilon}$

$$
\begin{aligned}
& \|Q_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}}(\psi_1) - Q_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}}(\psi_2)\|_{C^0_{\nu_1-2}} \\
\le \quad & c\,\varepsilon^{\frac{N}{k(N+2)}}[\Theta^{\frac{4}{N-2}} + \varepsilon^{\frac{4(N-1)}{k(N^2-4)}} + \kappa_2^{\frac{4}{N-2}}\varepsilon^{\frac{4N}{k(N^2-4)}}]\|\psi_1 - \psi_2\|_{C^0_{\nu_1}}
\end{aligned} \tag{87}
$$

By Lemma 3, we can define the map

$$T_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}} : \mathcal{E}_{ext,1,\varepsilon} \longrightarrow \mathcal{E}_{ext,1,\varepsilon}$$

by $T_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}}(w) := \psi$ where ψ is the solution of

$$-\Delta\psi = q + Q_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{k}}(w).$$

It follows from the estimates (85) to (87) we could choose $\kappa_2 > 0$ (depending on Θ) in such a way that the mapping $T_{\Gamma,\varphi_{i,2},\mathbf{b},\mathbf{a}',\mathbf{s}}(w)$ is well defined and is a contraction, provided ε is chosen small enough, say $\varepsilon \in (0, \varepsilon_2)$. In particular, this mapping has a unique fixed point in $\mathcal{E}_{ext,1,\varepsilon}$ which yields a solution of (82). Therefore, we have proved the following :

Proposition 8 *Given $\mathbf{a}' \in \Omega_1 \times \cdots \times \Omega_m$, $\mathbf{b} \in (\mathbb{R}^N)^m$, $\Psi_2 \in (\mathcal{E}_2)^m$ and $\mathbf{s} \in P_{\varepsilon,2}$, there exists $u_{ext,i}$ solution of equation (82) of the form (83) in $\Omega_{ext,i,\varepsilon}$, satisfying*

$$u_{ext,i} = -(-1)^{l_i+k}\Gamma_i\varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}}\, G_i(\frac{\cdot - a_i}{\varepsilon}, a_i') + T_{b_i,a_i+\varepsilon a_i'}(\varphi_{i,2}) + \left\langle s_i, T_{b_i,a_i+\varepsilon a_i'}(\cdot - a_i - \varepsilon a_i')\right\rangle$$

on $\partial T_{-b_i,a_i+\varepsilon a_i'}(B(a_i + \varepsilon a_i', r_{\varepsilon,2}))$ for all $1 \le i \le m$ and $u_{ext,i} = 0$ on $\partial(a_i + \varepsilon\Omega_i)$. Furthermore, the function $u_{ext,i}$ can be expanded as

$$\begin{aligned} u_{ext,i} &= -(-1)^{l_i+k}\Gamma_i\varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}}\, G_i(\frac{\cdot - a_i}{\varepsilon}, a_i') + V_{T_{b_i,a_i+\varepsilon a_i'}}(\varphi_{i,2}) \\ &\quad + \left\langle s_i, T_{b_i,a_i+\varepsilon a_i'}(\cdot - a_i - \varepsilon a_i')\right\rangle + \mathcal{O}(\varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}+\frac{N}{k(N+2)}}) \end{aligned} \tag{88}$$

in $T_{-b_i,a_i+\varepsilon a_i'}(B(a_i + \varepsilon a_i', r_{\varepsilon,2}) - B(a_i + \varepsilon a_i', r_{\varepsilon,2}/2))$.

Similarly,

$$\|r_{\varepsilon,2}\,\partial_n T_{-b_i,a_i+\varepsilon a_i'}(w_{ext,i}))\|_{C^{1,\alpha}(S^{N-1})} \le c_6\varepsilon^{\frac{N-2}{4k}-\frac{N-2}{2}+\frac{N}{k(N+2)}} \tag{89}$$

where n is the outside unit normal vector on the boundary of $B(a_i + \varepsilon a_i', r_{\varepsilon,2})$ and c_6 is a constant independent of ε (but depending on Θ). We define the function u_{ext} to be equal to $u_{ext,i}$ on $\Omega_{ext,i,\varepsilon}$ for all $0 \le i \le m$. In the following consideration we will fix some $\alpha \in (0, 1)$.

6.5 The Cauchy data mapping

We explain how the free parameters in Propositions 6, 7 and 8 can be chosen so that the functions u_{int} and u_{ext} can be glued together to obtain a solution of problem (1) in Ω_ε.

