

A NEW CONFORMAL INVARIANT ON 3-DIMENSIONAL MANIFOLDS

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ABSTRACT. By improving the analysis developed in the study of σ_k -Yamabe problem, we prove in this paper that the De Lellis-Topping inequality is true on 3-dimensional Riemannian manifolds of nonnegative scalar curvature. More precisely, if (M^3, g) is a 3-dimensional closed Riemannian manifold with non-negative scalar curvature, then

$$\int_M \left| Ric - \frac{\bar{R}}{3}g \right|^2 dv(g) \leq 9 \int_M \left| Ric - \frac{R}{3}g \right|^2 dv(g),$$

where $\bar{R} = vol(g)^{-1} \int_M R dv(g)$ is the average of the scalar curvature R of g . Equality holds if and only if (M^3, g) is a space form. We in fact study the following new conformal invariant

$$\tilde{Y}([g_0]) := \sup_{g \in \mathcal{C}_1([g_0])} \frac{vol(g) \int_M \sigma_2(g) dv(g)}{\left(\int_M \sigma_1(g) dv(g) \right)^2},$$

where $\mathcal{C}_1([g_0]) := \{g = e^{-2u}g_0 \mid R > 0\}$ and prove that $\tilde{Y}([g_0]) \leq 1/3$, which implies the above inequality.

1. INTRODUCTION

Very recently, De Lellis and Topping proved an interesting result about a generalization of Schur Lemma

Theorem A. [Almost Schur Lemma [6]] *For $n \geq 3$, if (M^n, g) is an n -dimensional closed Riemannian manifold with non-negative Ricci tensor, then*

$$(1) \quad \int_M \left| Ric - \frac{\bar{R}}{n}g \right|^2 dv(g) \leq \frac{n^2}{(n-2)^2} \int_M \left| Ric - \frac{R}{n}g \right|^2 dv(g),$$

where $\bar{R} = vol(g)^{-1} \int_M R dv(g)$ is the average of the scalar curvature R of g .

The result can be seen as a quantitative version or a stability result of the Schur Lemma. It was proved in [6] that the constant in inequality (1) is optimal and the non-negativity of the Ricci tensor can not be removed in general: When $n \geq 5$ there are examples of metrics on \mathbb{S}^n which make the ratio of the left hand side of (1) to the right hand side of (1) arbitrarily large. When $n = 3$, they found manifolds which makes the ratio arbitrarily large. An interesting question remains open: *Inequalities of this form may hold for $n = 3$ and $n = 4$ with constants depending on the topology of M .*

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With an observation that the De Lellis-Topping inequality is equivalent to an inequality in terms of σ_k -scalar curvature

$$(2) \quad \left(\int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} \text{vol}(g) \int_M \sigma_2(g) dv(g),$$

we proved in [10] that (1) holds for 4-dimensional manifolds of nonnegative scalar curvature, by using an argument of Gursky [16].

Theorem B. [10] *Let (M^4, g) is a 4-dimensional closed Riemannian manifold with non-negative scalar curvature, then*

$$(3) \quad \int_M \left| \text{Ric} - \frac{\bar{R}}{4} g \right|^2 dv(g) \leq 4 \int_M \left| \text{Ric} - \frac{R}{4} g \right|^2 dv(g),$$

where $\bar{R} = \text{vol}(g)^{-1} \int_M R dv(g)$ is the average of the scalar curvature R of g . Or equivalently, we have

$$(4) \quad \frac{8}{3} \text{vol}(g) \int_M \sigma_2(g) dv(g) \leq \left(\int_M \sigma_1(g) dv(g) \right)^2.$$

In fact, one can find inequality (4) in the argument of Gursky [16]. This argument uses a crucial property of σ_2 -scalar curvature that $\int_M \sigma_2(g) dv(g)$ is a conformal invariant, which is only true on 4-dimensional manifolds. Nevertheless, inspired by our previous work in [7] we conjectured in [10] that this is true for 3-dimensional manifolds. In this paper, by improving the analysis developed in the study of σ_k -Yamabe problem, we give an affirmative answer to this conjecture. Namely we will show that Theorem A holds under the condition of non-negativity of the scalar curvature for dimension $n = 3$.

Theorem 1. *Let (M^3, g) is a 3-dimensional closed Riemannian manifold with non-negative scalar curvature. We have*

$$(5) \quad \int_M \left| \text{Ric} - \frac{\bar{R}}{3} g \right|^2 dv(g) \leq 9 \int_M \left| \text{Ric} - \frac{R}{3} g \right|^2 dv(g).$$

Moreover, equality holds if and only if (M^3, g) is a space form.

Without the condition of non-negativity of the scalar curvature, Theorem 1 is not true. Examples can be found in [6]. When $n > 4$, Theorem A is also not true under a weaker condition that the scalar curvature is positive. For various problems related to the De Lellis-Topping inequality, see [11].

Our proof is based on the study of a new conformal invariant. From now, let $n = 3$. We define

$$(6) \quad \tilde{Y}([g_0]) := \sup_{g \in \mathcal{C}_1([g_0])} \frac{\text{vol}(g) \int_M \sigma_2(g) dv(g)}{\left(\int_M \sigma_1(g) dv(g) \right)^2},$$

where $\mathcal{C}_1([g_0]) := \{g = e^{-2u}g_0 \mid R > 0\}$ and $[g_0] := \{g = e^{-2u}g_0\}$. We define the first Yamabe constant on 3-dimensional manifolds by

$$Y_1([g_0]) := \inf_{\tilde{g} \in [g_0]} \frac{\int_M \sigma_1(\tilde{g}) dv(\tilde{g})}{(\text{vol}(\tilde{g}))^{\frac{1}{3}}}.$$

Since $\sigma_1(g) = R/2(n-1)$, the first Yamabe constant $Y_1([g_0])$ is a positive constant multiple of the ordinary Yamabe constant. Theorem 1 follows from the observation mentioned above and the following

Theorem 2. *Let (M^3, g) is a 3-dimensional closed Riemannian manifold with positive Yamabe constant $Y_1([g_0]) > 0$, then*

$$(7) \quad \tilde{Y}([g_0]) \leq \frac{1}{3}.$$

To show Theorem 2 we will study a fully nonlinear Yamabe type equation (10), which is closely related to the σ_k -Yamabe problem initiated in [23], [2] and studied by many mathematicians. (See for example [13] and [24]) Though the fully nonlinearity, the corresponding σ_k Yamabe equation shares very nice properties. (See [14] and [21]) A nice application of the analysis developed in the study of the σ_k -Yamabe problem is the 4-dimensional sphere theorem obtained by Chang-Gursky-Yang in [3]. As another application, with C.-S. Lin we obtained in [7] a 3-dimensional sphere theorem. Another proof was given by Catino-Djadli in [5]. See also [2], [8], [12], [19], [20], [25], and especially a survey paper [17] for other applications. This paper can be seen as a new application of this analysis. However, comparing to the ordinary Yamabe problem and σ_k -Yamabe problem we encounter an extra difficulty, without a corresponding Sobolev inequality, which is in fact inequality (7) that we want to prove.

Theorem 1 gives also a new characterization of three-dimensional spherical space forms. Another related characterization of three-dimensional space forms was recently given by Gursky and Viaclovsky in [18].

The paper is organized as follows. In Section 2 we consider the new conformal invariant \tilde{Y} and its related energy functional. The critical point of this energy functional satisfies a Yamabe type equation (10) below. We show in Lemma 2 that any critical point satisfies $\tilde{Y} \leq 1/3$. Hence to prove Theorem 2 we only need to prove that \tilde{Y} is achieved. This is in fact a new Yamabe type problem, with a new difficulty -without a corresponding Sobolev inequality. This problem is difficult and still remains open. Instead of attacking this problem directly we consider a suitable perturbed problem. This perturbed equation, to find it is a very delicate issue, is introduced in Section 3. In Section 4 we prove first local C^2 estimates and then global C^2 estimates for the flow, by using the local estimates. The uniform parabolicity of the flow is proved in Section 5. One of key estimates (Lemma 6) and main Theorems are proved in Section 6. Related problems and Conjectures are proposed in Section 7.

