THE GAUSS-BONNET-CHERN MASS OF CONFORMALLY FLAT MANIFOLDS

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Abstract. In this paper we show positive mass theorems and Penrose type inequalities for the Gauss-Bonnet-Chern mass, which was introduced recently in [20], for asymptotically flat conformally flat manifolds and its rigidity.

1. Introduction

Recently motivated by the Einstein-Gauss-Bonnet theory [9, 43] and the pure Lovelock theory [36, 15], we introduced in [20] (and [21]) the Gauss-Bonnet-Chern mass by using the Gauss-Bonnet curvature

$$L_k := \frac{1}{2^k}\delta^{i_1 i_2\cdots i_{2k-1}i_{2k}}_{j_1 j_2\cdots j_{2k-1}j_{2k}} R_{i_1 i_2}{}^{j_1 j_2}\cdots R_{i_{2k-1}i_{2k}}{}^{j_{2k-1}j_{2k}}, \tag{1.1}$$

where $\delta^{i_1 i_2\cdots i_{2k-1}i_{2k}}_{j_1 j_2\cdots j_{2k-1}j_{2k}}$ is the generalized Kronecker delta defined in (2.2) below. When $k = 1$, L_1 is just the scalar curvature R. When $k = 2$, it is the (second) so-called the Gauss-Bonnet curvature

$$L_2 = R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2,$$

which appeared first in the paper of Lanczos [32] in 1938. For general k it is the Euler integrand in the Gauss-Bonnet-Chern theorem [13, 14] if $n = 2k$ and is therefore called the dimensional continued Euler density in physics if $k < n/2$. Here n is the dimension. In this paper we are interested in the case $k < n/2$. The Gauss-Bonnet-Chern mass introduced in [20] is defined

$$m_k = m_{GBC} = c(n,k)\lim_{r\to\infty}\int_{S_r} P_{(k)}^{ijlm}\partial_m g_{jl}\nu_i dS, \tag{1.2}$$

with

$$c(n,k) = \frac{(n-2k)!}{2^{k-1}(n-1)!\omega_{n-1}},$$

where ω_{n-1} is the volume of $(n-1)$-dimensional standard unit sphere and S_r is the Euclidean coordinate sphere, dS is the volume element on S_r induced by the Euclidean metric, ν is the outward unit normal to S_r in $\mathbb{R}^n$ and the derivative is the ordinary partial derivative. Here the tensor $P_{(k)}$ is decided by the decomposition

$$L_k = P_{(k)}^{ijlm}R_{ijlm}. \tag{1.3}$$

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In this paper we use the Einstein summation convention. The tensor $P_{(k)}$ has a crucial property of divergence-free, which guarantees the Gauss-Bonnet-Chern mass is well-defined and is a geometric invariant, under a suitable decay condition. See Section 2 below or [20]. When $k=1$,

$$P_{(1)}^{ijlm} = \frac{1}{2}(g^{il}g^{jm} - g^{im}g^{jl}),$$

and m_1 is just the ADM mass introduced by Arnowitt, Deser, and Misner [1] for asymptotically flat Riemannian manifolds. For a similar mass see also [33].

A complete manifold $(\mathcal{M}^n, g)$ is said to be an asymptotically flat (AF) of order τ (with one end) if there is a compact set K such that $\mathcal{M} \setminus K$ is diffeomorphic to $\mathbb{R}^n \setminus B_R(0)$ for some $R>0$ and in the standard coordinates in $\mathbb{R}^n$, the metric g has the following expansion

$$g_{ij} = \delta_{ij} + \sigma_{ij},$$

with

$$|\sigma_{ij}| + r|\partial \sigma_{ij}| + r^2 |\partial^2 \sigma_{ij}| = O(r^{-\tau}),$$

where r and ∂ denote the Euclidean distance and the standard derivative operator on $\mathbb{R}^n$ respectively. The condition that the Gauss-Bonnet-Chern mass be well-defined is

$$\tau > \frac{n-2k}{k+1}, \tag{1.4}$$

and L_k is integrable over $\mathcal{M}$. In this case, the Gauss-Bonnet-Chern mass is a geometric invariant, which is a generalization of the work of Bartnik for the ADM mass m_1 [2].

The positive mass theorem for the ADM mass $m_{ADM} = m_1$, which plays an important role in differential geometry, was proved by Schoen and Yau [38] for $3 \le n \le 7$ and by Witten for general spin manifolds. See also [34, 35]. Its refinement, the Penrose inequality, was proved by Huisken-Ilmanen [27] and Bray [3] for $n=3$ and Bray-Lee [7] for $n \le 7$. Recently there are many interesting works on special, but interesting classes of asymptotically flat manifolds. In [31] Lam showed the positive mass theorem and the Penrose inequality for asymptotically flat graphs in $\mathbb{R}^{n+1}$ by using an elementary, but elegant proof. See also the generalizations of Lam's work in [16, 17, 28, 29]. A Penrose type inequality was proved for conformally flat manifolds by Freire-Schwartz [18], Jauregui [30] and Schwartz [39] by using the relation between mass and the capacity. This relation was used already in the proof of Penrose inequality in [3]. For this relation, see also [5] and [8]. It is interesting to see that there is a deep relation between the ADM mass and various geometric objects.

We are interested in generalizing the above results to our Gauss-Bonnet-Chern $m_{GBC} = m_k$ ($k \ge 2$). Motivated by the work of Lam [31], we showed a positive mass theorem and an optimal Penrose inequality when $\mathcal{M}$ is an asymptotically flat graphs in $\mathbb{R}^{n+1}$ in [20]. This Penrose inequality establishes a relationship between the mass m_{GBC} and more geometric objects [20]. In this paper we are interested in studying m_{GBC} mass on conformally flat manifolds.

A *conformally flat* manifold with or without boundary, CF manifold for short, is a manifold $(\mathcal{M}^n, g) = (\mathbb{R}^n/\Omega, e^{-2u}\delta)$, where δ is the canonical Euclidean metric on $\mathbb{R}^n$, Ω is a smooth bounded (possibly empty, not necessarily connected) open set and u is smooth. A CF manifold $(\mathcal{M}^n, g)$ is called an asymptotically flat CF manifold of decay order τ if

$$|u| + |x||\nabla u| + |x|^2 |\nabla^2 u| = O(|x|^{-\tau}). \tag{1.5}$$

In this paper we always assume that $k < \frac{n}{2}$, $\tau > \frac{n-2k}{k+1}$ and L_k is integrable.

First we have a positive mass theorem.

Theorem 1.1. *Let $(\mathcal{M}^n, g) = (\mathbb{R}^n, e^{-2u}\delta)$ be an asymptotically flat CF manifold. Assume further that $L_j(g) \geq 0$ for all $j \leq k$. Then the mass $m_k \geq 0$. Moreover, equality holds if and only if $u \equiv 0$, i.e., $\mathcal{M}$ is the Euclidean space.*

The condition $L_j(g) \geq 0$ for any $j \leq k$ here is equivalent to $g \in \Gamma_k$, which will be discussed in Section 2 below. A similar result was announced by Li-Nguyen in [33].

For the Gauss-Bonnet-Chern mass, m_{2j+1} has different behavior from m_{2j}. The former behaves like the ADM mass m_1 and the latter like m_2. For k even, we have also a positive mass theorem for metrics in a non-positive cone.

Theorem 1.2. *Let k be even and $(\mathcal{M}^n, g) = (\mathbb{R}^n, e^{-2u}\delta)$ be an asymptotically flat CF manifold. Assume $(-1)^j L_j \geq 0$ for all $j \leq k$. Then the mass $m_k \geq 0$. Moreover, equality holds if and only if $u \equiv 0$, i.e., $\mathcal{M}$ is the Euclidean space.*

Theorem 1.1 and Theorem 1.2 provide a support for our conjecture on the positivity of the Gauss-Bonnet-Chern mass in [20]. Furthermore, from our proof we have a Penrose type inequality.