We want to choose the suitable parameters

$$\Xi := (\Xi_1, \Xi_2, \Xi_3) = ((\mathbf{b}, \mathbf{a}', \Lambda, \Gamma, A, D), (\mathbf{g}, \mathbf{s}), (\Psi_1, \Psi_2, \Phi_1, \Phi_2))$$

so that u_{int} and u_{ext} have the same Cauchy data on each $\partial T_{-b_i,a_i+\varepsilon a_i'}(B(a_i+\varepsilon a_i', r_{\varepsilon,1}))$ and $\partial T_{-b_i,a_i+\varepsilon a_i'}(B(a_i+\varepsilon a_i', r_{\varepsilon,2}))$ or equivalently, on each $\partial B(a_i+\varepsilon a_i', r_{\varepsilon,1})$ and $\partial B(a_i+\varepsilon a_i', r_{\varepsilon,2})$, $T_{-b_i,a_i+\varepsilon a_i'}(u_{int})$ and $T_{-b_i,a_i+\varepsilon a_i'}(u_{ext})$ have the same Cauchy data. Once this is done, the function defined by $u = u_{int}$ in $\Omega_{int,\varepsilon}$ and $u = u_{ext}$ in $\Omega_{ext,\varepsilon}$ will be $\mathcal{C}^1$ and solution of

(1) away from the $\partial\Omega_{int,\varepsilon} \cap \partial\Omega_{ext,\varepsilon}$. Elliptic regularity theory will then imply that it is a solution in Ω. Moreover, it will follow from the construction itself that u has the desired behavior near each a_i and this will complete the proof of Theorem 1.

Therefore, it remains to solve, for all $i = 1, \ldots, m$, the system

$$\left\{ \begin{array}{rcl} T_{-b_i, a_i+\varepsilon a'_i}(u_{int}) & = & T_{-b_i, a_i+\varepsilon a'_i}(u_{ext}), \\ \partial_n T_{-b_i, a_i+\varepsilon a'_i}(u_{int}) & = & \partial_n T_{-b_i, a_i+\varepsilon a'_i}(u_{ext}), \end{array} \right. \tag{90}$$

on $\partial B(a_i + \varepsilon a'_i, r_{\varepsilon,1}) \cup \partial B(a_i + \varepsilon a'_i, r_{\varepsilon,2})$.

We denote by Π_j the $L^2(S^{n-1})$-projection onto $\mathrm{Span}\{e_j\}$, and

$$\Pi(\phi) := \phi - \sum_{j=0}^{N} \Pi_j(\phi)$$

For all $i = 1, \ldots, m$, the $L^2(S^{n-1})$-projection of (90) over the orthogonal complement of $\mathrm{Span}\{e_0, \ldots, e_N\}$ yields the system of equations

$$\begin{array}{rcl} \varphi_{i,1} & = & \phi_{i,1} + \varepsilon^{\frac{N-2}{4k}} F_{i,1}(\Xi), \\ r_{\varepsilon,1}\, \partial_n V_{\varphi_{i,1}} & = & r_{\varepsilon,1}\, \partial_n W_{\phi_{i,1}} + \varepsilon^{\frac{N-2}{4k}} F_{i,2}(\Xi), \\ \varphi_{i,2} & = & \phi_{i,2} + \varepsilon^{\frac{N-2}{4k} + \frac{N-2}{k(N+2)} - \frac{N-2}{2}} F_{i,3}(\Xi), \\ r_{\varepsilon,2}\, \partial_n V_{\varphi_{i,2}} & = & r_{\varepsilon,2}\, \partial_n W_{\phi_{i,2}} + \varepsilon^{\frac{N-2}{4k} + \frac{N-2}{k(N+2)} - \frac{N-2}{2}} F_{i,4}(\Xi) \end{array} \tag{91}$$