2. A NEW CONFORMAL INVARIANT AND A RELATED FLOW

Let us first recall the definition of the k -scalar curvature, which was first introduced by Viaclovsky [23] and has been intensively studied by many mathematicians, see for example the references in [7] and two survey papers [13] and [24]. Let

$$S_g = \frac{1}{n-2} \left(Ric - \frac{R}{2(n-1)} \cdot g \right)$$

be the Schouten tensor of g . For an integer k with $1 \leq k \leq n$ let σ_k be the k -th elementary symmetric function in \mathbb{R}^n . The k -scalar curvature is defined by

$$\sigma_k(g) := \sigma_k(\Lambda_g),$$

where Λ_g is the set of eigenvalue of the matrix $g^{-1} \cdot S_g$. In particular,

$$\sigma_1(g) = \frac{R}{2(n-1)}, \quad \sigma_2(g) = \frac{1}{2(n-2)^2} \left\{ -|Ric|^2 + \frac{n}{4(n-1)} R^2 \right\}.$$

We have in [10] the following observation.

Lemma 1. ([10]) *Inequality (1) is equivalent to*

$$\left(\int_M \sigma_1(g) dv(g) \right)^2 \geq \frac{2n}{n-1} vol(g) \int_M \sigma_2(g) dv(g).$$

Let g_0 be a metric on M^3 with positive scalar curvature and $\mathcal{C}_1([g_0]) := \{g \in [g_0] \mid \sigma_1(g) > 0\}$. Define an energy functional

$$(8) \quad \mathcal{E}(g) := \frac{vol(g) \int_M \sigma_2(g) dv(g)}{\left(\int_M \sigma_1(g) dv(g) \right)^2}.$$

and

$$(9) \quad \tilde{Y}([g_0]) := \sup_{g \in \mathcal{C}_1([g_0])} \mathcal{E}(g).$$

$\tilde{Y}([g_0])$ is a new conformal invariant. To show Theorem 2 is equivalent to show that this invariant is always less than or equal to $1/3$. A critical point of \mathcal{E} in $\mathcal{C}_1([g_0])$ satisfies a new Yamabe type equation

$$(10) \quad \frac{\sigma_2(g) - 3r_2(g)}{\sigma_1(g)} = -2s(g),$$

where $r_2(g)$ is the average of $\sigma_2(g)$ and $s(g)$ the average of $\frac{\sigma_2(g)}{\sigma_1(g)}$ with respect to the measure $\sigma_1(g)dv(g)$ are defined by

$$r_2(g) := \frac{\int_M \sigma_2(g) dv(g)}{vol(g)} \quad \text{and} \quad s(g) := \frac{\int_M \sigma_2(g) dv(g)}{\int_M \sigma_1(g) dv(g)}.$$

We observe that solutions of (10) have an interesting property.

Lemma 2. *Every solution $g \in \mathcal{C}_1([g_0])$ of (10) satisfies*

$$(11) \quad \mathcal{E}(g) \leq 1/3,$$

and equality if and only if g is an Einstein metric.

Proof. From the Newton inequality

$$\frac{\sigma_2(g)}{\sigma_1(g)} \leq \frac{1}{3}\sigma_1(g),$$

we have

$$\begin{aligned} \frac{1}{3} \left(\int_M \sigma_1(g) dv(g) \right)^2 &\geq \int_M \frac{\sigma_2(g)}{\sigma_1(g)} dv(g) \int_M \sigma_1(g) dv(g) \\ &= 3r_2(g) \int_M \frac{1}{\sigma_1(g)} dv(g) \int_M \sigma_1(g) dv(g) - 2s(g) \text{vol}(g) \int_M \sigma_1(g) dv(g) \\ &\geq 3r_2(g) (\text{vol}(g))^2 - 2s(g) \text{vol}(g) \int_M \sigma_1(g) dv(g) \\ &= \int_M \sigma_2(g) dv(g) \text{vol}(g). \end{aligned}$$

In the first equality we have used Equation (10) and in the second inequality the Cauchy-Schwarz inequality. It is clear to see that equality holds if and only if

$$\frac{\sigma_2(g)}{\sigma_1(g)} = \frac{1}{3}\sigma_1(g),$$

and hence if and only if (M^3, g) is an Einstein manifold. ■

Therefore, to prove Theorem 2 we only need to prove the existence of the maximum of functional \mathcal{E} in \mathcal{C}_1 . This is a new Yamabe type problem. However to prove the existence of the the maximum of functional \mathcal{E} is very difficult. One would meet not only the typical difficulty -loss of the compactness- of the ordinary Yamabe problem (and many other geometric variational problems, for example, harmonic maps, Yang-Mills fields), the fully nonlinearity of the σ_k -Yamabe problem, but also a new problem that we have not a corresponding (optimal) Sobolev inequality yet. This corresponding Sobolev inequality is

$$\sup \mathcal{E}(g) < \infty, \text{ or } \sup \mathcal{E}(g) \leq \frac{1}{3}.$$

This is in fact what we want to show. Hence we need to consider certain suitable perturbed functionals.

3. A PERTURBED PROBLEM AND ITS FLOW

As mentioned in the Introduction, to find a suitable perturbed problem is a delicate issue. Let $\varepsilon > 0$ be some small constant and $g \in \mathcal{C}_1([g_0])$. We define

$$(12) \quad \mathcal{E}_\varepsilon(g) := \frac{\left(\int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) \right) \left(\int_M e^{\varepsilon u} dv(g) - \varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^{3-\varepsilon} \right)}{\left(\int_M (\sigma_2(g) - \frac{\varepsilon}{2} e^{4u}) dv(g) \right)^\varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^2}.$$

This perturbed function is well defined in a smaller space

$$\mathcal{C}_{1,\varepsilon}([g_0]) := \{g \in \mathcal{C}_1([g_0]) \mid \int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) > 0, \int_M e^{\varepsilon u} dv(g) - \varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^{3-\varepsilon} > 0\}.$$

This functional looks quite complicated. But it satisfies all properties we want to have. Denote the maximum of \mathcal{E}_ε in $\mathcal{C}_{1,\varepsilon}([g_0])$ by

$$M_\varepsilon := \sup_{g \in \mathcal{C}_{1,\varepsilon}([g_0])} \mathcal{E}_\varepsilon(g).$$

For this perturbed energy functional, the corresponding Euler-Lagrange equation could be written as follows

$$(13) \quad \frac{\sigma_2(g) - (\nu_1(g)e^{\varepsilon u} + \nu_2(g)e^{4u})}{\sigma_1(g)} + \mu(g) = 0$$

where $\nu_1(g)$, $\nu_2(g)$ and $\mu(g)$ are given respectively

$$\nu_1(g) := (3 - \varepsilon) \frac{k(g) \left(\int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) \right)}{\int_M e^{\varepsilon u} dv(g) - \varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^{3-\varepsilon}},$$

$$\nu_2(g) := \varepsilon k(g) \left(1 - \frac{\varepsilon \int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g)}{2 \int_M (\sigma_2(g) - \frac{\varepsilon}{2} e^{4u}) dv(g)} \right),$$

$$\mu(g) := k(g) \int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) \left(\frac{2}{\int_M \sigma_1(g) dv(g)} + \frac{\varepsilon(3 - \varepsilon) \left(\int_M \sigma_1(g) dv(g) \right)^{2-\varepsilon}}{\int_M e^{\varepsilon u} dv(g) - \varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^{3-\varepsilon}} \right),$$

with

$$\begin{aligned} k(g) &:= \frac{\int_M (\sigma_2(g) - \frac{\varepsilon}{2} e^{4u}) dv(g)}{\int_M (\sigma_2(g) - \frac{\varepsilon}{2} e^{4u}) dv(g) - \varepsilon \int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g)} \\ &= \frac{\int_M (\sigma_2(g) - \frac{\varepsilon}{2} e^{4u}) dv(g)}{(1 - \varepsilon) \int_M (\sigma_2(g) - \frac{\varepsilon(1 - 2\varepsilon)}{2(1 - \varepsilon)} e^{4u}) dv(g)} \geq 1. \end{aligned}$$

By definition we have

Lemma 3. *We have*

- (i) $\frac{\nu_1(g)}{\mu(g)} \leq \frac{1}{\varepsilon \left(\int \sigma_1(g) dv(g) \right)^{2-\varepsilon}}.$
- (ii) $\tilde{Y}([g_0]) \leq \limsup_{\varepsilon \rightarrow 0} M_\varepsilon.$

Proof. The proof is easy to check. ■

We want to show that

- (1) M_ε is achieved by some $g_\varepsilon \in \mathcal{C}_{1,\varepsilon}([g_0])$ for $\varepsilon > 0$, which certainly satisfies (13).
- (2) Every solution g of (13) satisfies an estimate

$$(14) \quad \mathcal{E}_\varepsilon(g) \leq \left(\frac{2}{C\varepsilon}\right)^\varepsilon \frac{1}{3(1-\varepsilon)},$$

where C is a constant independent of ε .

