Regularity of optimal transport maps on locally nearly spherical manifolds

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Abstract. Given a compact connected n-dimensional Riemannian manifold, we investigate the smoothness of the optimal transport map between the smooth densities with respect to the squared Riemannian distance cost. The optimal map is characterized by $\exp(\operatorname{grad} u)$, where the potential function u satisfies a Monge-Ampère type equation. Delanoë [5] showed the smoothness of u on the Riemannian surfaces when the scalar curvature is close to 1 in C^2 norm. In this work, we study the regularity issue on Riemannian manifolds with curvature sufficiently close to curvature of round sphere in C^2 norm in all dimensions and prove that the $\mathcal{C}$-curvature on such Riemannian manifolds satisfies an improved Ma-Trudinger-Wang condition and the Jacobian of the exponential map is positive. As a consequence, we imply the smoothness of the optimal transport map by the continuity method.

Régularité de l'application du transport optimal sur les variétés riemanniennes localement proches de la sphère

Résumé. Etant donné une variété riemannienne compacte connexe de dimension n, nous étudions la régularité de l'application du transport optimal entre les densités lisses par rapport au coût de la distance riemannienne au carré. L'application du transport optimal est caractérisée par exp(grad u), où la fonction potentielle u satisfait une équation de type Monge-Ampère. Delanoë [5] a montré la régularité de u sur les surfaces riemanniennes lorsque la courbure scalaire est proche de 1 dans la norme C^2. Dans ce travail, nous étudions le problème de régularité sur les variétés riemanniennes avec courbure suffisamment proche de la courbure de la sphère usuelle dans la norme C^2 en toutes les dimensions et prouvons que la $\mathcal{C}$-courbure sur de telles variétés riemanniennes satisfait une condition Ma-Trudinger-Wang améliorée et le jacobien de l'application exponentielle est strictement positive. Par conséquent, nous impliquons la régularité de l'application du transport optimal par la méthode de continuité.

1. Introduction and main results

1.1. Background

Let (M, g) be a compact connected Riemannian manifold without boundary of dimension $n \geqslant 2$. For short, we call such (M, g) as a closed Riemannian manifold. Let $d(\cdot, \cdot)$ be geodesic distance on M and denote $dvol$ the Riemannian volume form. We consider the optimal transportation problem on M with the cost $c(\cdot, \cdot) = \frac{1}{2} d^2(\cdot, \cdot)$. The problem of optimal transportation is to find the most efficient strategy to transport an assigned mass distribution to another one. Precisely, let $\mu_0 = \rho_0 dvol$ and $\mu_1 = \rho_1 dvol$ be two positive Borel probability measures on M with the density ρ_0 and ρ_1 with respect to the volume form $dvol$. The problem consists in minimizing the total cost functional

$$\text{Cost}(G) = \int_M c(x, G(x)) d\mu_0$$

among all Borel measurable maps $G : M \to M$ which push forward μ_0 to μ_1 in the sense that

$$\mu_1(E) = \mu_0(G^{-1}(E)), \forall E \subset M \; Borel \; set.$$

The minimizers are called optimal transport maps. This problem is first posed by Monge [34] in 1781 with the Euclidean distance cost $c(x, y) = |x-y|$. One and a half centuries later, Kantorovich [20] reduced the problem to an

infinite dimensional linear program. For the squared Euclidean distance cost $c(x, y) = \frac{1}{2}|x - y|^2$, Brenier [1] showed the existence and uniqueness of the optimal transport map which can be characterized as the gradient of some convex function. McCann [32] developed Brenier's theory on Riemannian manifolds. He showed the optimal transport map is unique and takes the form $G(m) = \exp_m(\nabla u(m))$ where u is some c-convex function, that is, $\forall x \in M$, $u(x) = \sup_{y \in M}(-c(x, y) - v(y))$ for some function v on M. Such function u is called the potential function of the optimal transport map G. The aim of this paper is to show the smoothness of the optimal transport maps G, or equivalently, the smoothness of the optimal transport potential u. Ma-Trudinger-Wang [31] introduced for the first time the MTW tensor which is crucial in the study of the regularity theory of the optimal transport maps. Later on, Kim-McCann [22] interpreted the MTW tensor as a curvature tensor of some pseudo-Riemannian metric. There are a lot of developments in the past decade for example $[5, 8, 9, 11\text{--}13, 15, 17, 24, 26\text{--}28, 30, 36]$ etc. For more references, see the book of Villani [37].

We recall the definition of the $\mathcal{C}$-curvature. Given $m \in M$, denote by $\mathrm{Cut}_m \subset M$ the cut locus of M at m. The closed subset Cut of TM is defined by
$$\mathrm{Cut} = \{(m, \nu) \in M \times T_m M, \exp_m \nu \in \mathrm{Cut}_m\}.$$
We consider the open connected component of $TM \backslash \mathrm{Cut}$ containing the zero section and denote it by
$$\mathrm{NoCut} = \{(m, \nu), \forall t \in [0, 1] \text{ and } (m, t\nu) \notin \mathrm{Cut}\}.$$
Given $(m, \nu) \in \mathrm{NoCut}, (\xi, \eta) \in T_m M \times T_m M$, we define the $\mathcal{C}$-curvature by
$$\mathcal{C}(m, \nu)(\xi, \eta) = -\frac{3}{2} \frac{\partial^2}{\partial s^2}\Big|_{s=0} \frac{\partial^2}{\partial t^2}\Big|_{t=0} c(\exp_m t\xi, \exp_m(\nu + s\eta)). \qquad (1.1)$$

For the more intrinsic geometric interpretation, we use the aforementioned pseudo-Riemannian metric h on $M \times M$ and set Sect_h for its sectional curvature tensor viewed as a field of quadratic forms on $\bigwedge^2 T(M \times M)$, for each $(m, \overline{m}) \in M \times M \setminus \mathrm{Cut}_M$ with $\mathrm{Cut}_M = \exp(\mathrm{Cut})$ the cut locus of M and each $(\xi, \bar{\xi}) \in T_m M \times T_{\overline{m}} M$, the associated *cross-curvature* is defined in [22]:
$$\mathrm{cross}_{(m, \overline{m})}(\xi, \bar{\xi}) := \mathrm{Sect}_h[(\xi \oplus 0) \wedge (0 \oplus \bar{\xi})].$$
Kim and McCann [22] observed that it must vanish for some choice of $(\xi, \bar{\xi})$. Trudinger *et al* noted [31, p.164] that one identically recovers $\frac{1}{2} \mathrm{cross}_{(m, \overline{m})}(\xi, \bar{\xi})$

at $\overline{m} = \exp_m(V)$ with $(m, V) \in \mathrm{NoCut}$ and $\bar{\xi} = d(\exp_m)(V)(\nu)$, by calculating the quantity:

$$\mathcal{C}(m, V)(\xi, \nu) := -\frac{D^2}{d\lambda^2}\left[A(m, V + \lambda\nu)(\xi)\right]_{\lambda=0} \qquad (1.2)$$

where $A(m, V)(\xi) := \nabla d[p \to c(p, \exp_m(V))]_{p=m}(\xi, \xi)$ with ∇ the Levi–Civita connection of the Riemannian metric g and where D stands for the canonical flat connection of T_mM.

When $\nu = 0$, Loeper [28] observed that the $\mathcal{C}$-curvature is just the sectional curvature. In fact, it follows from the Taylor expansion for the geodesic distance [33, p.5] that

$$
\begin{aligned}
d^2(\exp_m t\xi, \exp_m s\eta) &= |\xi|_m^2 t^2 - 2g_m(\xi, \eta)ts + |\eta|_m^2 s^2 - \\
&\quad \frac{1}{3}R_m(\xi, \eta, \xi, \eta)t^2s^2 + o((t^2 + s^2)^2).
\end{aligned}
$$

Thus we see $\mathcal{C}(m, 0)(\xi, \eta) = R_m(\xi, \eta, \xi, \eta)$. More generally, we have the asymptotic expansion [24] for the $\mathcal{C}$-curvature

$$\mathcal{C}(m, \nu)(\xi, \eta) = R_m(\xi, \eta, \xi, \eta) + \frac{1}{2}(\nabla_\eta R)(\xi, \nu, \xi, \eta) + \frac{1}{4}(\nabla_\nu R)(\xi, \eta, \xi, \eta) + o(|\nu|).$$

We introduce several curvature conditions [23, 31, 36]

(i) A3S condition is satisfied if there exists some $\kappa_0 > 0$ such that $\mathcal{C}(m, \nu)(\xi, \eta) \geqslant \kappa_0|\xi|^2|\eta|^2, \forall(m, \nu) \in \mathrm{NoCut}, \forall(\xi, \eta) \in T_mM \times T_mM$ with $\langle\xi, \eta\rangle = 0$;

(ii) A3W condition is satisfied if $\mathcal{C}(m, \nu)(\xi, \eta) \geqslant 0, \forall(m, \nu) \in \mathrm{NoCut}, \forall(\xi, \eta) \in T_mM \times T_mM$ with $\langle\xi, \eta\rangle = 0$;

(iii) the non-negatively c-curved (or NNCC) condition is satisfied if $\mathcal{C}(m, \nu)(\xi, \eta) \geqslant 0, \forall(m, \nu) \in \mathrm{NoCut}, \forall(\xi, \eta) \in T_mM \times T_mM$;

(iv) the almost-positively c-curved (or APCC) condition is satisfied if NNCC condition is satisfied and $\mathcal{C}(m, \nu)(\xi, \eta) = 0$ if and only if the span of the vectors (ν, ξ, η) has dimension at most 1.

The A3S condition is satisfied on the round sphere [29] and on nearly spherical manifolds [8, 16, 17, 30]. The NNCC condition is stable under Riemannian products, unlike A3W condition. Each of the above conditions is stable under Riemannian submersion [23]. The APCC condition holds on spheres [23] (see also [8, 16, 29]), on projective spaces $\mathbb{CP}^k$ and $\mathbb{HP}^k$ [23] and on positively curved Riemannian locally symmetric space [7]. The APCC condition is satisfied on surfaces close to 2-sphere [9]. On the other hand, the A3W condition can imply nonnegative sectional curvature [28], but the inverse is

not true [18, 21, 30].

For the regularity issue, the A3W condition is necessary for the continuity of the optimal transport map [28] and also sufficient under some suitable assumptions [36]. There are many works related to the C^1 regularity of potential u (or continuity of optimal transport maps) for all measures μ_0, μ_1 (possibly not smooth). For the instance, see references [15–18, 23] etc. Here we are interested in the high order regularity on closed manifolds. Such regularity result holds on flat manifolds [11], on spheres [29], on complexe or quaternionic projective spaces [5, 13, 23] (see also [27]), on product of spheres [13, 14, 23], on nearly spherical manifolds with topology [8, 30] and on 2 dimensional simply connected manifolds or positively curved Riemannian locally symmetric spaces [5].

1.2. Main results and outline of the paper

Let us recall some notations (see [2–4, 35]). Let X, Y, Z, W be smooth vector fields on M. The (3,1)-type Riemann curvature tensor of the Riemaniann manifold (M, g) is defined by

$$R(X, Y)Z = \nabla_X \nabla_Y Z - \nabla_Y \nabla_X Z - \nabla_{[X,Y]} Z$$

where ∇ denotes the Levi-Civita connection of g. We set Riem for the associated (4,0)-type Riemann curvature tensor[1], that is.

$$\text{Riem}(X, Y, Z, W) = \langle R(Z, W)Y, X \rangle$$

Here, we adopt the Einstein summation convention over repeated indexes. In a local coordinate system $\{x^1, \cdots, x^n\}$, the components of Riemann curvature tensor are given by $R(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})\frac{\partial}{\partial x^i} = R^l_{ijk}\frac{\partial}{\partial x^l}$ and $R_{ijkl} = g_{ip}R^p_{jkl}$ respectively. The Ricci tensor is obtained by the contraction $\text{Ric}_{ij} = g^{kl}R_{ikjl}$ and the scalar curvature by $\text{Scal} = g^{ij}\text{Ric}_{ij}$. The Riemannian metric induces norms on all the tensor bundles. More precisely, the squared norm of (r, s)-tensor field T in the coordinate system $x = (x^1, \cdots, x^n)$ is given by

$$|T|^2 = g_{i_1 k_1} \cdots g_{i_s k_s} g^{j_1 l_1} \cdots g^{j_r l_r} T^{i_1 \cdots i_s}_{j_1 \cdots j_r} T^{k_1 \cdots k_s}_{l_1 \cdots l_r},$$

Let $K : \text{Gr}_2(M) \to \mathbb{R}$ be the sectional curvature defined on the Grassmann bundle of tangent 2-planes (see [5]). We always assume the sectional

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[1] We use $g(\cdot, \cdot)$ and $\langle \cdot, \cdot \rangle$ interchangeably.

curvatures satisfy

$$\min_{\mathrm{Gr}_2(M)} K = 1. \tag{1.3}$$

and we define for the Riemann curvature tensor when $n \geqslant 3$

$$\varepsilon := \|\mathrm{Riem} - \frac{\mathrm{Scal}}{2n(n-1)} g \oslash g\|_{C^2(M,g)} \tag{1.4}$$

Here the Kulkarni-Nomizu product $\oslash$ is defined as follows: given two 2-covariant tensors h and k, the Kulkarni-Nomizu product is a 4-covariant tensor, determined by

$$\begin{aligned}(h \oslash k)(X_1, X_2, X_3, X_4) := \quad &h(X_1, X_3)k(X_2, X_4) + k(X_1, X_3)h(X_2, X_4) \\ &-h(X_1, X_4)k(X_2, X_3) - k(X_1, X_4)h(X_2, X_3),\end{aligned}$$

where X_j for $1 \leqslant j \leqslant 4$ are tangent vectors. If necessary, we could identify the contravariant tensor and the covariant one via the Riemannian metric g. In two dimension, the tensor on the right hand identically vanishes and we replace the quantity on the right hand in (1.4) by

$$\varepsilon := \|K - 1\|_{C^2(M,g)} \tag{1.5}$$

While the dimension $n \geqslant 3$, the quantity (1.4) vanishes if and only if (M, g) is space form. It follows from [8] that we consider an equivalent term to (1.4) when $n \geqslant 3$ or (1.5) when $n = 2$

$$\bar{\varepsilon} := \|\mathrm{Riem} - \frac{1}{2} g \oslash g\|_{C^2(M,g)}. \tag{1.6}$$

It is known on the round spheres $\mathbb{S}^n$ that for $\nu \neq 0$

$$\bar{A}(m, \nu)(\xi) = |\xi|_m^2 - (1 - |\nu|_m \cot |\nu|_m)(|\xi|_m^2 - g_m(\xi, \frac{\nu}{|\nu|_m})^2) \tag{1.7}$$

and we can calculate the $\mathcal{C}$-curvature

$$\overline{\mathcal{C}}(m, \nu)(\xi, \eta) = -\frac{3}{2} \frac{d^2}{ds^2}|_{s=0} \bar{A}(m, \nu + s\eta)(\xi). \tag{1.8}$$

It follows from [23] that the APCC condition holds on spheres $\mathbb{S}^n$. Our first main result concerns about the stability of the APCC condition.

THEOREM 1. — *Let (M, g) be a closed n-dimensional Riemannian manifold satisfying (1.3). Then there exists some universal positive constants ε_0, κ_0 such that if $\varepsilon < \varepsilon_0$, that is, when $n \geqslant 3$*

$$\|Riem - \frac{Scal}{2n(n-1)} g \oslash g\|_{C^2(M,g)} < \varepsilon_0$$

when $n = 2$

$$\|K - 1\|_{C^2(M,g)} < \varepsilon_0$$

Then for all $(m, \nu) \in NoCut$ and all tangent vectors ξ, η in $T_m M$

$$\mathcal{C}(m, \nu)(\xi, \eta) \geqslant \kappa_0(|\xi \wedge \eta|_m^2 + |\xi|_m^2 |\eta \wedge \nu|_m^2 + |\xi \wedge \nu|_m^2 |\eta|_m^2), \qquad (1.9)$$

where $|\xi \wedge \eta|_m^2 = |\xi|_m^2 |\eta|_m^2 - g_m(\xi, \eta)^2, |\eta \wedge \nu|_m^2 = |\eta|_m^2 |\nu|_m^2 - g_m(\eta, \nu)^2, |\xi \wedge \nu|_m^2 = |\xi|_m^2 |\nu|_m^2 - g_m(\xi, \nu)^2.$

Since the NNCC condition is stable for the Riemannian products, a direct consequence of the above theorem can be read as the following.

COROLLARY 1. — *Assume M and N satisfy the assumptions as in Theorem 1. Then the A3W condition holds on the product manifold $M \times N$.*

Once we prove the A3S condition, we could study the regularity of the optimal transport maps. For this purpose, we use the PDE setting. If the potential function u is C^2, it satisfies the following fully nonlinear PDE

$$\det(\mathrm{Hess}^{(c)}(u)(x)) = \mathcal{B}(x, \nabla_x u), \qquad (1.10)$$

where

$$\mathrm{Hess}^{(c)}(u)(x) := \quad [\nabla d\, c(., q)]_{[x, \exp_x(\nabla u(x))]} \\ + \nabla d\, u(x),$$

and

$$\mathcal{B}(x, \nabla_x u) := \frac{\rho_0(x)}{\rho_1(G(x)) \det(\partial_p \exp_x(p)(x, \nabla_x u))}$$

An observation due to Delanoë [5, Lemma 3.5] is the stay away property, that is, if the potential u is C^2, then for all point $m \in M$ the image point $G(m)$ of the optimal transport map does not meet the cut locus Cut_m. This is some kind of the first order estimate for the potential u. With the help of Theorem 1, we prove the following regularity result.

THEOREM 2. — *Let (M, g) be a closed n-dimensional Riemannian manifold satisfying (1.3). Then there exists some universal positive constant $\bar{\varepsilon}_1$ such that if when $n \geqslant 3$*

$$\|Riem - \frac{Scal}{2n(n-1)} g \oslash g\|_{C^2(M,g)} < \bar{\varepsilon}_1$$

when $n = 2$

$$\|K - 1\|_{C^2(M,g)} < \bar{\varepsilon}_1$$

then for all given $(\rho_0 dvol, \rho_1 dvol)$ of $C^{k,\alpha}$ positive Borel probability measures on M with $k \geqslant 2$ and $\alpha \in (0, 1)$, the potential function of the optimal transport map is $C^{k+2,\alpha}$.

We use the continuity method to get the result. For this purpose, we need to establish the suitable estimates à priori. The main difficulties come from two parts: on one hand, we need to deduce the suitable C^2 estimates. This is solved by Ma-Trudinger-Wang [31] under the suitable positivity of $\mathcal{C}$-curvature. Thus, it is done with the help of result in Theorem 1; on the other hand, the optimal transport map G needs to avoid the cut locus, that is, for all $m \in M$, $G(m)$ is uniformly away from the cut-locus of m. For this aim, we choose a suitable new test function to obtain the maximum principle. Delanöe [6] communicates kindly to us that he has obtained some stability result on A3S condition on the homogenous manifolds with positive sectional curvature and proved smoothness result on such manifolds.

The paper is organized as follows. Section 1 presents the main results of the paper. Section 2 is devoted to recall some notions of Riemannian geometry. In Section 3 we prove Theorem 1, that is, the $\mathcal{C}$-curvature satisfies APCC condition. The proof relies on a careful analysis together with the perturbative arguments comparing to the constant curvature case. It is divided into three cases: near the origin, the intermediate case and near the focalization. The asymptotic expansion of the $\mathcal{C}$-curvature gives the improved MTW condition near the origin. In the intermediate case, the fact that the improved MTW condition holds on the sphere gives rise to the improved MTW condition on M. We adapt the method [17] to verify improved MTW condition near the cut-locus. The last section is devoted to the proof of Theorem 2 by continuity method. We collect some known results and prove some technical results related to Jocobi fields in Appendix.

2. Preliminaries

In this section, we collecte some notions in riemannian geometry [2–4,35] (see also [8,37]).

2.1. Basic notations and conventions

Recall a fact that the tensor $g \oslash g$ is parallel, i.e.

$$\nabla(g \oslash g) = 0. \tag{2.1}$$

By the condition (1.3) and taking trace of the Riemann curvature tensor one and two times we get the following relations

$$\|\mathrm{Ric} - (n-1)g\|_{C^2(M,g)} \simeq \varepsilon, \|\mathrm{Scal} - n(n-1)\|_{C^2(M,g)} \simeq \varepsilon. \tag{2.2}$$

The (3,1)-form of $\frac{1}{2}g \oslash g$ is denoted by $\bar{R}$, i.e.

$$\bar{R}(X,Y)Z = \langle Y, Z \rangle X - \langle X, Z \rangle Y,$$

which is also the curvature tensor on the sphere $\mathbb{S}^n$. The components of $\bar{R}$ are given by $\bar{R}(\frac{\partial}{\partial x^j}, \frac{\partial}{\partial x^k})\frac{\partial}{\partial x^i} = \bar{R}^l_{ijk}\frac{\partial}{\partial x^l}$.

By Bonnet Myers theorem [4], the normalization (1.3) implies (M, g) is compact and there is conjugate point along every geodesic. Moreover, there is cut point along every geodesic [2]. Given $m \in M$, $\forall \nu \in T_m M$ with $|\nu|_m = 1$, let $t_C(m, \nu)$ be the distance from point m to the cut point of m along the geodesic $\exp_m(t\nu)$, i.e.

$$t_C(m, \nu) = \sup\{t \geqslant 0; \exp_m(s\nu)|_{0 \leqslant s \leqslant t} \text{ is a minimizing geodesic}\}.$$

The injectivity domain at m is denoted by $I(m)$, i.e.

$$I(m) = \{t\nu; 0 \leqslant t < t_C(m, \nu), \nu \in T_m M \backslash \{0\}\}.$$

The focal time $t_F(m, \nu)$ is defined by

$$t_F(m, \nu) = \inf\{t \geqslant 0; \exp_m(t\nu) \text{is conjugate to } m\}.$$

We recall that the cut time is smaller than the focal time, the injectivity domain is an open subset contains the origin in $T_m M$ and star-shaped with respect to origin. Moreover, $M = \exp_m(I(m)) \bigsqcup \text{Cut}_m$, where $\bigsqcup$ means disjoint union. The exponential map $\exp_m : I(m) \to M \backslash \text{Cut}_m$ is a diffeomorphism. We denote by $\exp_m^{-1}(y)$ all the velocities $\nu \in T_m M$ such that the geodesic $\exp_m(s\nu)|_{0 \leqslant s \leqslant 1}$ is minimizing and $\exp_m \nu = y$.

The geometry of injectivity domain is complicated. But on some special manifolds they have special geometric properties.

We state $\frac{1}{2}d^2(\cdot, \cdot)$ is smooth in $M \times M \backslash \text{Cut}_M$. For any $y \notin \text{Cut}_m$, the Gauss lemma implies that

$$\nabla_m \frac{d^2}{2}(y, m) = -\exp_m^{-1} y. \tag{2.3}$$

Given a real smooth function u defined on M, the Hessian of u at m is given by the linear operator from $T_m M$ to $T_m M$ defined

$$\forall \xi \in T_m M, \nabla_m^2 u(\xi) := \nabla_\xi(\text{grad } u).$$

It is easy to see that the Hessian is a self adjoint and can be calculated as follows

$$\langle \nabla_m^2 u(\xi), \xi \rangle = \frac{d^2}{ds^2}|_{s=0} u(\gamma(s)), \tag{2.4}$$

where γ is a geodesic such that $\gamma(0) = m$ and $\dot{\gamma}(0) = \xi$, where $\dot{}$ is the derivative with respect to the real variable s.

2.2. Jacobi fields

DEFINITION 1. — *Given $m \in M$ and $\nu \in T_m M \backslash \{0\}$, let $\{E_1, E_2, \cdots, E_n\}$ be an orthonormal basis of $T_m M$ with $E_1 = \nu/|\nu|_m$. Let $\gamma(\cdot)$ be a geodesic with initial point m and initial velocity ν and $\{e_1, e_2, \cdots, e_n\}$ be the parallel transport of $\{E_1, E_2, \cdots, E_n\}$ along γ. We define $J_0(m, \nu, t), J_1(m, \nu, t)$ as the matrix valued solutions of the second order equation*

$$\begin{cases} \ddot{J}_a + R J_a = 0, a = 0, 1, \\ J_0(m, \nu, 0) = 0, \dot{J}_0(m, \nu, 0) = I_n, \\ J_1(m, \nu, 0) = I_n, \dot{J}_1(m, \nu, 0) = 0. \end{cases} \tag{2.5}$$

where the elements of R are given by

$$R_{ij}(t) = \langle R(e_i(t), \dot{\gamma}(t)) \dot{\gamma}(t), e_j(t) \rangle. \tag{2.6}$$

As same as on the sphere, we define $\bar{J}_a(m, \nu, t)$ as the matrix-valued of the second order equation

$$\ddot{\bar{J}}_a + \bar{R} \bar{J}_a = 0, a = 0, 1, \tag{2.7}$$

with the initial condition $\bar{J}_0(m, \nu, 0) = 0, \dot{\bar{J}}_0(m, \nu, 0) = I_n$ and $\bar{J}_1(m, \nu, 0) = I_n, \dot{\bar{J}}_1(m, \nu, 0) = 0$. The elements of $\bar{R}$ are given by

$$\bar{R}_{ij}(m, \nu, t) = \langle \bar{R}(e_i(t), \dot{\gamma}(t)) \dot{\gamma}(t), e_j(t) \rangle.$$

It is easy to see that $\bar{J}_0$ and $\bar{J}_1$ in the orthonormal basis $\{e_1, e_2, \cdots, e_n\}$ are given respectively by

$$\bar{J}_0 = \begin{bmatrix} t & 0 \\ 0 & \frac{\sin(|\nu|t)}{|\nu|} I_{n-1} \end{bmatrix}, \bar{J}_1 = \begin{bmatrix} 1 & 0 \\ 0 & \cos(|\nu|t) I_{n-1} \end{bmatrix}.$$

From the homogeneity of a geodesic (see [2, p.64]), we get the homogeneity of the Jacobi fields $\lambda J_0(m, \lambda\nu, t) = J_0(m, \nu, \lambda t), J_1(m, \lambda\nu, t) = J_1(m, \nu, \lambda t)$, $\forall \lambda > 0$. For $t \in [0, 1]$, we extended J_a by continuity at $\nu = 0$ by $J_1(m, 0, t) = t I_n$ and $J_1(m, 0, t) = I_n$. For simplicity, the Jacobi fields $J_a(m, \nu, t)$ are abbreviated to $J_a(t)$ unless otherwise specified. By the definition of conjugate points, the matrix $J_0(t)$ is invertible for $\forall t \in (0, t_F(m, \nu))$. Moreover, the continuity of the Jacobi fields implies [37] $\det J_0(t) > 0$ for $\forall t \in (0, t_F(m, \nu))$. We first state the fundamental formula which will be used in section 3 to calculate the $\mathcal{C}$-curvature, that is, the hessian of the squared distance can be expressed in terms of Jacobi fields and we have the representation formula for the inhomogeneous Jacobi equations. We recall some results in [17, 37].