Next, we make the expansion of $G_i(\frac{x-a_i}{\varepsilon}, a'_i)$ around $\partial T_{-b_i, a_i+\varepsilon a'_i}(B(a_i + \varepsilon a'_i, r_{\varepsilon,2}))$ for all $i = 1, \cdots, m$

$$\begin{array}{rl} & G_i(\dfrac{x-a_i}{\varepsilon}, a'_i) \\ = & 1 - \dfrac{\varepsilon^{N-2} H_i(a'_i, a'_i)}{|x - a_i - \varepsilon a'_i|^{N-2}} \\ & - \dfrac{\varepsilon^{N-1} \langle \nabla_1 H_i(a'_i, a'_i), x - a_i - \varepsilon a'_i \rangle}{|x - a_i - \varepsilon a'_i|^{N}} + \dfrac{\varepsilon^{N-2}}{|x - a_i - \varepsilon a'_i|^{N-2}} \mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}) \end{array} \tag{92}$$

so that around $\partial B(a_i + \varepsilon a'_i, r_{\varepsilon,2})$

$$\begin{array}{rl} & T_{-b_i, a_i+\varepsilon a'_i}(G_i(\dfrac{x-a_i}{\varepsilon}, a'_i)) \\ = & 1 - \dfrac{\varepsilon^{N-2} H_i(a'_i, a'_i)}{|x - a_i - \varepsilon a'_i|^{N-2}} \\ & - \dfrac{\varepsilon^{N-1} \langle \nabla_1 H_i(a'_i, a'_i), x - a_i - \varepsilon a'_i \rangle}{|x - a_i - \varepsilon a'_i|^{N}} + \dfrac{\varepsilon^{N-2}}{|x - a_i - \varepsilon a'_i|^{N-2}} \mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}) \end{array} \tag{93}$$

On the other hand, around $\partial B(a_i+\varepsilon a_i', r_{\varepsilon,1})$ for all $i=1,\cdots,m$, we have by the same arguments,

$$\begin{aligned} & T_{-b_i,a_i+\varepsilon a_i'}(G(x,a_i+\varepsilon a_i')) \\ = \;& -H(a_i,a_i)+|x-a_i-\varepsilon a_i'|^{2-N} \\ & +\langle (N-2)H(a_i,a_i)b_i-\nabla_1 H(a_i,a_i), x-a_i-\varepsilon a_i'\rangle+\mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}) \end{aligned} \tag{94}$$

and for all $j\neq i$

$$\begin{aligned} & T_{-b_i,a_i+\varepsilon a_i'}(G(x,a_j+\varepsilon a_j')) \\ = \;& G(a_i,a_j)-\langle (N-2)G(a_i,a_j)b_i-\nabla_1 G(a_i,a_j), x-a_i-\varepsilon a_i'\rangle+\mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}). \end{aligned} \tag{95}$$

Together with the expansions obtained in Propositions 6 to 8, we infer the $L^2(S^{n-1})$-projection of (90) over Span$\{e_0\}$

$$\begin{aligned} & D_i e^{(2-N)A_i/2}=H(a_i,a_i)\Lambda_i-\sum_{j\neq i}(-1)^{l_j-l_i}G(a_j,a_i)\Lambda_j+F_{i,5}(\Xi), \\ & D_i\, e^{(N-2)A_i/2}=\Lambda_i+\varepsilon^{\frac{N-2}{k(N+2)}}F_{i,6}(\Xi), \\ & \Gamma_i=D_i^{1-2k}e^{(2-N)A_i/2}e^{(N-2)k\,C_{N,\eta_i}/2}+\varepsilon^{\frac{N-2}{k(N+2)}}F_{i,7}(\Xi), \\ & \Gamma_i H_i(a_i',a_i')=D_i^{1+2k}e^{(N-2)A_i/2}e^{(2-N)k\,C_{N,\eta_i}/2}+F_{i,8}(\Xi), \end{aligned} \tag{96}$$

Finally, the $L^2(S^{n-1})$-projection of (90) over Span$\{e_1,\ldots,e_N\}$ yields

$$\begin{aligned} g_i &= \varepsilon^{\frac{N-2}{4k}+\frac{N-1}{k(N+2)}}\,F_{i,9}(\Xi), \\ b_i &= \frac{\nabla_1 H(a_i,a_i)\Lambda_i-\sum_{j\neq i}(-1)^{l_j-l_i}\nabla_1 G(a_j,a_i)\Lambda_j}{(N-2)(H(a_i,a_i)\Lambda_i-\sum_{j\neq i}(-1)^{l_j-l_i}G(a_j,a_i)\Lambda_j)} \\ &\quad +\varepsilon^{\frac{-1}{k(N+2)}}\,F_{i,10}(\Xi), \\ s_i &= \varepsilon^{\frac{N-2}{4k}-\frac{N}{2}+\frac{N-1}{k(N+2)}}\,F_{i,11}(\Xi), \\ \nabla_1 H_i(a_i',a_i')\Gamma_i &= \varepsilon^{\frac{-1}{k(N+2)}}\,F_{i,12}(\Xi) \end{aligned} \tag{97}$$

