Locally nearly spherical surfaces are almost-positively c-curved*

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Abstract

The c-curvature of a complete surface with Gauss curvature close to 1 in C^2 norm is almost-positive (in the sense of Kim–McCann). Our proof goes by a careful case by case analysis combined with perturbation arguments from the constant curvature case, keeping track of an estimate on the closeness curvature condition.

1 Introduction and main results

Monge's problem, in optimal transport theory, goes back to [19]. In its general formulation, one is looking for an optimal map $f : (M, \mu) \to (\overline{M}, \bar{\mu})$ between two Polish probability spaces. The optimality criterion consists in minimizing the total cost functional $\int_M c(x, f(x))\, d\mu(x)$ among measurable maps which push μ to $\bar{\mu}$, where the cost function $c : M \times \overline{M} \to \mathbb{R} \cup \{+\infty\}$ is given lower semi-continuous with some additional properties (see *e.g.* [21] and references therein). In the emblematic case of the Brenier–McCann cost function: $M = \overline{M}, c = \frac{1}{2}d^2$, where M stands for a complete Riemannian manifold with associated distance function d, this problem was solved under mild assumptions on the given probability measures μ and $\bar{\mu}$ [2, 17]. In that case, the optimal map must read $f = \exp(\operatorname{grad} u)$ for some c-convex potential function u such that the pushing condition $f_{\#}\mu = \bar{\mu}$ becomes a PDE of Monge–Ampère type satisfied by u in a weak sense. Neil Trudinger and his co-workers observed that a similar solution scheme exists for a class of more general cost functions c for which, given smooth data, they analyzed the smoothness of the corresponding potential function u [16]. For the purpose of a one-sided interior estimate on an expression of second order (in u), they were lead to formulate a fourth-order two-points condition on the cost function c, called (A3S) condition. A weak form of the latter, called (A3W), was proved necessary (for the smoothness of u) by Loeper [14]; in particular, in the Brenier–McCann case, he interpreted (A3W) read on the diagonal of $M \times M$ as the non-negativity of the sectional curvature of M. Lately, still with $c = \frac{1}{2}d^2$, Cédric Villani and his co-workers were able to relate some variants of (A3S), checked stable at round spheres under C^4 small deformations of the standard round metric, with the convexity of the tangential domain of injectivity of the exponential map [15, 8, 10]. However, the very geometrical status of the

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fourth-order expression (in c) occuring in condition (A3S) was not understood untill Kim and McCann interpreted it [11] as a genuine, though quite special, curvature expression arising on the product manifold $M\times\overline{M}$ endowed with the pseudo-Riemannian metric: $h=-\frac{1}{2}\frac{\partial^2 c}{\partial x^i\partial\bar{x}^j}(dx^i\otimes d\bar{x}^j+d\bar{x}^j\otimes dx^i)$. They also defined an extended version of (A3S), stronger than (A3W), called non-negative cross-curvature condition (NNCC, for short) and proved that it is stable under Cartesian product[1] as well as, in the Brenier–McCann case, under Riemannian submersion [12]. Actually, in that case, they defined a stronger condition called almost-positive cross-curvature condition (APCC, for short) shown as stable as NNCC [12]. So, with $c=\frac{1}{2}d^2$, the stability of APCC under products and submersions enables to construct new APCC examples out of known ones – like the standard sphere [12]. In the present paper, we will prove the stability of APCC at the standard 2-sphere; specifically, we will check the APCC condition for $c=\frac{1}{2}d^2$ on a complete surface with Gauss curvature C^2 close to a positive constant. This result complements the stability one of [8] on the 2-sphere.

In order to state our result, let us first recall some definitions, restricting to connected complete Riemannian manifolds $M=\overline{M}$ with the cost function $c=\frac{1}{2}d^2$ defined on $M\times\overline{M}\setminus\mathrm{Cut}$, where Cut stands for the cut locus. Using the aforementioned pseudo-Riemannian metric h on $M\times\overline{M}$ and setting Sect_h for its sectional curvature tensor viewed as a field of quadratic forms on $\bigwedge^2 T(M\times\overline{M})$, for each $(m,\overline{m})\in M\times\overline{M}\setminus\mathrm{Cut}$ and each $(\xi,\bar{\xi})\in T_mM\times T_{\overline{m}}\overline{M}$, the associated *cross-curvature* is defined by [11]:

$$\mathrm{cross}_{(m,\overline{m})}(\xi,\bar{\xi}):=\mathrm{Sect}_h[(\xi\oplus 0)\wedge(0\oplus\bar{\xi})].$$

Kim and McCann observed that it must vanish for some choice of $(\xi,\bar{\xi})$ [12]. If it is identically non-negative, we say that the manifold M is NNCC. For instance, the standard n-sphere is NNCC [12, 8] and if a manifold M endowed with a Riemannian metric g is so, its sectional curvature tensor Sect_g must be non-negative because, at $\overline{m}=m$, we have: $\mathrm{cross}_{(m,m)}(\xi,\bar{\xi})\equiv\frac{4}{3}\mathrm{Sect}_g(\xi\wedge\bar{\xi})$, as first observed by Loeper [14]. Pulling back by the exponential map:

$$(m,V)\in\mathrm{NoCut}\longrightarrow(m,\exp_m(V))\in M\times\overline{M}\setminus\mathrm{Cut}\ ,$$

where NoCut is the domain of TM defined by:

$$\mathrm{NoCut}:=\{(m,V)\in TM,\ \forall t\in[0,1],\exp_m(tV)\notin\mathrm{Cut}_m\}$$

(and Cut_m, the cut locus of the point m), Trudinger *et al* noted [16, p.164] that one identically recovers $\frac{1}{2}\mathrm{cross}_{(m,\overline{m})}(\xi,\bar{\xi})$ at $\overline{m}=\exp_m(V)$ with $(m,V)\in\mathrm{NoCut}$ and $\bar{\xi}=d(\exp_m)(V)(\nu)$, by calculating the quantity:

$$\mathcal{C}(m,V)(\xi,\nu):=-\frac{D^2}{d\lambda^2}\left[A(m,V+\lambda\nu)(\xi)\right]_{\lambda=0}\tag{1}$$

where: $A(m,V)(\xi):=\nabla d[p\rightarrow c(p,\exp_m(V))]_{p=m}(\xi,\xi)$ with ∇ the Levi–Civita connection of the Riemannian metric g and where D stands for the canonical flat connection of T_mM. In [7], we performed a stepwise calculation of $A(m,V)(\xi)$

[1] unlike (A3S), or even (A3W) alone as soon as a factor is not NNCC

and its first and second derivatives with respect to V, in a Fermi chart along the geodesic $t \in [0,1] \to \exp_m(tV) \in M$. This calculation just requires that (m,V) belong to NoConj, denoting so the domain[2] of TM which consists of tangent vectors $(m,W) \in TM$ such that the geodesic segment $t \in [0,1] \to \exp_m(tW)$ contains no conjugate points, a fact conceptualized in [8] using the Hamiltonian flow (see also [10]). Neil Trudinger suggested that one calls the quantity $\mathcal{C}(m,V)(\xi,\nu)$ defined by (1), now with $(m,V) \in \mathrm{NoConj}$, the c-curvature[3] of M at (m,V,ξ,ν). It is known to vanish if $\mathrm{rank}(V,\xi,\nu) \leq 1$ [7, 12]. Now, the definition given in [12] of an APCC (resp. NNCC) manifold reads in terms of the c-curvature as follows:

Definition 1 *Let M be a connected complete Riemannian manifold with cost function $c = \frac{1}{2}d^2$. We say that M is non-negatively c-curved, or NNCC, if $\mathcal{C}(m,V)(\xi,\nu) \geq 0$ for each $(m,V) \in \mathrm{NoConj}$ and each couple (ξ,ν) in T_mM. If M is NNCC and such that: $\mathcal{C}(m,V)(\xi,\nu) = 0$ if and only if the span of (V,ξ,ν) has dimension at most 1, we call it almost-positively c-curved, or APCC.*

Let us call, for short, a *surface* any smooth connected complete 2-dimensional Riemannian manifold without boundary. We aim at the following result:

Theorem 1 *Let S be a surface with Gauss curvature K such that $\min_S K = 1$. There exists a small universal constant $\eta > 0$ such that, if $|K-1|_{C^2(S)} \leq \eta$, then S is APCC.*

Here, the C^2 norm of a function $f : S \to \mathbb{R}$ is defined (using the Riemannian norm $|.|$ on tensors) by: $|f|_{C^2(S)} := \sup_S |f| + \sup_S |df| + \sup_S |\nabla df|$.
With $\eta = 0$ (constant curvature), the result is proved in [12] (see also [8] for NNCC). If $V = 0$, the result is obvious (due to the cross-curvature interpretation when $m = \overline{m}$), so we will assume $V \neq 0$ with no loss of generality.

Remark 1 Let

$$\mathrm{D}_c = \sup\{|V|_m,\ (m,V) \in \mathrm{NoConj}\}$$

be the diameter of conjugacy of S. Since $K \geq 1$, the Bonnet–Myers theorem [1, 4, 6, 18] implies: $\mathrm{D}_c \leq \pi$ (in particular, the diameter of S is at most equal to π and S is compact).

Actually, we will prove a stronger result, namely:

Theorem 2 *Let S be a surface with $\min_S K = 1$. There exists small universal positive constants η, ς such that, if $|K-1|_{C^2(S)} \leq \eta$, for any $(m,V) \in \mathrm{NoConj}$ and any couple (ξ,ν) of vectors in T_mS, the following inequality holds:*

$$\mathcal{C}(m,V)(\xi,\nu) \geq \varsigma\ \mathcal{A}_2(m,V,\xi,\nu)\ , \tag{2}$$

where $\mathcal{A}_2(m,V,\xi,\nu)$ stands for the squared quadratic mean of the areas of the parallelograms repectively defined in T_mS by the couples $(\xi,\nu),(V,\xi),(V,\nu)$, in other words:

$$\mathcal{A}_2(m,V,\xi,\nu) = |\xi|^2|\nu|^2 - g(\xi,\nu)^2 + |V|^2|\xi|^2 - g(V,\xi)^2 + |V|^2|\nu|^2 - g(V,\nu)^2.$$

[2] as well-known [4, 6], NoConj is the maximum rank domain for the exponential map which contains NoCut

[3] we will use this short denomination instead of 'extended MTW tensor' as in [15, 9, 10] or 'Ma–Trudinger–Wang curvature' as in [9, 13]

The outline of the paper essentially coincides with that of the proof. We present a quick derivation of the c-curvature expression in Section 2 and related perturbative estimates for that expression, based on the assumption that the C^2 norm of $(K-1)$ is small, in Section 3. Using the latter, we prove successively Theorem 2 under the additional assumption that the point $\exp_m(V)$ lies, either near the first conjugate point m^* of m along the geodesic $t \in \mathbb{R}^+ \to \exp_m(tV) \in S$ (Section 4), or near m (Section 5), or in-between (Section 6). The proof of Theorem 2 itself, as a whole, is provided in Section 7, by synthetizing the various, sometimes redundant, smallness assumptions made in the previous sections on $|K-1|_{C^2(S)}$, ς and an extra parameter δ used to locate $\exp_m(V)$ with respect to m and m^* as just described. The proof of the main perturbation lemma is deferred to Appendix A, but Section 3 includes a straightforward application of it to a uniform convexity estimate for the boundary of NoConj.

Finally, a warning must be made about some notations and conventions used below. Starting from Lemma 1 (Section 3), we will abbreviate $|K-1|_{C^2(S)}$ merely as ε. In Section 4 (resp. Section 5), we will set $\delta d(m,m^*)$ (resp. δ) for the maximal distance assumed between $\exp_m(V)$ and the first conjugate point m^* (resp. and the point m); consistently in Section 6, we will set $\frac{1}{2}\delta d(m,m^*)$ (resp. $\frac{1}{2}\delta$) for the minimal distance at which $\exp_m(V)$ must stay away from m^* (resp. from m) on that geodesic. In the course of the proof, starting from Lemma 1, we will require various (fairly explicit, universal) smallness conditions on ε or δ. Furthermore, in each case or subcase distinguished below for (m,V,ξ,ν), we will find a different value of the (small positive) constant ς occuring in (2); the actual value to be taken for ς in the statement of Theorem 2 will be, of course, the *smallest* among them. The various universal[4] constants and smallness conditions arising in the paper are listed in Appendix B to which the reader should systematically refer.

2 c-curvature expression in dimension 2

Henceforth, we fix a surface S, a point $m_0 \in S$ and three non-zero tangent vectors (V_0,ξ,ν) in $T_{m_0}S$ with $(m_0,V_0) \in$ NoCut and (ξ,ν) linearly independent. We wish to calculate the c-curvature $\mathcal{C}(m_0,V_0)(\xi,\nu)$.

