Institut Camille Jordan
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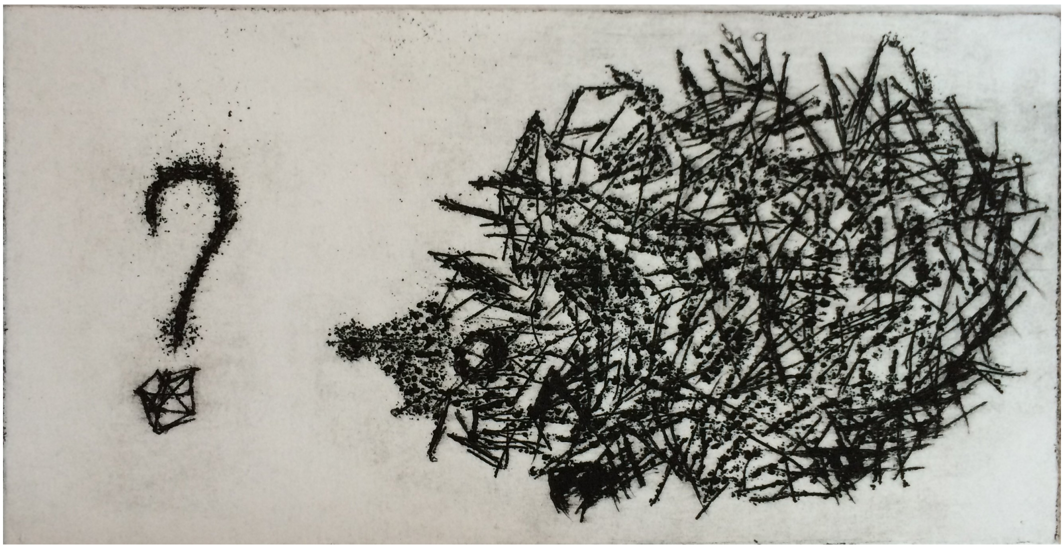
Autour des singularités d'applications vectorielles en physique de la matière condensée



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Thèse de doctorat

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Chapitre 1

Introduction

Avant-propos

Le manuscrit de cette thèse de doctorat, effectuée sous la direction de Petru Mironescu, se décompose en trois parties (auxquelles correspondent les trois premières sections de l'introduction). Les deux premières parties ont en commun l'étude de modèles physiques faisant intervenir des systèmes elliptiques dont les solutions présentent des « singularités » : la première partie est consacrée aux cristaux liquides nématiques, dans le cadre de la théorie de Landau-de Gennes ; la seconde partie rassemble quelques résultats obtenus autour de la théorie de Ginzburg-Landau pour la supraconductivité. Enfin la troisième partie est dédiée à certaines propriétés de la caractérisation « par convolution » des espaces fonctionnels de Besov.

Les différents travaux rassemblés dans cette thèse ont fait l'objet des publications suivantes.

- Chapitre 2 : *Uniaxial symmetry in nematic liquid crystals*, Ann. Inst. H. Poincaré Anal. Non Linéaire (à paraître).
- Chapitre 3 : *Biaxial escape in nematics at low temperature*, avec A. Contreras, soumis.
- Chapitre 4 : *Minimizers of the Landau-de Gennes energy around a spherical colloid particle*, avec S. Alama et L. Bronsard, soumis.
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- Chapitre 6 : *Existence of critical points with semi-stiff boundary conditions for singular perturbation problems in simply connected planar domains*, avec P. Mironescu, J. Math. Pures Appl. (2014).
- Chapitre 7 : *Persistence of superconductivity in thin shells beyond Hc1*, avec A. Contreras, Commun. Contemp. Math. (à paraître).
- Chapitre 8 : *Vortex structure in p-wave superconductors* , avec S. Alama et L. Bronsard, soumis.
- Chapitre 9 : *Characterization of function spaces via low regularity mollifier*, avec P. Mironescu, Discrete Contin. Dyn. Syst. Ser. A (à paraître).

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1.1 Cristaux liquides nématiques

Les cristaux liquides correspondent à des états de la matière intermédiaires entre liquide isotrope et cristal. Contrairement aux gaz et aux liquides, leurs propriétés physiques ne sont pas invariantes par tout déplacement (translation et rotation) de l'espace : ils présentent une **brisure de symétrie**. Toutefois cette brisure de symétrie est moindre que dans les cristaux, qui ne sont invariants que par des groupes discrets engendrés par certaines translations et rotations. Dans les cristaux liquides, la symétrie translationnelle (invariance par translation) est conservée – au moins dans une direction. Dans les phases nématique et cholestérique, seule la symétrie orientationnelle (invariance par rotation) est brisée : ces phases présentent un **ordre d'orientation**, mais pas d'ordre de position. Dans les phases smectiques et colonnaires la symétrie translationnelle est brisée, mais seulement dans une ou deux directions. Les **propriétés optiques remarquables** des cristaux liquides, dues à ces brisures de symétrie, sont à l'origine de leurs nombreuses applications industrielles.

On s'intéressera ici exclusivement aux cristaux liquides nématiques. Ils se composent de **molécules allongées**, qui **tendent à s'aligner** dans une direction commune : c'est ainsi qu'apparaît l'ordre d'orientation. Les centres de gravité des molécules sont distribués « au hasard » : il n'y a pas d'ordre de position. La phase nématique (du grec $\nu\eta\mu\alpha$, fil) doit son nom aux textures « à fils » qu'on peut y observer. Il s'agit de **défauts** optiques. L'ordre d'orientation peut en effet présenter des singularités, linéaires ou ponctuelles. Comprendre l'apparition et la structure de ces singularités est un des objectifs majeurs de l'étude théorique des cristaux liquides nématiques. Trois principales théories ont attiré l'attention des mathématiciens : les théories d'Oseen-Frank, Ericksen et Landau-de Gennes.

Les modèles d'Oseen-Frank et d'Ericksen

Pour un matériau nématique contenu dans un domaine $\Omega \subset \mathbb{R}^3$, la théorie d'Oseen-Frank décrit l'ordre d'orientation à l'aide d'un champ de directeur

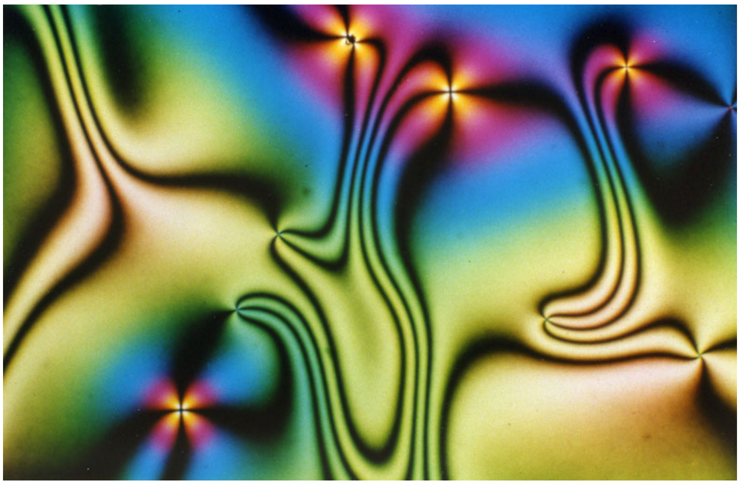


FIGURE 1.1 – Défauts optiques dans un cristal liquide nématique [77].

$n\colon \Omega \to \mathbb{S}^2$. Le vecteur unitaire $n(x)$ représente la direction moyenne d'alignement des molécules autour du point $x \in \Omega$. Un cas simplifié du modèle d'Oseen-Frank correspond au problème des applications harmoniques à valeurs dans $\mathbb{S}^2$ – c'est-à-dire les applications minimisant l'énergie de Dirichlet $\int |\nabla n|^2$ – et a été très largement étudié. Schoen et Uhlenbeck [123] ont prouvé qu'une application harmonique minimisante est analytique en dehors d'un ensemble discret de singularités. Brezis, Coron et Lieb [32] ont précisé la structure de ces défauts : ils sont de degré topologique ± 1, et le champ de directeur y est approximativement à symétrie radiale.

Remarque 1.1. Dans le cas général non simplifié, l'énergie élastique ne se réduit pas à la fonctionnelle de Dirichlet. Hardt, Kinderlehrer et Lin [59] ont démontré que l'ensemble singulier est alors de dimension de Hausdorff strictement inférieure à un. La régularité optimale reste un problème ouvert.

Pour une description plus précise du cœur des défauts, la théorie d'Ericksen couple le champ de directeur $n\colon \Omega \to \mathbb{S}^2$ à un champ scalaire $s\colon \Omega \to \mathbb{R}$. Le paramètre d'ordre scalaire $s(x)$ est une mesure du degré d'alignement le long de la direction moyenne $n(x)$. Les défauts correspondent au lieu de « fusion isotrope » $\{s = 0\}$, où l'ordre d'orientation est détruit. Dans un cas simplifié du modèle d'Ericksen, on est ramené à un problème d'applications harmoniques à valeurs dans un cône [63].

Le modèle de Landau-de Gennes

Une des limitations des modèles d'Oseen-Frank et d'Ericksen est de décrire l'ordre d'orientation à travers une *seule* direction d'alignement. En pratique, l'alignement des molécules est rarement à symétrie axiale autour d'une direction principale, et il est nécessaire de préciser une direction secondaire d'alignement. Pour prendre ce phénomène en compte, la théorie de Landau-de Gennes décrit l'ordre d'orientation à l'aide d'un paramètre d'ordre tensoriel, le Q-tenseur, à valeur dans l'espace

$$\mathcal{S}_0 = \{Q \in M_3(\mathbb{R}) \colon Q_{ij} = Q_{ij},\ \operatorname{tr} Q = 0\},$$

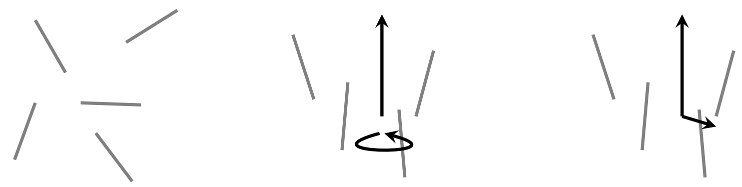
des matrices symétriques 3×3 de trace nulle. Les vecteurs propres de Q représentent les directions moyennes d'alignement, et les valeurs propres mesurent le degré d'alignement : plus une valeur propre est grande, meilleur est l'alignement dans la direction correspondante.

Remarque 1.2. Cette interprétation des vecteurs et valeurs propres de Q correspond à une définition microscopique du Q-tenseur comme moment statistique d'ordre deux de la direction des molécules. Plus précisément, les directions des molécules sont distribuées suivant une mesure de probabilité μ sur la sphère $\mathbb{S}^2$, et on a :

$$Q = \int_{\mathbb{S}^2} p \otimes p \, d\mu(p) - \frac{1}{3} I.$$

Le second moment est le premier moment non trivial (les molécules n'étant pas orientées, on a $d\mu(p) = d\mu(-p)$ et le premier moment $\int p \, d\mu(p)$ s'annule). La matrice $I/3$ correspond au cas d'une distribution uniforme $d\mu_0 = (4\pi)^{-1} d\mathcal{H}^2$, de sorte que Q est une mesure de la déviation de l'ordre d'orientation par rapport au cas isotrope (où il n'y a *pas* d'ordre d'orientation). Le Q-tenseur peut aussi être interprété – et mesuré – au niveau macroscopique : par exemple comme une fonction affine de la susceptibilité magnétique du matériau [45].

Un Q-tenseur peut décrire trois phases différentes, correspondant à différents degrés de symétrie :
— la phase **isotrope**, où $Q = 0$;
— la phase **uniaxe**, où Q a deux valeurs propres égales ;
— enfin la phase **biaxe**, où Q a toutes ses valeurs propres distinctes.



(a) état isotrope (b) état uniaxe (c) état biaxe

FIGURE 1.2 – Les différents degrés de symétrie dans $\mathcal{S}_0$.

Une distribution des orientations μ à symétrie axiale autour d'un directeur $n \in \mathbb{S}^2$ correspondra a un état uniaxe. Les Q-tenseurs uniaxes peuvent s'écrire sous la forme

$$Q = s\left(n \otimes n - \frac{1}{3} I\right), \qquad s \in \mathbb{R},\ n \in \mathbb{S}^2. \tag{1.1}$$

Le nombre réel s est un paramètre d'ordre scalaire qui mesure le degré d'alignement le long du directeur. Le cas $s = 0$ correspond à l'état isotrope. On peut interpréter le modèle d'Ericksen, où l'ordre d'orientation est décrit par le couple

(s, n), comme une restriction du modèle de Landau-de Gennes à l'état uniaxe. Le modèle d'Oseen-Frank correspond à un alignement uniaxe pour lequel le paramètre d'ordre scalaire aurait une valeur fixe $s = s_* > 0$.

La configuration d'un matériau nématique contenu dans un domaine $\Omega \subset \mathbb{R}^3$ est décrite par une application $Q \colon \Omega \to \mathcal{S}_0$. A une telle configuration on associe une énergie libre dépendant d'une constante élastique $L > 0$,

$$\mathcal{F}(Q) = \mathcal{F}_L(Q) = \int_\Omega \left[\frac{L}{2} |\nabla Q|^2 + f(Q) \right] dx. \tag{1.2}$$

Le potentiel $f \colon \mathcal{S}_0 \to \mathbb{R}$ est invariant par changement de référentiel, ce qui implique qu'il soit de la forme $f(Q) = \varphi(\operatorname{tr}(Q^2), \operatorname{tr}(Q^3))$.

Remarque 1.3. Il serait pertinent de considérer dans l'énergie libre une contribution élastique plus générale que $L/2\,|\nabla Q|^2$, mais on s'attend à ce que l'énergie simplifiée (1.2) fournisse déjà une bonne description qualitative du matériau.

En pratique, on suppose que le potentiel est de la forme

$$f(Q) = -\frac{a}{2} \operatorname{tr}(Q^2) - \frac{b}{3} \operatorname{tr}(Q^3) + \frac{c}{4} \operatorname{tr}(Q^2)^2, \tag{1.3}$$

où $b, c > 0$ et $a \geq 0$ sont des coefficients dépendant du matériau et de la température.

Un rôle important est bien sûr joué par l'ensemble des matrices $Q \in \mathcal{S}_0$ qui minimisent le potentiel $f(Q)$. Il s'agit exactement des Q-tenseurs uniaxes de paramètre d'ordre scalaire fixé $s_* > 0$:

$$\mathcal{U}_* := \operatorname*{arg\,min}_{Q \in \mathcal{S}_0} f(Q) = \left\{ s_* \left(n \otimes n - \frac{1}{3} I \right) : n \in \mathbb{S}^2 \right\}, \tag{1.4}$$

$$s_* = \frac{b + \sqrt{b^2 + 24ac}}{4c}.$$

En particulier, la théorie d'Oseen-Frank correspond à contraindre Q à prendre des valeurs dans $\mathcal{U}_*$, c'est-à-dire des valeurs minimisant $f(Q)$.

Dans la suite, on considérera essentiellement des minimiseurs de l'énergie libre $\mathcal{F}(Q)$, à donnée au bord fixée à valeurs dans $\mathcal{U}_*$: conditions d'**ancrage** fort. Les équations d'Euler-Lagrange correspondantes sont de la forme

$$L\Delta Q = \nabla f(Q) = -a\,Q - b \left(Q^2 - \frac{1}{3} \operatorname{tr}(Q^2) I \right) + c \operatorname{tr}(Q^2) Q, \tag{1.5}$$

et leurs solutions dans H^1 sont analytiques.

La limite de London

A la limite de London $L \to 0$, le potentiel $f(Q)$ est le terme dominant dans l'énergie $\mathcal{F}_L(Q)$ (1.2), et on s'attend donc à retrouver le modèle d'Oseen-Frank (qui correspond à prendre des valeurs minimisant le potentiel). En effet, de toute suite de minimiseurs Q_L, on peut extraire une sous-suite convergeant en norme H^1 vers une application harmonique minimisante à valeurs dans $\mathcal{U}_*$. Dans un domaine simplement connexe on peut identifier cette limite à une application harmonique minimisante à valeurs dans $\mathbb{S}^2$ [14], qui a un nombre fini de singularités. La convergence est en fait uniforme loin de ces défauts [95, 105].

On a donc une situation similaire au modèle simplifié de Ginzburg-Landau étudié par Bethuel, Brezis et Hélein [25] : dans leur cas, une suite d'applications régulières $u_\varepsilon \colon B^2 \to \mathbb{R}^2$ converge vers $u_* \colon B^2 \to \mathbb{S}^1$, et la convergence est uniforme loin des singularités de la limite. Chaque singularité correspond à une classe d'homotopie non triviale dans $\pi_1(\mathbb{S}^1)$, ce qui force u_ε à s'annuler près de ce défaut.

Ici, la suite d'applications Q_L à valeur dans $\mathcal{S}_0$ converge vers n_* à valeurs dans $\mathbb{S}^2$. Une singularité de n_* correspond à une classe non triviale de $\pi_2(\mathbb{S}^2)$. Contrairement à la situation de Ginzburg-Landau, puisque $\mathcal{S}_0$ est de dimension 5, cette topologie non triviale ne force pas *a priori* Q_L à s'annuler près des défauts.

En accord avec cette observation, la littérature physique prédit deux comportements possibles près d'un défaut : la « fusion isotrope » (*isotropic melting*, évoquée plus haut en lien avec le modèle d'Ericksen) et la « fuite biaxe » (*biaxial escape* [130]). La fusion isotrope correspond à l'annulation du paramètre d'ordre au cœur du défaut (comme dans le cas de Ginzburg-Landau). La fuite biaxe consiste à éviter cette annulation en s'éloignant fortement de la contrainte d'uniaxie, tirant ainsi parti des degrés de liberté supplémentaires offerts par l'espace $\mathcal{S}_0$. Suivant les régimes – c'est-à-dire suivant les valeurs des constantes matérielles a, b et c qui caractérisent le potentiel $f(Q)$ (1.3) – on s'attend à observer l'un ou l'autre de ces deux comportements.