PROPOSITION 1. — *Under the above assumptions, we have*

(a) *Given $\nu \in T_m M \backslash \{0\}$, let $J(t)$ be the Jacobi field along the geodesic $\exp_m(t\nu)$ determined by the conditions $J(0) = \xi, J(1) = 0$ and $J(t) \neq 0$ for all $0 < t < 1$. Then $J(t) = -J_0(t)J_0^{-1}(1)J_1(1)(\xi) + J_1(t)(\xi)$;*

(b) *For $t \in [0, t_F(m, \nu))$, let $S(m, \nu, t)$ be the linear operator from $T_m M$ to $T_m M$ whose matrix in the orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ is given by $tJ_0(t)^{-1}J_1(t)$. Then the linear operator $S(m, \nu, t) : T_m M \to T_m M$ is self adjoint. Moreover, if $\nu \in I(m)$, then for $\forall \xi \in T_m M$,*

$$\langle \nabla_m^2 c(\cdot, \exp_m \nu)(\xi), \xi \rangle = \langle S(m, \nu, 1)(\xi), \xi \rangle. \tag{2.8}$$

(c) *(Representation formula)The solution of the matrix valued inhomogeneous Jacobi equation*

$$\ddot{J}(t) + R(t)J(t) = B(t)$$

is given by the formula

$$J(t) = J_0(t)\dot{J}(0) + J_1(t)J(0) + J_0(t)\int_0^t J_1^* B ds - J_1(t)\int_0^t J_0^* B ds, \tag{2.9}$$

where J_a^* is the transpose (or adjoint) of the matrix J_a for $a = 0, 1$.

REMARK 1. — *1) (Homogeneity)From the homogeneity of the Jacobi fields, we have $S(m, \lambda\nu, t) = S(m, \nu, \lambda t)$ for $\lambda > 0$. Then we can extended S by continuity at $\nu = 0$ by $S(m, 0, t) = I_n$.*
2) The linear operator $S(m, \nu, t)$ has explicit formula on space forms [25], for instance, on the round sphere $\mathbb{S}^n$, $\bar{S}(m, \nu, t)(\xi) = \xi - (1 - t|\nu| \cot(t|\nu|))(\xi - \langle \xi, \frac{\nu}{|\nu|} \rangle \frac{\nu}{|\nu|})$.
3) Given some C^2 real valued function $f : I \subset \mathbb{R} \to \mathbb{R}$ defined on some interval I, we consider some ODE

$$\ddot{f} + f = \phi,$$

where $\dot{}$ is the derivative with respect to the real variable. Hence, it is known that the representation formula holds

$$f(t) = f(0) \cos t + \dot{f}(0) \sin t + \sin t \int_0^t \phi(s) \cos s ds - \cos t \int_0^t \phi(s) \sin s ds. \tag{2.10}$$

2.3. Fermi coordinate system

In this work, we use extensively the Fermi chart in the calculus. Here we recall the definition and some results in [8].

DEFINITION 2. — *(Fermi coordinate system) Let (M,g) be a n-dimensional Riemannian manifold. Given $(m_0, v_0) \in NoCut$ with $v_0 \neq 0$, let $I = [0, |v_0|] \subset \mathbb{R}$ be a compact interval. Let $\gamma : I \to M$ such that $\gamma(t) = \exp_{m_0}(tv_0/|v_0|)$ be a geodesic and $\{e_1(t), e_2(t), \cdots, e_n(t)\}$ be a parallel orthonormal moving frame of vector fields along the geodesic γ with $e_1(t) = \frac{\dot{\gamma}(t)}{|\dot{\gamma}(t)|}$. The Fermi coordinate system $\{x^1, \cdots, x^n\}$ are defined by*

$$x^1(\exp_{\gamma(t)}(\sum_{\beta=2}^{n} \lambda^\beta e_\beta(t))) = \frac{t}{|\dot{\gamma}(t)|},$$

$$x^\alpha(\exp_{\gamma(t)} \sum_{\beta=2}^{n} \lambda^\beta e_\beta(t)) = \lambda^\alpha, 2 \leqslant \alpha \leqslant n, t \in I.$$

where λ^β are sufficiently small so that the exponential maps are defined.

The Fermi coordinate system is generalization of the normal coordinate system. To see this, along the axis we have

$$\forall x^1 \in I, \forall i, j, k \in \{1, 2, \cdots, n\}, g_{ij}(x^1, 0) = \delta_{ij}, \partial_k g_{ij}(x^1, 0) = 0, \quad (2.11)$$

where δ is Kronecker symbol. In the following, the Latin indices run over $1, \cdots, n$ and the Greek indices run over $2, \cdots, n$. We calculate higher order derivatives of the metric and Christoffel symbols in such chart.

LEMMA 1. — *The following identities hold on the axis*

$$\partial_{ij}^2 g_{11} = -2R_{1i1j}, \partial_{\alpha\beta}^2 g_{1\mu} = -\frac{2}{3}(R_{\alpha 1 \beta \mu} + R_{\alpha\mu\beta 1}), \quad (2.12)$$

$$\partial_{\alpha\beta}^2 g_{\rho\mu} = -\frac{1}{3}(R_{\beta\rho\alpha\mu} + R_{\beta\mu\alpha\rho}), \quad (2.13)$$

$$\partial_k \Gamma_{1j}^i = R_{jk1}^i, \partial_\alpha \Gamma_{\beta\mu}^i = \frac{1}{3}(R_{\beta\alpha\mu}^i + R_{\mu\alpha\beta}^i), \quad (2.14)$$

$$\partial_{\alpha\beta}^2 \Gamma_{11}^i = \nabla_1 R_{\beta\alpha 1}^i + \nabla_\alpha R_{1\beta 1}^i, \quad (2.15)$$

$$\partial_{\alpha\beta}^2 \Gamma_{1\mu}^1 = \frac{1}{3}(\nabla_1 R_{\beta\alpha\mu}^1 - \nabla_1 R_{\mu\beta\alpha}^1) - \nabla_\alpha R_{\mu 1 \beta}^1, \quad (2.16)$$

$$\partial_{\alpha\beta}^2 \Gamma_{1\mu}^\rho = \frac{1}{2}(\nabla_\alpha R_{\mu\beta 1}^\rho + \nabla_\beta R_{\mu\alpha 1}^\rho) + \frac{1}{6}(\nabla_1 R_{\alpha\beta\mu}^\rho + \nabla_1 R_{\beta\alpha\mu}^\rho). \quad (2.17)$$

Proof. All identities are proved in [8, Lemma 2] except (2.13). However, its proof is same. We leave the detail for the readers. □

3. Proof of Theorem 1

We give first the expression of the $\mathcal{C}$-curvature. We adapt the presentation in [17]. The proof of theorem relies on a careful analysis together with the perturbative arguments comparing to the constant curvature case.

3.1. $\mathcal{C}$-curvature calculation in dimension n

In this subsection we calculate the $\mathcal{C}$-curvature. Fix $m_0 \in M, \nu_0 \in I(m_0)\backslash\{0\}$ and $(\xi, \eta) \in T_{m_0}M \times T_{m_0}M$. Since the $\mathcal{C}$-curvature is homogeneous with degree 2 in both ξ and η, it suffices to assume that $|\xi|_{m_0} = |\eta|_{m_0} = 1$. Fix an orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ of the tangent space $T_{m_0}M$ so that $\nu_0 = |\nu_0|_{m_0}E_1, \xi = \xi_1 E_1 + \xi_2 E_2 + \xi_3 E_3, \eta = \eta_1 E_1 + \eta_2 E_2$ and identify the tangent vectors at m_0 with their coordinates in this basis. Then the metric at the point m_0 is given by the canonical scalar product of $\mathbb{R}^n$. We denote $\tau = |\nu_0|$. It will be implicitly understood throughout the calculations that the inner product and the Riemann curvature are evaluated at the point m_0.

Combining (1.2) and (2.8), we have

$$\begin{aligned}
\mathcal{C}(m_0, \nu_0)(\xi, \eta) &= -\frac{3}{2}\frac{d^2}{ds^2}|_{s=0}\langle \nabla^2_{m_0} c(., \exp_{m_0}(\nu_0 + s\eta))(\xi), \xi\rangle_{m_0} \\
&= -\frac{3}{2}\frac{d^2}{ds^2}|_{s=0}\langle S(m_0, \nu_0 + s\eta, 1)(\xi), \xi\rangle_{m_0}.
\end{aligned}$$

For any $s \in \mathbb{R}$ small enough, we can write $\nu_0 + s\eta = t(s)(\cos\theta E_1 + \sin\theta E_2)$, where

$$t(s) = |\nu_0 + s\eta|_{m_0}, \theta(s) = \tan^{-1}\left(\frac{s\eta_2}{\tau + s\eta_1}\right).$$

Together with Remark 1, it follows that

$$\mathcal{C}(m_0, \nu_0)(\xi, \eta) = -\frac{3}{2}\frac{d^2}{ds^2}|_{s=0}\langle S(m_0, \cos\theta(s)E_1 + \sin\theta(s)E_2, t(s))(\xi), \xi\rangle_{m_0}.$$

We give some notations now. Let $\gamma_\theta(t)$ be the geodesic with initial point m_0 and initial velocity $\cos\theta E_1 + \sin\theta E_2$. For $|\theta|$ small, let $\{e_1(\theta, t), e_2(\theta, t), \cdots, e_n(\theta, t)\}$ be the parallel transport along the geodesic $\gamma_\theta(t)$ with $e_1(\theta, 0) = \cos\theta E_1 + \sin\theta E_2, e_2(\theta, 0) = -\sin\theta E_1 + \cos\theta E_2, e_i(\theta, 0) = E_i$ for $i \geqslant 3$. Let $J_0(\theta, t), J_1(\theta, t)$ be the solutions of the Jacobi equation (2.5) and $R(\theta, t)$ the curvature matrix by (2.6) along the geodesic $\gamma_\theta(t)$. The matrix of $S(m_0, e_1(\theta, 0), t)$ in the orthonormal basis $\{e_1(\theta, 0), e_2(\theta, 0), \cdots, e_n(\theta, 0)\}$ is given by

$$S(\theta, t) = tJ_0(\theta, t)^{-1}J_1(\theta, t).$$

Let

$$Q(\theta) = \begin{bmatrix} \cos\theta & \sin\theta & \\ -\sin\theta & \cos\theta & \\ & & I_{n-2} \end{bmatrix}.$$

Then the matrix of $S(m_0, e_1(\theta, 0), t)$ in the orthonormal basis $\{E_1, E_2, \cdots, E_n\}$ is $Q(\theta)^T S(\theta, t) Q(\theta)$, i.e.

$$\langle S(m_0, e_1(\theta, 0), t)(\xi), \xi \rangle = \langle S(\theta, t) Q(\theta)\xi, Q(\theta)\xi \rangle. \tag{3.1}$$

Here, the dot stands for the derivative with respect to t and the prime for the derivative with respect to θ.

Differentiating (3.1) once and twice with respect to s successively, we have

$$\begin{aligned} \frac{d}{ds}\langle S(m_0, e_1(\theta, 0), t)(\xi), \xi \rangle &= [\langle S'Q\xi, Q\xi \rangle + 2\langle SQ\xi, Q'\xi \rangle]\frac{d\theta}{ds} + \\ &\quad \langle \dot{S}Q\xi, Q\xi \rangle \frac{dt}{ds}, \end{aligned}$$

$$\begin{aligned} &\frac{d^2}{ds^2}\langle S(m_0, e_1(\theta, 0), t)(\xi), \xi \rangle \\ =\ & [\langle S''Q\xi, Q\xi \rangle + 4\langle S'Q\xi, Q'\xi \rangle + 2\langle SQ'\xi, Q'\xi \rangle + 2\langle SQ\xi, Q''\xi \rangle](\frac{d\theta}{ds})^2 \\ &+ [2\langle \dot{S}'Q\xi, Q\xi \rangle + 4\langle \dot{S}Q\xi, Q'\xi \rangle]\frac{dt}{ds}\frac{d\theta}{ds} + \langle \ddot{S}Q\xi, Q\xi \rangle(\frac{dt}{ds})^2 \\ &+ [\langle S'Q\xi, Q\xi \rangle + 2\langle SQ\xi, Q'\xi \rangle]\frac{d^2\theta}{ds^2} + \langle \dot{S}Q\xi, Q\xi \rangle\frac{d^2t}{ds^2}. \end{aligned} \tag{3.2}$$

The direct computations lead to

$$\begin{aligned} \frac{dt}{ds} &= \frac{s(\eta_1^2 + \eta_2^2) + \tau\eta_1}{|\nu_0 + s\eta|}, \frac{d^2t}{ds^2} = \frac{\tau^2\eta_2^2}{|\nu_0 + s\eta|^3}, \\ \frac{d\theta}{ds} &= \frac{\tau\eta_2}{s^2(\eta_1^2 + \eta_2^2) + 2s\tau\eta_1 + \tau^2}, \frac{d^2\theta}{ds^2} = -\frac{2\tau\eta_2[s(\eta_1^2 + \eta_2^2) + \tau\eta_1]}{[s^2(\eta_1^2 + \eta_2^2) + 2s\tau\eta_1 + \tau^2]^2}, \end{aligned}$$

so that at $s = 0$, we have

$$t = \tau, \frac{dt}{ds} = \eta_1, \frac{d^2t}{ds^2} = \frac{\eta_2^2}{\tau}, \theta = 0, \frac{d\theta}{ds} = \frac{\eta_2}{\tau}, \frac{d^2\theta}{ds^2} = -\frac{2\eta_1\eta_2}{\tau^2}. \tag{3.3}$$

Writing

$$P\xi = (\xi_1, \xi_2, 0, \cdots, 0)^T, P^\perp\xi = (\xi_2, -\xi_1, 0, \cdots, 0)^T,$$

then at $s = 0$, we have

$$Q\xi = \xi, Q'\xi = P^\perp\xi, Q''\xi = -P\xi. \tag{3.4}$$

Gathering (3.2), (3.3) and (3.4), we obtain

$$\frac{d^2}{ds^2}|_{s=0}\langle S(m_0, e_1(\theta, 0), t)(\xi), \xi\rangle$$

$$= \ [\langle S''\xi, \xi\rangle + 4\langle S'\xi, P^\perp\xi\rangle + 2\langle SP^\perp\xi, P^\perp\xi\rangle - 2\langle S\xi, P\xi\rangle]\frac{\eta_2^2}{\tau^2} +$$

$$[2\langle \dot{S}'\xi, \xi\rangle + 4\langle \dot{S}\xi, P^\perp\xi\rangle]\frac{\eta_1\eta_2}{\tau} + \langle \ddot{S}\xi, \xi\rangle\eta_1^2 +$$

$$[\langle S'\xi, \xi\rangle + 2\langle S\xi, P^\perp\xi\rangle](-\frac{2\eta_1\eta_2}{\tau^2}) + \langle \dot{S}\xi, \xi\rangle\frac{\eta_2^2}{\tau}$$

$$= \ \langle \ddot{S}\xi, \xi\rangle\eta_1^2 + [\frac{2}{\tau}\langle \dot{S}'\xi, \xi\rangle + \frac{4}{\tau}\langle \dot{S}\xi, P^\perp\xi\rangle - \frac{2}{\tau^2}\langle S'\xi, \xi\rangle -$$

$$\frac{4}{\tau^2}\langle S\xi, P^\perp\xi\rangle]\eta_1\eta_2 + [\frac{1}{\tau^2}\langle S''\xi, \xi\rangle + \frac{1}{\tau}\langle \dot{S}\xi, \xi\rangle +$$

$$\frac{4}{\tau^2}\langle S'\xi, P^\perp\xi\rangle + \frac{2}{\tau^2}\langle SP^\perp\xi, P^\perp\xi\rangle - \frac{2}{\tau^2}\langle S\xi, P\xi\rangle]\eta_2^2$$

which yields

$$\mathcal{C}(m_0, \nu_0)(\xi, \eta)$$
$$= \ -\frac{3}{2}\langle \ddot{S}\xi, \xi\rangle\eta_1^2 + 3[-\frac{1}{\tau}\langle \dot{S}'\xi, \xi\rangle - \frac{2}{\tau}\langle \dot{S}\xi, P^\perp\xi\rangle + \frac{1}{\tau^2}\langle S'\xi, \xi\rangle +$$
$$\frac{2}{\tau^2}\langle S\xi, P^\perp\xi\rangle]\eta_1\eta_2 + \frac{3}{2}[-\frac{1}{\tau^2}\langle S''\xi, \xi\rangle - \frac{1}{\tau}\langle \dot{S}\xi, \xi\rangle - \frac{4}{\tau^2}\langle S'\xi, P^\perp\xi\rangle -$$
$$\frac{2}{\tau^2}\langle SP^\perp\xi, P^\perp\xi\rangle + \frac{2}{\tau^2}\langle S\xi, P\xi\rangle]\eta_2^2$$
$$= \ a_{11}(m_0, \nu_0, \xi)\eta_1^2 + a_{12}(m_0, \nu_0, \xi)\eta_1\eta_2 + a_{22}(m_0, \nu_0, \xi)\eta_2^2,$$

where

$$a_{11}(m_0, \nu_0, \xi) := \ -\frac{3}{2}\langle \ddot{S}\xi, \xi\rangle$$
$$a_{12}(m_0, \nu_0, \xi) := \ 3[-\frac{1}{\tau}\langle \dot{S}'\xi, \xi\rangle - \frac{2}{\tau}\langle \dot{S}\xi, P^\perp\xi\rangle + \frac{1}{\tau^2}\langle S'\xi, \xi\rangle + \quad \frac{2}{\tau^2}\langle S\xi, P^\perp\xi\rangle]$$
$$a_{22}(m_0, \nu_0, \xi) := \ \frac{3}{2}[-\frac{1}{\tau^2}\langle S''\xi, \xi\rangle - \frac{1}{\tau}\langle \dot{S}\xi, \xi\rangle$$
$$-\frac{4}{\tau^2}\langle S'\xi, P^\perp\xi\rangle - \frac{2}{\tau^2}\langle SP^\perp\xi, P^\perp\xi\rangle + \frac{2}{\tau^2}\langle S\xi, P\xi\rangle]$$

3.2. Proof of Theorem 1 and Corollary 1

The proof of Theorem 1 is divided into three cases: near the origin, the intermediate case and near the focalization. The asymptotic behaviour gives the improved MTW condition near the origin. In the intermediate case, we use the similar strategy in [8]: the fact that the improved MTW condition holds on the sphere gives rise to the improved MTW condition on M. We adapt the method [17] to verify improved MTW condition near the cut-locus.

3.2.1. Asymptotic behaviour near the origin

With the help of the above preparations, we can obtain our main result in this subsection as follows.

THEOREM 3. — *Under the same assumptions as in Theorem 1, there exist some universal positive constants $\tilde{\varepsilon}_0, \delta_1, \kappa_1$ such that if $\varepsilon < \tilde{\varepsilon}_0$, then for all $(m_0, \nu_0) \in NoCut$ with $\tau = |\nu_0| < \delta_1$ and all tangent vectors ξ, η in $T_{m_0} M$*

$$\mathcal{C}(m_0, \nu_0)(\xi, \eta) \geqslant \kappa_1 (|\xi \wedge \eta|^2_{m_0} + |\xi|^2_{m_0} |\eta \wedge \nu_0|^2_{m_0} + |\xi \wedge \nu_0|^2_{m_0} |\eta|^2_{m_0}). \quad (3.5)$$

Proof. We deduce first the expansion of the $\mathcal{C}$-curvature near the origin. For this purpose, by (B.25)(B.26)(B.27) below, we have

$$\mathcal{C}(m_0, \nu_0)(\xi, \eta)$$

$$= \quad [R(\xi, E_1, \xi, E_1) + \tfrac{3}{4}\tau(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 +$$
$$\nabla_1 R_{1313}\xi_3^2) + \tfrac{2}{5}\tau^2(\xi_2^2 + \xi_3^2) + O(\varepsilon\tau^2 + \tau^4)(\xi_2^2 + \xi_3^2)]\eta_1^2 +$$
$$\{2R(\xi, E_1, \xi, E_2) + \tau[\tfrac{1}{2}\nabla_2 R_{1212}\xi_2^2 + (\nabla_1 R_{1223} +$$
$$\nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \tfrac{1}{2}\nabla_2 R_{1313})\xi_3^2 -$$
$$\nabla_1 R_{1212}\xi_1\xi_2 - \nabla_1 R_{1213}\xi_1\xi_3] - \tfrac{2}{5}\tau^2\xi_1\xi_2 +$$
$$O(\varepsilon\tau^2 + \tau^3)(\xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3)\}\eta_1\eta_2 +$$
$$\{R(\xi, E_2, \xi, E_2) + \tau[\tfrac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \tfrac{1}{2}\nabla_2 R_{1212}\xi_1\xi_2 -$$
$$\tfrac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 + \tfrac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 +$$
$$(\tfrac{1}{4}\nabla_1 R_{2323} + \tfrac{1}{2}\nabla_2 R_{1323})\xi_3^2] + \tfrac{\tau^2}{15}(\xi_1^2 + \xi_2^2 + 2\xi_3^2) +$$
$$O(\varepsilon\tau^2 + \tau^4)\}\eta_2^2$$

$$= \quad R(\xi, E_1, \xi, E_1)\eta_1^2 + 2R(\xi, E_1, \xi, E_2)\eta_1\eta_2 + R(\xi, E_2, \xi, E_2)\eta_2^2 +$$
$$\tfrac{3}{4}\tau(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \nabla_1 R_{1313}\xi_3^2)\eta_1^2 +$$
$$\tau[\tfrac{1}{2}\nabla_2 R_{1212}\xi_2^2 + (\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 +$$
$$(\nabla_1 R_{1323} + \tfrac{1}{2}\nabla_2 R_{1313})\xi_3^2 - \nabla_1 R_{1212}\xi_1\xi_2 - \nabla_1 R_{1213}\xi_1\xi_3]\eta_1\eta_2 +$$
$$\tau[\tfrac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \tfrac{1}{2}\nabla_2 R_{1212}\xi_1\xi_2 - \tfrac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 +$$
$$\tfrac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + (\tfrac{1}{4}\nabla_1 R_{2323} + \tfrac{1}{2}\nabla_2 R_{1323})\xi_3^2]\eta_2^2 +$$
$$\tau^2(\tfrac{1}{15}\xi_1^2\eta_2^2 - \tfrac{2}{5}\xi_1\xi_2\eta_1\eta_2 + \tfrac{2}{5}\xi_2^2\eta_1^2 + \tfrac{1}{15}\xi_2^2\eta_2^2 + \tfrac{2}{5}\xi_3^2\eta_1^2 + \tfrac{2}{15}\xi_3^2\eta_2^2) +$$
$$O(\varepsilon\tau^2 + \tau^4)(\xi_1\xi_2\eta_1\eta_2 + \xi_1\xi_3\eta_1\eta_2 + \xi_2^2\eta_1^2 +$$
$$\xi_2^2\eta_1\eta_2 + \xi_3^2\eta_1^2 + \xi_3^2\eta_1\eta_2 + \eta_2^2)$$

$$= \quad I_1 + II_1 + III_1 + IV_1, \quad (3.6)$$

where I_1 (resp. II_1, III_1, IV_1) collects all terms containing 0-power (resp. 1-power, 2-power and higher order powers) in τ.

We estimate term by term from I_1 to IV_1. The key point is the fact that the combination of the zero order term and the second order term controls all others.