2.1 General case

A chart $x=(x^1,x^2)$ of S centered at m_0 such that the local components $g_{ij}(x)$ of the metric satisfy: $g_{ij}(0)=\delta_{ij},\ dg_{ij}(0)=0$, is called *normal* at m_0; let x be such a chart. We set $v=(v^1,v^2)$ for the fiber coordinates of $TS \to S$ naturally associated to x, use Einstein's convention and abbreviate partial derivatives as follows:

$$\partial_i = \frac{\partial}{\partial x^i}, \partial_{ij} = \frac{\partial^2}{\partial x^i \partial x^j}, \ldots; D_i = \frac{\partial}{\partial v^i}, D_{ij} = \frac{\partial^2}{\partial v^i \partial v^j}, \ldots$$

For each $(m,V) \in$ NoCut with m in the domain of the chart x, we set:

$$X = X(x,v,t) = \big(X^1(x,v,t), X^2(x,v,t)\big) = x\left(\exp_m(tV)\right),$$

[4] thus, in particular, independent of $(m,V) \in$ NoConj

where $x = x(m)$ and $V = v^i \partial_i$. For $V \in T_{m_0}S$ such that $(m_0, V) \in \mathrm{NoCut}$, and setting $\xi = \xi^i \partial_i$, we recall from [7] that the quadratic form $A(m_0, V)(\xi)$ defined in the introduction is equal to $A_{ij}(v)\xi^i\xi^j$ with:

$$A_{ij}(v) = Y^i_k(v)\ \partial_j X^k(0, v, 1) \tag{3}$$

and the matrix $Y^i_k(v)$ given by: $Y^i_k(v)\ D_j X^k(0, v, 1) = \delta^i_j$. Given $V = v^i\partial_i$ as above, it is convenient to compute the right-hand side of (3) by choosing for x a particular normal chart at m_0 (unique up to $x^1 \to -x^1$), namely:

Definition 2 *A Fermi chart along V is a normal chart x at m_0 such that $V = r\partial_2$ (with $r = |V|$) and the Riemannian metric reads:*

$$g = dx^1 \otimes dx^1 + G(x^1, x^2)\ dx^2 \otimes dx^2,\ \textit{with}\ G(0, x^2) = 1,\ \partial_1 G(0, x^2) = 0.$$

Let x be a Fermi chart along V. The geodesic $t \in [0, 1] \to m_t = \exp_m(tV) \in S$ (called the *axis* of the chart) simply reads $t \mapsto X((0, 0),(0, r), t) = (0, tr)$ and, for fixed x^2, the paths which read $t \mapsto (t, x^2)$ are geodesics of S as well, orthogonal to the axis. The Christoffel symbols are given by:

$$\Gamma^1_{22} = -\frac{1}{2}\partial_1 G,\ \Gamma^2_{12} = \frac{\partial_1 G}{2G},\ \Gamma^2_{22} = \frac{\partial_2 G}{2G},\ \text{others vanish},$$

and the Gauss curvature, by $K = -\dfrac{\partial_{11}\sqrt{G}}{\sqrt{G}}$. We thus get for the derivatives of the Christoffel symbols on the axis, intrinsic expressions given in terms of K at $x = (0, x^2)$ by:

$$\partial_1\Gamma^1_{22} = -\partial_1\Gamma^2_{12} = K,\ \partial_{11}\Gamma^1_{22} = -\partial_{11}\Gamma^2_{12} = \partial_1 K,\ \partial_1\Gamma^2_{22} = 0,\ \partial_{11}\Gamma^2_{22} = -\partial_2 K.$$

With these formulas at hand, we readily find:

$$\partial X((0, 0),(0, r), t) = \begin{pmatrix} f_0(t) & 0 \\ 0 & 1 \end{pmatrix},\ DX((0, 0),(0, r), t) = \begin{pmatrix} f_1(t) & 0 \\ 0 & t \end{pmatrix},$$

where $f_i(t) = f_i((0, 0),(0, r), t)$ for $i \in \{0, 1\}$; here, $f_i(x, w, t)$ are the expressions in the chart x of the solutions for $t \in [0, 1]$ of the Jacobi equation:

$$\ddot{f} + |W|^2\ K(\exp_m(tW))\, f = 0 \tag{4}$$

(where $x = x(m), W = w^i\partial_i$ with $(m, W) \in \mathrm{NoConj}$, and we use the dot notation: $\dot{f} = \dfrac{df}{dt}$, $\ddot{f} = \dfrac{d^2 f}{dt^2}$), satisfying the initial condition:

$$f_i(0) = \delta_{i0},\ \dot{f}_i(0) = \delta_{i1}\ .$$

Remark 2 For later use, we observe that, for $t \in (0, 1]$ and $(m, W) \in \mathrm{NoConj}$, we have: $0 < f_1(x, w, t)$. Moreover, Sturm comparison theorem [4] combined with Remark 1 provides the pinching:

$$\frac{\sin\left(\sqrt{\max_S K}|W|t\right)}{\sqrt{\max_S K}|W|} \le f_1(x, w, t) \le \frac{\sin(|W|t)}{|W|},$$

which yields $f_1(x, w, t) \le t \le 1$ and $\lim\limits_{|W|\downarrow 0} f_1(x, w, 1) = 1$.

Back to $(m,W)=(m_0,V)$, applying (3) in our Fermi chart along V, we get:

$$A(m_0,V)(\xi)=|\xi|^2-\left(1-\frac{f_0(1)}{f_1(1)}\right)|\xi-g(\xi,U)U|^2,\ \text{with } U=\frac{V}{|V|}.$$

Here comes a key observation, also made in [9] (and extended to the higher dimensional setting in [10], see also [13]): the right-hand side of the preceding equation is intrinsic because so is (4). We may thus use a single Fermi chart x, along the sole tangent vector V_0 at m_0, and write for each $V=v^i\partial_i\in T_{m_0}S$ close to V_0:

$$A(m_0,V)(\xi)=|\xi|^2-\left(1-\frac{f_0(0,v,1)}{f_1(0,v,1)}\right)|\xi-g(\xi,U)U|^2. \tag{5}$$

We will now calculate the c-curvature $\mathcal{C}(m_0,V_0)(\xi,\nu)$ in that Fermi chart (fixed once for all), by combining (1) with (5). Letting henceforth ξ and ν be unit vectors and orienting the tangent plane $T_{m_0}S$ by the local volume form $dx^1\wedge dx^2$, we denote by ϑ (resp. φ) the angle in $[0,2\pi)$ by which a direct rotation brings ξ (resp. ν) to $U_0=\dfrac{V_0}{|V_0|}=\partial_2$; in other words, we set:

$$\xi=\sin\vartheta\ \partial_1+\cos\vartheta\ \partial_2,\ \nu=\sin\varphi\ \partial_1+\cos\varphi\ \partial_2.$$

A lengthy but routine calculation yields:

$$\begin{aligned}\mathcal{C}(m_0,V_0)(\xi,\nu)= & -\ \sin^2\vartheta\left(\frac{f_0''}{f_1}-\frac{f_0f_1''}{f_1^2}-\frac{2f_0'f_1'}{f_1^2}+\frac{2f_0(f_1')^2}{f_1^3}\right)\\ & +\ \frac{2}{r_0^2}\left(\cos^2\vartheta-\cos^2(\vartheta+\varphi)\right)\left(1-\frac{f_0}{f_1}\right)\\ & +\ \frac{4}{r_0}\cos\vartheta\sin\vartheta\sin\varphi\left(\frac{f_0'}{f_1}-\frac{f_0f_1'}{f_1^2}\right),\end{aligned} \tag{6}$$

where we have set, for short: $f_a'=\nu^iD_if_a(0,v_0,1)$, $f_a''=\nu^i\nu^jD_{ij}f_a(0,v_0,1)$, for $a=0,1$, and $v_0=(0,r_0)$ with $r_0=|V_0|$.

2.2 Constant curvature case recalled

Setting for short $\kappa=K(m_0)$ and $\bar r=\sqrt{\kappa}\ r$, let us recall the expressions which occur for f_0,f_1 in case $K\equiv\kappa$, labelling them all with a bar:

$$\bar f_0(0,v,t)=\cos(\bar r t),\ \bar f_1(0,v,t)=\frac{\sin(\bar r t)}{\bar r},\ \text{where } r=\sqrt{(v^1)^2+(v^2)^2}.$$

At $(v,t)=(v_0,1)$, with $v_0=(0,r_0)$ and $\bar r_0=\sqrt{\kappa}\ r_0$, we infer correspondingly:

$$\bar f_0'=-\sqrt{\kappa}\ \sin\bar r_0\ \cos\varphi,\quad \bar f_0''=\kappa\left(-\frac{\sin\bar r_0}{\bar r_0}+\left(\frac{\sin\bar r_0}{\bar r_0}-\cos\bar r_0\right)\cos^2\varphi\right),$$

$$\bar f_1'=\frac{\sqrt{\kappa}}{\bar r_0}\left(\cos\bar r_0-\frac{\sin\bar r_0}{\bar r_0}\right)\cos\varphi\ ,$$

$$\bar f_1''=\frac{\kappa}{\bar r_0^2}\left(\cos\bar r_0-\frac{\sin\bar r_0}{\bar r_0}+\left(3\left(\frac{\sin\bar r_0}{\bar r_0}-\cos\bar r_0\right)-\bar r_0\sin\bar r_0\right)\cos^2\varphi\right),$$

hence:

$$\begin{aligned} \frac{1}{\kappa}\,\overline{\mathcal{C}}(m_0,V_0)(\xi,\nu) &= \sin^2\vartheta\ \sin^2\varphi\ \frac{\bar r_0^2+\bar r_0\cos\bar r_0\sin\bar r_0-2\sin^2\bar r_0}{\bar r_0^2\sin^2\bar r_0} \\ &+ 2\sin^2\vartheta\ \cos^2\varphi\ \frac{\sin\bar r_0-\bar r_0\cos\bar r_0}{\sin^3\bar r_0} \\ &+ 2\cos^2\vartheta\ \sin^2\varphi\ \frac{\sin\bar r_0-\bar r_0\cos\bar r_0}{\bar r_0^2\sin\bar r_0} \\ &+ 4\cos\vartheta\ \sin\vartheta\ \cos\varphi\ \sin\varphi\ \frac{\sin^2\bar r_0-\bar r_0^2}{\bar r_0^2\sin^2\bar r_0}. \end{aligned} \tag{7}$$

3 Perturbative tools

In the sequel of the paper, dropping the first argument $x = x(m)$ since it is fixed, equal to $(0,0) = x(m_0)$, we simply write: $f_a = f_a(v,t)$, $X = X(v,t)$ and, abusively with the same letter: $K(X(v,t)) = K\left(\exp_{m_0}(tV)\right)$, where $V = v^i\partial_i$. Moreover, anytime the second argument v is equal to $v_0 = (0, r_0)$, we will also drop it and just write: $f_a = f_a(t)$ and so on.
Given a real number $\omega > 0$, we will require the linear map:

$$f \in C^0([0,1],\mathbb{R}) \longrightarrow \mathcal{S}_\omega(f) \in C^0([0,1],\mathbb{R})$$

defined as the solution map $f \mapsto u$ of the linear initial value problem:

$$\ddot u + \omega^2 u = f,\ \ u(0) = \dot u(0) = 0.$$

The representation formula : $\mathcal{S}_\omega(f)(t) = \displaystyle\int_0^t \frac{\sin(\omega(t-\tau))}{\omega}\ f(\tau)\ d\tau$ is well known. Setting $\|v\| = \sup\limits_{t\in[0,1]} |v(t)|$, it yields for $\mathcal{S}_\omega$ the contraction estimate:

$$\|\mathcal{S}_\omega(f)\| \le \frac{1}{2}\|f\|, \tag{8}$$

easily obtained by writing:

$$u(t) = \int_0^t \dot u(\tau)d\tau = \int_0^t\int_0^\tau \cos(\omega(\tau-\theta))\ f(\theta)\ d\theta d\tau.$$

We will also require the following formulas (written at $t = 1$, for $f(t) = t$ and $f(t) = t^2$):

$$\mathcal{S}_{\bar r_0}(t)(1) = \frac{\bar r_0 - \sin\bar r_0}{\bar r_0^3},\ \ \mathcal{S}_{\bar r_0}(t^2)(1) = \frac{\bar r_0^2 + 2(\cos\bar r_0 - 1)}{\bar r_0^4}. \tag{9}$$

We are now ready to state our main perturbation lemma, the proof of which is deferred to Appendix A:

Lemma 1 *If* $|K-1|_{C^2(S)} \le \dfrac{1}{\pi^2}$*, there exists universal constants* $B_{1ka}, B_{2ka}, B_{3ka}$*, for* $a \in \{0,1\}$ *and* $k \in \{0,1,2\}$*, such that the following estimates hold:*

$$\|D_\nu^k f_a\| \le B_{1ka},\ \ \|D_\nu^k(f_a - \bar f_a)\| \le B_{2ka}\ \varepsilon\ r_0^{2-k},$$

$$\|D_\nu^k(f_a-\bar f_a)+r_0^{3-k}\ \psi_k\ \mathcal{S}_{\bar r_0}(t^{a+1})\|\le B_{3ka}\ \varepsilon\ r_0^{4-k}\ ,$$

where, for short, $\varepsilon:=|K-1|_{C^2(S)}$ and:

$$\psi_0:=\partial_2K(0),\ \ \psi_1:=3\cos\varphi\ \partial_2K(0)+\sin\varphi\ \partial_1K(0),$$

$$\psi_2:=(2+4\cos^2\varphi)\ \partial_2K(0)+4\sin\varphi\cos\varphi\ \partial_1K(0)$$

(from now on, we will freely use to these abbreviations).

Remark 3 Let us stress that the bounds:

$$\forall a=0,1,\ \ \|D_{12}f_a\|\le 2B_{12a},\ \ \|D_{12}(f_a-\bar f_a)\|\le 2B_{22a}\varepsilon,$$

follow from thoses on $\|D_{\nu\nu}f_a\|$ and $\|D_{\nu\nu}(f_a-\bar f_a)\|$ by letting $\nu=\frac{1}{\sqrt2}(\partial_1+\partial_2)$.

The first line of conclusion of Lemma 1 will be used to prove Theorem 1 near[5] the first conjugate point (Section 4). Uniformly away from that point, and crucially for $r_0\downarrow 0$, the proof requires the second line of conclusion through a Maclaurin type approximation estimate for the c-curvature, namely:

Corollary 1 *If $|K-1|_{C^2(S)}\le\dfrac{1}{\pi^2}$ and $\bar r_0<\pi$, there exists a universal constant C_1 such that the absolute value of the following expression:*

$$\begin{aligned}\frac{f_1^3}{\bar f_1^3}\,\mathcal{C}(m_0,V_0)(\xi,\nu)\quad &-\quad \overline{\mathcal{C}}(m_0,V_0)(\xi,\nu)-\frac{r_0\psi_2\sin^2\vartheta}{\bar f_1}\left(\mathcal{S}_{\bar r_0}(t)(1)-\frac{\bar f_0\mathcal{S}_{\bar r_0}(t^2)(1)}{\bar f_1}\right)\\ &+\quad \frac{2r_0\psi_0\mathcal{S}_{\bar r_0}(t^2-t)(1)}{\bar f_1}\left(\cos^2\vartheta-\cos^2(\vartheta+\varphi)\right)\\ &+\quad \frac{4r_0\psi_1\cos\vartheta\sin\vartheta\sin\varphi}{\bar f_1}\left(\mathcal{S}_{\bar r_0}(t)(1)-\frac{\bar f_0\mathcal{S}_{\bar r_0}(t^2)(1)}{\bar f_1}\right)\end{aligned}$$

is bounded above by:

$$\frac{1}{\bar f_1^3}C_1^3\pi^8\varepsilon r_0^2(338\sin^2\vartheta+268\cos^2\vartheta\sin^2\varphi).$$

Proof of the corollary. For each $a\in\{0,1\}$ and $k\in\{0,1,2\}$, we split $D_\nu^kf_a$ identically into three summands: $D_\nu^kf_a=S_1^{(k,a)}+S_2^{(k,a)}+S_3^{(k,a)}$ given by:

$$S_1^{(k,a)}=D_\nu^k\bar f_a,\ \ S_2^{(k,a)}=-r_0^{3-k}\psi_k\mathcal{S}_{\bar r_0}(t^{a+1})(1).$$

