AN ALMOST SCHUR THEOREM ON 4-DIMENSIONAL MANIFOLDS

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Abstract. In this small note we prove that the almost Schur theorem, introduced by De Lellis-Topping, is true on 4-dimensional Riemannian manifolds of nonnegative scalar curvature and discuss some related problems on other dimensional manifolds.

1. Introduction

Very recently, De Lellis and Topping proved an interesting result about a generalization of Schur theorem

Theorem 1 (Almost Schur Theorem [1]). *For $n \geq 3$, if (M^n, g) is a closed Riemannian manifold with non-negative Ricci tensor, then*

$$\int_M |Ric - \frac{\overline{R}}{n} g|^2 dv(g) \leq \frac{n^2}{(n-2)^2} \int_M |Ric - \frac{R}{n} g|^2 dv(g), \tag{1}$$

where $\overline{R} = vol(g)^{-1} \int_M R dv(g)$ is the average of the scalar curvature R of g.

It is clear that the Schur theorem follows directly from the Theorem. The latter can be seen as a quantitative version or a stability result of the Schur Theorem. In [1] they also showed that the constant in inequality (1) is optimal and the non-negativity of the Ricci tensor can not be removed in general: When $n \geq 5$ they gave examples of metrics on $\mathbb{S}^n$ which make the radio of the left hand side of (1) to the right hand side of (1) arbitrarily large. When $n = 3$, they found manifolds which makes the ratio arbitrarily. At the end they left an open question: *Inequalities of this form may hold for $n = 3$ and $n = 4$ with constants depending on the topology of M.*

In this small note we will show that the Theorem 1 holds under the condition of non-negativity of the scalar curvature for dimension $n = 4$.

Theorem 2. *If $n = 4$, and if (M^4, g) is a closed Riemannian manifold with non-negative scalar curvature, then (1) holds. Moreover, equality holds if and only if (M^4, g) is an Einstein manifold.*

We first observe that inequality (1) is equivalent to

$$\left(\int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} vol(g) \int_M \sigma_2(g) dv(g), \tag{2}$$

where $\sigma_k(g)$ is the k-scalar curvature of metric g. Its definition will be recalled in Section 2. Then we prove this inequality for $n = 4$ by using an argument given by Gursky [3].

2. Proof of Theorem 2

Let us first recall the definition of the k-scalar curvature, which was first introduced by Viaclovsky [4] and has been intensively studied by many mathematicians. Let

$$S_g = \frac{1}{n-2}\left(Ric_g - \frac{R_g}{2(n-1)}\cdot g\right)$$

be the Schouten tensor of g. For an integer k with $1 \le k \le n$ let σ_k be the k-th elementary symmetric function in $\mathbb{R}^n$. The k-scalar curvature is

$$\sigma_k(g) := \sigma_k(\Lambda_g),$$

where Λ_g is the set of eigenvalue of the matrix $g^{-1}\cdot S_g$. In particular, $\sigma_1(g) = \operatorname{tr} S$ and $\sigma_2 = \frac{1}{2}((\operatorname{tr} S)^2 - |S|^2)$. It is trivial to see that

$$\begin{aligned} \sigma_1(g) &= \frac{R}{2(n-1)}, \\ \sigma_2(g) &= \frac{1}{2(n-2)^2}\left\{-|Ric|^2 + \frac{n}{4(n-1)}R^2\right\}, \\ \left|Ric - \frac{R}{n}g\right|^2 &= |Ric|^2 - \frac{R^2}{n}. \end{aligned}$$

From above it is easy to have the following observation.

Observation. *Inequality (1) is equivalent to (2).*

Hence, instead of proving Theorem 3 we actually prove

Theorem 3. *If $n = 4$, and if (M^n, g) is a closed Riemannian manifold with non-negative scalar curvature, then (2) holds. Moreover, equality holds if and only if (M, g) is an Einstein metric.*

The proof of Theorem 3 follows closely a nice argument of Gursky [3].

Lemma 1. *For any $n \ge 3$ and any closed Riemannian manifold (M^n, g), there exists a conformal metric $g_1 \in [g]$ satisfying*

$$\frac{2n}{n-1}\frac{\int_M \sigma_2(g_1)dv(g_1)}{(vol(g_1))^{\frac{n-4}{n}}} \le Y_1([g])^2, \tag{3}$$

where $Y_1([g])$ is the first Yamabe invariant defined by

$$Y_1([g]) := \inf_{g\in[g]} \frac{\int_M \sigma_1(g)dv(g)}{(vol(g))^{\frac{n-2}{n}}} \tag{4}$$

and $[g]$ is the conformal class of the metric to g.

Here our definition of the Yamabe constant is different from the standard one by a multiple factor $\frac{1}{2(n-1)}$.

Proof of Lemma 1. The proof follows closely an argument given by Gursky in [3]. Let g_1 a solution of Yamabe problem. Thus the scalar curvature, and hence $\sigma_1(g)$ is constant. We have a simple fact: for any $n \times n$ symmetric matrix A

$$(\sigma_1(A))^2 \geq \frac{2n}{n-1}\sigma_2(A)$$

equality holds if and only if the matrix is a multiple of the identity one. Now the following calculations lead to

$$\frac{2n}{n-1}vol(g_1)\int_M \sigma_2(g_1)dv(g_1) \leq vol(g_1)\int_M (\sigma_1(g_1))^2 dv(g_1) = (\int_M \sigma_1(g_1)dv(g_1))^2. \tag{5}$$

Here we have used the fact that $\sigma_1(g_1)$ is a constant. Therefore,

$$\frac{2n}{n-1}\frac{\int_M \sigma_2(g_1)dv(g_1)}{(vol(g_1))^{\frac{n-4}{n}}} \leq \left(\frac{\int_M \sigma_1(g_1)dv(g_1)}{vol(g_1)^{\frac{n-2}{n}}}\right)^2 = Y_1([g])^2,$$

since g_1 is a Yamabe solution. ■

Proof of Theorem 3. In the case of dimension $n = 4$, it is well-known that $\int_M \sigma_2(g)dv(g)$ is constant in any given conformal class. Hence by Lemma 1 we have

$$\begin{aligned}\frac{2n}{n-1}\int_M \sigma_2(g)dv(g) &= \frac{2n}{n-1}\int_M \sigma_2(g_1)dv(g_1) \leq Y_1([g])^2 \\ &\leq \left(\frac{\int_M \sigma_1(g)dv(g)}{vol(g)^{\frac{1}{2}}}\right)^2.\end{aligned}$$

In the last inequality we have used the condition $\sigma_1(g) \geq 0$, which implies that $Y_1([g]) \geq 0$. The equality holds if and only if the Schouten tenser S_g is proportional to the metric g, i.e., g is an Einstein metric. ■

We conjecture that Theorem 2 is true for $n = 3$. To attack this conjecture one needs to study a corresponding Yamabe type problem. The methods developed, especially in [2], for σ_k-Yamabe problem would be helpful to study this problem.

References

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