Le hérisson radial

Une première étape dans la compréhension de ce phénomène consiste à étudier le problème « modèle » du hérisson radial. Dans une boule de rayon R avec ancrage radial (donnée au bord à symétrie sphérique) les équations d'Euler-Lagrange (1.5) admettent une solution symétrique de la forme

$$Q(x) = s(r)\left(\frac{x}{|x|}\otimes\frac{x}{|x|}-\frac{I}{3}\right),\quad s(0)=0,\ s(R)=s_*.$$

Le profil radial $s(r)$ satisfait une équation différentielle ordinaire. Cette solution fournit un example typique de fusion isotrope : l'ordre d'orientation reste purement uniaxe, et le paramètre d'ordre scalaire s'annule au cœur du défaut.

On s'intéresse plus particulièrement au hérisson radial dans l'espace entier $\mathbb{R}^3$ (correspondant au cas $R=\infty$), et on se pose la question de sa stabilité. Différents régimes ont été identifiés, notamment suivant la valeur de $a\geq 0$. Pour $a\to+\infty$, le hérisson radial est instable : on peut exhiber une perturbation explicite [53] qui induise dans l'énergie une seconde variation négative. Pour $a\sim 0$, le hérisson radial est stable [73] : la seconde variation de l'énergie est positive. Ce résultat de stabilité repose sur une habile décomposition permettant de séparer les variables, alliée à de fines estimations du profil radial.

Pour $a\to+\infty$, le potentiel $f(Q)$ dégénère (après une renormalisation adéquate) vers un potentiel de type Ginzburg-Landau $f(Q)=(|Q|^2-1)^2$. Des techniques développées pour le modèle de Ginzburg-Landau [99] peuvent alors être adaptées [67, 68] pour montrer que près d'un défaut, un minimiseur de l'énergie libre (1.2) a un profil radial. Ceci, à condition de supposer qu'on a un minimiseur **purement uniaxe**, c'est-à-dire de la forme

$$Q(x) = s(x)\left(n(x)\otimes n(x)-\frac{1}{3}I\right).$$

L'instabilité du hérisson radial permet alors de conclure que les minimiseurs de l'énergie libre (1.2) ne sont pas purement uniaxes [94]. Ce résultat fournit une interprétation du phénomène de fuite biaxe lorsque $a \gg 1$. Toutefois la contrainte de pure uniaxie est extrêmement forte : elle surdétermine le problème (cf. § 1.1.1). Il parait plus pertinent physiquement d'interpréter la fuite biaxe à travers des régions de **forte** biaxie – approche mise en œuvre par Canevari [33] dans un cadre bidimensionnel (cf. § 1.1.2).

Autour de la biaxie et des défauts

Dans la suite de cette section, on présente les résultats obtenus durant cette thèse sur les cristaux liquides nématiques : la non-existence de solutions purement uniaxes non-triviales, dans différents contextes (§ 1.1.1) ; la fuite biaxe pour $a \gg 1$ (§ 1.1.2) ; la configuration d'un matériau nématique autour d'une particule colloïdale (§ 1.1.3) ; enfin, l'analyse d'une cellule frustrée où un mécanisme impliquant des états fortement biaxes est étudié en détail (§ 1.1.4) ;

1.1.1 La contrainte de symétrie uniaxe

Dans le chapitre 2, on s'intéresse aux solutions purement uniaxes des équations d'Euler-Lagrange (1.5), c'est-à-dire les solutions qui peuvent s'écrire sous la forme

$$Q(x) = s(x)\left(n(x) \otimes n(x) - \frac{1}{3}I\right). \tag{1.6}$$

L'*ansatz* (1.6) correspond à une contrainte de symétrie locale. Le phénomène de fuite biaxe peut être interprété, à l'instar de [67, 94], comme la brisure de cette symétrie.

Remarque 1.4. Dans [94] les auteurs démontrent que dans la limite $a \to +\infty$ les minimiseurs ne peuvent pas être purement uniaxe. Les résultats du chapitre 2 indiquent qu'en fait il n'y a en général pas de solutions purement uniaxes aux équations d'Euler-Lagrange : et ce, indépendamment de la valeur de a et de la minimisation de l'énergie.

On connaît deux types « triviaux » de solutions purement uniaxes :
— les solutions constantes, ou à directeur constant ;
— les solutions à symétrie radiale $Q(x) = s(r)(e_r \otimes e_r - I/3)$, où $e_r = x/|x|$.

On démontre que ce sont essentiellement les seules : il n'existe pas en général de solutions purement uniaxes non-triviales. Plus précisément, dans le cas bidimensionel on sait classifier toutes les solutions uniaxes :

Théorème. *Si* $\Omega \subset \mathbb{R}^3$ *est un ouvert connexe et* $Q \in H^1_{loc}(\Omega; \mathcal{S}_0)$ *est une solution purement uniaxe* (1.6) *des équations d'Euler-Lagrange* (1.5) *vérifiant*

$$\partial_3 Q \equiv 0,$$

alors Q *est à directeur constant.*

Pour des configurations tridimensionnelles, on sait traiter le cas modèle d'une boule B_R avec ancrage radial :

Théorème. *Si* $Q \in H^1(B_R;\mathcal{S}_0)$ *est une solution purement uniaxe* (1.6) *des équations d'Euler-Lagrange* (1.5) *vérifiant*

$$Q(x) = s_0 \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3} I \right) \quad \textit{pour } x \in \partial B_R,$$

avec $s_0 \neq 0$, *alors* Q *est à symétrie radiale.*

Il est crucial de noter que ces résultats portent sur des solutions purement uniaxes des équations d'Euler-Lagrange,

— sans aucune hypothèse de minimisation d'énergie,

— et sans restriction majeure sur la forme du potentiel $f(Q)$.

En particulier, ceci contraste fortement avec le résultat de symétrie radiale démontré dans [67, 68] pour des solutions purement uniaxes minimisant l'énergie, dans la limite $a \to \infty$. Si une morale doit être tirée de ces résultats, c'est que la contrainte de pure uniaxie est extrêmement rigide, et que l'interprétation de la fuite biaxe comme violation de cette contrainte uniaxe paraît donc limitée.

Les deux résultats de rigidité ci-dessus ont comme point de départ l'écriture des équations satisfaites par le paramètre d'ordre scalaire $s(x)$ et le directeur $n(x)$. Un premier système d'équations peut être obtenu comme équations d'Euler-Lagrange de la fonctionnelle $\mathcal{F}(s,n)$ obtenue en injectant l'*ansatz* uniaxe (1.1) dans l'énergie libre $\mathcal{F}(Q)$. Ce système elliptique exprime la criticalité par rapport aux perturbations préservant la symétrie :

$$(S) \qquad \begin{cases} \Delta s = 3|\nabla n|^2 s + \dfrac{1}{L} f'(s), \\ s\Delta n + 2(\nabla s \cdot \nabla) n = -s|\nabla n|^2 n, \end{cases}$$

Pour certaines contraintes de symétrie, il arrive que la criticalité par rapport aux perturbations brisant la symétrie n'apporte aucune information supplémentaire (principe de criticalité symétrique [108]). Il en va par exemple ainsi de la contrainte de symétrie sphérique : un minimiseur parmi les configurations radiales est automatiquement solution des équations d'Euler-Lagrange complètes. Ici, aucune simplification n'a lieu, et la criticalité non symétrique fournit une équation supplémentaire :

$$(SB) \qquad 2\sum_{k=1}^{3} \partial_k n \otimes \partial_k n = |\nabla n|^2 \left(I - n \otimes n \right).$$

Cette équation supplémentaire ne porte que sur le directeur $n(x)$. Elle est d'ordre un, et relativement dégénérée car quadratique en ∇n. Malgré cette dégénérescence, on s'attend intuitivement à ce que satisfaire à la fois (S) et (SB) soit un problème surdéterminé.

Dans le cas bidimensionnel, des manipulations algébriques de (S) et (SB) alliées à l'hypothèse $\partial_3 n \equiv 0$ permettent en effet d'obtenir que

$$\partial_1 n \cdot \partial_2 n = 0, \quad |\partial_1 n| = |\partial_2 n| \equiv c_0.$$

La constante c_0 ne saurait être non nulle, car on obtiendrait alors une isométrie locale $n \colon \mathbb{R}^2 \to \mathbb{S}^2$ contredisant le *theorema egregium* de Gauss. On en déduit donc que n est constant.

Dans le cas d'une boule avec ancrage radial, la preuve de la symétrie radiale repose sur l'analyticité des solutions. Dans l'esprit du théorème de Cauchy-Kovaleskaya, on tire profit des conditions de Dirichlet combinées aux équations (S) et (SB) pour déterminer la donnée au bord « à l'ordre suivant ». Plus spécifiquement, on détermine les dérivées radiales à la surface :

$$\partial_r n \equiv 0, \quad \partial_r s \equiv s_1 \qquad \text{sur } \partial B_R.$$

De cette information on déduit la symétrie radiale. La dégénérescence de l'équation (SB) fait qu'on est précisément dans le cas où le théorème de Cauchy-Kovaleskaya ne s'applique pas. Ainsi la détermination des dérivées radiale d'ordre un requiert de prendre quatre dérivées radiales successives de (SB) et deux dérivées de (S) pour obtenir assez d'information (ce qui rend la démonstration relativement fastidieuse, bien qu'élémentaire).

1.1.2 Fuite biaxe à basse température (avec A. Contreras)

Rappelons que près d'un défaut, deux comportements possibles sont suggérés par la littérature physique : la fusion isotrope et la fuite biaxe. Ces deux comportements correspondent à des topologies différentes pour l'ensemble $\mathcal{V}$ (contenant $\mathcal{U}_*$) des valeurs prises par le paramètre d'ordre : suivant la forme de $\mathcal{V}$, on peut – ou non – étendre continûment *et sans point d'annulation* une donnée au bord correspondant à une classe non triviale de $\pi_2(\mathcal{U}_*)$.

Dans le cas des applications purement uniaxes, on autorise Q à prendre ses valeurs dans le cône uniaxe $\mathcal{U} := \mathbb{R}\cdot\mathcal{U}_*$ et toute extension continue d'une donnée au bord topologiquement non triviale doit s'annuler, car $\pi_2(\mathcal{U}\setminus\{0\}) \simeq \pi_2(\mathcal{U}_*)$. Dans [33], Canevari remarque que cela est encore vrai même si on autorise « un peu » de biaxie.

Plus précisément, le **paramètre de biaxie** introduit dans [75],

$$\beta(Q) = 1 - 6\frac{(\mathrm{tr}(Q^3))^2}{|Q|^6}, \quad 0 \leq \beta(Q) \leq 1,$$

est tel que : Q est uniaxe si et seulement si $\beta(Q) = 0$. On interprète alors les Q-tenseurs **maximalement biaxes** comme ceux qui vérifient $\beta(Q) = 1$. La remarque cruciale de Canevari [33] concerne le cône $\mathcal{C} := \{\beta < 1\}$ des Q-tenseurs non maximalement biaxes : on a $\pi_2(\mathcal{C}\setminus\{0\}) \simeq \pi_2(\mathcal{U}_*)$. Ainsi, même si on autorise le Q-tenseur à prendre ses valeurs dans $\mathcal{C}$, toute extension continue d'une donnée au bord topologiquement non-triviale devra s'annuler.

En conséquence, démontrer que la fusion isotrope *n'a pas lieu* fournit une interprétation physiquement satisfaisante du phénomène de fuite biaxe : une donnée au bord topologiquement non triviale force le paramètre d'ordre à prendre des valeurs maximalement biaxes. Dans le chapitre 3 on s'intéresse au régime $a \gg 1$, où a est le coefficient apparaissant dans le terme quadratique du potentiel $f(Q)$ (1.3), et on démontre le résultat suivant :

Théorème. *Pour $a \gg 1$, si une configuration Q minimise l'énergie libre* (1.2)*, alors elle* ***ne s'annule pas****.*

Ce résultat implique évidemment le fait précédemment connu [94] que les minimiseurs ne peuvent être purement uniaxes, mais va bien plus loin puisqu'il exclut aussi des minimiseurs à valeurs non maximalement biaxes. Dans [33], Canevari démontre un résultat similaire dans un cadre bidimensionnel. La preuve du théorème ci-dessus repose sur des arguments très différents (qui pourraient s'adapter au cas bidimensionnel).

La propriété essentielle de la limite $a \to +\infty$ est la dégénérescence du potentiel $f(Q)$ vers un potentiel de type Ginzburg-Landau. Plus précisément, après une renormalisation adéquate, on a

$$f(Q) \approx a \left(|Q|^2 - 1\right)^2 + \sqrt{a}\, g(Q), \qquad (a \gg 1),$$

où le terme non-dominant $g(Q) \geq 0$ est nul exactement pour Q uniaxe de norme un. Ainsi $f(Q)$ pénalise bien la biaxie, mais cette pénalisation est de moins en moins forte lorsque a croît. Cette dégénérescence permet de relier le profil d'un minimiseur autour d'un défaut, à une application harmonique minimisante à valeur dans $\mathbb{S}^4 \subset \mathcal{S}_0$. Un *blow-up* à la bonne échelle « efface » en effet le terme $g(Q)$ du potentiel. Puis un *blow-down* fournit une application harmonique à valeurs dans $\mathbb{S}^4$. La régularité des applications harmoniques [123] permet alors d'écarter la possibilité d'un point d'annulation. L'étape cruciale de la convergence du *blow-down* repose sur l'analyse délicate effectuée dans [97, 86].

1.1.3 Suspension colloïdale (avec S. Alama et L. Bronsard)

L'immersion d'une particule étrangère dans un cristal liquide nématique crée des défauts topologiques conduisant à de fascinants phénomènes d'auto-assemblement [113, 104] aux nombreuses applications potentielles [125, 112]. Cette sensibilité à l'inclusion d'un corps étranger peut aussi être exploitée à des fins biomédicales [141], notamment pour concevoir de nouveaux capteurs biologiques percevant en temps réel la présence de microbes [129, 66, 71].

L'ancrage du nématique à la surface des particules joue un rôle crucial dans la structure d'une suspension colloïdale. Dans le chapitre 4 on étudie une particule sphérique avec ancrage homéotrope (*i.e.* normal), plongée dans un matériau nématique uniformément aligné. Il s'agit d'un pas crucial vers la compréhension de colloïdes nématiques plus complexes. Durant les deux dernières décennies, différents auteurs ont abordé ce problème [136, 80, 91, 131, 115] à travers des approximations heuristiques et des calculs numériques . La charge topologique créée par l'ancrage homéotrope est compensée par l'apparition de défauts. Les travaux cités distinguent deux types de configurations :

— quadrupolaire, avec défaut linéaire en « anneau de Saturne »,
— ou dipolaire, avec défaut ponctuel de degré -1.

La plupart de ces travaux [136, 80, 91, 131] utilisent le modèle d'Oseen-Frank qui présente l'inconvénient d'associer une énergie infinie aux défauts linéaire : l'énergie d'une configuration quadrupolaire à anneau de Saturne doit donc être renormalisée.

Dans le chapitre 4 on justifie rigoureusement l'observation numérique [131, 115] de configurations quadrupolaires pour les petites particules, et dipolaires pour les grandes particules. Pour une petite particule on obtient de plus le rayon exact de l'anneau de Saturne, pour lequel les différentes valeurs calculées dans [136, 80, 91, 131, 115] ne concordaient pas.

On note $r_0 > 0$ le rayon de la particule colloïdale : le matériau nématique est contenu dans un domaine $\Omega_{r_0} := \mathbb{R}^3 \setminus \overline{B}_{r_0}$. Loin de la particule, l'alignement

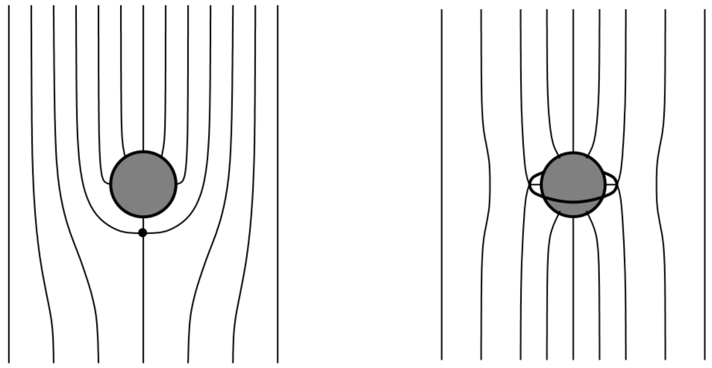


FIGURE 1.3 – Configurations dipolaire et quadrupolaire [131].

uniforme est supposé conservé :

$$\lim_{|x|\to\infty} Q(x) = Q_\infty := s_* \left(e_z \otimes e_z - \frac{1}{3} I \right), \qquad e_z = (0,0,1).$$

Pour assurer l'existence de minimiseurs, on complète cette condition par une contrainte d'intégrabilité : $Q - Q_\infty \in L^2(|x|^{-2}\, dx)$.

L'ancrage homéotrope à la surface de la particule est imposé à travers l'addition à l'energie libre $\mathcal{F}(Q)$ d'un terme d'énergie surfacique

$$\mathcal{F}_s(Q) = \frac{W}{2} \int_{\partial B_{r_0}} |Q_s - Q|^2 \, d\mathcal{H}^2, \qquad Q_s := s_* \left(\frac{x}{|x|} \otimes \frac{x}{|x|} - \frac{1}{3} I \right),$$

pour une certaine force d'ancrage $W > 0$. Cet ancrage homéotrope « faible » est plus réaliste physiquement que l'ancrage fort (conditions de Dirichlet) correspondant à $W = +\infty$. Il induit les conditions au bord

$$\frac{L}{W} \frac{\partial Q}{\partial \nu} = Q_s - Q \quad \text{sur } \partial\Omega_{r_0},$$

où ν est la normale extérieure à Ω_{r_0}, et L la constante élastique apparaissant dans l'énergie libre (1.2).

