SIGN CHANGING TOWER OF BUBBLES FOR AN ELLIPTIC PROBLEM AT THE CRITICAL EXPONENT IN PIERCED NON-SYMMETRIC DOMAINS

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ABSTRACT. We consider the problem $\Delta u+|u|^{\frac{4}{N-2}}u=0$ in Ω_ε, $u=0$ on $\partial\Omega_\varepsilon$, where $\Omega_\varepsilon:=\Omega\setminus\{B(a,\varepsilon)\bigcup B(b,\varepsilon)\}$, with Ω a bounded smooth domain in $\mathbb{R}^N$, $N\geq 3$, $a\neq b$ two points in Ω, and ε is a positive small parameter. As ε goes to zero, we construct sign changing solutions with multiple blow up both at a and at b.

Keywords: critical Sobolev exponent, blowing-up solution, tower of bubbles, Robin's function.

AMS subject classification: 35J20, 35J60.

1. Introduction

Let D be a smooth bounded domain in $\mathbb{R}^N$, $N\geq 3$. Consider the following nonlinear elliptic problem

$$\Delta u+|u|^{\frac{4}{N-2}}u=0 \quad \text{in} \quad D, \quad u=0 \quad \text{on} \quad \partial D. \tag{1.1}$$

It is well know that the Sobolev embedding $H_0^1(D)\hookrightarrow L^{\frac{2N}{N-2}}(D)$ is not compact and for this reason solvability of (1.1) is a quite delicate issue. Pohozaev's identity [33] shows that problem (1.1) has only the trivial solution if the domain D is assumed to be strictly starshaped. On the other hand, if D is an annulus then (1.1) has a (unique) positive solution in the class of functions with radial symmetry [22]. In the nonsymmetric case, Coron [13] found via variational methods that (1.1) is solvable under the assumption that D is a domain exhibiting a small hole. Substantial improvement of this result was obtained by Bahri and Coron [4], showing that if some homology group of D with coefficients in $\mathbf{Z}_2$ is not trivial, then (1.1) has at least one positive solution (see also [3, 8, 10, 19, 28, 25, 35] for related results). If the domain D has several holes, then a multiplicity result for positive solutions to (1.1) is obtained in [34]. On the other hand, in [12] the authors found a second solution in Coron's setting (one small hole), but they were unable to say if the second solution was positive or changed sign. Existence and qualitative behavior of sign changing solutions for elliptic problems with critical nonlinearity have been investigated by several authors in the last years (see [5, 6, 9, 11, 20, 21, 26, 27]). A large number of sign changing solutions to (1.1) in the presence of a single hole has been proved in [29].

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More precisely, in [29] the authors assume that $D = \Omega \setminus B(0,\varepsilon)$, where Ω is a bounded domain, which contains the origin and is symmetric with respect to the origin. They prove the existence of an arbitrary number of sign changing solutions for (1.1), if the radius ε of the removed ball is small enough. The shape of such solution is a superposition of blowing up bubbles with alternate sign concentrating around the center 0 of the removed ball $B(0,\varepsilon)$.
A bubble is a function defined in $\mathbb{R}^N$ of the form

$$U_{\mu,\xi}(x) = \alpha_N \left(\frac{\mu}{\mu^2 + |x-\xi|^2} \right)^{\frac{N-2}{2}} \tag{1.2}$$

where $\alpha_N := [N(N-2)]^{\frac{N-2}{4}}$, μ is any positive parameter and ξ a point in $\mathbb{R}^N$. These functions are all and the only positive bounded solutions of problem (1.1) in the whole space $\mathbb{R}^N$ [1, 36].
The result in [29], as well as in other related problems where construction of tower of bubbles is obtained [14, 15, 16, 32], rely strongly on the assumption of symmetry of the domain. On the other hand, even if delicate, removing the symmetry assumption can be done. The first contribution in this direction is due to [17], where the authors generalize the construction of tower of bubbles for the slightly super critical Brezis-Nirenberg problem obtained in [14] for a general non symmetric domain. They obtained this result under a further non degeneracy condition: if ξ_0 is a non degenerate critical point of the Robin's function of the domain it is possible to construct a tower of bubbles concentrating at ξ_0. Even if generic, this non degeneracy assumption is hard to check: the only result about that is contained in [18], where the author shows that the origin is a non degenerate critical point of the Robin's function if the domain is convex and axially symmetric with respect to the origin. Let us mention that recently in [31] the authors drop both the assumptions of symmetry of the domain and of non degeneracy of the Robin's function. The proof in [17] uses a gluing technique developed in [23] in some other context. The proof in [31] is based on the use of a Liapunov-Schmidt reduction.

The aim of the present work is to remove the assumption of symmetry on the pierced domain Ω. Let us be more precise.
Let Ω be a bounded domain with smooth boundary and a be a given point in Ω. Given a parameter $\varepsilon > 0$ small, we remove from Ω the ball centered at a with radius $r_a\varepsilon$. Here r_a is a positive fixed number. We are interested in constructing solutions with the shape of a tower of bubbles around the removed ball for the problem at the critical exponent

$$\begin{cases} \Delta u + |u|^{\frac{4}{N-2}} u = 0 & \text{in } \Omega \setminus B(a, r_a\varepsilon), \\ u = 0 & \text{on } \partial\left(\Omega \setminus B(a, r_a\varepsilon)\right). \end{cases} \tag{1.3}$$

The result we prove is the following

Theorem 1.1. *For any integer $k \geq 1$, there exists $\varepsilon_k > 0$ such that for any $\varepsilon \in (0, \varepsilon_k)$ there exists a pair of solutions u_ε and $-u_\varepsilon$ to problem (1.3) such that*

$$u_\varepsilon(x) = \alpha_N \sum_{i=1}^{k} (-1)^{i+1} \left(\frac{M_i \varepsilon^{\frac{2i-1}{2k}}}{M_i^2 \varepsilon^{2\frac{2i-1}{2k}} + |x-a|^2} \right)^{\frac{N-2}{2}} + \Theta_\varepsilon(x),$$

where $M_1, \dots, M_k$ are positive constants depending only on N and k and $\|\Theta_\varepsilon\|_{\mathrm{H}^1_0(\Omega \setminus B(a, r_a\varepsilon))} \to 0$ as $\varepsilon \to 0$.

The second result we get reads as follows. Let a, b be two given points in Ω with $a \neq b$. Given a parameter $\varepsilon > 0$ small, we remove from Ω two balls of centers a and b and radius respectively $r_a\varepsilon$ and $r_b\varepsilon$. Here r_a and r_b are two positive fixed numbers. We construct solutions with the shape of two towers of bubbles around the removed balls for the problem at the critical exponent

$$\begin{cases} \Delta u + |u|^{\frac{4}{N-2}}\, u = 0 & \text{in } \Omega \setminus \{\, B(a, r_a\varepsilon) \cup B(b, r_b\varepsilon) \,\}, \\ u = 0 & \text{on } \partial \left(\Omega \setminus \{\, B(a, r_a\varepsilon) \cup B(b, r_b\varepsilon) \,\} \right). \end{cases} \tag{1.4}$$

The result we prove is the following

Theorem 1.2. *For any integer $k \geq 1$, there exists $\varepsilon_k > 0$ such that for any $\varepsilon \in (0, \varepsilon_k)$ there exists a pair of solutions u_ε and $-u_\varepsilon$ to problem (1.4) such that*

$$u_\varepsilon(x) = \alpha_N \left[\sum_{i=1}^{k} (-1)^{i+1} \left(\frac{M_i \varepsilon^{\frac{2i-1}{2k}}}{M_i^2 \varepsilon^{2\frac{2i-1}{2k}} + |x-a|^2} \right)^{\frac{N-2}{2}} - \sum_{i=1}^{k} (-1)^{i+1} \left(\frac{N_i \varepsilon^{\frac{2i-1}{2k}}}{N_i^2 \varepsilon^{2\frac{2i-1}{2k}} + |x-b|^2} \right)^{\frac{N-2}{2}} \right] + \Theta_\varepsilon(x),$$

where $M_1, \dots, M_k$, $N_1, \dots, N_k$ are positive constants depending only on N and k and $\|\Theta_\varepsilon\|_{\mathrm{H}_0^1(\Omega \setminus \{B(a, r_a\varepsilon) \cup B(b, r_b\varepsilon)\})} \to 0$ as $\varepsilon \to 0$.

Observe that in the above construction, the first elements in the two towers have opposite sign. On the other hand, in case that the two towers are build upon bubbles of the same sign, an extra condition on the position of the centers a and b of the holes is required. This condition is on the sign of a certain combination of the Green function of Ω and its regular part. We thus need to recall their definitions. We denote by $G(x, y)$ the Green function of the Laplace operator in Ω with zero Dirichlet boundary condition and we denote by $H(x, y)$ its regular part, namely

$$G(x, y) = \gamma_N \left(\frac{1}{|x-y|^{N-2}} - H(x, y) \right), \tag{1.5}$$

with $\gamma_N := \frac{1}{(N-2)|\partial B|}$, where $|\partial B|$ denotes the surface area of the unit sphere in $\mathbb{R}^N$. Thus for all $y \in \Omega$, $H(x, y)$ satisfies

$$-\Delta H(x, y) = 0 \quad \text{in} \quad \Omega, \quad H(x, y) = \frac{1}{|x-y|^{N-2}} \quad x \in \partial\Omega. \tag{1.6}$$

The Robin's function is defined as $H(x, x)$, $x \in \Omega$.

Our last result is the following

Theorem 1.3. *Assume*

$$H^{1/2}(a, a) H^{1/2}(b, b) - G(a, b) > 0. \tag{1.7}$$

For any integer $k \geq 1$, there exists $\varepsilon_k > 0$ such that for any $\varepsilon \in (0, \varepsilon_k)$ there exists a pair of solutions u_ε and $-u_\varepsilon$ to problem (1.4) such that

$$u_\varepsilon(x) = \alpha_N \left[\sum_{i=1}^{k} (-1)^{i+1} \left(\frac{A_i \varepsilon^{\frac{2i-1}{2k}}}{A_i^2 \varepsilon^{2\frac{2i-1}{2k}} + |x-a|^2} \right)^{\frac{N-2}{2}} \right.$$
$$\left. + \sum_{i=1}^{k} (-1)^{i+1} \left(\frac{B_i \varepsilon^{\frac{2i-1}{2k}}}{B_i^2 \varepsilon^{2\frac{2i-1}{2k}} + |x-b|^2} \right)^{\frac{N-2}{2}} \right] 1 + \Theta_\varepsilon(x),$$

where $A_1, \dots, A_k,\ B_1, \dots, B_k$ are positive constants depending only on N and k and $\|\Theta_\varepsilon\|_{\mathrm{H}_0^1(\Omega\setminus\{B(a, r_a\varepsilon)\cup B(b, r_b\varepsilon)\})} \to 0$ as $\varepsilon \to 0$.

Theorem 1.2 and Theorem 1.3 extend the results obtained in [34] and in [30] in the case of two holes when $k = 1$: our results claim that on top of solutions found in [34] and [30] one can put two towers of sign changing bubbles.
Let us mention that natural extensions of the results obtained in Theorems 1.2 and 1.3 can be obtained in the case of several holes removed.
We will prove our results with the aim of a Liapunov-Schmidt reduction, which we describe, together with the scheme of the proof, in Section 2.

2. Proof of Theorem 1.2, Theorem 1.3 and Theorem 1.1

We will describe the steps of the proof of Theorem 1.2. The proof of Theorem 1.3 and Theorem 1.1 can be carried out in a similar way.
For any $\varepsilon > 0$ fixed, set $\Omega_\varepsilon := \Omega \setminus \{ B(a, r_a\varepsilon) \cup B(b, r_b\varepsilon) \}$. Let $H_0^1(\Omega_\varepsilon)$ be the usual Sobolev space equipped with the scalar product $< u, v >= \int\limits_{\Omega_\varepsilon} \nabla u \nabla b$, which induces the norm $\|u\| = (\int_{\Omega_\varepsilon} |\nabla u|^2\, dx)^{\frac{1}{2}}$. Let $L^q(\Omega_\varepsilon)$ be the space equipped with the norm $|u|_q = (\int_{\Omega_\varepsilon} |u|^q\, dx)^{\frac{1}{q}}$. By Sobolev Embedding Theorem we have the existence of a positive constant S, depending only on N, such that $|u|_{\frac{2N}{N-2}} \leq S\|u\| \quad$ for all $u \in H_0^1(\Omega_\varepsilon)$. Consider now the adjoint operator of the above embedding $i : H_0^1(\Omega_\varepsilon) \hookrightarrow L^{\frac{2N}{N-2}}(\Omega_\varepsilon)$, namely the map $i^* : L^{\frac{2N}{N+2}}(\Omega_\varepsilon) \to H_0^1(\Omega_\varepsilon)$ defined as follows: if $w \in L^{\frac{2N}{N+2}}(\Omega_\varepsilon)$ then $u = i^*(w)$ in $H_0^1(\Omega_\varepsilon)$ is the unique solution of the equation $-\Delta u = w$ in Ω_ε, $u = 0$ on $\partial\Omega_\varepsilon$. We have the existence of a positive constant c, which depends only on the dimension N, such that

$$\|i^*(w)\| \leq c\,|w|_{\frac{2N}{N+2}} \qquad \text{for all } u \in L^{\frac{2N}{N+2}}(\Omega_\varepsilon). \tag{2.1}$$

Using the above definitions and notations, problem (1.4) can be re-written as follows

$$u = i^*[f(u)], \quad u \in H_0^1(\Omega_\varepsilon), \tag{2.2}$$

where $f(u) := |u|^{p-1}u$ and $p = \frac{N+2}{N-2}$.
We next describe the shape of the solutions we are looking for. We start with the definition of the two towers, centered respectively around a and b. We define

$$V_a(x) = \sum_{j=1}^{k} (-1)^{j+1} u_{ja}(x), \quad V_b(x) = \sum_{j=1}^{k} (-1)^{j+1} u_{jb}(x) \tag{2.3}$$

where

$$u_{ja}(x) = P_\varepsilon U_{\mu_{j\varepsilon}, a_{j\varepsilon}}(x), \quad u_{jb}(x) = P_\varepsilon U_{\delta_{j\varepsilon}, b_{j\varepsilon}}(x) \tag{2.4}$$

In (2.4) P_ε denotes the projection onto $H_0^1(\Omega_\varepsilon)$, namely for a given function defined on all $\mathbb{R}^N$, $P_\varepsilon u$ is the unique solution in of the problem $\Delta P_\varepsilon u = \Delta u$ in Ω_ε and $P_\varepsilon u = 0$ on $\partial\Omega_\varepsilon$. Furthermore, in (2.4) we assume that

$$\mu_{j\varepsilon} = \varepsilon^{\frac{2j-1}{2k}}\mu_j \quad \text{and} \quad \delta_{j\varepsilon} = \varepsilon^{\frac{2j-1}{2k}}\delta_j \tag{2.5}$$

for some positive numbers μ_j and δ_j, and

$$a_{j\varepsilon} = a + \mu_{j\varepsilon}\tau_j \quad \text{and} \quad b_{j\varepsilon} = b + \delta_{j\varepsilon}\sigma_j \tag{2.6}$$

for some points τ_j and σ_j in $\mathbb{R}^N$. We will assume the following bounds on the parameters and points appearing in (2.5) and (2.6): given $\delta > 0$ small

$$\eta < \mu_j,\, \delta_j < \eta^{-1}, \quad |\tau_j|, |\sigma_j| < \eta \quad \text{for all } j = 1, \dots, k. \tag{2.7}$$

To refer to the parameters above, we will use the compact notation

$$\begin{aligned} &\bar\tau = (\tau_1, \dots, \tau_k), \quad \bar\sigma = (\sigma_1, \dots, \sigma_k) \in \mathbb{R}^{Nk}, \text{ and} \\ &\bar\mu = (\mu_1, \dots, \mu_k), \quad \bar\delta = (\delta_1, \dots, \delta_k) \in \mathbb{R}^k_+. \end{aligned} \tag{2.8}$$

The solution predicted by Theorem 1.2 has the form

$$u(x) = V(x) + \phi(x), \quad \text{where} \quad V(x) = V_a(x) - V_b(x). \tag{2.9}$$

Here the term ϕ has to be thought as a smaller perturbation of V.

We next describe the term ϕ in (2.9). To do so, let us recall (see [7]) that, for all $\delta > 0$ and $\zeta \in \mathbb{R}^N$, every bounded solution to the linear equation

$$-\Delta\psi = f'\left(U_{\delta,\zeta}\right)\psi \text{ in } \mathbb{R}^N$$

is a linear combination of the functions

$$Z^j_{\delta,\zeta}(x) := \partial_{\zeta_j} U_{\delta,\zeta}(x) = \alpha_N (N-2)\delta^{\frac{N-2}{2}} \frac{x_j - \zeta_j}{(\delta^2 + |x-\zeta|^2)^{N/2}},\ j = 1, \dots, N$$

and

$$Z^0_{\delta,\zeta}(x) := \partial_\delta U_{\delta,\zeta}(x) = \alpha_N \frac{N-2}{2}\delta^{\frac{N-4}{2}} \frac{|x-\zeta|^2 - \delta^2}{(\delta^2 + |x-\zeta|^2)^{N/2}}.$$

We define the subspace of $H_0^1(\Omega_\varepsilon)$

$$K := \text{span}\left\{P_\varepsilon Z^h_{\mu_{j\varepsilon}, a_{j\varepsilon}}\ , P_\varepsilon Z^h_{\delta_{j\varepsilon}, b_{j\varepsilon}} :\ h = 0, 1, \dots, N,\ j = 1, \dots, k\right\},$$

where P_ε is the projection onto $H_0^1(\Omega_\varepsilon)$ as defined before, and

$$K^\perp := \left\{\phi \in H_0^1(\Omega_\varepsilon)\ :\ \left\langle \phi, P_\varepsilon Z^h_{\mu_{j\varepsilon}, a_{j\varepsilon}}\right\rangle = \left\langle \phi, P_\varepsilon Z^h_{\delta_{j\varepsilon}, b_{j\varepsilon}}\right\rangle = 0, \right.$$
$$\left. h = 0, 1, \dots, N,\ j = 1, \dots, k\right\}.$$

Let $\Pi : H_0^1(\Omega_\varepsilon) \to K$ and $\Pi^\perp : H_0^1(\Omega_\varepsilon) \to K^\perp$ be the orthogonal projections.