The term I_1 It is clear that $I_1 = R_{m_0}(\xi, \eta, \xi, \eta)$. Recall the curvature approximation (1.3). Thus the term I_1 has the lower bound

$$
\begin{aligned}
I_1 &\geqslant |\xi|^2|\eta|^2 - \langle \xi, \eta \rangle^2 \qquad\qquad (3.7)\\
&= (\xi_1\eta_2 - \xi_2\eta_1)^2 + \xi_3^2(\eta_1^2 + \eta_2^2).
\end{aligned}
$$

The term II_1 The term II_1 involves the first order parts $\tau\xi_1^2\eta_2^2, \tau\xi_2^2\eta_1^2, \tau\xi_1\xi_2\eta_2^2$ and $\tau\xi_2^2\eta_1\eta_2$. They cann't be directly controlled by the associated terms in the second order, but the combination of them composes good terms, more precisely, the term II_1 can be stated as

$$
\begin{aligned}
II_1 &= \frac{3}{4}\tau\nabla_1 R_{1212}\xi_2^2\eta_1^2 - \tau\nabla_1 R_{1212}\xi_1\xi_2\eta_1\eta_2 + \frac{\tau}{4}\nabla_1 R_{1212}\xi_1^2\eta_2^2 +\\
&\quad \frac{\tau}{2}\nabla_2 R_{1212}(\xi_2^2\eta_1\eta_2 - \xi_1\xi_2\eta_2^2) + \frac{3}{2}\tau\nabla_1 R_{1213}\xi_2\xi_3\eta_1^2 + \frac{3}{4}\tau\nabla_1 R_{1313}\xi_3^2\eta_1^2 +\\
&\quad \tau[(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \frac{1}{2}\nabla_2 R_{1313})\xi_3^2 -\\
&\quad \nabla_1 R_{1213}\xi_1\xi_3]\eta_1\eta_2 + \tau[-\frac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 +\\
&\quad \frac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + (\frac{1}{4}\nabla_1 R_{2323} + \frac{1}{2}\nabla_2 R_{1323})\xi_3^2]\eta_2^2 +\\
&= \frac{\tau}{4}\nabla_1 R_{1212}(\xi_2\eta_1 - \xi_1\eta_2)^2 + \frac{\tau}{2}\nabla_1 R_{1212}(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_1 +\\
&\quad \frac{\tau}{2}\nabla_2 R_{1212}(\xi_2\eta_1 - \xi_1\eta_2)\xi_2\eta_2 + \frac{3}{2}\tau\nabla_1 R_{1213}\xi_2\xi_3\eta_1^2 + \frac{3}{4}\tau\nabla_1 R_{1313}\xi_3^2\eta_1^2 +\\
&\quad \tau[(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \frac{1}{2}\nabla_2 R_{1313})\xi_3^2 -\\
&\quad \nabla_1 R_{1213}\xi_1\xi_3]\eta_1\eta_2 + \tau[-\frac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 +\\
&\quad \frac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + (\frac{1}{4}\nabla_1 R_{2323} + \frac{1}{2}\nabla_2 R_{1323})\xi_3^2]\eta_2^2
\end{aligned}
$$

Using the curvature approximation (1.6), Cauchy Schwartz inequality and the parallel property (2.1), we estimate the term II_1 as follows

$$
\begin{aligned}
II_1 &\geqslant -\varepsilon(\frac{\tau}{4} + \frac{1}{2})(\xi_1\eta_2 - \xi_2\eta_1)^2 - \varepsilon\tau^2\xi_1^2\eta_2^2 - 2\varepsilon\tau^2\xi_2^2\eta_1^2 -\\
&\quad \frac{1}{2}\varepsilon\tau^2\xi_2^2\eta_2^2 - \varepsilon(\frac{5}{4} + \frac{3}{2}\tau)\xi_3^2\eta_1^2 - \varepsilon(\frac{7}{4} + \frac{3}{2}\tau)\xi_3^2\eta_2^2.
\end{aligned}
$$

Here we drop the upper indicator for $\bar{\varepsilon}$ in (1.6). Now assume $0 < \tau < 2$, the following inequality holds

$$II_1 \geqslant -\varepsilon(\xi_1\eta_2 - \xi_2\eta_1)^2 - \varepsilon\tau^2\xi_1^2\eta_2^2 - \frac{5}{4}\varepsilon\tau^2\xi_2^2\eta_1^2 - \qquad (3.8)$$
$$\frac{1}{2}\varepsilon\tau^2\xi_2^2\eta_2^2 - \varepsilon(\frac{5}{4} + \frac{3}{2}\tau)\xi_3^2\eta_1^2 - \varepsilon(\frac{7}{4} + \frac{3}{2}\tau)\xi_3^2\eta_2^2.$$

The term IV_1 The remainder error term is negligible comparing with the second order of τ when ε and τ are small. Let ε_1 be some small positive real number to be fixed later. Assume $0 < \varepsilon < \frac{\varepsilon_1}{2}, 0 < \tau < \frac{\varepsilon_1}{2}$ such that

$$IV_1 = O(\varepsilon\tau^2 + \tau^4)(\xi_1^2\eta_2^2 + \xi_2^2\eta_1^2 + \xi_2^2\eta_2^2 + \xi_3^2\eta_1^2 + \xi_3^2\eta_2^2 + \eta_2^2) \quad (3.9)$$
$$\geqslant -\varepsilon_1\tau^2(\xi_2^2\eta_1^2 + \xi_3^2\eta_1^2 + \eta_2^2).$$

Here we use the fact $\xi_1^2 + \xi_2^2 + \xi_3^2 = 1$. Gathering (3.6), (3.7), (3.8) and (3.9), we estimate

$$
\begin{aligned}
\mathcal{C}(m_0, \nu_0)(\xi, \eta) &\geqslant (\xi_1\eta_2 - \xi_2\eta_1)^2 + \xi_3^2(\eta_1^2 + \eta_2^2) - \frac{1}{2}\varepsilon_1(\xi_1\eta_2 - \xi_2\eta_1)^2 - \\
&\quad \frac{1}{2}\varepsilon_1\tau^2\xi_1^2\eta_2^2 - \frac{5}{8}\varepsilon_1\tau^2\xi_2^2\eta_1^2 - \frac{1}{4}\varepsilon_1\tau^2\xi_2^2\eta_2^2 - \varepsilon_1(\frac{5}{8} + \frac{3}{4}\tau)\xi_3^2\eta_1^2 - \\
&\quad \varepsilon_1(\frac{7}{8} + \frac{3}{4}\tau)\xi_3^2\eta_2^2 + \tau^2(\frac{1}{15}\xi_1^2\eta_2^2 - \frac{2}{5}\xi_1\xi_2\eta_1\eta_2 + \frac{2}{5}\xi_2^2\eta_1^2 + \\
&\quad \frac{1}{15}\xi_2^2\eta_2^2 + \frac{2}{5}\xi_3^2\eta_1^2 + \frac{2}{15}\xi_3^2\eta_2^2) - \varepsilon_1\tau^2(\xi_2^2\eta_1^2 + \xi_3^2\eta_1^2 + \eta_2^2) \\
&= (1 - \frac{\varepsilon_1}{2})(\xi_1\eta_2 - \xi_2\eta_1)^2 + (\frac{1}{15} - \frac{\varepsilon_1}{2})\tau^2\xi_1^2\eta_2^2 - \frac{2}{5}\tau^2\xi_1\xi_2\eta_1\eta_2 + \\
&\quad (\frac{2}{5} - \frac{5}{8}\varepsilon_1)\tau^2\xi_2^2\eta_1^2 + (\frac{1}{15} - \frac{\varepsilon_1}{4})\tau^2\xi_2^2\eta_2^2 + \\
&\quad [\frac{2}{5}\tau^2 - \varepsilon_1(\frac{5}{8} + \frac{3}{4}\tau) + 1]\xi_3^2\eta_1^2 + \\
&\quad [\frac{2}{15}\tau^2 - \varepsilon_1(\frac{7}{8} + \frac{3}{4}\tau) + 1]\xi_3^2\eta_2^2 - \varepsilon_1\tau^2(\xi_2^2\eta_1^2 + \xi_3^2\eta_1^2 + \eta_2^2) \\
&= [1 - \varepsilon_1 + (\frac{1}{15} - 2\varepsilon_1)\tau^2]\xi_1^2\eta_2^2 - 2(1 - \varepsilon_1 + \frac{\tau^2}{5})\xi_1\xi_2\eta_1\eta_2 + \\
&\quad [1 - \varepsilon_1 + (\frac{2}{5} - 2\varepsilon_1)\tau^2]\xi_2^2\eta_1^2 + \frac{\varepsilon_1}{2}(\xi_1\eta_2 - \xi_2\eta_1)^2 + \\
&\quad [1 - \varepsilon_1(\frac{5}{8} + \frac{3}{4}\tau + \tau^2)]\xi_3^2\eta_1^2 + [1 - \varepsilon_1(\frac{7}{8} + \frac{3}{4}\tau)]\xi_3^2\eta_2^2 + \\
&\quad \frac{3}{2}\varepsilon_1\tau^2\xi_1^2\eta_2^2 + \frac{3}{8}\varepsilon_1\tau^2\xi_2^2\eta_1^2 + (\frac{1}{15} - \frac{\varepsilon_1}{4})\tau^2\xi_2^2\eta_2^2 + \\
&\quad \frac{2}{5}\tau^2\xi_3^2\eta_1^2 + \frac{2}{15}\tau^2\xi_3^2\eta_2^2 - \varepsilon_1\tau^2\eta_2^2.
\end{aligned}
$$

The discriminant of the quadratic polynomial $h_{\varepsilon,\tau}(t) = [1 - \varepsilon + (\frac{1}{15} - 2\varepsilon)\tau^2]t^2 - 2(1 - \varepsilon + \frac{\tau^2}{5})t + 1 - \varepsilon + (\frac{2}{5} - 2\varepsilon)\tau^2$ is polynomial with arguments (ε, τ). By continuity , there exists a small positive constant $\tilde{\varepsilon}_1$ such that the discriminant is non-positive in $[0, \tilde{\varepsilon}_1] \times [0, \tilde{\varepsilon}_1]$. Together with the fact $\xi_1^2 + \xi_2^2 + \xi_2^3 = 1$, we have

$$
\begin{aligned}
\mathcal{C}(m_0, \nu_0)(\xi, \eta) \geqslant\ & \frac{\varepsilon_1}{2}(\xi_1\eta_2 - \xi_2\eta_1)^2 + [1 - \varepsilon_1(\frac{5}{8} + \frac{3}{4}\tau + \tau^2)]\xi_3^2\eta_1^2 + \\
& [1 - \varepsilon_1(\frac{7}{8} + \frac{3}{4}\tau)]\xi_3^2\eta_2^2 + \frac{1}{2}\varepsilon_1\tau^2\xi_1^2\eta_2^2 + \frac{3}{8}\varepsilon_1\tau^2\xi_2^2\eta_1^2 + \\
& (\frac{1}{15} - \frac{5\varepsilon_1}{4})\tau^2\xi_2^2\eta_2^2 + \frac{2}{5}\tau^2\xi_3^2\eta_1^2 + (\frac{2}{15} - \varepsilon_1)\tau^2\xi_3^2\eta_2^2.
\end{aligned}
$$

Thus, we could choose some small $\varepsilon_1 < \tilde{\varepsilon}_1$ such that for any $\tau < \frac{\varepsilon_1}{2}$ there holds $1 - \varepsilon_1(\frac{5}{8} + \frac{3}{4}\tau + \tau^2) > 0$ and $1 - \varepsilon_1(\frac{7}{8} + \frac{3}{4}\tau) > 0$. As a consequence, there exist the constants $\tilde{\varepsilon}_0, \kappa_1$ and δ_1 such that $\forall \varepsilon < \tilde{\varepsilon}_0$ and $\forall \nu_0 \in T_{m_0}M$ with $|\nu_0| < \delta_1$, we have

$$
\mathcal{C}(m_0, \nu_0)(\xi, \eta) \geqslant \kappa_1 \left[(|\xi|^2|\eta|^2 - \langle\xi, \eta\rangle^2) + |\nu_0|^2(\xi_2^2 + \xi_3^2)|\eta|^2 + |\nu_0|^2|\xi|^2\eta_2^2\right].
$$

Finally, we prove Theorem 3. $\qquad\square$

3.2.2. Behaviour near the focalization

Under the curvature assumption (1.3) and (1.4) (or (1.5)) with ε small enough, we can obtain by the method in [17] near the focalization.

THEOREM 4. — *Under the same assumptions as in Theorem 1, there exist some universal positive constants $\tilde{\varepsilon}_{0,1}, \delta_2 \in (\frac{3\pi}{4}, \pi), \kappa_{1,1}$ such that if $\varepsilon < \tilde{\varepsilon}_{0,1}$, then for all $(m_0, \nu_0) \in NoCut$ with $\delta_2 < \tau = |\nu_0| < t_F(m_0, \nu_0)$ and for all tangent vectors ξ, η in $T_{m_0}M$*

$$
\mathcal{C}(m_0, \nu_0)(\xi, \eta) \geqslant \kappa_{1,1}(|\xi \wedge \eta|_{m_0}^2 + |\xi|_{m_0}^2|\eta \wedge \nu_0|_{m_0}^2 + |\xi \wedge \nu_0|_{m_0}^2|\eta|_{m_0}^2). \tag{3.10}
$$

Proof. In view of [17, (5.22)], there exist positive numbers $\kappa_2, \varepsilon_2, \delta_2 \in (\frac{3\pi}{4}, \pi)$ such that if $\varepsilon < \varepsilon_2$, then for all $(m_0, \nu_0) \in \text{NoCut}$ with $\delta_2 < \tau = |\nu_0| < t_F(m_0, \nu_0)$ and for all tangent vectors ξ, η in $T_{m_0}M$ with $|\xi| = |\eta| = 1$, the $\mathcal{C}$-curvature has the following estimate

$$
\mathcal{C}(m_0, \nu_0)(\xi, \eta) \geqslant \kappa_2(|S^\perp\xi|^2|\eta|^2 + \xi_1^2\eta_2^2), \tag{3.11}
$$

where $S^\perp$ denotes the orthogonal projection of S on the orthogonal subspace $\nu_0^\perp$.

From the curvature assumption (1.3), the Hessian comparison theorem [4] infers

$$-S^\perp \geqslant -\frac{|\nu_0|\cos|\nu_0|}{\sin|\nu_0|}I_{n-1}.$$

Hence the term $|S^\perp\xi|^2$ controls $\xi_2^2 + \xi_3^2$ if $|\nu_0| \geqslant \frac{3\pi}{4}$, i.e. $|S^\perp\xi|^2 \geqslant 2(\xi_2^2 + \xi_3^2)$. By Cauchy-Schwarz inequality, we estimate

$$\frac{1}{2}(\xi_1^2\eta_2^2 + \xi_2^2\eta_1^2) \geqslant \frac{1}{4}(\xi_1\eta_2 - \xi_2\eta_1)^2.$$

Thus, it follows from (3.11),

$$
\begin{aligned}
\mathcal{C}(m_0,\nu_0)(\xi,\eta) &\geqslant \kappa_2\left[\frac{1}{2}\xi_1^2\eta_2^2 + \frac{1}{4}(\xi_1\eta_2 - \xi_2\eta_1)^2 + (\xi_2^2 + \xi_3^2)|\eta|^2\right]\\
&\geqslant \kappa_2\left[\frac{1}{2}\xi_1^2\eta_2^2 + \frac{1}{4}\xi_2^2\eta_1^2 - \frac{1}{2}\xi_1\xi_2\eta_1\eta_2 + (\xi_2^2 + \xi_3^2)|\eta|^2\right]\\
&= \kappa_2\left[\frac{1}{4}(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2) + \frac{1}{4}\xi_1^2\eta_2^2 + \xi_2^2|\eta|^2 + \frac{3}{4}\xi_3^2|\eta|^2\right]\\
&\geqslant \kappa_2\left[(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2) + (\xi_2^2 + \xi_3^2)|\eta|^2 + |\xi|^2\eta_2^2\right]\\
&\geqslant C\kappa_2\left[(|\xi|^2|\eta|^2 - \langle\xi,\eta\rangle^2) + |\nu_0|^2(\xi_2^2 + \xi_3^2)|\eta|^2 + |\nu_0|^2|\xi|^2\eta_2^2\right].
\end{aligned}
$$

Therefore, there exist positive constants $\tilde\varepsilon_{0,1} < \varepsilon_2, \kappa_{1,1} < C\kappa_2$ such that (3.10) holds. Finally, the proof of Theorem 4 is complete. $\square$

3.2.3. Behaviour in the intermediate case

In this subsection we assume $\delta_1 \leqslant |\nu_0| \leqslant \delta_2 < t_F(m_0,\nu_0)$, where the positive constant δ_2 is strictly smaller than π and given in Theorem 4, and the positive constant δ_1 is given in Theorem 3. We adapt the same strategy in [8, Theorem 2] to get the stability result. For convenience of readers, we recast as follows.

PROPOSITION 2. — *Let (M,g) be a closed n-dimensional Riemannian manifold satisfying (1.3) and (1.4) (or (1.5)) with $\varepsilon < \gamma$. Let $(m_0,\nu_0) \in NoCut$ with $\tau = |\nu_0| \leqslant \delta_2$. Assume γ is small enough such that $\delta_2 < t_F(m_0,\nu_0)$ and*

$$\frac{|\nu_0|}{\sin|\nu_0|}\gamma \leqslant \frac{1}{4\sqrt{n-1}}.$$

Then there exists a positive constant $C_1 \geqslant 1$ under control (independent of (m_0,ν_0,γ)) such that for all $\xi,\eta \in T_{m_0}M$ the following inequality holds

$$|\mathcal{C}(m_0,\nu_0)(\xi,\eta) - \bar{\mathcal{C}}(m_0,\nu_0)(\xi,\eta)| \leqslant C_1\left(\frac{|\nu_0|}{\sin|\nu_0|}\right)^4\gamma(|\xi^\perp|^2 + |\eta^\perp|^2),$$

where $\xi^\perp$ (resp. $\eta^\perp$) is orthogonal projection of ξ (resp. η) on the orthogonal subspace $\nu_0^\perp$.

Proof. The $\mathcal{C}$-curvature difference has the following expression

$$\mathcal{C}(m_0, \nu_0)(\xi, \eta) - \bar{\mathcal{C}}(m_0, \nu_0)(\xi, \eta) \tag{3.12}$$

$$= -\frac{3}{2}\langle(\ddot{S} - \dot{\bar{S}})\xi, \xi\rangle\eta_1^2 + 3[-\frac{1}{\tau}\langle\dot{S}'\xi, \xi\rangle + \frac{1}{\tau^2}\langle S'\xi, \xi\rangle - \frac{2}{\tau}\langle(\dot{S} - \dot{\bar{S}})\xi, P\xi^\perp\rangle +$$

$$\frac{2}{\tau^2}\langle(S - \bar{S})\xi, P\xi^\perp\rangle]\eta_1\eta_2 + \frac{3}{2}[-\frac{1}{\tau^2}\langle S''\xi, \xi\rangle - \frac{1}{\tau}\langle(\dot{S} - \dot{\bar{S}})\xi, \xi\rangle -$$

$$\frac{4}{\tau^2}\langle S'P\xi^\perp, \xi\rangle - \frac{2}{\tau^2}\langle(S - \bar{S})P\xi^\perp, P\xi^\perp\rangle + \frac{2}{\tau^2}\langle(S - \bar{S})\xi, P\xi\rangle]\eta_2^2.$$

Plugging (B.28) and (B.29) into (3.12) we get the result. Therefore, Proposition 2 is proved. $\square$

As a consequence, we get the stability result as follows.

THEOREM 5. — *Under the same assumptions as in Theorem 1, there exist some universal positive constants $\tilde{\varepsilon}_{0,2}, \kappa_{1,2}$ such that if $\varepsilon < \tilde{\varepsilon}_{0,2}$, then for all $(m_0, \nu_0) \in NoCut$ with $\delta_1 \leqslant \tau = |\nu_0| \leqslant \delta_2 < t_F(m_0, \nu_0)$ and for all tangent vectors ξ, η in $T_{m_0}M$*

$$\mathcal{C}(m_0, \nu_0)(\xi, \eta) \geqslant \kappa_{1,2}(|\xi \wedge \eta|_{m_0}^2 + |\xi|_{m_0}^2|\eta \wedge \nu_0|_{m_0}^2 + |\xi \wedge \nu_0|_{m_0}^2|\eta|_{m_0}^2). \tag{3.13}$$

Proof. Assume $0 < \varepsilon < \frac{\sin\delta_2}{4\delta_2\sqrt{n-1}}$. We note the function $\frac{t}{\sin t}$ is nondecreasing in the interval $[0, \pi)$, and the condition in Proposition 2 is satisfied. Thus, we have

$$\mathcal{C}(m_0, \nu_0)(\xi, \eta) \;\geqslant\; \bar{\mathcal{C}}(m_0, \nu_0)(\xi, \eta) - C_1(\frac{\delta_2}{\sin\delta_2})^4\varepsilon(\xi_2^2 + \xi_3^2 + \eta_2^2)$$

$$\geqslant\; \bar{\kappa}_2(|\xi|^2|\eta|^2 - \langle\xi, \eta\rangle^2) + (\tilde{\kappa}_0\tau^2 - \frac{C_1\delta_2^4}{\sin^4\delta_2}\varepsilon)(\xi_2^2 + \xi_3^2 + \eta_2^2)$$

$$\geqslant\; \bar{\kappa}_2(|\xi|^2|\eta|^2 - \langle\xi, \eta\rangle^2) + (\tilde{\kappa}_0 - \frac{C_1\delta_2^4}{\delta_1^2\sin^4\delta_2}\varepsilon)\tau^2(\xi_2^2 + \xi_3^2 + \eta_2^2)$$

$$\geqslant\; \frac{\bar{\kappa}_2}{2}[|\xi|^2|\eta|^2 - \langle\xi, \eta\rangle^2 + \tau^2(\xi_2^2 + \xi_3^2 + \eta_2^2)]$$

$$\geqslant\; \kappa_{1,2}\left[(|\xi|^2|\eta|^2 - \langle\xi, \eta\rangle^2) + |\nu_0|^2(\xi_2^2 + \xi_3^2)|\eta|^2 + |\nu_0|^2|\xi|^2\eta_2^2\right].$$

Here we choose $\varepsilon < \tilde{\varepsilon}_{1,2} := \min(\frac{\bar{\kappa}_2\delta_1^3\sin^4\delta_2}{2C_1\delta_2^4}, \frac{\sin\delta_2}{4\delta_2\sqrt{n-1}})$ and $\kappa_{1,2} < \frac{\bar{\kappa}_2}{2}$. Finally, we prove the result. $\square$

3.2.4. Proof of Theorem 1

Proof of Theorem 1. It is a direct consequence of Theorems 3, 4 and 5. □

3.2.5. Proof of Corollary 1

Proof of Corollary 1. Thanks to Theorem 1, NNCC conidtion holds. Such condition is stable for Riemannian product. On the other hand, NNCC condition implies A3W condition. Thus, the desired result yields. □

4. Proof of Theorem 2

In this section, we are going to prove Theorem 2. Let $(\rho_0 dvol, \rho_1 dvol)$ be $C^{k,\alpha}$ positive Borel probability measures on M. The curvatures of M satisfy (1.3) and (1.4) (or (1.5)). Fix any couple $(k, \alpha) \in \mathbb{N} \times (0, 1)$, with $k \geqslant 2$. Let $\mathrm{Hess}^{(c)}u$ be c-Hessian of u, namely,

$$\mathrm{Hess}^{(c)}u = \nabla_m^2 u + \nabla_m^2 c(\cdot, \exp_m \nabla_m u).$$

On the one hand, a C^2 potential function u of the optimal transport map $G(m) = \exp_m \nabla_m u$ pushing forward $\rho_0 dvol$ to $\rho_1 dvol$ satisfies the following Monge-Ampère equation

$$\det(d_{\nabla_m u} \exp_m)\det \mathrm{Hess}^{(c)}u = \frac{\rho_0(m)}{\rho_1(\exp_m \nabla_m u)}. \qquad (4.1)$$

On the other hand, a classical C^2 solution of the above equation is the potential function of the optimal transport map $\exp_m \nabla_m u$ pushes forward $\rho_0 dvol$ to $\rho_1 dvol$. To establish Theorem 2, it suffices to prove the Monge-Ampère equation admits a classical $C^{k+2,\alpha}$ solution. We attack it by the continuity method. Here we consider only regularity issue on simply connected manifold M. For the general case, we reduce to simply connected manifold by covering arguments (see [8]).

Let $\mathcal{I}$ be the set of the parameter $t \in [0, 1]$ for which there exists a $C^{k+2,\alpha}$ solution u_t of the equation (4.1) with ρ_1 replaced by $\rho_t = (1 - t)\rho_0 + t\rho_1$. It is clear that $0 \in \mathcal{I}$, so the set $\mathcal{I}$ is not empty. The openness is derived by the implicit function theorem [19]. If $\mathcal{I}$ is closed, the connectedness of the set

$[0, 1]$ shows $\mathcal{I} = [0, 1]$ and we prove the equation (4.1) admits a $C^{k,\alpha}$ solution. From Theorem 1, we know that on the Riemannian manifold with the curvature conditions (1.3) and (1.4) (or (1.5)), A3S condition holds. [5, Theorem 6.1] (see also [8]) reduced the closedness of the set $\mathcal{I}$ to the following estimate

Main estimate. There exists a positive number δ_0 (depending on the densities) under control such that

$$\det d_{\nabla_m u_t} \exp_m \geqslant \delta_0, \tag{4.2}$$

for each $(t, m) \in \mathcal{I} \times M$.

Fix $t \in \mathcal{I}$. From [5, Theorem 3.1], we know that ∇u_t at m locates in the injectivity domain at m. Recalling the curvature assumption (1.3), by Rauch comparison theorem [2], the length $|\nabla_m u_t|$ is strictly less than π. The Jacobian is related to Jacobi field, i.e.

$$\det d_{\nabla_m u_t} \exp_m = \det J_0(m, \nabla_m u_t, 1)$$

By Bishop's theorem [4], $\det d_{\nabla_m u_t} \exp_m$ is uniformly bounded above by 1 if M has nonnegative Ricci curvature. It is known that $\det d_{\nabla_m u_t} \exp_m$ is positive. But $\det d_{\nabla_m u_t} \exp_m$ may not has a positive lower bound. Since $\det J_0(m, \nu, 1)$ vanishes if (and only if) $\exp_m \nu$ is conjugate to m, so the estimate (4.2) is not obvious. For instance, on the sphere $\mathbb{S}^n$ with the standard metric, $\det d_{\nabla_m u_t} \exp_m = (\frac{\sin |\nabla_m u_t|}{|\nabla_m u_t|})^{n-1}$ is close to zero when $|\nabla_m u_t|$ is close to π. Making use of Lemma 3 in Appendix, the estimate (4.2) is obvious if $\max\{|\nabla_m u_t| : m \in M\} \leqslant \frac{3\pi}{4}$, provided ε is small enough. Thus without loss of generality, we assume there exists at least a point such that the length of gradient $|\nabla u_t|$ at that point is not smaller than $\frac{3\pi}{4}$. To prove (4.2) we need to construct an appropriate test function.