Dans le régime de petite particule $r_0^2 \ll L$, les solutions d'énergie finie des équations d'Euler-Lagrange convergent vers une limite explicite :

Théorème. *Lorsque $r_0^2/L \to 0$ et $r_0 W/L \to w \in (0, +\infty]$, on a*

$$Q(r_0 x) \to Q_0(x) = \frac{w}{1+w} \frac{1}{r^3} Q_s + \left(1 - \frac{w}{1+w} \frac{1}{r} \right) Q_\infty,$$

localement uniformément en $x \in \overline{\Omega}_1$.

On peut alors interpréter l'anneau de Saturne comme lieu uniaxe de Q_0 : la plus grande valeur propre de $Q_0(x)$ est double si et seulement si x appartient à un cercle horizontal, dont le rayon rayon $r_w > 1$ vérifie

$$r^3 - \frac{w}{1+w} r^2 - \frac{w}{3+w} = 0.$$

Ainsi pour un ancrage fort $w = +\infty$, le rapport du rayon de l'anneau de Saturne au rayon de la particule est $r \approx 1,47$. Ce rapport décroît avec la force de l'ancrage, et sous la valeur critique $w = \sqrt{3}$ l'anneau de Saturne disparaît complètement.

L'analyse du régime de grande particule $r_0^2 \gg L$ est plus délicate. On obtient à la limite un problème d'application harmonique à valeurs dans $\mathbb{S}^2$. On se restreint au cas d'ancrage fort $W = +\infty$: condition de Dirichlet radiale à la surface de la particule. Le degré topologique impose alors à l'application harmonique d'avoir au moins un défaut de degré -1. Déterminer le nombre exact de défauts d'une application harmonique minimisante est en général un problème très difficile, et il existe peu de situations où l'on sait que ce nombre correspond au degré topologique. C'est pourquoi on fait une hypothèse simplificatrice supplémentaire de symétrie axiale autour de l'axe vertical (en un sens précisé dans le chapitre 4). On peut alors démontrer qu'il n'y a bien qu'un seul défaut :

Théorème. *Soit Q un minimiseur de l'énergie libre parmi les configurations à symétrie axiale, avec condition de Dirichlet radiale $Q = Q_s$ sur $\partial\Omega_{r_0}$. Lorsque $r_0^2/L \to +\infty$, on a le long d'une sous-suite*

$$Q(r_0 x) \longrightarrow s_* \left(n(x) \otimes n(x) - \frac{1}{3} I \right),$$

localement uniformément en $x \in \overline{\Omega}_1 \setminus \{p_0\}$, où p_0 est l'unique singularité de l'application harmonique $n(x)$.

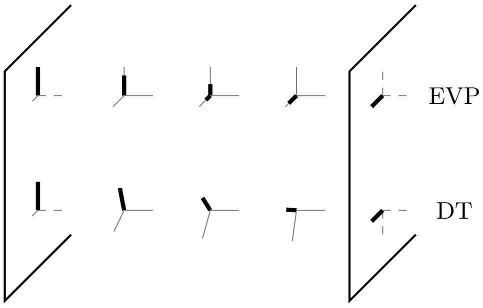
Les applications harmoniques à symétrie axiale ont été étudiées dans [60, 64]. Le point central du théorème ci-dessus est la détermination du nombre de défauts de l'application harmonique limite (en l'occurrence un seul), justifiant l'observation de configurations dipolaires.

1.1.4 Une cellule nématique frustrée

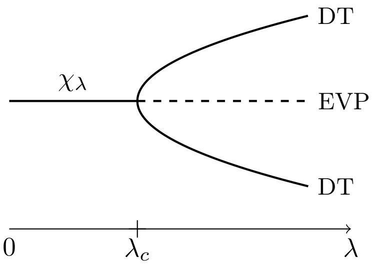
On s'intéresse dans le chapitre 5 à une cellule hybride imposant une frustration à un matériau nématique. Il s'agit de deux plaques parallèles, entre lesquelles se trouve le cristal liquide. La frustration est créée par des conditions d'ancrage antagonistes : le directeur spécifié à la surface de l'une des plaques est orthogonal au directeur spécifié sur l'autre plaque.

Deux paramètres peuvent varier : une température réduite θ, et une longueur typique λ proportionnelle à la largeur de la cellule. A température fixée, et lorsque la largeur de la cellule varie, des études numériques [26, 109] ont mis en évidence une bifurcation « fourche » avec deux types de solutions : à *échange de valeurs propres* (EVP), et à *directeur tournant* (DT). Le mécanisme d'échange de valeurs propres des solutions EVP n'est possible qu'à travers des états fortement biaxes.

Le but de l'analyse contenue dans le chapitre 5 est de justifier rigoureusement la forme du diagramme de la figure 1.4b. Les symétries du problèmes rendent naturelles les deux restrictions suivantes, déjà imposées dans [26, 109] : on considère des configurations ne dépendant que d'une coordonnée (perpendiculaire aux plaques) ; et on impose une direction propre du Q-tenseur (orthogonale aux directions d'ancrage au bord). Après une renormalisation adaptée,



(a) Configurations EVP et DT : vecteurs propres de Q à travers la cellule, avec en exergue celui correspondant à la plus grande valeur propre.



(b) Bifurcation fourche décrite dans [26, 109].

FIGURE 1.4 – Solutions EVP/DT et diagramme de bifurcation

on est ramené à l'étude d'applications $Q\colon [-1,1] \to \mathcal{S}_0$ satisfaisant le système d'équations différentielles ordinaires

$$\frac{1}{\lambda^2} Q'' = \frac{\theta}{3} Q - 2\left(Q^2 - \frac{|Q|^2}{3} I\right) + \frac{1}{2}|Q|^2 Q.$$

Les conditions d'ancrage à la surface des plaques fournissent des conditions de Dirichlet en $x = \pm 1$.

Pour une cellule très large ($\lambda \gg 1$), les minimiseurs de l'énergie convergent vers deux configurations uniaxes correspondant à un directeur tournant dans un sens ou dans l'autre. Pour une cellule très étroite ($\lambda \ll 1$), des considérations générales sur une classe de systèmes elliptiques perturbés (cf. annexe 5.B) permettent de montrer l'unicité de la solution, qui est nécessairement de type EVP. On dispose alors d'une branche lisse $(0, \lambda_*) \ni \lambda \mapsto \chi_\lambda$ de solutions EVP, pour un certain $\lambda_* = \lambda_*(\theta)$.

On montre ensuite l'apparition d'une bifurcation fourche, au moins dans un certain intervalle de température autour de $\theta = -8$, température à laquelle les calculs numériques dans [26] sont effectués car le système d'équations y est simplifié.

Théorème. *Soit $\theta \approx -8$. Il existe $\lambda_c = \lambda_c(\theta) \in (0, \lambda_*)$ tel que la solution χ_λ est stable pour $\lambda < \lambda_c$, instable pour $\lambda > \lambda_c$, et la perte de stabilité s'accompagne d'une bifurcation fourche symétrique.*

Dans le cas $\theta = -8$ on peut se ramener, pour l'étude des solutions χ_λ, à une seule équation au lieu d'un système de deux équations. Cette simplification permet une analyse très précise de ces solutions EVP. On montre ainsi que :

— la branche de solutions $\lambda \mapsto \chi_\lambda$ est définie pour tout $\lambda > 0$,

— elle perd sa stabilité pour un certain $\lambda_c > 0$, et cette perte de stabilité est non dégénérée (la plus petite valeur propre de l'opérateur linéarisé change de signe avec une dérivée strictement négative).

La non-dégénérescence de la perte de stabilité permet ensuite d'appliquer un théorème de bifurcation classique [42]. De plus elle permet d'appliquer le théorème des fonctions implicites pour traiter le cas perturbé $\theta \approx -8$.

1.2 Supraconductivité

Dans le cadre de la théorie de Ginzburg-Landau, l'état d'un matériau supraconducteur est décrit par un paramètre d'ordre u à valeurs complexes. Ce paramètre d'ordre peut-être interprété comme une fonction d'onde, dont le module au carré $|u|^2$ est la densité de paires d'électrons supraconducteurs. Ainsi l'état supraconducteur correspond à $|u| \approx 1$, alors que l'état normal correspond à $|u| \approx 0$.

L'application d'un champ magnétique $H_{ex}\colon \mathbb{R}^3 \to \mathbb{R}^3$ à un matériau supraconducteur $S \subset \mathbb{R}^3$ induit un potentiel magnétique $A\colon \mathbb{R}^3 \to \mathbb{R}^3$. L'énergie du système dépend alors du potentiel A et de la fonction d'onde $u\colon S \to \mathbb{C}$. Cette énergie est donnée par la fonctionnelle de Ginzburg-Landau

$$GL_\kappa^{3d}(u,A) = \frac{1}{2}\int_S \left\{ |(\nabla - iA)u|^2 + \frac{\kappa^2}{2}(|u|^2-1)^2 \right\} + \frac{1}{2}\int_{\mathbb{R}^3} |\operatorname{curl} A - H_{ex}|^2,$$

où $\kappa > 0$ est le paramètre de Ginzburg-Landau, qui dépend tu type de supraconducteur considéré.

Le cas d'un supraconducteur de forme cylindrique dans le régime $\kappa = 1/\varepsilon \gg 1$ a particulièrement attiré l'attention de la communauté mathématique. Supposons $S = \Omega \times \mathbb{R}$ avec une section $\Omega \subset \mathbb{R}^2$, et que le champ magnétique appliqué $H_{ex} = h_{ex} e_z$ ($h_{ex} \in \mathbb{R}$) est parallèle à l'axe du cylindre. On est alors ramené à l'étude de la fonctionelle

$$GL_\varepsilon^{2d}(u,A) = \frac{1}{2}\int_\Omega \left\{ |(\nabla - iA)u|^2 + \frac{1}{2\varepsilon^2}(|u|^2-1)^2 \right\} + \frac{1}{2}\int_{\mathbb{R}^2} |\operatorname{curl} A - h_{ex}|^2,$$

où $u\colon \Omega \to \mathbb{C}$ et $A\colon \mathbb{R}^2 \to \mathbb{R}^2$. Un zéro isolé du paramètre d'ordre u, avec un degré topologique non nul, est appelé *vortex*.

Une large littérature a été consacrée à la description des *vortex*. Au début des années 90, Bethuel, Brezis et Hélein [25] considèrent un modèle simplifié sans champ magnétique

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega \left\{ |\nabla u|^2 + \frac{1}{2\varepsilon^2}(|u|^2-1)^2 \right\}.$$

L'effet d'un champ magnétique élevé y est remplacé par l'imposition d'un degré topologique à travers une condition de Dirichlet $g\colon \partial\Omega \to \mathbb{S}^1$ topologiquement non triviale. Bien que cette condition n'ait pas de réalité physique, leur description des vortex correspond a des phénomènes observés par les physiciens. En particulier, la vorticité est quantifiée par le degré de la donnée au bord, l'énergie se concentre dans les vortex, et les interactions vortex-vortex et vortex-bord sont décrites par une énergie renormalisée. Les phénomènes liés aux variations de l'intensité du champ magnétique h_{ex} nécessitent bien sûr de s'intéresser au modèle non simplifié $GL_\varepsilon^{2d}(u,A)$. Dans une série d'articles suivis du monographe [119], Sandier et Serfaty démontrent rigoureusement l'existence de seuils critiques pour le champ magnétique h_{ex}, et analysent en détails la répartition des vortex. Enfin l'étude de la disparition totale de supraconductivité lorsque le champ magnétique est très élevé, liée à l'analyse semi-classique du Laplacien magnétique, est présentée dans [51].

Dans la suite de cette section, on présente les résultats obtenus durant cette thèse autour du modèle de Ginzburg-Landau : l'existence de points critiques de l'énergie simplifiée $E_\varepsilon(u)$ lorsque la condition de Dirichlet est remplacée par la seule prescription d'un degré au bord (§ 1.2.1) ; la persistance de la supraconductivité dans une fine coque supraconductrice au-delà du premier champ critique (§ 1.2.2) ; enfin la structure d'un vortex dans un modèle de supraconductivité non conventionnelle couplant deux paramètres d'ordre complexes (§ 1.2.3).

1.2.1 Un problème non-compact : conditions semi-rigides (avec P. Mironescu)

Dans le modèle simplifié $E_\varepsilon(u)$ considéré par Bethuel, Brezis et Hélein [25], c'est la condition de Dirichlet $u_{|\partial\Omega} = g$, où $g\colon \partial\Omega \to \mathbb{S}^1$ a un degré topologique non nul, qui entraîne la formation de vortex. Plus précisément, il est démontré dans [25] que les minimiseurs u_ε ont des zéros isolés à l'intérieur de Ω, qui convergent lorsque $\varepsilon \to 0$ vers les singularités d'une configuration limite $u_\star$ à valeurs dans $\mathbb{S}^1$. En l'absence de conditions de Dirichlet, on n'observe plus de vortex : les minimiseurs de E_ε sont constants, et tout point critique u non constant est instable pour ε petit [126].

Il est naturel de considérer un modèle intermédiaire où l'on prescrit seulement le degré du paramètre d'ordre : on s'intéresse dans le chapitre 6 aux points critiques de l'énergie $E_\varepsilon(u)$ dans l'espace

$$\mathcal{E}_d := \left\{ u \in H^1(\Omega;\mathbb{C})\colon\ |u| = 1 \text{ sur } \partial\Omega,\ \deg(u,\partial\Omega) = d \right\}, \quad d \in \mathbb{Z}.$$

Cela revient à imposer une condition de Dirichlet sur le module, et une condition de Neumann sur la phase du paramètre d'ordre : conditions au bord semi-rigides

$$|u| = 1 \quad \text{et} \quad u \wedge \frac{\partial u}{\partial \nu} = 0 \quad \text{sur } \partial\Omega.$$

La principale difficulté liée à ce modèle est le fait que les classes $\mathcal{E}_d$ ne sont pas stables par convergence faible H^1 : le problème de minimisation correspondant $\min_{\mathcal{E}_d} E_\varepsilon$ est non compact.

La question de l'existence d'un minimiseur est liée à la forme du domaine Ω. Dans un domaine simplement connexe, l'infimum de E_ε dans $\mathcal{E}_d$ n'est jamais atteint [19] (pour $d \neq 0$). Dans un domaine multiplement connexe (où l'on prescrit alors le degré sur chaque composante du bord), l'existence d'un minimiseur dépend de la capacité du domaine [55, 19, 18], toutefois il est toujours possible de construire des minimiseurs locaux pour ε petit [21, 49]. Cette construction ne semble pas pouvoir s'adapter au cas des domaines simplement connexes. Pour ε grand, l'existence de point critiques de E_ε dans $\mathcal{E}_d$ dans un domaine simplement connexe est démontrée dans [20] à l'aide d'une construction de type min-max. Reste la question de l'existence de points critiques dans un domaine simplement connexe pour ε petit, traitée dans le chapitre 6 par des techniques de perturbation singulière.

L'idée utilisée dans le chapitre 6 repose sur la construction par Pacard et Rivière [107] de points critiques de E_ε avec condition de Dirichlet. Ces points critiques u_ε sont construits comme perturbations d'une configuration limite $u_\star$, supposée « non dégénérée ». De manière similaire, nous obtenons l'existence de point critiques de E_ε à degré prescrit, sous l'hypothèse que le domaine Ω est – en un certain sens – non dégénéré.

Rappelons [25] qu'une configuration limite $u_\star$ est une application harmonique à valeurs dans $\mathbb{S}^1$, ayant en $a_1, \ldots, a_k \in \Omega$ des singularités de degrés respectifs $d_1, \dots, d_k \in \mathbb{Z}$ (dont la somme est égale à d). Supposons les degrés d_j fixés. A toute condition de Dirichlet $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$ on peut associer l'énergie renormalisée $W_g(a_1, \dots, a_k)$ de la configuration $u_\star$ ayant une singularité en chaque point a_j, et valant g au bord. Pour la construction de Pacard et Rivière [107], une configuration limite $u_\star$ est non dégénérée si $a = (a_1, \dots, a_k)$ est un point critique non dégénéré de W_g.

Dans notre cas, on s'intéresse aux conditions au bord semi-rigides : la configuration limite $u_\star$ doit satisfaire la condition de Neumann pour la phase

$$u_\star \wedge \frac{\partial u_\star}{\partial \nu} = 0.$$

L'application $u_\star$ est uniquement déterminée (modulo $\mathbb{S}^1$) par $a = (a_1, \dots, a_k)$, le k-uplet de ses singularités. On note $g^a := \mathrm{tr}(u_\star, \partial\Omega)$ la condition de Dirichlet correspondante. Si on suppose que

$$(ND1) \qquad a \text{ est un point critique non dégénéré de } W_{g^a},$$

on peut définir, pour $g \approx g^a$, une application harmonique $u_{\star,g} \approx u_\star$ qui soit non dégénérée au sens de Pacard et Rivière, et un opérateur

$$T_\star \colon C^{1,\beta}(\partial\Omega; \mathbb{S}^1)/\mathbb{S}^1 \to \dot{C}^\beta(\partial\Omega; \mathbb{R}), \quad g \mapsto u_{\star,g} \wedge \frac{\partial}{\partial \nu} u_{\star,g},$$

qui soit C^1 et Fredholm d'indice 0. On dit alors que la configuration limite $u_\star$ est non dégénérée pour les conditions semi-rigides, si cet opérateur vérifie :

$$(ND2) \qquad DT_\star(g^a) \text{ est inversible.}$$

S'il existe une configuration limite non dégénérée, il est possible de la perturber pour obtenir des solutions à notre problème :

Théorème. *Si $d_j \in \{\pm 1\}$, et si le domaine simplement connexe Ω est non dégénéré, au sens qu'il existe une configuration limite vérifiant $(ND1) - (ND2)$, alors pour ε assez petit l'énergie E_ε admet un point critique u_ε à degré prescrit $d = d_1 + \dots + d_k$.*

La démonstration de ce théorème repose sur le fait qu'on peut rendre la construction $u_{\varepsilon,g}$ de Pacard et Rivière localement uniforme en la donnée au bord g, et ce dans un cadre fonctionnel assez faible pour que la dérivée normale de la phase de $u_{\varepsilon,g}$ soit une perturbation compacte de la configuration limite (en tant qu'opérateur appliqué à g). Il est alors possible de choisir une donnée au bord g_ε telle que la solution u_ε correspondante vérifie les conditions semi-rigides.