In order to solve problem (1.4) we will solve the couple of equations

$$\Pi^\perp\left\{V + \phi - i^*\left[f\left(V + \phi\right)\right]\right\} = 0 \tag{2.10}$$

$$\Pi\left\{V + \phi - i^*\left[f\left(V + \phi\right)\right]\right\} = 0. \tag{2.11}$$

Given $\bar\tau$, $\bar\sigma$, $\bar\mu$ and $\bar\delta$ (see (2.8)) whose components satisfy conditions (2.7), one can solve uniquely equation (2.10) in $\phi \in K^\perp$. This solution ϕ is the *lower order term* in the description of the ansatz (2.9). This is the content of

Proposition 2.1. *For any $\eta > 0$, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\bar\tau, \bar\sigma \in \mathbb{R}^{Nk}$, for any $\bar\mu, \bar\sigma \in \mathbb{R}^k_+$, satisfying (2.7) and for any $\varepsilon \in (0, \varepsilon_0)$ there exists a unique $\phi = \phi(\bar\tau, \bar\sigma, \bar\mu, \bar\delta) \in K^\perp$ which solves equation (2.10). Moreover*

$$\|\phi\| \le \begin{cases} c\varepsilon^{\frac{N-2}{2k}\frac{p}{2}} \text{ if } N \ge 7, \\ c\varepsilon^{\frac{N-2}{2k}} |\ln \varepsilon| \text{ if } N = 6, \\ c\varepsilon^{\frac{N-2}{2k}} \text{ if } 3 \le N \le 5. \end{cases} \tag{2.12}$$

Finally, $(\bar\tau, \bar\sigma, \bar\mu, \bar\delta) \to \phi(\bar\tau, \bar\sigma, \bar\mu, \bar\delta)$ is a C^1-map.

Roughly speaking, the solution ϕ to (2.10) is found with a fixed point argument, which works thanks to two fundamental ingredients: the existence and estimates of the inverse of the linear operator obtained linearizing problem (1.4) around V in the space $K^\perp$ (see Section 5) and the study of the error term

$$R := \Pi^\perp \{i^*[f(V)] - V\}. \tag{2.13}$$

This last estimate is carried out in Section 6.

We are left now to solve equation (2.11), more precisely to find points $\bar\tau$, $\bar\sigma$ in $\mathbb{R}^{Nk}$, and parameters $\bar\mu$, $\bar\sigma$ in $\mathbb{R}^k_+$ so that (2.11) is satisfied. It happens that this problem has a variational structure, in the sense that solving (2.11) reduces to find critical points to some given explicit finite dimensional functional. Let us introduce the energy associated to problem (1.4)

$$J_\varepsilon(u) = \frac{1}{2}\int_{\Omega_\varepsilon} |\nabla u|^2\, dx - \frac{1}{p+1}\int_{\Omega_\varepsilon} |u|^{p+1}. \tag{2.14}$$

Furthermore, we define the function $\widetilde J_\varepsilon : \mathbb{R}^{kN} \times \mathbb{R}^{kN} \times \mathbb{R}^k_+ \times \mathbb{R}^k_+ \to \mathbb{R}$ by

$$\widetilde J_\varepsilon(\bar\tau, \bar\sigma, \bar\mu, \bar\delta) := J_\varepsilon(V + \phi). \tag{2.15}$$

Next result contains two fundamental statements to conclude the proof of our Theorem 1.2. First it states that solving equation (2.11) is equivalent to finding critical points $(\bar\tau_\varepsilon, \bar\sigma_\varepsilon, \bar\mu_\varepsilon, \bar\delta_\varepsilon)$ of the finite dimensional function defined in (2.15). Second it computes the asymptotic expansion, as $\varepsilon \to 0$, of the function $\widetilde J_\varepsilon(\bar\tau, \bar\sigma, \bar\mu, \bar\delta)$, for points and parameters satisfying (2.7). More precisely, in the above region the function $\widetilde J_\varepsilon(\bar\tau, \bar\sigma, \bar\mu, \bar\delta)$ is uniformly close, together with its derivatives, to $J_\varepsilon(V)$. The proof of these facts are contained in Section 7. Furthermore, we can expand explicitly $J_\varepsilon(V)$ and prove that it is closed in a C^1 sense to a constant plus an function $\Psi(\bar\tau, \bar\sigma, \bar\mu, \bar\delta)\varepsilon^{\frac{N-2}{2k}}$ plus a lower order term $o(\varepsilon^{\frac{N-2}{2k}})$. This fact is proved in section 3.
In the whole paper we will use the notation $O(1)$ or $o(1)$ to denote a continuous function of the parameters μ_j, δ_j, τ_j and σ_j, which is bounded or approaching to zero as ε goes to zero uniformly in the range described by constraint (2.7).

Proposition 2.2. *The following facts hold.*

Part 1. *If $(\bar\tau_\varepsilon, \bar\sigma_\varepsilon, \bar\mu_\varepsilon, \bar\delta_\varepsilon)$ is a critical point of $\widetilde J_\varepsilon$, then the function $V+\phi$ is a solution to problem (1.4).*

Part 2. *For any $\eta > 0$, there exists $\varepsilon_0 > 0$ such that for any $\varepsilon \in (0, \varepsilon_0)$ it holds*

$$\widetilde J_\varepsilon(\bar\tau, \bar\sigma, \bar\mu, \bar\delta) = 2c_1\frac{\alpha_N^{p+1}}{N}k + \frac{\alpha_N^{p+1}}{2}\Psi(\bar\tau, \bar\sigma, \bar\mu, \bar\delta)\varepsilon^{\frac{N-2}{2k}}(1 + o(1)), \tag{2.16}$$

C^1-uniformly with respect to points and parameters $\bar\tau, \bar\sigma, \bar\mu, \bar\delta$ satisfying (2.7). The functions Ψ is defined as follows

$$\Psi(\bar\tau,\bar\sigma,\bar\mu,\bar\delta) = c_2\left[H(a,a)\mu_1^{N-2} + H(b,b)\delta_1^{N-2} + 2G(a,b)\mu_1^{\frac{N-2}{2}}\delta_1^{\frac{N-2}{2}}\right] + \frac{\Gamma(\tau_k)}{(1+|\tau_k|^2)^{\frac{N-2}{2}}}\frac{r_a^{N-2}}{\mu_k^{N-2}} + \frac{\Gamma(\sigma_k)}{(1+|\sigma_k|^2)^{\frac{N-2}{2}}}\frac{r_b^{N-2}}{\delta_k^{N-2}} + 2\sum_{j=1}^{k-1}\left[\Gamma(\tau_j)\left(\frac{\mu_{j+1}}{\mu_j}\right)^{\frac{N-2}{2}} + \Gamma(\sigma_j)\left(\frac{\delta_{j+1}}{\delta_j}\right)^{\frac{N-2}{2}}\right]. \tag{2.17}$$

Here

$$c_1 = \int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}\,dz, \quad c_2 = \int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\,dz. \tag{2.18}$$

and $F : \mathbb{R}^N \to \mathbb{R}$ is the smooth function defined by

$$\Gamma(x) := \int_{\mathbb{R}^N}\frac{1}{(1+|y-x|^2)^{\frac{N+2}{2}}}\frac{1}{|y|^{N-2}}dy, \quad x\in\mathbb{R}^N. \tag{2.19}$$

We have now all the tools to give the

Proof of Theorem 1.2. In virtue of (i) of Proposition 4.1 there exists a nondegenerate critical point $(0,0,\bar\mu_0,\bar\delta_0)$ of the function Ψ introduced in (2.17), which is stable with respect to C^1-perturbation. Therefore, taking into account the expansion (2.16) in Proposition 2.2, Part 2, we deduce that if ε is small enough the function $\widetilde{J}_\varepsilon$ (see (2.15)) has a critical point $(\bar\tau_\varepsilon,\bar\sigma_\varepsilon,\bar\mu_\varepsilon,\bar\delta_\varepsilon)$ such that $\bar\tau_\varepsilon,\bar\sigma_\varepsilon\to 0$, $\bar\mu_\varepsilon\to\bar\mu_0$ and $\bar\delta_\varepsilon\to\bar\delta_0$ as ε goes to 0. Finally, from Proposition 2.2, Part 1, and from formula (**??**), it follows that $V+\phi$, where V is defined in (2.9) and ϕ is the function whose existence is guaranteed by Proposition 2.1, is the solution predicted by Theorem 1.2. □

Proof of Theorem 1.3. We look for a solution to (1.4) of the form $u(x) = W(x) + \phi(x)$ where $W(x) = V_a(x) + V_b(x)$ (instead of $V_a(x) - V_b(x)$). Here V_a, V_b are defined as in (2.3) and satisfy (2.5), (2.6), (2.7). The rest term ϕ is a lower order term which is constructed exactly as in Proposition 2.1. Arguing as in the proof of Theorem 1.2 we are lead to find a critical point of the reduced energy, whose expansion is given in (2.16) where in this case the function $\Psi = \Psi^*$ becomes

$$\Psi^*(\bar\tau,\bar\sigma,\bar\mu,\bar\delta) = c_2\left[H(a,a)\mu_1^{N-2} + H(b,b)\delta_1^{N-2} - 2G(a,b)\mu_1^{\frac{N-2}{2}}\delta_1^{\frac{N-2}{2}}\right] + \frac{\Gamma(\tau_k)}{(1+|\tau_k|^2)^{\frac{N-2}{2}}}\frac{r_a^{N-2}}{\mu_k^{N-2}} + \frac{\Gamma(\sigma_k)}{(1+|\sigma_k|^2)^{\frac{N-2}{2}}}\frac{r_b^{N-2}}{\delta_k^{N-2}} + 2\sum_{j=1}^{k-1}\left[\Gamma(\tau_j)\left(\frac{\mu_{j+1}}{\mu_j}\right)^{\frac{N-2}{2}} + \Gamma(\sigma_j)\left(\frac{\delta_{j+1}}{\delta_j}\right)^{\frac{N-2}{2}}\right]. \tag{2.20}$$

Let us point out that in this case the interaction between the first two bubbles of the towers is negative and is given by $-2G(a,b)$, while in the case of Theorem 1.2 it is positive and is given by $+2G(a,b)$. Finally, using (ii) of Proposition 4.1, the proof follows the same argument of the proof of Theorem 1.2. □

Proof of Theorem 1.1. We look for a solution to (1.3) of the form $u(x) = V_a(x) + \phi(x)$, where V_a is defined as in (2.3) and satisfy (2.5), (2.6), (2.7). The rest term ϕ is a lower order term which is constructed exactly as in Proposition 2.1. Arguing as in the proof of Theorem 1.2 we are lead to find a critical point of the reduced energy, whose expansion is given in (2.16) where in this case the function Ψ reduces to

$$\Psi(\bar\tau,\bar\mu) = c_2 H(a,a)\mu_1^{N-2} + \frac{\Gamma(\tau_k)}{(1+|\tau_k|^2)^{\frac{N-2}{2}}}\frac{r_a^{N-2}}{\mu_k^{N-2}} + 2\sum_{j=1}^{k-1}\Gamma(\tau_j)\left(\frac{\mu_{j+1}}{\mu_j}\right)^{\frac{N-2}{2}}.$$

Arguing as in Proposition 4.1, we can prove that ϕ has a non degenerate critical point $(0,\bar\mu_0)$. Finally, the proof follows the same argument of the proof of Theorem 1.2. □

3. Expansion of the energy functional

This section is devoted to the computation of the expansion of $J_\varepsilon(V)$, where J_ε is the functional defined in (2.14) and V is defined in (2.9).
The main result of this section is contained in the following

Theorem 3.1. *For any $\eta > 0$, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\bar\tau,\bar\sigma$ in $\mathbb{R}^{Nk}$ and any $\bar\mu,\bar\delta$ in $\mathbb{R}^k_+$ satisfying (2.7) and for any $\varepsilon\in(0,\varepsilon_0)$, we have*

$$\begin{aligned}J_\varepsilon(V_a - V_b) &= 2c_1\frac{\alpha_N^{p+1}}{N}k\\ &+\frac{\alpha_N^{p+1}}{2}\Bigg\{c_2\left[H(a,a)\mu_1^{N-2} + H(b,b)\delta_1^{N-2} + 2G(a,b)(\mu_1\delta_1)^{\frac{N-2}{2}}\right]\\ &+\frac{\Gamma(\tau_k)}{(1+|\tau_k|^2)^{\frac{N-2}{2}}}\frac{r_a^{N-2}}{\mu_k^{N-2}} + \frac{\Gamma(\sigma_k)}{(1+|\sigma_k|^2)^{\frac{N-2}{2}}}\frac{r_b^{N-2}}{\delta_k^{N-2}}\\ &+2\sum_{j=1}^{k-1}\left[\Gamma(\tau_j)\left(\frac{\mu_{j+1}}{\mu_j}\right)^{\frac{N-2}{2}} + \Gamma(\sigma_j)\left(\frac{\delta_{j+1}}{\delta_j}\right)^{\frac{N-2}{2}}\right]\Bigg\}\varepsilon^{\frac{N-2}{2k}}\\ &+o\left(\varepsilon^{\frac{N-2}{2k}}\right),\end{aligned} \tag{3.1}$$

C^1−uniformly with respect to μ_j, δ_j, τ_j and σ_j, satisfying (2.7). Here the positive constants c_1 and c_2 are given in (2.18) *and the function F is defined in (2.19).*

Of fundamental importance to carry out the proof of the above expansion are the two Lemmas that follows. The first one gives a description of the basic element of each one of our towers, namely the projection onto $H^1_0(\Omega_\varepsilon)$ of the standard bubble $U_{\delta,\xi}$, for proper election of δ and ξ. The second Lemma is a direct consequence of the first one.
We start with

Lemma 3.1. *Assume that $\xi = a + \mu\tau$, with $\mu\to 0$ as $\varepsilon\to 0$ and $\varepsilon = o(\mu)$ as $\varepsilon\to 0$. Then, if we define*

$$R(x) := P_\varepsilon U_{\mu,\xi}(x) - U_{\mu,\xi}(x) + \alpha_N\mu^{\frac{N-2}{2}}H(x,\xi) + \alpha_N\frac{1}{\mu^{\frac{N-2}{2}}(1+|\tau|^2)^{\frac{N-2}{2}}}\frac{(r_a\varepsilon)^{N-2}}{|x-a|^{N-2}},$$

there exists a positive constant c such that for any $x \in \Omega \setminus (B(a, r_a\varepsilon) \cup B(b, r_b\varepsilon))$

$$|R(x)| \leq c\mu^{\frac{N-2}{2}} \left[\frac{\varepsilon^{N-2}(1+\varepsilon\mu^{-N+1})}{|x-a|^{N-2}} + \mu^2 + (\frac{\varepsilon}{\mu})^{N-2} \right] \tag{3.2}$$

$$|\partial_\mu R(x)| \leq c\mu^{\frac{N-4}{2}} \left[\frac{\varepsilon^{N-2}(1+\varepsilon\mu^{-N+1})}{|x-a|^{N-2}} + \mu^2 + (\frac{\varepsilon}{\mu})^{N-2} \right] \tag{3.3}$$

$$|\partial_{\tau_i} R(x)| \leq c\mu^{\frac{N}{2}} \left[\frac{\varepsilon^{N-2}(1+\varepsilon\mu^{-N})}{|x-a|^{N-2}} + \mu^2 + \frac{\varepsilon^{N-2}}{\mu^{N-1}} \right] \tag{3.4}$$

Proof. We scale as follows: $\hat{R}(y) = \mu^{-\frac{N-2}{2}}\alpha_N^{-1} R(r_a\varepsilon y + a)$. Thus $-\Delta\hat{R} = 0$ in $\hat{\Omega}_\varepsilon$, where

$$\hat{\Omega}_\varepsilon = \left(\frac{\Omega - a}{r_a\varepsilon} \right) \setminus \left(B(0,1) \cup B(\frac{b-a}{r_a\varepsilon}, \frac{r_b}{r_a}) \right).$$

It is easy to check that $\hat{\Omega}_\varepsilon \to \mathbb{R}^N \setminus B(0,1)$ as $\varepsilon \to 0$, and that if $y \in \partial B(0,1)$

$$\hat{R}(y) = -\frac{1}{\mu^{N-2}(1+|\frac{r_a\varepsilon}{\mu}y - \tau|^2)^{\frac{N-2}{2}}} + H(r_a\varepsilon y + a, \xi) + \frac{1}{\mu^{N-2}(1+|\tau|^2)^{\frac{N-2}{2}}}$$

and if $y \in \partial\left(\frac{\Omega-a}{r_a\varepsilon}\right)$

$$\hat{R}(y) = -\frac{1}{(\mu^2 + |r_a\varepsilon y - \mu\tau|^2)^{\frac{N-2}{2}}} + \frac{1}{|r_a\varepsilon y - \mu\tau|^{N-2}} + \frac{1}{\mu^{N-2}(1+|\tau|^2)^{\frac{N-2}{2}}|y|^{N-2}}.$$

Thus we get the estimates

$$|\hat{R}(y)| \leq C(1 + \frac{1}{\mu^{N-2}}\frac{\varepsilon}{\mu}) \quad \text{for all } y \in \partial B(0,1),$$

and

$$|\hat{R}(y)| \leq C(\mu^2 + (\frac{\varepsilon}{\mu})^{N-2}) \quad \text{for all } y \in \partial\left(\frac{\Omega - a}{r_a\varepsilon}\right).$$

A comparison argument for harmonic functions implies that

$$|\hat{R}(y)| \leq C\left[\frac{1+\varepsilon\mu^{1-N}}{|y|^{N-2}} + \mu^2 + (\frac{\varepsilon}{\mu})^{N-2}\right].$$

This fact gives (3.2).