Theorem 1.3. *Let $(\mathcal{M}^n, g) = (\mathbb{R}^n \setminus \Omega, e^{-2u}\delta)$ be an asymptotically flat CF manifold. Assume that Ω is convex, $\partial\mathcal{M} = (\Omega, e^{-2u}\delta)$ is a horizon of $(\mathcal{M}, g)$ (i.e. $\partial\mathcal{M} = \partial\Omega \subset \mathcal{M}$ is minimal) and u is constant on $\partial\Omega$. Assume further that $L_j(g) \geq 0$ for any $j \leq k$. Then we have Penrose type inequalities*

$$m_k \geq \left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}}. \tag{1.6}$$

Moreover, if $k \geq 2$, we have the following strengthened Penrose type inequality

$$m_k \geq \left(\frac{\int_{\partial\Omega} R}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-2k}{n-3}}, \tag{1.7}$$

where R is the scalar curvature of $\partial\Omega$ as a hypersurface in $\mathbb{R}^n$.

The assumptions on the boundary $\partial\Omega$ can be reduced by the result of Guan-Li [24] and the results could be slightly strengthened. For more details see Section 4 below. Unlike the Penrose inequality obtained in [20], this Penrose inequality is not optimal. Our Penrose inequality is motivated by the work of Jauregui in [30], who obtained (1.6) for $k = 1$. The idea is to express the mass via various integral identities.

Before ending the introduction, we would like to give remarks on the assumptions on metrics. Instead of discussing the assumption that the metric g satisfies $L_j(g) \geq 0$ for any $j \leq k$, it would be better to discuss a stronger assumption that the metric g satisfies

$$L_j(g) > 0, \text{ for any } j \leq k. \tag{1.8}$$

Condition (1.8) is well-known as an ellipticity condition in the study of a fully nonlinear Yamabe problem. See for example [42] and [23]. When $k \geq n/2$, it is a rather restrictive condition. In fact, in this case, it implies that the metric has positive Ricci tensor. See [26]. In this paper, we consider the case $k < n/2$, in which it is not as strong as it looks like. For a fixed k, the larger the dimension n is, the weaker this condition is. In the lowest dimension we consider, i.e., $n = 2k+1$, this condition is quite similar to the condition requiring a metric with positive scalar curvature in the 3-dimensional case. For example, the results of Gromov-Lawson and Schoen-Yau on gluing of metrics of positive scalar curvature can be extended to our case. In [25] it was proved that if $k < n/2$, and M_1 and M_2 are two compact manifolds (not necessarily locally conformally flat) with condition (1.8), then the connected sum $M_1 \# M_2$ also admits a metric with condition (1.8). If, in addition, M_1 and M_2 are locally conformally flat, then $M_1 \# M_2$ admits a locally conformally flat metric with condition (1.8). With this result, one can construct a family of non-symmetric metrics with condition (1.8) from a rotationally symmetric metric with condition (1.8), such that this family of metrics concentrated in a suitable sense at finitely many given points. Another condition that u is constant on $\partial\Omega$ in Theorem 1.3 is rather restrictive. However, here we need not an "outermost" condition on the horizons, as in the ordinary Penrose inequality, which is quite difficult to handle mathematically. This condition that u is constant on $\partial\Omega$ was also used in [17, 18, 30, 29, 31, 20, 39].

The rest of the paper is organized as follows. In Section 2 we recall the definitions of the Gauss-Bonnet curvature L_k and the σ_k-scalar curvature and their relationship when the underlying manifolds are locally conformally flat. In Section 3 we prove the positive mass theorems, Theorem 1.1 and Theorem 1.2. Theorem 1.3 is proved in Section 4.

2. The Gauss-Bonnet curvatures and the σ_k-scalar curvatures

We recall the definition of generalized *k-th Gauss-Bonnet curvature*

$$L_k := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-1} i_{2k}}_{j_1 j_2 \cdots j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-1} i_{2k}}{}^{j_{2k-1} j_{2k}}. \tag{2.1}$$

Here the generalized Kronecker delta is defined by

$$\delta^{j_1 j_2 \cdots j_r}_{i_1 i_2, \cdots i_r} = \det \begin{pmatrix} \delta^{j_1}_{i_1} & \delta^{j_2}_{i_1} & \cdots & \delta^{j_r}_{i_1} \\ \delta^{j_1}_{i_2} & \delta^{j_2}_{i_2} & \cdots & \delta^{j_r}_{i_2} \\ \vdots & \vdots & \vdots & \vdots \\ \delta^{j_1}_{i_r} & \delta^{j_2}_{i_r} & \cdots & \delta^{j_r}_{i_r} \end{pmatrix}. \tag{2.2}$$

When $k = 2$, we can write

$$\begin{aligned} L_2 &= R_{\mu\nu\rho\sigma}R^{\mu\nu\rho\sigma} - 4R_{\mu\nu}R^{\mu\nu} + R^2 \\ &= |W|^2 + \frac{n-3}{n-2}\left(\frac{n}{n-1}R^2 - 4|Ric|^2\right) \\ &= |W|^2 + 8(n-2)(n-3)\sigma_2(A_g) \\ &= R_{ijkl}P^{ijkl}_{(2)}, \end{aligned} \tag{2.3}$$

where

$$P_{(2)}^{ijkl} = R^{ijkl} + R^{jk}g^{il} - R^{jl}g^{ik} - R^{ik}g^{jl} + R^{il}g^{jk} + \frac{1}{2}R(g^{ik}g^{jl} - g^{il}g^{jk}), \tag{2.4}$$

W denotes the Weyl tensor, Ric the Ricci tensor, R the scalar curvature and

$$A_g := \frac{1}{n-2}\left(Ric - \frac{R}{2(n-1)}g\right),$$

the Schouten tensor and σ_2 the 2-th elementary symmetric function defined below. $P_{(2)}$ is the divergence-free part of the Riemann curvature tensor $Riem$. For the general L_k-curvature, the corresponding $P_{(k)}$ curvature is

$$P_{(k)}^{stlm} := \frac{1}{2^k}\delta^{i_1 i_2 \cdots i_{2k-3} i_{2k-2} st}_{j_1 j_2 \cdots j_{2k-3} j_{2k-2} j_{2k-1} j_{2k}} R_{i_1 i_2}{}^{j_1 j_2} \cdots R_{i_{2k-3} i_{2k-2}}{}^{j_{2k-3} j_{2k-2}} g^{j_{2k-1} l} g^{j_{2k} m}. \tag{2.5}$$

Recall that $L_k = P_{(k)}^{ijlm} R_{ijlm}$ and the tensor $P_{(k)}$ has the following crucial property.

Proposition 2.1. *The tensor $P_{(k)}$ has the same symmetry and anti-symmetry as the Riemann curvature tensor and satisfies*

$$\nabla_i P_{(k)}^{ijlm} = 0.$$

Proof. The case $k = 1$ is trivial. We have proved the $k = 2$ case in [20]. For the general case, it follows from the symmetry of the Riemann curvature tensor and the differential Bianchi identity. And this result does not appear to actually be used in the paper, we skip the proof here. □

Now we consider the case that $(\mathcal{M}^n, g)$ is a conformally flat manifold of dimension $n \geq 5$. Namely, $(\mathcal{M}^n, g) = (\mathbb{R}^n, e^{-2u}\delta)$, where δ is the canonical Euclidean metric on $\mathbb{R}^n$. In this case, we will show the curvature L_k is just the σ_k-scalar curvature (up to a constant multiple), which was considered by Viaclovsky in [41] and has been intensively studied in the σ_k Yamabe problem.