Let $\mathcal{J}(m, \nabla_m u_t) = -|\nabla_m u_t|^2 S^{-1}(m, \nabla_m u_t, 1)$. We consider the minimization problem

$$\min\{\langle \mathcal{J}\xi, \xi \rangle : (m, \xi) \in TM, \frac{3\pi}{4} \leqslant |\nabla u_t|_m, |\xi|_m = 1, \xi \perp \nabla_m u_t\}.$$

Let (m_0, ξ_0) be the minimum point. We consider the test function

$$h(m, \xi) = \frac{\langle \mathcal{J}\xi, \xi \rangle + \langle \xi, \nabla u_t \rangle^2}{|\xi|^2 - \frac{\langle \xi, \nabla u_t \rangle^2}{|\nabla u_t|^2}}.$$

Then h attains the minimum at the point (m_0, ξ_0) in a neighborhood of the point (m_0, ξ_0) in TM. To see this, let $\xi^\perp = \xi - \langle \xi, \nabla u_t \rangle \frac{\nabla u_t}{|\nabla u_t|^2}$ be the

orthogonal projection of ξ on $(\nabla_m u)^\perp$. Then

$$\frac{\langle \mathcal{J}\xi^\perp, \xi^\perp \rangle}{|\xi^\perp|^2} = h(m, \xi).$$

By continuity, we obtain h attains local minimum at the point (m_0, ξ_0). The minimum $h(m_0, \xi_0)$ has a nice explanation– the second eigenvalue of $\mathcal{J}$. Specifically, as h is bilinear with respect to ξ on the orthogonal subspace $(\nabla_m u)^\perp$, the minimum $h(m_0, \xi_0)$ is the second eigenvalue of the self adjoint operator $\mathcal{J}(m_0, \nabla_{m_0} u)$ with the associated eigenvector ξ_0. As a consequence of the above explanation, a necessary condition for the main estimate (4.2) is that the minimum $h(m_0, \xi_0)$ has a positive lower bound. At first the minimum $h(m_0, \xi_0)$ has to be positive. To see this, from the Hessian Comparison Theorem, we know that $-S^\perp$ is positive definite. Thus the minimum $h(m_0, \xi_0)$ is positive and has a lower bound under control. To differentiate the test function h, it needs to rule out the boundary case. Proposition 5 infers

$$h(m_0, \xi_0) \geqslant -\frac{|\nabla_{m_0} u_t| \sin |\nabla_{m_0} u_t|}{\cos |\nabla_{m_0} u_t|} - C_8 \varepsilon,$$

provided $\varepsilon < \gamma_1$. Since the function $-\frac{r \sin r}{\cos r}$ is non-increasing in $(\frac{\pi}{2}, \pi)$, we can assume $|\nabla_{m_0} u_t| > \pi - \delta, 0 < \delta < \frac{\pi}{4}$. Henceforth, we drop the subscript t and set for short $\tau = |\nabla_{m_0} u|$.

Some local notations We take the Fermi coordinate system $x = (x^1, x^2, \cdots, x^n)$ along the geodesic $\exp_{m_0}(s\nabla_{m_0} u)$ as the one constructed in section 2 and $v = (v^1, v^2, \cdots, v^n)$ be the fiber coordinates of $TM \to M$ naturally associated to x. We abbreviate the partial derivatives as follows

$$\partial_i = \frac{\partial}{\partial x^i}, \partial_{ij}^2 = \frac{\partial^2}{\partial x^i \partial x^j}, D_i = \frac{\partial}{\partial v^i}, D_{ij}^2 = \frac{\partial^2}{\partial v^i \partial v^j}.$$

Components of tensors will be denoted by

$$\mathrm{grad} u = \nabla^i u(m) \frac{\partial}{\partial x^i}, \nabla_m^2 u = \nabla_j^i u(m) dx^j \otimes \frac{\partial}{\partial x^i}, S = S_j^i(m, \nu, 1) dx^j \otimes \frac{\partial}{\partial x^i},$$

$$\mathcal{J} = \mathcal{J}_j^i(m, \nu) dx^j \otimes \frac{\partial}{\partial x^i}, \mathcal{H} = \mathcal{H}_j^i(m, \nu) dx^j \otimes \frac{\partial}{\partial x^i}, \mathcal{F} = \mathcal{F}_j^i(m, \nu) dx^j \otimes \frac{\partial}{\partial x^i},$$

where $\mathcal{H}_j^i = \nabla_j^i u + S_j^i, \mathcal{H}_k^i \mathcal{F}_j^k = \delta_j^i$.

We denote $D^k = g^{kl} D_l$ and $\partial^k = g^{kl} \partial_l$ the lifting indices. Let us now describe the behavior of the components $\mathcal{J}_\beta^\alpha$.

PROPOSITION 3. — *Let $(m_0, \nu_0) \in NoCut$ and $|\nu_0| \geqslant \pi - \delta$ with $0 < \delta < \frac{\pi}{4}$. Let x be the Fermi coordinate system associated to the geodesic $\exp_{m_0} s\nu_0$ and v be the fiber coordinates of $TM \to M$ naturally associated to*

x. Then there exists a positive constant C under control such that the absolute value of the first and second partial derivatives of the components $\mathcal{J}_\beta^\alpha, \alpha, \beta \in \{2, \cdots, n\}$ with respect to (x, v), evaluated at the point $(0, (|\nu_0|, 0), 1)$, are all bounded above by $C\varepsilon$, except the following partial derivatives

$$D_1 \mathcal{J}_\alpha^\alpha, \partial_{\beta\beta} \mathcal{J}_\alpha^\alpha, D_{11} \mathcal{J}_\alpha^\alpha, D_{\beta\beta} \mathcal{J}_\alpha^\alpha, D_{\varphi\psi} \mathcal{J}_\iota^\phi, \psi \neq \varphi, \iota \neq \phi.$$

and the following estimates hold

$$\begin{align}
|D_1 \mathcal{J}_\alpha^\alpha + \pi| &\leqslant C(\varepsilon + \delta), & (4.3) \\
|\partial_{\beta\beta} \mathcal{J}_\alpha^\alpha - \pi^2| &\leqslant C(\varepsilon + \delta), & (4.4) \\
|D_{11} \mathcal{J}_\alpha^\alpha + 2| &\leqslant C(\varepsilon + \delta), & (4.5) \\
|D_{\beta\beta} \mathcal{J}_\alpha^\alpha + 1 + 2\delta_{\alpha\beta}| &\leqslant C(\varepsilon + \delta), & (4.6) \\
|D_{\varphi\psi} \mathcal{J}_\iota^\phi + \delta_{\phi\varphi}\delta_{\iota\psi} + \delta_{\phi\psi}\delta_{\iota\varphi}| &\leqslant C(\varepsilon + \delta), \psi \neq \varphi, \iota \neq \phi. & (4.7)
\end{align}$$

Proof. Recalling Proposition 5, we only need to calculate the following components

$$\partial_x \bar{\mathcal{J}}_\beta^\alpha, D_v \bar{\mathcal{J}}_\beta^\alpha, \partial_{xx} \bar{\mathcal{J}}_\beta^\alpha, D_{xv} \bar{\mathcal{J}}_\beta^\alpha, D_{vv} \bar{\mathcal{J}}_\beta^\alpha$$

where $\bar{\mathcal{J}} = -|\nu|^2 \bar{S}^{-1}(m, \nu, 1)$. By differentiating the equation

$$\langle \bar{\mathcal{J}}\xi, \xi \rangle = f(|\nu|)|\xi|^2 - (|\nu|^2 + f(|\nu|))\langle \xi, \frac{\nu}{|\nu|} \rangle^2, \text{where } f(|\nu|) = -\frac{|\nu| \sin |\nu|}{|\nu|}.$$

Evaluated at the point (m_0, ν_0, ξ) with $\xi \perp \nu_0$ and $|\xi| = 1$, we get

$$\begin{align}
\partial_i \bar{\mathcal{J}}_\beta^\alpha \xi_\alpha \xi^\beta &= 0, D_i \bar{\mathcal{J}}_\beta^\alpha \xi^\alpha \xi^\beta = \dot{f}\delta_{1i}, \\
\partial_{ij} \bar{\mathcal{J}}_\beta^\alpha \xi_\alpha \xi^\beta &= -|\nu_0|\dot{f}(|\nu_0|), \partial_i D_j \bar{\mathcal{J}}_\beta^\alpha \xi^\alpha \xi^\beta = 0, \\
D_{ij} \bar{\mathcal{J}}_\beta^\alpha \xi^\alpha \xi^\beta &= \ddot{f}\delta_{1i}\delta_{1j} + \frac{\dot{f}}{|\nu_0|}(\delta_{ij} - \delta_{1i}\delta_{1j}) - \\
&\quad 2(1 - \delta_{1i})(1 - \delta_{1j})(1 - \frac{\sin |\nu_0|}{|\nu_0| \cos |\nu_0|})\xi^i \xi^j.
\end{align}$$

Using the symmetry of $\bar{\mathcal{J}}$, we get the desired results. $\qquad \square$

REMARK 2. — *Let $(m_0, \nu_0) \in NoCut$. Assume x is the Fermi coordinate system associated to the geodesic $\exp_{m_0} s\nu_0$ and v is the fiber coordinates of $TM \to M$ naturally associated to x. It is well-known*

$$p_2 = \exp_{p_1}[-grad_{p_1} c(., p_2)], \tag{4.8}$$

whenever $p_2 \in M$ is not cut point with respect to $p_1 \in M$. Assume the points p_a for $a = 1, 2$ are in the domain of the Fermi coordinate system and set $x =$

$x(p_1)$. *Suppose $m = p_1$ and $m_2 = \exp_m(\nu)$ and denote $v = (v^i), v^i = dx^i(\nu)$. Differentiating (4.8) with respect to the coordinates x at $x(m)$, we have*

$$D_k X^i(x, v, 1)\nabla^k_j c(m, \exp_m \nu) = \delta^\nabla_j X^i(x, v, 1), \qquad (4.9)$$

where $\delta^\nabla_j X^i = \partial_j X^i - \Gamma^p_{jl}(x)v^l D_p X^i$ is defined in an intrinsic way (see [5]).

In the following, all terms are evaluated at the point $(x, v) = (0, (\tau, 0))$. It is implicitly understood throughout the calculations.

The critical point condition. By differentiating the test function with respect to x^i, the first derivative condition on critical point could be read as

$$(\partial_i \mathcal{J}^\alpha_\beta + \nabla^k_i u D_k \mathcal{J}^\alpha_\beta)(\xi_0)_\alpha (\xi_0)^\beta = 0. \qquad (4.10)$$

Under the curvature assumption (1.3), by the Hessian comparison theorem we have $h(m_0, \xi_0) > 0$. The components of ξ_0 are denoted by $(\xi_0)^i$, i.e., $\xi_0 = (\xi_0)^i \partial_i$.
Differentiating two times on the test function h with respect to x, the second derivative condition on extrema can be read as follows

$$0 \leqslant I_2 + II_2 + III_2 + IV_2 + V_2, \qquad (4.11)$$

where

$$
\begin{aligned}
I_2 &= -\langle \mathcal{J}\xi_0, \xi_0 \rangle \mathcal{F}^i_j g^{jl} \partial^2_{il} g_{\alpha\beta}(\xi_0)^\alpha (\xi_0)^\beta + \mathcal{F}^i_j g^{jl} \partial^2_{il} g_{\alpha k} \mathcal{J}^k_\beta(\xi_0)^\alpha(\xi_0)^\beta, \\
II_2 &= -\tau \mathcal{F}^i_j \partial^j \Gamma^k_{1i} D_k \mathcal{J}^\alpha_\beta(\xi_0)_\alpha(\xi_0)^\beta + \mathcal{F}^i_j g^{jl} \partial^2_{il} \mathcal{J}^\alpha_\beta(\xi_0)_\alpha(\xi_0)^\beta, \\
III_2 &= 2\mathcal{F}^i_j \nabla^j_k u \partial_i D_k \mathcal{J}^\alpha_\beta(\xi_0)_\alpha(\xi_0)^\beta, \\
IV_2 &= 2(1 + \frac{1}{\tau^2}\langle \mathcal{J}\xi_0, \xi_0 \rangle)\mathcal{F}^i_j \nabla^\alpha_i u \nabla^j_\beta u(\xi_0)_\alpha(\xi_0)^\beta + \\
&\quad \mathcal{F}^i_j \nabla^k_i u \nabla^j_l u D_k D^l \mathcal{J}^\alpha_\beta(\xi_0)_\alpha(\xi_0)^\beta, \\
V_2 &= \mathcal{F}^i_j \partial^j \nabla^k_i u D_k \mathcal{J}^\alpha_\beta(\xi_0)_\alpha(\xi_0)^\beta.
\end{aligned}
$$

Here $(\xi_0)_i = g_{ij}(\xi_0)^j$ are coordinates of the corresponding co-vector related to ξ_0. The potential function u evaluated at the point m_0 satisfies the equation

$$det J_0 det(H^i_j) = \frac{\rho_0}{\rho_t \circ G}.$$

with the matrix (H^i_j) positive definite and $J_0 = D_v X$. The positive definiteness of the matrix (H^i_j) implies that $\nabla^1_1 u$ is larger than -1. We also write the expression

$$S(0, (\tau, 0), 1) = \begin{bmatrix} 1 & 0 \\ 0 & S^\alpha_\beta \end{bmatrix}.$$

Moreover, we have

$$-S^\perp \geqslant -\frac{\tau \cos \tau}{\sin \tau} I_{n-1} \tag{4.12}$$

We calculate each of the terms I_2 to V_2.

The term I_2 Recall $g_{ij} = \delta_{ij}$ on the axis and $\mathcal{J}_\alpha^1 = 0$ at the point $(0, (\tau, 0))$ so that the term I_2 can be recast as

$$I_2 = -\langle \mathcal{J}\xi_0, \xi_0 \rangle \mathcal{F}_\psi^\varphi g^{\psi\gamma} \partial_{\varphi\gamma}^2 g_{\alpha\beta}(\xi_0)^\alpha (\xi_0)^\beta + \mathcal{F}_\psi^\varphi g^{\psi\gamma} \partial_{\varphi\gamma}^2 g_{\alpha\iota} \mathcal{J}_\beta^\iota(\xi_0)^\alpha (\xi_0)^\beta.$$

Then we calculate the term I_2 by Lemma 1 and find

$$I_2 = \frac{2}{3} \langle \mathcal{J}\xi_0, \xi_0 \rangle \mathcal{F}_\psi^\varphi R_{\varphi\alpha}{}^\psi{}_\beta (\xi_0)^\alpha (\xi_0)^\beta - \frac{2}{3} \mathcal{F}_\psi^\varphi R_{\varphi\alpha}{}^\psi{}_\iota \mathcal{J}_\beta^\iota (\xi_0)^\alpha (\xi_0)^\beta.$$

Using the curvature assumption (1.4) (or (1.5)), Proposition 5, the positive definiteness of $(\mathcal{F}_j^i)$ and the Cauchy-Schwarz inequality, there exists a universal constant $C > 0$ such that the following upper bound holds

$$I_2 \leqslant C\varepsilon \mathcal{F}_\alpha^\alpha. \tag{4.13}$$

The term II_2 Using (1.4) (or (1.5)), Lemma 1, Proposition 5 and the Cauchy-Schwarz inequality, there exists a universal constant $C > 0$ (maybe different value than previous one) such that the following upper bound holds

$$II_2 \leqslant C(\varepsilon + \delta) \mathcal{F}_i^i.$$

Recalling $\mathcal{F}_j^i \nabla_i^1 u \nabla_1^j u = \nabla_1^1 u - 1 + \mathcal{F}_1^1$, we have $\mathcal{F}_1^1 \leqslant 2 + \mathcal{F}_j^i \nabla_i^1 u \nabla_1^j u$ since $\nabla_1^1 u \geqslant -1$. Thus, we infer

$$II_2 \leqslant C(\varepsilon + \delta) \mathcal{F}_\alpha^\alpha + C(\varepsilon + \delta) \mathcal{F}_j^i \nabla_i^k u \nabla_k^j u + C. \tag{4.14}$$

The term III_2 Using $\mathcal{F}_k^i \nabla_j^k u = \delta_j^i - \mathcal{F}_k^i S_j^k$, we estimate

$$III_2 = 2\partial_k D^k \mathcal{J}_\beta^\alpha(\xi_0)_\alpha (\xi_0)^\beta + 2\mathcal{F}_j^i S_k^j \partial_i Dk \mathcal{J}_\beta^\alpha(\xi_0)_\alpha (\xi_0)^\beta.$$

From the Proposition 5, there exists a universal constant $C > 0$ such that

$$III_2 \leqslant C\varepsilon(1 + \mathcal{F}_j^i S_k^j S_i^k).$$

Let us observe that the following identities hold: $\forall k, l = \{1, \cdots, n\}$

$$\mathcal{F}_j^i \nabla_i^k u \nabla_l^j u = \nabla_l^k u - S_l^k + \mathcal{F}_j^i S_i^k S_l^j = \mathcal{H}_l^k - 2S_l^k + \mathcal{F}_j^i S_i^k S_l^j. \tag{4.15}$$

The positive definiteness of $(\mathcal{H}_j^i), (-S^\perp)$ and the facts $S_1^1 = 1$ and $S_\alpha^1 = 0$ imply

$$III_2 \leqslant C\varepsilon(\mathcal{F}_j^i \nabla_i^k u \nabla_k^j u + 1). \tag{4.16}$$

The term IV_2 Splitting the negative term $\mathcal{F}_j^i \nabla_i^k u \nabla_l^j u D_k D^l \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta$ into four parts, we have

$$
\begin{aligned}
IV_2 &= \mathcal{F}_j^i \nabla_i^1 u \nabla_l^j u D_1 D^1 \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta + 2\mathcal{F}_j^i \nabla_i^1 u \nabla_\iota^j u D_1 D^\iota \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta + \\
&\quad \mathcal{F}_j^i \nabla_i^\varphi u \nabla_\varphi^j u D_\varphi D^\varphi \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta + \sum_{\varphi \neq \psi} \mathcal{F}_j^i \nabla_i^\varphi u \nabla_\psi^j u D_\varphi D^\psi \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta \\
&\quad + 2(1 + \frac{1}{\tau^2} \langle \mathcal{J}\xi_0, \xi_0 \rangle) \mathcal{F}_j^i \nabla_i^\alpha u \nabla_\beta^j u (\xi_0)_\alpha (\xi_0)^\beta.
\end{aligned}
$$

Using Proposition 5, there exists a universal constant $C > 0$ such that the following upper bound holds

$$
\begin{aligned}
IV_2 &\leqslant [C(\varepsilon + \delta) - 2]\mathcal{F}_j^i \nabla_i^1 u \nabla_1^j u + [C(\varepsilon + \delta) - 1]\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u \quad (4.17) \\
&\quad + \frac{2}{\tau^2} \langle \mathcal{J}\xi_0, \xi_0 \rangle \mathcal{F}_j^i \nabla_i^\alpha u \nabla_\beta^j u (\xi_0)_\alpha (\xi_0)^\beta.
\end{aligned}
$$

The term V_2 The term V_2 involves the third derivatives of u. We use the equation (4.1). After commuting the third derivatives of u, the term V_2 can be estimated

$$
V_2 = \mathcal{F}_j^i (\partial^k \nabla_i^j u + g^{jm} \tau R_{m1i}^k) D_k \mathcal{J}_\beta^\alpha (\xi_0)^\alpha (\xi_0)^\beta.
$$

By taking the logarithm and differentiating the equation (4.1) with respect to the variable x^k, we obtain the following simplified expressions

$$
\begin{aligned}
V_2 &= [\frac{\partial_k \rho_0}{\rho_0} - \frac{\partial_l \rho_t}{\rho_t} (J_1)_k^l - (J_1^{-1})_j^i \partial_k (J_1)_i^j - (\mathcal{J}^{-1})_j^i \partial_k \mathcal{J}_i^j - \\
&\quad \mathcal{F}_j^i \partial_k S_i^j] D^k \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta + \\
&\quad [-\frac{\partial_i \rho_t}{\rho_t} (J_0)_l^i - (J_1^{-1})_j^i D_l (J_1)_i^j - (\mathcal{J}^{-1})_j^i D_l \mathcal{J}_i^j - \\
&\quad \mathcal{F}_j^i D_l S_i^j] \nabla_k^l u D^k \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta + \\
&\quad \frac{2n}{\tau} \nabla_k^1 u D^k \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta + \tau g^{jm} \mathcal{F}_j^i R_{i1m}^k D_k \mathcal{J}_\beta^\alpha (\xi_0)_\alpha (\xi_0)^\beta.
\end{aligned}
$$

We observe that $\partial S = -S(\partial S^{-1})S$ and $DS = -S(DS^{-1})S$. Using the critical condition (4.10), Lemma 4, Proposition 5, the identities (4.15) and Cauchy-Schwarz inequality, there exists a positive constant $C > 0$ such that there holds

$$
\begin{aligned}
V_2 &\leqslant C \max_{t \in [0,1]} \{|dlog\rho_t|\} + C(\varepsilon + \delta)\mathcal{F}_j^i \nabla_i^k u \nabla_k^j u + [C(\varepsilon + \delta) \quad (4.18) \\
&\quad - \pi^2]\mathcal{F}_\alpha^\alpha + C.
\end{aligned}
$$

Gathering (4.11), (4.13),(4.14),(4.16),(4.17) and (4.18), we obtain the following inequality

$$
\begin{aligned}
0 \quad \leqslant \quad & C \max_{t \in [0,1]} \{|dlog\rho_t|\} + [C(\varepsilon + \delta) - \pi^2]\mathcal{F}_\alpha^\alpha + [C(\varepsilon + \delta) - 2]\mathcal{F}_j^i \nabla_i^1 u \nabla_1^j u + \\
& [C(\varepsilon + \delta) + \frac{2}{\tau^2}\langle \mathcal{J}\xi_0, \xi_0\rangle - 1]\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u + C.
\end{aligned}
$$

Fix $\varepsilon < \frac{1}{8C}, \delta < \frac{1}{8C}$. Recalling (4.10), (4.12), (4.15), Proposition 5 and using the fact the function $r \to -\frac{r \sin r}{\cos r}$ is decreasing in the interval $[\frac{3\pi}{4}, \pi]$, we obtain the inequality

$$
\begin{aligned}
0 \quad \leqslant \quad & C \max_{t \in [0,1]} \{|dlog\rho_t|\} - \frac{n-1}{2}\frac{(\pi - \delta)\cos\delta}{\sin\delta} \\
& + (\frac{2}{\tau^2}\langle \mathcal{J}\xi_0, \xi_0\rangle - \frac{1}{4})\mathcal{F}_j^i \nabla_i^\alpha u \nabla_\alpha^j u + C.
\end{aligned}
\tag{4.19}
$$

We take $\delta < \min\{\frac{1}{8C}, \frac{3\sqrt{2}\pi(n-1)}{16C(\max_{t \in [0,1]}\{|dlog\rho_t|\}+1)}\}$, the minimum $h(m_0, \xi_0)$ must be bounded below by

$$
\frac{1}{8}(\pi - \min\{\frac{1}{8C}, \frac{3\sqrt{2}\pi(n-1)}{16C(\max_{t \in [0,1]}\{|dlog\rho_t|\}+1)}\})^2.
$$

Finally, the proof of Theorem 2 is complete.

Appendix A. The geodesic motion and applications

A.1. The derivatives up to third order of geodesic motion

In this subsection, we collect some results in [8]. Fix $m_0 \in M, \nu_0 \in I(m_0)\backslash\{0\}$ and take the associated Fermi coordinate system along the geodesic $\exp_{m_0}(t\nu_0)$ and $v = (v^1, v^2, \cdots, v^n)$ be the fiber coordinates of $TM \to M$ naturally associated to x. The curvatures of the Riemaniann manifold satisfy the assumptions (1.3) and (1.4) (or (1.5)). For $m \in M, \nu \in I(m)$ with m in the domain of the Fermi coordinate system x, let $X(x, v, t)$ be the coordinates of the geodesic $\exp_m t\nu$. Then $X(x, v, t)$ is the solution of the Cauchy problem

$$
\ddot{X}^k + \Gamma_{ij}^k(X)\dot{X}^i \dot{X}^j = 0, X^k(x, v, 0) = x^k, \dot{X}^k(x, v, 0) = v^k.
\tag{A.1}
$$

In the Fermi coordinate system, (m_0, ν_0) corresponds to $(0, v_0)$ where $v_0 = (|\nu_0|, 0)$. On the axis, set for short $X_0(t) := X(0, v_0, t)$. We recall some results in [8]. Let $J_0 = D_v X, J_1 = \partial_x X$ be Jacobi fields in Section 2. Differentiating

(A.1) once with respect to the variable x (or v), on the axis, we get the following equation

$$\ddot{J}_a^i + \partial_l \Gamma_{jk}^i(X) \dot{X}^j \dot{X}^k J_a^l + 2\Gamma_{jk}^i(X) \dot{X}^j \dot{J}_a^k = 0, \tag{A.2}$$

with the initial conditions, namely either

$$\partial_a X^i(0) = \delta_a^i, \partial_a \dot{X}^i(0) = 0,$$

or

$$D_a X^i(0) = 0, D_a \dot{X}^i(0) = \delta_a^i.$$

We note that the equation (A.2) is equivalent to (2.5) on the axis. Thus, we have

LEMMA 2. — *[8, Lemma4] There exists a positive constant $C_2 > 0$ such that on the axis, for each $t \in [0, 1]$, the terms*

$$|\partial_x X(0, v_0, t)|, |\partial_x \dot{X}(0, v_0, t)|, |D_v X(0, v_0, t)|, |D_v \dot{X}(0, v_0, t)|,$$

are all bounded above by C_2.