Let us now denote by $R_\mu(x) = \partial_\mu R(x)$ and define $\hat{R}_\mu(y) = \mu^{-\frac{N-4}{2}} R(r_a\varepsilon y + a)$. A direct computation shows that

$$|\hat{R}_\mu(y)| \leq C(1 + \frac{1}{\mu^{N-2}}\frac{\varepsilon}{\mu}) \quad \text{for all } y \in \partial B(0,1),$$

and

$$|\hat{R}_\mu(y)| \leq C(\mu^2 + (\frac{\varepsilon}{\mu})^{N-2}) \quad \text{for all } y \in \partial\left(\frac{\Omega - a}{r_a\varepsilon}\right).$$

This fact gives (3.3).

Finally, let $R_i(x) = \partial_{\tau_i} R(x)$ and $\hat{R}_i(y) = \mu^{-\frac{N}{2}} R_i(r_a\varepsilon y + a)$. We get the following estimates

$$|\hat{R}_i(y)| \leq C(1 + \frac{\varepsilon}{\mu^N}) \quad \text{for all } y \in \partial B(0,1),$$

and

$$|\hat{R}_i(y)| \leq C(\mu^2 + \frac{\varepsilon^{N-2}}{\mu^{N-1}}) \quad \text{for all } y \in \partial\left(\frac{\Omega - a}{r_a\varepsilon}\right).$$

This fact gives (3.4).

□

Lemma 3.2. *Under the same assumption of Lemma 3.1 we have the validity of the following estimate*

$$\int\limits_{\Omega_\varepsilon} U_{\mu,\xi}^{\frac{4}{N-2}} \left(P_\varepsilon U_{\mu,\xi} - U_{\mu,\xi}\right)^2 = O\left(\mu^N + (\varepsilon/\mu)^N\right) \quad \text{if} \quad N \geq 5,$$

$$= O\left(\mu^4 |\log \mu| + (\varepsilon/\mu)^4 |\log(\varepsilon/\mu)|\right) \quad \text{if} \quad N = 4,$$

$$= O\left(\mu^2 + (\varepsilon/\mu)^2\right) \quad \text{if} \quad N = 3.$$

Proof. As direct consequence of Lemma 3.1, we have to estimate

$$\int\limits_{\Omega_\varepsilon} \frac{\mu^2}{(\mu^2 + |x-\xi|^2)^2} \left(\mu^{N-2} + \frac{\varepsilon^{2(N-2)}\mu^{-(N-2)}}{|x-a|^{2(N-2)}}\right) dx.$$

Now, we have if $N \geq 5$

$$\int\limits_{\Omega_\varepsilon} \frac{\mu^2}{(\mu^2 + |x-\xi|^2)^2} dx = 0 \left(\mu^2 \int\limits_{\Omega} \frac{1}{|x-a|^4} dx\right)$$

and if $N = 3$ (setting $x - \xi = \mu y$)

$$\int\limits_{\Omega_\varepsilon} \frac{\mu^2}{(\mu^2 + |x-\xi|^2)^2} dx = 0 \left(\mu \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^2} dy\right).$$

Moreover, we have if $N \geq 5$ (setting $x - a = \varepsilon y$)

$$\int\limits_{\Omega_\varepsilon} \frac{\mu^2}{(\mu^2 + |x-\xi|^2)^2} \frac{1}{|x-a|^{2(N-2)}} = 0 \left(\varepsilon^{-(N-4)} \mu^{-2} \int\limits_{\{|y| \geq 1\}} \frac{1}{|y|^{2(N-2)}} dy\right)$$

and if $N = 3$ (setting $x - \xi = \mu y$)

$$\int\limits_{\Omega_\varepsilon} \frac{\mu^2}{(\mu^2 + |x-\xi|^2)^2} \frac{1}{|x-a|^2} = 0 \left(\mu^{-1} \int\limits_{\mathbb{R}^N} \frac{1}{(1+|y|^2)^2} \frac{1}{|y-\tau|^2} dy\right).$$

The case $N = 4$ can be treated in a similar way.
Collecting all the previous estimates, the claim follows.

□

Proof of Theorem 3.1. .
We write

$$J_\varepsilon(V_a - V_b) = J_\varepsilon(V_a) + J_\varepsilon(V_b) + J_\varepsilon^{a,b} \tag{3.5}$$

where

$$J_\varepsilon^{a,b} := -\int_{\Omega_\varepsilon} \nabla V_a \nabla V_b - \frac{1}{p+1} \int_{\Omega_\varepsilon} \left(|V_a - V_b|^{p+1} - |V_a|^{p+1} - |V_b|^{p+1}\right) dx. \tag{3.6}$$

We start to estimate $J_\varepsilon(V_a)$ in (3.5). In a very similar way, the estimate of the term $J_\varepsilon(V_b)$ will follow.

Recall that $V_a(x) = \sum_{j=1}^{k}(-1)^{j+1}u_{ja}(x)$. For simplicity of notation, while computing the expansion of $J_\varepsilon(V_a)$, we will write u_j instead of u_{ja}. Then, using the fact that $\int_{\Omega_\varepsilon} \nabla u_i \nabla u_j\,dx = \int_{\Omega_\varepsilon} u_j^p\,u_i\,dx$, we have

$$J_\varepsilon(V_a) = \sum_{j=1}^{k} J_\varepsilon(u_j) \;+\; J_\varepsilon^1 \tag{3.7}$$

where

$$J_\varepsilon^1 := -\frac{1}{p+1}\int_{\Omega_\varepsilon}\left[|\sum_{j=1}^{k}(-1)^{j+1}u_j|^{p+1}\,dx - \sum_{j=1}^{k}|u_j|^{p+1} - (p+1)\sum_{i>j}(-1)^{i+j}u_i^p u_j\right]dx. \tag{3.8}$$

Let us fix j in $\{1,\dots,k\}$. To simplify again the notation, we will use U_j to denote the function $U_{\mu_{j\varepsilon},a_{j\varepsilon}}$. Since $\Delta u_j = U_j^p$ in Ω_ε and $u_j = 0$ on $\partial\Omega_\varepsilon$, we see that, for some $0 \le s \le 1$,

$$\begin{aligned}
J_\varepsilon(u_j) &= \frac{1}{N}\int_{\Omega_\varepsilon} U_j^{p+1}\,dx + \frac{1}{2}\int_{\Omega_\varepsilon} U_j^p(u_j - U_j)\,dx - \frac{1}{p+1}\int_{\Omega_\varepsilon}[|u_j|^{p+1} - U_j^{p+1}]\,dx\\
&= \frac{1}{N}\int_{\Omega_\varepsilon} U_j^{p+1}\,dx - \frac{1}{2}\int_{\Omega_\varepsilon} U_j^p(u_j - U_j)\,dx - p\int_{\Omega_\varepsilon}[U_j + s(u_j - U_j)]^{p-1}\,[u_j - U_j]^2\,dx\\
&= A_j + B_j + C_j.
\end{aligned} \tag{3.9}$$

It is useful to point out that $\mu_{j\varepsilon}, \frac{\mu_{j\varepsilon}}{\varepsilon} = O\left(\varepsilon^{\frac{1}{2k}}\right)$, because of (2.5).

First we observe that Lemma 3.2 implies that

$$|C_j| = o\left(\varepsilon^{\frac{N-2}{2k}}\right). \tag{3.10}$$

If we perform the change of variables $x - a = \mu_{j\varepsilon}z$, the domain Ω_ε gets transformed into

$$\tilde{\Omega}_\varepsilon = \left(\frac{\Omega\setminus\{a\}}{\mu_{j\varepsilon}}\right)\setminus\left(B(0,\frac{r_a\varepsilon}{\mu_{j\varepsilon}})\bigcup B(b-a,\frac{r_b\varepsilon}{\mu_{j\varepsilon}})\right). \tag{3.11}$$

Since $\frac{\varepsilon}{\mu_{j\varepsilon}} \to 0$ as $\varepsilon \to 0$, the set $\tilde{\Omega}_\varepsilon$ converges to the whole space $\mathbb{R}^N$ and we get

$$A_j = \frac{1}{N}\alpha_N^{p+1}\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^N}\,dz + O\left(\varepsilon^{\frac{2j-1}{2k}N}\right), \quad \text{for all } j = 1,\dots,k. \tag{3.12}$$

We observe for later purpose that $|\varepsilon^{\frac{2j-1}{2k}N}| \le \varepsilon^{\frac{N}{2k}}$.

Using the notations introduced in Lemma 3.1, we write

$$B_j = \frac{1}{2}(B_{j1} + B_{j2} + B_{j3}) \tag{3.13}$$

where

$$B_{j1} = \alpha_N^{p+1}\mu_{j\varepsilon}^{\frac{N-2}{2}}\int_{\Omega_\varepsilon}\left(\frac{\mu_{j\varepsilon}}{{\mu_{j\varepsilon}}^2 + |x - a_{j\varepsilon}|^2}\right)^{\frac{N+2}{2}} H(x, a_{j\varepsilon})\,dx \tag{3.14}$$

$$B_{j2} = \alpha_N^{p+1}r_a^{N-2}\frac{\varepsilon^{N-2}{\mu_{j\varepsilon}}^{-\frac{N-2}{2}}}{(1+|\tau_j|^2)^{\frac{N-2}{2}}}\int_{\Omega_\varepsilon}\left(\frac{\mu_{j\varepsilon}}{{\mu_{j\varepsilon}}^2 + |x - a_{j,\varepsilon}|^2}\right)^{\frac{N+2}{2}}\frac{1}{|x-a|^{N-2}}\,dx \tag{3.15}$$

and

$$B_{j3} = -\alpha_N^{p+1}\int_{\Omega_\varepsilon}\left(\frac{\mu_{j\varepsilon}}{{\mu_{j\varepsilon}}^2 + |x - a_{j,\varepsilon}|^2}\right)^{\frac{N+2}{2}} R(x)\,dx. \tag{3.16}$$

Using again the change of variables $x-a=\mu_{j\varepsilon}z$, the domain Ω_ε gets transformed into $\tilde\Omega_\varepsilon$ (3.11) and we get

$$\begin{aligned} B_{j1} &= \alpha_N^{p+1}{\mu_{j\varepsilon}}^{N-2}\int_{\tilde\Omega_\varepsilon}\left(\frac{1}{1+|z+\tau_j|^2}\right)^{\frac{N+2}{N-2}} H(a+\mu_{j\varepsilon}z, a+\mu_{j\varepsilon}\tau_j)\,dz \\ &= \alpha_N^{p+1}(\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}\,dz)\,H(a,a)\,\mu_j^{N-2}\varepsilon^{\frac{2j-1}{2k}(N-2)}(1+o(1)) \end{aligned} \tag{3.17}$$

and

$$\begin{aligned} B_{j2} &= \alpha_N^{p+1}\frac{r_a^{N-2}\varepsilon^{N-2}}{{\mu_{j\varepsilon}}^{N-2}(1+|\tau_j|^2)^{\frac{N-2}{2}}}\int_{\tilde\Omega_\varepsilon}\left(\frac{1}{1+|z-\tau_j|^2}\right)^{\frac{N+2}{N-2}}\frac{1}{|z|^{N-2}}\,dz \qquad (3.18)\\ &= \alpha_N^{p+1}\frac{r_a^{N-2}}{(1+|\tau_j|^2)^{\frac{N-2}{2}}}(\int_{\mathbb{R}^N}\frac{1}{|z|^{N-2}(1+|z-\tau_j|^2)^{\frac{N+2}{2}}}\,dz)\,\frac{\varepsilon^{\frac{(N-2)(2k-2j+1)}{2k}}}{\mu_j^{N-2}}(1+o(1)). \end{aligned}$$

Finally, using the result in Lemma 3.1, we have

$$|B_{j3}| = o(\varepsilon^{\frac{2j-1}{2k}(N-2)}+\varepsilon^{(N-2)\frac{2(k-j)-1}{2k}}), \qquad \text{for all } j=1,\dots,k. \tag{3.19}$$

Thus we conclude from (3.9)–(3.19) that

$$\sum_{j=1}^k J_\varepsilon(u_j) = kc_1\frac{\alpha_N^{p+1}}{N}+\frac{\alpha_N^{p+1}}{2}\left[c_2H(a,a)\mu_1^{N-2}+\frac{r_a^{N-2}\Gamma(\tau_k)}{(1+|\tau_k|^2)^{\frac{N-2}{2}}}\frac{1}{\mu_k^{N-2}}\right]\varepsilon^{\frac{N-2}{2k}}(1+o(1)). \tag{3.20}$$

Next we estimate the term J_ε^1 (3.8) in (3.7). Assume $B(a,\rho)\bigcap B(b,\rho)=\emptyset$ for some $\rho>0$. Thus we write

$$-(p+1)\,J_\varepsilon^1 = (\int_{\Omega_\varepsilon\setminus B(a,\rho)}+\int_{\Omega_\varepsilon\bigcap B(a,\rho)})G_\varepsilon^1(x)\,dx, \tag{3.21}$$

with

$$G_\varepsilon^1 = \left(|\sum_{j=1}^k(-1)^{j+1}u_j|^{p+1}\,dx-\sum_{j=1}^k|u_j|^{p+1}-(p+1)\sum_{i>j}(-1)^{i+j}u_i^pu_j\right).$$

The first integral in (3.21) is lower order respect to the first one. Indeed we have

$$\begin{aligned} \left|\int_{\Omega_\varepsilon\setminus B(a,\rho)}G_\varepsilon^1\right| &\le C\left[\sum_{j=1}^k\int_{\Omega_\varepsilon\setminus B(a,\rho)}U_j^{p+1}+\sum_{i\neq j}\int_{\Omega_\varepsilon\setminus B(a,\rho)}U_i^p\,U_j\right] \\ &\le C\left[\sum_j{\mu_{j\varepsilon}}^N+\sum_{i\neq j}{\mu_{i\varepsilon}}^{\frac{N+2}{2}}\,{\mu_{j\varepsilon}}^{\frac{N-2}{2}}\right] = O\left(\varepsilon^{\frac{N}{2k}}\right). \end{aligned}$$

To deal with the second integral in (3.21), we will decompose the set $\Omega_\varepsilon\bigcap B(a,\rho)=B(a,\rho)\setminus B(a,r_a\varepsilon)$ into the union of non-overlapping annuli. More precisely, we write

$$B(a,\rho)\setminus B(a,r_a\varepsilon)=\bigcup_{l=1}^k\mathcal{A}_l \tag{3.22}$$

where for all $l=1,\dots,k$,

$$\mathcal{A}_l := B(a,\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}})\setminus B(a,\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}).$$

with $\mu_{0\varepsilon}:=\mu_{1\varepsilon}^{-1}\rho^2$ and $\mu_{k+1\varepsilon}:=\mu_{k\varepsilon}^{-1}r_a^2\varepsilon^2$.

Thus we write

$$\int_{\Omega_\varepsilon \bigcap B(a,\rho)} G_1^\varepsilon\, dx = \sum_{l=1}^{k} \int_{\mathcal{A}_l} G_1^\varepsilon\, dx \tag{3.23}$$

Fix now l. We write

$$\int_{\mathcal{A}_l} G_1^\varepsilon\, dx = \int_{\mathcal{A}_l} \left[\left| \sum_j (-1)^{j+1} u_j \right|^{p+1} - u_l^{p+1} - (p+1) u_l^p \sum_{i \neq l} (-1)^{i+l} u_i \right]$$
$$- \sum_{i\neq l} \int_{\mathcal{A}_l} u_i^{p+1} - (p+1) \int_{\mathcal{A}_l} \left[\sum_{i>j} (-1)^{i+j} u_i^p u_j - u_l^p \sum_{i\neq l} (-1)^{i+l} u_i \right].$$

Now we further decompose the last integral above as follows

$$- (p+1) \int_{\mathcal{A}_l} \left[\sum_{i>j} (-1)^{i+j} u_i^p u_j - u_l \sum_{i\neq l} (-1)^{i+l} u_i \right]$$
$$= -(p+1) \left[- \sum_{j>l} (-1)^{l+j} \int_{\mathcal{A}_l} u_l^p u_j + \sum_{i>j, i\neq l} (-1)^{i+j} \int_{\mathcal{A}_l} u_i^p u_j \right]$$
$$= (p+1) \left[\sum_{j>l} (-1)^{j+l} \int_{\mathcal{A}_l} U_l^p U_j + \sum_{j>l} (-1)^{j+l} \int_{\mathcal{A}_l} [(u_l^p - U_l^p) U_j] + [u_l^p (u_j - U_j)] \right.$$
$$\left. - \sum_{i>j,\, i\neq l} (-1)^{j+i} \int_{\mathcal{A}_l} u_i^p u_j \right]$$

Summarizing the above information and putting in evidence the principal term, we write

$$\int_{\mathcal{A}_l} G_1^\varepsilon\, dx = (p+1) \sum_{j>l} (-1)^{l+j} \int_{\mathcal{A}_l} U_l^p U_j\, dx + r_l \tag{3.24}$$

where $r_l = \sum_{j=1}^4 r_{jl}$ with

$$r_{1l} = \int_{\mathcal{A}_l} \left[\left| \sum_j (-1)^{j+1} u_j \right|^{p+1} - u_l^{p+1} - (p+1) u_l^p \sum_{i\neq l} (-1)^{i+l} u_i \right],$$

$$r_{2l} = - \sum_{i\neq l} \int_{\mathcal{A}_l} u_i^{p+1},$$

$$r_{3l} = (p+1) \sum_{j>l} (-1)^{j+l} \int_{\mathcal{A}_l} \left\{ [(u_l^p - U_l^p) U_j] + [u_l^p (u_j - U_j)] \right\},$$

$$r_{4l} = (p+1) \sum_{i>j,\, i\neq l} (-1)^{j+i} \int_{\mathcal{A}_l} u_i^p u_j.$$

We first deal with the main term in (3.24), namely $(p+1) \sum_{j>l} (-1)^{l+j} \int_{\mathcal{A}_l} U_l^p U_j\, dx$. Hence we are interested in computing $\int_{\mathcal{A}_l} U_l^p U_j\, dx$ for $l = 1, \dots, k-1$. In the region

$\mathcal{A}_l$ we perform the change of variables $x-a=\mu_{l\varepsilon}z$. Thus the transformed domains are

$$\tilde{\mathcal{A}}_l=\{z\in\mathbb{R}^N\,:\,\sqrt{\frac{\mu_{l+1\varepsilon}}{\mu_{l\varepsilon}}}\le|z|\le\sqrt{\frac{\mu_{l-1\varepsilon}}{\mu_{l\varepsilon}}}\}\quad\text{if}\quad l=1,\dots,k-1.$$