For the convenience of the reader, we recall some basic properties on the elementary symmetric functions (see for example [22, 11, 41]). For $1 \leq k \leq n$ and $\lambda = (\lambda_1, \cdots, \lambda_n) \in \mathbb{R}^n$, the k-th elementary symmetric function is defined as

$$\sigma_k(\lambda) := \sum_{i_1 < i_2 < \cdots i_k} \lambda_{i_1} \cdots \lambda_{i_k}.$$

The definition can be extended to symmetric matrices. For a symmetric matrix B, denote $\lambda(B) = (\lambda_1(B), \cdots, \lambda_n(B))$ be the eigenvalues of B. We set

$$\sigma_k(B) := \sigma_k(\lambda(B)).$$

We define also $\sigma_0(B) = 1$. Let I be the identity matrix. Then we have for any $t \in \mathbb{R}$,

$$\sigma_n(I + tB) = \det(I + tB) = \sum_{i=0}^{n} \sigma_i(B) t^i.$$

We recall the definition of the Gårding cone: for $1 \leq k \leq n$, let Γ_k^+ (resp. Γ_k) is a cone in $\mathbb{R}^n$ determined by

$$\Gamma_k^+ = \{\lambda \in \mathbb{R}^n : \quad \sigma_1(\lambda) > 0, \cdots, \sigma_k(\lambda) > 0\}.$$
$$(\text{resp. } \Gamma_k = \{\lambda \in \mathbb{R}^n : \quad \sigma_1(\lambda) \geq 0, \cdots, \sigma_k(\lambda) \geq 0\}).$$

A symmetric matrix B is called belong to Γ_k^+ (resp. Γ_k) if $\lambda(B) \in \Gamma_k^+$ (resp. $\lambda(B) \in \Gamma_k$). The k-th Newton transformation is defined as follows

$$(T_k)^i_j(B) := \frac{\partial \sigma_{k+1}}{\partial b^i_j}(B), \tag{2.6}$$

where $B = (b^i_j)$. If there is no confusion, we omit the index k. We recall some basic properties about σ_k and T.

$$\sigma_k(B) = \frac{1}{k!}\delta^{i_1\cdots i_k}_{j_1\cdots j_k} b^{j_1}_{i_1}\cdots b^{j_k}_{i_k} = \frac{1}{k}\mathrm{tr}(T_{k-1}(B)B), \tag{2.7}$$

$$(T_k)^i_j(B) = \frac{1}{k!}\delta^{ii_1\cdots i_k}_{jj_1\cdots j_k} b^{j_1}_{i_1}\cdots b^{j_k}_{i_k}. \tag{2.8}$$

T_k can be also described with induction

$$T_k = \sigma_r I - BT_{k-1} \quad \text{and} \quad T_0 = I, \tag{2.9}$$

which yields

$$T_k = \sum_{i=0}^k \sigma_{k-i}(B)(-B)^i = \sigma_k(B)I - \sigma_{k-1}(B)B + \cdots + (-1)^k B^k.$$

It is well-known that $\sigma_k^{1/k}$ is concave in Γ_k, which implies that

$$\sigma_k(A+B) \geq \sigma_k(A) + \sigma_k(B), \qquad \text{for any } A, B \in \Gamma_k. \tag{2.10}$$

The σ_k-scalar curvature $\sigma_k(g)$ is defined in [41] by

$$\sigma_k(g) := \sigma_k(g^{-1}A_g),$$

where A_g is the Schouten tensor of g.

Proposition 2.2. *Let $(\mathcal{M}^n, g)$ be a locally conformally flat metric of dimension n. Assume $2k < n$. Then*

$$L_k = 2^k k! \frac{(n-k)!}{(n-2k)!}\sigma_k(g). \tag{2.11}$$

Proof. We recall the decomposition of the Riemann curvature tensor

$$Riem = W + A \owedge g.$$

As $W \equiv 0$, we have

$$R_{i_1 i_2}{}^{j_1 j_2} = A_{i_1}{}^{j_1}\delta_{i_2}{}^{j_2} + \delta_{i_1}{}^{j_1}A_{i_2}{}^{j_2} - A_{i_1}{}^{j_2}\delta_{i_2}{}^{j_1} - \delta_{i_1}{}^{j_2}A_{i_2}{}^{j_1}. \tag{2.12}$$

It follows that

$$\begin{aligned} L_k &= \tfrac{1}{2^k}\delta^{i_1 i_2\cdots i_{2k-1}i_{2k}}_{j_1 j_2\cdots j_{2k-1}j_{2k}} R_{i_1 i_2}{}^{j_1 j_2}\cdots R_{i_{2k-1}i_{2k}}{}^{j_{2k-1}j_{2k}} \\ &= 2^k \delta^{i_1 i_2\cdots i_{2k-1}i_{2k}}_{j_1 j_2\cdots j_{2k-1}j_{2k}} A_{i_1}{}^{j_1}\delta_{i_2}{}^{j_2}\cdots A_{i_{2k-1}}{}^{j_{2k-1}}\delta_{i_{2k}}{}^{j_{2k}} \\ &= 2^k (n-k)\cdots(n-2k+1)\delta^{i_1 i_3\cdots i_{2k-1}}_{j_1 j_3\cdots j_{2k-1}} A_{i_1}{}^{j_1}A_{i_3}{}^{j_3}\cdots A_{i_{2k-1}}{}^{j_{2k-1}} \\ &= 2^k k!(n-k)\cdots(n-2k+1)\sigma_k(A). \end{aligned}$$

Here in the second equality we use the facts

$$\begin{aligned}\delta^{i_1i_2\cdots i_{2k-1}i_{2k}}_{j_1j_2\cdots j_{2k-1}j_{2k}}A_{i_1}{}^{j_1}\delta_{i_2}{}^{j_2} &= \delta^{i_1i_2\cdots i_{2k-1}i_{2k}}_{j_1j_2\cdots j_{2k-1}j_{2k}}\delta_{i_1}{}^{j_1}A_{i_2}{}^{j_2}\\ &= -\delta^{i_1i_2\cdots i_{2k-1}i_{2k}}_{j_1j_2\cdots j_{2k-1}j_{2k}}A_{i_1}{}^{j_2}\delta_{i_2}{}^{j_1} = -\delta^{i_1i_2\cdots i_{2k-1}i_{2k}}_{j_1j_2\cdots j_{2k-1}j_{2k}}\delta_{i_1}{}^{j_2}A_{i_2}{}^{j_1},\end{aligned}$$

and in the third equality we use the basic property of generalized Kronecker delta

$$\delta^{i_1i_2\cdots i_{p-1}i_p}_{j_1j_2\cdots j_{p-1}j_p}\delta_{i_p}{}^{j_p} = (n-p+1)\delta^{i_1i_2\cdots i_{p-1}}_{j_1j_2\cdots j_{p-1}},$$

which follows from the Laplace expansion of determinant

$$\delta^{i_1i_2\cdots i_p}_{j_1j_2\cdots j_p} = \sum_{s=1}^{p}(-1)^{p+s}\delta^{i_p}_{j_s}\delta^{i_1\cdots i_s\cdots \hat{i}_p}_{j_1\cdots \hat{j}_s\cdots j_p}.$$

□

For the special case $k = \frac{n}{2}$ of Proposition 2.2 see [41]. Another important property will be the following.

Proposition 2.3. *([40]) Let $(\mathcal{M}^n, g)$ be a locally conformally flat manifold of dimension n. Then $T_{k-1}(A)$ is divergence-free.*

Proof. See the proof of Proposition 2.2 in [40]. □

Without the conformal flatness Proposition 2.3 still holds for $k = 2$, i.e., T_1 is divergence-free, which was also proved in [40].