From (9), we define the constants c_6,c_7 as in Appendix B. From Lemma 1, we know that

$$\left|S_1^{(k,a)}\right|\le B_{1ka},\ \ \left|S_3^{(k,a)}\right|\le B_{3ka}\varepsilon r_0^{4-k},$$

and from the obvious bounds:

$$|\psi_0|\le\varepsilon,\ \ |\psi_1|\le 4\varepsilon,\ \ |\psi_2|\le 8\varepsilon,\tag{10}$$

we further know that

$$\left|S_2^{(k,a)}\right|\le 8c_{6+a}\varepsilon r_0^{3-k}\ .$$

[5]where $\overline{\mathcal{C}}(m_0,V_0)(\xi,\nu)$ could blow up since $\bar r_0$ could exit from $(0,\pi)$ for $\varepsilon\ne0$

Let us consider the expression (6) of the c-curvature, multiply it by f_1^3 and, using the preceding splittings and bounds, let us estimate the Maclaurin approximation of each of the three auxiliary expressions:

$$E_1 := f_1^2 f_0'' - f_0 f_1 f_1'' - 2 f_1 f_0' f_1' + 2 f_0 (f_1')^2$$

$$E_2 := \frac{2}{r_0^2} f_1^2 (f_1 - f_0)\,,\ E_3 := \frac{4}{r_0} f_1 (f_1 f_0' - f_0 f_1')\,,$$

which occur in $f_1^3 \mathcal{C}(m_0, V_0)(\xi, \nu)$ as coefficients, respectively of:

$$-\sin^2\vartheta,\ \left(\cos^2\vartheta - \cos^2(\vartheta+\varphi)\right),\ \cos\vartheta\sin\vartheta\sin\varphi\ .$$

Setting $\overline{E}_1, \overline{E}_2, \overline{E}_3$, for the corresponding quantities defined with $\bar{f}_0, \bar{f}_1$ instead of f_0, f_1, and proceeding stepwise, with careful intermediate calculations[6], we get for the $(E_\ell - \overline{E}_\ell)$'s the following analogues of the second line of conclusion of Lemma 1:

$$\left|E_1 - \overline{E}_1 + r_0 \psi_2 \bar{f}_1 \left[\bar{f}_1 \mathcal{S}_{\bar{r}_0}(t)(1) - \bar{f}_0 \mathcal{S}_{\bar{r}_0}(t^2)(1)\right]\right| \le 154\pi^8 C_1^3 \varepsilon r_0^2\ ,$$

$$\left|E_2 - \overline{E}_2 + 2 r_0 \psi_0 \bar{f}_1^2 \mathcal{S}_{\bar{r}_0}(t^2 - t)(1)\right| \le 84\pi^8 C_1^3 \varepsilon r_0^2\ ,$$

$$\left|E_3 - \overline{E}_3 + 4 r_0 \psi_1 \bar{f}_1 \left[\bar{f}_1 \mathcal{S}_{\bar{r}_0}(t)(1) - \bar{f}_0 \mathcal{S}_{\bar{r}_0}(t^2)(1)\right]\right| \le 200\pi^8 C_1^3 \varepsilon r_0^2\ ,$$

where the constant C_1 is defined[7] in Appendix B, as well as three other constants c_8, c_9, c_{10}, and where, recalling Remark 1, π^8 is used as an upper bound for $\max\left(1, r_0^{p-2}\right)$ with[8] $2 \le p \le 10$. Since $\bar{r}_0 < \pi$, we may divide by $\bar{f}_1^3 > 0$ the resulting Maclaurin approximation estimate for $f_1^3 \mathcal{C}(m_0, V_0)(\xi, \nu)$ and, using the general inequalities:

$$\begin{aligned} \left|\cos^2\vartheta - \cos^2(\vartheta+\varphi)\right| &\le \sin^2\vartheta + 2\cos^2\vartheta\sin^2\varphi\ , \\ 2|\cos\vartheta\sin\vartheta\sin\varphi| &\le \sin^2\vartheta + \cos^2\vartheta\sin^2\varphi\ , \end{aligned} \tag{11}$$

we obtain the estimate of Corollary 1.

Quick digression on the convexity of NoConj**.** The reader may wish to skip the rest of this section, devoted to a quick digression from our main topic. Indeed, let us pause and provide a uniform convexity estimate on the tangential domains

$$\mathrm{NoConj}_m = \{W \in T_m S, (m, W) \in \mathrm{NoConj}\},$$

obtained in terms of $|K-1|_{C^2(S)}$ as a direct consequence of Lemma 1, and stated as follows:

Corollary 2 *Let S be a surface as above with:* $\min_S K = 1$. *There exists universal positive constants β, γ, C, with $\beta \le \dfrac{1}{\pi^2}$ and $\gamma \le C$, such that, if $|K-1|_{C^2(S)} \le \beta$, for each $m_0 \in S$ and $V_0 \in \partial\,\mathrm{NoConj}_{m_0}$, the curvature of the boundary curve $\partial\,\mathrm{NoConj}_{m_0}$ at V_0 is pinched between γ and C.*

[6] in particular, for counting numbers of terms which are $O(\varepsilon r_0^2)$

[7] using, in particular, the bounds $\sqrt{\kappa} \le 1 + \frac{1}{2\pi^2} < \frac{19}{18}$ and $\kappa \le 1 + \frac{1}{\pi^2} < \frac{10}{9}$

[8] for instance, $p = 2$ (resp. $p = 10$) for $\left(S_1^{(0,1)}\right)^2 S_3^{(2,0)}$ (resp. $\left(S_3^{(0,1)}\right)^2 S_3^{(2,0)}$) in the first term of E_1

Qualitative proofs of the uniform convexity of NoConj are given in [5, 10] for C^4 perturbations of the standard n-sphere. Let us further note that, combining Corollary 2 with Theorem 1, one can readily show that NoCut is convex for small enough β by arguing as in [10], here just with a linear path $t \in [0,1] \to V_t = tV_1 + (1-t)V_0$ in $T_{m_0}S$, with V_0 and V_1 in NoCut_{m_0}.

Proof. Fix (m_0, V_0) as stated and take a Fermi chart x along V_0, sticking to the above notations. From the vanishing of $f_1(0, v_0, t)$ at $t = 1$ combined with its positivity for $t \in (0,1)$ and the uniqueness of the solution of the initial (here final, rather) value problem [20], we infer that $\dot{f}_1(0, v_0, 1) < 0$ hence also $D_2 f_1(0, v_0, 1) < 0$, since $f_1(0, v_0, t) = f_1((0,0),(0,r_0),t) \equiv t f_1((0,0),(0,tr_0),1)$. Therefore, near V_0, the curve $\partial\,\mathrm{NoConj}_{m_0}$ admits the equation $v^2 = h(v^1)$ with the function h implicitly given by:

$$f_1\left((0,0),(v^1,h(v^1)),1\right) = 0, \text{ and } h(0) = r_0.$$

Now, classically [3], the curvature $\mathbf{k}$ of $\partial\,\mathrm{NoConj}_{m_0}$ at V_0 is equal to:

$$\mathbf{k} = \frac{-h''(0)}{\left(1 + h'(0)^2\right)^{3/2}} \equiv -\frac{D_{11}f_1(D_2f_1)^2 - 2D_{12}f_1(D_1f_1)(D_2f_1) + D_{22}f_1(D_1f_1)^2}{\left((D_1f_1)^2 + (D_2f_1)^2\right)^{3/2}}.$$

Considering this formula, and since with $\varepsilon = 0$ we would have $r_0 = \pi, f_1 = \bar{f}_1$ and $\mathbf{k} = \frac{1}{\pi^2}$, the timeliness of Lemma 1 for our purpose is fully conceivable. For an effective proof, we first observe that, by Sturm theorem [4], r_0 is pinched between $\pi/\sqrt{1+\varepsilon}$ and π; in particular, we have:

$$\pi\left(1 - \frac{\varepsilon}{2}\right) \le \bar{r}_0 \le \pi\left(1 + \frac{\varepsilon}{2}\right). \tag{12}$$

At $(x, v, t) = (0, v_0, t)$, Lemma 1 and the formulas of Section 2.2 imply:

$$D_2 f_1 \le \frac{1}{r_0}\left(\cos\bar{r}_0 - \frac{\sin\bar{r}_0}{\bar{r}_0}\right) + B_{211}\varepsilon r_0\ ,$$

$$D_{11} f_1 \le \frac{1}{r_0^2}\left(\cos\bar{r}_0 - \frac{\sin\bar{r}_0}{\bar{r}_0}\right) + B_{221}\varepsilon\ ,$$

which, combined with the pinching of r_0 and standard bounds on the cosine and sine, yields:

$$D_2 f_1 \le \frac{1}{r_0}\left(-1 + \varepsilon\left(\frac{1}{2} + B_{211}\pi^2\right) + \frac{\varepsilon^2\pi^2}{8}\right),$$

$$D_{11} f_1 \le \frac{1}{r_0^2}\left(-1 + \varepsilon\left(\frac{1}{2} + B_{221}\pi^2\right) + \frac{\varepsilon^2\pi^2}{8}\right).$$

So $D_2 f_1 \le -\frac{1}{2r_0}$, hence in particular $|Df_1| \ge \frac{1}{2r_0} \ge \frac{1}{2\pi}$, and $D_{11}f_1 \le -\frac{1}{2r_0^2}$, provided ε is small enough. Moreover, still by Lemma 1 and Section 2.2, we have at $(0, v_0, t)$:

$$|D_1 f_1| \le B_{211}\varepsilon r_0 \le B_{211}\pi\varepsilon\ .$$

The combination of Lemma 1 (including Remark 3) with the preceding bounds yields, on the one hand:

$$\mathbf{k} \le (2\pi)^3 6 B_{121} B_{111}^2 =: C,$$

on the other hand:

$$\mathbf{k} \geq \frac{1}{2B_{111}^3\sqrt{2}}\left(\frac{1}{8r_0^4} - \varepsilon\pi B_{211}B_{121}\left(4B_{111} + \varepsilon\pi B_{211}\right)\right)$$

so that $\mathbf{k} \geq \gamma := \dfrac{1}{32\pi^4\sqrt{2}B_{111}^3}$ for ε small enough. Altogether, this pinching of $\mathbf{k}$ holds provided we require $\varepsilon \leq \beta$ with β the smallest among the positive roots of the quadratic equations:

$$\frac{\pi^2}{8}\beta^2 + \left(\frac{1}{2} + B_{221}\pi^2\right)\beta - \frac{1}{2} = 0 \ ,$$

$$\pi^2 B_{121}B_{211}^2\beta^2 + 4\pi B_{111}B_{121}B_{211}\beta - \frac{1}{16\pi^4} = 0 \ .$$

4 c-curvature almost-positivity near conjugacy

In this section, we prove Theorem 2 at $(m_0, V_0) \in \mathrm{NoConj}$ and (ξ, ν) unit vectors of $T_{m_0}S$ in case the point $\exp_{m_0}(V_0)$ is close to the first conjugate point m_0^* of m_0 along the geodesic $t \in \mathbb{R}^+ \to \exp_{m_0}(tV_0) \in S$. Specifically, setting l_0 for the length of that geodesic curve from m_0 up to m_0^*, we establish the following proposition:

Proposition 1 *There exists a triple of small (strictly) positive real numbers $(\eta_1, \delta_1, \varsigma_1)$ such that $\mathcal{C}(m_0, V_0)(\xi, \nu)$ satisfies the lower bound (2) with $\varsigma \leq \varsigma_1$, provided $\varepsilon = |K-1|_{C^2(S)} \leq \eta_1$ and $(1-\delta_1)l_0 \leq |V_0| < l_0$.*

Proof. Sticking to previous notations and recalling (12), we infer from the pinching of $|V_0|$ the following ones (dropping the subscript of δ_1):

$$\left(1 - \frac{\varepsilon}{2} - \delta\right)\pi \leq r_0 \leq \pi \ \text{and}\ \left(1 - \frac{\varepsilon}{2} - \delta\right)\pi \leq \bar{r}_0 < \left(1 + \frac{\varepsilon}{2}\right)\pi \ . \tag{13}$$

We will assume:

$$r_0 \in \left(\frac{5\pi}{6}, \pi\right) \ \text{and}\ \bar{r}_0 \in \left(\frac{5\pi}{6}, \frac{7\pi}{6}\right) \tag{14}$$

with no loss of generality (it holds under the smallness condition posed below[9] on ε and δ, see (40) of Appendix B). Combining (13) with Remark 2, the formulas of Section 2.2 and the first line of conclusion of Lemma 1, we derive the following set of inequalities:

$$-1 - \varepsilon\pi^2 B_{200} \leq f_0 \leq -1 + \varepsilon\pi^2 B_{200} + \left(\frac{\varepsilon}{2} + \delta\right)^2 \frac{\pi^2}{2} \tag{15}$$

$$0 < f_1 \leq \frac{\sin r_0}{r_0} \leq \frac{\frac{\varepsilon}{2} + \delta}{1 - \left(\frac{\varepsilon}{2} + \delta\right)} \tag{16}$$

$$|f_0'| \leq \left(\frac{\varepsilon}{2} + \delta\right)\left(1 + \frac{\varepsilon}{2}\right)\pi + \varepsilon\pi B_{210}. \tag{17}$$

Furthermore, we derive two important lower bounds, namely:

[9] we will say, for short, that they are (40)-small

Lemma 2 *If* $\cos\varphi \neq 0$, *and* ε *and* δ *satisfy the relative smallness condition*[10] *(39) (see Appendix B), the lower bound:*

$$|f_1'| \geq \frac{|\cos\varphi|}{\pi} - \varepsilon\pi B_{211} - \frac{\frac{\varepsilon}{2}}{\pi\left(1-\left(\frac{\varepsilon}{2}+\delta\right)\right)} - \left(\frac{\varepsilon}{2}+\delta\right)^2 \frac{\pi}{2}$$

holds true, as well as the sign condition: $-f_1'\cos\varphi > 0$. *If* $|\cos\varphi| \leq \dfrac{1}{2}$ *and* ε *and* δ *are (40)-small, the following lower bound is valid instead:*

$$f_0 f_1'' \geq \frac{1}{8\pi^2} - \varepsilon\left(B_{221} + \frac{1}{8}B_{200}\right) - \frac{1}{16}\left(\frac{\varepsilon}{2}+\delta\right)^2 .$$

Proof. If $\cos\varphi \neq 0$, using $|f_1' - \bar{f}_1'| \leq \varepsilon\pi B_{211}$ combined with the lower bound:

$$-\frac{\bar{f}_1'}{\cos\varphi} \geq \frac{1}{\pi}\left(1 - \frac{\frac{\varepsilon}{2}}{1-\left(\frac{\varepsilon}{2}+\delta\right)} - \frac{\pi^2}{2}\left(\frac{\varepsilon}{2}+\delta\right)^2\right),$$

one can readily check the first part of the lemma. For the second part, we first note that $\bar{f}_1''$ is bounded above by the expression:

$$\frac{1}{r_0^2}\left[-1 + \frac{\frac{\varepsilon}{2}}{1-\left(\frac{\varepsilon}{2}+\delta\right)} + \frac{\pi^2}{2}\left(\frac{\varepsilon}{2}+\delta\right)^2 + \cos^2\varphi\left(\frac{3}{1-\left(\frac{\varepsilon}{2}+\delta\right)} + \frac{\varepsilon}{2}\left(1+\frac{\varepsilon}{2}\right)\pi^2\right)\right].$$

If $|\cos\varphi| \leq \frac{1}{2}$, it implies $\bar{f}_1'' \leq -\frac{1}{8\pi^2}$ provided ε and δ are taken (40)-small. By Lemma 1, the inequality

$$f_1'' \leq -\frac{1}{8\pi^2} + \varepsilon B_{221}$$

follows. Combined with (15), it yields the second part of the lemma.