It is immediate to see that (2.5) gives that the transformed domain $\tilde{\mathcal{A}}_l$ converges to the whole space $\mathbb{R}^N$ as $\varepsilon\to0$.
With this in mind and using the fact that $j>l$ and $l=1,\dots,k-1$, we have

$$\int_{\mathcal{A}_l}U_l^pU_j\,dx=\left(\frac{\mu_{j\varepsilon}}{\mu_{l\varepsilon}}\right)^{\frac{N-2}{2}}\int_{\tilde{\mathcal{A}}_l}\frac{\alpha_N^{p+1}}{(1+|z-\tau_l|^2)^{\frac{N+2}{2}}}\frac{1}{[(\frac{\mu_{j\varepsilon}}{\mu_{l\varepsilon}})^2+|z-\frac{\mu_{j\varepsilon}}{\mu_{l\varepsilon}}\tau_j|^2]^{\frac{N-2}{2}}}\,dz$$

$$=\alpha_N^{p+1}(\int_{\mathbb{R}^N}\frac{1}{|z|^{N-2}(1+|z-\tau_l|^2)^{\frac{N+2}{2}}}\,dz)\left(\frac{\mu_j}{\mu_l}\right)^{\frac{N-2}{2}}\varepsilon^{\frac{(N-2)(j-l)}{2k}}(1+o(1)).\quad(3.25)$$

Since $\varepsilon^{\frac{N-2}{2k}+\frac{1}{k}}=\varepsilon^{\frac{N}{2k}}$, we thus conclude that, for all $l=1,\dots,k-1$,

$$\sum_{j>l}(-1)^{l+j}\int_{\mathcal{A}_l}U_l^pU_j\,dx=-\alpha_N^{p+1}\Gamma(\tau_l)\left(\frac{\mu_{l+1}}{\mu_l}\right)^{\frac{N-2}{2}}\varepsilon^{\frac{(N-2)}{2k}}(1+o(1)),\qquad(3.26)$$

where F is defined in 2.19. To get the estimate of $\int_{\mathcal{A}_l}G_1^\varepsilon\,dx$ we are left to show that the term r_l in (3.24) is negligible. We claim that this fact will be consequence of two fundamental computations

$$\int_{\mathcal{A}_l}U_j^{p+1}\,dx=O\left(\varepsilon^{\frac{N}{2k}}\right)\qquad\text{for all } j\ne l,\qquad(3.27)$$

$$\int_{\mathcal{A}_l}U_i^pU_jdx=O\left(\varepsilon^{\frac{N}{2k}}\right)\qquad\text{for all } j\ne l,\quad\text{for all } i\ne l,\qquad(3.28)$$

and

$$\int_{\mathcal{A}_l}U_l^pU_jdx=O\left(\varepsilon^{\frac{|l-j|(N-2)}{2k}}\right)\qquad\text{for all } j\ne l.\qquad(3.29)$$

To get (3.27), we perform the change of variable $x-a=\mu_{j\varepsilon}z$ to get

$$\int_{\mathcal{A}_l}U_j^{p+1}\,dx=\int_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{j\varepsilon}}<|z|<\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{j\varepsilon}}}\frac{1}{(1+|z-\tau_j|^2)^N}\,dz.$$

If $j>l$ then $\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{j\varepsilon}}\to\infty$ and so, for some positive constant C,

$$|\int_{\mathcal{A}_l}U_j^{p+1}\,dx|\le C\int_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{j\varepsilon}}}^\infty t^{-N-1}\,dt=C\left(\frac{\mu_{j\varepsilon}}{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}\right)^N=O(\varepsilon^{\frac{N}{2k}}).$$

If $j<l$ then $\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{j\varepsilon}}\to0$ and so, for some positive constant C,

$$|\int_{\mathcal{A}_l}U_j^{p+1}\,dx|\le C\left(\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{j\varepsilon}}-\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{j\varepsilon}}\right)^N\le C\left(\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{j\varepsilon}}\right)^N=O(\varepsilon^{\frac{N}{2k}}).$$

These facts give the validity of (3.27).
Estimate (3.28) is a direct consequence of (3.27) and Holder inequality, since

$$\left|\int_{\mathcal{A}_l}U_i^pU_jdx\right|\le\left(\int_{\mathcal{A}_l}U_i^{p+1}\,dx\right)^{\frac{p}{p+1}}\left(\int_{\mathcal{A}_l}U_j^{p+1}\,dx\right)^{\frac{1}{p+1}}\le CO(\varepsilon^{\frac{N}{2k}}).$$

Finally (3.29) is a direct consequence of the computations contained in (3.25) when $j > l$. Assume now that $j < l$. Perform the change of variable $x - a = \mu_{l\varepsilon} z$, one gets

$$\int_{\mathcal{A}_l} U_l^p U_j \, dx = \mu_{l\varepsilon}^{\frac{N-2}{2}} \mu_{j\varepsilon}^{\frac{N-2}{2}} \int_{\tilde{\mathcal{A}}_l} \frac{\alpha_N^{p+1}}{(1+|z-\tau_l|^2)^{\frac{N+2}{2}}} \frac{1}{[\mu_{j\varepsilon}^2 + |\mu_{l\varepsilon} z - \mu_{j\varepsilon}\tau_j|^2]^{\frac{N-2}{2}}} \, dz$$
$$= (\int_{\mathbb{R}^N} \frac{\alpha_N^{p+1}}{|z-\tau_l|^{N-2}(1+|z-\tau_j|^2)^{\frac{N+2}{2}}} \, dz) \left(\frac{\mu_l}{\mu_j}\right)^{\frac{N-2}{2}} \varepsilon^{\frac{(N-2)(l-j)}{2k}}$$
$$= O\left(\varepsilon^{\frac{(N-2)(l-j)}{2k}}\right).$$

¿From this we conclude (3.29).
Let us now estimate the terms that define r_l (see (3.24)). First we have

$$|r_{1l}| \le C \left(\sum_{j \ne l} \int_{\mathcal{A}_l} U_l^{p-1} U_j^2 + \sum_{i,j \ne l} \int_{\mathcal{A}_l} U_i^{p-1} U_j^2 \right) \le C \varepsilon^{\frac{N-2}{2k}(1+\frac{2}{N+2})}$$

since, if $j \ne l$,

$$\int_{\mathcal{A}_l} U_l^{p-1} U_j^2 \le C \left(\int_{\mathcal{A}_l} U_l^p U_j \right)^{\frac{p-1}{p}} \left(\int_{\mathcal{A}_l} U_j^{p+1} \right)^{\frac{1}{p}} \le C \varepsilon^{\frac{N-2}{2k}(1+\frac{N}{N+2})}.$$

and, for $i \ne l$ and $j \ne l$,

$$\int_{\mathcal{A}_l} U_i^{p-1} U_j^2 \le C \left(\int_{\mathcal{A}_l} U_i^p U_j \right)^{\frac{p-1}{p}} \left(\int_{\mathcal{A}_l} U_j^{p+1} \right)^{\frac{1}{p}} \le C \varepsilon^{\frac{N}{2k}}.$$

An immediate consequence of (3.27) is that $|r_{2l}| \le C\varepsilon^{\frac{N}{2k}}$, while from (3.28) we have that $|r_{4l}| \le C\varepsilon^{\frac{N}{2k}}$.
We are left to estimate r_{3l}. We thus fix $j > l$. In particular we just take $l \ne j$. A consequence of Lemma 3.1 is that in $\mathcal{A}_l$ we have

$$|u_j(x) - U_j(x)| \le C \frac{\varepsilon^{N-2}}{\mu_{j\varepsilon}{}^{\frac{N-2}{2}} |x-a|^{N-2}}.$$

Hence, using again the change of variables $x - a = \mu_{l\varepsilon} z$, we see that the first terms in the expression of r_{3l} can be estimated as follows

$$\left| \int_{\mathcal{A}_l} u_l^p (u_j - U_j) \, dx \right| \le C \frac{\varepsilon^{N-2}}{\mu_{j\varepsilon}^{\frac{N-2}{2}}} \int_{\mathcal{A}_l} U_l^p \frac{1}{|x-a|^{N-2}} \, dx$$
$$\le C \frac{\varepsilon^{N-2}}{(\mu_{j\varepsilon}\mu_{l\varepsilon})^{\frac{N-2}{2}}} \int_{\tilde{\mathcal{A}}_l} \frac{1}{(1+|z-\tau_l|^2)^{\frac{N+2}{2}}} \frac{1}{|z|^{N-2}} \, dz$$
$$\le C \varepsilon^{\frac{N-2}{2k}(2k-j-l+1)} \le C \varepsilon^{\frac{N-2}{k}}.$$

The remaining terms in the definition of r_{3l} can be estimated as follows. We have for $j > l$ and using again the change of variable in $\mathcal{A}_l$ given by $x - a = \mu_{l\varepsilon} z$,

$$\begin{aligned}\left|\int_{\mathcal{A}_l}(u_l^p - U_l^p)U_j\,dx\right| &\le C\int_{\mathcal{A}_l} U_l^{p-1}|u_l - U_l|U_j\,dx \le C\frac{\varepsilon^{N-2}}{\mu_{l\varepsilon}^{\frac{N-2}{2}}}\int_{\mathcal{A}_l}\frac{U_l^{p-1}U_j}{|x-a|^{N-2}}\,dx\\ &\le C\frac{\varepsilon^{N-2}\mu_{j\varepsilon}^{\frac{N-2}{2}}}{\mu_{l\varepsilon}^{\frac{N-2}{2}}}\int_{\tilde{\mathcal{A}}_l}\frac{1}{(1+|z-\tau_l|^2)^2}\frac{1}{|z|^{N-2}}\frac{1}{(\mu_{j\varepsilon}^2 + |\mu_{l\varepsilon}z - \mu_{j\varepsilon}\tau_j|^2)^{\frac{N-2}{2}}}\,dz\\ &\le C\frac{\varepsilon^{N-2}}{\mu_{l\varepsilon}^{\frac{N-2}{2}}\mu_{j\varepsilon}^{\frac{N-2}{2}}}\int_{\mathbb{R}^N}\frac{1}{(1+|z-\tau_l|^2)^2}\frac{1}{|z|^{(N-2)}}\,dz \le C\varepsilon^{\frac{N-2}{k}}.\end{aligned}$$

By all the previous estimates we get

$$J_\varepsilon^1 = \alpha_N^{p+1}\sum_{l=1}^{k-1}\Gamma(\tau_l)\left(\frac{\mu_{l+1}}{\mu_l}\right)^{\frac{N-2}{2}}\varepsilon^{\frac{N-2}{2k}}(1+o(1)). \tag{3.30}$$

By (3.7), (3.20) and (3.30) we conclude that

$$\begin{aligned}J_\varepsilon(V_a) &= kc_1\frac{\alpha_N^{p+1}}{N}\\ &+ \frac{\alpha_N^{p+1}}{2}\left\{c_2 H(a,a)\mu_1^{N-2} + \frac{r_a^{N-2}\Gamma(\tau_k)}{(1+|\tau_k|^2)^{\frac{N-2}{2}}}\frac{1}{\mu_k^{N-2}} + \sum_{l=1}^{k-1}\Gamma(\tau_l)\left(\frac{\mu_{l+1}}{\mu_l}\right)^{\frac{N-2}{2}}\right\}\varepsilon^{\frac{N-2}{2k}}\\ &+ o\left(\varepsilon^{\frac{N-2}{2k}}\right).\end{aligned} \tag{3.31}$$

In a very similar way one gets the expansion of $J_\varepsilon(V_b)$ in (3.5), that is

$$\begin{aligned}J_\varepsilon(V_b) &= kc_1\frac{\alpha_N^{p+1}}{N}\\ &+ \frac{\alpha_N^{p+1}}{2}\left\{c_2 H(b,b)\delta_1^{N-2} + \frac{r_b^{N-2}\Gamma(\sigma_k)}{(1+|\sigma_k|^2)^{\frac{N-2}{2}}}\frac{1}{\delta_k^{N-2}} + \sum_{l=1}^{k-1}\Gamma(\sigma_l)\left(\frac{\delta_{l+1}}{\delta_l}\right)^{\frac{N-2}{2}}\right\}\varepsilon^{\frac{N-2}{2k}}\\ &+ o\left(\varepsilon^{\frac{N-2}{2k}}\right).\end{aligned} \tag{3.32}$$

We are now left with the estimate of $J_\varepsilon^{a,b}$ in (3.6) to complete the expansion of (3.5).

Standard arguments (see [2] and [3]) prove that

$$\begin{aligned}&\int_{\Omega_\varepsilon}\nabla P_\varepsilon U_{\mu_{i\varepsilon},a_{i\varepsilon}}\nabla P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}}\\ &= \alpha_N^{p+1}\left(\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}dz\right)G(a_{i\varepsilon},b_{j\varepsilon})\mu_{i\varepsilon}^{\frac{N-2}{2}}\delta_{j\varepsilon}^{\frac{N-2}{2}}(1+o(1))\\ &= \alpha_N^{p+1}\left(\int_{\mathbb{R}^N}\frac{1}{(1+|z|^2)^{\frac{N+2}{2}}}dz\right)G(a,b)\mu_i^{\frac{N-2}{2}}\delta_j^{\frac{N-2}{2}}\varepsilon^{\frac{(j+i-1)(N-2)}{2k}}(1+o(1)).\end{aligned}$$

Therefore

$$\int_{\Omega_\varepsilon} \nabla V_a \nabla V_b = \alpha_N^{p+1} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} dz \right) G(a,b) \left(\mu_1 \delta_1\right)^{\frac{N-2}{2}} \varepsilon^{\frac{N-2}{2k}} (1+o(1)). \tag{3.33}$$

Let now $\rho > 0$ be such that $B(a,\rho) \cap B(b,\rho) = \emptyset$. Define

$$G_{2\varepsilon} = |V_a - V_b|^{p+1} - V_a^{p+1} - V_b^{p+1}.$$

Taking into account that $D\left(|x|^{p+1}\right) = (p+1)x|x|^{p-1}$, a Taylor expansion gives

$$\begin{aligned}
\int_{\Omega_\varepsilon} G_{2\varepsilon} &= \int_{\Omega_\varepsilon \cap B(a,\rho)} G_{2\varepsilon} + \int_{\Omega_\varepsilon \cap B(b,\rho)} G_{2\varepsilon} + O(\mu_{1\varepsilon}^N + \delta_{1\varepsilon}^N) \\
&= -(p+1) \left[\int_{\Omega_\varepsilon \cap B(a,\rho)} V_a^p V_b + \int_{\Omega_\varepsilon \cap B(b,\rho)} V_b^p V_a \right] \\
&\quad + \frac{p(p+1)}{2} \left[\int_{\Omega_\varepsilon \cap B(a,\rho)} (V_a + sV_b)^{p-1} V_b^2 + \int_{\Omega_\varepsilon \cap B(b,\rho)} (V_b + sV_a)^{p-1} V_a^2 \right] \\
&\quad + O(\mu_{1\varepsilon}^N + \delta_{1\varepsilon}^N) \\
&= -(p+1) \sum_{j=1}^k \left[\int_{\Omega_\varepsilon \cap B(a,\rho)} U^p_{\mu_{j\varepsilon}, a_{j\varepsilon}} V_b + \int_{\Omega_\varepsilon \cap B(b,\rho)} U^p_{\delta_{j\varepsilon}, b_{j\varepsilon}} V_a \right] \\
&\quad + I_1 + I_2 + O\left(\varepsilon^{\frac{N}{2k}}\right),
\end{aligned} \tag{3.34}$$

where

$$I_1 := -\left[\int_{\Omega_\varepsilon \cap B(a,\rho)} \left(V_a^p - \sum_j U^p_{\mu_{j\varepsilon}, a_{j\varepsilon}} \right) V_b + \int_{\Omega_\varepsilon \cap B(b,\rho)} \left(V_b^p - \sum_j U^p_{\delta_{j\varepsilon}, b_{j\varepsilon}} \right) V_a \right]$$

and

$$I_2 := -\frac{p(p+1)}{2} [\int_{\Omega_\varepsilon \cap B(a,\rho)} (V_a + sV_b)^{p-1} V_b^2 + \int_{\Omega_\varepsilon \cap B(b,\rho)} (V_b + sV_a)^{p-1} V_a^2].$$

It is straightforward to see that $I_1, I_2 = O\left(\varepsilon^{\frac{N}{2k}}\right)$. Furthermore, it is by now standard (see [2] and [3]) that

$$\begin{aligned}
&\int_{\Omega_\varepsilon \cap B(a,\rho)} U^p_{\mu_{j\varepsilon}, a_{j\varepsilon}} P_\varepsilon U_{\delta_{i\varepsilon}, b_{i\varepsilon}} \, dx \\
&= \alpha_N^{p+1} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} dz \right) G(a_{i\varepsilon}, b_{j\varepsilon}) \mu_{i\varepsilon}^{\frac{N-2}{2}} \delta_{j\varepsilon}^{\frac{N-2}{2}} (1+o(1)) \\
&= \alpha_N^{p+1} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} dz \right) G(a,b) \mu_i^{\frac{N-2}{2}} \delta_j^{\frac{N-2}{2}} \varepsilon^{\frac{(j+i-1)(N-2)}{2k}} (1+o(1)).
\end{aligned} \tag{3.35}$$

By (3.34) and (3.35) we deduce

$$\int_{\Omega_\varepsilon} G_{2\varepsilon} = -2(p+1)\alpha_N^{p+1} \left(\int_{\mathbb{R}^N} \frac{1}{(1+|z|^2)^{\frac{N+2}{2}}} dz \right) G(a,b) \left(\mu_1 \delta_1\right)^{\frac{N-2}{2}} \varepsilon^{\frac{N-2}{2k}} (1+o(1).$$

We thus conclude that

$$J_\varepsilon^{a,b} = c_2 \alpha_N^{p+1} G(a,b) \left(\mu_1 \delta_1\right)^{\frac{N-2}{2}} \varepsilon^{\frac{N-2}{2k}} (1+o(1)). \tag{3.36}$$

Finally by (3.5), (3.31), (3.32) and (3.36) the C^0-estimate in (3.1) follows. Arguing in a similar way, we can also prove the C^1-estimate. □

4. The reduced function

This section is devoted to guarantee that the functions Ψ and Ψ^* defined in (2.17) and (2.20) have critical points which are stable under C^1-perturbation of them.