3. Positive Mass Theorem for CF manifolds and Rigidity

In this section we prove Theorem 1.1 and Theorem 1.2. For the proof we need one more well-known property.

Proposition 3.1. *Let $u : \mathbb{R}^n \to \mathbb{R}$ be some smooth function. Denote $D^2u = (u_{ij})$ be the hessian matrix of u with respect to Euclidean metric. Then $T_k(D^2u)$ is divergence-free, that is,*

$$\partial_i T_k^{ij}(D^2u) = \partial_j T_k^{ij}(D^2u) = 0.$$

Remark 3.2. *Note that in Proposition 3.1 the divergence-free is with respect to the standard euclidean metric δ and in Proposition 2.3 the divergence-free is with respect to the metric $g = e^{-2u}\delta$.*

For an asymptotically flat CF manifold, we first have an equivalent form of Gauss-Bonnet-Chern mass defined by (1.2). By (1.3), (2.12) together with Proposition 2.1, we have

$$L_k = 4P_{(k)}^{ijlm}A_{il}g_{jm} = -4P_{(k)}^{ijjl}A_{il}e^{-2u}.$$

On the other hand, from (2.7) and (2.11) we have

$$L_k = 2^k(k-1)!\frac{(n-k)!}{(n-2k)!}(T_{k-1}(A))^{il}A_{il}.$$

For the Gauss-Bonnet-Chern mass (1.2) we have

$$\begin{aligned} m_k &:= \frac{(n-2k)!}{2^{k-1}(n-1)!\,\omega_{n-1}} \lim_{r\to\infty}\int_{S_r} P_{(k)}^{ijlm}\partial_m g_{jl}\nu_i dS \\ &= \frac{(n-2k)!}{2^{k-1}(n-1)!\,\omega_{n-1}} \lim_{r\to\infty}\int_{S_r} -2e^{-2u}P_{(k)}^{ijjl}u_l\nu_i dS. \end{aligned}$$

Combining all together, we thus obtain the following equivalent form of (1.2),

$$m_k = \lim_{r\to\infty}\frac{(k-1)!(n-k)!}{(n-1)!\,\omega_{n-1}}\int_{S_r}(T_{k-1}(A))^{ij}u_j\nu_i dS. \tag{3.1}$$

This formula will be useful in the computation of the Gauss-Bonnet-Chern mass.

Now we start to prove Theorem 1.1. For the convenience of readers, we propose a remark first.

Remark 3.3. *In the following proof of Theorem 1.1, the calculations before (3.7) are with respect to the Euclidean metric δ, namely $\sigma_k(A)$ means $\sigma_k(\delta^{-1}A)$. Hence from (2.11) that $L_k = 2^k k!\frac{(n-k)!}{(n-2k)!}e^{2ku}\sigma_k(A)$, which has be used in (3.7).*

Proof of Theorem 1.1. Since $g = e^{-2u}\delta$, a direct computation gives

$$\begin{aligned} Ric &= (n-2)(D^2u + \frac{1}{n-2}(\Delta u)\delta + du\otimes du - |\nabla u|^2\delta), \\ R &= e^{2u}(2(n-1)\Delta u - (n-1)(n-2)|\nabla u|^2), \end{aligned}$$

which imply

$$A_g := \frac{1}{n-2}\left(Ric - \frac{Rg}{2(n-1)}\right) = D^2u - \frac{|\nabla u|^2}{2}\delta + du\otimes du. \tag{3.2}$$

Here ∇ and Δ are operators with respect to the Euclidean metric δ and D^2 is the Hessian operator. Since

$$T_{k-1}(D^2u) = T_{k-1}(A) + O(|x|^{-k\tau-2k+2}),$$

which follows from (1.5) and (2.8), we have by (3.1)

$$m_k = \lim_{r\to\infty}\frac{(k-1)!(n-k)!}{(n-1)!\,\omega_{n-1}}\int_{S_r}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS. \tag{3.3}$$

Applying Proposition 3.1 and Green's formula, we obtain

$$\int_{S_r}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS = \int_{B_r}(T_{k-1}(D^2u))^{ij}u_{ij}dx = k\int_{B_r}\sigma_k(D^2u)dx. \tag{3.4}$$

Now, we write

$$D^2u = A + \frac{|\nabla u|^2}{2}I - du\otimes du.$$

It is crucial to see that the matrix $\frac{|\nabla u|^2}{2}I - du\otimes du$ has one eigenvalue $-\frac{|\nabla u|^2}{2}$ and $n-1$ eigenvalues $\frac{|\nabla u|^2}{2}$. Therefore, $B := \frac{|\nabla u|^2}{2}I - du\otimes du \in \Gamma_k^+$ for $k < n/2$, for

$$\sigma_j(B) = \frac{(n-1)!(n-2j)}{2^j j!(n-j)!}|\nabla u|^{2j} \quad \text{for any } j \le k < n/2.$$

It follows from (2.10) that

$$\sigma_k(D^2u) = \sigma_k(A+B) \tag{3.5}$$

$$\geq \sigma_k(A)+\sigma_k(B) = \sigma_k(A) + \frac{(n-1)!(n-2k)}{2^k k!(n-k)!}|\nabla u|^{2k}. \tag{3.6}$$

Finally, we infer

$$\begin{aligned} m_k \geq \quad & \frac{(n-2k)!}{2^k(n-1)!\omega_{n-1}}\int_{\mathcal{M}} e^{(n-2k)u}L_k(g)dvol_g \\ & + \frac{n-2k}{2^k}\int_{\mathcal{M}} e^{(n-2k)u}|\nabla u|_g^{2k}dvol_g. \end{aligned} \tag{3.7}$$

This yields the positivity of the mass m_k. Moreover, if $m_k = 0$, we have $\nabla u \equiv 0$. Hence $u \equiv 0$, that is, g is the Euclidean metric. We finish the proof of the Theorem. □

Proof of Theorem 1.2. Let $v := e^u$. Thus, the conformal metric is written as $g = v^{-2}\delta$. For such a representation of the metric, the Schouten tensor (3.2) can be written as

$$A = \frac{D^2v}{v} - \frac{|\nabla v|^2\delta}{2v^2}.$$

Let $\alpha \in \mathbb{R}$ be some sufficiently negative number to be fixed later. As in the proof of Theorem 1.1, it follows from the decay condition (1.5) of u that

$$(T_{k-1}(D^2u))^{ij}u_j\nu_i = v^\alpha(T_{k-1}(D^2v))^{ij}v_j\nu_i + O(|x|^{-(k+1)\tau-2k+1}),$$

which implies from (2.8) and (3.1)

$$m_k = \lim_{r\to\infty}\frac{(k-1)!(n-k)!}{(n-1)!\,\omega_{n-1}}\int_{S_r} v^\alpha(T_{k-1}(D^2v))^{ij}v_j\nu_i dS. \tag{3.8}$$

Thus, integration by parts leads to

$$\begin{aligned} m_k = \frac{(k-1)!(n-k)!}{(n-1)!\,\omega_{n-1}} \Big\{ & \int_{\mathbb{R}^n} v^\alpha(T_{k-1}(D^2v))^{ij}v_{ji}dx \\ & + \int_{\mathbb{R}^n} v^\alpha(T_{k-1}(D^2v))^{ij}{}_{,i}v_j dx \\ & + \alpha\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-1}(D^2v))^{ij}v_iv_j dx\Big\}. \end{aligned}$$

On the other hand, it follows from Proposition (3.1) that $(T_{k-1}(D^2v))^{ij}{}_{,i} = 0$ and also

$$(T_{k-1}(D^2v))^{ij}v_{ji} = k\sigma_k(D^2v).$$

Therefore, we have

$$\begin{aligned} m_k = & \frac{k!(n-k)!}{(n-1)!\,\omega_{n-1}}\int_{\mathbb{R}^n} v^\alpha\sigma_k(D^2v)dx \\ & + \frac{(k-1)!(n-k)!\alpha}{(n-1)!\,\omega_{n-1}}\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-1}(D^2v))^{ij}v_iv_j dx. \end{aligned}$$

We will try to write the integral of the right hand in terms of $\sigma_i(D^2v)$ and $|\nabla v|^{2i}$, then in terms of $\sigma_i(A)$ and $|\nabla v|^{2i}$ for $0\le i\le k$.