We require the notation $\overline{\partial_x X}(t)$ and $\overline{D_v X}(t)$ for the solution $\bar{J}_a$ of the unperturbed equation

$$\ddot{\bar{J}}_a^i + |\nu_0|^2 \bar{R}_{1\beta 1}^i \bar{J}_a^\beta = 0,$$

with the initial conditions, namely either

$$\overline{\partial_a X^i}(0) = \delta_a^i, \overline{\dot{\partial_a X^i}}(0) = 0,$$

or

$$\overline{D_a X^i}(0) = 0, \overline{\dot{D_a X^i}}(0) = \delta_a^i.$$

It is clear that $\overline{\partial_x X}(t)$ and $\overline{D_v X}(t)$ correspond to $\bar{J}_1$ and $\bar{J}_0$ respectively on the axis. Thus, we have

LEMMA 3. — *[8, Lemma 5] There exists a positive constant $C_3 > 0$ such that on the axis, for each $t \in [0, 1]$, we have*

$$|\partial_x X(0, v_0, t) - \overline{\partial_x X}(t)| \leqslant C_3 \varepsilon,$$
$$|\partial_x \dot{X}(0, v_0, t) - \overline{\dot{\partial_x X}}(t)| \leqslant C_3 \varepsilon,$$
$$|D_v X(0, v_0, t) - \overline{D_v X}(t)| \leqslant C_3 \varepsilon,$$
$$|D_v \dot{X}(0, v_0, t) - \overline{\dot{D_v X}}(t)| \leqslant C_3 \varepsilon.$$

REMARK 3. — *For later use, dealing with $|J_0 - \bar{J}_0|$, the constant C_3 can be taken value $2\sqrt{n-1}$ (see [8, Remark 5]).*

Let J_{ab} be $\partial^2_{ab}X, \partial_a D_b X, D_a \partial_b X$ or $D^2_{ab}X$. Differentiating the Cauchy problem (A.1) twice with respect to the parameters x and v

$$\ddot{J}^i_{ab} + \partial_l \Gamma^i_{jk}\dot{X}^j \dot{X}^k J^l_{ab} + 2\Gamma^i_{jk}\dot{X}^j \dot{J}^k_{ab} = \tag{A.3}$$
$$-\partial^2_{lp}\Gamma^i_{jk}\dot{X}^j \dot{X}^k J^l_a J^p_b - 2\partial_l \Gamma^i_{jk}\dot{X}^j(\dot{J}^k_b J^l_a + \dot{J}^k_a J^l_b) - 2\Gamma^i_{jk}\dot{J}^j_b \dot{J}^k_a.$$

and the homogenuous initial conditions

$$J^i_{ab}(0) = \dot{J}^i_{ab}(0) = 0. \tag{A.4}$$

On the axis, recalling (2.15), equation (A.3) is reduced to

$$\ddot{J}^i_{ab} + |\nu_0|^2 R^i_{1\alpha1}(X_0)J^\alpha_{ab} = -|\nu_0|^2 \partial^2_{lp}\Gamma^i_{11}J^l_a J^p_b - 2|\nu_0|R^i_{k\beta1}(X_0)(\dot{J}^k_b J^\beta_a + \dot{J}^k_a J^\beta_b).$$

By Lemma 1, we have (see [8, Lemma9])

LEMMA 4. — *There exists a positive constant $C_4 > 0$ such that on the axis, for each $t \in [0,1]$, the terms*

$$|\partial^2_{xx}X(0,v_0,t)|, |\partial^2_{xx}\dot{X}(0,v_0,t)|,$$
$$|\partial_x D_v X(0,v_0,t)|, |\partial_x D_v \dot{X}(0,v_0,t)|,$$
$$|D^2_{vv}X(m_0,v_0,t)|, |D^2_{vv}\dot{X}(0,v_0,t)|$$

are all bounded by C_4.

Let us introduce the solutions $\overline{\partial^2_{xx}X}, \overline{\partial_x D_v X}, \overline{D_v \partial_x X}$ and $\overline{D^2_{vv}X}$ along the axis of the unperturbed equation

$$\ddot{\bar{J}}^i_{ab} + |\nu_0|^2 \delta^i_\alpha \bar{J}^\alpha_{ab} = -2|\nu_0|(\delta^i_\beta \delta^1_k - \delta^i_1 \delta^\beta_k)(\bar{J}^\beta_a \dot{\bar{J}}^k_b + \bar{J}^\beta_b \dot{\bar{J}}^k_a), \tag{A.5}$$

with null initial conditions

$$\bar{J}^i_{ab}(0) = \dot{\bar{J}}^i_{ab}(0) = 0. \tag{A.6}$$

We have the following perturbation result (see [8, Lemma10]).

LEMMA 5. — *There exists a positive constant $C_5 > 0$ such that on the axis, for each $t \in [0,1]$, we have*

$$|\partial^2_{xx}X(0,v_0,t) - \overline{\partial^2_{xx}X}(t)| \leqslant C_5\varepsilon,$$
$$|\partial^2_{xx}\dot{X}(0,v_0,t) - \overline{\dot{\partial^2_{xx}X}}_0(t)| \leqslant C_5\varepsilon,$$
$$|\partial_x D_v X(0,v_0,t) - \overline{\partial_x D_v X}(t)| \leqslant C_5\varepsilon,$$
$$|\partial_x D_v \dot{X}(0,v_0,t) - \overline{\dot{\partial_x D_v X}}(t)| \leqslant C_5\varepsilon,$$
$$|D^2_{vv}X(0,v_0,t) - \overline{D^2_{vv}X}(t)| \leqslant C_5\varepsilon,$$
$$|D^2_{vv}\dot{X}(0,v_0,t) - \overline{\dot{D^2_{vv}X}}(t)| \leqslant C_5\varepsilon.$$

Furthermore, $|\partial_{1x}^2 X(0, v_0, t)| \leqslant C_5\varepsilon$, $|\partial_1 D_v X(0, v_0, t)| \leqslant C_5\varepsilon$.

Proof. The first 6 statements are the results in (or comes directly from) [8, Lemma10]. The last two approximations are just the consequences of the facts $\overline{\partial_{1x}^2 X}(t) = 0$ and $\overline{\partial_1 D_v X}(t) = 0$ which follows from the equation (A.5) and the initial conditions (A.6). $\square$

Let $J_{abc}^i(t)$ equal to $\partial_{xxx}^3 X(0, v_0, t), \partial_{xx}^2 D_v X(0, v_0, t), \partial_x D_{vv}^2 X(0, v_0, t)$ or $D_{vvv}^3 X(0, v_0, t)$. Differentiating (A.1) three times with respect to the variables x and v

$$
\begin{aligned}
\ddot{J}_{abc}^i + |\nu_0|^2 R_{1\alpha 1}^i(X_0) J_{abc}^\alpha &= -|\nu_0|^2 \partial_{lpq}^3 \Gamma_{11}^i J_a^l J_b^p J_c^q - |\nu_0|^2 \partial_{lp}^2 \Gamma_{11}^i \sum_{(a,b,c)} J_{ab}^l J_c^p \\
&\quad -2|\nu_0| \partial_{lp}^2 \Gamma_{1k}^i \sum_{(a,b,c)} \dot{J}_a^k J_b^l J_c^p \\
&\quad -2|\nu_0| R_{k\beta 1}^i(X_0) \sum_{(a,b,c)} (\dot{J}_a^k J_{bc}^\beta + \dot{J}_{ab}^k J_c^\beta) \\
&\quad -2\partial_\beta \Gamma_{jk}^i \sum_{(a,b,c)} \dot{J}_a^j \dot{J}_b^k J_c^\beta.
\end{aligned}
$$

with null initial conditions

$$
J_{abc}^i(0) = \dot{J}_{abc}^i(0) = 0.
$$

Here $\sum_{(a,b,c)}$ means circular summation on (a, b, c). It is stated (or with the same arguments) in [8, Lemma12].

LEMMA 6. — *There exists a positive constant* $C_6 > 0$ *such that on the axis, for each* $t \in [0, 1]$, *the terms*

$$
\begin{aligned}
&|\partial_{xxx}^3 X(0, v_0, t)|, |\partial_{xxx}^3 \dot{X}(0, v_0, t)|, \\
&|\partial_{xx}^2 D_v X(0, v_0, t)|, |\partial_{xx}^2 D_v \dot{X}(0, v_0, t)|, \\
&|\partial_x D_{vv}^2 X(0, v_0, t)|, |\partial_x D_{vv}^2 \dot{X}(0, v_0, t)|, \\
&|D_{vvv}^3 X(0, v_0, t)|, |D_{vvv}^3 \dot{X}(0, v_0, t)|
\end{aligned}
$$

are all bounded above a universal positive constant C_6.

Let us introduce the solutions $\overline{\partial^3_{xxx}X}, \overline{\partial^2_{xx}D_vX}, \overline{\partial_xD_v\partial_xX}, \overline{D_v\partial^2_{xx}X}, \overline{\partial_xD^2_{vv}X},$ $\overline{D_v\partial_xD_vX}, \overline{D^2_{vv}\partial_xX}$ and $\overline{D^3_{vvv}X}$ along the axis of the unperturbed equation

$$\ddot{\bar{J}}^i_{abc} + |\nu_0|^2\delta^i_\alpha\bar{J}^\alpha_{abc} \tag{A.7}$$
$$= \frac{4}{3}(\delta^i_k - \delta^i_1\delta^1_k) \sum_{(a,b,c)} (|\nu_0|^2\bar{J}^k_a\bar{J}^\beta_b\bar{J}^\beta_c - 2\dot{\bar{J}}^\beta_a\dot{\bar{J}}^\beta_b\bar{J}^k_c) -$$
$$2|\nu_0|(\delta^i_\beta\delta^1_k - \delta^i_1\delta^\beta_k) \sum_{(a,b,c)} (\dot{\bar{J}}^k_a\bar{J}^\beta_{bc} + \dot{\bar{J}}^k_{ab}\bar{J}^\beta_c) -$$
$$2(\delta^i_k - \delta^i_1\delta^1_k) \sum_{(a,b,c)} \dot{\bar{J}}^1_a\dot{\bar{J}}^1_b\bar{J}^k_c +$$
$$[2\delta^i_1\delta^1_k + \frac{2}{3}(\delta^i_k - \delta^i_1\delta^1_k)] \sum_{(a,b,c)} (\dot{\bar{J}}^k_a\dot{\bar{J}}^\beta_b\bar{J}^\beta_c + \dot{\bar{J}}^\beta_a\dot{\bar{J}}^k_b\bar{J}^\beta_c).$$

with null initial conditions

$$\bar{J}^i_{abc}(0) = \dot{\bar{J}}^i_{abc}(0) = 0.$$

It is shown (or with the same arguments) in [8, Lemma13].

LEMMA 7. — *There exists a positive constant $C_7 > 0$ such that on the axis, for each $t \in [0,1]$, we have*

$$|\partial^3_{xxx}X(0,v_0,t) - \overline{\partial^3_{xxx}X}(t)| \leqslant C_7\varepsilon,$$
$$|\partial^2_{xx}D_vX(0,v_0,t) - \overline{\partial^2_{xx}D_vX}(t)| \leqslant C_7\varepsilon,$$
$$|\partial_xD^2_{vv}X(0,v_0,t) - \overline{\partial_xD^2_{vv}X}(t)| \leqslant C_7\varepsilon,$$
$$|D^3_{vvv}X(0,v_0,t) - \overline{D^3_{vvv}X}(t)| \leqslant C_7\varepsilon.$$

Appendix B. The behaviour of Jacobi fields and applications

B.1. The behaviour of the curvature matrix

In this subsection, we study the behaviour of the curvature matrix. Given $m_0 \in M, \nu_0 \in I(m_0)\backslash\{0\}$ and some orthonormal basis $\{E_1 = \frac{\nu_0}{|\nu_0|}, E_2, \cdots, E_n\}$ of the tangent space $T_{m_0}M$, let $\gamma_\theta(t)$ be the geodesic with the initial point m_0 and the initial velocity $\cos\theta E_1 + \sin\theta E_2$. For $|\theta|$ small enough, let $\{e_1(\theta,t), e_2(\theta,t), \cdots, e_n(\theta,t)\}$ be the parallel transport along the geodesic $\gamma_\theta(t)$ with $e_1(\theta,0) = \cos\theta E_1 + \sin\theta E_2$, $e_2(\theta,0) = -\sin\theta E_1 + \cos\theta E_2$, and $e_i(\theta,0) = E_i$ for $i \geqslant 3$. Then $\{e_1(0,t), e_2(0,t), \cdots, e_n(0,t)\}$ is the parallel orthonormal moving frame along the geodesic $\gamma_0(t)$ with $e_1(0,t) = \dot{\gamma}_0(t)$.

Let $X(\theta, t) = (X^1(\theta, t), X^2(\theta, t), \cdots, X^n(\theta, t))$ denote the coordinate of the geodesic $\gamma_\theta(t)$ in the Fermi coordinate system, i.e.

$$X^k(\theta, t) = X^k(0, (\cos\theta, \sin\theta, 0, \cdots, 0), t).$$

Along the geodesic $\gamma_\theta(t)$, there are two bases in the tangent space: the natural basis $\{\frac{\partial}{\partial x^i}\}$ and the orthonormal basis $\{e_1, \cdots, e_n\}$. Set $(Y_i^j(\theta, t))$ for the coordinates of the orthonormal chart $\{e_1, \cdots, e_n\}$, i.e. $e_i(\theta, t) = Y_i^j(\theta, t)\partial_j$. It is clear that $Y_1^i(\theta, t) = \dot{X}^i(\theta, t)$. Recall the dot stands for the derivative with respect to t and the prime for the derivative with respect to θ. Given two real function $f(t)$ and $h(t)$, we write $f(t) = \mathcal{B}(h(t))$ if there exists a positive constant C (under control) such that $|f(t)| \leqslant C|h(t)|$ for all t in a given range. The third derivative of $f(t)$ with respect to t will be denoted by $\dddot{f}(t)$. The derivatives of the elements of the $n \times k$ matrix-valued function $(a_j^i(\theta, t))$ with respect to t and θ will be denoted by $\dot{a}_j^i(\theta, t), a_j'^i(\theta, t), \cdots$, etc. For short, we drop the indices x and v if there is no confusion in the context.

B.1.1. More asymptotic behaviour of the geodesic motion

We investigate the geodesic motion. In particular, we study its asymptotic behaviour near the origin.

LEMMA 8. — *Under the curvature assumptions (1.3) and (1.4) (or (1.5)) with $\varepsilon < 1$, for all $t \in [0, \tau]$ with $\tau = |\nu_0|$, the geodesic motion on the axis satisfies in Fermi chart*

1) $X'(0, t) = (0, \sin t + \mathcal{B}(\varepsilon t^3), \mathcal{B}(\varepsilon t^3), \cdots, \mathcal{B}(\varepsilon t^3))^T$,
   $\dot{X}'(0, t) = (0, \cos t + \mathcal{B}(\varepsilon t^2), \mathcal{B}(\varepsilon t^2), \cdots, \mathcal{B}(\varepsilon t^2))^T$,
   $\ddot{X}'(0, t) = (0, -\sin t + \mathcal{B}(\varepsilon t), \mathcal{B}(\varepsilon t), \cdots, \mathcal{B}(\varepsilon t))^T$,
   where ^{T} is the transpose;
2) $X''(0, t) = (-\sin t \cos t + \mathcal{B}(\varepsilon t^3), \mathcal{B}(\varepsilon t^3), \cdots, \mathcal{B}(\varepsilon t^3))^T$,
   $\dot{X}''(0, t) = (-\cos(2t) + \mathcal{B}(\varepsilon t^2), \mathcal{B}(\varepsilon t^2), \cdots, \mathcal{B}(\varepsilon t^2))^T$,
   $\ddot{X}''(0, t) = (4\sin t \cos t + \mathcal{B}(\varepsilon t), \mathcal{B}(\varepsilon t), \cdots, \mathcal{B}(\varepsilon t))^T$;
3) $\dddot{X}'(0, 0) = (0, -R_{121}^2(0), \cdots, -R_{121}^n(0))^T$,
   $\dddot{X}''(0, 0) = (-4R_{221}^1(0), -4R_{221}^2(0), \cdots, -4R_{221}^n(0))^T$.

Proof. The coordinates of the geodesic $\exp_{m_0} t(\cos\theta E_1 + \sin\theta E_2)$ are the solution of the following Cauchy problem

$$\begin{cases} \ddot{X}^i + \Gamma^i_{jk}(X)\dot{X}^j\dot{X}^k = 0, \\ X(\theta,0) = 0, \dot{X}(\theta,0) = (\cos\theta, \sin\theta, 0, \cdots, 0)^T. \end{cases} \tag{B.1}$$

On the axis, since the Christoffel symbols vanish, we have

$$X(0,t) = (t,0,\cdots,0)^T.$$

1) Differentiating (B.1) with respect to θ, we obtain

$$\ddot{X'}^i + \partial_p\Gamma^i_{jk}\dot{X}^j\dot{X}^k X'^p + 2\Gamma^i_{jk}\dot{X}'^j\dot{X}^k = 0,$$

with the initial condition

$$X'^i(0,0) = 0, \dot{X}'^i(0,0) = \delta^i_2.$$

It follows from (2.14) and (2.17) that on the axis we have

$$\begin{cases} \ddot{X'}^i + R^i_{1\alpha1}X'^\alpha = 0, \\ X'(0,0) = 0, \dot{X}'(0,0) = (0,1,0,\cdots,0)^T. \end{cases} \tag{B.2}$$

It is clear that $X'^1(0,t) \equiv 0$. For $i > 1$, we first establish the following standard estimation.

Claim: For any $t \in [0,\tau] \subset [0,\pi]$,

$$\max\{|X'(0,t)|, |\dot{X}'(0,t)|\} \leqslant e^{\frac{\pi}{2}}.$$

Proof of the Claim: Let $f = |X'|^2 + |\dot{X}'|^2$ with $f(0) = 1$. The derivative of f has the form $\dot{f} = 2(\overline{R}^\beta_{1\alpha1}(X) - R^\beta_{1\alpha1}(X))X'^\alpha\dot{X}'^\beta$. Recall $\varepsilon \leqslant 1$. Using the Cauchy-Schwarz inequality, we get $\dot{f} \leqslant f$. We conclude that $f(t) \leqslant e^t \leqslant e^\pi$. Thus claim is proved.

We define $\overline{X}'(t) = (\overline{X}'^1(t), \cdots, \overline{X}'^n(t))$ so that

$$\begin{cases} \ddot{\overline{X}}'^i + \overline{R}^i_{1\alpha1}\overline{X}'^\alpha = 0, \\ \overline{X}'(0,0) = 0, \dot{\overline{X}}'(0,0) = (0,1,0,\cdots,0)^T. \end{cases}$$

Denote $\mathcal{E}^i$ be the difference $X'^i - \overline{X}'^i$. We rewrite the equation (B.2) in the perturbative form

$$\ddot{\mathcal{E}}^i + \mathcal{E}^i = (\overline{R}^i_{1\alpha1} - R^i_{1\alpha1})X'^\alpha, \tag{B.3}$$

with the homogeneous initial conditions

$$\mathcal{E}^i(0,0) = \dot{\mathcal{E}}^i(0,0) = 0.$$

By the representation formula (2.10), we get

$$\mathcal{E}^i(0,t) \;\;=\;\; \sin t \int_0^t (\overline{R}^i_{1\alpha 1} - R^i_{1\alpha 1}) X'^{\alpha} \cos s\, ds - \cos t \int_0^t (\overline{R}^i_{1\alpha 1} - R^i_{1\alpha 1}) X'^{\alpha} \sin s\, ds.$$

which yields by the above claim $|\mathcal{E}^i(0,t)| \leqslant \frac{3}{2}\varepsilon t^2 e^{\frac{\pi}{2}}$ for all $t \in [0,\tau]$. More precisely, we have for all $t \in [0,\tau]$

$$|X'^2(0,t) - \sin t| \leqslant \frac{3}{2}\varepsilon t^2 e^{\frac{\pi}{2}}, \;\; |X'^j(0,t)| \leqslant \frac{3}{2}\varepsilon t^2 e^{\frac{\pi}{2}}, \forall j \geqslant 3.$$

Using (B.3), we have $|\ddot{\mathcal{E}}^i(0,t)| \leqslant (\frac{9}{2}\pi e^{\frac{\pi}{2}}+1)\varepsilon t$. Therefore, $\dot{\mathcal{E}}(0,t) = \int_0^t \ddot{\mathcal{E}}(0,s) ds = \mathcal{B}(\varepsilon t^2), \mathcal{E}(0,t) = \int_0^t \dot{\mathcal{E}}(0,s) ds = \mathcal{B}(\varepsilon t^3)$. Hence, we prove the first part.

2) Differentiating (B.1) twice with respect to θ, we obtain

$$\ddot{X}''^i + \partial_p \Gamma^i_{jk} \dot{X}^j \dot{X}^k X''^p + \partial^2_{pq} \Gamma^i_{jk} \dot{X}^j \dot{X}^k X'^p X'^q +$$
$$4\partial_p \Gamma^i_{jk} \dot{X}'^j \dot{X}^k X'^p + 2\Gamma^i_{jk}(\dot{X}''^j \dot{X}^k + \dot{X}'^j \dot{X}'^k) = 0,$$

with the initial condition

$$X''^i(0,0) = 0, \dot{X}''^i(0,0) = -\delta^i_1.$$

Using (2.14), (2.16) and (2.17), we get on the axis

$$\begin{cases} \ddot{X}''^i + R^i_{1\alpha 1} X''^{\alpha} + (\nabla_\alpha R^i_{1\beta 1} + \nabla_1 R^i_{\beta\alpha 1}) X'^{\alpha} X'^{\beta} + 4R^i_{\beta\alpha 1} X'^{\alpha} \dot{X}'^{\beta} = 0, \\ X''(0,0) = 0, \dot{X}''(0,0) = (-1,0,\cdots,0)^T. \end{cases}$$

Claim: There exists a positive universal constant C such that, for any $t \in [0,\tau] \subset [0,\pi]$,

$$\max\{|X''(0,t)|, |\dot{X}''(0,t)|\} \leqslant e^{C\pi}.$$

Proof of the Claim Let $f_1(t) = |X''(0,t)|^2 + |\dot{X}''(0,t)|^2$. Then

$$\begin{aligned} \dot{f}_1(t) \;\;&=\;\; 2X'''^i \dot{X}''^i + 2\dot{X}''^i \ddot{X}''^i \\ &=\;\; 2X'''^i \dot{X}''^i - 2R^i_{1\alpha 1} X''^{\alpha} \dot{X}''^i - 2[(\nabla_\alpha R^i_{1\beta 1} + \nabla_1 R^i_{\beta\alpha 1}) X'^{\alpha} X'^{\beta} + \\ &\quad\;\; 4R^i_{\beta\alpha 1} X'^{\alpha} \dot{X}'^{\beta}]\dot{X}''^i \\ &\leqslant\;\; C_1 f_1(t) + C_1, \end{aligned}$$

which gives $f_1(t) \leqslant C_1 e^{C_1 t} \leqslant e^{C\pi}$. Thus the claim is proved.