Proposition 4.1. (i) *There exist $\bar\mu_0, \bar\delta_0 \in R_+^k$ such that $(0,0,\bar\mu_0,\bar\delta_0)$ is a non degenerate critical point of the function Ψ defined in* (2.17).
(ii) *If* (1.7) *holds, there exist $\bar\mu_0, \bar\delta_0 \in R_+^k$ such that $(0,0,\bar\mu_0,\bar\delta_0)$ is a non degenerate critical point of the function Ψ^* defined in* (2.20).

Proof. Let us rewrite the functions Ψ and Ψ^* as

$$\begin{aligned}\Phi(\bar\tau,\bar\sigma,\bar\mu,\bar\delta) &:= h_a\mu_1^2 + h_b\delta_1^2 + 2h_{ab}\mu_1\delta_1 + g(\tau_k)\frac{1}{\mu_k^2} + g(\sigma_k)\frac{1}{\delta_k^2}\\ &\quad + \left[f(\tau_1)\frac{\mu_2}{\mu_1} + \cdots + f(\tau_{k-1})\frac{\mu_k}{\mu_{k-1}}\right] + \left[f(\sigma_1)\frac{\delta_2}{\delta_1} + \cdots + f(\sigma_{k-1})\frac{\delta_k}{\delta_{k-1}}\right],\end{aligned}$$

where we replaced $\mu_i^{\frac{N-2}{2}}$ and $\delta_i^{\frac{N-2}{2}}$ with μ_i and δ_i, respectively, and we also set $h_a := c_2 H(a,a)$, $h_b := b_2 H(b,b)$, $h_{a,b} := \pm c_2 G(a,b)$

$$g_a(x) := \frac{r_a^{N-2}\Gamma(x)}{(1+|x|^2)^{\frac{N-2}{2}}},\ g_b(x) := \frac{r_b^{N-2}\Gamma(x)}{(1+|x|^2)^{\frac{N-2}{2}}},\ f(x) := 2\Gamma(x).$$

First of all, we point out that if we fix $\bar\tau = \bar\sigma = 0$ the function $(\bar\mu,\bar\delta) \to \Phi(0,0,\bar\mu,\bar\delta)$ has a minimum point $(\bar\mu_0,\bar\delta_0)$. In fact, the quadratic form $(\mu_1,\delta_1) \to h_a\mu_1^2 + h_b\delta_1^2 + 2h_{ab}\mu_1\delta_1$ is strictly positively definite: this is trivial if $h_{ab} = +2G(a,b)$ and it follows by (1.7) if $h_{ab} = -2G(a,b)$.
We are going to show that $(0,0,\bar\mu_0,\bar\delta_0)$ is a nondegenerate critical point of Φ. The claim immediately follows.
Let us remark that

$$\mathcal{H}\Phi(0,0,\bar\mu_0,\bar\delta_0) = \begin{pmatrix} \mathcal{H}_{\bar\tau,\bar\sigma}\Phi(0,0,\bar\mu_0,\bar\delta_0) & 0 \\ 0 & \mathcal{H}_{\bar\mu,\bar\delta}\Phi(0,0,\bar\mu_0,\bar\delta_0)\end{pmatrix}.$$

By Lemma 4.2 we easily deduce that $\left|\mathcal{H}_{\bar\tau,\bar\sigma}\Phi(0,0,\bar\mu_0,\bar\delta_0)\right| \neq 0$. It remains to prove that

$$\left|\mathcal{H}_{\bar\mu,\bar\delta}\Phi(0,0,\bar\mu_0,\bar\delta_0)\right| \neq 0. \tag{4.1}$$

Let us compute $\nabla\Phi(\bar\tau,\bar\sigma,\bar\mu,\bar\delta)$ in a generic point:

$$\begin{aligned}
\partial_{\mu_1}\Phi &= 2h_a\mu_1 + 2h_{ab}\delta_1 - f(\tau_1)\frac{\mu_2}{\mu_1^2}\\
\partial_{\mu_i}\Phi &= \frac{f(\tau_{i-1})}{\mu_{i-1}} - f(\tau_i)\frac{\mu_{i+1}}{\mu_i^2},\ i=2,\dots,k-1\\
\partial_{\mu_k}\Phi &= -2\frac{g_a(\tau_k)}{\mu_k^3} + \frac{f(\tau_{k-1})}{\mu_{k-1}}.\\
\partial_{\delta_1}\Phi &= 2h_b\delta_1 + 2h_{ab}\mu_1 - f(\sigma_1)\frac{\delta_2}{\delta_1^2}\\
\partial_{\delta_i}\Phi &= \frac{f(\sigma_{i-1})}{\delta_{i-1}} - f(\sigma_i)\frac{\delta_{i+1}}{\delta_i^2},\ i=2,\dots,k-1\\
\partial_{\delta_k}\Phi &= -2\frac{g_b(\sigma_k)}{\delta_k^3} + \frac{f(\sigma_{k-1})}{\delta_{k-1}}.
\end{aligned}$$

If $\nabla\Phi(\bar\tau,\bar\sigma,\bar\mu,\bar\delta)=0$, in particular we get

$$\alpha_a := A\mu_1 = f(\tau_1)\frac{\mu_2}{\mu_1} = \dots = f(\tau_{k-1})\frac{\mu_k}{\mu_{k-1}} = \frac{2g_a(\tau_k)}{\mu_k^2}, \quad A := (2h_a\mu_1 + 2h_{ab}\delta_1), \tag{4.2}$$

$$\alpha_b := B\delta_1 = f(\sigma_1)\frac{\delta_2}{\delta_1} = \dots = f(\delta_{k-1})\frac{\delta_k}{\delta_{k-1}} = \frac{2g_b(\sigma_k)}{\delta_k^2}, \quad B := (2h_b\delta_1 + 2h_{ab}\mu_1). \tag{4.3}$$

Now let $\bar\tau=\bar\sigma=0$ and set $\beta := f(0)$. Then we have:

$$\mathcal{H}_{\bar\mu,\bar\delta}\Phi(0,0,\bar\mu_0,\bar\delta_0) = \begin{pmatrix}
2h_a + \frac{2\beta\mu_2}{\mu_1^3} & -\frac{\beta}{\mu_1^2} & \dots & 0 & 2h_{ab} & 0 & \dots & 0\\
-\frac{\beta}{\mu_1^2} & \frac{2\beta\mu_3}{\mu_2^3} & \dots & 0 & 0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & \frac{6g_a(0)}{\mu_k^4} & 0 & 0 & \dots & 0\\
2h_{ab} & 0 & \dots & 0 & 2h_b + \frac{2\beta\mu_2}{\delta_1^3} & -\frac{\beta}{\delta_1^2} & \dots & 0\\
0 & 0 & \dots & 0 & -\frac{\beta}{\delta_1^2} & \frac{2\beta\mu_3}{\delta_2^3} & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & \frac{6g_b(0)}{\delta_k^4}
\end{pmatrix}.$$

By (4.2) and (4.3) we get

$$
\begin{aligned}
&\left|\mathcal{H}_{\bar\mu,\bar\delta}(0,0,\bar\mu_0,\bar\delta_0)\right| \\
&=\left|\begin{pmatrix}
2h_a\mu_1^2+2\alpha_a & -\beta & \dots & 0 & 2h_{ab}\mu_1^2 & 0 & \dots & 0\\
-\frac{\alpha_a^2}{\beta} & 2\alpha_a & \dots & 0 & 0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 3\alpha_a & 0 & 0 & \dots & 0\\
2h_{ab}\delta_1^2 & 0 & \dots & 0 & 2h_b\delta_1^2+2\alpha_b & -\beta & \dots & 0\\
0 & 0 & \dots & 0 & -\frac{\alpha_b^2}{\beta} & 2\alpha_b & \dots & 0\\
\vdots & \vdots & \ddots & \vdots & \vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 0 & 0 & 0 & \dots & 3\alpha_b
\end{pmatrix}\right| \\
&=\left|\begin{pmatrix}
\mathcal{A} & 2h_{ab}\mu_1^2\mathcal{L}\\
2h_{ab}\delta_1^2\mathcal{L} & \mathcal{B}
\end{pmatrix}\right|
\end{aligned}
$$

where

$$
\mathcal{A}:=\begin{pmatrix}
2h_a\mu_1^2+2\alpha_a & -\beta & \dots & 0\\
-\frac{\alpha_a^2}{\beta} & 2\alpha_a & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 3\alpha_a
\end{pmatrix},
$$

$$
\mathcal{B}:=\begin{pmatrix}
2h_b\delta_1^2+2\alpha_b & -\beta & \dots & 0\\
-\frac{\alpha_b^2}{\beta} & 2\alpha_b & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 3\alpha_b
\end{pmatrix}
$$

and

$$
\mathcal{L}:=\begin{pmatrix}
1 & 0 & \dots & 0\\
0 & 0 & \dots & 0\\
\vdots & \vdots & \ddots & \vdots\\
0 & 0 & \dots & 0
\end{pmatrix}.
$$

In order to prove (4.1) we shall show that

$$
\begin{cases}
\mathcal{A}x+2h_{ab}\mu_1^2\mathcal{L}y=0\\
2h_{ab}\delta_1^2\mathcal{L}x+\mathcal{B}y=0
\end{cases}
\quad\Longrightarrow\quad x=y=0.
$$

By the first equation we deduce

$$
x=-2h_{ab}\mu_1^2\left(\mathcal{A}^{-1}\mathcal{L}\right)y,
$$

because by Remark 4.3 and by (4.2) we get

$$
|\mathcal{A}|=\alpha_a^{k-1}\left(8kh_a\mu_1^2+2h_{ab}(2k+1)\mu_1\delta_1\right)\neq 0.
$$

Therefore, by the second equation we get

$$
\left[\mathcal{B}-4h_{ab}^2\mu_1^2\delta_1^2\left(\mathcal{L}\mathcal{A}^{-1}\mathcal{L}\right)\right]y=0.
$$

We point out that

$$\mathcal{B}-4h_{ab}^2\mu_1^2\delta_1^2\left(\mathcal{L}\mathcal{A}^{-1}\mathcal{L}\right)=\begin{pmatrix} 2h_b\delta_1^2+2\alpha_b-4h_{ab}^2\mu_1^2\delta_1^2a_{11} & -\beta & \dots & 0\\ -\frac{\alpha_b^2}{\beta} & 2\alpha_b & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 3\alpha_b\end{pmatrix},$$

where a_{11} is the element in the first row and in the first column of the matrix $\mathcal{A}^{-1}$, namely

$$a_{11}=\frac{\alpha_a^{k-1}(2k-1)}{|\mathcal{A}|}=\frac{2k-1}{8kh_a\mu_1^2+2h_{ab}(2k+1)\mu_1\delta_1}.$$

Finally, by Remark 4.3 and by (4.3) we get

$$\left|\mathcal{B}-4h_{ab}^2\mu_1^2\delta_1^2\left(\mathcal{L}\mathcal{A}^{-1}\mathcal{L}\right)\right|=\left|\begin{pmatrix} 2h_b\delta_1^2+2\alpha_b-4h_{ab}^2\mu_1^2\delta_1^2a_{11} & -\beta & \dots & 0\\ -\frac{\alpha_b^2}{\beta} & 2\alpha_b & \dots & 0\\ \vdots & \vdots & \ddots & \vdots\\ 0 & 0 & \dots & 3\alpha_b\end{pmatrix}\right|$$

$$=\alpha_b^{k-1}\left[\left(2h_b\delta_1^2+2\alpha_b-4h_{ab}^2\mu_1^2\delta_1^2a_{11}\right)(2k-1)-\alpha_b(2k-3)\right]$$

$$=\frac{\left(64k^2h_ah_b+32kh_{ab}^2\right)\mu_1^2\delta_1^2}{8kh_a\mu_1^2+2h_{ab}(2k+1)\mu_1\delta_1}\neq 0.$$

That proves our claim. □

Lemma 4.2. *$x=0$ is a non degenerate critical point of the function Γ defined in* (2.19).

Proof. Let us compute the Hessian matrix $\mathcal{H}\Gamma(0)$. We have

$$\partial_{x_i}\Gamma(x)=-(N+2)\int\limits_{\mathbb{R}^N}\frac{y_i+x_i}{\left(1+|y+x|^2\right)^{\frac{N+4}{2}}}\frac{1}{|y|^{N-2}}$$

and

$$\partial^2_{x_ix_j}\Gamma(x)=-(N+2)\int\limits_{\mathbb{R}^N}\left[-(N+4)\frac{(y_i+x_i)(y_j+x_j)}{\left(1+|y+x|^2\right)^{\frac{N+6}{2}}}+\frac{\delta_{ij}}{\left(1+|y+x|^2\right)^{\frac{N+4}{2}}}\right]\frac{1}{|y|^{N-2}}.$$

In particular $\partial^2_{x_ix_j}\Gamma(0)=0$ if $i\neq j$ and

$$\partial^2_{x_ix_i}\Gamma(0)=-(N+2)\int\limits_{\mathbb{R}^N}\left[-(N+4)\frac{y_i^2}{\left(1+|y|^2\right)^{\frac{N+6}{2}}}+\frac{1}{\left(1+|y|^2\right)^{\frac{N+4}{2}}}\right]\frac{1}{|y|^{N-2}}.$$

Taking into account that

$$\int\limits_{\mathbb{R}^N}\frac{y_i^2}{\left(1+|y|^2\right)^{\frac{N+6}{2}}}\frac{1}{|y|^{N-2}}=\frac{1}{N}\int\limits_{\mathbb{R}^N}\frac{|y|^2}{\left(1+|y|^2\right)^{\frac{N+6}{2}}}\frac{1}{|y|^{N-2}}$$

we have

$$\partial^2_{x_ix_i}\Gamma(0)=-\frac{N+2}{N}\int\limits_{\mathbb{R}^N}\frac{N-4y_1^2}{\left(1+|y|^2\right)^{\frac{N+6}{2}}}\frac{1}{|y|^{N-2}}.$$

We are going to prove that

$$\int\limits_{\mathbb{R}^N} \frac{N-4|y|^2}{(1+|y|^2)^{\frac{N+6}{2}}} \frac{1}{|y|^{N-2}} \neq 0.$$

The claim immediately follows.
It holds

$$\begin{aligned}
&\int\limits_{\mathbb{R}^N} \frac{N-4|y|^2}{(1+|y|^2)^{\frac{N+6}{2}}} \frac{1}{|y|^{N-2}} = \omega_N \int\limits_0^{+\infty} r \frac{N-4r^2}{(1+r^2)^{\frac{N+6}{2}}} dr \\
&= \omega_N (N+4) \int\limits_0^{+\infty} \frac{r}{(1+r^2)^{\frac{N+6}{2}}} dr - 4\omega_N \int\limits_0^{+\infty} \frac{r}{(1+r^2)^{\frac{N+4}{2}}} dr \\
&= -\omega_N \left(\frac{1}{(1+r^2)^{\frac{N+4}{2}}} \right)\Bigg|_0^{+\infty} + \frac{4}{N+2} \omega_N \left(\frac{1}{(1+r^2)^{\frac{N+2}{2}}} \right)\Bigg|_0^{+\infty} \\
&= \omega_N \frac{N-2}{N+2}.
\end{aligned}$$

□

Remark 4.3. *It holds*

$$\left| \begin{pmatrix} \gamma & -\beta & 0 & \dots & 0 \\ -\frac{\alpha^2}{\beta} & 2\alpha & -\beta & \dots & 0 \\ 0 & -\frac{\alpha^2}{\beta} & 2\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 3\alpha \end{pmatrix} \right| = \alpha^{k-1} \left[\gamma(2k-1) - \alpha(2k-3) \right], \tag{4.4}$$

where k denotes the dimension of the above matrix.