In order to investigate the sign of the c-curvature expression (6), we will have to recast this expression in appropriate forms, namely, either:

$$\begin{aligned}
\mathcal{C}(m_0,V_0)(\xi,\nu) = & - \sin^2\vartheta\left(\frac{f_0''}{f_1} - \frac{f_0 f_1''}{f_1^2} - \frac{2f_0'f_1'}{f_1^2}\right) \\
& - \frac{2f_0}{f_1}\left(\frac{f_1'}{f_1}\sin\vartheta + \frac{\sin\varphi\cos\vartheta}{r_0}\right)^2 \\
& + \frac{2}{r_0^2}\left(1-\frac{f_0}{f_1}\right)\left(2\cos\vartheta\cos\varphi\sin\vartheta\sin\varphi - \sin^2\vartheta\sin^2\varphi\right) \\
& + \frac{2}{r_0^2}\cos^2\vartheta\sin^2\varphi + \frac{4}{r_0}\cos\vartheta\sin\vartheta\sin\varphi\frac{f_0'}{f_1} ,
\end{aligned} \tag{18}$$

or:

$$\begin{aligned}
\mathcal{C}(m_0,V_0)(\xi,\nu) = & - \sin^2\vartheta\left(\frac{f_0''}{f_1} - \frac{f_0 f_1''}{f_1^2} - \frac{2f_0'f_1'}{f_1^2} + \frac{2f_0(f_1')^2}{f_1^3} + \frac{2}{r_0^2}\left(1-\frac{f_0}{f_1}\right)\right) \\
& + \frac{2}{r_0^2}\left(1-\frac{f_0}{f_1}\right)(\cos\varphi\sin\vartheta + \cos\vartheta\sin\varphi)^2 \\
& + \frac{4}{r_0}\cos\vartheta\sin\vartheta\sin\varphi\left(\frac{f_0'}{f_1} - \frac{f_0 f_1'}{f_1^2}\right),
\end{aligned} \tag{19}$$

[10] to be used only in Section 4.2 below, with $|\cos\varphi|$ bounded away from 0 by a (small) universal constant *i.e.* with $|\cos\varphi|$ *replaced* by that constant

We will also have to distinguish cases, depending on the size of $|\cos\varphi|$, then on the relative size of further arising quantities. In each case, relying on Lemma 2 and treating f_1 as a small parameter in intermediate steps thanks to (16), we will be able to find a leading term blowing up positively as ε and δ go to zero and argue with it.

We are now ready to continue the proof of Proposition 1 and start out for a case by case discussion of the sign of the c-curvature.

4.1 Case $|\cos\varphi|$ small enough

4.1.1 Subcase $\left|\sin\vartheta\frac{f_1'}{f_1}\right| \leq \frac{|\sin\varphi\cos\vartheta|}{2r_0}$

In this subcase, the assumption $|\cos\varphi| \leq \frac{1}{2}$ will suffice. We note the estimate:

$$\left(\frac{f_1'}{f_1}\sin\vartheta + \frac{\sin\varphi\cos\vartheta}{r_0}\right)^2 \geq \frac{\sin^2\varphi\cos^2\vartheta}{4r_0^4}$$

and use it to derive from (18) the inequality:

$$\begin{aligned}\mathcal{C}(m_0,V_0)(\xi,\nu) \quad &\geq \quad \frac{\sin^2\vartheta}{f_1^2}\left[f_0f_1'' + 2f_0'f_1' - f_1\left(f_0'' + \frac{2}{r_0^2}(f_1-f_0)\sin^2\varphi\right)\right] \\ &- \quad \frac{2f_0}{f_1}\frac{\sin^2\varphi\cos^2\vartheta}{4r_0^2} \\ &- \quad \frac{4}{f_1}|\cos\vartheta\sin\vartheta\sin\varphi|\left(\frac{f_1-f_0}{r_0^2} + \frac{|f_0'|}{r_0}\right) .\end{aligned}$$

The right-hand side will be handled relying on the second part of Lemma 2 combined with the pinching (14) of r_0 and previous estimates on the various D^kf_a terms which arise apart from f_0f_1''. Doing so, we can establish for $\mathcal{C}(m_0,V_0)(\xi,\nu)$ the lower bound:

$$\begin{aligned}\mathcal{C}(m_0,V_0)(\xi,\nu) \quad &\geq \quad \frac{\sin^2\vartheta}{f_1^2}\left(\frac{1}{12\pi^2} - R_1(\varepsilon,\delta)\right) + \frac{\sin^2\varphi\cos^2\vartheta}{8\pi^2 f_1} \\ &+ \quad \frac{\sin^2\vartheta}{24\pi^2 f_1^2} + \frac{\sin^2\varphi\cos^2\vartheta}{8\pi^2 f_1} - 4\left(1+\frac{5}{\pi^2}\right)\frac{|\cos\vartheta\sin\vartheta\sin\varphi|}{f_1} ,\end{aligned} \tag{20}$$

provided ε and δ are (41)-small, where $R_1(\varepsilon,\delta)$ stands for the rational function of (ε,δ) vanishing at $(0,0)$ given by the right-hand side of the smallness condition (41). We claim that the second line of the right-hand side of (20) is non-negative for small enough ε and δ. Indeed, from the identity $a^2+b^2 \geq 2|ab|$ used with $a = \frac{\sin\vartheta}{2\pi\sqrt{6}f_1}$ and $b = \frac{\sin\varphi\cos\vartheta}{2\pi\sqrt{2f_1}}$, we infer that this line is bounded below by:

$$\frac{|\cos\vartheta\sin\vartheta\sin\varphi|}{f_1\sqrt{f_1}}\left(\frac{1}{4\sqrt{3}\pi^2} - 4\left(1+\frac{5}{\pi^2}\right)\sqrt{f_1}\right),$$

and the claim follows by taking ε and δ (42)-small. Eventually, for ε and δ (41)(42)-small, we obtain:

$$\mathcal{C}(m_0,V_0)(\xi,\nu) \geq \frac{\sin^2\vartheta}{24\pi^2 f_1^2} + \frac{\sin^2\varphi\cos^2\vartheta}{8\pi^2 f_1} .$$

Combining this lower bound with Remark 2 and the useful, easily established[11], inequality:

$$\sin^2\vartheta + \cos^2\vartheta \sin^2\varphi \geq \frac{1}{4\pi^2}\mathcal{A}_2(m_0, V_0, \xi, \nu), \qquad (21)$$

we get (2) at (m_0, V_0, ξ, ν) with $\varsigma = \dfrac{1}{96\pi^4}$.

4.1.2 Subcase $\left|\sin\vartheta \frac{f_1'}{f_1}\right| > \frac{|\sin\varphi\cos\vartheta|}{2r_0}$

The second line of the right-hand side of (18) is non-negative due to (15)(16). So we may write:

$$\begin{aligned}\mathcal{C}(m_0, V_0)(\xi, \nu) \quad &\geq \quad \frac{\sin^2\vartheta}{f_1^2}\left[f_0 f_1'' + 2f_0'f_1' - f_1\left(f_0'' + \frac{2}{r_0^2}(f_1 - f_0)\sin^2\varphi\right)\right] \\ &- \quad \frac{4}{f_1}|\cos\vartheta\sin\vartheta\sin\varphi|\left(\frac{f_1 - f_0}{r_0^2}|\cos\varphi| + \frac{|f_0'|}{r_0}\right),\end{aligned}$$

hence also:

$$\begin{aligned}\mathcal{C}(m_0, V_0)(\xi, \nu) \quad &\geq \quad \frac{\sin^2\vartheta}{f_1^2}\left[f_0 f_1'' + 2f_0'f_1' - f_1\left(f_0'' + \frac{2}{r_0^2}(f_1 - f_0)\sin^2\varphi\right)\right] \\ &- \quad \frac{8\sin^2\vartheta}{f_1^2}|f_1'|\left(\frac{f_1 - f_0}{r_0}|\cos\varphi| + |f_0'|\right)\end{aligned}$$

by applying our subcase assumption. If $|\cos\varphi| \leq \frac{1}{2}$, repeating the above argument, we see that the first line of the preceding right-hand side is larger than $\dfrac{\sin^2\vartheta}{f_1^2}\left(\dfrac{1}{8\pi^2} - R_1(\varepsilon, \delta)\right)$, while the second line is bounded below by:

$$-8B_{111}\frac{\sin^2\vartheta}{f_1^2}\left[\left(\frac{\varepsilon}{2} + \delta\right)\left(1 + \frac{\varepsilon}{2}\right)\pi + \varepsilon\pi B_{210} + \frac{1 + \varepsilon\pi^2 B_{200} + \frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}}{\pi\left(1 - \frac{\varepsilon}{2} - \delta\right)}|\cos\varphi|\right]$$

as shown by combining Lemma 1 with (13)(15)(16)(17). Altogether, we may write:

$$\begin{aligned}\mathcal{C}(m_0, V_0)(\xi, \nu) \quad &\geq \quad \frac{\sin^2\vartheta}{f_1^2}\left(\frac{1}{12\pi^2} - R_2(\varepsilon, \delta)\right) \\ &+ \quad \frac{\sin^2\vartheta}{f_1^2}\left(\frac{1}{24\pi^2} - 8B_{111}\frac{1 + \varepsilon\pi^2 B_{200} + \frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}}{\pi\left(1 - \frac{\varepsilon}{2} - \delta\right)}|\cos\varphi|\right)\end{aligned}$$

with $R_2(\varepsilon, \delta)$ given by the right-hand side of (43). We get from (42):

$$\frac{1 + \varepsilon\pi^2 B_{200} + \frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}}{\left(1 - \frac{\varepsilon}{2} - \delta\right)} \leq \frac{768(\pi^2+5)^2 + 2\pi^2 + 1}{768(\pi^2+5)^2 - 1} < 1.00013\ ;$$

[11] hint: use Remark 1

besides, we have: $\pi B_{111} = 5 + \pi\sqrt{2} + 3\pi^2 \simeq 39,05 < 40$. So the smallness conditions:

$$|\cos\varphi| \leq \frac{1}{7704} \tag{22}$$

and (43) imply that $\mathcal{C}(m_0, V_0)(\xi, \nu) \geq \dfrac{\sin^2\vartheta}{24\pi^2 f_1^2}$. In our present subcase, the latter inequality yields:

$$\mathcal{C}(m_0, V_0)(\xi, \nu) \geq \frac{\sin^2\vartheta}{48\pi^2 f_1^2} + \frac{\cos^2\vartheta \sin^2\varphi}{192\pi^2 r_0^2 f_1'^2} \ .$$

Combining Lemma 1 with (13), we have: $r_0|f_1'| \leq \varepsilon\pi^2 B_{211} + 2|\cos\varphi|$. So we can arrange to have $r_0|f_1'| \leq \dfrac{1}{2}$ (say) by taking ε (44)-small. Now, we may write

$$\mathcal{C}(m_0, V_0)(\xi, \nu) \geq \frac{1}{48\pi^2}\left(\sin^2\vartheta + \cos^2\vartheta \sin^2\varphi\right)$$

and, from (21), conclude that (2) holds at (m_0, V_0, ξ, ν), indeed, with $\varsigma = \dfrac{1}{192\pi^4}$.

4.2 Case $|\cos\varphi| > \frac{1}{7704}$

In this case, the first part of Lemma 2 implies:

$$|f_1'| \geq \frac{1}{15408\pi} \text{ with } -f_1'\cos\varphi > 0 \ , \tag{23}$$

provided ε and δ are (45)-small. Furthermore, if the latter are (40)(41)-small, we infer from (15) the pinching:

$$\frac{1}{2} \leq -f_0 \leq \frac{3}{2} \ . \tag{24}$$

which will be used repeatedly.

4.2.1 Subcase $\cos\vartheta\cos\varphi\sin\vartheta\sin\varphi \leq 0$

Working with the expression (19) of $\mathcal{C}(m_0, V_0)(\xi, \nu)$, the second line of which is non-negative, and combining (23) with (24), (14) and Lemma 1, we get the inequality:

$$\begin{aligned}
\mathcal{C}(m_0, V_0)(\xi, \nu) \quad &\geq \quad \frac{\sin^2\vartheta}{f_1^3}\left(\frac{1}{15408^2\pi^2} - f_1\left(B_{120} + \frac{3}{2}B_{121} + 2B_{110}B_{111} + \frac{36}{5\pi^2}\right)\right) \\
&+ \quad \frac{2}{r_0^2}\left(1 - \frac{f_0}{f_1}\right)(\cos\varphi\sin\vartheta + \cos\vartheta\sin\varphi)^2 \\
&+ \quad \frac{2}{r_0 f_1^2}|\cos\vartheta\sin\vartheta\sin\varphi|\left(\frac{1}{15408\pi} - 2f_1 B_{110}\right).
\end{aligned}$$

Recalling (16) (24) and assuming that ε and δ are (46)-small, we infer the lower bound:

$$\begin{aligned}
\mathcal{C}(m_0, V_0)(\xi, \nu) \quad &\geq \quad \frac{\sin^2\vartheta}{2\pi^2 15408^2 f_1^3} + \frac{2}{r_0^2}\left(1 + \frac{1}{2f_1}\right)\cos^2\vartheta\sin^2\varphi \\
&+ \quad \frac{1}{15408\pi r_0 f_1^2}|\cos\vartheta\sin\vartheta\sin\varphi|\left(1 - \frac{4}{r_0}15408\pi f_1(f_1 - f_0)\right),
\end{aligned}$$

the second line of the right-hand side of which is non-negative, as checked by combining Remark 2 with (13) (16) (24) and (42). Using Remark 2 and (13) to treat its first line, we obtain the inequality

$$\mathcal{C}(m_0, V_0)(\xi, \nu) \geq \frac{\sin^2 \vartheta}{2\pi^2 15408^2} + \frac{3}{\pi^2} \cos^2 \vartheta \sin^2 \varphi$$

which, recalling (21), implies (2) at (m_0, V_0, ξ, ν) with $\varsigma = \dfrac{1}{8\pi^4 15408^2}$.