This implies that $\tilde{Y}([g_0]) \leq \limsup_{\varepsilon \rightarrow 0} M_\varepsilon \leq 1/3$. Estimate (14) will be proved in Lemma 6 below. To study the achievement of M_ε , we introduce a conformal flow, which is different from the Yamabe flow considered in [7].

$$(15) \quad \frac{du}{dt} = -\frac{1}{2}g^{-1} \cdot \frac{d}{dt}g := e^{-2u} \frac{\sigma_2(g) - (\nu_1(g)e^{\varepsilon u} + \nu_2(g)e^{4u})}{\sigma_1(g)} + \mu(g)e^{-2u} + m(g),$$

where $m(g)$ is chosen by

$$\int_M \sigma_1(g) \left(e^{-2u} \frac{\sigma_2(g) - (\nu_1(g)e^{\varepsilon u} + \nu_2(g)e^{4u})}{\sigma_1(g)} + \mu(g)e^{-2u} + m(g) \right) dv(g) = 0.$$

Proposition 1. *Let $n = 3$. Flow (15) preserves $\int_M \sigma_1(g)dv(g)$, while it increases $\mathcal{E}_\varepsilon(g)$, provided $g(t) \in \mathcal{C}_{1,\varepsilon}([g_0])$.*

Proof. It is clear that the flow preserves $\int_M \sigma_1(g)dv(g)$. By a direct computation we have

$$\frac{d}{dt}\mathcal{E}_\varepsilon(g) = \frac{\mathcal{E}_\varepsilon(g) \int_M e^{-2u} \sigma_1(g) \left(\frac{\sigma_2(g) - (\nu_1(g)e^{\varepsilon u} + \nu_2(g)e^{4u})}{\sigma_1(g)} + \mu(g) \right)^2}{k(g) \int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g)} \geq 0.$$

■

Since the flow increases $\mathcal{E}_\varepsilon(g)$, the flow preserves the properties $\int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) > 0$, $\int_M e^{\varepsilon u} dv(g) - \varepsilon (\int_M \sigma_1(g) dv(g))^{3-\varepsilon} > 0$. We will show below that the flow preserves $\mathcal{C}_1([g_0])$, and hence $\mathcal{C}_{1,\varepsilon}([g_0])$. This is certainly one of crucial properties of the flow.

4. C^2 ESTIMATES

In this section, we will establish a priori estimates for flow (15). Local estimates for this class of fully nonlinear conformal equations were first given in [14]. Since then there are many extensions. See for instance [4] and the survey paper [24]. Let Γ_k^+ be a convex open cone -the Garding cone- defined by

$$\Gamma_k^+ = \{\Lambda = (\lambda_1, \lambda_2, \dots, \lambda_n) \in \mathbb{R}^n \mid \sigma_j(\Lambda) > 0, \forall j \leq k\}.$$

Similarly, we say a symmetric matrix $W \in \Gamma_k^+$ if the set of eigenvalues of W belongs to Γ_k^+ . By $g \in \mathcal{C}_k^+$ we mean that $g^{-1} \cdot S_g(x)$ belongs to Γ_k^+ for any $x \in M$. If $g = e^{-2u}g_0$, we have the transformation formula of the Schouten tensor

$$S_g = \nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2} g_0 + S_{g_0}.$$

Therefore, $g = e^{-2u}g_0 \in \mathcal{C}_k$ if and only if

$$(\nabla^2 u + du \otimes du - \frac{|\nabla u|^2}{2}g_0 + S_{g_0})(x) \in \Gamma_k^+, \quad \forall x \in M.$$

To establish a priori estimates, we first need a technical key lemma.

Lemma 4. *For $1 < k \leq n$ set $F = \frac{\sigma_k}{\sigma_{k-1}}$. We have*

- 1) *the matrix $(F^{ij})(W)$ is semi-positive definite at $W \in \Gamma_{k-1}^+$ and is positive definite at $W \in \Gamma_{k-1}^+ \setminus \mathcal{R}_1$, where \mathcal{R}_1 is the set of matrices of rank 1.*
- 2) *The function F is concave in the cone Γ_{k-1}^+ . When $k = 2$, for all $W \in \Gamma_1^+$ and for all $R = (r_{ij}) \in \mathcal{S}_n$, we have*

$$(16) \quad \sum_{ijkl} \frac{\partial^2}{\partial w_{ij} \partial w_{kl}} \left(\frac{\sigma_2(W)}{\sigma_1(W)} \right) r_{ij} r_{kl} = - \frac{\sum_{ij} (\sigma_1(W) r_{ij} - \sigma_1(R) w_{ij})^2}{\sigma_1^3(W)}.$$

Proof. For the proof, see [7]. ■

Assume $g_1 \in \mathcal{C}_{1,\varepsilon}([g_0])$. We consider flow (15) with the initial metric g_1 . Lemma 4 implies that (15) is parabolic. By the standard implicit function theorem we have the short-time existence result. Let $T^* \in (0, \infty]$ so that $[0, T^*)$ is the maximum interval for the existence of the flow $g(t) \in \mathcal{C}_{1,\varepsilon}([g_0])$.

Theorem 3. *Assume that $n = 3$, and $g(0) = g_1 \in \mathcal{C}_{1,\varepsilon}([g_0])$. Let u be a solution of (15) in a geodesic ball $B_R \times [0, T]$ for $T < T^*$ and $R < \tau_0$, the injectivity radius of M . Then there is a constant C depending only on (B_R, g_0) and independent of T such that for any $(x, t) \in B_{R/2} \times [0, T]$*

$$(17) \quad \begin{aligned} |\nabla u|^2 + |\nabla^2 u| &\leq C \left(1 + \frac{\nu_1(g)}{\mu(g)} e^{-(2-\varepsilon) \inf_{B_R} u} \right) \\ &\leq C \left(1 + \frac{1}{\varepsilon (\int \sigma_1(g) dv(g))^{2-\varepsilon}} e^{-(2-\varepsilon) \inf_{B_R} u} \right). \end{aligned}$$

Proof. In the proof, C (resp. c) is a constant independent of T , which may vary from line to line. Let $W = (w_{ij})$ be an $n \times n$ matrix with

$$w_{ij} = u_{ij} + u_i u_j - \frac{|\nabla u|^2}{2} (g_0)_{ij} + (S_{g_0})_{ij}.$$

Here u_i and u_{ij} are the first and second derivatives of u with respect to the background metric g_0 . Define

$$\nu := \nu_1(g) e^{-(4-\varepsilon)u} + \nu_2(g), \quad \bar{\nu} := \nu_1(g) e^{-(4-\varepsilon)u}$$

and

$$F(W, u) := \frac{\sigma_2(W) - \nu}{\sigma_1(W)}.$$

Set

$$(18) \quad \begin{aligned} (F^{ij}(W, u)) &:= \left(\frac{\partial F}{\partial w_{ij}}(W) \right) \\ &= \left(\frac{\sigma_1(W)T^{ij} - \sigma_2(W)\delta^{ij} + \nu\delta^{ij}}{\sigma_1^2(W)} \right) \end{aligned}$$

where $(T^{ij}) = (\sigma_1(W)\delta^{ij} - w^{ij})$ is the first Newton transformation associated with W , and δ^{ij} is the Kronecker symbol. From Proposition 1, we know $\nu_1(g) > 0$, $\nu_2(g) > 0$ and $\mu(g) > 0$. In view of Lemma 4 we know that (F^{ij}) is positive definite and F is concave in $W \in \Gamma_1^+$. Moreover, we have

$$(19) \quad \sum_{ijkl} \frac{\partial^2 (F(W, u))}{\partial w_{ij} \partial w_{kl}} r_{ij} r_{kl} \leq -2 \frac{\nu(\sum_i r_{ii})^2}{\sigma_1^3(W)}.$$

Let $S(TM)$ denote the unit tangent bundle of M with respect to the background metric g_0 . We define a function $\tilde{G} : S(TM) \times [0, T] \rightarrow \mathbb{R}$

$$(20) \quad \tilde{G}(e, t) = (\nabla^2 u + |\nabla u|^2 g_0)(e, e).$$

Without loss of generality, we assume $R = 1$. Let $\rho \in C_0^\infty(B_1)$ be a cut-off function defined as in [14] such that

$$(21) \quad \begin{aligned} \rho &\geq 0, & \text{in } B_1, \\ \rho &= 1, & \text{in } B_{1/2}, \\ |\nabla \rho(x)| &\leq 2b_0 \rho^{1/2}(x), & \text{in } B_1, \\ |\nabla^2 \rho| &\leq b_0, & \text{in } B_1. \end{aligned}$$