Here $F_{i,j}(\Xi)$ for $i=1,\ldots,m$ and $j=1,\ldots,12$ are continuous maps satisfying

$$|F_{i,l}(\Xi)|=\mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}). \tag{98}$$

We define "Dirichlet to Neumann map" for any

$$\mathcal{S}:\Pi(\mathcal{C}^{2,\alpha}(S^{N-1}))\longrightarrow \Pi(\mathcal{C}^{1,\alpha}(S^{N-1}))$$

by

$$\mathcal{S}(\psi)=r\,(\partial_n V_\psi-\partial_n W_\psi),$$

where V_ψ (resp. W_ψ) is the harmonic extension in the ball $B(0,r)$ (resp. in $\mathbb{R}^N - B(0,r)$) defined in Lemma 1 and Lemma 2. It is well known that $\mathcal{S}$ is an isomorphism [27] the norm of whose inverse does not depend on r.

Using the expansion (9), we can write (96) in the following form :

$$\begin{aligned}
&\Lambda_i^{-\frac{2k+1}{2k-1}} e^{\frac{(N-2)kC_N}{2k-1}} H_i(a_i', a_i')^{\frac{1}{2k-1}} \\
&\qquad = H(a_i,a_i)\Lambda_i - \sum_{j\neq i}(-1)^{l_j-l_i} G(a_j,a_i)\Lambda_j + \mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}), \\
&D_i\, e^{(N-2)A_i/2} = \Lambda_i + \mathcal{O}(\varepsilon^{\frac{N}{k(N+2)}}), \\
&\Gamma_i = \frac{D_i^{2-2k}\, e^{(N-2)k\,C_N/2}}{\Lambda_i} + \mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}), \\
&D_i^2 = \Lambda_i^{-\frac{2}{2k-1}} e^{\frac{(N-2)kC_N}{2k-1}} H_i(a_i', a_i')^{\frac{1}{2k-1}} + \mathcal{O}(\varepsilon^{\frac{2}{k(N+2)}}),
\end{aligned} \tag{99}$$

Recall a_i^* the critical point in (**A1**) and denote $\Lambda^* = (\Lambda_1^*, \cdots, \Lambda_m^*)$ the non-degenerate critical point in (**A2**). Set

$$\begin{aligned}
D_i^* &:= (\Lambda_i^*)^{-\frac{1}{2k-1}} e^{\frac{(N-2)kC_N}{2(2k-1)}} H_i(a_i^*, a_i^*)^{\frac{1}{2(2k-1)}}, \\
\Gamma_i^* &:= (\Lambda_i^*)^{-\frac{1}{2k-1}}\, e^{\frac{(N-2)k\,C_N}{2(2k-1)}}\, H_i(a_i^*, a_i^*)^{\frac{1-k}{2k-1}}, \\
A_i^* &:= \frac{2}{N-2}\log(\frac{\Lambda_i^*}{D_i^*}) = \frac{2}{N-2}\log\left((\Lambda_i^*)^{\frac{2k}{2k-1}} e^{-\frac{(N-2)kC_N}{2(2k-1)}} H_i(a_i^*, a_i^*)^{-\frac{1}{2(2k-1)}}\right), \\
b_i^* &:= \frac{\nabla_1 H(a_i,a_i)\Lambda_i^* - \sum_{j\neq i}(-1)^{l_j-l_i}\nabla_1 G(a_j,a_i)\Lambda_j^*}{(N-2)(H(a_i,a_i)\Lambda_i^* - \sum_{j\neq i}(-1)^{l_j-l_i} G(a_j,a_i)\Lambda_j^*)}
\end{aligned}$$

In view of (97), we have

$$\nabla_1 H_i(a_i', a_i') = \mathcal{O}(\varepsilon^{\frac{1}{k(N+2)}})$$

so that it follows from the assumption (**A1**)

$$a_i' - a_i^* = \mathcal{O}(\varepsilon^{\frac{1}{k(N+2)}})$$

and

$$H_i(a_i', a_i') = H_i(a_i^*, a_i^*) + \mathcal{O}(\varepsilon^{\frac{1}{k(N+2)}})$$