4.2.2 Subcase $\cos\vartheta \cos\varphi \sin\vartheta \sin\varphi > 0$

Here, since $-f_1' \cos\varphi > 0$, we know that the expressions $\sin\vartheta \dfrac{f_1'}{f_1}$ and $\dfrac{\sin\varphi\cos\vartheta}{r_0}$ have *opposite* signs.

Case $\left|\sin\vartheta \dfrac{f_1'}{f_1}\right| \leq \dfrac{4|\sin\varphi\cos\vartheta|}{5r_0}$ **or** $\left|\sin\vartheta \dfrac{f_1'}{f_1}\right| \geq \dfrac{5|\sin\varphi\cos\vartheta|}{4r_0}$**:** If a and b are two real numbers such that: $ab < 0$ and $|a| \leq \frac{4}{5}|b|$ or $|b| \leq \frac{4}{5}|a|$, one can readily verify that they satisfy: $(a+b)^2 \geq \frac{1}{50}(a^2+b^2)$. Using the expression (18) of $\mathcal{C}(m_0, V_0)(\xi, \nu)$, we apply the preceding estimate with $a = \sin\vartheta \dfrac{f_1'}{f_1}, b = \dfrac{\sin\varphi\cos\vartheta}{r_0}$, and find the c-curvature bounded below by:

$$\sin^2\vartheta \left(\frac{-f_0 (f_1')^2}{25 f_1^3} - \frac{f_0''}{f_1} + \frac{f_0 f_1'' + 2 f_0' f_1'}{f_1^2} - \frac{2}{r_0^2} \sin^2\varphi \left(1 - \frac{f_0}{f_1}\right)\right)$$
$$+ \frac{4|\cos\vartheta \sin\vartheta \sin\varphi|}{f_1} \left(-\frac{f_0}{7704 r_0^2} - \frac{|f_0'|}{r_0} \right) - \frac{f_0}{25 f_1 r_0^2} \cos^2\vartheta \sin^2\varphi \ ,$$

hence also, combining Lemma 1 with (14)(23)(24) and (17), by:

$$\frac{\sin^2\vartheta}{f_1^3} \left(\frac{1}{50\pi^2 15408^2} - f_1 \left(B_{120} + \frac{3}{2} B_{121} + 2 B_{110} B_{111} + \frac{36}{5\pi^2} \right) \right)$$
$$+ \frac{4|\cos\vartheta\sin\vartheta\sin\varphi|}{f_1} \left(\frac{1}{15408\pi^2} - \frac{6}{5} \left(\left(\frac{\varepsilon}{2} + \delta \right) \left(1 + \frac{\varepsilon}{2} \right) + \varepsilon B_{210} \right) \right) + \frac{1}{50\pi^2 f_1} \cos^2\vartheta \sin^2\varphi.$$

Recalling (16), we infer that:

$$\mathcal{C}(m_0, V_0)(\xi, \nu) \geq \frac{\sin^2\vartheta}{100\pi^2 15408^2 f_1^3} + \frac{1}{50\pi^2 f_1} \cos^2\vartheta \sin^2\varphi \ ,$$

provided ε and δ are (47)-small. Recalling Remark 2 and (21), it yields (2) at (m_0, V_0, ξ, ν) with $\varsigma = \dfrac{1}{400\pi^4 15408^2}$.

Case $\dfrac{4|\sin\varphi\cos\vartheta|}{5r_0} < \left|\sin\vartheta \dfrac{f_1'}{f_1}\right| < \dfrac{5|\sin\varphi\cos\vartheta|}{4r_0}$**:** This case is more difficult because we cannot use the square occuring in the second line of (18) any more;

all we can do now from (18) is write:

$$\begin{aligned}\mathcal{C}(m_0,V_0)(\xi,\nu) &\geq \sin^2\vartheta\left(-\frac{f_0''}{f_1}+\frac{f_0f_1''+2f_0'f_1'}{f_1^2}-\frac{2}{r_0^2}\sin^2\varphi\left(1-\frac{f_0}{f_1}\right)\right)\\ &+ \frac{4}{r_0^2}|\sin\varphi\cos\vartheta||\cos\varphi\sin\vartheta|\left(1-\frac{f_0}{f_1}\right)\\ &- \frac{4}{r_0}|\sin\varphi\cos\vartheta||\sin\vartheta|\frac{|f_0'|}{f_1}\end{aligned}$$

and, from our present assumption, infer for $\mathcal{C}(m_0,V_0)(\xi,\nu)$ the lower bound:

$$\begin{aligned}\sin^2\vartheta &\left(-\frac{f_0''}{f_1}+\frac{f_0f_1''+2f_0'f_1'}{f_1^2}-\frac{2}{r_0^2}\sin^2\varphi\left(1-\frac{f_0}{f_1}\right)\right)\\ - &\frac{16f_0}{5r_0f_1^2}\sin^2\vartheta|f_1'\cos\varphi|-5\sin^2\vartheta\frac{|f_1'f_0'|}{f_1^2}\ .\end{aligned}\tag{25}$$

We will factorize $\dfrac{\sin^2\vartheta}{f_1^2}$ as leading blowing up term in this expression and seek a positive coefficient for it. Doing so, we focus on the terms:

$$-\frac{f_0\sin^2\vartheta}{f_1^2}\left(-f_1''+\frac{16}{5r_0}|f_1'\cos\varphi|\right),$$

thus carefully investigate the sign of the latter parenthesis. Using Lemma 1, we find it bounded below by:

$$\left(-\bar{f}_1''+\frac{16}{5r_0}|\bar{f}_1'\cos\varphi|\right)-\varepsilon\left(B_{221}+\frac{16}{5}B_{211}\right).$$

Now, a direct calculation of $\left(-\bar{f}_1''+\dfrac{16}{5r_0}|\bar{f}_1'\cos\varphi|\right)$, using the expressions of $\bar{f}_1'$ and $\bar{f}_1''$ given in Section 2.2, shows that it is equal to:

$$\frac{1}{r_0^2}\left[|\cos\bar{r}_0|\left(1+\frac{1}{5}\cos^2\varphi\right)+\frac{\sin\bar{r}_0}{\bar{r}_0}\left(1+\left(\bar{r}_0^2+\frac{1}{5}\right)\cos^2\varphi\right)\right];$$

recalling (13), we see that it will meet the required positivity. Back to the lower bound (25), rewritten as $\mathcal{C}(m_0,V_0)(\xi,\nu)\geq\dfrac{\sin^2\vartheta}{f_1^2}E$ with E equal to:

$$(-f_0)\left(-f_1''+\frac{16}{5r_0}|f_1'\cos\varphi|\right)-7|f_0'f_1'|-f_1\left(|f_0''|+\frac{2\sin^2\varphi}{r_0^2}(f_1-f_0)\right),$$

the preceding argument, combined with Lemma 1, Remark 2 and (13) (14) (16) (17) (24), implies that $\mathcal{C}(m_0,V_0)(\xi,\nu)\geq\dfrac{\sqrt{3}\sin^2\vartheta}{8\pi^2f_1^2}$ provided ε and δ are (48)-small. In the present subcase, the latter inequality implies:

$$\mathcal{C}(m_0,V_0)(\xi,\nu)\geq\frac{\sqrt{3}}{16\pi^2}\left(\frac{\sin^2\vartheta}{f_1^2}+\frac{16\cos^2\vartheta\sin^2\varphi}{25r_0^2(f_1')^2}\right).$$

Recalling that $r_0|f_1'| \leq \frac{1}{2}$ due to (44) and using Remark 2, we obtain

$$\mathcal{C}(m_0, V_0)(\xi, \nu) \geq \frac{\sqrt{3}}{16\pi^2}\left(\sin^2\vartheta + \cos^2\vartheta \sin^2\varphi\right)$$

which, combined with (21), yields (2) at (m_0, V_0, ξ, ν) with $\varsigma = \frac{\sqrt{3}}{64\pi^4}$.

5 c-curvature almost-positivity near the origin

In this section, we prove Theorem 2 at (m_0, V_0, ξ, ν) when $d\left(m_0, \exp_{m_0}(V_0)\right)$ is small.

Proposition 2 *There exists a triple of small (strictly) positive real numbers $(\eta_2, \delta_2, \varsigma_2)$ such that $\mathcal{C}(m_0, V_0)(\xi, \nu)$ satisfies the lower bound (2) with $\varsigma \leq \varsigma_2$, provided $\varepsilon = |K-1|_{C^2(S)} \leq \eta_2$ and $|V_0| \leq \delta_2$.*

Proof. As already observed, we may take $V_0 \neq 0$ with no loss of generality. Dropping the subscript of δ_2, we take $\bar{r}_0 \leq \frac{\pi}{2}$ by assuming ε and δ (49)-small. We use the Maclaurin type approximation of $\frac{f_1^3}{\bar{f}_1^3}\,\mathcal{C}(m_0, V_0)(\xi, \nu)$ obtained in Corollary 1 and proceed to specify it further as $r_0 \downarrow 0$. As regards its first summand, namely $\overline{\mathcal{C}}(m_0, V_0)(\xi, \nu)$, the expression (7) prompts us to define constants $c_{11}, \ldots, c_{14}$ as done in Appendix B. These definitions imply at once that the absolute value of:

$$\overline{\mathcal{C}}(m_0, V_0)(\xi, \nu) - \frac{2\kappa\bar{r}_0^2}{45}\sin^2\vartheta\sin^2\varphi - \frac{2\kappa}{3}\left(1 + \frac{2\bar{r}_0^2}{5}\right)\sin^2\vartheta\cos^2\varphi$$

$$-\frac{2\kappa}{3}\left(1 + \frac{2\bar{r}_0^2}{15}\right)\cos^2\vartheta\sin^2\varphi - \frac{4\kappa}{3}\left(1 + \frac{\bar{r}_0^2}{5}\right)\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi$$

is bounded above by:

$$\kappa\bar{r}_0^3(c_{11}\sin^2\vartheta\sin^2\varphi + c_{12}\sin^2\vartheta\cos^2\varphi$$

$$+c_{13}\cos^2\vartheta\sin^2\varphi + c_{14}|\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi|).$$

Let us now focus on the second summand, namely on the expression

$$E_4 := \frac{r_0\psi_2\sin^2\vartheta}{\bar{f}_1}\left(\mathcal{S}_{\bar{r}_0}(t)(1) - \frac{\bar{f}_0\mathcal{S}_{\bar{r}_0}(t^2)(1)}{\bar{f}_1}\right)$$

$$-\frac{2r_0\psi_0\mathcal{S}_{\bar{r}_0}(t^2-t)(1)}{\bar{f}_1}\left(\cos^2\vartheta - \cos^2(\vartheta+\varphi)\right)$$

$$-\frac{4r_0\psi_1\cos\vartheta\sin\vartheta\sin\varphi}{\bar{f}_1}\left(\mathcal{S}_{\bar{r}_0}(t)(1) - \frac{\bar{f}_0\mathcal{S}_{\bar{r}_0}(t^2)(1)}{\bar{f}_1}\right)$$

and rewrite, on the one hand:

$$\frac{r_0}{\bar{f}_1}\left(\mathcal{S}_{\bar{r}_0}(t)(1) - \frac{\bar{f}_0\mathcal{S}_{\bar{r}_0}(t^2)(1)}{\bar{f}_1}\right)$$

as: $r_0\mathcal{S}_{\bar{r}_0}(t-t^2)(1)+\sqrt{\kappa}r_0^2\left[A_1(\bar{r}_0)\mathcal{S}_{\bar{r}_0}(t)(1)-A_2(\bar{r}_0)\mathcal{S}_{\bar{r}_0}(t^2)(1)\right]$, where[12]:

$$A_1(\tau):=\frac{\tau-\sin\tau}{\tau\sin\tau},\ A_2(\tau):=\frac{\tau^2\cos\tau-\sin^2\tau}{\tau\sin^2\tau}$$

(and note that two additional constants c_{15}, c_{16} are defined accordingly as in Appendix B), on the other hand:

$$\frac{r_0}{\bar{f}_1}\mathcal{S}_{\bar{r}_0}(t^2-t)(1)=r_0\mathcal{S}_{\bar{r}_0}(t^2-t)(1)+\sqrt{\kappa}r_0^2A_1(\bar{r}_0)\mathcal{S}_{\bar{r}_0}(t^2-t)(1).$$

Furthermore, the Maclaurin expansion of $\mathcal{S}_{\bar{r}_0}(t^2-t)(1)$ prompts us to write:

$$r_0\mathcal{S}_{\bar{r}_0}(t^2-t)(1)=-\frac{r_0}{12}+\kappa r_0^3A_3(\bar{r}_0)$$

(defining so the auxiliary function A_3 and, accordingly, a constant c_{17} as in Appendix B). Gathering terms of same order and recalling (11), we obtain that the absolute value of:

$$E_4-\frac{r_0}{6}\left[2\sin\vartheta\sin\varphi\sin(\vartheta-\varphi)\partial_1K(0)+\left(2\sin\vartheta\cos\varphi\sin(\vartheta-\varphi)+\sin^2(\vartheta-\varphi)\right)\partial_2K(0)\right]$$

is bounded above by:

$$\begin{aligned}&2\sqrt{\kappa}\varepsilon r_0^2[(8(c_{15}c_6+c_{16}c_7)+(c_6+c_7)c_{15})\sin^2\vartheta\\&+(4(c_{15}c_6+c_{16}c_7)+2(c_6+c_7)c_{15})\cos^2\vartheta\sin^2\varphi]\\&+2c_{17}\ \kappa\varepsilon r_0^3\left(9\sin^2\vartheta+6\cos^2\vartheta\sin^2\varphi\right)\ .\end{aligned}$$