Muni de ce résultat abstrait, on doit ensuite s'assurer qu'il est bien applicable en pratique : existe-t-il des domaines non dégénérés ? Dans le cas où $d = 1$ et où il n'y a qu'une seule singularité, on montre en effet que « la plupart » des domaines sont non dégénérés, et on obtient le

Théorème. *Supposons $d = 1$.*

(i) Si Ω est « assez proche » d'un disque, alors E_ε admet des points critiques à degré prescrit 1 pour ε petit.

(ii) Tout domaine simplement connexe Ω peut être « approché » par des domaines non dégénérés, dans lesquels E_ε admet donc des points critiques à degré prescrit 1 pour ε petit.

Ici la notion de domaines « proches » s'exprime à travers la proximité de représentations conformes du disque dans ces domaines simplement connexes.

1.2.2 Fines coques superconductrices (avec A. Contreras)

Lorsque le matériau supraconducteur est une fine coque fermée, on l'assimile à une surface compacte $\mathcal{M} \subset \mathbb{R}^3$ homéomorphe à une sphère. Etant donné un champ de vecteurs $\mathbf{A}$ sur $\mathcal{M}$, on considère la fonctionnelle de Ginzburg-Landau

$$G_\varepsilon(u) = \int_{\mathcal{M}} \left(|(\nabla_{\mathcal{M}} - ih\mathbf{A})u|^2 + \frac{1}{2\varepsilon^2}(|u|^2 - 1)^2 \right) d\mathcal{H}^2_{\mathcal{M}}.$$

Le paramètre $h > 0$ mesure l'intensité du potentiel magnétique $h\mathbf{A}$, induit par un champ externe appliqué $h\mathbf{H}_{ex}$ dans $\mathbb{R}^3$. Cette réduction dimensionnelle est justifiée rigoureusement par Contreras et Sternberg [41] qui obtiennent la fonctionnelle G_ε comme Γ-limite de l'énergie complète GL^{3d}_κ. Contrairement au cas de la réduction dimensionnelle cylindrique GL^{2d}_ε, ici le potentiel magnétique est fixé. D'autre part, le champ magnétique induit $H = \operatorname{curl} \mathbf{A} \in C^1(\mathcal{M}; \mathbb{R})$ n'est *a priori* pas constant, ni même de signe constant, ce qui modifie et complique l'analyse des vortex.

Dans [41, 40], la valeur asymptotique de $h = h(\varepsilon)$ à laquelle les vortex commencent à apparaître – le « premier champ critique » – est identifiée : $h_{c_1} \sim \lambda_c |\ln \varepsilon|$, où λ_c dépend explicitement du champ magnétique H. La topologie de $\mathcal{M}$ implique que le degré global est nul : les vortex apparaissent par paires, un vortex de degré positif contre-balançant un vortex de degré négatif.

Dans le chapitre 7 on étudie le régime $h \sim \lambda |\ln \varepsilon|$ pour $\lambda > \lambda_c$. Dans le cas cylindrique étudié par Sandier et Serfaty [119], ce régime voit grandir une région où les vortex sont uniformément répartis, jusqu'à ce que la supraconductivité ne persiste plus que dans une fine couche près du bord pour $\lambda \gg 1$. Dans notre cas, la surface $\mathcal{M}$ n'ayant pas de bord, l'analogue de ce comportement n'est *a priori* pas évident. Une autre différence cruciale est liée à la forme du champ magnétique H : dans notre cas il peut s'annuler, et changer de signe. L'analyse spectrale dans [102] suggère que la région où la supraconductivité persiste soit concentrée près de l'ensemble $\{H = 0\}$. Dans le cas du modèle bidimensionnel cylindrique et d'une intensité h bien supérieure au régime considéré ici, si le champ magnétique externe peut varier et s'annuler, alors la supraconductivité est en effet concentrée près du lieu d'annulation du champ magnétique [110].

L'analyse du chapitre 7 repose sur la réduction à un modèle de champ moyen, dont la démonstration rigoureuse dans [118] peut s'adapter au cas considéré ici. La répartition des vortex pour $\varepsilon \ll 1$ est alors décrite par une mesure de vorticité μ, liée à un problème d'obstacle bilatéral. Dans [118] le problème d'obstacle est unilatéral car le champ magnétique ne change pas de signe, mais ici les deux obstacles doivent être pris en compte.

Ce problème d'obstacle, et donc la mesure μ, dépendent du paramètre

$$\beta := \frac{1}{\lambda} = \lim_{\varepsilon \to 0} \frac{|\ln \varepsilon|}{h}.$$

La supraconductivité persiste dans la région SC_β où il n'y a pas de vortex, c'est à dire en dehors du support de μ. On démontre que pour $\beta \to 0$, cette région correspond génériquement à une voisinage de taille $\beta^{1/3}$ du lieu où H s'annule :

Théorème. *Si H est non dégénéré, au sens où ∇H ne s'annule pas sur $\{H = 0\}$, alors il existe $C > 0$ indépendant de β, tel que la région supraconductrice*

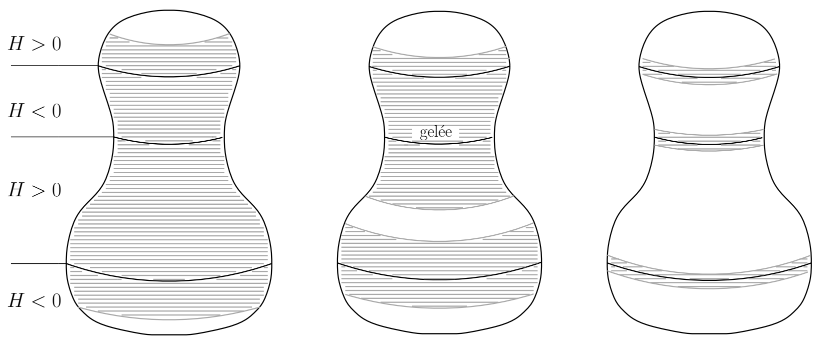
SC_β vérifie

$$\left\{\mathrm{dist}_{\{H=0\}} \leq C^{-1}\beta^{1/3}\right\} \subset SC_\beta \subset \left\{\mathrm{dist}_{\{H=0\}} \leq C\beta^{1/3}\right\}.$$

L'hypothèse générique sur H implique que l'ensemble $\{H=0\}$ est une union finie de courbe régulières disjointes. La démonstration du théorème repose sur la construction de fonctions de comparaison pour le problème d'obstacle bilatéral. Contrairement au problème d'obstacle unilatéral [120, Annexe A], on ne peut se contenter d'obtenir des sur- et sous-solutions de l'équation elliptique correspondante. On utilise à la place un principe de comparaison pour les problèmes d'obstacles bilatéraux [44], et les fonctions de comparaison sont des solutions de problèmes d'obstacles modifiés.

On a ainsi une image relativement claire de la région supraconductrice lorsque $\beta \sim 0$, et en particulier SC_β et $\{H=0\}$ ont le même nombre de composantes connexes. D'autre part lorsque $\beta \sim 1/\lambda_c$ et les vortex commencent à apparaître, SC_β a génériquement une composante connexe. Si $\{H=0\}$ possède strictement plus d'une composante connexe, il existe donc nécessairement des régimes intermédiaires marqués par des transitions dans la nature topologique de SC_β.

L'étude de ces régimes intermédiaires semblant hors de notre portée en toute généralité, on se concentre sur un problème « modèle » à symétrie cylindrique : la surface $\mathcal{M}$ est une surface de révolution, et $\{H=0\}$ est une union de cercles (en nombre nécessairement impair). On fait de plus l'hypothèse qu'il y a exactement trois cercles, premier cas non trivial où l'on observe des transitions intéressantes. (Cette hypothèse simplifie les notations mais ne change pas le fond du problème.) On obtient alors l'existence de trois régimes séparés par des valeurs critiques $1/\lambda_c > \beta_1^* > \beta_2^* > 0$. Suivant la position de β, la région supraconductrice SC_β a une, deux ou bien trois composantes connexes. Ce résultat (cf. Proposition 7.15) est illustré dans la figure 1.5 ci-dessous.



(a) $1/\lambda_c > \beta > \beta_1^*$ (b) $\beta_1^* > \beta > \beta_2^*$ (c) $\beta_2^* > \beta > 0$

FIGURE 1.5 – La région SC_β dans les trois régimes intermédiaires

Dans le régime où SC_β a deux composantes connexes, on observe un intéressant phénomène de « gel » de la frontière libre : une des deux composantes ne dépend pas de β. Ce phénomène n'est pas particulier au cas symétrique, et on identifie ensuite les propriétés qui conduisent au gel de la frontière libre dans un cadre général (cf. Proposition 7.17).

1.2.3 Supraconductivité à symétrie 'p' : structure d'un vortex (avec S. Alama et L. Bronsard)

Certains matériaux supraconducteurs « non conventionnels » ne peuvent être décrits par la fonctionnelle classique GL_κ^{3d}. Ce phénomène est lié aux propriétés de symétrie de l'appariement des électrons supraconducteurs. Un exemple de supraconductivité non conventionnelle est donnée par Sr_2RuO_4, dont l'état supraconducteur à symétrie 'p' (*'p-wave'*) est décrit par une fonction d'onde $\eta=(\eta_+,\eta_-)$ à deux composantes complexes [65]. La symétrie 'p' se traduit par une fonctionnelle d'énergie fortement anisotropique, couplant les gradients des deux composantes. Négligeons dans un premier temps le champ magnétique et certains termes très asymétriques. En notant $\Pi_\pm=\partial_x\pm i\partial_y$, on considère alors l'énergie simplifiée

$$\begin{aligned} E(\eta) &= \int \left[e_{kin}(\eta)+\kappa^2 e_{pot}(\eta)\right]\,dx, \\ \text{où}\quad e_{kin}(\eta) &= |\nabla\eta_+|^2+|\nabla\eta_-|^2+(\Pi_-\eta_+)\cdot(\Pi_+\eta_-), \\ \text{et}\quad e_{pot}(\eta) &= \frac{1}{2}(|\eta_+|^2-1)^2+\frac{1}{2}(|\eta_-|^2-1)^2+2|\eta_+|^2|\eta_-|^2. \end{aligned}$$

Les équations d'Euler-Lagrange correspondantes sont données par

$$\begin{cases} 2\Delta\eta_- + \Pi_-^2\eta_+ = \kappa^2\left(2\eta_-(|\eta_-|^2-1)+4\eta_-\,|\eta_+|^2\right), \\ 2\Delta\eta_+ + \Pi_+^2\eta_- = \kappa^2\left(2\eta_+(|\eta_+|^2-1)+4\eta_+\,|\eta_-|^2\right). \end{cases}$$

Le terme « cinétique » $e_{kin}(\eta)$ est positif, mais pas coercif : première indication des difficultés de ce modèle. Le terme « potentiel » $e_{pot}(\eta)$ est minimisé exactement lorsque l'une des deux composantes est nulle, et l'autre de module un. Ainsi, loin des vortex on s'attend à observer une composante « dominante » et une composante « secondaire », par exemple $|\eta_-|\approx 1$ et $|\eta_+|\approx 0$.

Dans le chapitre 8 on étudie la structure des vortex isolés dans ce modèle de Ginzburg-Landau à symétrie 'p'. On montre tout d'abord que le manque de coercivité de la fonctionnelle d'énergie peut être compensé par certaines conditions de Dirichlet (cf. Théorème 8.1). En particulier on en déduit l'existence de solutions aux équations d'Euler-Lagrange dans un disque, avec conditions de Dirichlet $\eta_\pm=\alpha_\pm e^{in_\pm\theta}$ si $\alpha_+\neq\alpha_-$. Ceci permet ensuite de démontrer l'existence de solutions entières « équivariantes » :

Théorème. *Les équations d'Euler-Lagrange admettent une solution entière de la forme*

$$\eta_\pm = f_\pm(r)e^{\pm i\theta},\quad \textit{avec}\ \lim_{+\infty} f_- = 1,\ \lim_{+\infty} f_+ = 0.$$

De plus on a les développements asymptotiques

$$f_- = 1-\frac{1}{2r^2}-\frac{7}{4r^4}+O(r^{-6}),\qquad f_+ = -\frac{1}{2r^2}-\frac{13}{4r^4}+O(r^{-6}),$$

lorsque $r\to+\infty$.

On s'attend naturellement à ce que les fonctions $f_\pm(r)$ aient un signe fixé. Au vu de leurs développements asymptotiques, on conjecture que $f_-\geq 0$ et

$f_+ \leq 0$. Toutefois le couplage des dérivées dans l'énergie cinétique (qui se traduit dans les équations d'Euler-Lagrange par un couplage des termes d'ordre deux) empêche d'invoquer les arguments classiques qui permettraient de démontrer cette conjecture.

Pour obtenir un résultat dans cette direction, on s'intéresse à un système perturbé, dans lequel un petit coefficient $t \approx 0$ multiplie les termes de couplage d'ordre deux :

Théorème. *Il existe $t_0 > 0$ tel que, pour tout $t \in (0, t_0)$, le système perturbé*

$$\begin{cases} 2\Delta\eta_- + t\,\Pi_-^2\eta_+ = \kappa^2\left(2\eta_-(|\eta_-|^2 - 1) + 4\eta_-\,|\eta_+|^2\right), \\ 2\Delta\eta_+ + t\,\Pi_+^2\eta_- = \kappa^2\left(2\eta_+(|\eta_+|^2 - 1) + 4\eta_+\,|\eta_-|^2\right). \end{cases}$$

admette une solution entière $\eta_\pm^t = f_\pm^t(r)e^{\pm i\theta}$ avec $f_-^t(\infty) = 1$ et $f_+^t(\infty) = 0$, et vérifiant de plus

$$0 < f_-(r) < 1, \qquad f_+(r) < 0,$$

pour tout $r > 0$.

Ces solutions sont obtenues comme perturbations d'une solution f^0 telle que $f_+^0 = 0$ et f_-^0 est le profil radial du vortex de degré 1 dans le modèle classique de Ginzburg-Landau. La structure du système linéarisé autour de cette solution permet d'obtenir les bornes $0 < f_-(r) < 1$, et $f_+(r) < 0$, pour tout t assez petit. Pour obtenir des estimations uniformes en r, on allie l'étude du système linéarisé à un développement asymptotique *a priori* des solutions du système perturbé.

1.3 Espaces de Besov et noyaux de convolution peu réguliers (avec P. Mironescu)

La régularité d'une fonction $f\colon \mathbb{R}^n \to \mathbb{R}$ peut être caractérisée à travers la vitesse de convergence de la convolution $f * \rho_\varepsilon$ vers f, où $\rho_\varepsilon = \varepsilon^{-n}\rho(\cdot/\varepsilon)$ pour un noyau de convolution ρ assez régulier vérifiant $\int \rho = 1$. Par exemple, si le noyau ρ appartient à la classe de Schwartz $\mathcal{S}$ des fonctions lisses à décroissance rapide, on a alors, pour tout $s > 0$ non entier et tout $p \in [1, \infty)$, l'équivalence des normes

$$\|f\|_{W^{s,p}}^p \sim \|f\|_{L^p}^p + \int_0^1 \|f - f * \rho_\varepsilon\|_{L^p}^p \,\frac{d\varepsilon}{\varepsilon^{sp+1}},$$

sur l'espace de Sobolev fractionnaire $W^{s,p}(\mathbb{R}^n)$.

Quelles hypothèses minimales sur ρ assurent la validité d'une telle équivalence ? C'est la question qu'on se pose dans le chapitre 9. On utilise pour y répondre des techniques relativement élémentaires reposant sur la décomposition de Littlewood-Paley. Ces techniques sont naturellement adaptées à une classe plus générale d'espaces fonctionnels, les espaces de Besov $B_{p,q}^s$. Rappelons qu'une décomposition de Littlewood-Paley (inhomogène) est de la forme $f = \sum_{j\geq 0} f_j$, où les fonctions lisses f_j « isolent » certains modes de Fourier : la transformée de Fourier de f_{j+1} est à support dans un anneau $\{c2^j \leq |\xi| \leq C2^{j+1}\}$. Une fonction $f \in L^p$ est de classe $B_{p,q}^s$ si et seulement si $(2^{sj}\,\|f_j\|_{L^p})_{j\geq 0} \in \ell^q$.

De manière surprenante, on montre qu'un taux de convergence adapté de $f * \rho_\varepsilon$ implique bien la régularité $B^s_{p,q}$, sans aucune hypothèse de régularité sur le noyau ρ :

Théorème. *Soit $\rho \in L^1$ tel que $\int \rho = 1$. Pour tous $s > 0$, $1 \leq p, q \leq \infty$, on a*

$$\|f\|^q_{B^s_{p,q}} \lesssim \|f\|^q_{L^p} + \int_0^1 \|f - f * \rho_\varepsilon\|^q_{L^p} \frac{d\varepsilon}{\varepsilon^{sq+1}}.$$

Cette estimation repose sur le fait que les modes « assez hauts » peuvent être estimés en terme de $(f - f * \rho_\varepsilon)$, tandis que les modes bas sont simplement dominés par la norme L^p.