Hence, (91), (97) and (99) are equivalent to the following system

$$\begin{aligned}
&b_i = b_i^* + G_{i,1}(\Xi), \qquad a_i = a_i^* + G_{i,2}(\Xi), \\
&\Lambda_i = \Lambda_i^* + G_{i,3}(\Xi), \qquad \Gamma_i = \Gamma_i^* + G_{i,4}(\Xi), \\
&A_i = A_i^* + G_{i,5}(\Xi), \qquad D_i = D_i^* + G_{i,6}(\Xi), \\
&g_i = \varepsilon^{\frac{N-2}{4k}+\frac{N}{k(N+2)}} G_{i,7}(\Xi), \qquad s_i = \varepsilon^{\frac{N-2}{4k}-\frac{N}{2}+\frac{N}{k(N+2)}} G_{i,8}(\Xi), \\
&\varphi_{i,1} = \varepsilon^{\frac{N-2}{4k}+\frac{1}{k(N+2)}} G_{i,9}(\Xi), \qquad \phi_{i,1} = \varepsilon^{\frac{N-2}{4k}+\frac{1}{k(N+2)}} G_{i,10}(\Xi), \\
&\varphi_{i,2} = \varepsilon^{\frac{N-2}{4k}+\frac{N-1}{k(N+2)}-\frac{N-2}{2}} G_{i,11}(\Xi), \qquad \phi_{i,2} = \varepsilon^{\frac{N-2}{4k}+\frac{N-1}{k(N+2)}-\frac{N-2}{2}} G_{i,12}(\Xi)
\end{aligned} \tag{100}$$

where $G_{i,l}(\Xi)$ for all $l = 1, \dots, 12$ and for all $i = 1, \dots, m$ are continuous maps satisfying

$$|G_{i,l}(\Xi)| = \mathcal{O}(\varepsilon^{\frac{1}{k(N+2)}})$$

Moreover, elliptic regularity Theory shows that all $G_{i,l}(\Xi)$ are compact operators.

We set $\Xi_1^* = (\mathbf{b}^*, \mathbf{a}^*, \Lambda^*, \Gamma^*, A^*, D^*)$. We consider the set

$$\mathcal{A} = \overline{B(\Xi_1^*, r_0)} \times P_{1,\varepsilon} \times P_{2,\varepsilon} \times (\mathcal{E}_1)^{2m} \times (\mathcal{E}_2)^{2m}$$

where r_0 is some fixed small positive number and $B(\Xi_1^*, r_0) \subset \mathbb{R}^{(2N+4)m}$. Now set $\Theta = |\Xi_1^*| + r_0 + 2\sum_i \frac{1}{D_i^*} + \frac{1}{\Lambda_i^*}$ and we can write formally the system (100) as

$$\Xi = \Upsilon(\Xi),$$

It follows from the above analysis that $\Upsilon : \mathcal{A} \to \mathcal{A}$ is a continuous compact map. According to Schauder fixed point theorem, Υ has a fixed point in $\mathcal{A}$. This completes the proof of Theorem 1.

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References

[1] A. Bahri, J.M. Coron: *On a nonlinear elliptic equation involving the critical Sobolev exponent: the effect of the topology of the domain*, Comm. Pure Appl. Math. **41** (1988), no. 3, 253–294.

[2] A.Bahri, Y. Li, O. Rey: *On a variational problem with lack of compactness: The topological effect of the critical points at infinity*, Calc. Var. Partial Differ. Equ. 3 no.1 (1995), 67-93.

[3] T. Bartsch, T. Weth: *A note on additional properties of sign changing solutions to superlinear elliptic equations*, Top. Meth. Nonlin. Anal. **22** (2003), 1–14.

[4] T. Bartsch, A.M. Micheletti, A. Pistoia: *On the existence and the profile of nodal solutions of elliptic equations involving critical growth*, Calc. Var. Partial Differential Equations **26** no. 3 (2006), 265–282.

[5] H. Brezis, L. Nirenberg: *Positive solutions of non linear elliptic equations involving critical Sobolev expronents*, Comm. Pure Appl. Math. **36** (1983), 437-477.

[6] A. Byde: *Gluing theorems for constant scalar curvature manifolds*, Indiana Univ. Math. J. **52** (2003), 1147–1199.

[7] M. Clapp, T. Weth: *Minimal nodal solutions of the pure critical exponent problem on a symmetric domain*, Calc. Var. Partial Differential Equations **21** no. 1 (2004), 1–14.

[8] E. A. Coddington and N. Levinson, *Theory of ordinary differential equations*, McGill-Hill, New York (1955).