Here $b_0 > 1$ is a constant. Since $e^{-2u}g_0 \in \mathcal{C}_1$, to bound $|\nabla u|$ and $|\nabla^2 u|$ we only need to bound $(\nabla^2 u + |\nabla u|^2 g_0)(e, e)$ from above for all $e \in S(TM)$ and for all $t \in [0, T]$. For this purpose, consider $G(e, t) = \rho(x)\tilde{G}(e, t)$. Assume $(e_1, t_0) \in S(T_{x_0}M) \times (0, T]$ such that

$$(22) \quad G(e_1, t_0) = \max_{S(TM) \times [0, T]} G(e, t).$$

We may further assume that

$$(23) \quad G(e_1, t_0) > n \max_{B_1} \sigma_1(g_0).$$

Let (e_1, \dots, e_n) be an orthonormal basis at point (x_0, t_0) . Now choose the normal coordinates around x_0 such that at point x_0

$$\frac{\partial}{\partial x_1} = e_1$$

and consider the function G on $M \times [0, T]$ defined by

$$G(x, t) := \rho(x)(u_{11} + |\nabla u|^2)(x, t).$$

Clearly, (x_0, t_0) is a maximum point of $G(x, t)$ on $M \times [0, T]$. At (x_0, t_0) , we have

$$(24) \quad 0 \leq G_t = \rho(u_{11t} + 2 \sum_l u_l u_{lt}),$$

$$(25) \quad 0 = G_j = \frac{\rho_j}{\rho} G + \rho(u_{11j} + 2 \sum_{l \geq 1} u_l u_{lj}), \quad \text{for any } j,$$

$$(26) \quad 0 \geq (G_{ij}) = \left(\frac{\rho \rho_{ij} - 2\rho_i \rho_j}{\rho^2} G + \rho(u_{11ij} + \sum_{l \geq 1} (2u_{li} u_{lj} + 2u_l u_{lij})) \right).$$

Recall that (F^{ij}) is definite positive. Hence, we have

$$(27) \quad \begin{aligned} 0 &\geq \sum_{i,j \geq 1} F^{ij} G_{ij} - G_t \\ &\geq \sum_{i,j \geq 1} F^{ij} \frac{\rho \rho_{ij} - 2\rho_i \rho_j}{\rho^2} G + \rho \sum_{i,j \geq 1} F^{ij} (u_{11ij} + \sum_{l \geq 1} (2u_{li} u_{lj} + 2u_l u_{lij})) \\ &\quad - \rho(u_{11t} + 2 \sum_{l \geq 1} u_l u_{lt}). \end{aligned}$$

First, from the definition of ρ , we have

$$(28) \quad \sum_{i,j \geq 1} F^{ij} \frac{\rho \rho_{ij} - 2\rho_i \rho_j}{\rho^2} G \geq -C \sum_{i,j \geq 1} |F^{ij}| \frac{1}{\rho} G,$$

and

$$(29) \quad \begin{aligned} \sum_{i,j \geq 1} |F^{ij}| &\geq \sum_i F^{ii} \\ &= \left(n - 1 - \frac{n\sigma_2(W)}{\sigma_1^2(W)} \right) + \frac{n\nu}{\sigma_1^2(W)} \geq C \sum_{i,j \geq 1} |F^{ij}|, \end{aligned}$$

since W is positive definite. From (29) we have

$$(30) \quad \sum_i F^{ii} \geq \frac{n-1}{2} + \frac{n\nu}{\sigma_1^2(W)} = 1 + \frac{3\nu}{\sigma_1^2(W)}.$$

Using the facts that

$$(31) \quad u_{kij} = u_{ijk} + \sum_m R_{mikj} u_m,$$

$$(32) \quad u_{kkij} = u_{ijkk} + \sum_m (2R_{mikj} u_{mk} - Ric_{mj} u_{mi} - Ric_{mi} u_{mj} - Ric_{mi,j} u_m + R_{mikj,k} u_m)$$

and

$$(33) \quad \left(\sum_l u_l^2 \right)_{11} = 2 \sum_l (u_{11l} u_l + u_{1l}^2) + O(|\nabla u|^2),$$

we have

$$(34) \quad \sum_{i,j \geq 1} F^{ij} u_{11ij} \geq \sum_{i,j \geq 1} F^{ij} \left(w_{ij11} - (u_{11})_i u_j - u_i (u_{11})_j + \sum_{l \geq 1} (u_{1l}^2 + u_{11l} u_l) (g_0)_{ij} \right) - 2 \sum_{i,j \geq 1} F^{ij} u_{i1} u_{j1} - C(1 + |\nabla^2 u| + |\nabla u|^2) \sum_{i,j \geq 1} |F^{ij}|$$

and

$$(35) \quad \sum_{i,j,l} F^{ij} u_l u_{lij} \geq \sum_{i,j,l} F^{ij} u_l w_{ijl} - \sum_{i,j,l} F^{ij} (u_l u_{il} u_j + u_l u_i u_{jl}) + \frac{1}{2} \sum_{i,j} F^{ij} \langle \nabla u, \nabla (|\nabla u|^2) \rangle (g_0)_{ij} - C(1 + |\nabla u|^2) \sum_{i,j \geq 1} |F^{ij}|.$$

Combining (34) and (35), we deduce

$$(36) \quad \begin{aligned} & \sum_{i,j \geq 1} F^{ij} (u_{11ij} + 2 \sum_{l \geq 1} (u_l u_{lj} + u_l u_{lij})) \\ \geq & \sum_{i,j \geq 1} F^{ij} (w_{ij11} + 2 \sum_{l \geq 1} w_{ijl} u_l) + 2 \sum_{i,j \geq 1} F^{ij} \sum_{l \geq 2} u_l u_{lj} + \sum_{i,j,l \geq 1} u_{1l}^2 F^{ij} (g_0)_{ij} \\ & - \sum_{i,j} F^{ij} [(u_{11} + |\nabla u|^2)_i u_j + u_i (u_{11} + |\nabla u|^2)_j - \langle \nabla u, \nabla (u_{11} + |\nabla u|^2) \rangle (g_0)_{ij}] \\ & - C(1 + |\nabla^2 u| + |\nabla u|^2) \sum_{i,j \geq 1} |F^{ij}| \\ \geq & \sum_{i,j} F^{ij} (w_{ij11} + 2 \sum_l w_{ijl} u_l) + u_{11}^2 \sum_{i,j} F^{ij} (g_0)_{ij} \\ & + \sum_{i,j} F^{ij} (\rho_i u_j + \rho_j u_i - \langle \nabla \rho, \nabla u \rangle (g_0)_{ij}) \frac{G}{\rho^2} - C(1 + |\nabla^2 u| + |\nabla u|^2) \sum_{i,j \geq 1} |F^{ij}|. \end{aligned}$$

In the last inequality we have used (25). Now, we want to estimate $\sum_{i,j,l} F^{ij} w_{ijl} u_l$ and $\sum_{i,j} F^{ij} w_{ij11}$ respectively. By differentiating F we get

$$(37) \quad \sum_l F_l u_l = \sum_{i,j,l} F^{ij} w_{ijl} u_l + \sum_l \frac{\partial F}{\partial u} u_l^2 = \sum_{i,j,l} F^{ij} w_{ijl} u_l + \sum_l \frac{(4 - \varepsilon) \bar{\nu} u_l^2}{\sigma_1(W)}.$$

By differentiating F twice and using the concavity (19) of F in W , we have

$$\begin{aligned}
(38) \quad \sum_{i,j} F^{ij} w_{ij11} &= F_{11} - \sum_{i,j,k,m} \frac{\partial^2 F}{\partial w_{ij} \partial w_{km}} w_{ij1} w_{km1} \\
&\quad - 2 \sum_{i,j} \frac{\partial^2 F}{\partial w_{ij} \partial u} w_{ij1} u_1 - \frac{\partial^2 F}{\partial^2 u} u_1^2 - \frac{\partial F}{\partial u} u_{11} \\
&\geq F_{11} + \frac{2\nu(\sum_i w_{ii1})^2}{(\sigma_1(W))^3} + \frac{2(4-\varepsilon)\bar{\nu}(\sum_i w_{ii1})u_1}{(\sigma_1(W))^2} + \frac{(4-\varepsilon)^2\bar{\nu}u_1^2}{\sigma_1(W)} - \frac{(4-\varepsilon)\bar{\nu}u_{11}}{\sigma_1(W)} \\
&\geq F_{11} + \frac{(4-\varepsilon)^2\bar{\nu}u_1^2}{2\sigma_1(W)} - \frac{(4-\varepsilon)\bar{\nu}u_{11}}{\sigma_1(W)}.
\end{aligned}$$