When $i = 1$, let $f_2(t) = X''^1(0,t) + \sin t \cos t$. Thus

$$
\begin{aligned}
\ddot{f}_2(t) &= -\nabla_1 R_{\beta\alpha 1}^1 X'^\alpha X'^\beta + 4(\overline{R}_{\beta\alpha 1}^1 - R_{\beta\alpha 1}^1) X'^\alpha \dot{X}'^\beta - \\
&\quad 4\overline{R}_{\beta\alpha 1}^1 X'^\alpha \dot{X}'^\beta - 4\sin t \cos t \\
&= 4X'^\alpha \dot{X}'^\alpha - 4\sin t \cos t + \mathcal{B}(\varepsilon t) \\
&= 4X'^2 \dot{X}'^2 - 4\sin t \cos t + \mathcal{B}(\varepsilon t) \\
&= 4(X'^2 - \sin t)\dot{X}'^2 + 4(\dot{X}'^2 - \cos t)\sin t + \mathcal{B}(\varepsilon t) \\
&= \mathcal{B}(\varepsilon t).
\end{aligned}
$$

When $i > 1$, X''^i satisfies the following equation

$$
\ddot{X}''^i + X''^i = (\overline{R}_{1\alpha 1}^i - R_{1\alpha 1}^i) X''^\alpha - (\nabla_\alpha R_{1\beta 1}^i + \nabla_1 R_{\beta\alpha 1}^i) X'^\alpha X'^\beta - 4R_{\beta\alpha 1}^i X'^\alpha \dot{X}'^\beta,
$$

with the initial condition

$$
X''^i(0,0) = \dot{X}''^i(0,0) = 0.
$$

By the representation formula (2.10), we get

$$
\begin{aligned}
X''^i(0,t) &= \sin t \int_0^t [(\overline{R}_{1\alpha 1}^i - R_{1\alpha 1}^i) X''^\alpha - (\nabla_\alpha R_{1\beta 1}^i + \nabla_1 R_{\beta\alpha 1}^i) X'^\alpha X'^\beta - \\
&\quad 4R_{\beta\alpha 1}^i X'^\alpha \dot{X}'^\beta] \cos s\, ds - \cos t \int_0^t [(\overline{R}_{1\alpha 1}^i - R_{1\alpha 1}^i) X''^\alpha - \\
&\quad (\nabla_\alpha R_{1\beta 1}^i + \nabla_1 R_{\beta\alpha 1}^i) X'^\alpha X'^\beta - 4R_{\beta\alpha 1}^i X'^\alpha \dot{X}'^\beta] \sin s\, ds.
\end{aligned}
$$

Note that $\overline{R}_{1\alpha 1}^i - R_{1\alpha 1}^i = \mathcal{B}(\varepsilon), \nabla_\alpha R_{1\beta 1}^i = \mathcal{B}(\varepsilon), \nabla_1 R_{\beta\alpha 1}^i = \mathcal{B}(\varepsilon), R_{\beta\alpha 1}^i = \mathcal{B}(\varepsilon)$ and $X', \dot{X}', X''$ are uniformly bounded shown by two previous claims. Thus we have $X''^i(0,t) = \mathcal{B}(\varepsilon t)$. Gathering the above results, we infer

$$
\ddot{X}''(0,t) = (4\sin t \cos t, 0, \cdots, 0)^T + \mathcal{B}(\varepsilon t).
$$

Integrating this equality, we obtain

$$
\begin{aligned}
\dot{X}''(0,t) &= (-\cos(2t), 0, \cdots, 0)^T + \mathcal{B}(\varepsilon t^2), \\
X''(0,t) &= (-\sin t \cos t, 0, \cdots, 0)^T + \mathcal{B}(\varepsilon t^3).
\end{aligned}
$$

3) Differentiating (B.1) once with respect to θ and t respectively, we have

$$
\begin{aligned}
&\dddot{X}'^i + \partial_{pq}^2 \Gamma_{jk}^i \dot{X}^j \dot{X}^k \dot{X}^p X'^q + \partial_p \Gamma_{jk}^i (\ddot{X}^j \dot{X}^k X'^p + \dot{X}^j \ddot{X}^k X'^p + \dot{X}^j \dot{X}^k \dot{X}'^p) + \\
&2\partial_p \Gamma_{jk}^i \dot{X}'^j \dot{X}^k \dot{X}^p + 2\Gamma_{jk}^i (\ddot{X}'^j \dot{X}^k + \dot{X}'^j \ddot{X}^k) = 0.
\end{aligned}
$$

Using the facts $\dot{X}(0,0) = (1,0,\cdots,0)^T$, $X'(0,0) = 0$, $\dot{X}'(0,0) = (0,1,0,\cdots,0)^T$ and $\partial_1\Gamma^k_{ij}(0) = 0$ and together with (2.17), we get

$$\overset{..}{X}'^i(0,0) = -R^i_{121}(0).$$

Differentiating (B.1) twice with respect to θ and once with respect to t respectively

$$\overset{...}{X}''^i + \partial^2_{pq}\Gamma^i_{jk}\dot{X}^j\dot{X}^k\dot{X}^p X''^q + \partial_p\Gamma^i_{jk}(2\ddot{X}^j\dot{X}^k X''^p + \dot{X}^j\dot{X}^k X''^p) +$$

$$\partial^3_{pql}\Gamma^i_{jk}\dot{X}^j\dot{X}^k X'^p X'^q\dot{X}^l + 2\partial^2_{pq}\Gamma^i_{jk}(\ddot{X}^j\dot{X}^k X'^p X'^q + \dot{X}^j\dot{X}^k \dot{X}'^p X'^q) +$$

$$4\partial^2_{pq}\Gamma^i_{jk}\dot{X}'^j\dot{X}^k\dot{X}^p X'^q + 4\partial_p\Gamma^i_{jk}(\ddot{X}'^j\dot{X}^k X'^p + \dot{X}'^j\ddot{X}^k X'^p + \dot{X}'^j\dot{X}^k\dot{X}'^p) +$$

$$2\partial_p\Gamma^i_{jk}(\dot{X}''^j\dot{X}^k + \dot{X}'^j\dot{X}'^k)\dot{X}^p + 2\Gamma^i_{jk}(\ddot{X}''^j\dot{X}^k + \dot{X}''^j\ddot{X}^k + 2\ddot{X}'^j\dot{X}'^k) = 0.$$

Again by (2.17) and together with the facts $\dot{X}(0,0) = (1,0,\cdots,0)^T$, $X'(0,0) = X''(0,0) = (0,\cdots,0)^T$, $\dot{X}'(0,0) = (0,1,0\cdots,0)^T$, $\dot{X}''(0,0) = (-1,0\cdots,0)^T$ and $\Gamma^k_{ij}(0) = \partial_1\Gamma^k_{ij}(0) = 0$, we get

$$\overset{...}{X}''^i(0,0) = -4R^i_{221}(0).$$

Finally, the Lemma is proved. $\qquad\square$

B.1.2. The orthonormal basis motion

We write $e_i(\theta,t) = Y^j_i(\theta,t)\partial_j$ for all i. Since the orthonormal moving basis $\{e_1,\cdots,e_n\}$ is parallel, we consider its equation in Fermi chart

$$\dot{Y}^i_j + \Gamma^i_{kl}(X)\dot{X}^l Y^k_j = 0, \tag{B.4}$$

with the initial condition

$$Y(\theta,0) = \begin{bmatrix} \cos\theta & -\sin\theta & \\ \sin\theta & \cos\theta & \\ & & I_{n-2} \end{bmatrix}.$$

We remark when j is equal to 1, $Y^i_1(\theta,t) = \dot{X}^i(\theta,t)$. It is obvious that $Y(0,t) = I_n$.

LEMMA 9. — *Under the curvature assumptions (1.3) and (1.4) (or (1.5)) with $\varepsilon < 1$, then in the Fermi chart, the derivatives of the basis motion along the axis satisfy the following estimates, $\forall t \in [0,\tau]$ with $\tau = |\nu_0|$*

1) $Y'^i_j(0,t) = -(\delta^i_1\delta^2_j - \delta^i_2\delta^1_j)\cos t + \mathcal{B}(\varepsilon t^2)$,

   $\dot{Y}'^i_j(0,t) = (\delta^i_1\delta^2_j - \delta^i_2\delta^1_j)\sin t + \mathcal{B}(\varepsilon t);$

2) $Y''^i_j(0,t) = -\delta^i_1 \delta^1_j \cos(2t) - \delta^i_2 \delta^2_j \cos^2 t +$
$\frac{1}{3}(\delta^i_j - \delta^i_2 \delta^2_j - \delta^j_1 \delta^i_j) \sin^2 t + \mathcal{B}(\varepsilon t^2),$
$\dot{Y}''^i_j(0,t) = 4\delta^i_1 \delta^1_j \sin t \cos t + 2\delta^i_2 \delta^2_j \sin t \cos t +$
$\frac{2}{3}(\delta^i_j - \delta^i_2 \delta^2_j - \delta^j_1 \delta^i_j) \sin t \cos t + \mathcal{B}(\varepsilon t);$

3) $\dot{Y}''^i_j(0,0) = 0,$
$\ddot{Y}''^i_j(0,0) = 2\delta^2_j R^i_{121}(0) - \frac{2}{3}(1 + 5\delta^1_j) R^i_{22j}(0).$

Proof 1) Differentiating equation (B.4) with respect to θ, we get

$$\dot{Y}'^i_j + \partial_p \Gamma^i_{kl} \dot{X}^l X'^p Y^k_j + \Gamma^i_{kl}(\dot{X}'^l Y^k_j + \dot{X}^l Y'^k_j) = 0,$$

which implies by Lemma 1

$$\dot{Y}'^i_j + R^i_{j\alpha 1} X'^\alpha = 0,$$

with initial condition

$$Y'^i_j(0,0) = \delta^i_2 \delta^1_j - \delta^i_1 \delta^2_j.$$

Thus, we infer

$$
\begin{aligned}
\dot{Y}'^i_j(0,t) &= -R^i_{j\alpha 1} X'^\alpha \\
&= (\bar{R}^i_{j\alpha 1} - R^i_{j\alpha 1})X'^\alpha - \bar{R}^i_{j\alpha 1}X'^\alpha.
\end{aligned}
$$

By Lemma 8, we have

$$
\begin{aligned}
\dot{Y}'^i_j(0,t) &= -\bar{R}^i_{j\alpha 1} X'^\alpha + \mathcal{B}(\varepsilon t) \\
&= -\bar{R}^i_{j21} X'^2 + \mathcal{B}(\varepsilon t) \\
&= (\delta^i_1 \delta^2_j - \delta^i_2 \delta^1_j)\sin t + \mathcal{B}(\varepsilon t).
\end{aligned}
$$

Integrating this equality, we deduce

$$
\begin{aligned}
Y'^i_j(0,t) &= Y'^i_j(0,0) + \int_0^t \dot{Y}'^i_j(0,s)ds \\
&= \delta^i_2 \delta^1_j - \delta^i_1 \delta^2_j + (\delta^i_1 \delta^2_j - \delta^i_2 \delta^1_j)\int_0^t \sin s\, ds + \mathcal{B}(\varepsilon t^2) \\
&= (\delta^i_2 \delta^1_j - \delta^i_1 \delta^2_j)\cos t + \mathcal{B}(\varepsilon t^2).
\end{aligned}
$$

2) Differentiating equation (B.4) twice with respect to the parameter θ, we obtain

$$\dot{Y}''^i_j + 2\partial_p \Gamma^i_{kl} \dot{X}^l X'^p Y'^k_j + \partial_p \Gamma^i_{kl} \dot{X}^l X''^p Y^k_j + 2\partial_p \Gamma^i_{kl} \dot{X}'^l X'^p Y^k_j +$$
$$\partial^2_{pq} \Gamma^i_{kl} \dot{X}^l X'^p X'^q Y^k_j + \Gamma^i_{kl}(\dot{X}''^l Y^k_j + 2\dot{X}'^l Y'^k_j + \dot{X}^l Y''^k_j) = 0,$$

with the initial condition

$$Y'''^i_j(0,0) = -\delta^i_1\delta^1_j - \delta^i_2\delta^2_j.$$

which yields on the axis

$$\dot{Y}''^i_j + 2R^i_{k\alpha 1}X'^\alpha Y'^k_j + R^i_{j\alpha 1}X''^\alpha + 2\partial_\alpha \Gamma^i_{\beta j}X'^\alpha \dot{X}'^\beta + \partial^2_{\alpha\beta}\Gamma^i_{1j}X'^\alpha X'^\beta = 0,$$

We consider the cases $j > 1$. Using (2.16),(2.17) and Lemma 8, we get

$$\begin{aligned}
\dot{Y}''^i_j &= -2R^i_{k\alpha 1}X'^\alpha Y'^k_j - \frac{2}{3}(R^i_{\beta\alpha j} + R^i_{j\alpha\beta})X'^\alpha \dot{X}'^\beta + \mathcal{B}(\varepsilon t) \\
&= -2\bar{R}^i_{k\alpha 1}X'^\alpha Y'^k_j - \frac{2}{3}(\bar{R}^i_{\beta\alpha j} + \bar{R}^i_{j\alpha\beta})X'^\alpha \dot{X}'^\beta + \mathcal{B}(\varepsilon t) \\
&= -2\delta^2_j \bar{R}^i_{121}X^{2'}Y^{1'}_2 - \frac{2}{3}(\bar{R}^i_{22j} + \bar{R}^i_{j22})X'^2 \dot{X}'^2 + \mathcal{B}(\varepsilon t) \\
&= 2\delta^2_j \bar{R}^i_{121}\sin t \cos t - \frac{2}{3}\bar{R}^i_{22j}\sin t \cos t + \mathcal{B}(\varepsilon t) \\
&= 2\delta^i_2\delta^2_j \sin t \cos t + \frac{2}{3}(\delta^i_j - \delta^i_2\delta^2_j)\sin t \cos t + \mathcal{B}(\varepsilon t).
\end{aligned}$$

which gives by integration

$$\begin{aligned}
Y'''^i_j(0,t) &= Y'''^i_j(0,0) + \int_0^t \dot{Y}''^i_j(0,s)ds \\
&= -\delta^i_2\delta^2_j + 2\delta^i_2\delta^2_j \int_0^t \sin s \cos s\, ds + \\
&\quad \frac{2}{3}(\delta^i_j - \delta^i_2\delta^2_j)\int_0^t \sin s \cos s\, ds + \mathcal{B}(\varepsilon t^2) \\
&= -\delta^i_2\delta^2_j + \delta^i_2\delta^2_j \sin^2 t + \frac{1}{3}(\delta^i_j - \delta^i_2\delta^2_j)\sin^2 t + \mathcal{B}(\varepsilon t^2) \\
&= -\delta^i_2\delta^2_j \cos^2 t + \frac{1}{3}(\delta^i_j - \delta^i_2\delta^2_j)\sin^2 t + \mathcal{B}(\varepsilon t^2).
\end{aligned}$$

Similarly, when $j = 1$, it follows from Lemmas 1 and 8

$$\begin{aligned}
\dot{Y}''^i_1 &= -2R^i_{k\alpha 1}X'^\alpha Y'^k_j - R^i_{1\alpha 1}X''^\alpha - 2\partial_\alpha \Gamma^i_{\beta 1}X'^\alpha \dot{X}'^\beta + \partial^2_{\alpha\beta}\Gamma^i_{11}X'^\alpha X'^\beta \\
&= -4R^i_{221}\sin t \cos t + \mathcal{B}(\varepsilon t^2) \\
&= -4R^i_{221}(0)\sin t \cos t + \mathcal{B}(\varepsilon t) \\
&= 4\delta^i_1 \sin t \cos t + \mathcal{B}(\varepsilon t)
\end{aligned}$$

Recalling the initial condition $Y'''^i_1(0,0) = -\delta^i_1$, this yields

$$Y'''^i_1(0,t) = -\cos 2t\delta^i_1 + \mathcal{B}(\varepsilon t^2).$$

We prove 2).

3) The first part comes directly 2).

For the second part, differentiating equation (B.4) twice with respect to θ and once with respect to t respectively

$$\ddot{Y}''^{i}_{j} + 2\partial^2_{pq}\Gamma^i_{kl}\dot{X}^l\dot{X}^p X'^q Y'^k_j + 2\partial_p\Gamma^i_{kl}(\ddot{X}^l X'^p Y'^k_j + \dot{X}^l\dot{X}'^p Y'^k_j +$$

$$\dot{X}^l X'^p \dot{Y}'^k_j) + \partial^2_{pq}\Gamma^i_{kl}\dot{X}^l\dot{X}^p X''^q Y^k_j + \partial_p\Gamma^i_{kl}(\ddot{X}^l X''^p Y^k_j + \dot{X}^l\dot{X}''^p Y^k_j +$$

$$\dot{X}^l X''^p \dot{Y}^k_j) + 2\partial^2_{pq}\Gamma^i_{kl}\dot{X}'^l\dot{X}^p X'^q Y^k_j + 2\partial_p\Gamma^i_{kl}(\ddot{X}'^l X'^p Y^k_j + \dot{X}'^l\dot{X}'^p Y^k_j +$$

$$\dot{X}'^l X'^p \dot{Y}^k_j) + \partial^3_{apq}\Gamma^i_{kl}\dot{X}^l\dot{X}^a X'^p X'^q Y^k_j + \partial^2_{pq}\Gamma^i_{kl}(\ddot{X}^l X'^p X'^q Y^k_j +$$

$$\dot{X}^l\dot{X}'^p X'^q Y^k_j + \dot{X}^l X'^p \dot{X}'^q Y^k_j + \dot{X}^l X'^p X'^q \dot{Y}^k_j) + \partial_p\Gamma^i_{kl}(\dot{X}''^l Y^k_j +$$

$$2\dot{X}'^l Y'^k_j + \dot{X}^l Y''^k_j)\dot{X}^p + \Gamma^i_{kl}(\ddot{X}''^l Y^k_j + \dot{X}''^l \dot{Y}^k_j + 2\ddot{X}'^l Y'^k_j +$$

$$2\dot{X}'^l\dot{Y}'^k_j + \ddot{X}^l Y''^k_j + \dot{X}^l\dot{Y}''^k_j) = 0,$$

Combining (2.17) with the relations $\dot{X}(0,0) = (1,0\cdots,0)^T$, $X'(0,0) = X''(0,0) = (0\cdots,0)^T$, $\dot{X}'(0,0) = (0,1,0\cdots,0)^T$, $\dot{X}''(0,0) = (-1,0\cdots,0)^T$, $Y(0,0) = I_n$ and $\Gamma^k_{ij}(0) = \partial_1\Gamma^k_{ij}(0) = 0$, we get

$$
\begin{aligned}
\ddot{Y}''^i_j(0,0) &= -2R^i_{k21}(0)Y'^k_j(0,0) - 2\partial_2\Gamma^i_{2j}(0)\\
&= -2\delta^1_j R^i_{221}(0) + 2\delta^2_j R^i_{121}(0) - 2\delta^1_j R^i_{221}(0) - \frac{2}{3}(1-\delta^1_j)R^i_{22j}(0)\\
&= -\frac{10}{3}\delta^1_j R^i_{221}(0) + 2\delta^2_j R^i_{121}(0) - \frac{2}{3}R^i_{22j}(0)\\
&= 2\delta^2_j R^i_{121}(0) - \frac{2}{3}(1+5\delta^1_j)R^i_{22j}(0).
\end{aligned}
$$

We complete the proof of Lemma 9. $\qquad\square$

B.1.3. The behaviour of curvature matrix

In this subsection, we take account of the derivatives of the curvature matrix. Combining (2.6) and equality $\dot{\gamma}_\theta(t) = e_1(\theta,t)$, by the anti-symmetry of the Riemann curvature tensor, $R_{ij}(\theta,t) = 0$ when $i = 1$ or $j = 1$. Our main result in this subsection can be read as

PROPOSITION 4. — *Under the curvature assumptions (1.3) and (1.4) (or (1.5)) with $\varepsilon < 1$, then in the Fermi chart, we have on the axis: for all $t \in [0,\tau]$ with $\tau = |\nu_0|$*

1) $\dot{R}_{ij}(0,t) = \nabla_1 R_{1i1j}(X(0,t))$,
   $R'_{ij}(0,t) = (R_{1i2j}+R_{1j2i}+R_{121j}\delta^1_i+R_{121i}\delta^1_j)\cos t + \nabla_2 R_{1i1j}\sin t + \mathcal{B}(\varepsilon t^2)$;

2) When $i, j > 1$,

$\ddot{R}_{ij}(0, t) = \nabla_{11}^2 R_{1i1j}(X(0, t))$,

$\dot{R}'_{ij}(0, t) = (\nabla_1 R_{1i2j} + \nabla_1 R_{1j2i} + \nabla_2 R_{1i1j}) \cos t + \mathcal{B}(\varepsilon t)$,

$R''_{ij}(0, t) = [2R_{2i2j} - 2R_{1i1j} + \delta_i^2 R_{121j} + \delta_j^2 R_{121i} + 2\delta_i^1 R_{122j} + 2\delta_j^1 R_{122i} + 2R_{1212}\delta_i^1 \delta_j^1] \cos^2 t + (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + 2\nabla_2 R_{2j1i} + 2\nabla_2 R_{121i}\delta_j^1 + 2\nabla_2 R_{121j}\delta_i^1) \sin t \cos t + \mathcal{B}(\varepsilon t^2)$;

3) When $i, j > 1$,

$\dot{R}''_{ij}(0, 0) = 2\nabla_1 R_{2i2j}(0) - 3\nabla_1 R_{1i1j}(0) + \delta_i^2 \nabla_1 R_{121j(0)} + \delta_j^2 \nabla_1 R_{121i}(0) + 2\delta_i^1 \nabla_1 R_{122j}(0) + 2\delta_j^1 \nabla_1 R_{122i}(0) + 2\nabla_1 R_{1212}(0)\delta_i^1 \delta_j^1 + 2\nabla_2 R_{2i1j}(0) + 2\nabla_2 R_{2j1i}(0) + 2\nabla_2 R_{121i}(0)\delta_j^1 + 2\nabla_2 R_{121j}(0)\delta_i^1$,

$\ddot{R}''_{ij}(0, 0)$
$= -[2R_{2i2j}(0) - 2R_{1i1j}(0) + \delta_i^2 R_{121j}(0) + \delta_j^2 R_{121i}(0) + 2\delta_i^1 R_{122j}(0) + 2\delta_j^1 R_{122i}(0) + 2R_{1212}(0)\delta_i^1 \delta_j^1] + [2\nabla_1\nabla_1 R_{2i2j}(0) - 4\nabla_1\nabla_1 R_{1i1j}(0) + \delta_i^2 \nabla_1\nabla_1 R_{121j}(0) + \delta_j^2 \nabla_1\nabla_1 R_{121i}(0) + 2\delta_i^1 \nabla_1\nabla_1 R_{122j}(0) + 2\delta_j^1 \nabla_1\nabla_1 R_{122i}(0) + 2\nabla_1\nabla_1 R_{1212}(0)\delta_i^1 \delta_j^1] + 2[2\nabla_1\nabla_2 R_{2i1j}(0) + 2\nabla_1\nabla_2 R_{2j1i}(0) + 2\nabla_1\nabla_2 R_{121i}(0)\delta_j^1 + 2\nabla_1\nabla_2 R_{121j}(0)\delta_i^1] + \mathcal{B}(\varepsilon)$
$= \mathcal{B}(\varepsilon)$.

Proof. As we see $R_{ij}(\theta, t) = 0$ when $i = 1$ or $j = 1$, we study the cases $i, j > 1$.

Recall $e_i(\theta, t) = Y_i^j(\theta, t)\partial_j \ \forall i$, in particular $e_1(\theta, t) = \dot{X}^j(\theta, t)\partial_j$. By (2.6), we can write

$$\begin{aligned} R_{ij}(\theta, t) &= \langle R(e_i(\theta, t), e_1(\theta, t))e_1(\theta, t), e_j(\theta, t)\rangle \\ &= R_{dcba}\dot{X}^a Y_i^b \dot{X}^c Y_j^d \\ &= R_{abcd}\dot{X}^a Y_i^b \dot{X}^c Y_j^d. \end{aligned} \tag{B.5}$$

Here the last equality follows from the symmetry of the Riemann curvature tensor.