Combining the latter inequality with the one derived above for the first summand $\overline{\mathcal{C}}(m_0,V_0)(\xi,\nu)$ of the expansion of $\mathcal{C}(m_0,V_0)(\xi,\nu)$ given in Corollary 1, we infer that, if we consider the decomposition:

$$\frac{f_1^3}{\bar{f}_1^3}\ \mathcal{C}(m_0,V_0)(\xi,\nu)=I+II+III$$

with

$$\begin{aligned}I:=&\frac{\kappa}{3}\left(1+\frac{23\bar{r}_0^2}{30}\right)\sin^2\vartheta\cos^2\varphi+\frac{\kappa}{3}\left(1+\frac{\bar{r}_0^2}{10}\right)\cos^2\vartheta\sin^2\varphi\\&-\frac{2\kappa}{3}\left(1+\frac{2\bar{r}_0^2}{5}\right)\sin\vartheta\cos\vartheta\sin\varphi\cos\varphi\ ,\end{aligned}$$

and

$$\begin{aligned}II:=&\frac{\kappa}{3}\sin^2(\vartheta-\varphi)+\frac{\kappa\bar{r}_0^2}{180}\left(\sin^2\vartheta\cos^2\varphi+\cos^2\vartheta\sin^2\varphi+4\sin^2\vartheta\sin^2\varphi\right)\\&+\frac{r_0}{6}\ [2\sin\vartheta\sin\varphi\sin(\vartheta-\varphi)\partial_1K(0)\\&+\left(2\sin\vartheta\cos\varphi\sin(\vartheta-\varphi)+\sin^2(\vartheta-\varphi)\right)\partial_2K(0)]\end{aligned}$$

[12] so that: $A_1(\bar{r}_0)=\frac{1}{\bar{r}_0}\left(\frac{1}{\bar{f}_1}-1\right)$, $A_2(\bar{r}_0)=\frac{1}{\bar{r}_0}\left(\frac{\bar{f}_0}{\bar{f}_1^2}-1\right)$

and III so defined, then the quantity:

$$\left| III - \frac{\kappa \bar r_0^2}{180}\left(\sin^2\vartheta\cos^2\varphi + \cos^2\vartheta\sin^2\varphi + 4\sin^2\vartheta\sin^2\varphi\right)\right|$$

is altogether bounded above by:

$$\begin{aligned}
&\varepsilon r_0^2 \sin^2\vartheta\left(\frac{338C_1^3\pi^8}{\bar f_1^3} + 2\sqrt{\kappa}\,[8(c_{15}c_6 + c_{16}c_7) + (c_6+c_7)c_{15}]\right)\\
&+\varepsilon r_0^2\cos^2\vartheta\sin^2\varphi\left(\frac{268C_1^3\pi^8}{\bar f_1^3} + 2\sqrt{\kappa}\,[4(c_{15}c_6 + c_{16}c_7) + 2(c_6+c_7)c_{15}]\right)\\
&+\varepsilon r_0^3\ 2\kappa c_{17}\left(9\sin^2\vartheta + 6\cos^2\vartheta\sin^2\varphi\right)\\
&+\kappa \bar r_0^3\ (c_{11}\sin^2\vartheta\sin^2\varphi + c_{12}\sin^2\vartheta\cos^2\varphi)\\
&+\kappa\bar r_0^3\ (c_{13}\cos^2\vartheta\sin^2\varphi + c_{14}|\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi|).
\end{aligned}$$

Now, let us discuss separately the positivity of each summand I, II, III. Noting that

$$I \geq \frac{2\kappa}{3}|\cos\vartheta\sin\vartheta\cos\varphi\sin\varphi|\left(\sqrt{\left(1+\frac{\bar r_0^2}{10}\right)\left(1+\frac{23\bar r_0^2}{30}\right)} - \left(1+\frac{2\bar r_0^2}{5}\right)\right),$$

we find $I \geq 0$ provided $\bar r_0 \leq \frac{2}{\sqrt{5}}$ which holds if ε and δ are (50)-small. Next, we have:

$$\begin{aligned}
II \geq \frac{\kappa}{3}\sin^2(\vartheta-\varphi) &+ \frac{\kappa\bar r_0^2}{180}\left(\sin^2\vartheta + \sin^2\varphi + 2\sin^2\vartheta\sin^2\varphi\right)\\
&-\frac{\varepsilon r_0}{6}\ \left(4|\sin\vartheta| + |\sin(\vartheta-\varphi)|\right)|\sin(\vartheta-\varphi)|\ ,
\end{aligned}$$

hence

$$\begin{aligned}
II \geq \frac{\kappa}{9}\sin^2(\vartheta-\varphi) &+ \frac{\kappa\bar r_0^2}{360}\left(\sin^2\vartheta+\sin^2\varphi\right) + \sin^2(\vartheta-\varphi)\left(\frac{\kappa}{9} - \frac{\varepsilon r_0}{6}\right)\\
&+\frac{\kappa}{9}\sin^2(\vartheta-\varphi) + \frac{\kappa\bar r_0^2}{360}\sin^2\vartheta - \frac{2\varepsilon r_0}{3}|\sin\vartheta\sin(\vartheta-\varphi)|\ .
\end{aligned}$$

So, assuming provisionally $III \geq 0$, and under the further smallness conditions[13]:

$$\varepsilon\delta \leq \frac{2}{3},\ \varepsilon \leq \frac{1}{6\sqrt{10}},$$

the first of which implies $\left(\frac{\kappa}{9} - \frac{\varepsilon r_0}{6}\right) \geq 0$, the second of which ensures that the second line of our last lower bound on II is identically non-negative, we obtain:

$$\frac{f_1^3}{\bar f_1^3}\,\mathcal{C}(m_0, V_0)(\xi,\nu) \geq \frac{\kappa}{9}\sin^2(\vartheta-\varphi) + \frac{\kappa^2}{360}\ r_0^2\left(\sin^2\vartheta + \sin^2\varphi\right),$$

[13]implied, for instance, by (41) and (50)

which proves Proposition 2 with $\varsigma = \dfrac{1}{360}$ in (2). Finally, let us discuss the non-negativity of III. From $\bar{r}_0 \leq \dfrac{\pi}{2}$, we have $\bar{f}_1(\bar{r}_0) \geq \dfrac{2}{\pi}$; moreover, $\sqrt{\kappa}$ is bounded above by $1 + \dfrac{1}{12\sqrt{10}} < 1.027$ due to our last smallness assumption on ε. So the constants c_{18}, c_{19} defined in Appendix B can be used as upper bounds on the coefficients respectively of $\varepsilon r_0^2 \sin^2 \vartheta$ and $\varepsilon r_0^2 \cos^2 \vartheta \sin^2 \varphi$ in the lengthy expression which controls $\left| III - \dfrac{\kappa \bar{r}_0^2}{180} \ldots \right|$ (*cf. supra*). Using them and recalling (11), we infer from the control just mentionned that:

$$\begin{aligned}\frac{1}{r_0^2} III \quad &\geq \quad \sin^2 \vartheta \left[\frac{1}{180} - \varepsilon(c_{18} + 19 c_{17}\delta) - \frac{115}{100}\delta(c_{11} + c_{12} + \frac{1}{2}c_{14}) \right] \\ &+ \quad \cos^2 \vartheta \sin^2 \varphi \left[\frac{1}{180} - \varepsilon(c_{19} + 13 c_{17}\delta) - \frac{115}{100}\delta(c_{13} + \frac{1}{2}c_{14}) \right].\end{aligned}$$

Therefore $III \geq 0$ provided ε and δ are taken (51)(52)-small and Proposition 2 is proved.

6 c-curvature almost-positivity elsewhere

In this section, we prove Theorem 2 at (m_0, V_0, ξ, ν) when $\exp_{m_0}(V_0)$ stays away from m_0 and m_0^* as specified[14] in the

Proposition 3 *There exists a triple of small (strictly) positive real numbers* $(\eta_3, \delta_3, \varsigma_3)$ *such that* $\mathcal{C}(m_0, V_0)(\xi, \nu)$ *satisfies the lower bound (2) with* $\varsigma \leq \varsigma_3$, *provided* $\varepsilon = |K - 1|_{C^2(S)} \leq \eta_3$ *and* $\frac{1}{2}\delta_3 \leq |V_0| \leq \left(1 - \frac{1}{2}\delta_3\right) \ell_0$.

Proof. Dropping, for short, the subscript of δ_3, the following pinching holds:

$$\frac{1}{2}\delta\sqrt{1-\varepsilon} \leq \bar{r}_0 \leq \pi\left(1 - \frac{1}{2}\delta\right)\sqrt{1+\varepsilon}\ .$$

Recalling the assumption of Lemma 1, namely $\varepsilon \leq \frac{1}{\pi^2}$, and assuming that $\varepsilon \leq \frac{1}{2}\delta$, we readily infer the pinching:

$$\frac{9}{20}\delta \leq \bar{r}_0 \leq \left(1 - \frac{\delta}{4}\right)\pi \tag{26}$$

the right-hand side of which yields the estimate:

$$\frac{1}{\bar{f}_1} \leq \frac{\pi}{\sin\left(\frac{\delta}{4}\pi\right)} \tag{27}$$

recorded here for later use. Now, a quick way[15] of proving Proposition 3 is to apply Corollary 1 once on the whole range $\left[\frac{1}{2}\delta, \left(1 - \frac{1}{2}\delta\right)\ell_0\right]$ of r_0. Indeed,

[14] sticking to the notation ℓ_0 introduced just before Proposition 1

[15] at the expense, though, of the very strong smallness condition (53) below on ε

combining the latter with (10) (11) and $r_0 \leq \pi$, we find:

$$\begin{aligned}\frac{f_1^3}{\bar{f}_1^3}\mathcal{C}(m_0,V_0)(\xi,\nu) &\geq \overline{\mathcal{C}}(m_0,V_0)(\xi,\nu) - \frac{\varepsilon}{\bar{f}_1^3}\sin^2\vartheta\left(338C_1^3\pi^{10}+20\pi(c_6+c_7)\right)\\ &- \frac{\varepsilon}{\bar{f}_1^3}\cos^2\vartheta\sin^2\varphi\left(268C_1^3\pi^{10}+20\pi(c_6+c_7)\right) ,\end{aligned}$$

or else, using $\kappa \geq 1$ and (27):

$$\frac{f_1^3}{\bar{f}_1^3}\mathcal{C}(m_0,V_0)(\xi,\nu) \geq \frac{1}{\kappa}\overline{\mathcal{C}}(m_0,V_0)(\xi,\nu) - \frac{\varepsilon\pi^3 c_{20}}{\sin^3\left(\frac{\delta}{4}\pi\right)}(\sin^2\vartheta+\cos^2\vartheta\sin^2\varphi) . \quad (28)$$

Recalling (7), we split $\frac{1}{\kappa}\overline{\mathcal{C}}(m_0,V_0)(\xi,\nu)$ into two summands, namely, the square:

$$\overline{S}_1 = 2\left(\sin\vartheta\cos\varphi\sqrt{\frac{\bar{r}_0^2-\sin^2\bar{r}_0}{\bar{r}_0\sin^3\bar{r}_0}} - \cos\theta\sin\varphi\sqrt{\frac{\bar{r}_0^2-\sin^2\bar{r}_0}{\bar{r}_0^3\sin\bar{r}_0}}\right)^2 ,$$

and the remaining part, equal to:

$$\overline{S}_2 = \sin^2\vartheta\sin^2\varphi\frac{h_1(\bar{r}_0)}{\bar{r}_0^2\sin^2\bar{r}_0} + 8\sin^2\vartheta\cos^2\varphi\frac{\cos\frac{\bar{r}_0}{2}h_2(\bar{r}_0)}{\bar{r}_0\sin^3\bar{r}_0} + 8\cos^2\vartheta\sin^2\varphi\frac{\cos\frac{\bar{r}_0}{2}h_2(\bar{r}_0)}{\bar{r}_0^3\sin\bar{r}_0},$$

where

$$h_1(\tau) = \tau^2 + \tau\sin\tau\cos\tau - 2\sin^2\tau ,$$
$$h_2(\tau) = (\tau+\sin\tau\cos\tau)\sin\tau - 2\tau^2\cos\tau .$$

On the one hand, from (26), we note that $\cos\frac{\bar{r}_0}{2} \geq \sin\left(\frac{\delta}{8}\pi\right)$. On the other hand, a careful but elementary check[16] shows that the function h_1 (resp. h_2) is increasing on $[0,\pi]$ (resp. on $\left[0,\frac{\pi}{2}\right]$). We thus infer from (26) that

$$\overline{S}_2 \geq \gamma_1(\sin^2\vartheta+\cos^2\vartheta\sin^2\varphi)$$

with:

$$\gamma_1 = \min\left(\frac{1}{\pi^2}h_1\left(\frac{9}{20}\delta\right), \frac{8}{\pi^3}\sin\left(\frac{\delta}{8}\pi\right)h_2\left(\frac{9}{40}\delta\right)\right).$$

Here, lengthy but routine calculations show that:

$$h_1(\tau) = \frac{2}{315}\tau^6(7-\tau^2) + O(\tau^{10})$$
$$h_2(\tau) = \frac{4}{5}\tau^6\left(\frac{2}{9}-\frac{1}{21}\tau^2\right) + O(\tau^{10})$$

hence:

$$\gamma_1 = \frac{8}{\pi^3}\sin\left(\frac{\delta}{8}\pi\right)h_2\left(\frac{9}{40}\delta\right).$$

Back to (28), we obtain the lower bound:

$$\frac{f_1^3}{\bar{f}_1^3}\mathcal{C}(m_0,V_0)(\xi,\nu) \geq \frac{1}{2}\gamma_1(\sin^2\vartheta+\cos^2\vartheta\sin^2\varphi) ,$$

provided the smallness condition (53) is fulfilled by ε and δ. Combining this bound with Remark 2 and the estimates (21) (27), we conclude that $\mathcal{C}(m_0,V_0)(\xi,\nu)$ satisfies (2), indeed, with $\varsigma = \frac{1}{\pi^8}\sin\left(\frac{\delta}{8}\pi\right)\sin^3\left(\frac{\delta}{4}\pi\right)h_2\left(\frac{9}{40}\delta\right)$.