These estimates give

$$\begin{aligned}
(39) \quad \sum_{i,j \geq 1} F^{ij} (w_{ij11} + 2 \sum_{l \geq 1} w_{ijl} u_l) &\geq F_{11} + 2 \sum_l F_l u_l \\
&\quad - \frac{(4-\varepsilon)\bar{\nu}u_{11}}{\sigma_1(W)} - \sum_{l=1}^n \frac{2(4-\varepsilon)\bar{\nu}u_l^2}{\sigma_1(W)}.
\end{aligned}$$

Recall from (15) that

$$(40) \quad F = u_t - \mu(g)e^{-2u} - m(g).$$

Hence we have

$$(41) \quad F_{11} = u_{11t} - \mu(g)e^{-2u}(-2u_{11} + 4u_1^2),$$

$$(42) \quad F_l = u_{lt} - \mu(g)e^{-2u}(-2u_l), \quad \forall l = 1, \dots, n.$$

Gathering (27), (28), (29), (36) (39), (41) and (42), we obtain

$$\begin{aligned}
(43) \quad 0 &\geq -C \left(\sum_{i,j} |F^{ij}| \right) \frac{G}{\rho} + \rho \left(\sum_i F^{ii} \right) u_{11}^2 - C\rho \left(\sum_{i,j} |F^{ij}| \right) (1 + |\nabla u|^2 + |\nabla^2 u|) \\
&\quad + \sum_{i,j} F^{ij} (\rho_i u_j + \rho_j u_i - \langle \nabla \rho, \nabla u \rangle (g_0)_{ij}) \frac{G}{\rho} - \rho \mu(g) e^{-2u} \left(-2u_{11} - 4 \sum_{l=2}^n u_l^2 \right) \\
&\quad - \rho \frac{(4-\varepsilon)\bar{\nu}}{\sigma_1(W)} (u_{11} + 2 \sum_{l=1}^n u_l^2).
\end{aligned}$$

From the fact $W \in \Gamma_1^+$, we have that $u_{11}(x_0, t_0) \geq \frac{1}{20} |\nabla u|^2(x_0, t_0)$, and hence $G(x_0, t_0) \leq 21\rho(x_0)u_{11}(x_0, t_0)$ (see (44) in [7]). Multiplying (43) by ρ we deduce

$$(44) \quad 0 \geq \sum_i F^{ii} (-CG + (\frac{G}{21})^2 - CG^{\frac{3}{2}}) + \rho e^{-2u} (\mu(g) \frac{2G}{21} - 8\nu_1(g) e^{-(2-\varepsilon)u} \frac{G}{\sigma_1(W)}).$$

When $\frac{G}{\sigma_1(W)} \geq 2352 = 16 \times (21)^2/3$, it follows from (30) that

$$\frac{1}{2} \sum_i F^{ii} (\frac{G}{21})^2 - 8\nu_1(g) \rho e^{-(4-\varepsilon)u} \frac{G}{\sigma_1(W)} \geq \nu \left(\frac{G^2}{294\sigma_1^2(W)} - 8\rho \frac{G}{\sigma_1(W)} \right) \geq 0.$$

Together with (44), we have

$$0 \geq \sum_i F^{ii}(-CG + \frac{1}{2}(\frac{G}{21})^2 - CG^{\frac{3}{2}}),$$

from which we easily have

$$G(x_0, t_0) \leq C.$$

This gives the desired result. When $\frac{G}{\sigma_1(W)} < 2352$, the desired result follows from (44) and Lemma 3 (i). \blacksquare

Remark 1. Let $g = e^{-2u}g_0 \in \mathcal{C}_{1,\varepsilon}$ be a solution of (13) in a geodesic ball B_R and $R < \tau_0$, the injectivity radius of M . Then there is a constant C depending only on (B_R, g_0) such that for any $x \in B_{R/2}$ the estimate (17) holds.

Corollary 1. Under the same assumptions as in Theorem 3, there is a constant C depending only on g_0 (independent of T) such that for any $t \in [0, T]$

$$(45) \quad \|u\|_{C^2(M)} \leq C.$$

Proof. By Proposition 1, we may assume that $\int_M \sigma_1(g)dv(g) \equiv 1$ without loss of generality. Thus, we have a uniform volume bound, namely

$$(46) \quad \text{vol}(g) \leq (Y_1([g_0]))^{-3}.$$

Claim. There is a constant $C > 0$ independent of $T \in [0, T^*)$ such that $\forall t \in [0, T]$

$$(47) \quad u(t, x) \geq C.$$

Set $m(t) = \min_{x \in M} u(t, x)$ and $u(t, x_t) = m(t)$. We prove the claim by a contradiction argument and assume that there exists a sequence $\{t_n\}$ such that $t_n \rightarrow T$ and $m(t_n) \rightarrow -\infty$. Applying Theorem 3, we have for all $x \in M$ and $n \in \mathbb{N}$

$$|\nabla u(t_n, x)|^2 \leq \frac{C}{\varepsilon} e^{-(2-\varepsilon)m(t_n)},$$

which implies for all $x \in B(x_{t_n}, \sqrt{\varepsilon}e^{(1-\varepsilon/2)m(t_n)})$

$$|u(t_n, x) - m(t_n)| \leq C.$$

As a consequence, we infer

$$\text{vol}(g(t_n)) \geq C \int_{B(x_{t_n}, \sqrt{\varepsilon}e^{(1-\varepsilon/2)m(t_n)})} e^{-3m(t_n)} dv(g_0) \geq \varepsilon^{3/2} e^{(-3\varepsilon/2)m(t_n)} \rightarrow \infty$$

which contradicts our uniform volume bound (46). This contradiction yields the desired claim.

From Theorem 3 and the Claim, there is a constant $C > 0$, independent of $T \in [0, T^*)$ such that $\forall (t, x) \in [0, T] \times M$

$$(48) \quad |\nabla u(t, x)| + |\nabla^2 u(t, x)| \leq C.$$

Using the fact $\int_M \sigma_1(g)dv(g) \equiv 1$, we have $\forall (t, x) \in [0, T] \times M$

$$|u(t, x)| + |\nabla u(t, x)| + |\nabla^2 u(t, x)| \leq C.$$

Therefore, we finish the proof of Theorem. \blacksquare

Remark 2. *Our perturbed equation is so chosen such that the argument in Corollary 1 works and the estimate in Lemma 6 hold.*

Remark 3. *Under the same assumptions as in Remark 1, there is a constant C depending only on g_0 such that*

$$(49) \quad \|u\|_{C^2(M)} \leq C.$$

5. UNIFORM PARABOLICITY

We prove in this Section that our flow (15) preserves the positivity of the scalar curvature.

Proposition 2. *There is a constant $C_0 > 0$, independent of $T \in [0, T^*)$ such that $\sigma_1(g(t)) > C_0$ for any $t \in [0, T]$.*

Proof. The proof is a modification of the proof given in [15] and [9], with more attention on ν_1 and ν_2 , and their derivatives. Recall

$$\begin{aligned} W &= (w_{ij}) = (\nabla_{ij}^2 u + u_i u_j - \frac{|\nabla u|^2}{2}(g_0)_{ij} + (S_{g_0})_{ij}), \\ \nu &= \nu_1(g)e^{-(4-\varepsilon)u} + \nu_2(g). \end{aligned}$$

We define

$$F := \frac{\sigma_2(W) - \nu}{\sigma_1(W)} - \kappa e^{-2u}$$

for some sufficiently large κ to be fixed later. Hence, $F = u_t - (\kappa + \mu(g))e^{-2u} - m(g(t))$. By Corollary 1 one can show that there is a constant $c_1 > 0$ which is independent of $T > 0$ such that

$$(50) \quad \frac{1}{c_1} > \int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) > c_1, \quad \frac{1}{c_1} > \int_M e^{\varepsilon u} dv(g) - \varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^{3-\varepsilon} > c_1.$$

To show this, from Corollary 1 we first have that $\int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g)$ and $\int_M e^{\varepsilon u} dv(g) - \varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^{3-\varepsilon}$ are bounded from above by some positive constants. It follows from Proposition 1 that $\mathcal{E}_\varepsilon(g)$ is bounded from below by some positive constant and the fact that $\int \sigma_1(g) dv(g)$ is constant along the flow. Therefore, the second part in the inequalities yields. As a consequence, $\nu_1(g)$, $\nu_2(g)$ and $\mu(g)$ are bounded from above and from below by some positive constants. Again from Corollary 1, $m(g)$ is bounded.