Directly from (2.9), we know

$$T_i(D^2v)=\sigma_i(D^2v)I-T_{i-1}(D^2v)D^2v=\sigma_i(D^2v)I-D^2vT_{i-1}(D^2v).$$

It follows, together with the partial integration

$$\begin{aligned}
&\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-1}(D^2v))^{ij}v_iv_j dx\\
=&\int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2-\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-2}(D^2v))^{il}v_{jl}v_jv_i\\
=&\int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2-\frac{1}{2}\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-2}(D^2v))^{ij}(|\nabla v|^2)_jv_i\\
=&\int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2+\frac{\alpha-1}{2}\int_{\mathbb{R}^n} v^{\alpha-2}(T_{k-2}(D^2v))^{ij}|\nabla v|^2v_iv_j\\
&+\frac{1}{2}\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-2}(D^2v))^{ij}{}_{,j}|\nabla v|^2v_i+\frac{1}{2}\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-2}(D^2v))^{ij}|\nabla v|^2v_{ij}\\
=&\frac{k+1}{2}\int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2+\frac{\alpha-1}{2}\int_{\mathbb{R}^n} v^{\alpha-2}(T_{k-2}(D^2v))^{ij}|\nabla v|^2v_iv_j.
\end{aligned}$$

Here the boundary term at infinity vanishes in the integration by parts because of the asymptotical assumption (1.5). More generally, we have the following claim.

Claim. For all $1\le l\le k-2$, we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} v^{\alpha-1-l}(T_{k-1-l}(D^2v))^{ij}|\nabla v|^{2l}v_iv_j\\
=&\frac{k+l+1}{2(l+1)}\int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)}\\
&+\frac{\alpha-l-1}{2(l+1)}\int_{\mathbb{R}^n} v^{\alpha-2-l}(T_{k-2-l}(D^2v))^{ij}|\nabla v|^{2(l+1)}v_iv_j.
\end{aligned}\tag{3.9}$$

As above we have

$$\begin{aligned}
&\int_{\mathbb{R}^n} v^{\alpha-1-l}(T_{k-1-l}(D^2v))^{ij}|\nabla v|^{2l}v_iv_j dx\\
=&\int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)}-\int_{\mathbb{R}^n} v^{\alpha-1-l}(T_{k-2-l}(D^2v))^{ij}|\nabla v|^{2l}v_{im}v_mv_i\\
=&\int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)}-\frac{1}{2}\int_{\mathbb{R}^n} v^{\alpha-1-l}|\nabla v|^{2l}(T_{k-2-l}(D^2v))^{ij}(|\nabla v|^2)_jv_i.
\end{aligned}\tag{3.10}$$

On the other hand, we have

$$
\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^n} v^{\alpha-1-l}|\nabla v|^{2l}(T_{k-2-l}(D^2v))^{ij}(|\nabla v|^2)_j v_i\\
=\;& \frac{\alpha-1-l}{2}\int_{\mathbb{R}^n} v^{\alpha-2-l}(T_{k-2-l}(D^2v))^{ij}|\nabla v|^{2(l+1)}v_iv_j\\
&+\frac{k-1-l}{2}\int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)}\\
&+\frac{l}{2}\int_{\mathbb{R}^n} v^{\alpha-1-l}(T_{k-2-l}(D^2v))^{ij}|\nabla v|^{2l}(|\nabla v|^2)_j v_i,
\end{aligned}
$$

which implies

$$
\begin{aligned}
&-\frac{1}{2}\int_{\mathbb{R}^n} v^{\alpha-1-l}|\nabla v|^{2l}(T_{k-2-l}(D^2v))^{ij}(|\nabla v|^2)_j v_i\\
=\;& \frac{\alpha-1-l}{2(l+1)}\int_{\mathbb{R}^n} v^{\alpha-2-l}(T_{k-2-l}(D^2v))^{ij}|\nabla v|^{2(l+1)}v_iv_j\\
&+\frac{k-1-l}{2(l+1)}\int_{\mathbb{R}^n} v^{\alpha-1-l}\sigma_{k-1-l}(D^2v)|\nabla v|^{2(l+1)}.
\end{aligned}
$$

Going back to (3.10), the desired claim yields. Hence, we have by inductively using (3.9)

$$
\begin{aligned}
&\int_{\mathbb{R}^n} v^{\alpha-1}(T_{k-1}(D^2v))^{ij}v_iv_j dx\\
=\;& \frac{k+1}{2}\int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2+\frac{(\alpha-1)\cdots(\alpha-k+1)}{2^{k-1}(k-1)!}\int_{\mathbb{R}^n} v^{\alpha-k}|\nabla v|^{2k}\\
&+\sum_{l=2}^{k-1}\frac{(\alpha-1)\cdots(\alpha-l+1)(k+l)}{2^l l!}\int_{\mathbb{R}^n} v^{\alpha-l}|\nabla v|^{2l}\sigma_{k-l}(D^2v).
\end{aligned}
$$

Finally, we infer

$$
\begin{aligned}
&\frac{(n-1)!\,\omega_{n-1}}{(k-1)!(n-k)!}m_k\\
=\;& k\int_{\mathbb{R}^n} v^{\alpha}\sigma_k(D^2v)dx+\frac{(k+1)\alpha}{2}\int_{\mathbb{R}^n} v^{\alpha-1}\sigma_{k-1}(D^2v)|\nabla v|^2\\
&+\frac{\alpha(\alpha-1)\cdots(\alpha-k+1)}{2^{k-1}(k-1)!}\int_{\mathbb{R}^n} v^{\alpha-k}|\nabla v|^{2k}\\
&+\sum_{l=2}^{k-1}\frac{\alpha(\alpha-1)\cdots(\alpha-l+1)(k+l)}{2^l l!}\int_{\mathbb{R}^n} v^{\alpha-l}|\nabla v|^{2l}\sigma_{k-l}(D^2v).
\end{aligned}
\tag{3.11}
$$