L'estimation réciproque requiert bien, elle, une hypothèse sur le noyau. Une condition nécessaire évidente est que pour tout $\eta \in \mathcal{S}$ on ait

$$\int_0^1 \|\eta - \eta * \rho_\varepsilon\|^q_{L^p} \frac{d\varepsilon}{\varepsilon^{sq+1}} < \infty.$$

On montre en fait qu'il suffit qu'une fonction η fixée vérifie cette condition avec $p = q = 1$ pour que ρ soit un « bon noyau » :

Théorème. *Soient $\rho \in L^1$ tel que $\int \rho = 1$, et $s > 0$. Les deux propositions suivantes sont équivalentes :*

(i) Il existe $\eta \in \mathcal{S}$, $\int \eta \neq 0$, vérifiant

$$\int_0^1 \|\eta - \eta * \rho_\varepsilon\|_{L^1} \frac{d\varepsilon}{\varepsilon^{s+1}} < \infty.$$

(ii) Pour tous $1 \leq p, q \leq \infty$,

$$\|f\|^q_{B^s_{p,q}} \sim \|f\|^q_{L^p} + \int_0^1 \|f - f * \rho_\varepsilon\|^q_{L^p} \frac{d\varepsilon}{\varepsilon^{sq+1}} \tag{1.7}$$

Pour montrer que (i) implique bien (ii) on commence par élargir la validité de (i) à toute fonction $\eta \in \mathcal{S}$, grâce à des techniques classiques d'analyse harmonique. On peut alors choisir une bonne fonction η qui permette d'estimer les modes bas de $(f - f * \rho_\varepsilon)$ en utilisant (i). On conclut ensuite par une estimation générale sur la norme d'opérateurs à noyaux.

On attache un intérêt particulier aux cas où $f * \rho_\varepsilon$ est particulièrement simple à calculer, par exemple lorsque ρ est une fonction caractéristique. La condition (i) permet alors d'expliciter le degré de régularité qu'un tel noyau pourra caractériser :

Proposition. *Soient $s > 0$, $A \subset \mathbb{R}^n$ un ensemble borné de mesure finie, et $\rho = |A|^{-1} \mathbb{1}_A$. Le noyau ρ permet de caractériser les espaces $B^s_{p,q}$ – i.e. ρ vérifie* (1.7) *– si et seulement si :*

1. *l'ensemble A est centré, i.e. $\int_A y \, dy = 0$, et $s < 2$,*
2. *ou l'ensemble A n'est pas centré et $s < 1$.*

D'autre part, dans le cas d'un noyau ρ général, il peut-être intéressant d'avoir un critère plus explicite que la condition (i). Pour $0 < s < 1$ on obtient une condition suffisante simple :

Proposition. *Soient $\rho \in L^1$ tel que $\int \rho = 1$, et $0 < s < 1$. Si ρ vérifie*

$$\int |y|^s \, |\rho(y)| \, dy < \infty,$$

alors ρ vérifie (1.7).

Cette condition suffisante est optimale en un certain sens : pour un noyau positif $\rho \geq 0$ elle est aussi nécessaire (cf. Proposition 9.6). Pour caractériser des degrés de régularité supérieurs $s \geq 1$ il n'existe pas condition suffisante aussi simple, mais on peut requérir que le noyau ait ses premiers moments nuls (cf. Proposition 9.10).

1.4 Perspectives

Les problèmes étudiés dans cette thèse soulèvent un certain nombre de questions et conjectures naturelles. En voici quelques unes.

Solutions purement uniaxes

La rigidité des configurations nématiques purement uniaxes établie dans le chapitre 2 suggère que dans le cas général tridimensionnel, les seules solutions purement uniaxes à directeur non constant soient à symétrie radiale, et ce sans avoir à préciser de conditions au bord.

Fuite biaxe

Dans le chapitre 3 on étudie le phénomène de fuite biaxe dans un certain régime asymptotique $a \to +\infty$. Qu'en est-il dans un régime plus général ? La stabilité locale du hérisson radial pour $a \approx 0$ [73] n'apporte *a priori* aucune information sur les minimiseurs globaux. Les calculs numériques dans [100] semblent même indiquer qu'une configuration fortement biaxe est énergétiquement favorable pour tout $a \geq 0$.

Une autre question liée au comportement des minimiseurs près d'un défaut est celle de la symétrie des solutions entières : on sait qu'elles ne sont pas nécessairement radiales, mais il serait intéressant de démontrer qu'elles sont toutes à symétrie axiale.

Suspensions colloïdales

L'étude de l'ordre nématique autour d'une particule colloïdale dans le chapitre 4 laisse ouvert le cas de l'ancrage faible pour une très grande particule. Il est clair que pour une force d'ancrage nulle $w = 0$ l'alignement sera uniforme, mais peut-on décrire rigoureusement la transition entre configurations dipolaire et uniforme ? Une première étape dans cette direction pourrait consister à établir un lien entre la force d'ancrage w et des estimations de régularité au bord. Toujours dans le régime de très grande particule, il serait intéressant de justifier (ou d'infirmer) la contrainte de symétrie axiale.

Enfin ce problème ouvre la voie à l'étude de suspensions colloïdales plus complexes composées de plusieurs particules. On pourrait par exemple s'attaquer à une description des interactions induites entre particules par la structure nématique, dans le cas d'un grand nombre de très petites particules.

Cellule nématique frustrée : symétrie des solutions

Le système elliptique étudié dans le chapitre 5 possède une certaine symétrie par rapport au centre de la cellule. Cette symétrie soulève la question générale suivante. Soit deux fonctions $u, v\colon [-1,1] \to \mathbb{R}$ minimisant une fonctionnelle

$$\int_{-1}^{1} \left[(u')^2 + (v')^2 + f(u,v)\right] dx,$$

avec conditions au bord $u(\pm 1) = 0$, $v(\pm 1) = \pm 1$. Si l'on suppose que la fonction f est paire par rapport à sa deuxième variable, et que $(0, \pm 1)$ sont des points critiques de f, a-t-on nécessairement : la fonction u est paire et la fonction v est impaire ?

Conditions semi-rigides

Dans le chapitre 6 on construit, pour une équation de Ginzburg-Landau, des solutions avec conditions au bord semi-rigides. Dans le cas d'un seul vortex, les domaines simplement connexes « non dégénérés » dans lesquels notre construction s'applique sont génériques. La question de l'existence (et de la généricité) de domaines non dégénérés dans le cas de plusieurs vortex reste ouverte.

D'autre part, les techniques développées dans le chapitre 6 pourraient servir à construire dans des domaines multiplement connexes des solutions dont les vortex ne s'échappent pas vers le bord, contrairement aux minimiseurs locaux construit dans [21, 49].

Fines coques supraconductrices

Dans le chapitre 7 on décrit précisément, dans le cas d'une fine coque supraconductrice correspondant à une surface de révolution, différents régimes par lesquels passe la persistance de la supraconductivité au-delà du premier champ critique. Une description similaire est-elle possible pour des géométries plus générales ? Il s'agirait dans un premier temps de comprendre quelles quantités permettent d'identifier les valeurs critiques (du champ magnétique) séparant les différents régimes.

Supraconductivité à symétrie 'p'

Dans le chapitre 8 on s'intéresse, pour un modèle de type Ginzburg-Landau à deux composantes complexes, aux solutions entières de la forme $\eta_\pm = f_\pm(r)e^{in_\pm\theta}$. On montre leur existence dans le cas $n_\pm = \pm 1$, mais pour des exposants différents cette question reste ouverte : la preuve repose sur une borne uniforme *a priori* que nous n'avons pas su établir pour $n \neq \pm 1$.

D'autre part on s'attend à ce que les fonctions $f_\pm(r)$ aient un signe constant : on établit un résultat dans ce sens pour un système modifié, mais la question demeure pour le système original.

Espaces de Besov caractérisés par convolution

Dans le chapitre 9 on démontre, à $s > 0$ fixé, un critère pour qu'un noyau de convolution ρ permette de caractériser la régularité $B^s_{p,q}$ pour tous p et q. Reste la question plus fine de déterminer, à s, p et q fixés, quels noyaux caractérisent $B^s_{p,q}$.

Première partie

Cristaux liquides nématiques

Chapitre 2

Symétrie uniaxe

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2.1 Introduction

Nematic liquid crystals are composed of rigid rod-like molecules which tend to align in a common preferred direction. For a macroscopic description of such orientational ordering, several continuum theories are available, relying on different order parameters.

The state of alignment can be simply characterized by a director field n with values in the unit sphere $\mathbb{S}^2$, corresponding to the local preferred direction of orientation. Within such a description, topological constraints may force the appearance of defects: regions where the director field is not continuous. To obtain a finer understanding of such regions, one needs to introduce a scalar order parameter s, corresponding to the degree of alignment along the director n. However, the (s, n) description only accounts for uniaxial nematics, which correspond to a symmetric case of the more general biaxial nematic phase. To describe biaxial regions, a tensorial order parameter Q is needed. Biaxiality has been used to theoretically describe defect cores [100, 78, 111, 130, 47, 46] and material frustration [109, 26, 12], and has been observed experimentally [93, 1].

The n and (s, n) descriptions can both be interpreted within the Q-tensor description. The tensorial order parameter Q describes different degrees of symmetry: isotropic, uniaxial or biaxial. The isotropic case $Q = 0$ corresponds to the full symmetry group $G = SO(3)$. The uniaxial case corresponds to a broken symmetry group $H \approx O(2)$. And the biaxial case corresponds to a further broken symmetry group with 4 elements. The (s, n) description amounts to restricting the order parameter space to uniaxial or isotropic Q-tensor: only Q-tensors which are 'at least $O(2)$-symmetric' are considered. The n description arises in the London limit, since the space of degeneracy is $G/H \approx \mathbb{S}^2/\{\pm 1\}$ (see for instance [137, Section 2] for more details).

In physical systems presenting some symmetry, existence of symmetric equilibrium configurations is a common phenomenon: such configurations can be obtained by looking for a solution with a special symmetrical *ansatz*. In some cases this phenomenon can be formalized mathematically as a Principle of Symmetric Criticality [108]. In the present paper we investigate whether the same principle applies to uniaxial symmetry in nematic liquid crystals: do there exist uniaxial Q-tensor equilibrium configurations? or is the uniaxial symmetry always broken?

We consider a Landau-de Gennes free energy. We do not work with the usual four-terms expansion of the bulk free energy but with a general frame invariant bulk free energy.

We start by considering the case of one- or two-dimensional configurations: that is, configurations exhibiting translational invariance in at least one direction of space [109, 26, 12, 78, 130]. In Theorem 2.9 we describe completely the one- or two-dimensional uniaxial equilibrium configurations: these are essentially only the configurations with constant director field n. In particular, *even if the boundary conditions enhance uniaxial symmetry, the uniaxial order is destroyed in the whole system*, unless the director field is uniform.

The three dimensional case is more complex. While in one and two dimensions the uniaxial configurations are essentially trivial, there does exist a non trivial uniaxial configuration in three dimensions: namely, the so-called radial hedgehog [100, 124], which corresponds to a spherically symmetric configuration in a sperical droplet of nematic, with strong radial anchoring on the surface. In Theorem 2.12 we show that any uniaxial equilibrium configuration must be spherically symmetric, in this particular nematic system. Such a result constitutes a first step towards a complete characterization of three-dimensional uniaxial equilibrium configurations. We expect the radial hedgehog to be the only non trivial uniaxial equilibrium.

Our main results, Theorem 2.9 and Theorem 2.12, bring out the idea that the constraint of uniaxial symmetry is very restrictive and is in general not satisfied, except in very symmetric situations. These results shed a very new light on the phenomenon of 'biaxial escape' [130], and are *fundamentally different* from the previous related ones in the literature. Indeed, biaxiality was always shown to occur by means of free energy comparison methods, while we *only rely on the equilibrium equations*. In particular our results hold for all metastable configurations. Moreover, the appearance of biaxiality was usually related to special values of parameters such as the temperature [100] – which affects the bulk equilibrium –, or the size of the system [26] – which affects the director deformation. We show instead that biaxiality occurs for *any value of the temperature* (since the bulk energy density we work with is arbitrary) and *any kind of director deformation.* In short: escape to biaxiality appears in all possible situations, and the equilibrium equations themselves force this escape.

The plan of the paper is the following. In Section 2.2 we introduce the mathematical model describing orientational order. In Section 2.3 we derive the equilibrium equations for a configuration with uniaxial symmetry, and discuss the appearance of an extra equation corresponding to equilibrium with respect to symmetry-breaking perturbations. Sections 2.4 and 2.5 contain the main results of the paper: in Section 2.4 we deal with one- and two-dimensional configurations and prove Theorem 2.9, and in Section 2.5 we focus on a spherical nematic droplet with radial anchoring and prove Theorem 2.12.

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2.2 Description of the model

2.2.1 Order parameter and degrees of symmetry

In a nematic liquid crystal, the local state of alignment is described by an order parameter taking values in

$$\mathcal{S} = \left\{ Q \in M_3(\mathbb{R});\ Q = {}^t Q,\ \operatorname{tr} Q = 0 \right\}, \tag{2.1}$$

the set of all symmetric traceless 3×3 matrices. The space $\mathcal{S}$ is naturally endowed with the euclidean structure induced by the usual scalar product on $M_3(\mathbb{R})$:

$$\langle A, A' \rangle = \operatorname{tr}({}^t A A') = \sum_{ij} a_{ij} a'_{ij} \qquad \forall A, A' \in M_3(\mathbb{R}).$$

The group $G = SO(3)$ acts on the order parameter space $\mathcal{S}$: we denote by $\operatorname{Isom}(\mathcal{S})$ the group of linear isometries of $\mathcal{S}$, and the action is given by the group morphism

$$\rho\colon G \to \operatorname{Isom}(\mathcal{S}), \quad \rho(g)Q = gQ{}^t g.$$

Note that this action ρ is related to the natural action of G on $\mathbb{R}^3$: $\rho(g)Q$ is the order parameter one should observe after changing the coordinate frame by g in $\mathbb{R}^3$.

In the order parameter space $\mathcal{S}$ we may distinguish three types of elements, depending on their degree of symmetry. The degree of symmetry of an element $Q \in \mathcal{S}$ is given by its isotropy subgroup

$$H(Q) := \{ g \in G,\ \rho(g)Q = Q \},$$

which can be of three different kinds:

— If $Q = 0$, then $H(Q) = G$, and Q describes the *isotropic phase*.

— If Q has two equal (non zero) eigenvalues, then

$$Q = \lambda \left(n \otimes n - \frac{1}{3}\mathbf{I} \right), \quad \lambda \in \mathbb{R}^*,\ n \in \mathbb{S}^2,$$

and thus $Q = \lambda\rho(g)A_0$, where $A_0 = \mathbf{e_z} \otimes \mathbf{e_z} - \mathbf{I}/3$ and $g \in G$ maps $\mathbf{e_z}$ to n. Therefore $H(Q)$ is conjugate via g to

$$D_\infty := H(A_0) = \left\langle \{ r_{\mathbf{e_z},\theta} \}_{\theta \in \mathbb{R}}, r_{\mathbf{e}_y,\pi} \right\rangle \approx O(2),$$

where $r_{n,\theta}$ stands for the element of G corresponding to the rotation of axis n and angle θ. In this case, Q describes the *uniaxial phase*.

— If Q has three distinct eigenvalues, and $g \in G$ maps the canonical orthonormal basis $(\mathbf{e_x}, \mathbf{e_y}, \mathbf{e_z})$ to an orthonormal basis of eigenvectors of Q, then $H(Q)$ is conjugate via g to

$$D_2 = \langle r_{\mathbf{e_x},\pi}, r_{\mathbf{e_y},\pi} \rangle \approx \mathbb{Z}/2\mathbb{Z} \times \mathbb{Z}/2\mathbb{Z}.$$

In this case, Q describes the *biaxial phase*.

Hence there is a hierarchy in the breaking of symmetry that Q can describe:

$$\{0\} \subset \mathcal{U} \subset \mathcal{S},$$

where

$$\mathcal{U} = \left\{ s\left(n \otimes n - \frac{1}{3}\mathbf{I}\right) ;\ s \in \mathbb{R},\ n \in \mathbb{S}^2 \right\}, \tag{2.2}$$

is the set of order parameter which can describe a breaking of symmetry from G to D_∞. Elements of $\mathcal{U}$ are characterized by their *director* $n \in \mathbb{S}^2$ and their *scalar order parameter* $s \in \mathbb{R}$.

Remark 2.1. Note that the scalar order parameter s of a uniaxial tensor $Q \in \mathcal{U}$ is uniquely determined since $s = 0$ if $Q = 0$, and

$$s = 3\frac{\operatorname{tr}(Q^3)}{|Q|^2}$$

otherwise. On the other hand, the director is uniquely determined up to a sign if $Q \neq 0$, and not determined at all if $Q = 0$.

2.2.2 Equilibrium configurations

We consider a nematic liquid crystal contained in an open set $\Omega \subset \mathbb{R}^3$. The state of alignment of the material is described by a map

$$Q \colon \Omega \to \mathcal{S}.$$

At equilibrium, the configuration should minimize a free energy functional of the form

$$\mathcal{F}(Q) = \int_\Omega (f_{el} + f_b)\, dx,$$

where f_{el} is an elastic energy density, and f_b is the bulk free energy.

Here we consider the one constant approximation for the elastic term:

$$f_{el} = \frac{L}{2}|\nabla Q|^2,$$

and the most general frame invariant (i.e. invariant under the action ρ) bulk term:

$$f_b = \varphi(\operatorname{tr}(Q^2), \operatorname{tr}(Q^3)),$$

for some function

$$\varphi \colon \mathbb{R} \times \mathbb{R} \to \mathbb{R}_+,$$

which we assume to be smooth.

Remark 2.2. A fundamental property of the free energy density $f(Q) = f_{el} + f_b$ is its *frame invariance*: for any $Q \in H^1_{loc}(\mathbb{R}^3; \mathcal{S})$ it holds

$$f(g \cdot Q)(x) = f(Q)(g^{-1}x) \qquad \forall g \in G,$$

where $g \cdot Q$ denotes the natural action of G on maps Q, given by

$$(g \cdot Q)(x) = \rho(g)Q(g^{-1}x) = gQ(g^{-1}x)g^{-1}. \tag{2.3}$$

More general elastic terms f_{el} are physically relevant, as long as the frame invariance property is conserved.