Without loss of generality, we assume that the minimum of F is achieved at $(x_0, t_0) \in M \times (0, T]$. Near (x_0, t_0) , we have

$$\begin{aligned} (51) \quad \frac{d}{dt} F &= \sum_{ij} A^{ij} (\nabla_g^2(u_t))_{ij} + 2\kappa e^{-2u} u_t + \frac{(4-\varepsilon)\nu_1(g)e^{-(4-\varepsilon)u} u_t}{\sigma_1(W)} - \frac{\alpha(t)}{\sigma_1(W)} \\ &= \sum_{ij} A^{ij} [(\nabla_g^2(F))_{ij} + (\kappa + \mu(g))(\nabla_g^2(e^{-2u}))_{ij}] + 2\kappa e^{-2u} u_t + \frac{(4-\varepsilon)\nu_1(g)e^{-(4-\varepsilon)u} u_t}{\sigma_1(W)} \\ &\quad - \frac{\alpha(t)}{\sigma_1(W)}, \end{aligned}$$

where

$$\alpha(t) := \frac{d\nu_1(g)}{dt} e^{-(4-\varepsilon)u} + \frac{d\nu_2(g)}{dt}$$

and

$$A^{ij} := \frac{\partial F}{\partial w_{ij}} = \frac{(\sigma_1^2(W) - \sigma_2(W) + \nu)\delta^{ij} - \sigma_1(W)W^{ij}}{\sigma_1^2(W)}$$

is positive definite. We choose the normal coordinates so that W is a diagonal matrix at (x_0, t_0) . First we claim there exists some constant $c_2 > 0$ independent of T and κ such that for all $t \in [0, T]$

$$(52) \quad \left| \frac{d\nu_1(g)}{dt}(t) \right| \leq c_2 \left(1 + \kappa + \frac{1}{\sigma_1(W)(x_0, t_0)} \right),$$

$$(53) \quad \left| \frac{d\nu_2(g)}{dt}(t) \right| \leq c_2 \left(1 + \kappa + \frac{1}{\sigma_1(W)(x_0, t_0)} \right).$$

Using (15), (50) and Corollary 1, we can estimate

$$(54) \quad \left| \frac{d\nu_1(g)}{dt}(t) \right| \leq c \left(1 + \int_M \frac{1}{\sigma_1(W)(x, t)} dv(g_0) \right) \leq c \left(1 + \max_{x \in M} \frac{1}{\sigma_1(W)(x, t)} \right).$$

Since (x_0, t_0) is the minimum of F in $M \times [0, T]$, we have

$$(55) \quad \frac{\nu}{\sigma_1(W)}(x, t) \leq \frac{\sigma_2(W)}{\sigma_1(W)}(x, t) - \kappa e^{-2u(x, t)} - \frac{\sigma_2(W) - \nu}{\sigma_1(W)}(x_0, t_0) + \kappa e^{-2u(x_0, t_0)},$$

for any point $(x, t) \in M \times [0, T]$. Applying Corollary 1, we have that $\sigma_1(W)$, $\sigma_2(W)$ and e^{-2u} are bounded and ν is bounded from above and from below by some positive constants. Together with the fact $\sigma_2(W)(x, t) \leq \frac{1}{3}\sigma_1^2(W)(x, t)$, (55) implies there exists $c_3 > 0$ independent of T and κ such that for all $(x, t) \in M \times [0, T]$

$$(56) \quad \frac{1}{\sigma_1(W)(x, t)} \leq c_3 \left(1 + \kappa + \frac{1}{\sigma_1(W)(x_0, t_0)} \right),$$

which, in turn, together with (54), implies (52). Similarly, we have (53). Hence, we prove the desired claim. As a consequence, we have at the point (x_0, t_0)

$$(57) \quad \frac{|\alpha(t_0)|}{\sigma_1(W)} \leq c \left(\frac{1 + \kappa}{\sigma_1(W)} + \frac{1}{\sigma_1^2(W)} \right)$$

Since (x_0, t_0) is the minimum of F in $M \times [0, T]$, at this point we have $\frac{dF}{dt} \leq 0$, $F_l = 0 \forall l$ and (F_{ij}) is non-negative definite. Note that

$$(\nabla_g^2)_{ij}F = F_{ij} + u_i F_j + u_j F_i - \sum_l u_l F_l \delta_{ij} = F_{ij},$$

at (x_0, t_0) , where F_j and F_{ij} are the first and second derivatives with respect to the background metric g_0 . From the positivity of A and (51), we have

$$\begin{aligned}
(58) \quad 0 &\geq F_t - \sum_{i,j} A^{ij} F_{ij} \\
&\geq (\kappa + \mu(g)) \sum_{i,j} A^{ij} \{ (e^{-2u})_{ij} + u_i (e^{-2u})_j + u_j (e^{-2u})_i - \sum_l u_l (e^{-2u})_l \delta_{ij} \} \\
&\quad + 2\kappa e^{-2u} u_t + \frac{(4-\varepsilon)\nu_1(g)e^{-(4-\varepsilon)u} u_t}{\sigma_1(W)} - \frac{\alpha}{\sigma_1(W)} \\
&= (\kappa + \mu(g)) e^{-2u} \sum_{i,j} A^{ij} \{ -2w_{ij} + 2u_i u_j + 2S(g_0)_{ij} + |\nabla u|^2 \delta_{ij} \} \\
&\quad + 2\kappa e^{-2u} u_t + \frac{(4-\varepsilon)\nu_1(g)e^{-(4-\varepsilon)u} u_t}{\sigma_1(W)} - \frac{\alpha}{\sigma_1(W)} \\
&= (\kappa + \mu(g)) e^{-2u} \left(\frac{-2\sigma_2(W) - 2\nu}{\sigma_1(W)} \right) + 2\kappa e^{-2u} u_t + \frac{(4-\varepsilon)\nu_1(g)e^{-(4-\varepsilon)u} u_t}{\sigma_1(W)} \\
&\quad + (\kappa + \mu(g)) e^{-2u} \sum_{i,j} A^{ij} (2u_i u_j + 2S(g_0)_{ij} + |\nabla u|^2 \delta_{ij}) - \frac{\alpha}{\sigma_1(W)}.
\end{aligned}$$

Here we have used $\sum_{i,j} A^{ij} w_{ij} = \frac{\sigma_2(W) + \nu}{\sigma_1(W)}$. A direct computation gives

$$(59) \quad \sum_{i,j} A^{ij} S(g_0)_{ij} = \frac{(\sigma_1^2(W) - \sigma_2(W))\sigma_1(g_0)}{\sigma_1^2(W)} - \frac{1}{\sigma_1(W)} \sum_{i,j} W^{ij} S(g_0)_{ij} + \frac{\nu\sigma_1(g_0)}{\sigma_1^2(W)}.$$

Gathering (57), (58) and (59), we have

$$\begin{aligned}
(60) \quad 0 &\geq F_t - \sum_{i,j} A^{ij} F_{ij} \\
&\geq (\kappa + \mu(g)) e^{-2u} \left[\frac{-2\sigma_2(W) - 2\nu}{\sigma_1(W)} + \frac{2(\sigma_1^2(W) - \sigma_2(W))\sigma_1(g_0)}{\sigma_1^2(W)} \right. \\
&\quad \left. - \frac{2}{\sigma_1(W)} \sum_{i,j} W^{ij} S(g_0)_{ij} + \frac{2\nu\sigma_1(g_0)}{\sigma_1^2(W)} \right] \\
&\quad - C \left(1 + \frac{1+\kappa}{\sigma_1(W)} + \frac{1}{\sigma_1^2(W)} \right),
\end{aligned}$$

since (A^{ij}) is positive definite and $\kappa + \mu(g)$ is positive. Let us use $O(1)$ denote terms with a uniform bound. One can check again $\sigma_2(W) = O(1)$ for $\|u\|_{C^2}$ is uniformly bounded and $\sum_{i,j} W^{ij} S(g_0)_{ij} = O(1)$. Also the term $\sigma_1^2(W) - \sigma_2(W)$ is always non-negative. We

can choose κ such that

$$\frac{(\kappa + \mu(g))\nu\sigma_1(g_0)e^{-2u}}{\sigma_1^2(W)} \geq \frac{C}{\sigma_1^2(W)} \quad \text{and} \quad \kappa + \mu(g) \geq \frac{\kappa + 1}{2}$$

Fixing such κ , from (60) we conclude there holds at the point (x_0, t_0)

$$0 \geq \frac{\kappa}{\sigma_1^2(W)} - c_4\left(\frac{\kappa + 1}{\sigma_1(W)} + 1\right)$$

for some positive constants $c_4 > 0$ independent of T and κ . Consequently, there is a positive constant $c_5 > 0$ (independent of T) such that

$$\sigma_1(W)(x_0, t_0) \geq c_5.$$

Hence, from (56) and Corollary 1, there is a positive constant $c_6 > 0$, independent of T , such that for all $(x, t) \in M \times [0, T]$

$$\sigma_1(W)(x, t) \geq c_6.$$

This finishes the proof of the Theorem. \blacksquare

6. PROOF OF MAIN THEOREMS

Now we can show the convergence of flow (15).