Now we want to write m_k in terms of $\sigma_l(A)$ and $|\nabla v|^{2l}$. Recall

$$
D^2v = vA+\frac{|\nabla v|^2 I}{2v},
$$

so that for all $1\le l\le k$ we have

$$
\sigma_l(D^2v)=v^l\sigma_l(A+\frac{|\nabla v|^2 I}{2v^2})=v^l\sum_{j=0}^{l}C_{n-j}^{l-j}\sigma_j(A)\left(\frac{|\nabla v|^2}{2v^2}\right)^{l-j},
$$

where $C_{n-j}^{k-j}=\frac{(n-j)!}{(n-k)!(k-j)!}$. From (3.11), we deduce

$$
\begin{aligned}
&\frac{(n-1)!\,\omega_{n-1}}{(k-1)!(n-k)!}m_k\\
=\ & k\int_{\mathbb{R}^n} v^{\alpha+k}\sum_{j=0}^{k} C_{n-j}^{k-j}\sigma_j(A)\left(\frac{|\nabla v|^2}{2v^2}\right)^{k-j}\\
&+(k+1)\alpha\int_{\mathbb{R}^n} v^{\alpha+k}\sum_{j=0}^{k-1} C_{n-j}^{k-1-j}\sigma_j(A)\left(\frac{|\nabla v|^2}{2v^2}\right)^{k-j}\\
&+\frac{2\alpha(\alpha-1)\cdots(\alpha-k+1)}{(k-1)!}\int_{\mathbb{R}^n} v^{\alpha+k}\left(\frac{|\nabla v|^2}{2v^2}\right)^{k}\\
&+\sum_{l=2}^{k-1}\sum_{j=0}^{k-l}\frac{\alpha(\alpha-1)\cdots(\alpha-l+1)(k+l)}{l!}\int_{\mathbb{R}^n} v^{\alpha+k}C_{n-j}^{k-l-j}\sigma_j(A)\left(\frac{|\nabla v|^2}{2v^2}\right)^{k-j}\\
=\ & \int_{\mathbb{R}^n} v^{\alpha+k}\sum_{j=0}^{k} P_{k-j}(\alpha)\sigma_j(A)\left(\frac{|\nabla v|^2}{2v^2}\right)^{k-j}.
\end{aligned}
$$

Here for all $0\le j\le k$, $P_j(\alpha)$ is a polynomial of degree j in α with a leading coefficient equal to k when $j=0$, to $k+1$ when $j=1$, to $\frac{2k-j}{(k-j)!}$ when $2\le j\le k-1$ and to $\frac{2}{(k-1)!}$ when $j=k$. Therefore, we can choose sufficiently negative number $\alpha<0$ such that $(-1)^jP_j(\alpha)>0$ for all $0\le j\le k$. By the assumptions $(-1)^jL_j\ge 0$ for all $1\le j\le k$, which are equivalent to $(-1)^j\sigma_j(A)\ge 0$, we have

$$
P_{k-j}(\alpha)\sigma_j(A)=(-1)^{k-j}P_{k-j}(\alpha)(-1)^j\sigma_j(A)\ge 0,
$$

i.e., each term on the right hand side in the last inequality is non-negative. This gives $m_k\ge 0$. Here we need that k is even. Moreover, if $m_k=0$, we have $\nabla v\equiv 0$, and hence v is a constant 1 and $\mathcal{M}$ is the standard euclidean space. We finish the proof. □

4. Penrose type inequality

Let $(\mathcal{M}^n,g)=(\mathbb{R}^n\setminus\Omega,e^{-2u}\delta)$ be now a CF manifold, where Ω is a bounded domain such that each connected component of Ω is star-shaped such that the second fundamental form of the boundary $\partial\Omega$ is in the cone $\Gamma_{k-1}^+(\partial\Omega)$. As before, we assume $2k<n$, $g\in\Gamma_k$, L_k integrable and u satisfies the decay condition at the infinity

$$
|u|+|x||\nabla u|+|x|^2|\nabla^2 u|=O(|x|^{-\tau}),
$$

with $\tau>\frac{n-2k}{k+1}$. First, we assume Ω has just one connected component.

Theorem 4.1. *Let $(\mathcal{M},g)=(\mathbb{R}^n\backslash\Omega,e^{-2u}\delta)$ satisfy the above assumptions. Assume, in addition, that $\partial\mathcal{M}$ is a horizon on $(\mathcal{M},g)$ (i.e. $\partial\mathcal{M}=\partial\Omega\subset\mathcal{M}$ is minimal) and u is constant on $\partial\Omega$.*

Then we have the following Penrose type inequality

$$\begin{aligned} m_k &\geq \frac{(n-2k)!}{2^k(n-1)!\,\omega_{n-1}}\int_{\mathcal{M}} e^{(n-2k)u}L_k(g)dvol_g \\ &\quad + \frac{n-2k}{2^k}\int_{\mathcal{M}} e^{(n-2k)u}|\nabla u|_g^{2k}dvol_g + \left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}} \\ &\geq \left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}}. \end{aligned} \tag{4.1}$$

Moreover, if we assume the second fundamental form of $\partial\Omega$ is in the cone Γ_{2k-1} ($k \geq 2$), we have

$$\begin{aligned} m_k &\geq \frac{(n-2k)!}{2^k(n-1)!\,\omega_{n-1}}\int_{M} e^{(n-2k)u}L_k(g)dvol_g \\ &\quad + \frac{n-2k}{2^k}\int_{\mathcal{M}} e^{(n-2k)u}|\nabla u|_g^{2k}dvol_g + \left(\frac{\int_{\partial\Omega} R}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-2k}{n-3}} \\ &\geq \left(\frac{\int_{\partial\Omega} R}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-2k}{n-3}}. \end{aligned} \tag{4.2}$$

Here R is the scalar curvature of $\partial\Omega$ as a hypersurface in $\mathbb{R}^n$.

Proof. Applying Proposition 3.1 and Green's formula, we obtain

$$\int_{S_r}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS - \int_{\partial\Omega}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS = k\int_{B_r\setminus\Omega}\sigma_k(D^2u)dx, \tag{4.3}$$

for large $r > 0$. The argument given in the proof of Theorem 1.1, together with (3.5) to (3.8), implies

$$\begin{aligned} m_k &\geq \frac{(n-2k)!}{2^k(n-1)!\omega_{n-1}}\int_{\mathcal{M}} e^{(n-2k)u}L_k(g)dvol_g \\ &\quad + \frac{n-2k}{2^k}\int_{\mathcal{M}} e^{(n-2k)u}|\nabla u|_g^{2k}dvol_g \\ &\quad + \frac{(k-1)!(n-2k)!}{(n-1)!\,\omega_{n-1}}\int_{\partial\Omega}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS. \end{aligned} \tag{4.4}$$

Recall ν is the normal vector pointing to the infinity. Since $\partial\mathcal{M}$ is a horizon of $\mathcal{M}$, the mean curvature of $\partial\mathcal{M}$ is equal to zero at the boundary. We denote H the mean curvature of $\partial\Omega$ in $\mathbb{R}^n$. As g is a conformal metric, the mean curvature of $\partial\mathcal{M}$ is equal to $e^u(H-(n-1)\langle\nabla u,\nu\rangle)$. Therefore, on the boundary $\partial\Omega$ we have

$$H-(n-1)\langle\nabla u,\nu\rangle = 0. \tag{4.5}$$

In particular, $\langle\nabla u,\nu\rangle > 0$ on the boundary, since we assume the second fundamental form N is in the cone $\Gamma^+_{k-1}(\partial\Omega)$. On the other hand, from the non-negativity of the scalar curvature, we have

$$\Delta u \geq 0.$$

Hence, by the Maximum principle, we deduce $u \le 0$ in Ω. For all $x \in \partial\Omega$, we split $T_x\mathbb{R}^n = T_x\partial\Omega \oplus \mathbb{R}\nu$ as the sum of tangential part and normal part. Let e_β $(1 \le \beta \le n-1)$ a basis of $\partial\Omega$ and $e_n = \nu$. And Let $B = (D^2u(e_i, e_j))_{1\le i,j\le n}$ be the Hessian matrix and $B' = (D^2u(e_\alpha, e_\beta))_{1\le\alpha,\beta\le n-1}$ the first $(n-1)\times(n-1)$ block in B. Recall that u is a constant on the boundary $\partial\Omega$. We have for all $1 \le \alpha, \beta \le n-1$,

$$\begin{aligned} D^2u(e_\alpha, e_\beta) &= e_\alpha e_\beta(u) - \nabla_{e_\alpha} e_\beta(u) \\ &= -\nabla_{e_\alpha} e_\beta(u) \\ &= -\langle \nabla_{e_\alpha} e_\beta, \nu\rangle \nu(u) \\ &= \langle \nabla u, \nu\rangle N(e_\alpha, e_\beta), \end{aligned} \tag{4.6}$$

where N is the second fundamental form with respect to the normal vector $-\nu$. Hence, we can compute

$$(T_{k-1}(D^2u))^{ij}u_j\nu_i = \langle \nabla u, \nu\rangle \frac{\partial \sigma_k(B)}{\partial b_{nn}} = \langle \nabla u, \nu\rangle \sigma_{k-1}(B'). \tag{4.7}$$