An equilibrium configuration is described by a map $Q \in H^1(\Omega; \mathcal{S})$ satisfying the Euler-Lagrange equation

$$L\Delta Q = 2(\partial_1\varphi)Q + 3(\partial_2\varphi)\left(Q^2 - \frac{|Q|^2}{3}I\right), \tag{2.4}$$

associated to the free energy $\mathcal{F}$.

Classical elliptic regularity arguments ensure that any solution of (2.4) which lies in $H^1 \cap L^\infty$ is smooth. In fact, if in addition φ is analytic, any $H^1 \cap L^\infty$ solution of (2.4) is actually analytic [103, Theorem 6.7.6].

In the sequel we will always consider smooth solutions. We discuss next a very mild sufficient condition on φ which ensures boundedness – and therefore smoothness – of solutions.

In a bounded regular domain Ω, a natural assumption on φ which ensures that any H^1 solution of (2.4) with bounded boundary data is in fact bounded is the following one:

$$\exists M > 0 \text{ such that } \left(|Q| \geq M \Longrightarrow 2|Q|^2(\partial_1\varphi) + 3(\partial_2\varphi)\mathrm{tr}(Q^3) \geq 0\right). \tag{2.5}$$

See [82, Lemma B.3] for a proof that assumption (2.5) on φ implies indeed that any $Q \in H^1$ solution of (2.4) satisfies

$$\|Q\|_{L^\infty(\Omega)} \leq \max(M, \|Q\|_{L^\infty(\partial\Omega)}).$$

The fourth order approximation for f_b usually considered in the literature

$$f_b(Q) = -a\mathrm{tr}(Q^2) - b\mathrm{tr}(Q^3) + c\mathrm{tr}(Q^2)^2, \tag{2.6}$$

corresponds to

$$\varphi(x, y) = -ax - by + cx^2,$$

which satisfies indeed (2.5), as long as $c > 0$ (and is obviously analytic).

2.3 Uniaxial equilibrium

In the sequel, we investigate the existence of purely uniaxial equilibrium configurations, i.e. solutions Q of the equilibrium equations (2.4), which satisfy

$$Q(x) \in \mathcal{U} \qquad \forall x \in \Omega.$$

In other words, a purely uniaxial equilibrium configuration is a solution of (2.4) which can be written in the form

$$Q(x) = s(x)\left(n(x)\otimes n(x) - \frac{1}{3}\mathbf{I}\right), \tag{2.7}$$

for some scalar field $s\colon \Omega \to \mathbb{R}$ and unit vector field $n\colon \Omega \to \mathbb{S}^2$.

Remark 2.3. Here we do not require *a priori* that the scalar field s and the unit vector field n in *ansatz* (2.7) be smooth. Note that s is uniquely determined (see Remark 2.1) by

$$s(x) = 3\frac{\mathrm{tr}(Q(x)^3)}{|Q(x)|^2}.$$

Therefore if Q is smooth, then s is smooth in the set $\{Q \neq 0\} \subset \Omega$ of points where Q does not vanish, and continuous in Ω. On the other hand, n is not uniquely determined (see Remark 2.1). However, in $\{Q \neq 0\}$ one can choose locally a smooth unit vector field n. More precisely, if Q is smooth and $x_0 \in \Omega$ is such that $Q(x_0) \neq 0$, then there exists an open ball $B \subset \Omega$ centered at x_0, and a smooth map $n\colon B \to \mathbb{S}^2$ such that (2.7) holds. The local smooth n is obtained through the implicit function theorem (see the proof of Theorem 2.9 below for more details).

Remark 2.4. Uniaxiality can be characterized through

$$Q \in \mathcal{U} \iff |Q|^6 = 6\left[\mathrm{tr}(Q^3)\right]^2,$$

so that any analytic map $Q\colon \Omega \to \mathcal{S}$ which is uniaxial in some open subset of Ω is automatically uniaxial everywhere [95]. Thus, for analytic φ, Theorems 2.9 and 2.12 proved below are valid if we replace the assumption that Q be purely uniaxial, with the assumption that Q be uniaxial in some open set.

Remark 2.5. The spherically symmetric radial hedgehog [100] provides an example of purely uniaxial equilibrium (see also Section 2.5 below). However, in the particular case of the radial hedgehog, uniaxial symmetry is a consequence of spherical symmetry, for which Palais' Principle of Symmetric Criticality applies [108]. The Principle of Symmetric Criticality is a general tool which allows to prove existence of symmetric equilibria. Roughly speaking, if the free energy and the space of admissible configurations are 'symmetric', then the Principle asserts the following: any symmetric configuration which is an equilibrium with respect to symmetry-preserving perturbations is *automatically* an equilibrium with respect to symmetry-breaking perturbations also. Of course the meaning of 'symmetric' needs to be precised: see [108] for a rigorous mathematical framework in which this Principle is valid.

However, in general the Principle of Symmetric Criticality does not apply to uniaxial symmetry, as is suggested by the following result (see Remark 2.7 below).

Proposition 2.6. *Let $\omega \subset \mathbb{R}^3$ be an open set. Let $s\colon \omega \to \mathbb{R}$ and $n\colon \omega \to \mathbb{S}^2$ be smooth maps such that the corresponding uniaxial Q (2.7) satisfies the equilibrium equation (2.4). Then s and n satisfy*

$$\begin{cases} \Delta s = 3|\nabla n|^2 s + \dfrac{1}{L}(2s\partial_1\varphi + s^2\partial_2\varphi), \\ s\Delta n + 2(\nabla s\cdot\nabla)n = -s|\nabla n|^2 n, \end{cases} \tag{2.8}$$

and, in regions where s does not vanish, n satisfies the extra equation

$$2\sum_{k=1}^{3} \partial_k n \otimes \partial_k n = |\nabla n|^2 \left(I - n\otimes n\right). \tag{2.9}$$

Proof. Plugging the uniaxial *ansatz* (2.7) into the equilibrium equation (2.4), we find, after rearranging the terms,

$$M_1 + M_2 + M_3 = 0,$$

where

$$\begin{aligned}
M_1 &= \left[\Delta s - 3|\nabla n|^2 s - \frac{1}{L}(2s\partial_1\varphi + s^2\partial_2\varphi)\right]\left(n\otimes n - \frac{1}{3}\mathbf{I}\right),\\
M_2 &= 2n\odot\left(s\Delta n + 2(\nabla s\cdot\nabla)n + s|\nabla n|^2 n\right),\\
M_3 &= s\left[2\sum_k \partial_k n\otimes\partial_k n + |\nabla n|^2\left(n\otimes n - \mathbf{I}\right)\right].
\end{aligned}$$

Here $\odot$ denotes the symmetric tensor product: the (i,j) component of $n\odot m$ is $(n_i m_j + n_j m_i)/2$.

Using the fact that $|n|^2$ is constant equal to 1, which implies in particular $n\cdot\partial_j n = 0$ and $n\cdot\Delta n + |\nabla n|^2 = 0$, we find that

$$\begin{aligned}
M_1 &\in \mathrm{Span}\left(n\odot n - \frac{1}{3}\mathbf{I}\right),\\
M_2 &\in \mathrm{Span}\left\{n\odot v\colon v\in n^\perp\right\},\\
M_3 &\in \mathcal{S}\cap\mathrm{Span}\left\{v\odot w\colon v,w\in n^\perp\right\}.
\end{aligned}$$

Recall here that $\mathcal{S}$ is the order parameter space (2.1) of traceless symmetric matrices. In particular, M_1, M_2 and M_3 are pairwise orthogonal (for the usual scalar product on $M_3(\mathbb{R})$, recalled in Section 2.2.1), and we deduce that

$$M_1 = M_2 = M_3 = 0.$$

We conclude that (2.8) and (2.9) hold. □

Remark 2.7. The system (2.8) satisfied by (s,n) is nothing else than the Euler-Lagrange equation associated to the energy

$$F(s,n) = \mathcal{F}(Q) = \int\left[\frac{L}{2}\left(\frac{2}{3}|\nabla s|^2 + 2s^2|\nabla n|^2\right) + \varphi(2s^2/3, 2s^3/9)\right]dx,$$

under the constraint $|n|^2 = 1$. In other words (2.8) expresses the fact that Q is an equilibrium of $\mathcal{F}$ with respect to perturbations preserving the symmetry constraint $Q\in\mathcal{U}$. The minimization of the functional F has been studied in [88]. On the other hand, the extra equation (2.9) expresses the fact that Q is an equilibrium with respect to symmetry-breaking perturbations. Since (2.9) is not trivial, we see that Palais' Principle of Symmetric Criticality does not apply to uniaxial symmetry.

Remark 2.8. The extra equation (2.9) is of the form $M_3 = 0$, with M_3 taking its values in $\mathcal{S}$ of dimension 5: it contains 5 scalar equations. However, it has been shown during the proof of Proposition 2.6 that, due to the constraint $n \in \mathbb{S}^2$, it holds in fact

$$M_3 \in \mathcal{M} := \mathcal{S} \cap \operatorname{Span} \left\{ v \otimes w \colon v, w \in n^\perp \right\}.$$

Since $\mathcal{M}$ and $\mathbb{S}^2$ are two-dimensional, the information really carried by (2.9) corresponds to a system of two first order partial differential equations, with two unknown. Such a system should have, for generic Dirichlet boundary conditions, at most one solution (this is heuristically motivated by Cauchy-Kovaleskaya's theorem). Therefore, system (2.8) coupled with Dirichlet boundary conditions and the extra equation (2.9) is, heuristically speaking, overdetermined. We expect solutions to exist only in very 'symmetric' cases. The results presented in the sequel are indeed of such a nature.

2.4 In one and two dimensions

In this section we concentrate on one- and two-dimensional configurations, which occur in case of translational invariance in at least one direction. Such a symmetry assumption is actually relevant for many nematic systems that are interesting both theoretically and for application purposes. For instance, in nematic cells bounded by two parallel plates with competing anchoring, one usually looks for one-dimensional solutions [109, 26, 12]. Such hybrid nematic cells provide a model system for understanding the physics of frustration, and this kind of geometry occurs in several nematic based optical devices. Another relevant geometry is the cylindrical one, in which two dimensional configurations can be considered [78, 130, 47, 46], with applications to high performance fibers [36, 34, 74].

Our conclusion (see Theorem 2.9 below) is that a one- or two-dimensional equilibrium configuration can be purely uniaxial only if the director field is constant. Thus in the translation-invariant case, the system (2.8) coupled with (2.9) is so strongly overdetermined that it admits only trivial solutions.

Theorem 2.9. *Let $\Omega \subset \mathbb{R}^3$ be an open set and Q be a smooth solution of the equilibrium equation* (2.4)*. Assume that Q is invariant in one direction: there exists $\nu_0 \in \mathbb{S}^2$ such that $\nu_0 \cdot \nabla Q \equiv 0$.*

(i) If Q is purely uniaxial (i.e. takes values in $\mathcal{U}$) then Q has constant director in every connected component of $\{Q \neq 0\}$. That is, for every connected component ω of $\{Q \neq 0\}$, there exists a uniform director $n_0 = n_0(\omega) \in \mathbb{S}^2$ such that

$$Q(x) = s(x) \left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I} \right) \qquad \forall x \in \omega.$$

for some scalar vector field $s \colon \Omega \to \mathbb{R}$.

(ii) If in addition Q is analytic and Ω is connected, then Q has constant director in the whole domain Ω: there exists $n_0 \in \mathbb{S}^2$ such that

$$Q(x) = s(x) \left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I} \right) \qquad \forall x \in \Omega.$$

Remark 2.10. The one-dimensional case is of course contained in the two-dimensional one, but we find useful to present a specific, much simpler argument here. In one dimension the extra equation (2.9) becomes

$$2n' \otimes n' = |n'|^2(\mathbf{I} - n \otimes n), \tag{2.10}$$

which readily implies $n' \equiv 0$. Indeed, if $n' \neq 0$ then the left hand-side of (2.10) is a matrix of rank one, while the right-hand side has rank two. Thus in one dimension the conclusion of Theorem 2.9 is achieved using only the extra equation (2.9).

In two dimensions however, the proof of Theorem 2.9 is more involved. In particular, the extra equation (2.9) does admit non trivial solutions. For instance a cylindrically symmetric director field introduced by Cladis and Kléman [39] and studied further in [22], which is given in cylindrical coordinates by

$$n(r, \theta, z) = \cos\psi(r)\,\mathbf{e_r} + \sin\psi(r)\,\mathbf{e_z} \quad \text{with } r\frac{d\psi}{dr} = \cos\psi,$$

satisfies (2.9). But there cannot exist any scalar field s such that (s, n) solves (2.8).

Proof of Theorem 2.9. Since the free energy density is frame invariant (see Remark 2.2) we may assume that $\nu_0 = \mathbf{e_z}$, so that $\partial_3 Q \equiv 0$.

We start by proving assertion *(i)* of Theorem 2.9. Fix a connected component ω of $\{Q \neq 0\}$ and define the smooth map $s\colon \omega \to \mathbb{R}$ by the formula

$$s = 3\frac{\operatorname{tr}(Q^3)}{|Q|^2}.$$

Recall that $s(x)$ is the scalar order parameter of $Q(x) \in \mathcal{U}$ (see Remark 2.1). In particular, s does not vanish in ω. In the sequel we are going to show that the smooth map Q/s is locally constant in ω, which obviously implies *(i)*.

Let $x_0 \in \omega$. We claim that there exists an open ball $B \subset \omega$ centered at x_0 and a smooth map $n\colon B \to \mathbb{S}^2$ such that the formula for Q in terms of s and n (2.7) holds in B (as announced in Remark 2.3).

Indeed, fix a director $n_0 \in \mathbb{S}^2$ of $Q(x_0)$: it holds $Q(x_0)n_0 = s_0 n_0$, with $s_0 = s(x_0)$. Since the eigenvalue s_0 is simple and $Q(x_0)$ maps $n_0^\perp$ to $n_0^\perp$, the implicit function theorem can be applied to the map

$$\omega \times \mathbb{R} \times n_0^\perp \to \mathbb{R}^3, \quad (x, s, v) \mapsto (Q(x) - s)\,(n_0 + v)$$

to obtain smooth maps v and $\tilde{s}$ defined in a neighborhood of x_0 and solving uniquely

$$Q(x)(n_0 + v) = \tilde{s}(n_0 + v) \quad \text{for } \tilde{s} \approx s_0,\ v \approx 0 \in n_0^\perp.$$

Since, for x close enough to x_0, eigenvalues of $Q(x)$ distinct from $s(x)$ are far from s_0, it must hold $\tilde{s} = s$. Therefore $n = (n_0 + v)/|n_0 + v|$ provides a smooth map such that (2.7) holds in a neighborhood of x_0, which we may assume to be an open ball B.

To prove *(i)* it remains to show that n is constant in B, which obviously implies that Q/s is locally constant (since $x_0 \in \omega$ is arbitrary).

We start by noting that, since by assumption $\partial_3 Q = 0$, it holds

$$\partial_3 s = \frac{3}{2} n \cdot (\partial_3 Q) n = 0, \quad \partial_3 n = \frac{1}{s} (\partial_3 Q) n = 0.$$

Thus (2.9) becomes

$$A := 2\partial_1 n \otimes \partial_1 n + 2\partial_2 n \otimes \partial_2 n - \left(|\partial_1 n|^2 + |\partial_2 n|^2 \right) (I - n \otimes n) = 0.$$

We deduce that

$$\partial_1 n \cdot A \, \partial_2 n = |\nabla n|^2 \partial_1 n \cdot \partial_2 n = 0,$$

which implies

$$\partial_1 n \cdot \partial_2 n = 0 \quad \text{in } B. \tag{2.11}$$

Using this last fact, we compute

$$\begin{aligned} \partial_1 n \cdot A \, \partial_1 n &= |\partial_1 n|^2 \left(|\partial_1 n|^2 - |\partial_2 n|^2 \right) = 0, \\ \partial_2 n \cdot A \, \partial_2 n &= |\partial_2 n|^2 \left(|\partial_2 n|^2 - |\partial_1 n|^2 \right) = 0, \end{aligned}$$

from which we infer

$$|\partial_1 n|^2 = |\partial_2 n|^2 \quad \text{in } B. \tag{2.12}$$

As a first consequence of (2.11) and (2.12), we obtain that

$$\begin{aligned} \Delta n \cdot \partial_1 n &= \frac{1}{2} \partial_1 \left[|\partial_1 n|^2 - |\partial_2 n|^2 \right] + \partial_2 \left[\partial_1 n \cdot \partial_2 n \right] = 0, \\ \Delta n \cdot \partial_2 n &= \frac{1}{2} \partial_2 \left[|\partial_2 n|^2 - |\partial_1 n|^2 \right] + \partial_1 \left[\partial_1 n \cdot \partial_2 n \right] = 0. \end{aligned}$$

That is, the vector Δn is orthogonal to both vectors $\partial_1 n$ and $\partial_2 n$. Therefore, taking the scalar product of the second equation of (2.8) with $\partial_1 n$ and $\partial_2 n$ and making use of (2.11) and (2.12), we are left with

$$\partial_1 s |\nabla n|^2 = \partial_2 s |\nabla n|^2 = 0 \quad \text{in } B. \tag{2.13}$$

We claim that (2.13) implies in fact

$$|\nabla n|^2 = 0 \quad \text{in } B. \tag{2.14}$$

Assume indeed that (2.14) does not hold, so that $|\nabla n|^2 > 0$ in some open set $W \subset B$. Then by (2.13) the scalar field s is constant in W, and the first equation of (2.8) implies that $|\nabla n|^2$ is constant in W. Up to rescaling the variable, we have thus obtained a map n mapping an open subset of the plane $\mathbb{R}^2$ into the sphere $\mathbb{S}^2$ and satisfying

$$\partial_1 n \cdot \partial_2 n = 0, \quad |\partial_1 n|^2 = |\partial_2 n|^2 = 1.$$

That is, n is a local isometry. Since the plane has zero curvature while the sphere has positive curvature, the existence of such an isometry contradicts Gauss's *Theorema egregium*. Hence we have proved the claim (2.14), and n must be constant in B. This ends the proof of *(i)*.