Theorem 4. *For small $\varepsilon > 0$ flow (15) with an initial metric $g_1 \in \mathcal{C}_{1,\varepsilon}([g_0])$ converges to a metric g_∞ satisfying (13).*

Proof. With the C^2 estimates (Corollary 1) and the uniform parabolicity (Proposition 2), one can show the convergence like in [15]. \blacksquare

In order to estimate the value of M_ε we need the following

Lemma 5. *There exists some $C_0 > 0$ depending only on g_0 such that for any $g = e^{-2u}g_0 \in \mathcal{C}_1([g_0])$ satisfying $\int_M \sigma_2(g)dv(g) \geq 0$ there holds*

$$(61) \quad C_0 e^{\max u} \leq \int e^{4u} dv(g) \leq e^{\max u} \text{vol}(g_0).$$

Proof. The second inequality is clear. We prove the first inequality. As in [7], we have for all $g \in \mathcal{C}_1([g_0])$

$$(62) \quad \int \sigma_2(g)dv(g) \leq -\frac{1}{16} \int |\nabla u|_{g_0}^4 e^{4u} d(g) + c \int e^{4u} dv(g),$$

for some positive constant $c > 0$. Since $\int \sigma_2(g)dv(g)$ is non-negative, we have

$$(63) \quad 4^4 \int |\nabla e^{u/4}|_{g_0}^4 dv(g_0) = \int |\nabla u|_{g_0}^4 e^{4u} dv(g) \leq c \int e^{4u} dv(g) = c \int (e^{u/4})^4 dv(g_0),$$

which implies, with the help of Sobolev's embedding Theorem ($W^{1,4} \subset C^{1/4}$), for all $x, y \in M$

$$(64) \quad |e^{u(x)/4} - e^{u(y)/4}| \leq c \left(\int e^{4u} dv(g) \right)^{1/4} (d_{g_0}(x, y))^{1/4},$$

where $d_{g_0}(x, y)$ is the distance between x and y with respect to the metric g_0 . Set

$$(65) \quad \beta := e^{\max_M u} = e^{u(x_0)}$$

for some $x_0 \in M$. It follows from (64) that there exists some $r > 0$ independent of u such that for any $y \in B(x_0, r)$

$$(66) \quad e^{u(y)/4} \geq \frac{1}{2}\beta^{1/4}$$

Here the geodesic ball $B(x_0, r)$ is taken for the metric g_0 . Hence, we deduce

$$\int_M e^{4u} dv(g) \geq \int_{B(x_0, r)} e^{4u} dv(g) \geq c\beta.$$

Therefore, we have finished to prove the Lemma. \blacksquare

Now we estimate the value of M_ε . The proof likes one given for Lemma 2, with the help of Lemma 5.

Lemma 6. *Let $C_0 > 0$ be the constant given in Lemma 5. Any solution $g \in \mathcal{C}_{1, \varepsilon}$ of (13) satisfies (14), i.e.,*

$$\mathcal{E}_\varepsilon(g) \leq \left(\frac{2}{C_0\varepsilon}\right)^\varepsilon \frac{1}{3(1-\varepsilon)}.$$

Proof. Multiplying (13) by $e^{\varepsilon u}$ and integrating over M , we have

$$(67) \quad \int_M \frac{e^{\varepsilon u} \sigma_2(g)}{\sigma_1(g)} dv(g) + \mu(g) \int_M e^{\varepsilon u} dv(g) = \int_M \frac{(\nu_1(g)e^{2\varepsilon u} + \nu_2(g)e^{(4+\varepsilon)u})}{\sigma_1(g)} dv(g)$$

By the Cauchy-Schwarz inequality, we have

$$\begin{aligned} & \int_M \frac{(\nu_1(g)e^{2\varepsilon u} + \nu_2(g)e^{(4+\varepsilon)u})}{\sigma_1(g)} dv(g) \int_M \sigma_1(g) dv(g) \\ & \geq \int_M \frac{\nu_1(g)e^{2\varepsilon u}}{\sigma_1(g)} dv(g) \int_M \sigma_1(g) dv(g) \\ & \geq \nu_1(g) \left(\int_M e^{\varepsilon u} dv(g) \right)^2. \end{aligned}$$

The above two inequalities implies that

$$(68) \quad \begin{aligned} \int_M \frac{e^{\varepsilon u} \sigma_2(g)}{\sigma_1(g)} dv(g) & \geq \nu_1(g) \frac{(\int_M e^{\varepsilon u} dv(g))^2}{\int_M \sigma_1(g) dv(g)} - \mu(g) \int_M e^{\varepsilon u} dv(g) \\ & = (1-\varepsilon) \frac{k(g) \int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) \int_M e^{\varepsilon u} dv(g)}{\int_M \sigma_1(g) dv(g)}. \end{aligned}$$

In the last equality we have used the definitions of $\nu_1(g)$ and $\mu(g)$.

On the other hand, we recall the facts

$$\sigma_2(g) \leq \frac{1}{3}(\sigma_1(g))^2$$

and for all $g \in \mathcal{C}_{1, \varepsilon}([g_0])$

$$\varepsilon \int_M e^{4u} dv(g) \leq \int_M \sigma_2(g) dv(g).$$

Hence, we get from Lemma 5

$$(69) \quad \int_M \frac{e^{\varepsilon u} \sigma_2(g)}{\sigma_1(g)} dv(g) \leq \frac{1}{3} \int_M e^{\varepsilon u} \sigma_1(g) dv(g) \leq \frac{1}{3} e^{\varepsilon \max u} \int_M \sigma_1(g) dv(g) \\ \leq \frac{1}{3} \left(\int_M (\sigma_2(g) - \frac{\varepsilon}{2} e^{4u} dv(g))^\varepsilon \left(\frac{2}{C_0 \varepsilon} \right)^\varepsilon \int_M \sigma_1(g) dv(g) \right).$$

(68) and (69) give us

$$\frac{\int_M (\sigma_2(g) - \varepsilon e^{4u}) dv(g) \int_M e^{\varepsilon u} dv(g)}{\left(\int_M (\sigma_2(g) - \frac{\varepsilon}{2} e^{4u}) dv(g) \right)^\varepsilon \left(\int_M \sigma_1(g) dv(g) \right)^2} \leq \left(\frac{2}{C_0 \varepsilon} \right)^\varepsilon \frac{1}{3(1-\varepsilon)k(g)},$$

which implies

$$\mathcal{E}_\varepsilon(g) \leq \left(\frac{2}{C_0 \varepsilon} \right)^\varepsilon \frac{1}{3(1-\varepsilon)k(g)} \leq \left(\frac{2}{C_0 \varepsilon} \right)^\varepsilon \frac{1}{3(1-\varepsilon)},$$

since $k(g) \geq 1$. This yields the desired result. \blacksquare

Proof of Theorem 2. If any $g \in \mathcal{C}_1([g_0])$ satisfies $\int_M \sigma_2(g) \leq 0$, then $\tilde{Y}([g_0]) \leq 0 < 1/3$.

Hence we consider that there is $g \in \mathcal{C}_1([g_0])$ with $\int_M \sigma_2(g) > 0$. For such a metric g we can choose a small number $\varepsilon_0 > 0$ such that $g \in \mathcal{C}_{1,\varepsilon}([g_0])$ for any $\varepsilon \in (0, \varepsilon_0)$. Hence, we have

$$\mathcal{E}_\varepsilon(g) \leq M_\varepsilon.$$

Theorem 4 and Remark 3 imply that M_ε is achieved by a metric $\tilde{g} \in \mathcal{C}_1 \cap [g_0]$ satisfying (13). From Lemma 6 we have

$$M_\varepsilon = \mathcal{E}_\varepsilon(\tilde{g}) \leq \left(\frac{2}{C_0 \varepsilon} \right)^\varepsilon \frac{1}{3(1-\varepsilon)},$$

and hence

$$\mathcal{E}(g) \leq \frac{1}{3}.$$

Therefore we have

$$\tilde{Y}([g_0]) \leq \frac{1}{3}.$$

This finishes the proof of the Theorem. \blacksquare

We consider now the energy functional \mathcal{E} in a larger class

$$\overline{\mathcal{C}_1([g_0])} := \{g = e^{-2u} g_0 \mid R \geq 0\}$$

and define

$$(70) \quad \bar{Y}([g_0]) := \sup_{g \in \overline{\mathcal{C}_1([g_0])}} \frac{\text{vol}(g) \int_M \sigma_2(g) dv(g)}{\left(\int_M \sigma_1(g) dv(g) \right)^2}.$$

Note that in $\overline{\mathcal{C}_1([g_0])}$ there is no metric with $R \equiv 0$, if $g_0 \in \mathcal{C}_1$. We have the following result, which improves slightly Theorem 2.