1) Differentiating (B.5) with respect to t, we deduce

$$\begin{aligned} \dot{R}_{ij}(\theta, t) &= \partial_p R_{abcd}\dot{X}^a Y_i^b \dot{X}^c Y_j^d \dot{X}^p + R_{abcd}\partial_t(\dot{X}^a Y_i^b \dot{X}^c Y_j^d) \\ &= \partial_p R_{abcd}\dot{X}^a Y_i^b \dot{X}^c Y_j^d \dot{X}^p + R_{abcd}(\ddot{X}^a Y_i^b \dot{X}^c Y_j^d + \dot{X}^a \dot{Y}_i^b \dot{X}^c Y_j^d \\ &\quad + \dot{X}^a Y_i^b \ddot{X}^c Y_j^d + \dot{X}^a Y_i^b \dot{X}^c \dot{Y}_j^d). \end{aligned}$$

Using the relations $X(0, t) = (t, 0, \cdots, 0)^T, Y(0, t) = I_n$, we infer on the axis

$$\dot{R}_{ij}(0, t) = \partial_1 R_{1i1j}(X(0, t)).$$

Recall the first covariant derivative formula

$$\nabla_p R_{abcd} = \partial_p R_{abcd} - \Gamma_{pa}^h R_{hbcd} - \Gamma_{pb}^h R_{ahcd} - \Gamma_{pc}^h R_{abhd} - \Gamma_{pd}^h R_{abch}.$$

Since the Christoffel symbols vanish identically on the axis, we have

$$\dot{R}_{ij}(0,t) = \nabla_1 R_{1i1j}. \tag{B.6}$$

Differentiating (B.5) with respect to θ, we obtain

$$
\begin{aligned}
R'_{ij}(\theta,t) &= \partial_p R_{abcd} \dot{X}^a Y_i^b \dot{X}^c Y_j^d X'^p + R_{abcd} \partial_\theta (\dot{X}^a Y_i^b \dot{X}^c Y_j^d) \\
&= \partial_p R_{abcd} \dot{X}^a Y_i^b \dot{X}^c Y_j^d X'^p + R_{abcd}(\dot{X'}^a Y_i^b \dot{X}^c Y_j^d \\
&\quad + \dot{X}^a Y'^b_i \dot{X}^c Y_j^d + \dot{X}^a Y_i^b \dot{X'}^c Y_j^d + \dot{X}^a Y_i^b \dot{X}^c Y'^d_j).
\end{aligned} \tag{B.7}
$$

Using the relations $X(0,t) = (t,0,\cdots,0)^T, X'^1(0,t) = 0, Y(0,t) = I_n$, we infer

$$
\begin{aligned}
R'_{ij}(0,t) &= \partial_\alpha R_{1i1j} X^{\alpha'} + R_{\alpha i1j}\dot{X}^{\alpha'} + R_{1\alpha 1j} Y_i^{\alpha'} + R_{1i\alpha j}\dot{X}^{\alpha'} + R_{1i1\alpha} Y_j^{\alpha'} \\
&= \nabla_\alpha R_{1i1j} X^{\alpha'} + (R_{\alpha i1j} + R_{1i\alpha j})\dot{X}^{\alpha'} + R_{1\alpha 1j} Y_i^{\alpha'} + R_{1i1\alpha} Y_j^{\alpha'},
\end{aligned}
$$

since the Christoffel symbols vanish identically on the axis. From Lemma 8 and Lemma 9, we get

$$
\begin{aligned}
R'_{ij}(0,t) &= \nabla_\alpha R_{1i1j} X'^\alpha + (R_{\alpha i1j} + R_{1i\alpha j})\dot{X'}^\alpha + \mathcal{B}(\varepsilon t^2) \\
&= \nabla_2 R_{1i1j} X'^2 + (R_{2i1j} + R_{1i2j})\dot{X'}^2 + \mathcal{B}(\varepsilon t^2) \\
&= \nabla_2 R_{1i1j} \sin t + (R_{1i2j} + R_{1j2i}) \cos t + \mathcal{B}(\varepsilon t^2).
\end{aligned}
$$

2) Differentiating (B.6) with respect to t and using the relation $X(0,t) = (t,0,\cdots,0)^T$, we get

$$\ddot{R}_{ij}(0,t) = \partial_1 \nabla_1 R_{1i1j}(X(0,t)) = \nabla_{11}^2 R_{1i1j}(X(0,t)),$$

since the Christoffel symbols vanish identically on the axis. Differentiating (B.5) with respect to θ and t respectively, there holds

$$
\begin{aligned}
\dot{R}'_{ij}(\theta,t) &= \partial^2_{pq}R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j \dot{X}^q X'^p + \partial_p R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j \dot{X}'^p + \\
&\quad \partial_p R_{abcd}\partial_t(\dot{X}^a Y^b_i \dot{X}^c Y^d_j)X'^p + \partial_p R_{abcd}\partial_\theta(\dot{X}^a Y^b_i \dot{X}^c Y^d_j)\dot{X}^p + \\
&\quad R_{abcd}\partial^2_{t\theta}(\dot{X}^a Y^b_i \dot{X}^c Y^d_j) \\
&= \partial^2_{pq}R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j \dot{X}^q X'^p + \partial_p R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j \dot{X}'^p + \\
&\quad \partial_p R_{abcd}(\ddot{X}^a Y^b_i \dot{X}^c Y^d_j + \dot{X}^a \dot{Y}^b_i \dot{X}^c Y^d_j + \dot{X}^a Y^b_i \ddot{X}^c Y^d_j + \\
&\quad \dot{X}^a Y^b_i \dot{X}^c \dot{Y}^d_j)X'^p + \partial_p R_{abcd}(\dot{X}'^a Y^b_i \dot{X}^c Y^d_j + \dot{X}^a Y'^b_i \dot{X}^c Y^d_j + \\
&\quad \dot{X}^a Y^b_i \dot{X}'^c Y^d_j + \dot{X}^a Y^b_i \dot{X}^c Y'^d_j)\dot{X}^p + R_{abcd}(\ddot{X}'^a Y^b_i \dot{X}^c Y^d_j + \\
&\quad \dot{X}'^a \dot{Y}^b_i \dot{X}^c Y^d_j + \dot{X}'^a Y^b_i \ddot{X}^c Y^d_j + \dot{X}'^a Y^b_i \dot{X}^c \dot{Y}^d_j + \ddot{X}^a Y'^b_i \dot{X}^c Y^d_j + \\
&\quad \dot{X}^a \dot{Y}'^b_i \dot{X}^c Y^d_j + \dot{X}^a Y'^b_i \ddot{X}^c Y^d_j + \dot{X}^a Y'^b_i \dot{X}^c \dot{Y}^d_j + \ddot{X}^a Y^b_i \dot{X}'^c Y^d_j + \\
&\quad \dot{X}^a \dot{Y}^b_i \dot{X}'^c Y^d_j + \dot{X}^a Y^b_i \ddot{X}'^c Y^d_j + \dot{X}^a Y^b_i \dot{X}'^c \dot{Y}^d_j + \ddot{X}^a Y^b_i \dot{X}^c Y'^d_j + \\
&\quad \dot{X}^a \dot{Y}^b_i \dot{X}^c Y'^d_j + \dot{X}^a Y^b_i \ddot{X}^c Y'^d_j + \dot{X}^a Y^b_i \dot{X}^c \dot{Y}'^d_j).
\end{aligned}
$$

Using the relations $X(0,t) = (t,0,\cdots,0)^T, X'^1(0,t) = 0, Y(0,t) = I_n$, we infer on the axis

$$
\begin{aligned}
\dot{R}'_{ij}(0,t) &= \partial^2_{1\alpha}R_{1i1j}X'^\alpha + \partial_\alpha R_{1i1j}\dot{X}'^\alpha + \partial_1 R_{\alpha i1j}\dot{X}'^\alpha + \\
&\quad \partial_1 R_{1\alpha 1j}Y'^\alpha_i + \partial_1 R_{1i\alpha j}\dot{X}'^\alpha + \partial_1 R_{1i1\alpha}Y'^\alpha_j + \\
&\quad R_{\alpha i1j}\ddot{X}'^\alpha + R_{1\alpha 1j}\dot{Y}'^\alpha_i + R_{1i\alpha j}\ddot{X}'^\alpha + R_{1i1\alpha}\dot{Y}'^\alpha_j \\
&= \partial^2_{1\alpha}R_{1i1j}X'^\alpha + (\partial_\alpha R_{1i1j} + \partial_1 R_{\alpha i1j} + \partial_1 R_{1i\alpha j})\dot{X}'^\alpha + \\
&\quad (R_{\alpha i1j} + R_{1i\alpha j})\ddot{X}'^\alpha + \partial_1 R_{1\alpha 1j}Y'^\alpha_i + \partial_1 R_{1i1\alpha}Y'^\alpha_j + \\
&\quad + R_{1\alpha 1j}\dot{Y}'^\alpha_i + R_{1i1\alpha}\dot{Y}'^\alpha_j.
\end{aligned}
$$

Using the fact $\nabla_p R_{abcd} = \partial_p R_{abcd}$ on the axis, it follows from Lemma 8 and Lemma 9 that on the axis there holds

$$
\partial^2_{1p}R_{abcd} = \nabla^2_{1p}R_{abcd},
$$

and

$$
\begin{aligned}
\dot{R}'_{ij}(0,t) &= \nabla^2_{1\alpha}R_{1i1j}X'^\alpha + (\nabla_\alpha R_{1i1j} + \nabla_1 R_{\alpha i1j} + \nabla_1 R_{1i\alpha j})\dot{X}'^\alpha + \\
&\quad (R_{\alpha i1j} + R_{1i\alpha j})\ddot{X}'^\alpha + \nabla_1 R_{1\alpha 1j}Y'^\alpha_i + \nabla_1 R_{1i1\alpha}Y'^\alpha_j + \\
&\quad + R_{1\alpha 1j}\dot{Y}'^\alpha_i + R_{1i1\alpha}\dot{Y}'^\alpha_j \\
&= \nabla^2_{1\alpha}R_{1i1j}X'^\alpha + (\nabla_\alpha R_{1i1j} + \nabla_1 R_{\alpha i1j} + \nabla_1 R_{1i\alpha j})\dot{X}'^\alpha + \\
&\quad (R_{\alpha i1j} + R_{1i\alpha j})\ddot{X}'^\alpha + \mathcal{B}(\varepsilon t) \\
&= \nabla^2_{12}R_{1i1j}X'^2 + (\nabla_2 R_{1i1j} + \nabla_1 R_{2i1j} + \nabla_1 R_{1i2j})\dot{X}'^2 + \\
&\quad (R_{2i1j} + R_{1i2j})\ddot{X}'^2 + \mathcal{B}(\varepsilon t) \\
&= \nabla^2_{12}R_{1i1j}\sin t + (\nabla_2 R_{1i1j} + \nabla_1 R_{2i1j} + \nabla_1 R_{1i2j})\cos t - \\
&\quad (R_{2i1j} + R_{1i2j})\sin t + \mathcal{B}(\varepsilon t) \\
&= (\nabla_1 R_{1i2j} + \nabla_1 R_{1j2i} + \nabla_2 R_{1i1j})\cos t + \mathcal{B}(\varepsilon t).
\end{aligned}
$$

Here in the last equality we used the fact that $R_{1i2j} = \mathcal{B}(\varepsilon)$.

Differentiating (B.5) twice with respect to θ, we get

$$
\begin{aligned}
R''_{ij}(\theta,t) &= \partial^2_{pq}R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j X'^p X'^q + \partial_p R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j X^{p''} + \\
&\quad 2\partial_p R_{abcd}\partial_\theta(\dot{X}^a Y^b_i \dot{X}^c Y^d_j)X'^p + R_{abcd}\partial^2_{\theta\theta}(\dot{X}^a Y^b_i \dot{X}^c Y^d_j) \\
&= \partial^2_{pq}R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j X'^p X'^q + \partial_p R_{abcd}\dot{X}^a Y^b_i \dot{X}^c Y^d_j X''^p + \\
&\quad 2\partial_p R_{abcd}(\dot{X}'^a Y^b_i \dot{X}^c Y^d_j + \dot{X}^a Y'^b_i \dot{X}^c Y^d_j + \dot{X}^a Y^b_i \dot{X}'^c Y^d_j + \\
&\quad \dot{X}^a Y^b_i \dot{X}^c Y'^d_j)X'^p + R_{abcd}(\dot{X}''^a Y^b_i \dot{X}^c Y^d_j + 2\dot{X}'^a Y'^b_i \dot{X}^c Y^d_j + \\
&\quad 2\dot{X}'^a Y^b_i \dot{X}'^c Y^d_j + 2\dot{X}'^a Y^b_i \dot{X}^c Y'^d_j + \dot{X}^a Y''^b_i \dot{X}^c Y^d_j + \\
&\quad 2\dot{X}^a Y'^b_i \dot{X}'^c Y^d_j + 2\dot{X}^a Y'^b_i \dot{X}^c Y'^d_j + \dot{X}^a \dot{Y}^b_i \dot{X}''^c Y^d_j + \\
&\quad 2\dot{X}^a Y^b_i \dot{X}'^c Y'^d_j + \dot{X}^a Y^b_i \dot{X}^c Y''^d_j).
\end{aligned}
$$

Applying the relations $X(0,t) = (t,0,\cdots,0)^T, X'^1(0,t) = 0, Y(0,t) = I_n$, we deduce on the axis

$$
\begin{aligned}
R''_{ij}(0,t) &= \partial^2_{\alpha\beta}R_{1i1j}X'^{\alpha}X'^{\beta} + \partial_p R_{1i1j}X''^{p} + 2\partial_\alpha R_{\beta i1j}X'^{\alpha}\dot{X}'^{\beta} + \\
&\quad 2\partial_\alpha R_{1\beta 1j}X'^{\alpha}Y'^{\beta}_i + 2\partial_\alpha R_{1i\beta j}X'^{\alpha}\dot{X}'^{\beta} + 2\partial_\alpha R_{1i1\beta}X'^{\alpha}Y'^{\beta}_j + \\
&\quad R_{ai1j}\dot{X}''^{a} + 2R_{\alpha b1j}\dot{X}'^{\alpha}Y'^{b}_i + 2R_{\alpha i\beta j}\dot{X}'^{\alpha}\dot{X}'^{\beta} + 2R_{\alpha i1\beta}\dot{X}'^{\alpha}Y'^{\beta}_j + \\
&\quad R_{1\beta 1j}Y''^{\beta}_i + 2R_{1\beta\alpha j}\dot{X}'^{\alpha}Y'^{\beta}_i + 2R_{1\alpha 1\beta}Y'^{\alpha}_i Y'^{\beta}_j + R_{1iaj}\dot{X}''^{a} + \\
&\quad 2R_{1i\alpha b}\dot{X}'^{\alpha}Y'^{b}_j + R_{1i1\beta}Y''^{\beta}_j \\
&= \partial^2_{\alpha\beta}R_{1i1j}X'^{\alpha}X'^{\beta} + \partial_p R_{1i1j}X''^{p} + 2(\partial_\alpha R_{\beta i1j} + \\
&\quad \partial_\alpha R_{1i\beta j})X'^{\alpha}\dot{X}'^{\beta} + 2(\partial_\alpha R_{1\beta 1j}X'^{\alpha}Y'^{\beta}_i + \partial_\alpha R_{1i1\beta}X'^{\alpha}Y'^{\beta}_j) + \\
&\quad (R_{ai1j} + R_{1iaj})\dot{X}''^{a} + 2R_{\alpha i\beta j}\dot{X}'^{\alpha}\dot{X}'^{\beta} + 2(R_{\alpha b1j}\dot{X}'^{\alpha}Y'^{b}_i + \\
&\quad R_{1i\alpha b}\dot{X}'^{\alpha}Y'^{b}_j) + 2(R_{\alpha i1\beta}\dot{X}'^{\alpha}Y'^{\beta}_j + R_{1\beta\alpha j}\dot{X}'^{\alpha}Y'^{\beta}_i) + \\
&\quad 2R_{1\alpha 1\beta}Y'^{\alpha}_i Y'^{\beta}_j + R_{1\beta 1j}Y''^{\beta}_i + R_{1i1\beta}Y''^{\beta}_j.
\end{aligned}
$$

It is clear that on the axis $\nabla_p R_{abcd} = \partial_p R_{abcd}$. On the other hand, using Lemma 1, we deduce on the axis

$$
\begin{aligned}
\partial^2_{\alpha\beta}R_{1i1j} &= \nabla^2_{\alpha\beta}R_{1i1j} + \partial_\alpha\Gamma^p_{1\beta}R_{pi1j} + \partial_\alpha\Gamma^p_{i\beta}R_{1p1j} + \\
&\quad \partial_\alpha\Gamma^p_{1\beta}R_{1ipj} + \partial_\alpha\Gamma^p_{j\beta}R_{1i1p} \\
&= \nabla^2_{\alpha\beta}R_{1i1j} + \partial_\alpha\Gamma^p_{1\beta}(R_{pi1j} + R_{1ipj}) + \partial_\alpha\Gamma^p_{i\beta}R_{1p1j} + \\
&\quad \partial_\alpha\Gamma^p_{j\beta}R_{1i1p} \\
&= \nabla^2_{\alpha\beta}R_{1i1j} + R^p_{\beta\alpha 1}(R_{1ipj} + R_{1jpi}) + \frac{1}{3}(R^p_{i\alpha\beta} + R^p_{\beta\alpha i})R_{1p1j} + \\
&\quad \frac{1}{3}(R^p_{j\alpha\beta} + R^p_{\beta\alpha j})R_{1i1p}.
\end{aligned}
$$

Thus $\partial^2_{\alpha\beta}R_{1i1j}$ are uniformly bounded on the axis. Therefore

$$
\begin{aligned}
R''_{ij}(0,t) &= \partial^2_{\alpha\beta}R_{1i1j}X'^{\alpha}X'^{\beta} + \nabla_p R_{1i1j}X''^{p} + 2(\nabla_\alpha R_{\beta i1j} + \\
&\quad \nabla_\alpha R_{1i\beta j})X'^{\alpha}\dot{X}'^{\beta} + 2(\nabla_\alpha R_{1\beta 1j}X'^{\alpha}Y'^{\beta}_i + \nabla_\alpha R_{1i1\beta}X'^{\alpha}Y'^{\beta}_j) + \\
&\quad (R_{ai1j} + R_{1iaj})\dot{X}''^{a} + 2R_{\alpha i\beta j}\dot{X}'^{\alpha}\dot{X}'^{\beta} + 2(R_{\alpha b1j}\dot{X}'^{\alpha}Y'^{b}_i + \\
&\quad R_{1i\alpha b}\dot{X}'^{\alpha}Y'^{b}_j) + 2(R_{\alpha i1\beta}\dot{X}'^{\alpha}Y'^{\beta}_j + R_{1\beta\alpha j}\dot{X}'^{\alpha}Y'^{\beta}_i) + \\
&\quad 2R_{1\alpha 1\beta}Y'^{\alpha}_i Y'^{\beta}_j + R_{1\beta 1j}Y''^{\beta}_i + R_{1i1\beta}Y''^{\beta}_j.
\end{aligned}
$$

With the help of Lemma 1 and Lemma 8, we obtain

$$
\begin{aligned}
R''_{ij}(0,t) &= \partial^2_{22}R_{1i1j}X'^2X'^2 + \nabla_1 R_{1i1j}X''^1 + 2(\nabla_2 R_{2i1j} + \\
&\quad \nabla_2 R_{1i2j})X'^2\dot{X}'^2 + 2R_{1i1j}\dot{X}''^1 + 2R_{2i2j}\dot{X}'^2\dot{X}'^2 + \\
&\quad 2(\delta_{2i}R_{211j}\dot{X}'^2 Y'^1_2 + R_{1i21}\delta_{2j}\dot{X}'^2 Y'^1_2) + \\
&\quad R_{1i1j}Y''^i_i + R_{1i1j}Y''^j_j + \mathcal{B}(\varepsilon t^2) \\
&= \partial^2_{22}R_{1i1j}\sin^2 t - \nabla_1 R_{1i1j}\sin t \cos t + 2(\nabla_2 R_{2i1j} + \\
&\quad \nabla_2 R_{1i2j})\sin t \cos t - 2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + \\
&\quad 2(\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t - (\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \\
&\quad \frac{1}{3}(\delta_{i3}+\delta_{j3}+\cdots+\delta_{in}+\delta_{jn})R_{1i1j}\sin^2 t + \mathcal{B}(\varepsilon t^2) \\
&= \partial^2_{22}R_{1i1j}\sin^2 t + (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + \\
&\quad 2\nabla_2 R_{1i2j})\sin t \cos t - 2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + \\
&\quad (\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \\
&\quad \frac{1}{3}(\delta_{i3}+\delta_{j3}+\cdots+\delta_{in}+\delta_{jn})R_{1i1j}\sin^2 t + \mathcal{B}(\varepsilon t^2).
\end{aligned}
$$

In view of (2.17) and by the formula for the second covariant derivative, we have on the axis

$$
\begin{aligned}
\partial^2_{22}R_{1i1j} &= \nabla^2_{22}R_{1i1j} + \partial_2\Gamma^p_{21}R_{pi1j} + \partial_2\Gamma^p_{2i}R_{1p1j} + \partial_2\Gamma^p_{21}R_{1ipj} + \partial_2\Gamma^p_{2j}R_{1i1p} \\
&= \nabla^2_{22}R_{1i1j} + R^p_{221}(R_{pi1j} + R_{1ipj}) + \frac{1}{3}R^p_{22i}R_{1p1j} + \frac{1}{3}R^p_{22j}R_{1i1p} \quad \text{(B.8)} \\
&= -\frac{8}{3}R_{1i1j} + \frac{1}{3}(\delta_{2i}+\delta_{2j})R_{1i1j} + \mathcal{B}(\varepsilon).
\end{aligned}
$$

Therefore, we infer on the axis

$$
\begin{aligned}
R''_{ij}(0,t) &= -\frac{8}{3}R_{1i1j}\sin^2 t + \frac{1}{3}(\delta_{2i}+\delta_{2j})R_{1i1j}\sin^2 t + \\
&\quad (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + 2\nabla_2 R_{1i2j})\sin t \cos t - \\
&\quad 2R_{1i1j}\cos(2t) + 2R_{2i2j}\cos^2 t + (\delta_{2i}+\delta_{2j})R_{1i1j}\cos^2 t + \\
&\quad \frac{1}{3}(\delta_{i3}+\delta_{j3}+\cdots+\delta_{in}+\delta_{jn})R_{1i1j}\sin^2 t + \mathcal{B}(\varepsilon t^2).
\end{aligned}
$$

Since for $i, j > 1, \delta_{i3} + \cdots + \delta_{in} = 1 - \delta_{2i}, \delta_{j3} + \cdots + \delta_{jn} = 1 - \delta_{2j}$ so that there holds on the axis

$$
\begin{aligned}
R_{ij}''(0, t) &= -\frac{8}{3} R_{1i1j} \sin^2 t + \frac{1}{3}(\delta_{2i} + \delta_{2j}) R_{1i1j} \sin^2 t + \\
&\quad (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + 2\nabla_2 R_{1i2j}) \sin t \cos t - \\
&\quad 2R_{1i1j} \cos(2t) + 2R_{2i2j} \cos^2 t + (\delta_{2i} + \delta_{2j}) R_{1i1j} \cos^2 t + \\
&\quad \frac{1}{3}(2 - \delta_{2i} - \delta_{2j}) R_{1i1j} \sin^2 t + \mathcal{B}(\varepsilon t^2) \\
&= -2R_{1i1j} \sin^2 t + (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{2i1j} + \\
&\quad 2\nabla_2 R_{1i2j}) \sin t \cos t - 2R_{1i1j} \cos(2t) + 2R_{2i2j} \cos^2 t + \\
&\quad (\delta_{2i} + \delta_{2j}) R_{1i1j} \cos^2 t + \mathcal{B}(\varepsilon t^2) \\
&= 2(R_{2i2j} - R_{1i1j}) \cos^2 t + (-\nabla_1 R_{1i1j} + 2\nabla_2 R_{1i2j} + \\
&\quad 2\nabla_2 R_{1j2i}) \sin t \cos t + (\delta_{2i} + \delta_{2j}) R_{1i1j} \cos^2 t + \mathcal{B}(\varepsilon t^2).
\end{aligned}
$$

3) It is a direct consequence of the second part. This ends the proof of the Proposition 4. $\qquad\square$

As a direct consequence of Proposition 4, we have the following result.

COROLLARY 2. — *Under the curvature assumptions (1.3) and (1.4) (or (1.5)) with $\varepsilon < 1$, then in the Fermi chart, we have on the axis: for all $t \in [0, \tau]$ with $\tau = |\nu_0|$*

1)

$$
\dot{R}'(0, 0) = \begin{bmatrix} 0 & 0 \\ 0 & \nabla_1 R_{1i2j} + \nabla_1 R_{1j2i} + \nabla_2 R_{1i1j} \end{bmatrix};
$$

2)

$$
R'(0, t) = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B}(\varepsilon) \end{bmatrix}, \dot{R}'(0, t) = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B}(\varepsilon) \end{bmatrix}, R''(0, t) = \begin{bmatrix} 0 & 0 \\ 0 & \mathcal{B}(\varepsilon) \end{bmatrix}.
$$

B.1.4. Asymptotic behaviour near the origin

In this part, we deduce the asymptotic behavior of the $\mathcal{C}$-curvature near the origin. We first take account of the asymptotic behavior of the coefficient a_{11} which involves $\ddot{S}$.

LEMMA 10. — *Under the curvature assumptions (1.3) and (1.4) (or (1.5)) with $\varepsilon < 1$, we have on the axis for small $t \geqslant 0$*

$$S(0,t) = I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0;0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6), \quad \text{(B.9)}$$

$$\dot{S}(0,t) = -\frac{2}{3}tR(0,0) - \frac{t^2}{4}\dot{R}(0,0) - \frac{4}{45}t^3 A + O(\varepsilon t^3 + t^5), \quad \text{(B.10)}$$

$$\ddot{S}(0,t) = -\frac{2}{3}R(0,0) - \frac{t}{2}\dot{R}(0,0) - \frac{4}{15}t^2 A + O(\varepsilon t^2 + t^4), \quad \text{(B.11)}$$

$$S'(0,t) = -\frac{t^2}{3}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6), \quad \text{(B.12)}$$

$$\dot{S}'(0,t) = -\frac{2}{3}tR'(0,0) - \frac{t^2}{4}\dot{R}'(0,0) + O(\varepsilon t^3 + t^5), \quad \text{(B.13)}$$

$$S''(0,t) = -\frac{t^2}{3}R''(0,0) - \frac{t^3}{12}\dot{R}''(0,0) + O(\varepsilon t^4 + t^6). \quad \text{(B.14)}$$

where $A = \begin{bmatrix} 0 & 0 \\ 0 & I_{n-1} \end{bmatrix}$ and R is given by (2.6) (see also Section B.1.3)

REMARK 4. — *1) The formula (B.9) can also recover the expression of the $\mathcal{C}$-curvature in the special case $\nu_0 = 0$. Using a Riemannian normal coordinate system at m_0, we get from (1.1),*

$$\mathcal{C}(m_0,0)(\xi,\eta) = -\frac{3}{2}\frac{d^2}{ds^2}\big|_{s=0}\langle S(m_0,\eta,s)(\xi),\xi\rangle$$
$$= \langle R(0,0)\xi,\xi\rangle = R_{m_0}(\xi,\eta,\xi,\eta).$$

Proof. Let $\tilde{J}_a(t)$ be the solution of the following second order equation

$$\ddot{\tilde{J}}_a(t) + R(0,0)\tilde{J}_a(t) = 0, \forall a = 0, 1$$

with the initial conditions

$$\tilde{J}_0(0) = 0, \dot{\tilde{J}}_0(0) = I_n,$$
$$\tilde{J}_1(0) = I_n, \dot{\tilde{J}}_1(0) = 0.$$

From the representation formula (2.9), we derive

$$J_0(t) = \tilde{J}_0(t) + \tilde{J}_0(t)\int_0^t \tilde{J}_1^*(s)[R(0,0) - R(0,s)]J_0(s)ds$$
$$- \tilde{J}_1(t)\int_0^t \tilde{J}_0^*[R(0,0) - R(0,s)]J_0 ds.$$

It follows from the Taylor formula and Proposition 4,

$$R(0,t) = R(0,0) + t\dot{R}(0,0) + O(\varepsilon t^2).$$

On the other hand, we have $\tilde{J}_0(t) = tI_n - \frac{t^3}{6}R(0,0) + \frac{t^5}{120}(R(0,0))^2 + O(t^7)$ so that

$$
\begin{aligned}
J_0(0,t) &= \tilde{J}_0(t) - \frac{t^4}{12}\dot{R}(0,0) + O(\varepsilon t^5) \\
&= tI_n - \frac{t^3}{6}R(0,0) - \frac{t^4}{12}\dot{R}(0,0) + \frac{t^5}{120}(R(0,0))^2 + O(\varepsilon t^5 + t^7) \\
&= tI_n - \frac{t^3}{6}R(0,0) - \frac{t^4}{12}\dot{R}(0,0) + \frac{t^5}{120}A + O(\varepsilon t^5 + t^7).
\end{aligned}
$$

Here we use the fact $(R(0,0))^2 = A + O(\varepsilon)$. Similarly, we have

$$
J_1(0,t) = I_n - \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}A + O(\varepsilon t^4 + t^6).
$$

Gathering the above estimates, we infer

$$
\begin{aligned}
S(0,t) &= tJ_0(0,t)^{-1}J_1(0,t) \\
&= t[tI_n - \frac{t^3}{6}R(0,0) - \frac{t^4}{12}\dot{R}(0,0) + \frac{t^5}{120}A + O(\varepsilon t^5 + t^7)]^{-1}[I_n - \\
&\quad \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}A + O(\varepsilon t^4 + t^6)] \\
&= [I_n + \frac{t^2}{6}R(0,0) + \frac{t^3}{12}\dot{R}(0,0) + \frac{7}{360}t^4 A + O(\varepsilon t^4 + t^6)][I_n - \\
&\quad \frac{t^2}{2}R(0,0) - \frac{t^3}{6}\dot{R}(0,0) + \frac{t^4}{24}A + O(\varepsilon t^4 + t^6)] \\
&= I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{12}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6).
\end{aligned}
$$

As the term $\ddot{S}$ involves $J_0^{-1}\dot{J}_0$ and $J_0^{-1}\dot{J}_1$, we cosider the expansion of $J_0^{-1}\dot{J}_0$ and $J_0^{-1}\dot{J}_1$. Differentiating (2.5) with respect to t, $\dot{J}_0(0,t)$ satisfies

$$
\begin{cases}
\ddot{\dot{J}}_0 + R\dot{J}_0 = -\dot{R}J_0, \\
\dot{J}_0(0) = I_n, \ddot{J}_0(0) = 0.
\end{cases}
$$

With the help of the representation formula (2.9) again, we obtain

$$
\dot{J}_0(0,t) = J_1(0,t) - J_0(0,t)\int_0^t J_1^*\dot{R}J_0 ds + J_1(0,t)\int_0^t J_0^*\dot{R}J_0 ds,
$$

which implies

$$
tJ_0^{-1}\dot{J}_0(0,t) = tJ_0^{-1}J_1(0,t) - t\int_0^t J_1^*\dot{R}J_0 ds + tJ_0^{-1}J_1(0,t)\int_0^t J_0^*\dot{R}J_0 ds.
$$

It follows from Proposition 4

$$
\begin{aligned}
R(0,t) &= \dot{R}(0,0) + \int_0^t \ddot{R}(0,s)ds \\
&= \dot{R}(0,0) + \int_0^t \nabla^2_{11} R(0,s)ds \\
&= \dot{R}(0,0) + O(\varepsilon t).
\end{aligned}
$$

Hence, we deduce

$$
tJ_0^{-1}\dot{J}_0(0,t) = I_n - \frac{t^2}{3}R(0,0) - \frac{t^3}{4}\dot{R}(0,0) - \frac{t^4}{45}A + O(\varepsilon t^4 + t^6) \tag{B.15}
$$

Similarly, we infer

$$
J_0^{-1}\dot{J}_1(0,t) = -R(0,0) - \frac{t}{2}\dot{R}(0,0) + O(\varepsilon t^2 + t^4). \tag{B.16}
$$

Recall the first and second derivatives of S with respect to t

$$
\dot{S} = (I_n - tJ_0^{-1}\dot{J}_0)J_0^{-1}J_1 + tJ_0^{-1}\dot{J}_1,
$$

$$
\ddot{S} = 2J_0^{-1}\dot{J}_1 - 2tJ_0^{-1}\dot{J}_0 J_0^{-1}\dot{J}_1 + 2J_0^{-1}\dot{J}_0(tJ_0^{-1}\dot{J}_0 - I_n)J_0^{-1}J_1.
$$

Together with (B.15) and (B.16), we deduce (B.10) and (B.11).