[16] left here as an exercise

7 Proof of Theorem 2

The proof of Theorem 2 at (m_0, V_0, ξ, ν) goes by combining Propositions 1, 2 and 3. We thus assume that ε and δ fulfill *all* the smallness conditions stated on them in Sections 4, 5 and 6. Doing so, we first observe that the assumption made on $|V_0|$ in Proposition 3 (now with $\delta_3 \equiv \delta_2 \equiv \delta_1 =: \delta$) overlaps, as it should, the corresponding ones of Propositions 1 and 2. Next, we find by inspection that the conditions (47) and (53) imply all others. From (47), combined with the rough pinching

$$1341 \leq B_{120} + \frac{3}{2} B_{121} + 2B_{110}B_{111} + \frac{36}{5\pi^2} \leq 1420 \ ,$$

we get: $\frac{\varepsilon}{2} + \delta \leq 3.2 \times 10^{-15}$, quite a small bound, indeed. Besides, calculations yield:

$$C_1 \geq B_{120} \geq 483, \ c_6 = \frac{1}{6}, \ c_7 \geq \frac{1}{12},$$

hence also the lower bound: $c_{20} \geq 3.5 \times 10^{15}$. From the latter combined with (53), in which we plug the value of γ_1 given above, we infer:

$$\varepsilon \leq \frac{4}{\pi^6 \times 3.5 \times 10^{15}} \sin\left(\frac{\delta}{8}\pi\right) \sin^3\left(\frac{\delta}{2}\pi\right) h_2\left(\frac{9}{40}\delta\right) \ .$$

Using the Maclaurin expansion found above for the function h_2, we obtain the further bound:

$$\varepsilon \leq 4.3 \times 10^{-23} \ \delta^{10} + O(\delta^{12}).$$

The latter combined with the former leads to: $\varepsilon \leq O(10^{-173})$, our worst bound. Finally, we choose for ς the *smallest* among the values found for it in the Sections 4, 5, 6, namely:

$$\varsigma = \min\left(\frac{1}{400\pi^4 154408^2}, \frac{1}{\pi^8} \sin\left(\frac{\delta}{8}\pi\right) \sin^3\left(\frac{\delta}{4}\pi\right) h_2\left(\frac{9}{40}\delta\right)\right),$$

or else, from the previous estimate on δ:

$$\begin{aligned} \varsigma &= \frac{1}{\pi^8} \sin\left(\frac{\delta}{8}\pi\right) \sin^3\left(\frac{\delta}{4}\pi\right) h_2\left(\frac{9}{40}\delta\right) \\ &\leq 4.7 \times 10^{-10} \ \delta^{10} + O(\delta^{12}) \leq O(10^{-160}). \end{aligned}$$

A Proof of Lemma 1

We will proceed stepwise in the Fermi chart along V_0, using repeatedly the Maclaurin theorem, the solution map $\mathcal{S}_{\bar{r}_0}$ and its contraction property, to derive estimates at $((0, r_0), t)$, uniform in $t \in [0, 1]$, on the expressions appearing in the conclusion of Lemma 1 and also on $|D^2 X|$ and $|D^j \mathcal{K}|$ for $j = 1, 2$, where $\mathcal{K} = K \circ X$.

A.1 Estimates of order 0

A.1.1 Basic estimates

From Remark 2, we may take $B_{101} = 1$. Besides, we have:

$$\|\kappa - \mathcal{K}\| \leq \varepsilon \ \min(1, r_0) \ . \tag{29}$$

On the axis of the Fermi chart, the functions $\tilde{f}_a = f_a - \bar{f}_a$ (with $a = 0, 1$) satisfy:

$$\frac{d^2 \tilde{f}_a}{dt^2} + r_0^2 \kappa \ \tilde{f}_a = \phi_{0a} \text{ with } \phi_{0a} = r_0^2(\kappa - \mathcal{K}) f_a \ .$$

Combining the latter with (8) applied to $\mathcal{S}_{\bar{r}_0}$, and (29), we get:

$$\|\tilde{f}_a\| \leq \frac{\varepsilon}{2} \min(r_0^2, r_0^3) \ \|f_a\|.$$

If $a = 0$, since $\|f_0\| \leq \|\tilde{f}_0\| + \|\bar{f}_0\| \leq \|\tilde{f}_0\| + 1$, we infer:

$$\|\tilde{f}_0\| \leq \frac{\mu}{1-\mu} \text{ with } \mu = \frac{\varepsilon}{2} \min(r_0^2, r_0^3),$$

while if $a = 1$, recalling Remark 2, we get at once: $\|\tilde{f}_1\| \leq \mu$. Since $\varepsilon \leq \frac{1}{\pi^2}$, we have $\varepsilon r_0^2 \leq 1$ (recalling Remark 1), an inequality used throughout this appendix. So we readily obtain:

$$\|\tilde{f}_0\| \leq \varepsilon \ \min\left(r_0^2, \frac{r_0^3}{2 - \varepsilon r_0^3}\right).$$

In particular, regarding the first line of conclusion of the lemma for $k = 0$, we may take $B_{200} = 1, B_{201} = \frac{1}{2}$, which yields $B_{100} = 2$ after use of the triangle inequality. Similarly, setting $h_0 = 1$ and $h_1 = t$, we find on the axis:

$$f_a - h_a = \mathcal{S}_{\bar{r}_0}\left(-r_0^2 \ \mathcal{K} \ h_a + r_0^2(\kappa - \mathcal{K})(f_a - h_a)\right)$$

for $a = 0, 1$. Combining (8) with an argument as the one above for $\tilde{f}_0$ yields:

$$\|f_a - h_a\| \leq \frac{r_0^2 \|\mathcal{K}\|}{2 - \varepsilon r_0^2} \leq r_0^2 \|\mathcal{K}\|$$

hence the inequalities:

$$\|f_0 - 1\| \leq r_0^2(1 + \varepsilon), \ \|f_1 - t\| \leq r_0^2(1 + \varepsilon) \ , \tag{30}$$

recorded here for later use.

A.1.2 Estimates on Maclaurin approximations

The first order Maclaurin approximation of $\mathcal{K}$ at $t = 0$ satisfies the estimate:

$$\|\mathcal{K} - \kappa - tr_0 \ \partial_2 K(0)\| \leq \frac{1}{2} \varepsilon \ r_0^2 \ . \tag{31}$$

The latter combined with the triangle inequality is used to evaluate the remainder of the first non trivial Maclaurin approximation of ϕ_{0a} at $t = 0$, namely of $\phi_{0a} + t^{a+1} r_0^3\ \partial_2 K(0)$ written as:

$$\phi_{0a} + t^{a+1} r_0^3\ \partial_2 K(0) = -r_0^2 \left(\mathcal{K} - \kappa - tr_0\ \partial_2 K(0)\right) f_a + tr_0^3\ \partial_2 K(0)\ (h_a - f_a).$$

It leads us to the upper bound:

$$\|\phi_{0a} + t^{a+1} r_0^3\ \partial_2 K(0)\| \leq \frac{1}{2}\varepsilon\ r_0^4\ \|f_a\| + \varepsilon\ r_0^3\ \|f_a - h_a\|$$

which, combined with (30) and (8), yields for

$$\tilde{f}_a + r_0^3\ \psi_0\ \mathcal{S}_{\bar{r_0}}(t^{a+1}) \equiv \mathcal{S}_{\bar{r_0}}\left(\phi_{0a} + t^{a+1} r_0^3\ \partial_2 K(0)\right)$$

the desired second line of conclusion with $B_{30a} = \dfrac{1}{4} B_{10a} + \dfrac{\pi}{2}\left(1 + \dfrac{1}{\pi^2}\right)$.

A.2 Estimates of order 1

A.2.1 Basic estimates

From the definition of $\mathcal{K}$ and f_1, we have at (v_0, t):

$$D_1\mathcal{K} = f_1(t)\ (\partial_1 K)(0, tr_0),\ \ D_2\mathcal{K} = t\ (\partial_2 K)(0, tr_0).$$

Recalling Remark 2, we conclude:

$$\forall i = 1, 2, \forall t \in [0, 1],\ \ |D_i\mathcal{K}(v_0, t)| \leq \varepsilon,\ \text{ thus } \|D_\nu\mathcal{K}(v_0, .)\| \leq \sqrt{2}\varepsilon. \tag{32}$$

Besides, if we apply D_ν to the Jacobi equations:

$$\ddot{f} + |v|^2\mathcal{K}(v, t) f = 0 \text{ and } \ddot{f} + |v|^2\kappa f = 0, \tag{33}$$

then let $v = v_0 = (0, r_0)$, we readily infer for $\tilde{f}_a$ the equation (still abbreviating freely D_ν by a prime): $\dfrac{d^2\tilde{f}'_a}{dt^2} + r_0^2\kappa\ \tilde{f}'_a = \phi_{1a}$, with:

$$\phi_{1a} = r_0^2(\kappa - \mathcal{K}) f'_a - 2r_0 \cos\varphi\ \kappa\tilde{f}_a + 2r_0\cos\varphi\ f_a(\kappa - \mathcal{K}) - r_0^2\mathcal{K}' f_a,$$

and for $\bar{f}_a$ the equation: $\dfrac{d^2\bar{f}'_a}{dt^2} + r_0^2\kappa\ \bar{f}'_a = -2r_0\cos\varphi\ \kappa\bar{f}_a$. Recalling (8), we get from the latter the auxiliary bound:

$$\|\bar{f}'_a\| \leq r_0\kappa \leq c_1 \tag{34}$$

(see Appendix B), and from the former:

$$\|\tilde{f}'_a\| \leq \frac{1}{2} r_0^2 \|\kappa - \mathcal{K}\| \left(\|\tilde{f}'_a\| + \|\bar{f}'_a\|\right) + r_0\kappa\ \|\tilde{f}_a\| + r_0\|\kappa - \mathcal{K}\|\ \|f_a\| + \frac{1}{2} r_0^2 \|\mathcal{K}'\|\ \|f_a\|,$$

after use of the triangle inequality. Previous bounds, namely (29)(32)(34) and those of Lemma 1 for $k = 0$, yield:

$$\|\tilde{f}'_a\| \leq \frac{1}{1 - \frac{1}{2}\varepsilon r_0^2}\left(\frac{1}{2}\varepsilon\kappa r_0^3 + B_{20a}\varepsilon\kappa r_0^3 + B_{10a}\varepsilon r_0 + \frac{1}{2} B_{10a}\sqrt{2}\varepsilon r_0^2\right)$$

hence the conclusion of the first line of the lemma holds for $k = 1$ with:

$$B_{21a} = 1 + \pi^2 + 2B_{20a}(1 + \pi^2) + B_{10a}(2 + \pi\sqrt{2}),$$

and, combining the triangle inequality with the auxiliary bound on $\bar{f}'_a$, with:

$$B_{11a} = \pi + \frac{1}{\pi}(1 + B_{21a}).$$

A.2.2 Estimates on Maclaurin approximations

From the expression found above for $D\mathcal{K}(v_0, t)$, we may write:

$$D_\nu \mathcal{K}(v_0, t) = t\partial_\nu K(0, tr_0) + \sin\varphi\ \partial_1 K(0, tr_0)\ (f_1 - t).$$

So, using the straightforward bound: $|\partial_\nu K(0, tr_0) - \partial_\nu K(0)| \leq \varepsilon r_0$ combined with the triangle inequality and (30), we obtain:

$$\|\mathcal{K}' - t\partial_\nu K(0)\| \leq (1 + c_1)\varepsilon r_0\ . \tag{35}$$

We wish now to estimate the remainder of the first non trivial Maclaurin approximation of ϕ_{1a} at $t = 0$, namely the $\|.\|$ norm of the expression:

$$\phi_{1a} + 2t\ h_a r_0^2 \cos\varphi\ \partial_2 K(0) + t\ h_a r_0^2\ \partial_\nu K(0).$$

To do so, we recast the latter as follows:

$$= -2\kappa r_0 \cos\varphi\ \tilde{f}_a - 2r_0 \cos\varphi\ f_a\ (\mathcal{K} - \kappa - tr_0\ \partial_2 K(0)) + 2tr_0^2 \cos\varphi\ \partial_2 K(0)(h_a - f_a)$$
$$+ r_0^2(\kappa - \mathcal{K})(\tilde{f}'_a + \bar{f}'_a) - r_0^2\ (\mathcal{K}' - t\partial_\nu K(0))\ f_a - tr_0^2\ \partial_\nu K(0)(f_a - h_a)$$

and apply the triangle inequality combined with (29)(30)(31)(34)(35) and the bounds of the lemma on the $\|.\|$ norms of $\tilde{f}_a, \tilde{f}'_a$. Observing that, if we apply the map $\mathcal{S}_{\bar{r_0}}$ to the preceding expression and use (9), we recover $\tilde{f}'_a + r_0^2\ \psi_1\ \mathcal{S}_{\bar{r_0}}(t^{a+1})$, and recalling (8), we infer that $\|\tilde{f}'_a + r_0^2\ \psi_1\ \mathcal{S}_{\bar{r_0}}(t^{a+1})\|$ is bounded above by:

$$\varepsilon r_0^3 \left(\frac{1}{2}B_{10a}\left(2 + \pi + \frac{1}{\pi}\right) + B_{20a}\kappa + B_{21a}\frac{\varepsilon}{2} + \frac{3}{2}r_0(1 + \varepsilon) + \frac{\kappa}{2}\right).$$

The second line of conclusion of Lemma 1 for $k = 1$, indeed, follows with:

$$B_{31a} = \frac{1}{2}B_{10a}\left(2 + \pi + \frac{1}{\pi}\right) + B_{20a}\left(1 + \frac{1}{\pi^2}\right) + B_{21a}\frac{1}{2\pi^2} + \frac{(3\pi + 1)}{2}\left(1 + \frac{1}{\pi^2}\right).$$

A.3 Estimates of order 2

A.3.1 Basic estimates

As in [7], applying twice D_ν to the geodesic equation with initial conditions $(0, v)$, then letting $v = v_0 = (0, r_0)$, and recalling the 2-dimensional formulas given after Definition 2 for the derivatives of the Christoffel symbols on the axis of the Fermi chart, yields for $D_{\nu\nu}X^i(t) = D_{\nu\nu}X^i(0, v_0, t)$ the following equations, with zero initial conditions:

$$\begin{aligned}\frac{d^2}{dt^2}\left(D_{\nu\nu}X^1\right) + r_0^2\mathcal{K}\ D_{\nu\nu}X^1 = &- 4r_0 \cos\varphi \sin\varphi\ \mathcal{K}f_1 \\ &- r_0^2 \sin^2\varphi\ f_1^2\ ((\partial_1 K) \circ X) \\ &- 2r_0^2 \sin\varphi \cos\varphi\ tf_1\ ((\partial_2 K) \circ X)\,,\end{aligned}$$

$$\frac{d^2}{dt^2}\left(D_{\nu\nu}X^2\right) = 4r_0 \sin^2\varphi\ \mathcal{K} f_1 \dot{f}_1 + r_0^2 \sin^2\varphi\ f_1^2\ \left((\partial_2 K)\circ X\right).$$

To treat the first equation, we view $\mathcal{K}$ as a perturbation of κ and apply the solution map $\mathcal{S}_{\bar{r}_0}$ and the estimates (8) (29) and that on $\|f_1\|$; to treat the second equation, we use our estimates on $\|\mathcal{K}\|$ and $\|f_1\|$ and note the further one:

$$\dot{f}_1(t) = 1+\int_0^t \ddot{f}_1(\theta)d\theta \equiv 1-r_0^2\int_0^t \mathcal{K}(\theta)f_1(\theta)d\theta \Longrightarrow \|\dot{f}_1\| \le 1+r_0^2(1+\varepsilon) \le 2+\pi^2.$$