Now we turn to the proof of assertion *(ii)* of Theorem 2.9. We start by proving the following

Claim: for any open ball $B \subset \Omega$, Q has constant director in B: there exists $n_0 \in \mathbb{S}^2$ such that $Q = s(n_0 \otimes n_0 - \mathbf{I}/3)$ in B.

Note that this *Claim* is simply a consequence of *(i)* if $B \subset \{Q \neq 0\}$. The additional information here is that $B \cap \{Q \neq 0\}$ may not be connected.

If $Q \equiv 0$ in B the *Claim* is obvious, so we assume $Q(x_0) \neq 0$ for some $x_0 \in B$. Let $n_0 \in \mathbb{S}^2$ be such that

$$Q(x_0) = s(x_0)\left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I}\right). \tag{2.15}$$

We now prove the *Claim* by contradiction: assume that there exists $x_1 \in B$ such that

$$Q(x_1) \neq s(x_1)\left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I}\right). \tag{2.16}$$

In particular, $Q(x_1) \neq 0$. Consider the segment $S = [x_0, x_1]$ contained in B and therefore in Ω. Since Q is analytic and does not vanish identically on S, the set $S \cap \{Q = 0\}$ must be discrete (and thus finite by compactness).

Since (2.16) holds, the (locally constant) director is not the same in the respective connected components of x_0 and x_1 in $S \cap \{Q \neq 0\}$. As a consequence, there must exist $x_2 \in S$, $n_1 \in \mathbb{S}^2 \setminus \{\pm n_0\}$ and $\delta > 0$ such that:

$$\begin{aligned}
&\{Q = 0\} \cap S \cap B_\delta(x_2) = \{x_2\},\\
&Q(x) = s(x)\left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I}\right) \quad \forall x \in [x_2, x_0] \cap B_\delta(x_2),\\
&Q(x) = s(x)\left(n_1 \otimes n_1 - \frac{1}{3}\mathbf{I}\right) \quad \forall x \in [x_2, x_1] \cap B_\delta(x_2).
\end{aligned}$$

Hence for small enough ε, the analytic map

$$\widetilde{Q} \colon (-\varepsilon, \varepsilon) \ni t \mapsto Q(x_2 + t(x_0 - x_1)) \in \mathcal{U}$$

vanishes exactly at $t = 0$, has constant director n_0 for $t > 0$ and constant director n_1 for $t < 0$. The associated map $\tilde{s}(t)$ is smooth in $(-\varepsilon, \varepsilon) \setminus \{0\}$ and it holds

$$\widetilde{Q}'(t) = \begin{cases} \tilde{s}'(t)\left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I}\right) & \text{for } t > 0\\ \tilde{s}'(t)\left(n_1 \otimes n_1 - \frac{1}{3}\mathbf{I}\right) & \text{for } t < 0.\end{cases}$$

We deduce that $l^+ := \lim_{0^+} \tilde{s}'$ and $l^- := \lim_{0^-} \tilde{s}'$ exist and satisfy

$$\widetilde{Q}'(0) = l^+\left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I}\right) = l^-\left(n_1 \otimes n_1 - \frac{1}{3}\mathbf{I}\right).$$

Since $n_0 \neq \pm n_1$, it must hold $l^+ = l^- = 0$. Thus $\tilde{s}$ is in fact C^1 in $(-\varepsilon, \varepsilon)$ and satisfies $\tilde{s}'(0) = 0$.

For any integer $k \geq 0$ it holds

$$\widetilde{Q}^{(k)}(t) = \begin{cases} \tilde{s}^{(k)}(t)\left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I}\right) & \text{for } t > 0\\ \tilde{s}^{(k)}(t)\left(n_1 \otimes n_1 - \frac{1}{3}\mathbf{I}\right) & \text{for } t < 0,\end{cases}$$

and we may repeat the same argument as above to show by induction that $\tilde{s}$ is smooth in $(-\varepsilon, \varepsilon)$ and all its derivatives vanish at 0. In particular we find that

$$\widetilde{Q}^{(k)}(0) = 0 \qquad \forall k \geq 0,$$

which implies that $Q \equiv 0$ on S, since $\widetilde{Q}$ is analytic: we obtain a contradiction, and the above *Claim* is proved.

We may now complete the proof of assertion *(ii)* of Theorem 2.9. We assume that Q does not vanish identically, and fix $x_0 \in \Omega$ such that $Q(x_0) \neq 0$. There exists $n_0 \in \mathbb{S}^2$ such that

$$Q(x_0) = s(x_0) \left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I} \right).$$

Let $x \in \Omega$. Since Ω is open and connected (and thus path-connected), there exists a "chain of open balls" from x_0 to x. More explicitly: there exist points

$$x_0, x_1, \ldots, x_{N-1}, x_N = x \in \Omega,$$

and open balls

$$B_0 \ni x_0, B_1 \ni x_1, \ldots, B_N \ni x_N,$$

such that

$$B_k \cap B_{k+1} \neq \emptyset \qquad k = 0, \ldots, N-1.$$

In each ball B_k, the above *Claim* ensures that Q has constant director. In B_0, since $Q(x_0) \neq 0$, the constant director is uniquely determined up to a sign and we may choose it to be n_0. We denote by $n_k \in \mathbb{S}^2$ a constant director in B_k.

In the intersection $B_k \cap B_{k+1}$, the vectors n_k and n_{k+1} are both admissible constant directors. Since Q is analytic and not uniformly zero, it can not be uniformly zero in the non empty open set $B_k \cap B_{k+1}$. Therefore the constant director in $B_k \cap B_{k+1}$ is uniquely determined (up to a sign): it holds $n_k = \pm n_{k+1}$. Hence we can actually choose the directors n_k such that

$$n_0 = n_1 = n_2 = \cdots = n_N,$$

and in particular we find

$$Q(x) = s(x) \left(n_N \otimes n_N - \frac{1}{3}\mathbf{I} \right) = s(x) \left(n_0 \otimes n_0 - \frac{1}{3}\mathbf{I} \right).$$

The proof of *(ii)* is complete. □

Remark 2.11. As already pointed out in Section 2.2.2, the assumption that Q is smooth is very natural, since physically relevant solutions are bounded and therefore smooth. The additional assumption of analyticity in assertion *(ii)* is also natural, since it is satisfied whenever the bulk free energy is analytic (and this is the case for the bulk free energy usually considered).

2.5 In a spherical droplet with radial anchoring

In this section we consider a droplet of nematic subject to strong radial anchoring on the surface. Droplets of nematic play an important role in some electro-optic applications, like polymer dispersed liquid crystals (PDLC) devices (see the review article [90] and references therein). Moreover, this problem is important theoretically as a model problem for the study of point defects, due to the universal features it exhibits [79].

The droplet containing the nematic is modelled as an open ball

$$B_R = \left\{x \in \mathbb{R}^3 \colon |x| < R\right\},$$

and strong radial anchoring corresponds to Dirichlet boundary conditions of the form

$$Q(x) = s_0 \left(\frac{x}{R} \otimes \frac{x}{R} - \frac{1}{3} I\right) \qquad \text{for } |x| = R, \tag{2.17}$$

for some fixed $s_0 \neq 0$.

In this setting, the equilibrium equation (2.4) admits a particular symmetric solution of the form

$$Q(x) = s(r) \left(\frac{x}{r} \otimes \frac{x}{r} - \frac{1}{3} I\right) \qquad \forall x \in B_R, \tag{2.18}$$

where $r = |x|$, and $s : (0, R) \to \mathbb{R}$ solves

$$\frac{d^2 s}{dr^2} + \frac{2}{r}\frac{ds}{dr} - \frac{6}{r^2} s = \frac{1}{L}\left(2s\partial_1\varphi(2s^2/3, 2s^3/9) + s^2\partial_2\varphi(2s^2/3, 2s^3/9)\right), \tag{2.19}$$

with boundary conditions $s(0) = 0$, $s(R) = s_0$. We call such a solution *radial hedgehog*.

As already mentioned in Remark 2.5, the existence of such a solution is ensured by Palais' Principle of Symmetric Criticality [108]. In fact, $G = SO(3)$ acts linearly and isometrically on the affine Hilbert space

$$\mathcal{H} = \left\{Q \in H^1(B_R; \mathcal{S}) \colon Q \text{ satisfies (2.17)}\right\}$$

by change of frame: the action is given by formula (2.3). The free energy is frame invariant (see Remark 2.2): it holds

$$\mathcal{F}(g \cdot Q) = \mathcal{F}(Q) \qquad \forall g \in G,\ Q \in \mathcal{H}.$$

Denoting by $\Sigma \subset \mathcal{H}$ the subspace of symmetric configurations, i.e. of those maps Q which satisfy $g \cdot Q = Q$ for all rotations $g \in G$, the Principle of Symmetric Criticality [108, Section 2] can be stated as follows: if $Q \in \Sigma$ is a critical point of $\mathcal{F}_{|\Sigma}$, then Q is a critical point of $\mathcal{F}$, i.e. Q solves the equilibrium equation (2.4).

Since Σ consists precisely of those Q which are of the form (2.18), and since the existence of a minimizer of $\mathcal{F}_{|\Sigma}$ is ensured by the direct method of the calculus of variations [81], we obtain the existence of the radial hedgehog solution of (2.4) described above by (2.18)-(2.19).

Spherically symmetric solutions are in fact the only purely uniaxial solutions of this problem. This is the content of the next result.

Theorem 2.12. *Assume that φ is analytic and satisfies* (2.5)*. Let $Q \in H^1(B_R, \mathcal{S})$ solve the equilibrium equation* (2.4)*, with radial boundary conditions* (2.17)*.*

If Q is purely uniaxial (i.e. takes values in $\mathcal{U}$), then Q is necessarily spherically symmetric: it satisfies (2.18)-(2.19)*.*

Remark 2.13. A recent result of Henao and Majumdar [67, 68] is a direct corollary of Theorem 2.12. In [67, 68], the authors consider a spherical droplet with radial anchoring, with a bulk free energy f_b of the form (2.6) and study the low temperature limit $a \to \infty$. They assume the existence of a sequence of uniaxial minimizers of the free energy, and show convergence towards a spherically symmetric solution.

Remark 2.14. As pointed out by the anonymous referee of this article, the proof of Theorem 2.12 remains valid if the domain is an annulus instead of a ball. Moreover, in the case of the ball and of the physical bulk potential (2.6), the radial solution is known to be unique [72], so that Theorem 2.12 implies that there is a unique purely uniaxial solution of (2.4)-(2.17).

Proof of Theorem 2.12: The assumption (2.5) on φ ensures that Q is bounded and therefore analytic (see Section 2.2.2).

Since Q is smooth up to the boundary ∂B, and does not vanish on the boundary, we may proceed as in the proof of Theorem 2.9 to obtain, in a neighborhood of each point of the boundary ∂B_R, smooth maps s and n such that the *ansatz* (2.7) holds (see also Remark 2.3). The locally well-defined map n is determined up to a sign. We determine it uniquely via the boundary condition

$$n(x) = \frac{x}{R} \qquad \text{for } |x| = R.$$

Therefore we obtain, for some $\delta > 0$, smooth maps

$$s\colon B_R \setminus B_{(1-\delta)R} \to \mathbb{R}, \quad n\colon B_R \setminus B_{(1-\delta)R} \to \mathbb{S}^2,$$

such that

$$Q(x) = s(x)\left(n(x) \otimes n(x) - \frac{1}{3}I\right) \quad \text{for } (1-\delta)R < |x| < R.$$

The values of s and n on the boundary ∂B_R are determined:

$$s(x) = s_0, \quad n(x) = \frac{x}{R} \qquad \text{for } |x| = R. \tag{2.20}$$

We use the fact that s and n satisfy the system (2.8) and the extra constraint (2.9), to determine in addition their radial derivatives on the boundary:

Lemma 2.15. *It holds*

$$\partial_r n \equiv 0, \quad \partial_r s \equiv s_1, \qquad \text{on } \partial B_R,$$

for some constant $s_1 \in \mathbb{R}$.

Lemma 2.15 constitutes the heart of the proof of Theorem 2.12. The proof of Lemma 2.15 can be found below. We start by showing how Lemma 2.15 implies the conclusion of Theorem 2.12.

Let $\tilde{s}$ be a local solution of (2.19) with Cauchy data

$$\tilde{s}(R) = s_0, \ \frac{d\tilde{s}}{dr}(R) = s_1,$$

where s_1 is the constant value of $\partial_r s$ on ∂B_R according to Lemma 2.15. We fix $\eta > 0$ such that $\tilde{s}$ is defined on $[R, R+\eta]$, and define a map $\widetilde{Q}$ on $B_{R+\eta}$ by

$$\widetilde{Q}(x) = \begin{cases} Q(x) & \text{if } |x| \leq R, \\ \tilde{s}(r)\left(\frac{x}{r}\otimes\frac{x}{r} - \frac{1}{3}I\right) & \text{if } R < |x| < R+\eta. \end{cases}$$

Lemma 2.15 ensures that the boundary conditions on ∂B_R match well at order 0 and 1: the map $\widetilde{Q}$ belongs to $C^1(\overline{B}_{R+\eta})$. Moreover, the matching boundary conditions on ∂B_R ensure that $\widetilde{Q}$ is a weak solution of the Euler-Lagrange equation (2.4) in $B_{R+\eta}$. In particular, $\widetilde{Q}$ is analytic (see Section 2.2.2). Hence, for any rotation $g \in G$, the map

$$x \mapsto \widetilde{Q}(gx) - g\widetilde{Q}(x)^t g$$

is analytic and vanishes in $B_{R+\eta} \setminus B_R$ and must therefore vanish everywhere. We deduce that Q is spherically symmetric and the proof of Theorem 2.12 is complete. □

Proof of Lemma 2.15: During this proof we make use of spherical coordinates (r, θ, φ) and denote by $(\mathbf{e_r}, \mathbf{e}_\theta, \mathbf{e}_\varphi)$ the associated (moving) eigenframe.

For simplicity we assume $R = 1$ (the general case follows by rescaling the variable) and write B for B_1. We proceed in three steps: we start by showing that, on the boundary ∂B, it holds

— $\partial_r n = 0$,
— then $\partial_r^2 n = 0$,
— and eventually $\partial_\theta \partial_r s = \partial_\varphi \partial_r s = 0$.

Step 1: $\partial_r n = 0$ on ∂B.

This first step is obtained as a consequence of the boundary condition (2.20), and of the constraint (2.9). Indeed, on the boundary, (2.20) determines the partial derivatives of n in two directions $\partial_\theta n$ and $\partial_\varphi n$, and (2.9) determines the partial derivative in the remaining direction.

In spherical coordinates, (2.9) becomes

$$2\left(\partial_r n\otimes\partial_r n + \frac{1}{r^2}\partial_\theta n\otimes\partial_\theta n + \frac{1}{r^2\sin^2\theta}\partial_\varphi n \partial_\varphi n\right) = |\nabla n|^2(I - n\otimes n), \quad (2.21)$$

and

$$|\nabla n|^2 = |\partial_r n|^2 + \frac{1}{r^2}|\partial_\theta n|^2 + \frac{1}{r^2\sin^2\theta}|\partial_\varphi n|^2.$$

Since on the boundary ∂B it holds

$$n = \mathbf{e_r}, \ \partial_\theta n = \mathbf{e}_\theta, \ \partial_\varphi n = \sin\theta\,\mathbf{e}_\varphi,$$

we deduce from (2.21) that

$$2\partial_r n\otimes\partial_r n = |\partial_r n|^2(I - \mathbf{e_r}\otimes\mathbf{e_r}) \quad \text{for } r = 1,$$

which implies $\partial_r n = 0$ (as in Remark 2.10) and proves *Step 1.*

Step 2: $\partial_r^2 n = 0$ on ∂B.

This second step is obtained as a consequence of *Step 1* and of the second equation of (2.8), together with the boundary conditions (2.20). In fact, it holds

$$\Delta n = \partial_r^2 n + \Delta_{\mathbb{S}^2} n = \partial_r^2 n - 2\mathbf{e_r} \quad \text{for } r = 1,$$

since $\partial_r n = 0$ by *Step 1* and $n = \mathbf{e_r}$ for $r = 1$. Moreover, since s is constant on the boundary, it holds

$$(\nabla s \cdot) n = \partial_r s \partial_r n = 0 \quad \text{for } r = 1.$$

Thus the second equation of (2.8) becomes, on the boundary,

$$s_0 \partial_r^2 n - 2 s_0 \mathbf{e_r} = -s_0 |\nabla n|^2 \mathbf{e_r} = -2 s_0 \mathbf{e_r} \quad \text{for } r = 1.$$

Here we used again (2.20) and *Step 1* to compute $|\nabla n|^2$ for $r = 1$. The last equation completes the proof of *Step 2.*

Step 3: $\partial_\theta \partial_r s = \partial_\varphi \partial_r s = 0$ on ∂B.

To prove this third step, we consider Taylor expansions of s and n with respect to $r - 1 \approx 0$, and plug them into (2.8) and (2.9) to obtain more information about higher order radial derivatives and find eventually that $\partial_r s$ is constant on the boundary.