Theorem 5. *If (M^3, g_0) is a closed 3-dimensional manifold with positive Yamabe constant $Y_1([g_0]) > 0$, then*

$$(71) \quad \bar{Y}([g_0]) \leq \frac{1}{3}.$$

Moreover, equality holds if and only if (M^3, g_0) is space form.

Proof of Theorem 5. For any metric $g = e^{-2u}g_0 \in \overline{\mathcal{C}_1([g_0])}$, we consider $g_t = e^{-2tu}g_0$ for $0 < t < 1$. Clearly, $g_t \in \mathcal{C}_1([g_0])$. By the approximation arguments and Theorem 2, we have

$$\mathcal{E}(g) = \lim_{t \rightarrow 1} \mathcal{E}(g_t) \leq \frac{1}{3}.$$

Now we suppose $\mathcal{E}(g) = \frac{1}{3}$. Thus, g is an extremal metric in the class of $\overline{\mathcal{C}_1([g])}$ for the energy functional \mathcal{E} . Denote $M_1 := \{x \in M, \sigma_1(g)(x) = 0\}$ and $M_2 := \{x \in M, \sigma_1(g)(x) > 0\}$. We have $M = M_1 \cup M_2$ and (10) is verified in M_2 . On the other hand, if $x \in M_1$, then $\sigma_1(g)(x) = 0$ and $\sigma_2(g)(x) \leq 0$. Hence, we deduce in M_1

$$\sigma_2(g) - 3r_2(g) + 2s(g)\sigma_1(g) < 0$$

since $r_2(g) > 0$. On the other hand, by the definition of r_2 and s we know

$$\int_M (\sigma_2(g) - 3r_2(g) + 2s(g)\sigma_1(g)) dv(g) = 0$$

which implies $M = M_2$ and we have Equation (10). Therefore, from Lemma 2, we infer that M is an Einstein manifold and we finish the proof. \blacksquare

Proof of Theorem 1. Let (M^3, g) be a metric of non-negative scalar curvature. If its Yamabe constant is positive, then by Theorem 5 we have $\mathcal{E}(g) \leq \bar{Y}([g]) \leq 1/3$, which is equivalent to (1) by Lemma 1. Hence we only need to consider the case that g has zero Yamabe constant. In this case one can show that g has scalar curvature zero and hence (1) holds trivially.

It is trivial to see that an Einstein metric satisfies (1) with equality. Now assume that g is a metric of non-negative scalar curvature which satisfies (1) with equality. If $\int_M \sigma_1(g) dv(g) = 0$, then we have $\sigma_1(g) = 0$, which implies that $\bar{R} = R \equiv 0$. Using (1), g is a Ricci flat metric, and hence a flat metric. If $\int_M \sigma_1(g) dv(g) > 0$, by Lemma 1 we have $\mathcal{E}(g) = 1/3$. Hence (M^3, g) is a space form by Theorem 5. \blacksquare

In a recent joint work with Xia [11] we proved the rigidity of (1), namely under the conditions in Theorem A equality in (1) holds if and only if (M, g) is an Einstein metric.

7. PROBLEMS AND CONJECTURES

We end the paper by proposing several related problems and conjectures.

Conjecture 1. *Theorem 1 holds if g has a non-negative first Yamabe constant $Y_1([g])$.*

When g has a negative first Yamabe constant, Theorem 1 is not true. For example see [6].

Problem 1. $\tilde{Y}([g_0])$ is achieved.

This is a Yamabe type problem, but with a different property. From the analysis developed here, together with a classification result of blow-up solutions like in [22], one can expect that this conjecture is true if $\tilde{Y}([g_0]) < 1/3$. It is trivial to see that any metric g with constant sectional curvature satisfies $\mathcal{E}(g) = 1/3$, and hence

$$\tilde{Y}([g]) = \frac{1}{3} = \tilde{Y}([g_{\mathbb{S}^3}]),$$

where $g_{\mathbb{S}^3}$ is the standard round metric on \mathbb{S}^3 . We conjecture

Conjecture 2. Let (M^3, g_0) be a closed manifold with $g_0 \in \mathcal{C}_1$. If $\tilde{Y}([g_0]) = 1/3$, then (M^3, g_0) is conformally equivalent to a 3-dimensional spherical space form.

It is interesting to see that this conjecture, if it is true, gives a characterization of a conformal Einstein metric on a 3-dimensional manifold.

Let

$$J(g) := \int_M \sigma_1(g) dv(g) \cdot \int_M \sigma_2(g) dv(g).$$

Conjecture 3. Let (M^3, g_0) be a closed manifold with $g_0 \in \mathcal{C}_1$. The following statement

$$\tilde{Y}_{2,1}([g_0]) := \sup_{g \in \mathcal{C}_1([g_0])} J(g) \leq J(g_{\mathbb{S}^3})$$

is true.

This conjecture is closely related to a problem which was asked by Viaclovsky to us several years ago. Let

$$J_2(g) := \text{vol}(g)^{1/3} \cdot \int_M \sigma_2(g) dv(g).$$

He asked if $\sup_{g \in \mathcal{C}_2([g_0])} J_2(g)$ is bounded. We believe that it is true and we even believe more.

Conjecture 4. Let (M^3, g_0) be a closed manifold with $g_0 \in \mathcal{C}_1$. The following statement

$$\tilde{Y}_2([g_0]) := \sup_{g \in \mathcal{C}_1([g_0])} J_2(g) \leq J_2(g_{\mathbb{S}^3})$$

is true.

It is easy to see that Conjecture 3 implies Conjecture 4 and Conjecture 4 implies Theorem 2.

The Euler-Lagrange equation of J_2 is the so-called σ_2 -Yamabe equation

$$(72) \quad \sigma_2(g) = b,$$

for some constant b . A Lemma 2 type result is true for this equation. This in fact directly follows from a volume comparison result of Gursky-Viaclovsky [20], which in turn follows from a volume comparison Theorem of Bray [1].

Lemma 7. *Let $g \in \mathcal{C}_1$ be a metric on a 3-dimensional manifold M^3 satisfying (72) then*

$$J_2(g) \leq J_2(g_{\mathbb{S}^3}).$$

Proof. We need only to consider the case $b > 0$, otherwise the Lemma is trivial. We may assume that $b = \sigma_2(g_{\mathbb{S}^3})$, i.e, $\sigma_2(g) = \sigma_2(g_{\mathbb{S}^3})$. Theorem 1.2 in [20] implies

$$\text{vol}(g) \leq \text{vol}(g_{\mathbb{S}^3}).$$

Hence we have

$$J_2(g) \leq (\text{vol}(g_{\mathbb{S}^3}))^{\frac{4}{3}} \sigma_2(g_{\mathbb{S}^3}) = J_2(g_{\mathbb{S}^3}).$$

■

Hence, to show Conjecture 4, as inspired by the proof given above, one needs only either to show that J_2 is achieved, or to show a suitable perturbed functional has a maximum, together with a Lemma 6 type estimate. This is a difficult problem. There is even an extra difficulty that the corresponding flow is in general not parabolic. However for functional J there is no this extra difficulty. This is known from Lemma 4. Therefore, it may be better to study Conjecture 3 first.

The Euler-Lagrange equation of J is the so-called quotient equation

$$(73) \quad \frac{\sigma_2(g)}{\sigma_1(g)} = b.$$

With the same idea, we need the following comparison result.

Conjecture 5. *Let (M^3, g) be a closed 3-dimensional manifold with $g \in \mathcal{C}_1$. Assume that*

$$\frac{\sigma_2(g)}{\sigma_1(g)} \geq \frac{\sigma_2(g_{\mathbb{S}^3})}{\sigma_1(g_{\mathbb{S}^3})}.$$

Then

$$\int_M \sigma_1(g) dv(g) \leq \int_{\mathbb{S}^3} \sigma_1(g_{\mathbb{S}^3}) dv(g_{\mathbb{S}^3}).$$

If these conjectures are true, then it is natural to ask

Problem 2. *Are $\tilde{Y}_{2,1}([g_0])$ and $\tilde{Y}_2([g_0])$ achieved?*

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