Here we have used the fact $\nabla_\beta u = 0$ on the boundary. Gathering (4.5) to (4.7), we deduce

$$(T_{k-1}(D^2u))^{ij}u_j\nu_i = \langle \nabla u, \nu\rangle^k \sigma_{k-1}(N) = \frac{1}{(n-1)^k}\sigma_1(L)^k\sigma_{k-1}(N). \tag{4.8}$$

Recall that in the Garding cone Γ_m^+, we have the Newton-MacLaurin inequalities,

$$\frac{\sigma_{m-1}\sigma_{m+1}}{\sigma_m^2} \le \frac{m(n-m-1)}{(m+1)(n-m)}, \tag{4.9}$$

$$\frac{\sigma_1\sigma_{m-1}}{\sigma_m} \ge \frac{m(n-1)}{n-m}. \tag{4.10}$$

We have

$$T_{k-1}(D^2u))^{ij}u_j\nu_i \ge \left(\frac{(k-1)!}{(n-1)\cdots(n-k+1)}\right)^{\frac{k}{k-1}} \sigma_{k-1}(N)^{\frac{2k-1}{k-1}}.$$

From the Hölder inequality and the Aleksandrov-Fenchel inequality (see [37], [24] and [12] for example), we have

$$\begin{aligned} \int_{\partial\Omega} (T_{k-1}(D^2u))^{ij}u_j\nu_i dS &\ge \left(\frac{(k-1)!}{(n-1)\cdots(n-k+1)}\right)^{\frac{k}{k-1}} \int_{\partial\Omega} \sigma_{k-1}(N)^{\frac{2k-1}{k-1}} \\ &\ge \left(\frac{(k-1)!}{(n-1)\cdots(n-k+1)}\right)^{\frac{k}{k-1}} \left(\int_{\partial\Omega} \sigma_{k-1}(N)\right)^{\frac{2k-1}{k-1}} |\partial\Omega|^{\frac{-k}{k-1}} \\ &\ge \frac{(n-1)!}{(k-1)!(n-k)!}\omega_{n-1}^{\frac{2k-1}{n-1}} |\partial\Omega|^{\frac{n-2k}{n-1}}. \end{aligned}$$

Going back to (4.4), we get the desired inequality (4.1). Now, assume $N \in \Gamma_{2k-1}$, it follows from the Newton-MacLaurin inequality that

$$\frac{1}{(n-1)^k}\sigma_1(N)^k\sigma_{k-1}(N) \ge \frac{(2k-1)!(n-2k)!}{(k-1)!(n-k)!}\sigma_{2k-1}(N).$$

Hence, again by the Aleksandrov-Fenchel inequality, we get

$$\begin{aligned}\int_{\partial\Omega}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS &\geq \frac{(2k-1)!(n-2k)!}{(k-1)!(n-k)!}\int_{\partial\Omega}\sigma_{2k-1}(N)\\ &\geq \frac{(n-1)!}{(k-1)!(n-k)!}\omega_{n-1}^{\frac{2k-3}{n-3}}\left(\int_{\partial\Omega}\frac{2\sigma_2(N)}{(n-1)(n-2)}\right)^{\frac{n-2k}{n-3}}.\end{aligned}$$

In view of (4.4), we prove inequality (4.2) and finish the proof. □

Remark 4.2. *In (4.2), the scalar curvature R could be replaced by other high order curvature tensor of order small than k which establishes a relationship between the mass m_{GBC} and more geometric objects.*

Remark 4.3. *We remark that when $k=1$, our mass $m_1 = m_{ADM}$. In this case the Penrose inequality in Theorem 4.1 is*

$$m_1 \geq \left(\frac{|\partial\Omega|}{\omega_{n-1}}\right)^{\frac{n-2}{n-1}},$$

which was already proved in [30]. *In fact, our Penrose inequality is motivated by his work. Note that we have taken a different test function comparing with the paper* [30].

Let Ω_i be the components of Ω, $i=1,\cdots l$, and let $\Sigma_i = \partial\Omega_i$. If we assume that each Σ_i is a horizon, we have the following

Corollary 4.4. *With the same condition of Theorem 4.1, and the additional condition that each Σ_i is a horizon Then we have the the following Penrose type inequality*

$$\begin{aligned}m_k &\geq \frac{(n-2k)!}{2^k(n-1)!\,\omega_{n-1}}\int_{\mathcal{M}}e^{(n-2k)u}L_k(g)dvol_g\\ &\quad+\frac{n-2k}{2^k}\int_{\mathcal{M}}e^{(n-2k)u}|\nabla u|_g^{2k}dvol_g+\sum_{i=1}^{l}\left(\frac{|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}}\\ &\geq \sum_{i=1}^{l}\left(\frac{|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}} \geq \left(\frac{\sum_{i=1}^{l}|\Sigma_i|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}}.\end{aligned}$$

Moreover, if we assume the second fundamental form of $\partial\Omega$ is in the cone Γ_{2k-1} $(k\geq 2)$, we have

$$\begin{aligned}m_k &\geq \frac{(n-2k)!}{2^k(n-1)!\,\omega_{n-1}}\int_{\mathcal{M}}e^{(n-2k)u}L_k(g)dvol_g\\ &\quad+\frac{n-2k}{2^k}\int_{\mathcal{M}}e^{(n-2k)u}|\nabla u|_g^{2k}dvol_g+\sum_{i=1}^{l}\left(\frac{\int_{\Sigma_i}R}{(n-1)(n-2)\,\omega_{n-1}}\right)^{\frac{n-2k}{n-3}}\\ &\geq \sum_{i=1}^{l}\left(\frac{\int_{\Sigma_i}R}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-2k}{n-3}} \geq \left(\frac{\sum_{i=1}^{l}\int_{\Sigma_i}R}{(n-1)(n-2)\omega_{n-1}}\right)^{\frac{n-2k}{n-3}}.\end{aligned}$$

Here R is the scalar curvature of $\partial\Omega$ as a hypersurface in $\mathbb{R}^n$.

Example 4.5. *$(\mathcal{M}^n = I\times\mathbb{S}^{n-1}, g)$ with coordinates (ρ,θ), general Schwardschild metrics are given*

$$g^k_{\rm Sch} = (1-\frac{2m}{\rho^{\frac{n}{k}-2}})^{-1}d\rho^2+\rho^2 d\Theta^2,$$

where $d\Theta^2$ is the round metric in $\mathbb{S}^{n-1}$, $m\in\mathbb{R}$ is the "total mass" of corresponding black hole solutions in the Lovelock gravity [15, 10]. *When $k=1$ we recover the Schwarzschild solutions of the Einstein gravity.*

Motivated by the Schwarzschild solutions, the above metrics also have the following form of conformally flat which is more convenient for computation ([20]).

$$g^k_{\rm Sch} = (1-\frac{2m}{\rho^{\frac{n}{k}-2}})^{-1}\rho^2+\rho^2 d\Theta^2 = (1+\frac{m}{2r^{\frac{n}{k}-2}})^{\frac{4k}{n-2k}}(dr^2+r^2d\Theta^2).$$

For this metric the Gauss-Bonnet-Chern mass $m_k = m^k$ (one can check it by (4.11) below) and the black hole (i.e. the horizon) $\Sigma = \partial\Omega = \{r = r_0 = (\frac{m}{2})^{\frac{k}{n-2k}}\}$ and its area is

$$|\Sigma| = \omega_{n-1}r_0^{n-1},$$

hence

$$\begin{aligned} m_k &= m^k = (2r_0^{\frac{n-2k}{k}})^k \\ &= 2^k\left(\frac{|\Sigma|}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}} = \frac{1}{2^k}\left(\frac{|\Sigma|_{g^k_{\rm Sch}}}{\omega_{n-1}}\right)^{\frac{n-2k}{n-1}}. \end{aligned}$$

We remark that the Penrose inequality in Theorem 1.3 is not optimal, since in Theorem 1.3 the area of Σ is computed with the Euclidean metric δ, not with the metric $g = e^{-2u}\delta$ itself. In general, if $(\mathcal{M}^n, g)$ is spherically symmetric, we have the following result.