We readily find:

$$\|D_{\nu\nu}X^1\| \le c_2 r_0,\ \|D_{\nu\nu}X^2\| \le c_3 r_0. \tag{36}$$

Next, we calculate the expression of $D_{\nu\nu}\mathcal{K}(v_0,t)$ and obtain:

$$\begin{aligned} D_{\nu\nu}\mathcal{K}(v_0,t) &= \partial_1 K(0,tr_0)\ D_{\nu\nu}X^1 + \partial_2 K(0,tr_0)\ D_{\nu\nu}X^2 \\ &\quad +\partial_{11}K(0,tr_0)\ f_1^2\sin^2\varphi + 2\partial_{12}K(0,tr_0)\ tf_1\sin\varphi\cos\varphi \\ &\quad +\partial_{22}K(0,tr_0)\ t^2\cos^2\varphi\ , \end{aligned}$$

from what we infer, using (36) combined with Remark 2:

$$\|D_{\nu\nu}\mathcal{K}(v_0,.)\| \le (c_2+c_3)\varepsilon r_0 + 2\varepsilon \le c_4\varepsilon. \tag{37}$$

Now, we apply $D_{\nu\nu}$ to (33) and get, on the one hand:

$$\frac{d^2}{dt^2}\left(\bar{f}_a''\right) + r_0^2\kappa\bar{f}_a'' = -2\kappa\bar{f}_a - 4\kappa r_0\cos\varphi\ \bar{f}_a'\ ,$$

from what, recalling (34), we infer the auxiliary bound:

$$\|\bar{f}_a''\| \le c_5\ , \tag{38}$$

on the other hand:

$$\frac{d^2}{dt^2}\left(\tilde{f}_a''\right) + r_0^2\kappa\tilde{f}_a'' = \phi_{2a}\ ,$$

with:

$$\begin{aligned} \phi_{2a} &= r_0^2(\kappa-\mathcal{K})f_a'' + 2(\kappa\bar{f}_a - \mathcal{K}f_a) + 4r_0\cos\varphi(\kappa\bar{f}_a' - \mathcal{K}f_a') \\ &\quad -4r_0\cos\varphi\ f_a D_\nu\mathcal{K} - 2r_0^2 f_a' D_\nu\mathcal{K} - r_0^2 f_a D_{\nu\nu}\mathcal{K}. \end{aligned}$$

Finally, from (8) applied (with $\omega = \bar{r}_0$) to the latter equation, we routinely derive the first line of conclusion of Lemma 1 for $k=2$ with:

$$B_{12a} = c_5 + \frac{1}{\pi^2}B_{22a}\ ,$$

after use of the triangle inequality combined with (38), and:

$$B_{22a} = 6+\pi^2(4+c_5)+(4\sqrt{2}+\pi c_4)\pi B_{10a}+2\sqrt{2}\pi^2 B_{11a}+2(1+\pi^2)\left(B_{20a}+2B_{21a}\right),$$

after use of the triangle inequality combined with (29)(32)(34)(37) and the bounds of the same line of conclusion for $k=0,1$.

A.3.2 Estimates on Maclaurin approximations

Finally, in order to estimate the $\|.\|$ norm of $\tilde{f}''_a + r_0\psi_2\mathcal{S}_{\bar{r}_0}(t^{a+1})$, we note that the latter is equal to $\mathcal{S}_{\bar{r}_0}\left(\phi_{2a} + 2th_a r_0\partial_2 K(0) + 4r_0\cos\varphi\; th_a\partial_\nu K(0)\right)$, we recast the argument of $\mathcal{S}_{\bar{r}_0}$ as follows:

$$\phi_{2a}+2th_a r_0\partial_2 K(0)+4r_0\cos\varphi\; th_a\partial_\nu K(0) = r_0^2(\kappa-\mathcal{K})f''_a - 2r_0^2 f'_a D_\nu\mathcal{K} - r_0^2 f_a D_{\nu\nu}\mathcal{K}$$

$$-2\kappa\tilde{f}_a - 2tr_0\partial_2 K(0)(f_a-h_a) - 2\left(\mathcal{K}-\kappa - tr_0\partial_2 K(0)\right)f_a - 4r_0\cos\varphi\;\kappa\tilde{f}'_a$$

$$+4r_0\cos\varphi(\kappa-\mathcal{K})f'_a - 4r_0\cos\varphi\; f_a\left(D_\nu\mathcal{K} - t\partial_\nu K(0)\right) + 4tr_0\cos\varphi(h_a-f_a)\partial_\nu K(0)\;,$$

and we apply (8) with $\omega = \bar{r}_0$ to the right-hand expression, combined with the triangle inequality, the previous bounds of Lemma 1 and (29) (30)(31)(32)(35)(37). Doing so term by term, we obtain the second line of conclusion of Lemma 1 for $k=2$ with:

$$\begin{aligned} B_{32a} &= \frac{1}{2}B_{12a} + \sqrt{2}B_{11a} + \frac{1}{2}B_{10a}c_4 + \left(1+\frac{1}{\pi^2}\right)B_{20a} + c_1 \\ &\quad + \frac{1}{2}B_{10a} + 2\left(1+\frac{1}{\pi^2}\right)B_{21a} + 2B_{11a} + 2B_{10a}(1+c_1) + 2c_1 \\ &\equiv 3c_1 + \frac{1}{2}(5+4c_1+c_4)B_{10a} + (\sqrt{2}+2)B_{11a} + \frac{1}{2}B_{12a} \\ &\quad + \left(1+\frac{1}{\pi^2}\right)(B_{20a}+2B_{21a})\;. \end{aligned}$$

B Auxiliary universal constants and conditions

B.1 List of constants for Section 3

$$c_1 = \pi + \frac{1}{\pi},\; c_2 = 4\left(\frac{3}{4\pi} + \frac{1}{\pi^2} + 1\right),\; c_3 = \frac{1}{2\pi} + 2\left(1+\frac{1}{\pi^2}\right)\left(2+\pi^2\right),$$

$$c_4 = \frac{11}{2} + 10\pi + \frac{8}{\pi} + 2\pi^3,\; c_5 = 1 + \frac{1}{\pi^2} + 2c_1^2\;,$$

$$c_6 = \sup_{\tau\in[0,2\pi]}\left|\frac{\tau - \sin\tau}{\tau^3}\right|,\; c_7 = \sup_{\tau\in[0,2\pi]}\left|\frac{\tau^2 + 2(\cos\tau - 1)}{\tau^4}\right|,$$

$$c_8 = \sup_{\tau\in[0,2\pi]}\left|\frac{\tau\cos\tau - \sin\tau}{\tau^2}\right|,\; c_9 = \sup_{\tau\in[0,2\pi]}\left|\frac{\tau\cos\tau - \sin\tau}{\tau^3}\right|,$$

$$c_{10} = \sup_{\tau\in[0,2\pi]}\left|\frac{\cos\tau\sin\tau - \tau}{\tau^3}\right|,$$

$$C_1 = \max\left(\max_{a=0,1;k=0,1,2}\left(B_{1ka}, B_{3ka}\right), 8c_6, 8c_7, \frac{19}{18}c_8, \frac{10}{9}c_9, c_{10}\right).$$

B.2 List of conditions on ε and δ (Section 4)

$$\frac{|\cos\varphi|}{\pi}\left(1-\frac{\frac{\varepsilon}{2}}{1-\left(\frac{\varepsilon}{2}+\delta\right)}-\frac{\pi^2}{2}\left(\frac{\varepsilon}{2}+\delta\right)^2\right)-\varepsilon\pi B_{211}>0 \tag{39}$$

an inequality to be used only in subsection 4.2 with $|\cos\varphi|$ replaced by $\frac{1}{7704}$;

$$\frac{2\varepsilon}{1-\left(\frac{\varepsilon}{2}+\delta\right)}+2\pi^2\left(\frac{\varepsilon}{2}+\delta\right)^2+\frac{\varepsilon\pi^2}{2}\left(1+\frac{\varepsilon}{2}\right)+\frac{3\left(\frac{\varepsilon}{2}+\delta\right)}{1-\left(\frac{\varepsilon}{2}+\delta\right)}\le\frac{1}{2} \tag{40}$$

$$\begin{aligned}\frac{1}{24\pi^2} \ \ge\ & R_1(\varepsilon,\delta):=\varepsilon\left(B_{221}+\frac{1}{8}B_{200}\right)+\frac{1}{16}\left(\frac{\varepsilon}{2}+\delta\right)^2\\ &+\ 2\pi B_{111}\left(\left(\frac{\varepsilon}{2}+\delta\right)\left(1+\frac{\varepsilon}{2}\right)+\varepsilon B_{210}\right)\\ &+\ \frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}\left[B_{120}+\frac{2}{\pi^2\left(1-\frac{\varepsilon}{2}-\delta\right)^2}\left(1+\varepsilon\pi^2B_{200}+\frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}\right)\right]\end{aligned} \tag{41}$$

$$\frac{1}{16\sqrt{3}(\pi^2+5)}\ge\sqrt{\frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}} \tag{42}$$

$$\frac{1}{24\pi^2}\ge R_2(\varepsilon,\delta):=R_1(\varepsilon,\delta)+8B_{111}\left(\left(\frac{\varepsilon}{2}+\delta\right)\left(1+\frac{\varepsilon}{2}\right)\pi+\varepsilon\pi B_{210}\right) \tag{43}$$

$$\varepsilon\le\frac{1}{2\pi^2B_{211}}\le 0.0018 \tag{44}$$

$$\frac{1}{15408\pi}\ge\varepsilon\pi B_{211}+\frac{\frac{\varepsilon}{2}}{\pi\left(1-\left(\frac{\varepsilon}{2}+\delta\right)\right)}+\left(\frac{\varepsilon}{2}+\delta\right)^2\frac{\pi}{2} \tag{45}$$

$$\frac{1}{2\pi^2 15408^2}\ge\frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}\left(B_{120}+\frac{3}{2}B_{121}+2B_{110}B_{111}+\frac{36}{5\pi^2}\right) \tag{46}$$

$$\frac{1}{100\pi^2 15408^2}\ge\frac{\frac{\varepsilon}{2}+\delta}{1-\left(\frac{\varepsilon}{2}+\delta\right)}\left(B_{120}+\frac{3}{2}B_{121}+2B_{110}B_{111}+\frac{36}{5\pi^2}\right) \tag{47}$$

$$\begin{aligned}\frac{\sqrt{3}}{8\pi^2} \ \ge\ & \frac{\left(\frac{\varepsilon}{2}+\delta\right)}{\left(1-\frac{\varepsilon}{2}-\delta\right)^3\pi^2}\left(\frac{6}{5}+\left(1+\frac{\varepsilon}{2}\right)^2\pi^2\right)+\varepsilon\left(B_{221}+\frac{16}{5}B_{211}\right)\\ &+\ 7\pi B_{111}\left(\left(\frac{\varepsilon}{2}+\delta\right)\left(1+\frac{\varepsilon}{2}\right)+\varepsilon B_{210}\right)+\frac{\left(\frac{\varepsilon}{2}+\delta\right)}{1-\left(\frac{\varepsilon}{2}+\delta\right)}\left(B_{120}+\frac{36}{5\pi^2}\right)\end{aligned} \tag{48}$$

B.3 List of constants and conditions on ε and δ (Section 5)

$$\left(1+\frac{\varepsilon}{2}\right)\delta\le\frac{\pi}{2} \tag{49}$$

$$c_{11}=\sup_{\tau\in\left[0,\frac{\pi}{2}\right]}\left|\frac{\tau^2+\tau\cos\tau\sin\tau-2\sin^2\tau}{\tau^5\sin^2\tau}-\frac{2}{45\tau}\right|$$

$$c_{12}=\sup_{\tau\in\left[0,\frac{\pi}{2}\right]}\left|\frac{2(\sin\tau-\tau\cos\tau)}{\tau^3\sin^3\tau}-\frac{2}{3\tau^3}\left(1+\frac{2\tau^2}{5}\right)\right|$$

$$c_{13} = \sup_{\tau \in \left[0, \frac{\pi}{2}\right]} \left| \frac{2(\sin\tau - \tau\cos\tau)}{\tau^5 \sin\tau} - \frac{2}{3\tau^3}\left(1 + \frac{\tau^2}{15}\right) \right|$$

$$c_{14} = \sup_{\tau \in \left[0, \frac{\pi}{2}\right]} \left| \frac{4(\sin^2\tau - \tau^2)}{\tau^5 \sin^2\tau} + \frac{4}{3\tau^3}\left(1 + \frac{\tau^2}{5}\right) \right|$$

$$c_{15} = \sup_{\tau \in \left[0, \frac{\pi}{2}\right]} \left| \frac{\tau - \sin\tau}{\tau\sin\tau} \right|, \; c_{16} = \sup_{\tau \in \left[0, \frac{\pi}{2}\right]} \left| \frac{\tau^2\cos\tau - \sin^2\tau}{\tau\sin^2\tau} \right|,$$

$$c_{17} = \sup_{\tau \in \left[0, \frac{\pi}{2}\right]} \left| \frac{2\cos\tau - 2 + \tau\sin\tau}{\tau^6} + \frac{1}{12\tau^2} \right|$$

$$\left(1 + \frac{\varepsilon}{2}\right)\delta \le \frac{2}{\sqrt{5}} \tag{50}$$

$$c_{18} = \frac{338 C_1^3 \pi^{11}}{8} + \frac{206}{100}\left[8(c_{15}c_6 + c_{16}c_7) + (c_6 + c_7)c_{15}\right]$$

$$c_{19} = \frac{268 C_1^3 \pi^{11}}{8} + \frac{411}{100}\left[2(c_{15}c_6 + c_{16}c_7) + (c_6 + c_7)c_{15}\right]$$

$$\frac{1}{180} \ge \varepsilon\ (c_{18} + 19 c_{17}\ \delta) + \frac{115}{100}\ \delta\ (c_{11} + c_{12} + \frac{1}{2}c_{14}) \tag{51}$$

$$\frac{1}{180} \ge \varepsilon\ (c_{19} + 13 c_{17}\ \delta) + \frac{115}{100}\delta\ (c_{13} + \frac{1}{2}c_{14}) \tag{52}$$

B.4 A constant and a condition on ε and δ (for Section 6)

$$c_{20} = 338\pi^{10}C_1^3 + 20\pi(c_6 + c_7)$$

$$\varepsilon \le \frac{1}{2\pi^3 c_{20}}\gamma_1 \sin^3\left(\frac{\delta\pi}{2}\right) \tag{53}$$

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