Proof. Let us introduce the tridiagonal matrix of order n defined by

$$A_n := \begin{pmatrix} 2\alpha & -\beta & 0 & \dots & 0 \\ -\frac{\alpha^2}{\beta} & 2\alpha & -\beta & \dots & 0 \\ 0 & -\frac{\alpha^2}{\beta} & 2\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 2\alpha \end{pmatrix}.$$

Arguing by induction one can easily prove that $|A_n| = (n+1)\alpha^n$. An easy computation shows that

$$\begin{aligned}
&\left| \begin{pmatrix} \gamma & -\beta & 0 & \dots & 0 \\ -\frac{\alpha^2}{\beta} & 2\alpha & -\beta & \dots & 0 \\ 0 & -\frac{\alpha^2}{\beta} & 2\alpha & \dots & 0 \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & 0 & \dots & 3\alpha \end{pmatrix} \right| \\
&= \gamma \left[3\alpha |A_{k-2}| - \alpha^2 |A_{k-3}| \right] - \alpha^2 \left[3\alpha |A_{k-3}| - \alpha^2 |A_{k-4}| \right] \\
&= \gamma \alpha^{k-1}(2k-1) - \alpha^k (2k-3) = \alpha^{k-1} \left[\gamma(2k-1) - \alpha(2k-3) \right]
\end{aligned}$$

and the claim follows. □

5. The linear problem

Let us introduce the linear operator $L : K^{\perp} \to K^{\perp}$ defined by

$$L(\phi) := \Pi^{\perp} \{\phi - i^{*} [f'(V)\,\phi]\}, \tag{5.1}$$

where $f'(V) = p|V|^{p-1}$, V is defined in (2.9) and $p = \frac{N+2}{N-2}$. In what follows we study the invertibility of the map L, starting with an a-priori estimate for solutions $\phi \in K^{\perp}_{\bar d,\xi}$ of $L_{\bar d,\xi}(\phi) = h$, for some right hand side h with bounded $\|\cdot\|$-norm. We have the validity of the following

Lemma 5.1. *For any $\eta > 0$, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\bar\tau, \bar\sigma$ in $\mathbb{R}^{Nk}$ and any $\bar\mu, \bar\delta$ in $\mathbb{R}^k_+$ satisfying (2.7) and for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\|L(\phi)\| \geq c\|\phi\| \quad \text{for all } \phi \in K^{\perp}.$$

Proof. We argue by contradiction. Assume there exist sequences $\varepsilon_n \to 0$, $\bar\tau_n$, $\bar\sigma_n \in \mathbb{R}^{Nk}$, $\bar\mu_n$, $\bar\delta_n \in \mathbb{R}^k_+$ where $\tau_{in} \to \tau_i \in \mathbb{R}^N$, $\sigma_{in} \to \sigma_i$, with $|\tau_i|$, $|\sigma_i| \leq \delta$, for $i = 1, \ldots, k$, and $\mu_{jn} \to \mu_j > 0$, $\delta_{jn} \to \delta_j > 0$, for $j = 1, \ldots, k$, and functions $\phi_n, \psi_n \in K^{\perp}$ such that

$$L(\phi_n) = \psi_n, \ \|\phi_n\| = 1 \text{ and } \|\psi_n\| \to 0 \quad \text{as} \quad n \to \infty. \tag{5.2}$$

¿From the definition of (5.1), we get the existence of $\zeta_n \in K$ such that

$$\phi_n - i^{*} [f'(V)\,\phi_n] = \psi_n + \zeta_n \tag{5.3}$$

Step 1. We prove that

$$\|\zeta_n\| \to 0. \tag{5.4}$$

By definition, we write $\zeta_n = \sum\limits_{\substack{h=0,1,\ldots,N\\ i=1,\ldots,k}} \alpha_n^{ih} PZ^j_{\mu_{in},a_{in}} + \sum\limits_{\substack{h=0,1,\ldots,N\\ i=1,\ldots,k}} \beta_n^{ih} PZ^j_{\delta_{in},b_{in}}$. To prove (5.4) it is enough to show that $\mu_{in}\alpha_n^{ih} \to 0$ and $\delta_{in}\beta_n^{ih} \to 0$ as $n \to \infty$, for all i, h. We will do it for α_n^{ih}. Thus we multiply (5.3) by $PZ^h_{\mu_{ln},a_{ln}}$, we integrate in Ω and we get

$$\left\langle \zeta_n, PZ^h_{\mu_{ln},a_{ln}} \right\rangle = \int\limits_{\Omega_{\varepsilon_n}} f'(V)\,\phi_n PZ^h_{\mu_{ln},a_{ln}} dx. \tag{5.5}$$

By Lemma 5.2 we deduce that

$$\mu_{ln}^2 \left\langle \zeta_n, PZ^h_{\mu_{ln},a_{ln}} \right\rangle = \alpha_n^{lh}[c_h + o(1)] + o(1) \left(\sum_{j\neq h} \alpha_n^{lj} + \sum \beta_n^{lj} \right) \tag{5.6}$$

Moreover, using the orthogonality condition $\left\langle \phi, PZ^j_{\mu_{ln},a_{ln}}\right\rangle = 0$ we deduce

$$\int_{\Omega_{\varepsilon_n}} f'(V)\,\phi_n PZ^h_{\mu_{ln},a_{ln}} = \int_{\Omega_{\varepsilon_n}} f'(V)\,\phi_n\left(PZ^h_{\mu_{ln},a_{ln}} - Z^h_{\mu_{ln},a_{ln}}\right)$$
$$+ \int_{\Omega_{\varepsilon_n}} \left[f'(V) - pU^{p-1}_{\mu_{ln},a_{ln}}\right]\phi_n Z^h_{\mu_{ln},a_{ln}}$$
$$\le |f'(V)|_{\frac{N}{2}}\,|\phi_n|_{\frac{2N}{N-2}}\left|PZ^h_{\mu_{ln},a_{ln}} - Z^h_{\mu_{ln},a_{ln}}\right|_{\frac{2N}{N-2}}$$
$$+ \left|f'(V) - pU^{p-1}_{\mu_{ln},a_{ln}}\right|_{\frac{N}{2}}|\phi_n|_{\frac{2N}{N-2}}\left|Z^h_{\mu_{ln},a_{ln}}\right|_{\frac{2N}{N-2}}$$
$$= \frac{1}{\mu_{ln}}o(1) \tag{5.7}$$

Finally (5.4) follows by (5.5), (5.6) and (5.7).

Step 2. Let us define

$$u_n := \phi_n - \psi_n - \zeta_n, \text{ so that } \|u_n\| \to 1. \tag{5.8}$$

Then equation (5.3) gets rewritten as

$$\begin{cases} -\Delta u_n = f'(V)\,u_n + f'(V)\,(\psi_n + \zeta_n) & \text{in } \Omega_{\varepsilon_n}, \\ u_n = 0 & \text{on } \partial\Omega_{\varepsilon_n}. \end{cases} \tag{5.9}$$

We prove that

$$\liminf_n \int_{\Omega_{\varepsilon_n}} f'(V)\,u_n^2 = c^2 > 0. \tag{5.10}$$

We multiply (5.9) by u_n we deduce that

$$\|u_n\|^2 = \int_{\Omega_{\varepsilon_n}} f'(V)\,u_n^2 + \int_{\Omega_{\varepsilon_n}} f'(V)\,(\psi_n + \zeta_n)\,u_n. \tag{5.11}$$

By Hölder's inequality, (5.2) and (5.4) we get

$$\left|\int_{\Omega_{\varepsilon_n}} f'(V)\,(\psi_n + \zeta_n)\,u_n\right| \le |f'(V)|_{\frac{N}{2}}\,|\psi_n + \zeta_n|_{\frac{2N}{N-2}}\,|u_n|_{\frac{2N}{N-2}}$$
$$\le c\,\|\psi_n + \zeta_n\|\,\|u_n\| = o(1). \tag{5.12}$$

We conclude that (5.10) follows by (5.8), (5.11) and (5.12).

Step 3

Let us define smooth cut off functions around each annuli $\mathcal{A}_{ln}$ and $\mathcal{B}_{ln}$ defined in (3.22) around $B(a, r_a\varepsilon)$ and around $B(b, r_b\varepsilon)$, respectively. Namely

$$\mathcal{A}_{ln} := B(a, \sqrt{\mu_{ln}\mu_{l-1,n}})\backslash B(a, \sqrt{\mu_{ln}\mu_{l+1,n}}) \text{ and } \mathcal{B}_{ln} := B(b, \sqrt{\delta_{ln}\delta_{l-1,n}})\backslash B(b, \sqrt{\delta_{ln}\delta_{l+1,n}}),$$

with the convention that $\mu_{0n} = \mu_{1n}^{-1}\rho^2$ for some $\rho > 0$ small and $\mu_{k+1,n} = \mu_{kn}^{-1} r_a^2\varepsilon^2$ and that $\delta_{0n} = \delta_{1n}^{-1}\rho^2$ for some $\rho > 0$ small and $\delta_{k+1,n} = \delta_{kn}^{-1} r_a^2\varepsilon^2$.

For any $j = 1, \dots, k$, let $\chi^a_{j,n}$ be a smooth cut-off function such that

$$\begin{cases} \chi^a_{jn}(x) = 1 \ \text{ if } \ \sqrt{\mu_{jn}\mu_{j+1n}} \le |x-a| \le \sqrt{\mu_{jn}\mu_{j-1n}}, \\ \chi^a_{jn}(x) = 0 \ \text{ if } \ |x-a| \le \frac{\sqrt{\mu_{jn}\mu_{j+1n}}}{2} \ \text{ or } \ |x-a| \ge 2\sqrt{\mu_{jn}\mu_{j-1n}}, \\ |\nabla \chi^a_{jn}(x)| \le \frac{2}{\sqrt{\mu_{jn}\mu_{j-1n}}} \ \text{ and } |\nabla^2 \chi^j_n(x)| \le \frac{4}{\mu_{jn}\mu_{j-1n}}. \end{cases} \tag{5.13}$$

Furthermore $j = 1, \dots, k$, let $\chi^b_{j,n}$ be a smooth cut-off function such that

$$\begin{cases} \chi^b_{jn}(x) = 1 \ \text{ if } \ \sqrt{\delta_{jn}\delta_{j+1n}} \le |x-b| \le \sqrt{\delta_{jn}\delta_{j-1n}}, \\ \chi^b_{jn}(x) = 0 \ \text{ if } \ |x-b| \le \frac{\sqrt{\delta_{jn}\delta_{j+1n}}}{2} \ \text{ or } \ |x-b| \ge 2\sqrt{\delta_{jn}\delta_{j-1n}}, \\ |\nabla \chi^b_{jn}(x)| \le \frac{2}{\sqrt{\delta_{jn}\delta_{j-1n}}} \ \text{ and } |\nabla^2 \chi^j_n(x)| \le \frac{4}{\delta_{jn}\delta_{j-1n}}. \end{cases} \tag{5.14}$$

For any $j = 1, \dots, k$ we define

$$\hat{u}^a_{jn}(y) = \mu_{jn}{}^{\frac{N-2}{2}} u_n(\mu_{jn}y + a)\chi^a_{jn}(\mu_{jn}y + a)$$

and

$$\hat{u}^b_{jn}(y) = \delta_{jn}{}^{\frac{N-2}{2}} u_n(\delta_{jn}y + b)\chi^b_{jn}(\delta_{jn}y + b).$$

We will show that, for any $j = 1, \dots, k$,

$$\hat{u}^a_{jn}, \hat{u}^b_{jn} \to 0 \text{ weakly in } \mathrm{D}^{1,2}(\mathbb{R}^N) \text{ and strongly in } \mathrm{L}^q_{loc}(\mathbb{R}^N) \text{ for any } q \in [2, 2^*). \tag{5.15}$$

We will prove this fact for $\hat{u}^a_{jn}$. For simplicity of notation, in what is left of this step we will drop the dependence on a.
Furthermore, let $\rho > 0$ be such that $B(a,\rho) \bigcap B(b,\rho) = \emptyset$ and consider the annuli introduced in (3.22).
It is useful to point out that for $x = \mu_{jn}y + a$

$$\nabla \hat{u}_{jn}(y) = \mu_{jn}{}^{\frac{N}{2}} \left[\nabla u_n(x)\chi_{jn}(x) + u_n(x)\nabla\chi_{jn}(x)\right], \tag{5.16}$$

and

$$\Delta \hat{u}_{jn}(y) = \mu_{jn}{}^{\frac{N+2}{2}} \left[\Delta u_n(x)\chi_{jn}(x) + 2\nabla u_n(x)\nabla\chi_{jn}(x) + u_n(x)\Delta\chi_{jn}(x)\right]. \tag{5.17}$$

First of all, by (5.16) and (5.13) we easily deduce that $\|\hat{u}_{jn}\|_{\mathrm{D}^{1,2}(\mathbb{R}^N)} \le c$.
Therefore, up to a subsequence, $\hat{u}_{jn} \to \hat{u}_j$ weakly in $\mathrm{D}^{1,2}(\mathbb{R}^N)$ and strongly in $\mathrm{L}^q_{loc}(\mathbb{R}^N)$ for any $q \in [2, 2^*)$.
We will show that $\hat{u}_j$ solves the problem

$$\Delta \hat{u}_j + f'\left(U_{1,-\tau_j}\right)\hat{u}_j = 0 \text{ in } \mathbb{R}^N \tag{5.18}$$

and satisfies the orthogonality conditions

$$\int\limits_{\mathbb{R}^N} \nabla Z^h_{1,-\tau_j} \nabla \hat{u}_j = 0, \ h = 0, 1, \dots, N. \tag{5.19}$$

These two facts imply that $\hat{u}_j = 0$, namely (5.15).
We are thus led to prove (5.18) and (5.19). We start with (5.18).

Let us perform the change of variable $x = \mu_{jn} y + a$, $y \in \Omega_n^j := \frac{\Omega_{\varepsilon_n} - a}{\mu_{jn}}$. By (5.17) and (5.9) we get for any $\varphi \in C_0^\infty(\mathbb{R}^N)$

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N} \nabla \hat{u}_{jn}(y) \nabla \varphi(y) dy = \int\limits_{\mathbb{R}^N} {\mu_{jn}}^2 f'(V(\mu_{jn} y + a)) \hat{u}_{jn}(y) \varphi(y) dy \\
&+ \int\limits_{\mathbb{R}^N} \mu_{jn}^{\frac{N+2}{2}} f'\left(V(\mu_{jn} y + a)\right) \left(\psi_n(\mu_{jn} y + a) + \zeta_n(\mu_{jn} y + a)\right) \chi_n^j(\mu_{jn} y + a) \varphi(y) dy \\
&+ 2\mu_{jn}^{\frac{N+2}{2}} \int\limits_{\mathbb{R}^N} \left[\nabla u_n(\mu_{jn} y + a) \nabla \chi_{jn}(\mu_{jn} y + a) + u_n(x) \Delta \chi_{jn}(\mu_{jn} y + a)\right] \varphi(y) dy \\
&=: I_1 + I_2 + I_3 + I_4.
\end{aligned} \tag{5.20}
$$

It is easy to check that $I_2, I_3, I_4 \to 0$. Let us compute the limit of I_1. If we denote $a_{jn} = a + \mu_{jn}\tau_j$, for $\frac{\sqrt{\mu_{jn}\mu_{j+1n}}}{2} \le |\mu_{jn} y| \le 2\sqrt{\mu_{jn}\mu_{j-1n}}$ we have

$$
f'\left(V(\mu_{jn} y + a)\right) = \; = f'\left(\frac{1}{\mu_{jn}^{\frac{N-2}{2}}} U_{1,0}(y + \tau_j) + \sum_{i \ne j} U_{\mu_{in}, a_{in}}(\mu_{jn} y + a) + o(1)\right), \tag{5.21}
$$

with

$$
U_{\mu_{in}, a_{in}}(\mu_{jn} y + a) = \begin{cases} O\left(\dfrac{1}{{\mu_{in}}^{\frac{N-2}{2}}}\right) & \text{if } j > i \\ O\left(\dfrac{{\mu_{in}}^{\frac{N-2}{2}}}{{\mu_{jn}}^{N-2}} \dfrac{1}{|y|^{N-2}}\right) & \text{if } i > j. \end{cases} \tag{5.22}
$$

Therefore by (5.21) and (5.22), using the Lebesgue's dominated convergence Theorem we get that

$$
I_1 \to \int\limits_{\mathbb{R}^N} f'\left(U_{1,0}(y + \tau_j)\right) \hat{w}^j(y) \varphi(y) dy.
$$

Thus (5.18) follows by passing to the limit in (5.20).
Let us now prove (5.19). We have

$$
\begin{aligned}
&\int\limits_{\mathbb{R}^N} \nabla Z^h_{1,-\tau_j}(y) \nabla \hat{u}_{jn}(y) dy = \int\limits_{\mathbb{R}^N} f'(U_{1,-\tau_j}(y)) Z^h_{1,-\tau_j}(y) \hat{u}_{jn}(y) dy \\
&= \mu_{jn} \int\limits_{\sqrt{\frac{\mu_{jn}\mu_{j+1n}}{2}} \le |x-a| \le 2\sqrt{\mu_{jn}\mu_{j-1n}}} f'(U_{\mu_{jn}, a_{jn}}(x)) Z^h_{\mu_{jn}, a_{jn}}(x) u_n(x) \chi_{jn}(x) dx \\
&= \mu_{jn} \left[\int\limits_{\mathcal{A}_{jn}} f'\left(U_{\mu_{jn}, a_{jn}}(x)\right) Z^h_{\mu_{jn}, a_{jn}}(x) u_n(x) dx + o\left(1\right)\right].
\end{aligned} \tag{5.23}
$$

Now we observe that, by (5.4) and (5.8),

$$
\mu_{jn} \int\limits_{\Omega_{\varepsilon_n}} \nabla P Z^h_{\mu_{jn}, a_{jn}}(x) \nabla u_n(x) dx = o\left(1\right). \tag{5.24}
$$

On the other hand

$$\mu_{jn}\int\limits_{\Omega_{\varepsilon_n}}\nabla PZ^h_{\mu_{jn},a_{jn}}(x)\nabla u_n(x)dx=\mu_{jn}\int\limits_{\Omega_{\varepsilon_n}}f'\left(U_{\mu_{jn},a_{jn}}(x)\right)Z^h_{\mu_{jn},a_{jn}}(x)u_n(x)dx$$
$$=\mu_{jn}\int\limits_{\mathcal{A}_{jn}}f'\left(U_{\mu_{jn},a_{jn}}(x)\right)Z^h_{\mu_{jn},a_{jn}}(x)u_n(x)dx+o(1) \tag{5.25}$$

since

$$\left|\int\limits_{\Omega_{\varepsilon_n}\setminus B(a_{jn},\rho)}f'(U_{\mu_{jn},a_{jn}})Z^h_{\mu_{jn},a_{jn}}(x)u_n(x)dx\right|$$
$$\leq c\left|Z^h_{\mu_{jn},a_{jn}}\right|_{\frac{2N}{N-2}}|u_n|_{\frac{2N}{N-2}}\left(\int\limits_{\Omega_{\varepsilon_n}\setminus B(a,\rho)}U^{\frac{2N}{N-2}}_{\mu_{jn},a_{jn}}\right)^{\frac{2}{N}}=O(1),$$

and for $l\neq j$

$$\mu_{jn}\left|\int\limits_{\mathcal{A}_{ln}}f'(U_{\mu_{jn},a_{jn}})Z^h_{\mu_{jn},a_{jn}}(x)u_n(x)dx\right|$$
$$\leq c\left|Z^h_{\mu_{jn},a_{jn}}\right|_{\frac{2N}{N-2}}|u_n|_{\frac{2N}{N-2}}\left(\int\limits_{\mathcal{A}_{ln}}U^{\frac{2N}{N-2}}_{\mu_{jn},a_{jn}}\right)^{\frac{2}{N}}=o\left(1\right).$$

¿From (5.23), (5.24) and (5.25) we get (5.19).