[9] J.M. Coron: *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sci. Paris Sr. I Math. 299 (1984), no. 7, 209–212.

[10] M. del Pino, J. Dolbeault, M. Musso: *"Bubble-tower" radial solutions in the slightly supercritical Brezis-Nirenberg problem*, J. Differ. equation. 193 no.2, (2003), 280-306.

[11] M. del Pino, J. Dolbeault, M. Musso: *The Brezis-Nirenberg problem near criticality in dimension 3*, J. Math. Pures Appl. (9) **83** no. 12 (2004), 1405–1456.

[12] M. del Pino, M. Musso, A. Pistoia: *Super-critical boundary bubbling in a semilinear Neumann problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **22** no. 1 (2005), 45–82.

[13] M. del Pino, P. Felmer, M. Musso: *Multi-peak solutions for super-critical elliptic problems in domains with small holes*, J. Differ. Equation. 182, No.2, (2002), 511-540.

[14] M. del Pino, P. Felmer, M. Musso: *Multi-bubble solutions for slightly super-critical elliptic problems in domains with symmetries*, Bull. London Math. Soc. 35no. 4 (2003), 513-521.

[15] Y. Ge, R. Jing, F. Pacard: *Bubble towers for supercritical semilinear elliptic equations*, J. Funct. Anal. **221** no. 2 (2005), 251–302.

[16] Y. Ge, R. Jing, F. Zhou: *Bubble tower solutions of slightly supercritical elliptic equations and application in symmetric domains*, DCDS-A 17 (2007), 751-770.

[17] Y. Ge, M. Musso, A. Pistoia: *Sign changing tower of bubbles for an elliptic problem at the critical exponent in pierced non-symmetric domains*, accepted in Comm. Partial Differential Equations.

[18] E. Hebey, M. Vaugon: *Existence and multiplicity of nodal solutions for nonlinear elliptic equations with critical Sobolev growth*, J. Funct. Anal. **119** (1994), 298–318.

[19] J. Kazdan, F.W. Warner:*Remarks on some quasilinear elliptic equations*, Comm. Pure Appl. Math. **28** (1975), 567–597.

[20] Y. Li: *Prescribing scalar curvature on S^n and related problems. I.*, J. Differ. Equations 120, No.2 (1995), 319-410.

[21] R. Mazzeo, F. Pacard: *Constant scalar curvature metrics with isolated singularities*, J. Duke Math 99 no. 3 (1999), 353-418.

[22] A.M. Micheletti, A. Pistoia: *On the effect of the domain geometry on the existence of sign changing solutions to elliptic problems with critical and supercritical growth*, Nonlinearity **17** no. 3 (2004), 851–866.

[23] M. Musso, A. Pistoia: *Sign changing solutions to a nonlinear elliptic problem involving the critical Sobolev exponent in pierced domains*, J. Math. Pures Appl. (9) **86** no. 6 (2006), 510–528.

[24] M. Musso, A. Pistoia: *Sign changing solutions to a Bahri-Coron's problem in pierced domains*, Discrete Contin. Dyn. Syst. **21** (2008), no. 1, 295–306.

[25] M. Musso, A. Pistoia: *Persistence of Coron's solution in nearly critical problems*, Ann. Sc. Norm. Super. Pisa Cl. Sci. **6** (2007), no. 2, 331–357.

[26] M. Musso, A. Pistoia: *Tower of Bubbles for almost critical problems in general domains*, to apper in J. Math. Pure Appl.

[27] F. Pacard, T. Rivière: *Linear and nonlinear aspects of vortices. The Ginzburg-Landau model* Progress in Nonlinear Differential Equations and their Applications. 39. Boston, Birkhäuser, (2000).

[28] A. Pistoia, T. Weth: *Sign changing bubble tower solutions in a slightly subcritical semilinear Dirichlet problem*, Ann. Inst. H. Poincaré Anal. Non Linéaire **24** no. 2 (2007), 325–340.

[29] S.I. Pohozaev: *Eigenfunction of the equation* $\Delta u + \lambda f(u) = 0$, Dokl. Akad. Nauk SSSR 165 (1965), 36-39.

[30] O. Rey: *On a variational problem with lack of compactness: the effect of small holes in the domain*, C. R. Acad. Sci. Paris Sér. I Math. 308 (1989), no. 12, 349–352.

[31] O. Rey: *The role of the Green function in a nonlinear elliptic equation involving the critical Sobolev exponetion*, J. Funct. Anal. 89 (1990), 1-52.

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