Proposition 4.6. *Suppose $(\mathcal{M}^n, g)$ is asymptotically flat CF manifold with $g = e^{-2u(r)}\delta$, ie., $(\mathcal{M}^n, g)$ is spherically symmetric, then*

$$m_k = \lim_{r\to\infty}\frac{1}{\omega_{n-1}}\int_{S_r}\frac{(u_r)^k}{r^{k-1}}dS_r. \tag{4.11}$$

If k is even, we always have $m_k\geq 0$. Moreover, if we assume $L_k\geq 0$ in such a case, then $m_k = 0$ if and only if u is a constant, i.e., $\mathcal{M}$ is the Euclidean space.

Proof. We adopt the equivalent form (3.3) to calculate the Gauss-Bonnet-Chern mass. Denote the radial derivative of u by $u_r \triangleq \frac{\partial u}{\partial r}$. We consider $\Omega = B_r$ being the ball centered at the origin with radius equal to r. Thus Ω can be seen as a level set of u which enable us to use the formulas in the proof of Theorem 4.1. Let $(e_1,\cdots,e_{n-1})$ be an orthonormal basis of the boundary $\partial\Omega$. It follows from (4.6) that for all $1\leq\alpha,\beta\leq n-1$

$$D^2u(e_\alpha,e_\beta) = \frac{u_r}{r}\delta_{\alpha\beta},$$

since the second fundamental form on $\partial\Omega = S_r$ is equal to $\frac{1}{r}I$ where I is the identity map. By (4.7) we have

$$T_{k-1}(D^2u)^{ij}u_i\nu_j = \frac{(n-1)\cdots(n-k+1)}{(k-1)!r^{k-1}}u_r^k.$$

Going back to (3.3), we get the desired result (4.11). It is clear $m_k \geq 0$ when k is even and $m_k = 0$ when u is a constant. Now we want to prove the inverse, under an extra condition that $L_k \geq 0$. A direct computation gives

$$A_{ij} = u_{rr}\frac{x_i}{r}\frac{x_j}{r} + \frac{u_r}{r}(\delta_{ij} - \frac{x_i}{r}\frac{x_j}{r}) + u_r^2(\frac{x_i}{r}\frac{x_j}{r} - \frac{1}{2}\delta_{ij}).$$

Thus, A has an eigenvalue $u_{rr} + \frac{1}{2}u_r^2$ and $n-1$ eigenvalues $\frac{u_r}{r} - \frac{1}{2}u_r^2$. Hence, we get

$$\sigma_k(A) = C_{n-1}^k\left(\frac{u_r}{r} - \frac{1}{2}u_r^2\right)^{k-1}\left(\frac{u_r}{r} - \frac{1}{2}u_r^2 + \frac{k}{n-k}(u_{rr} + \frac{1}{2}u_r^2)\right).$$

Similarly, we infer

$$\sigma_k(D^2u) = C_{n-1}^k\left(\frac{u_r}{r}\right)^{k-1}\left(\frac{u_r}{r} + \frac{ku_{rr}}{n-k}\right).$$

From the decay condition (1.5), we have $u_r = O(|x|^{-\tau-1}) = o(|x|^{-1})$. Therefore, there exists some $R_1 > 0$ such that for any $r > R_1$, there holds $\frac{1}{r} - \frac{1}{2}u_r > 0$.

We claim for all $r > R_1$, $u_r \leq 0$. Otherwise, we suppose $\exists\, r_1, r_2 \in (R_1, +\infty]$ with $r_2 < r_1$ such that $u_r^k r^{n-k}|_{r=r_1} = 0$ (when $r_1 = +\infty$, this is just the assumption $m_k = 0$) and $u_r(r) > 0$ for all $r \in (r_2, r_1)$. Since $\sigma_k(A) \geq 0$, we have $\frac{u_r}{r} - \frac{1}{2}u_r^2 + \frac{k}{n-k}(u_{rr} + \frac{1}{2}u_r^2) \geq 0$, which implies $\frac{u_r}{r} + \frac{k}{n-k}u_{rr} > 0$ for all $r \in (r_2, r_1)$ since $n > 2k$. Thus, for all $r \in (r_2, r_1)$, we have $\sigma_k(D^2u) > 0$. Applying (4.3), we get

$$\int_{S_{r_1}}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS - \int_{S_r}(T_{k-1}(D^2u))^{ij}u_j\nu_i dS = k\int_{B_{r_1}\setminus B_r}\sigma_k(D^2u)dx > 0,$$

which implies

$$\int_{S_r}\frac{(u_r)^k}{r^{k-1}}dS_r < 0.$$

This contradiction yields the desired claim. Assume that u is not a constant on $(R_1, +\infty)$. $\exists r_1, r_2 \in (R_1, +\infty]$ with $r_2 < r_1$ such that $u_r^k r^{n-k}|_{r=r_1} = 0$ and $u_r(r) < 0$ for all $r \in (r_2, r_1)$. Thus, $\left(\frac{u_r}{r} - \frac{1}{2}u_r^2\right)^{k-1} < 0$ on (r_2, r_1), which implies that

$$\frac{u_r}{r} - \frac{1}{2}u_r^2 + \frac{k}{n-k}(u_{rr} + \frac{1}{2}u_r^2) \leq 0,$$

since $\sigma_k(A) \geq 0$. Using the decay condition (1.5), there exists some positive constant $C > 0$ such that on (r_2, r_1) there holds

$$\frac{n-k}{k}\frac{u_r}{r} + Cr^{-1-\tau}u_r + u_{rr} \leq 0.$$

Therefore, we have

$$(r^{\frac{n-k}{k}}e^{-\frac{C}{\tau r^\tau}}u_r)' \leq 0,$$

which gives

$$r^{\frac{n-k}{k}}e^{-\frac{C}{\tau r^\tau}}u_r \geq [r^{\frac{n-k}{k}}e^{-\frac{C}{\tau r^\tau}}u_r]|_{r=r_1} = 0.$$

Hence, $u_r(r) \geq 0$ which contradicts the assumption $u_r(r) < 0$. Therefore, we prove that u is constant, say c_0, on $(R_1, +\infty)$. Without loss of generality, we assume $R_1 = \min\{t > 0, u \equiv c_0 \text{ on } (t, +\infty)\}$. Suppose $R_1 > 0$. So there exists some small positive number $\varepsilon < R_1$ such that $\frac{u_r^2}{r} - \frac{1}{2}u_r^3 > 0$ on $(R_1 - \varepsilon, R_1)$. (XXXXXX why XXXX) Thus, we could repeat the above arguments to obtain the desired contradiction according $u_r > 0$ or $u_r < 0$ on $(R_1 - \varepsilon, R_1)$. Finally, we prove $u \equiv c_0$ on $[0, +\infty)$.

□

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