Step 4 We show that a contradiction arises with (5.10), by showing that

$$\int\limits_{\Omega_{\varepsilon_n}}f'_{\varepsilon_n}\left(V_{\bar{d}_n,\xi_n}\right)u_n^2=o(1). \tag{5.26}$$

This fact concludes the proof of this Lemma.
Let us prove (5.26). We have

$$\int\limits_{\Omega_{\varepsilon_n}}f'(V)u_n^2=\int\limits_{\Omega_{\varepsilon_n}\setminus\{B(a,\rho)\cup B(b,\rho)\}}f'(V)u_n^2+\sum_{j=1}^k\int\limits_{\mathcal{A}_{jn}}f'(V)u_n^2+\sum_{j=1}^k\int\limits_{\mathcal{B}_{jn}}f'(V)u_n^2.$$

Now, it holds

$$\int\limits_{\Omega_{\varepsilon_n}\setminus\{B(a,\rho)\cup B(b,\rho)\}}f'(V)u_n^2\leq c\sum_{i=1}^k\left(\mu_{in}^2+\delta_{in}^2\right)\int\limits_{\Omega_{\varepsilon_n}\setminus B(\xi,\rho)}u_n^2=o(1).$$

Finally, for any j, we scale $x = \mu_{jn} y + a$ and we get

$$\begin{aligned}
\int\limits_{\mathcal{A}_{jn}} f'(V)u_n^2 &\le c\sum_{i=1}^{k} \int\limits_{\mathcal{A}_{jn}} U^{p-1}_{\mu_{in},a_{in}} u_n^2 + c\sum_{i=1}^{k} \int\limits_{\mathcal{A}_{jn}} U^{p-1}_{\delta_{in},b_{in}} u_n^2 \\
&\le c\sum_{i=1}^{k} \mu_{in}^2 \int\limits_{\mathbb{R}^N} \left(\frac{\mu_{in}}{\mu_{in}^2 + \mu_{jn}^2 |y-\tau_i|^2}\right)^2 \hat{u}_{jn}^2 + o(1) \\
&\le c\sum_{i<j} \left(\frac{\mu_{jn}}{\mu_{in}}\right)^2 + c\int\limits_{\mathbb{R}^N} \left(\frac{1}{1+|y|^2}\right)^2 \hat{u}_{jn}^2 + c\sum_{i>j} \left(\frac{\mu_{in}}{\mu_{jn}}\right)^2 + o(1) \\
&= o(1),
\end{aligned}$$

where the last estimate follows from the fact that $\left(\frac{1}{1+|y|^2}\right)^2 \in \mathrm{L}^{\frac{N}{2}}(\mathbb{R}^N)$ and (5.15) holds. In a similar way we prove that $\int\limits_{\mathcal{B}_{jn}} f'(V)u_n^2 = o(1)$. That concludes the proof.

□

Next result states the invertibility of the operator defined in (5.1).

Proposition 5.2. *For any $\eta > 0$, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\bar\tau, \bar\sigma$ in $\mathbb{R}^{Nk}$ and any $\bar\mu, \bar\delta$ in $\mathbb{R}^k_+$ satisfying (2.7) and for any $h \in K^\perp_{\bar d,\xi}$ there exists a unique $\phi \in K^\perp_{\bar d,\xi}$ solution to $L(\phi) = h$, for any $\varepsilon \in (0, \varepsilon_0)$. Furthermore*

$$\|h\| \ge c\|\phi\|. \tag{5.27}$$

Proof. Notice that the problem $L(\phi) = h$ in ϕ gets re-written as

$$\phi + K(\phi) = \bar h \quad \text{in} \quad K^\perp_{\bar d,\xi} \tag{5.28}$$

where $\bar h$ is defined by duality and $K : K^\perp_{\bar d,\xi} \to K^\perp_{\bar d,\xi}$ is a linear compact operator. Using Fredholm's alternative, showing that equation (5.28) has a unique solution for each $\bar h$ is equivalent to showing that the equation has a unique solution for $\bar h = 0$, which in turn follows from Lemma 5.1. The estimate (5.27) follows directly from Lemma 5.1. This concludes the proof of Proposition 5.2.

□

Remark 5.2. It holds

$$\begin{aligned}
\langle PZ^j_{\mu_{i\varepsilon},a_{i\varepsilon}}, PZ^h_{\mu_{l\varepsilon},a_{l\varepsilon}}\rangle &= o\left(\frac{1}{\mu_{i\varepsilon}^2}\right) \text{ if } l > i, \\
\langle PZ^j_{\mu_{i\varepsilon},a_{i\varepsilon}}, PZ^h_{\mu_{i\varepsilon},a_{i\varepsilon}}\rangle &= o\left(\frac{1}{\mu_{i\varepsilon}^2}\right) \text{ if } j \ne h, \\
\langle PZ^j_{\mu_{i\varepsilon},a_{i\varepsilon}}, PZ^j_{\mu_{i\varepsilon},a_{i\varepsilon}}\rangle &= \frac{c_j}{\mu_{j\varepsilon}^2}(1+o(1)) \\
\langle PZ^j_{\mu_{i\varepsilon},a_{i\varepsilon}}, PZ^h_{\delta_{l\varepsilon},b_{l\varepsilon}}\rangle &= o\left(\frac{1}{\mu_{i\varepsilon}^2}\right),\ o\left(\frac{1}{\delta_{l\varepsilon}^2}\right)
\end{aligned}$$

for some positive constants c_0 and $c_1 = \cdots = c_N$.

6. Proof of Proposition 2.1

The main point to prove Proposition 2.1 is to estimate the $\|\cdot\|$-norm of the error term R defined in (2.13). This is the content of next

Lemma 6.1. *For any $\eta > 0$, there exists $\varepsilon_0 > 0$ and $c > 0$ such that for any $\bar\tau, \bar\sigma$ in $\mathbb{R}^{Nk}$ and any $\bar\mu, \bar\delta$ in $\mathbb{R}^k_+$ satisfying (2.7) and for any $\varepsilon \in (0, \varepsilon_0)$, we have*

$$\|R\| \le \begin{cases} c\varepsilon^{\frac{N-2}{2k}\frac{p}{2}} \text{ if } N \ge 7, \\ c\varepsilon^{\frac{N-2}{2k}} |\ln \varepsilon| \text{ if } N = 6, \\ c\varepsilon^{\frac{N-2}{2k}} \text{ if } 3 \le N \le 5. \end{cases}$$

Proof. Since $P_\varepsilon U_{\delta,\xi} = i^*\left(U^p_{\delta,\xi}\right) = i^*\left[f\left(U_{\delta,\xi}\right)\right]$ for any $\delta > 0$ and point $\xi \in \Omega_\varepsilon$, we can write

$$R = \Pi^\perp\left(i^*\left[f(V) - \sum_{j=1}^k (-1)^{j+1} f(U_{\mu_{j\varepsilon},a_{j\varepsilon}}) + \sum_{j=1}^k (-1)^{j+1} f(U_{\delta_{j\varepsilon},b_{j\varepsilon}})\right]\right).$$

Therefore by (2.1) we deduce

$$\|R\| \le c\left|f(V) - \sum_{j=1}^k (-1)^{j+1} f(U_{\mu_{j\varepsilon},a_{j\varepsilon}}) + \sum_{j=1}^k (-1)^{j+1} f(U_{\delta_{j\varepsilon},b_{j\varepsilon}})\right|_{\frac{2N}{N+2}}.$$

Let us call

$$I := \left|f(V) - \sum_{j=1}^k (-1)^{j+1} f(U_{\mu_{j\varepsilon},a_{j\varepsilon}}) + \sum_{j=1}^k (-1)^{j+1} f(U_{\delta_{j\varepsilon},b_{j\varepsilon}})\right|_{\frac{2N}{N+2}}.$$

The claim will follow if we prove that

$$I \le \begin{cases} c\varepsilon^{\frac{N-2}{2k}\frac{p}{2}} \text{ if } N \ge 7, \\ c\varepsilon^{\frac{N-2}{2k}} |\ln \varepsilon| \text{ if } N = 6, \\ c\varepsilon^{\frac{N-2}{2k}} \text{ if } 3 \le N \le 5. \end{cases} \tag{6.1}$$

To simplify notations, we call $q = \frac{2N}{N+2}$. We have

$$\begin{aligned} I &\le \left|f(V) - \sum_{j=1}^k (-1)^{j+1} f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}}) + \sum_{j=1}^k (-1)^{j+1} f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}})\right|_q \\ &+ \sum_{j=1}^k \left|f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}}) - f(U_{\mu_{j\varepsilon},a_{j\varepsilon}})\right|_q + \sum_{j=1}^k \left|f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}}) - f(U_{\delta_{j\varepsilon},b_{j\varepsilon}})\right|_q \\ &= A + B + C. \end{aligned} \tag{6.2}$$

We start with the estimate of A. Let $\rho > 0$ so that $B(a,\rho) \bigcap B(b,\rho) = \emptyset$. We write

$$\begin{aligned} A^q &= \int_{\Omega_\varepsilon \setminus (B(a,\rho) \bigcup B(b,\rho))} \dots + \int_{B(a,\rho)\setminus B(a, r_a\varepsilon)} \dots + \int_{B(b,\rho)\setminus B(b, r_b\varepsilon)} \dots \\ &= A_1 + A_2 + A_3 \end{aligned}$$

In $\Omega_\varepsilon \setminus (B(a,\rho)\bigcup B(b,\rho))$ the function V is uniformly bounded by $\varepsilon^{\frac{N-2}{4k}}$, so we get

$$\int\limits_{\Omega_\varepsilon\setminus(B(a,\rho)\bigcup B(b,\rho))} \left| f(V) - \sum_{j=1}^{k}(-1)^{j+1} f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}}) + \sum_{j=1}^{k}(-1)^{j+1} f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}}) \right|^q \leq C\varepsilon^{\frac{N-2}{2k}\frac{p}{2}q},$$

thus $A_1 = O(\varepsilon^{\frac{N-2}{2k}\frac{p}{2}q})$. We next estimate A_2.
Let us then introduce the annuli $\mathcal{A}_l$ already defined in (3.22), namely for all $l = 1,\dots,k$, $\mathcal{A}_l := B(a,\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}) \setminus B(a,\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}})$. with $\mu_{0\varepsilon} := \mu_{1\varepsilon}^{-1}\rho^2$ and $\mu_{k+1\varepsilon} := \mu_{k\varepsilon}^{-1} r_a^2\varepsilon^2$, so that $B(a,\rho)\setminus B(a,r_a\varepsilon) = \bigcup_{l=1}^{k}\mathcal{A}_l$. We have

$$A_2 = \sum_{l=1}^{k}\int_{\mathcal{A}_l}\left| f(V) - \sum_{j=1}^{k}(-1)^{j+1} f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}}) + \sum_{j=1}^{k}(-1)^{j+1} f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}}) \right|^q$$

To simplify again the notation, we will use U_j to denote the function $U_{\mu_{j\varepsilon},a_{j\varepsilon}}$. Fix l. We have

$$\begin{aligned}
&\int_{\mathcal{A}_l}\left| f(V) - \sum_{j=1}^{k}(-1)^{j+1} f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}}) + \sum_{j=1}^{k}(-1)^{j+1} f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}}) \right|^q \\
&\leq c\int_{\mathcal{A}_l}\left| f(V) - \sum_{j=1}^{k}(-1)^{j+1} f(P_\varepsilon U_j)\right|^q + O\left(\varepsilon^{\frac{N-2}{2k}\frac{p}{2}q}\right) \\
&\leq c\left(\sum_{i\neq l}\int_{\mathcal{A}_l}|U_l^{p-1}U_i|^q + \sum_{i\neq l}\int_{\mathcal{A}_l}U_i^{pq}\right) + O\left(\varepsilon^{\frac{N-2}{2k}\frac{p}{2}q}\right).
\end{aligned}$$

Since $pq = p+1$, arguing as in the proof of estimate (3.27) we obtain that $\int_{\mathcal{A}_l} U_i^{pq} = O(\varepsilon^{\frac{N-2}{2k}\frac{p}{2}q})$. On the other hand, if $N > 6$, we get

$$\begin{aligned}
&\int\limits_{\mathcal{A}_l}\left|U_l^{p-1}U_i\right|^q \leq c\int\limits_{\mathcal{A}_l}\left(\frac{\mu_{l\varepsilon}^2}{(\mu_{l\varepsilon}^2+|x-a_{l\varepsilon}|^2)^2}\right)^q\left(\frac{\mu_{i\varepsilon}^{\frac{N-2}{2}}}{(\mu_{i\varepsilon}^2+|x-a_{i\varepsilon}|^2)^{\frac{N-2}{2}}}\right)^q \\
&= c\mu_{i\varepsilon}^{N-\frac{N-2}{2}q}\mu_{l\varepsilon}^{2q}\int\limits_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{i\varepsilon}}\leq|y|\leq\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{i\varepsilon}}}\frac{1}{(\mu_{l\varepsilon}^2+\mu_{i\varepsilon}^2|y|^2)^{2q}}\frac{1}{(1+|y|^2)^{\frac{N-2}{2}q}} \\
&= \begin{cases} O\left(\mu_{i\varepsilon}^{N-\frac{N+6}{2}q}\mu_{l\varepsilon}^{2q}\right)\displaystyle\int\limits_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{i\varepsilon}}\leq|y|\leq\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{i\varepsilon}}}\frac{1}{|y|^{4q}}\frac{1}{(1+|y|^2)^{\frac{N-2}{2}q}}, & \text{if } l > i, \\
O\left(\mu_{i\varepsilon}^{N-\frac{N-2}{2}q}\mu_{l\varepsilon}^{-2q}\right)\displaystyle\int\limits_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{i\varepsilon}}\leq|y|\leq\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{i\varepsilon}}}\frac{1}{(1+|y|^2)^{\frac{N-2}{2}q}}, & \text{if } l < i,
\end{cases} \\
&= O\left(\varepsilon^{\frac{N-2}{2k}(\frac{4}{N-2}+\frac{p}{2})q}\right).
\end{aligned}$$

If $N<6$ we get

$$\int\limits_{\mathcal{A}_l}\left|U_l^{p-1}U_i\right|^q\le c\mu_{l\varepsilon}^{N-2q}\mu_{i\varepsilon}^{\frac{N-2}{2}q}\int\limits_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{l\varepsilon}}}\frac{1}{(1+|y|^2)^{2q}}\frac{1}{(\mu_{i\varepsilon}^2+\mu_{l\varepsilon}^2|y|^2)^{\frac{N-2}{2}q}}$$

$$=\begin{cases}O\left(\mu_{l\varepsilon}^{N-2q}\mu_{i\varepsilon}^{-\frac{N-2}{2}q}\right)\displaystyle\int\limits_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{l\varepsilon}}\le|y|\le\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{l\varepsilon}}}\frac{1}{(1+|y|^2)^{2q}}, & \text{if } l>i,\\ O\left(\mu_{l\varepsilon}^{N-Nq}\mu_{i\varepsilon}^{\frac{N-2}{2}q}\right)\displaystyle\int\limits_{\frac{\sqrt{\mu_{l\varepsilon}\mu_{l+1\varepsilon}}}{\mu_{l\varepsilon}}\le|y|\le\frac{\sqrt{\mu_{l\varepsilon}\mu_{l-1\varepsilon}}}{\mu_{l\varepsilon}}}\frac{1}{|y|^{(N-2)q}}\frac{1}{(1+|y|^2)^{2q}}, & \text{if } l<i,\end{cases}$$

$$=O\left(\varepsilon^{\frac{N-2}{2k}q}\right).$$

A similar arguments allows to prove that if $N=6$ then

$$\int\limits_{\mathcal{A}_l}\left|U_l^{p-1}U_i\right|^q=O\left(\varepsilon^{\frac{N-2}{2k}q}|\ln\varepsilon|^q\right).$$

We thus conclude that

$$A_2\le\begin{cases}c\varepsilon^{\frac{N-2}{2k}\frac{p}{2}q} & \text{if } N\ge 7,\\ c\varepsilon^{\frac{N-2}{2k}q}|\ln\varepsilon|^q & \text{if } N=6,\\ c\varepsilon^{\frac{N-2}{2k}q} & \text{if } 3\le N\le 5.\end{cases}$$

A similar estimate can be obtained for A_3. We proved that

$$A\le\begin{cases}c\varepsilon^{\frac{N-2}{2k}\frac{p}{2}} & \text{if } N\ge 7,\\ c\varepsilon^{\frac{N-2}{2k}}|\ln\varepsilon| & \text{if } N=6,\\ c\varepsilon^{\frac{N-2}{2k}} & \text{if } 3\le N\le 5.\end{cases}\tag{6.3}$$

Let us now estimate the term B in (6.2). For any fixed i, from Lemma 3.1 we have

$$\int\limits_{\Omega_\varepsilon}|(PU_i)^p-U_i^p|^q\le c\int\limits_{\Omega_\varepsilon}\left|U_i^{p-1}(PU_i-U_i)\right|^q+c\int\limits_{\Omega_\varepsilon}|PU_i-U_i|^{pq}$$

$$\le c\mu_{i\varepsilon}^{\frac{N-2}{2}q}\int\limits_{\Omega_\varepsilon}\left(\frac{\mu_{i\varepsilon}^2}{(\mu_{i\varepsilon}^2+|x-a_{i\varepsilon}|^2)^2}\right)^q$$

$$+c\frac{\varepsilon^{(N-2)q}}{\mu_{i\varepsilon}^{\frac{N-2}{2}q}}\int\limits_{\Omega_\varepsilon}\left(\frac{\mu_{i\varepsilon}^2}{(\mu_{i\varepsilon}^2+|x-a_{i\varepsilon}|^2)^2}\right)^q\frac{1}{|x-a|^{(N-2)q}}$$

$$+c\mu_{i\varepsilon}^{\frac{N+2}{2}q},$$

since

$$\int\limits_{\Omega_\varepsilon}\left(\frac{\mu_{i\varepsilon}^2}{(\mu_{i\varepsilon}^2+|x-a_{i\varepsilon}|^2)^2}\right)^q=\begin{cases}O\left(\mu_{i\varepsilon}^{2q}\right) & \text{if } N\ge 7,\\ O\left(\mu_{i\varepsilon}^{2q}|\ln\mu_{i\varepsilon}|^q\right) & \text{if } N=6,\\ O\left(\mu_{i\varepsilon}^{N-2q}\right) & \text{if } 3\le N\le 5.\end{cases}$$

Therefore

$$B \le \begin{cases} \varepsilon^{\frac{N-2}{2k}\frac{p}{2}} \text{ if } N \ge 7, \\ \varepsilon^{\frac{N-2}{2k}} |\ln \varepsilon| \text{ if } N = 6, \\ \varepsilon^{\frac{N-2}{2k}} \text{ if } 3 \le N \le 5. \end{cases} \tag{6.4}$$

In a very analogous way, one gets a similar estimate for C. Estimates (6.2), (6.3) and (6.4) conclude the proof. □

We have now the tools to give the

Proof of Proposition 2.1. First of all, we point out that in virtue of Proposition 5.2, solving problem (2.10) is equivalent to find a fixed point of the operator

$$T(\phi) := L^{-1}\left(N(\phi) + R\right),\ \phi \in K^{\perp},$$

where R is defined in (2.13) and and

$$N(\phi) := \Pi^{\perp}\left\{i^*\left[f\left(V+\phi\right) - f\left(V\right) - f'\left(V\right)\phi\right]\right\}.$$

By Lemma 5.1 we get

$$\|T(\phi)\| \le c\left(\|N(\phi)\| + \|R\|\right) \text{ and } \|T(\phi_1) - T(\phi_2)\| \le c\,\|N(\phi_1) - N(\phi_2)\|.$$

It is by now standard to prove that

$$\|N(\phi)\| \le c\,|\phi|_{\frac{2N}{N+2}}^{\min\{2,p\}} \text{ and } \|N(\phi_1) - N(\phi_2)\| \le l\,\|\phi_1 - \phi_2\|, \quad \text{for some } l \in (0,1).$$

At this point we consider the set $E = \{\phi : \|\phi\| \le r(\varepsilon)\}$, where

$$r(\varepsilon) = \begin{cases} c\varepsilon^{\frac{N-2}{2k}\frac{p}{2}} \text{ if } N \ge 7, \\ c\varepsilon^{\frac{N-2}{2k}} |\ln \varepsilon| \text{ if } N = 6, \\ c\varepsilon^{\frac{N-2}{2k}} \text{ if } 3 \le N \le 5. \end{cases}$$

We conclude then that, for c small, T is a contraction mapping from E to E, and so it has a unique fixed point ϕ in E. A standard argument shows that $(\bar d, \xi) \to \phi_{\varepsilon,\bar d,\xi}$ is a C^1-map. This concludes the proof.