We consider $J_0^{-1}J_0'$, $J_0^{-1}J_1'$. $J_a'(0,t)$ satisfies the following equations

$$
\begin{cases}
\ddot{J}_a' + RJ_a' = -R'J_a, & \forall a = 0,1, \\
J_a'(0,0) = 0 = \dot{J}_a'(0,0).
\end{cases}
$$

By the representation formula (2.9), we infer $\forall t$

$$
\begin{aligned}
J_a'(0,t) = \quad &-J_0(0,t)\int_0^t J_1^* R' J_a(0,s)ds \tag{B.17} \\
&+J_1(0,t)\int_0^t J_0^* R' J_a(0,s)ds.
\end{aligned}
$$

By Proposition 4, we have

$$
R'(0,t) = R'(0,0) + t\dot{R}'(0,0) + O(\varepsilon t^2).
$$

Hence, we infer

$$
\begin{aligned}
J_0^{-1}J_0'(0,t) &= -\int_0^t J_1^* R' J_a ds + J_0^{-1}J_1(0,t)\int_0^t J_0^* R' J_a ds \tag{B.18} \\
&= -\frac{t^2}{6}R'(0,0) - \frac{t^3}{12}\dot{R}'(0,0) + O(\varepsilon t^4 + t^6).
\end{aligned}
$$

With the same arguments, we get

$$J_0^{-1} J_1'(0,t) \; = \; -\frac{t}{2} R'(0,0) - \frac{t^2}{6} \dot{R}'(0,0) + O(\varepsilon t^3 + t^5). \quad \text{(B.19)}$$

On the other hand $S' = -t J_0^{-1} J_0' J_0^{-1} J_1 + t J_0^{-1} J_1'$. Together with (B.18) and (B.19), we prove (B.12). With the same arguments, we estimate on the axis

$$J_0^{-1} \dot{J}_0'(0,t) \; = \; -\frac{t}{2} R'(0,0) - \frac{t^2}{3} \dot{R}'(0,0) + O(\varepsilon t^3 + t^5), \quad \text{(B.20)}$$

$$J_0^{-1} \dot{J}_1'(0,t) \; = \; -R'(0,0) - \frac{t}{2} \dot{R}'(0,0) + O(\varepsilon t^2 + t^4), \quad \text{(B.21)}$$

which yields (B.13).

It is obvious the $J_a''(0,t)$ for $a = 1,2$ satisfies the following equations

$$\begin{cases} \ddot{J}_a'' + R J_a'' = -R'' J_a - 2R' J_a', \\ J_a''(0) = 0 = \dot{J}_a''(0). \end{cases}$$

Applying the representation formula (2.9), we infer on the axis

$$J_a''(0,t) \; = -J_0(0,t) \int_0^t J_1^*(R'' J_a + 2R' J_a')(0,s)ds$$

$$+ J_1(0,t) \int_0^t (R'' J_a + 2R' J_a')(0,s)ds. \quad \text{(B.22)}$$

By Proposition 4, we have $R''(0,t) = R''(0) + t\dot{R}''(0) + O(\varepsilon t^2)$ so that

$$J_0^{-1} J_0''(0,t) \; = \; -\int_0^t J_1^*(R'' J_a + 2R' J_a')ds + (J_0^{-1} J_1)(0,t) \int_0^t (R'' J_a + 2R' J_a')ds$$

$$= \; -\frac{t^2}{6} R''(0,0) - \frac{t^3}{12} \dot{R}''(0,0) + O(\varepsilon t^4 + t^6). \quad \text{(B.23)}$$

Similarly, we have

$$J_0^{-1} J_1''(0,t) \; = \; -\frac{t}{2} R''(0,0) - \frac{t^2}{6} \dot{R}''(0,0) + O(\varepsilon t^3 + t^5). \quad \text{(B.24)}$$

On the other hand, we remark $S'' = 2t J_0^{-1} J_0' J_0^{-1} J_0' J_1 - t J_0^{-1} J_0'' J_0^{-1} J_1 - 2t J_0^{-1} J_0' J_0^{-1} J_1' + t J_0^{-1} J_1''$. Together with (B.23) and (B.24), we deduce (B.14). We finish the proof of Lemma 10. □

As consequences of Lemma 10, the coefficients a_{11}, a_{12} and a_{12} in the $\mathcal{C}$-curvature (3.5) have the following expansion

COROLLARY 3. — *Under the same assumptions as in Lemma 10, we have on the axis*

$$a_{11}(m_0, \nu_0, \xi) = R(\xi, E_1, \xi, E_1) + \tfrac{3}{4}\tau(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \nabla_1 R_{1313}\xi_3^2) + \tfrac{2}{5}\tau^2(\xi_2^2 + \xi_3^2) + O(\varepsilon\tau^2 + \tau^4)(\xi_2^2 + \xi_3^2) \quad (B.25)$$

$$a_{12}(m_0, \nu_0, \xi) = 2R(\xi, E_1, \xi, E_2) + \tau[\tfrac{1}{2}\nabla_2 R_{1212}\xi_2^2 + (\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \tfrac{1}{2}\nabla_2 R_{1313})\xi_3^2 - \nabla_1 R_{1212}\xi_1\xi_2 - \nabla_1 R_{1213}\xi_1\xi_3] - \tfrac{2}{5}\tau^2\xi_1\xi_2 + \quad (B.26)$$
$$O(\varepsilon\tau^2 + \tau^3)(\xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3).$$

$$a_{22}(m_0, \nu_0, \xi) = R(\xi, E_2, \xi, E_2) + \tau[\tfrac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \tfrac{1}{2}\nabla_2 R_{1212}\xi_1\xi_2 - \tfrac{1}{2}(\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3 + \tfrac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + \quad (B.27)$$
$$(\tfrac{1}{4}\nabla_1 R_{2323} + \tfrac{1}{2}\nabla_2 R_{1323})\xi_3^2] + \tfrac{\tau^2}{15}(\xi_1^2 + \xi_2^2 + 2\xi_3^2) + O(\varepsilon\tau^2 + \tau^4).$$

Proof. We study first the coefficients a_{11}. In view of (B.11), we calculate

$$a_{11}(m_0, \nu_0, \xi) = \langle R(0,0)\xi, \xi \rangle + \tfrac{3}{4}\tau\langle \dot{R}(0,0)\xi, \xi \rangle + \tfrac{2}{5}\tau^2\langle A\xi, \xi \rangle + O(\varepsilon\tau^2 + \tau^4)(\xi_2^2 + \xi_3^2)$$
$$= R_{1212}\xi_2^2 + 2R_{1213}\xi_2\xi_3 + R_{1313}\xi_3^2 + \tfrac{3}{4}\tau(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \nabla_1 R_{1313}\xi_3^2) + \tfrac{2}{5}\tau^2(\xi_2^2 + \xi_3^2) + O(\varepsilon\tau^2 + \tau^4)(\xi_2^2 + \xi_3^2)$$
$$= R(\xi, E_1, \xi, E_1) + \tfrac{3}{4}\tau(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \nabla_1 R_{1313}\xi_3^2) + \tfrac{2}{5}\tau^2(\xi_2^2 + \xi_3^2) + O(\varepsilon\tau^2 + \tau^4)(\xi_2^2 + \xi_3^2).$$

Therefore, we prove (B.25).

Now we calculate the coefficients a_{12}. Noting that $\langle \xi, P\xi \rangle = 0$, thus the

coefficient a_{12} takes the form

$$
\begin{aligned}
a_{12}(m_0, \nu_0, \xi) \;=\; & -\frac{3}{\tau}\langle \dot{S}'\xi, \xi\rangle - \frac{6}{\tau}\langle \dot{S}\xi, P^\perp\xi\rangle + \frac{3}{\tau^2}\langle S'\xi, \xi\rangle + \\
& \frac{6}{\tau^2}\langle (S - I_n)\xi, P^\perp\xi\rangle.
\end{aligned}
$$

Plugging (B.9), (B.10), (B.12) and (B.13) into the above expression, we get

$$
\begin{aligned}
a_{12}(m_0, \nu_0, \xi) \;=\; & 2\langle R'(0)\xi, \xi\rangle + \frac{3}{4}\tau\langle \dot{R}'(0)\xi, \xi\rangle + \\
& 4\langle R(0)\xi, P^\perp\xi\rangle + \frac{3}{2}\tau\langle \dot{R}(0)\xi, P^\perp\xi\rangle + \frac{8}{15}\tau^2\langle A\xi, P^\perp\xi\rangle - \\
& \langle R'(0)\xi, \xi\rangle - \frac{\tau}{4}\langle \dot{R}'(0)\xi, \xi\rangle - \\
& 2\langle R(0)\xi, P^\perp\xi\rangle - \frac{\tau}{2}\langle \dot{R}(0)\xi, P^\perp\xi\rangle - \frac{2}{15}\tau^2\langle A\xi, P^\perp\xi\rangle + \\
& O(\varepsilon t^2 + t^4)(\xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3) \\
\;=\; & \langle R'(0)\xi, \xi\rangle + 2\langle R(0)\xi, P^\perp\xi\rangle + \\
& + \tau\big(\frac{1}{2}\langle \dot{R}'(0)\xi, \xi\rangle + \langle \dot{R}(0)\xi, P^\perp\xi\rangle\big) + \\
& \frac{2}{5}\tau^2\langle A\xi, P^\perp\xi\rangle + O(\varepsilon t^2 + t^4)(\xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3).
\end{aligned}
$$

In view of Proposition 4, we get

$$
\begin{aligned}
a_{12}(m_0, \nu_0, \xi) \;=\; & 2R_{1223}\xi_2\xi_3 + 2R_{1323}\xi_3^2 + 2(-R_{1212}\xi_1\xi_2 - R_{1213}\xi_1\xi_3) + \\
& \tau\big[\frac{1}{2}\nabla_2 R_{1212}\xi_2^2 + (\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_2\xi_3 + (\nabla_1 R_{1323} + \\
& \frac{1}{2}\nabla_2 R_{1313})\xi_3^2 - \nabla_1 R_{1212}\xi_1\xi_2 - \nabla_1 R_{1213}\xi_1\xi_3\big] - \\
& \frac{2}{5}\tau^2\xi_1\xi_2 + O(\varepsilon\tau^2 + \tau^3)(\xi_2^2 + \xi_3^2 + \xi_1\xi_2 + \xi_1\xi_3),
\end{aligned}
$$

which gives the desired result (B.26). Now we consider the coefficient a_{22}. We remark that $\langle P^\perp\xi, P^\perp\xi\rangle = \langle \xi, P\xi\rangle$ so that the coefficient a_{22} takes the form

$$
\begin{aligned}
a_{22}(m_0, \nu_0, \xi) \;=\; & -\frac{3}{2\tau^2}\langle S''\xi, \xi\rangle - \frac{3}{2\tau}\langle \dot{S}\xi, \xi\rangle - \frac{6}{\tau^2}\langle S'\xi, P^\perp\xi\rangle - \\
& \frac{3}{\tau^2}\langle (S - I_n)P^\perp\xi, P^\perp\xi\rangle + \frac{3}{\tau^2}\langle (S - I_n)\xi, P\xi\rangle].
\end{aligned}
$$

Plugging (B.9), (B.10), (B.12) and (B.14) into the above expression, we get

$$
\begin{aligned}
a_{22}(m_0, \nu_0, \xi) \;=\; & \frac{1}{2}\langle R''(0)\xi, \xi\rangle + \frac{\tau}{8}\langle \dot{R}''(0)\xi, \xi\rangle + \\
& \langle R(0)\xi, \xi\rangle + \frac{3\tau}{8}\langle \dot{R}(0)\xi, \xi\rangle + \frac{2}{15}\tau^2\langle A\xi, \xi\rangle + \\
& 2\langle R'(0)\xi, P^\perp\xi\rangle + \frac{\tau}{2}\langle \dot{R}'(0)\xi, P^\perp\xi\rangle + \\
& \langle R(0)P^\perp\xi, P^\perp\xi\rangle + \frac{\tau}{4}\langle \dot{R}(0)P^\perp\xi, P^\perp\xi\rangle + \frac{\tau^2}{15}\langle AP^\perp\xi, P^\perp\xi\rangle - \\
& \langle R(0)\xi, P\xi\rangle - \frac{\tau}{4}\langle \dot{R}(0)\xi, P\xi\rangle - \frac{\tau^2}{15}\langle A\xi, P\xi\rangle + O(\varepsilon\tau^2 + \tau^4) \\
=\; & \frac{1}{2}\langle R''(0)\xi, \xi\rangle + 2\langle R'(0)\xi, P^\perp\xi\rangle + \langle R(0)\xi, \xi\rangle + \\
& \langle R(0)P^\perp\xi, P^\perp\xi\rangle - \langle R(0)\xi, P\xi\rangle + \\
& \tau\Big[\frac{1}{8}\langle \dot{R}''(0)\xi, \xi\rangle + \frac{1}{2}\langle \dot{R}'(0)\xi, P^\perp\xi\rangle + \frac{3}{8}\langle \dot{R}(0)\xi, \xi\rangle + \\
& \frac{1}{4}\langle \dot{R}(0)P^\perp\xi, P^\perp\xi\rangle - \frac{1}{4}\langle \dot{R}(0)\xi, P\xi\rangle\Big] + \\
& \frac{\tau^2}{15}\big[2\langle A\xi, \xi\rangle + \langle AP^\perp\xi, P^\perp\xi\rangle - \langle A\xi, P\xi\rangle\big] + O(\varepsilon\tau^2 + \tau^4).
\end{aligned}
$$

By Proposition 4, we obtain

$$
\begin{aligned}
a_{22}(m_0, \nu_0, \xi) &= -R_{1213}\xi_2\xi_3 + (-R_{1313} + R_{2323})\xi_3^2 - 2R_{1223}\xi_1\xi_3 + \\
&\quad R_{1212}\xi_2^2 + 2R_{1213}\xi_2\xi_3 + R_{1313}\xi_3^2 + \\
&\quad R_{1212}\xi_1^2 - (R_{1212}\xi_2^2 + R_{1213}\xi_2\xi_3) + \\
&\quad \tau\{\frac{1}{8}[-\nabla_1 R_{1212}\xi_2^2 + 4(-\nabla_1 R_{1213} + \nabla_2 R_{1223})\xi_2\xi_3 + \\
&\quad (2\nabla_1 R_{2323} - 3\nabla_1 R_{1313} + 4\nabla_2 R_{1323})\xi_3^2] - \\
&\quad \frac{1}{2}[\nabla_2 R_{1212}\xi_1\xi_2 + (\nabla_1 R_{1223} + \nabla_2 R_{1213})\xi_1\xi_3] + \\
&\quad \frac{3}{8}(\nabla_1 R_{1212}\xi_2^2 + 2\nabla_1 R_{1213}\xi_2\xi_3 + \nabla_1 R_{1313}\xi_3^2) + \\
&\quad \frac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \frac{1}{4}(\nabla_1 R_{1212}\xi_2^2 + \nabla_1 R_{1213}\xi_2\xi_3)\} \\
&\quad + \frac{\tau^2}{15}(2\xi_2^2 + 2\xi_3^2 - \xi_2^2 + \xi_1^2) + O(\varepsilon\tau^2 + \tau^4) \\
&= R_{1212}\xi_1^2 - 2R_{1223}\xi_1\xi_3 + R_{2323}\xi_3^2 + \\
&\quad \tau[\frac{1}{4}\nabla_1 R_{1212}\xi_1^2 - \frac{1}{2}\nabla_2 R_{1212}\xi_1\xi_2 - \frac{1}{2}(\nabla_1 R_{1223} + \\
&\quad \nabla_2 R_{1213})\xi_1\xi_3 + \frac{1}{2}\nabla_2 R_{1223}\xi_2\xi_3 + (\frac{1}{4}\nabla_1 R_{2323} + \\
&\quad \frac{1}{2}\nabla_2 R_{1323})\xi_3^2] + \frac{\tau^2}{15}(\xi_1^2 + \xi_2^2 + 2\xi_3^2) + O(\varepsilon\tau^2 + \tau^4),
\end{aligned}
$$

which yields the desired (B.27). Therefore, we prove the result. $\square$

B.2. The inverse of the Hessian of the squared distance near the focalization

In this subsection we consider the approximation of the inverse of the squared distance and the associated derivatives.

PROPOSITION 5. — *Let (M, g) be a closed n-dimensional Riemannian manifold satisfying (1.3) and (1.4) (or (1.5)) with ε small enough. Set $(m_0, \nu_0) \in NoCut, |\nu_0| \geqslant \frac{3\pi}{4}$. Let x be the Fermi coordinate system associated to the geodesic $\exp_{m_0} t\nu_0$ for $t \in [0, 1]$ and v be the fiber coordinates of $TM \to M$ naturally associated to x. Then there exist positive numbers $\gamma_1, C_8 > 0$ such that for all $\varepsilon < \gamma_1$, we have*

1) $|S^{-1}(m_0, \nu_0, 1) - \bar{S}^{-1}(m_0, \nu_0, 1)| \leqslant C_8\varepsilon;$

2) $|\partial_x S^{-1}(m_0, \nu_0, 1) - \partial_x \bar{S}^{-1}(m_0, \nu_0, 1)| \leqslant C_8 \varepsilon,$
$|D_v S^{-1}(m_0, \nu_0, 1) - D_v \bar{S}^{-1}(m_0, \nu_0, 1)| \leqslant C_8 \varepsilon;$

3) $|\partial_{xx}^2 S^{-1}(m_0, \nu_0, 1) - \partial_{xx}^2 \bar{S}^{-1}(m_0, \nu_0, 1)| \leqslant C_8 \varepsilon,$
$|\partial_x D_v S^{-1}(m_0, \nu_0, 1) - \partial_x D_v \bar{S}^{-1}(m_0, \nu_0, 1)| \leqslant C_8 \varepsilon,$
$|D_{vv}^2 S^{-1}(m_0, \nu_0, 1) - D_{vv}^2 \bar{S}^{-1}(m_0, \nu_0, 1)| \leqslant C_8 \varepsilon.$

Proof. Thanks to Lemma 2 and Lemma 3, $\partial_x X(m_0, \nu_0, 1)$ is inversible provided $|\nu_0| \geqslant \frac{3\pi}{4}$ and ε small enough. We state

$$S^{-1}(m_0, \nu_0, 1) = (\partial_x X(m_0, \nu_0, 1))^{-1} D_v X(m_0, \nu_0, 1)$$

Thus, the desired results yield from Lemmas 2, 3, 4, 5, 6 and 7. Finally, we prove Proposition 5. □

B.3. Perturbative calculation of the Hessian of the squared distance

PROPOSITION 6. — *Let (M, g) be a closed n-dimensional Riemannian manifold satisfying (1.3) and (1.4) (or (1.5)) with $\varepsilon < \gamma$. Given any $\delta_2 \in (3\pi/4, \pi)$, let $(m_0, \nu_0) \in NoCut$ with $\tau = |\nu_0| \leqslant \delta_2$. Assume γ is small enough such that $\delta_2 < t_F(m_0, \nu_0)$ and*

$$\frac{|\nu_0|}{\sin |\nu_0|} \gamma \leqslant \frac{1}{4\sqrt{n-1}}.$$

Then there exists a positive constant $C \geqslant 1$ independent of (m_0, ν_0, γ) such that

$$|\ddot{S} - \ddot{\bar{S}}| \leqslant C \frac{\tau^4}{\sin^4 \tau} \gamma, |\dot{S} - \dot{\bar{S}}| \leqslant C \frac{\tau^4}{\sin^3 \tau} \gamma, \ |S - \bar{S}| \leqslant C \frac{\tau^4}{\sin^2 \tau} \gamma, \quad \text{(B.28)}$$

$$|S'| \leqslant C \frac{\tau^4}{\sin^2 \tau} \gamma, |\dot{S}'| \leqslant C \frac{\tau^4}{\sin^3 \tau} \gamma, |S''| \leqslant C \frac{\tau^5}{\sin^3 \tau} \gamma. \quad \text{(B.29)}$$

Proof. It is known that [8, section2]

$$|J_0^{-1} - \bar{J}_0^{-1}| \leqslant 4\sqrt{n-1} (\frac{\tau}{\sin \tau})^2 \gamma. \qquad \text{(B.30)}$$

$$|J_0^{-1}| \leqslant 2\sqrt{n-1} \frac{\tau}{\sin \tau}. \qquad \text{(B.31)}$$

We adapt the proof in [8]. We investigate $S - \bar{S}$. From the expression of $\ddot{\bar{S}}$, we have

$$
\begin{aligned}
\ddot{S} - \ddot{\bar{S}} &= 2(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_1 + 2\bar{J}_0^{-1}(\dot{J}_1 - \dot{\bar{J}}_1) - 2t(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}\dot{J}_1 - \\
&\quad 2t\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}\dot{J}_1 - 2t\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_1 - \\
&\quad 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(\dot{J}_1 - \dot{\bar{J}}_1) + 2t(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}\dot{J}_0 J_0^{-1}J_1 + \\
&\quad 2t\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}\dot{J}_0 J_0^{-1}J_1 + 2t\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}J_1 + \\
&\quad 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}J_1 + 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})J_1 + \\
&\quad 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(J_1 - \bar{J}_1) - 2t(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_0 J_0^{-1}\dot{J}_1 - \\
&\quad 2t\bar{J}_0^{-1}(\dot{J}_0 - \dot{\bar{J}}_0)J_0^{-1}\dot{J}_1 - 2t\bar{J}_0^{-1}\dot{\bar{J}}_0(J_0^{-1} - \bar{J}_0^{-1})\dot{J}_1 - \\
&\quad 2t\bar{J}_0^{-1}\dot{\bar{J}}_0\bar{J}_0^{-1}(J_1 - \bar{J}_1) + t(J_0^{-1} - \bar{J}_0^{-1})\ddot{J}_1 + t\bar{J}_0^{-1}(\ddot{J}_1 - \ddot{\bar{J}}_1).
\end{aligned}
$$

Together with (B.30)(B.31) and Lemmas 2 and 3, we infer the first estimate in (B.28). With the same arguments, we get the last two estimates in (B.28). On the other hand, using the representation formula (B.17) and (B.22) and Proposition 4, we get $|J_a'| = \mathcal{B}(\tau\gamma), |J_a''| = \mathcal{B}(\tau\gamma), |\dot{J}_0'| = \mathcal{B}(\tau\gamma)$ for $a = 0, 1$. From the expression of $S', \dot{S}', S''$, we get the desired estimates in (B.29). Therefore, Proposition 6 is proved. $\qquad\square$

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