□

7. Proof of Proposition 2.2

Given the result of Proposition 2.1 we conclude that $V + \phi$, with V defined in (2.9) and ϕ predicted by Proposition 2.1, is a solution to our original problem if we can find $(\bar\tau, \bar\sigma, \bar\mu, \bar\delta) \in \mathbb{R}^{2Nk} \times \mathbb{R}^{2k}_+$ satisfying constraints (2.7) to solve equation (2.11). But this is equivalent to finding critical points to the explicit finite dimensional functional $\widetilde J_\varepsilon$ defined in (2.15), as we prove next.

Proof of Proposition 2.2, Part 1. To simplify the notations, we set $Z^h_{j,a} = Z^h_{\mu_{jve}, a_{j\varepsilon}}$ and $Z^h_{j,b} = Z^h_{\delta_{j\varepsilon}, b_{j\varepsilon}}$. By (2.10) we get

$$\begin{aligned} \nabla \widetilde J_\varepsilon(\bar\tau, \bar\sigma, \bar\mu, \bar\delta) &= J'_\varepsilon\left(V+\phi\right)\left[\nabla V + \nabla\phi\right] \\ &= \sum_{l=0}^{N}\sum_{i=1}^{k} c_a^{li}\left\langle P_\varepsilon Z^l_{ia}, \nabla V + \nabla\phi\right\rangle + \sum_{l=0}^{N}\sum_{i=1}^{k} c_b^{li}\left\langle P_\varepsilon Z^l_{ib}, \nabla V + \nabla\phi\right\rangle, \end{aligned} \tag{7.1}$$

for some vectors c_a^{li} and c_b^{li}. Thus, if $(\bar\tau, \bar\sigma, \bar\mu, \bar\delta)$ is a critical point for $\widetilde J_\varepsilon$, we have

$$\sum_{l=0}^{N}\sum_{i=1}^{k} c_a^{li}\left\langle P_\varepsilon Z_{ia}^l, \nabla V+\nabla\phi\right\rangle+\sum_{l=0}^{N}\sum_{i=1}^{k} c_b^{li}\left\langle P_\varepsilon Z_{ib}^l, \nabla V+\nabla\phi\right\rangle=0. \tag{7.2}$$

Equation (7.2) is equivalent to a homogeneous system of $2(N+1)k$ equations in $2(N+1)k$ variables, the components of the vectors c_a^{li} and c_a^{li}. We shall prove that all the components of c_a^{li} and c_a^{li} are equal to zero, provided ε is small enough, showing that the matrix of coefficients is at main order invertible. This fact gives the proof of the statement.
We start with the following direct computation

$$\partial_{\mu_j} V=\varepsilon^{\frac{2j-1}{2k}} P_\varepsilon Z_{j,a}^0+\varepsilon^{\frac{2j-1}{2k}}\sum_{h=1}^{N} P_\varepsilon Z_{ja}^h \tau_{j\,h},$$

and

$$\nabla_{\tau_j} V=\mu_{j\varepsilon}(P_\varepsilon Z_{j,a}^1,\dots,P_\varepsilon Z_{j,a}^N).$$

And analogous formulas hold true for $\partial_{\delta_j} V$ and $\nabla_{\sigma_j} V$. Now, by Lemma 5.2 one easily gets that the system

$$\sum_{l=0}^{N}\sum_{i=1}^{k} c_a^{li}\left\langle P_\varepsilon Z_{ia}^l, \nabla V\right\rangle+\sum_{l=0}^{N}\sum_{i=1}^{k} c_b^{li}\left\langle P_\varepsilon Z_{ib}^l, \nabla V\right\rangle=0.$$

has, at main order, an invertible matrix as the matrix of coefficients. Thus to get the proof of the result, we need to show that the other part of system (7.2)

$$\sum_{l=0}^{N}\sum_{i=1}^{k} c_a^{li}\left\langle P_\varepsilon Z_{ia}^l, \nabla\phi\right\rangle+\sum_{l=0}^{N}\sum_{i=1}^{k} c_b^{li}\left\langle P_\varepsilon Z_{ib}^l, \nabla\phi\right\rangle=0$$

is of lower order. To get this fact, we need to estimate the scalar products $\left\langle P_\varepsilon Z_{ia}^l \partial_s\phi\right\rangle$ and $\left\langle P_\varepsilon Z_{ib}^l \partial_s\phi\right\rangle$, where ∂_s denotes one of the components of the gradient of ϕ. Now, since $\phi\in K^\perp$, one has $\left\langle P_\varepsilon Z_{ja}^h,\partial_s\phi\right\rangle=-\left\langle\partial_s P_\varepsilon Z_{ja}^h,\phi\right\rangle$. Since $\|\partial_s P_\varepsilon Z_{ja}^h\|=O(\frac{1}{\varepsilon^{\frac{2j-1}{2k}}})$, one easily gets $\left\langle P_\varepsilon Z_{ja}^h,\partial_s\phi\right\rangle=o\left(\left|\left\langle P_\varepsilon Z_{ja}^h,\partial_s V\right\rangle\right|\right)$. A similar argument shows that $\left\langle P_\varepsilon Z_{jb}^h,\partial_s\phi\right\rangle=o\left(\left|\left\langle P_\varepsilon Z_{jb}^h,\partial_s V\right\rangle\right|\right)$. This facts give the result.

□

Remark 7.1. Following the proof and using the estimates contained in the proof of Proposition 2.2, Part 1, above, one gets the following estimate for the components of the vectors c_a^{hj} and c_b^{hj}, for any h and j

$$|c_a^{hj}|\le C\mu_{j\varepsilon}\|\phi\|,\quad |c_b^{hj}|\le C\delta_{j\varepsilon}\|\phi\|. \tag{7.3}$$

To get now the proof of Proposition 2.2, Part 2, we need to estimate the C^1 closeness of $J_\varepsilon\left(V+\phi\right)$ with $J_\varepsilon\left(V\right)$. This is the content of next

Lemma 7.2. *For any $\eta>0$, there exists $\varepsilon_0>0$ such that for any $\varepsilon\in(0,\varepsilon_0)$, we have*

$$J_\varepsilon\left(V+\phi\right)=J_\varepsilon\left(V\right)+o(\varepsilon^{\frac{N-2}{2k}}),$$

C^1-uniformly for any $\bar\tau,\bar\sigma$ in $\mathbb{R}^{Nk}$ and any $\bar\mu,\bar\delta$ in $\mathbb{R}_+^k$ satisfying (2.7).

Proof. We write

$$J_\varepsilon(V+\phi)-J_\varepsilon(V)=\frac{1}{2}\|\phi\|^2 - \int_{\Omega_\varepsilon}[f(V)-\sum_{j=1}^{k}(-1)^{j+1}f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}})+\sum_{j=1}^{k}(-1)^{j+1}f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}})]\phi - \int_{\Omega}[F(V+\phi)-F(V)-f(V)\phi], \tag{7.4}$$

where $F(u):=\frac{1}{p+1}|u|^{p+1}$. Using Hölder inequality and estimates (6.1) and (2.12)

$$\left|\int_{\Omega_\varepsilon}[f(V)-\sum_{j=1}^{k}(-1)^{j+1}f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}})+\sum_{j=1}^{k}(-1)^{j+1}f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}})]\phi\right| \le \left|f(V)-\sum_{j=1}^{k}(-1)^{j+1}f(P_\varepsilon U_{\mu_{j\varepsilon},a_{j\varepsilon}})+\sum_{j=1}^{k}(-1)^{j+1}f(P_\varepsilon U_{\delta_{j\varepsilon},b_{j\varepsilon}})\right|_{\frac{2N}{N+2}}|\phi|_{\frac{2N}{N-2}} = o(\varepsilon^{\frac{N-2}{2k}}). \tag{7.5}$$

On the other hand, by the mean value theorem we get for some $t\in[0,1]$

$$\left|\int_{\Omega_\varepsilon}[F(V+\phi)-F(V)-f(V)\phi]\right| \le \int_{\Omega_\varepsilon}\left|f'(V+t\phi)\phi^2\right| \le c\int_{\Omega_\varepsilon}|V|^{p-1}\phi^2+c\int_{\Omega_\varepsilon}|\phi|^{p+1} \le c\left||V|^{p-1}\right|_{\frac{N}{2}}|\phi|^2_{\frac{2N}{N-2}}+c|\phi|^{p+1}_{\frac{2N}{N-2}}=o(\varepsilon^{\frac{N-2}{2k}}), \tag{7.6}$$

using again (2.12) and taking into account that $\left||V|^{p-1}\right|_{\frac{N}{2}}=O(1)$. Therefore the C^0 closeness follows.

We need to show now that

$$\nabla J_\varepsilon(V+\phi)-\nabla J_\varepsilon(V)=o(\varepsilon^{\frac{N-2}{2k}}). \tag{7.7}$$

The proof of the above estimate is very similar to the proof of Lemma 8.1 in [31]. For completeness, we briefly sketch the principal steps below.

We write

$$\nabla J_\varepsilon(V+\phi)-\nabla J_\varepsilon(V)=[J'_\varepsilon(V+\phi)-J'_\varepsilon(V)][\nabla V]+J'_\varepsilon(V+\phi)[\nabla\phi]. \tag{7.8}$$

Let us use the notation ∂_s to denote one of the partial derivatives in the gradient. As computed in the Proof of Proposition 2.2, Part 1, the function $\partial_s V$ is a linear combination of $\varepsilon^{\frac{2j-1}{2k}}P_\varepsilon Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}$ and $\varepsilon^{\frac{2j-1}{2k}}P_\varepsilon Z^h_{\delta_{j\varepsilon}b_{j\varepsilon}}$, with coefficients uniformly bounded as $\varepsilon\to0$ for for any $\bar\tau,\bar\sigma$ in $\mathbb{R}^{Nk}$ and any $\bar\mu,\bar\delta$ in $\mathbb{R}^k_+$ satisfying (2.7). Thus, in order to estimate the first term in (7.8) it is enough to estimate, for instance

$$[J'(V+\phi)-J'(V)]\left[\varepsilon^{\frac{2j-1}{2k}}P_\varepsilon Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}\right]. \tag{7.9}$$

We write

$$\begin{aligned}&\left[J'\left(V+\phi\right)-J'\left(V\right)\right]\left[\varepsilon^{\frac{2j-1}{2k}}P_\varepsilon Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}\right]\\&=-\int\limits_{\Omega_\varepsilon} f'\left(V\right)\phi\varepsilon^{\frac{2j-1}{2k}}\left[P_\varepsilon Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}-Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}\right]\\&-\int\limits_{\Omega_\varepsilon}\left[f'\left(V\right)-f'\left(U_{\mu_{j\varepsilon}a_{j\varepsilon}}\right)\right]\phi\varepsilon^{\frac{2j-1}{2k}}Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}\\&-\int\limits_{\Omega_\varepsilon}\left[f\left(V+\phi\right)-f\left(V\right)-f'\left(V\right)\phi\right]\varepsilon^{\frac{2j-1}{2k}}P_\varepsilon Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}\\&=: I_1+I_2+I_3,\end{aligned}$$

because $\phi\in K^\perp$. It is immediate to check that $I_1=o\left(\varepsilon^{\frac{N-2}{2k}}\right)$. Let us estimate I_2. Since $\left|\varepsilon^{\frac{2j-1}{2k}}P_\varepsilon Z^h_{\mu_{j\varepsilon}a_{j\varepsilon}}\right|\le cU_{\mu_{j\varepsilon}a_{j\varepsilon}}$ we have

$$\begin{aligned}|I_2|&\le c\int_{\Omega_\varepsilon}\left|V^{p-1}-U^{p-1}_{\mu_{j\varepsilon}a_{j\varepsilon}}\right||\phi|U_{\mu_{j\varepsilon}a_{j\varepsilon}}\\&=c\int_{\Omega_\varepsilon\setminus B(a,\rho)}\dots+c\sum_{\substack{i=1\\i\neq j}}^k\int_{\mathcal{A}_i}\dots+c\int_{\mathcal{A}_j}\dots\\&=c\int_{\mathcal{A}_j}\dots+o\left(\varepsilon^{\frac{N-2}{2k}}\right),\end{aligned}$$

where $\mathcal{A}_i$ are the annuli defined in (3.22). Observe now that if $N\ge 7$

$$\begin{aligned}&\int_{\mathcal{A}_j}\left|V^{p-1}-U^{p-1}_{\mu_{j\varepsilon}a_{j\varepsilon}}\right||\phi|U_{\mu_{j\varepsilon}a_{j\varepsilon}}\\&\le c\int_{\mathcal{A}_j}U^{p-1}_{\mu_{j\varepsilon}a_{j\varepsilon}}\left|\left(P_\varepsilon U_{\mu_{j\varepsilon}a_{j\varepsilon}}-U_{\mu_{j\varepsilon}a_{j\varepsilon}}\right)+\sum_{i\neq j}P_\varepsilon U_{\mu_{i\varepsilon}a_{i\varepsilon}}+\sum_i P_\varepsilon U_{\delta_{i\varepsilon}b_{i\varepsilon}}\right||\phi|\\&\le c\left|U^{p-1}_{\mu_{j\varepsilon}a_{j\varepsilon}}\right|_{\frac N2}\left|P_\varepsilon U_{\mu_{j\varepsilon}a_{j\varepsilon}}-U_{\mu_{j\varepsilon}a_{j\varepsilon}}\right|_{\frac{2N}{N-2}}|\phi|_{\frac{2N}{N-2}}\\&+c\sum_{i\neq j}\left|U^{p-1}_{\mu_{j\varepsilon}a_{j\varepsilon}}\right|_{\frac N2}\left|U_{\mu_{i\varepsilon}a_{i\varepsilon}}\right|_{\mathrm{L}^{\frac{2N}{N-2}}(\mathcal{A}_j)}|\phi|_{\frac{2N}{N-2}}\\&+c\sum_i\left|U^{p-1}_{\mu_{j\varepsilon}a_{j\varepsilon}}\right|_{\frac N2}\left|U_{\delta_{i\varepsilon}b_{i\varepsilon}}\right|_{\mathrm{L}^{\frac{2N}{N-2}}(\Omega_\varepsilon\setminus B(a,\rho))}|\phi|_{\frac{2N}{N-2}}=o\left(\varepsilon^{\frac{N-2}{2k}}\right),\end{aligned}$$

where we use estimate (3.27). Thus we conclude that $I_1=o(\varepsilon^{\frac{N-2}{2k}})$. The case $3\le N\le 6$ can be treated similarly. Using similar arguments, we also obtain that $I_3=o(\varepsilon^{\frac{N-2}{2k}})$.

We are left with the estimate of $J'_\varepsilon(V+\phi)[\nabla\phi]$ in (7.8). By definition we have

$$J'_\varepsilon(V+\phi)[\nabla\phi]=\sum_{l=0}^N\sum_{i=1}^k c_a^{li}\left\langle P_\varepsilon Z^l_{\mu_{i\varepsilon}a_{i\varepsilon}},\nabla\phi\right\rangle+\sum_{l=0}^N\sum_{i=1}^k c_b^{li}\left\langle P_\varepsilon Z^l_{\delta_{i\varepsilon}b_{i\varepsilon}},\nabla\phi\right\rangle$$

Taking into account estimate (7.3), we get that

$$|J'_\varepsilon(V+\phi)[\nabla\phi]| = O(|\phi|^2_{\frac{2N}{N-2}}) = o(\varepsilon^{\frac{N-2}{2k}})$$

since one has, for instance,

$$|\left\langle P_\varepsilon Z^l_{\mu_{i\varepsilon}a_{i\varepsilon}}, \nabla\phi\right\rangle| \leq C|Z^l_{\mu_{i\varepsilon}a_{i\varepsilon}}|_{\frac{2N}{N-2}}|\phi|_{\frac{2N}{N-2}} \leq C\mu_{i\varepsilon}|\phi|_{\frac{2N}{N-2}}.$$

This concludes the proof. □

Proof of Proposition 2.2, Part 2. It follows from Theorem 3.1 and Lemma 7.2. □

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