Using *Step 1* and *Step 2*, we may write, for $r = |x| \in [1-\delta, 1]$ and $\omega = x/r \in \mathbb{S}^2$,

$$n = \mathbf{e_r} + (r-1)^3 m_1(\omega) + (r-1)^4 m_2(\omega) + O((r-1)^5) \tag{2.22}$$

$$s = s_0 + (r-1) s_1(\omega) + (r-1)^2 s_2(\omega) + O((r-1)^3), \tag{2.23}$$

where $6m_1 = \partial_r^3 n_{|\mathbb{S}^2}$, $24 m_2 = \partial_r^4 n_{|\mathbb{S}^2}$, $s_1 = \partial_r s_{|\mathbb{S}^2}$, and $2 s_2 = \partial_r^2 s_{|\mathbb{S}^2}$ are smooth functions of $\omega \in \mathbb{S}^2$, and

$$O((r-1)^k) = (r-1)^k \times \text{ some smooth function of } (r, \omega).$$

In the sequel, we plug the Taylor expansions above into (2.8) and (2.9) in order to conclude that s_1 is constant. The computations are elementary but tedious. In order to clarify them, we start by sketching the main steps without going into details. The complete proof follows below.

Sketch of the main steps: Plugging (2.22) into (2.9) leads to an equation of the form

$$0 = (r-1)^3 A_3 + (r-1)^4 A_4 + O((r-1)^5), \tag{2.24}$$

where

$$A_3 = A_3(m_1, \partial_\theta m_1, \partial_\varphi m_1),$$
$$A_4 = A_4(m_1, m_2, \partial_\theta m_1, \partial_\varphi m_1, \partial_\theta m_2, \partial_\varphi m_2).$$

At this point, a first simplification occurs, since A_4 is actually of the form

$$A_4 = -2A_3 + \widetilde{A}_4(m_2, \partial_\theta m_2, \partial_\varphi m_2),$$

so that from (2.24) we deduce

$$\widetilde{A}_4(m_2, \partial_\theta m_2, \partial_\varphi m_2) = 0. \tag{2.25}$$

Next we make use of (2.8). Plugging (2.22) and (2.23) into (2.8), we obtain equations of the form

$$\begin{cases} 0 = \alpha_0 + O(r-1) \\ 0 = (r-1)v_1 + (r-1)^2 v_2 + O((r-1)^3, \end{cases} \tag{2.26}$$

where

$$\alpha_0 = \alpha_0(s_0, s_1, s_2),$$
$$v_1 = v_1(s_0, m_1, \partial_\theta s_1, \partial_\varphi s_1)$$
$$v_2 = v_2(s_0, s_1, m_1, m_2, \partial_\theta s_1, \partial_\varphi s_1, \partial_\theta s_2, \partial_\varphi s_2).$$

The first equation in (2.26) implies that $\alpha_0 = 0$. Solving $\alpha_0 = 0$, we obtain an expression of s_2 in terms of s_1 and s_0, which we plug into v_2. Here a new simplification arises: it holds

$$v_2 = \tilde{v}_2(s_0, s_1, m_1, m_2) - 3v_1.$$

Thus (2.26) implies that $v_1 = \tilde{v}_2 = 0$. Solving $\tilde{v}_2 = 0$ we find an expression

$$m_2 = m_2(s_0, s_1, m_1),$$

which we plug into (2.25) to obtain an equation of the form

$$A_4^*(s_0, s_1, m_1, \partial_\theta s_1, \partial_\varphi s_1, \partial_\theta m_1, \partial_\varphi m_1) = 0.$$

Using the equation $A_3 = 0$ from (2.24), we are able to simplify the last expression of A_4^* into one which does not involve derivatives of m_1:

$$\widehat{A}_4(s_0, s_1, m_1, \partial_\theta s_1, \partial_\varphi s_1) = 0. \tag{2.27}$$

Eventually we use the equation $v_1 = 0$ to express m_1 in terms of s_0, $\partial_\theta s_1$ and $\partial_\varphi s_1$. Plugging that expression of m_1 into (2.27) leads us to a system of the form

$$A_4^\sharp(s_0, \partial_\theta s_1, \partial_\varphi s_1) = 0.$$

The above equation forces $\partial_\theta s_1 = \partial_\varphi s_1 = 0$ and thus allows to conclude.

Complete proof: It holds

$$\partial_r n = 3(r-1)^2 m_1 + 4(r-1)^3 m_2 + O((r-1)^4),$$
$$\partial_r^2 n = 6(r-1)m_1 + 12(r-1)^2 m_2 + O((r-1)^3),$$
$$\partial_\theta n = \mathbf{e}_\theta + (r-1)^3 \partial_\theta m_1 + (r-1)^4 \partial_\theta m_2 + O((r-1)^5),$$
$$\partial_\varphi n = \sin\theta \mathbf{e}_\varphi + (r-1)^3 \partial_\varphi m_1 + (r-1)^4 \partial_\varphi m_2 + O((r-1)^5),$$
$$\Delta_{\mathbb{S}^2} n = -2\mathbf{e_r} + O((r-1)^3),$$

and thus

$$\begin{aligned}
\Delta n &= \partial_r^2 n + \frac{2}{r}\partial_r n + \frac{1}{r^2}\Delta_{\mathbb{S}^2} n \\
&= \partial_r^2 n + 2(1+O((r-1))\partial_r n \\
&\quad + \left(1 - 2(r-1) + 3(r-1)^2 + O((r-1)^3\right)\left(-2\mathbf{e_r} + O((r-1)^3)\right) \\
&= 6(r-1)m_1 + 12(r-1)^2 m_2 + 6(r-1)^2 m_1 \\
&\quad - 2\mathbf{e_r} + 4(r-1)\mathbf{e_r} - 6(r-1)^2\mathbf{e_r} + O((r-1)^3) \\
&= -2\mathbf{e_r} + (r-1)\left[6m_1 + 4\mathbf{e_r}\right] + (r-1)^2\left[12m_2 + 6m_1 - 6\mathbf{e_r}\right] \\
&\quad + O((r-1)^3).
\end{aligned}$$

Hence we compute

$$\begin{aligned}
s\Delta n = -2s_0\mathbf{e_r} &+ (r-1)\left[6s_0m_1 + 4s_0\mathbf{e_r} - 2s_1\mathbf{e_r}\right] \\
&+ (r-1)^2\left[12s_0m_2 + 6s_0m_1 - 6s_0\mathbf{e_r} + 6s_1m_1 + 4s_1\mathbf{e_r} - 2s_2\mathbf{e_r}\right] \\
&+ O((r-1)^3).
\end{aligned}$$

Next we want to compute

$$(\nabla s \cdot \nabla)n = \partial_r s \partial_r n + \frac{1}{r^2}\partial_\theta s \partial_\theta n + \frac{1}{r^2\sin^2\theta}\partial_\varphi s \partial_\varphi n.$$

We calculate each term:

$$\begin{aligned}
\partial_r s \partial_r n &= \left(s_1 + 2(r-1)s_2 + O((r-1)^2)\right)\left(3(r-1)^2 m_1 + O((r-1)^3)\right) \\
&= 3s_1(r-1)^2 m_1 + O((r-1)^3),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{r^2}\partial_\theta s \partial_\theta n &= \frac{1}{r^2}\left((r-1)\partial_\theta s_1 + (r-1)^2\partial_\theta s_2 + O((r-1)^3)\right) \\
&\quad \times \left(\mathbf{e}_\theta + O((r-1)^3)\right) \\
&= (1 - 2(r-1) + O(r-1)) \\
&\quad \times \left((r-1)\partial_\theta s_1 \mathbf{e}_\theta + (r-1)^2\partial_\theta s_2\mathbf{e}_\theta + O((r-1)^3)\right) \\
&= (r-1)\partial_\theta s_1\mathbf{e}_\theta \\
&\quad + (r-1)^2\left[\partial_\theta s_2\mathbf{e}_\theta - 2\partial_\theta s_1\mathbf{e}_\theta\right] + O((r-1)^3),
\end{aligned}$$

$$\begin{aligned}
\frac{1}{r^2\sin^2\theta}\partial_\varphi s \partial_\varphi n &= (r-1)\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi + (r-1)^2\left[\frac{\partial_\varphi s_2}{\sin\theta}\mathbf{e}_\varphi - 2\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi\right] \\
&\quad + O((r-1)^3).
\end{aligned}$$

Thus it holds:

$$
\begin{aligned}
s\Delta n + 2(\nabla s \cdot \nabla)n = -2s_0\mathbf{e_r}& \\
&+ (r-1)\Big[6s_0m_1 + 4s_0\mathbf{e_r} - 2s_1\mathbf{e_r} \\
&\qquad + 2\partial_\theta s_1\mathbf{e}_\theta + 2\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi\Big] \\
&+ (r-1)^2\Big[12s_0m_2 + 6s_0m_1 - 6s_0\mathbf{e_r} \\
&\qquad + 12s_1m_1 + 4s_1\mathbf{e_r} - 2s_2\mathbf{e_r} + 2\partial_\theta s_2\mathbf{e}_\theta \\
&\qquad - 4\partial_\theta s_1\mathbf{e}_\theta + 2\frac{\partial_\varphi s_2}{\sin\theta}\mathbf{e}_\varphi - 4\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi\Big] \\
&+ O((r-1)^3).
\end{aligned}
$$

Our next step is to compute the symmetric matrix

$$
M = \partial_r n \otimes \partial_r n + \frac{1}{r^2}\partial_\theta n \otimes \partial_\theta n + \frac{1}{r^2\sin^2\theta}\partial_\varphi n \otimes \partial_\varphi n.
$$

We compute each term:

$$
\begin{aligned}
\partial_r n \otimes \partial_r n &= 9(r-1)^4 m_1 \otimes m_1 + O((r-1)^5), \\
\frac{1}{r^2}\partial_\theta n \otimes \partial_\theta n &= \big(1 - 2(r-1) + 3(r-1)^2 - 4(r-1)^3 + 5(r-1)^4\big) \\
&\quad \times \big(\mathbf{e}_\theta \otimes \mathbf{e}_\theta + 2(r-1)^3\partial_\theta m_1 \odot \mathbf{e}_\theta + 2(r-1)^4\partial_\theta m_2 \odot \mathbf{e}_\theta\big) \\
&\quad + O((r-1)^5) \\
&= \mathbf{e}_\theta \otimes \mathbf{e}_\theta - 2(r-1)\mathbf{e}_\theta \otimes \mathbf{e}_\theta + 3(r-1)^2\mathbf{e}_\theta \otimes \mathbf{e}_\theta \\
&\quad + (r-1)^3\left[-4\mathbf{e}_\theta \otimes \mathbf{e}_\theta + 2\partial_\theta m_1 \odot \mathbf{e}_\theta\right] \\
&\quad + (r-1)^4\left[5\mathbf{e}_\theta \otimes \mathbf{e}_\theta - 4\partial_\theta m_1 \odot \mathbf{e}_\theta + 2\partial_\theta m_2 \odot \mathbf{e}_\theta\right] \\
&\quad + O((r-1)^5),
\end{aligned}
$$

$$
\begin{aligned}
\frac{1}{r^2\sin^2\theta}\partial_\varphi n \otimes \partial_\varphi n = \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi& - 2(r-1)\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + 3(r-1)^2\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi \\
&+ (r-1)^3\left[-4\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi + \frac{2}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi\right] \\
&+ (r-1)^4\Big[5\mathbf{e}_\varphi \otimes \mathbf{e}_\varphi - \frac{4}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi \\
&\qquad + \frac{2}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi\Big] \\
&+ O((r-1)^5).
\end{aligned}
$$

Hence we have

$$
M = M_0 + (r-1)M_1 + \cdots + (r-1)^4 M_4 + O((r-1)^5),
$$

where

$$
\begin{aligned}
M_0 &= \mathbf{e}_\theta \otimes \mathbf{e}_\theta + \mathbf{e}_\varphi \otimes \mathbf{e}_\varphi = I - \mathbf{e_r} \otimes \mathbf{e_r},\\
M_1 &= -2(I - \mathbf{e_r} \otimes \mathbf{e_r}),\\
M_2 &= 3(I - \mathbf{e_r} \otimes \mathbf{e_r}),\\
M_3 &= -4(I - \mathbf{e_r} \otimes \mathbf{e_r}) + 2\partial_\theta m_1 \odot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi,\\
M_4 &= 9m_1 \otimes m_1 + 5(I - \mathbf{e_r} \otimes \mathbf{e_r}) - 4\partial_\theta m_1 \odot \mathbf{e}_\theta - \frac{4}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi\\
&\quad + 2\partial_\theta m_2 \odot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi.
\end{aligned}
$$

Using the fact that $|\nabla n|^2 = \operatorname{tr} M$, we obtain in particular

$$
\begin{aligned}
|\nabla n|^2 = {} & 2 - 4(r-1) + 6(r-1)^2\\
& + (r-1)^3\Big[-8 + 2\partial_\theta m_1 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi\Big]\\
& + (r-1)^4\Big[9|m_1|^2 + 10 - 4\partial_\theta m_1 \cdot \mathbf{e}_\theta - \frac{4}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi\\
&\qquad + 2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \cdot \mathbf{e}_\varphi\Big]\\
& + O((r-1)^5),
\end{aligned}
$$

and

$$
\begin{aligned}
|\nabla n|^2 n = {} & 2\mathbf{e_r} - 4(r-1)\mathbf{e_r} + 6(r-1)^2\mathbf{e_r}\\
& + (r-1)^3\Big[(-8 + 2\partial_\theta m_1 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi)\mathbf{e_r} + 2m_1\Big]\\
& + (r-1)^4\Big[\Big\{9|m_1|^2 + 10 - 4\partial_\theta m_1 \cdot \mathbf{e}_\theta - \frac{4}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi\\
&\qquad + 2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \cdot \mathbf{e}_\varphi\Big\}\mathbf{e_r} - 4m_1 + 2m_2\Big]\\
& + O((r-1)^5),
\end{aligned}
$$

$$
\begin{aligned}
s|\nabla n|^2 n = {} & 2s_0\mathbf{e_r} + (r-1)\big[2s_1 - 4s_0\big]\mathbf{e_r} + (r-1)^2\big[6s_0 - 4s_1 + 2s_2\big]\mathbf{e_r}\\
& + O((r-1)^3),
\end{aligned}
$$

$$
\begin{aligned}
|\nabla n|^2 n \otimes n = {} & 2\mathbf{e_r} \otimes \mathbf{e_r} - 4(r-1)\mathbf{e_r} \otimes \mathbf{e_r} + 6(r-1)^2\mathbf{e_r} \otimes \mathbf{e_r}\\
& + (r-1)^3\Big[(-8 + 2\partial_\theta m_1 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi)\mathbf{e_r} \otimes \mathbf{e_r}\\
&\qquad + 4m_1 \odot \mathbf{e_r}\Big]\\
& + (r-1)^4\Big[\Big\{9|m_1|^2 + 10 - 4\partial_\theta m_1 \cdot \mathbf{e}_\theta - \frac{4}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi\\
&\qquad + 2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \cdot \mathbf{e}_\varphi\Big\}\mathbf{e_r} \otimes \mathbf{e_r}\\
&\qquad - 8m_1 \odot \mathbf{e_r} + 4m_2 \odot \mathbf{e_r}\Big]\\
& + O((r-1)^5),
\end{aligned}
$$

$$
\begin{aligned}
&|\nabla n|^2(I - n \otimes n) \\
&\quad = 2(I - \mathbf{e_r} \otimes \mathbf{e_r}) - 4(r-1)(I - \mathbf{e_r} \otimes \mathbf{e_r}) \\
&\qquad + 6(r-1)^2(I - \mathbf{e_r} \otimes \mathbf{e_r}) \\
&\qquad + (r-1)^3\Big[(-8 + 2\partial_\theta m_1 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi)(I - \mathbf{e_r} \otimes \mathbf{e_r}) \\
&\qquad\qquad - 4m_1 \odot \mathbf{e_r}\Big] \\
&\qquad + (r-1)^4\Big[\Big\{9|m_1|^2 + 10 - 4\partial_\theta m_1 \cdot \mathbf{e}_\theta - \frac{4}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi \\
&\qquad\qquad + 2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \cdot \mathbf{e}_\varphi\Big\}(I - \mathbf{e_r} \otimes \mathbf{e_r}) \\
&\qquad\qquad + 8m_1 \odot \mathbf{e_r} - 4m_2 \odot \mathbf{e_r}\Big] \\
&\qquad + O((r-1)^5).
\end{aligned}
$$

Eventually, we have:

$$
\begin{aligned}
&s\Delta n + 2(\nabla s \cdot \nabla)n + s|\nabla n|^2 n \\
&\quad = (r-1)\Big[6s_0 m_1 + 2\partial_\theta s_1 \mathbf{e}_\theta + 2\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi\Big] \\
&\qquad + (r-1)^2\Big[12 s_0 m_2 + 6 s_0 m_1 + 12 s_1 m_1 + 2\partial_\theta s_2 \mathbf{e}_\theta \\
&\qquad\qquad - 4\partial_\theta s_1 \mathbf{e}_\theta + 2\frac{\partial_\varphi s_2}{\sin\theta}\mathbf{e}_\varphi - 4\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi\Big] \\
&\qquad + O((r-1)^3),
\end{aligned}
$$

and

$$
2M - |\nabla n|^2(I - n \otimes n) = (r-1)^3 A_3 + (r-1)^4 A_4 + O((r-1)^5),
$$

where

$$
\begin{aligned}
A_3 &= 4\partial_\theta m_1 \odot \mathbf{e}_\theta + \frac{4}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi \\
&\quad - \Big[2\partial_\theta m_1 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi\Big](I - \mathbf{e_r} \otimes \mathbf{e_r}) \\
&\quad + 4m_1 \odot \mathbf{e_r},
\end{aligned}
$$

$$
\begin{aligned}
A_4 &= 18 m_1 \otimes m_1 - 8\partial_\theta m_1 \odot \mathbf{e}_\theta - \frac{8}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi \\
&\quad + 4\partial_\theta m_2 \odot \mathbf{e}_\theta + \frac{4}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi \\
&\quad - \Big[9|m_1|^2 - 4\partial_\theta m_1 \cdot \mathbf{e}_\theta - \frac{4}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi \\
&\qquad\qquad + 2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi\Big](I - \mathbf{e_r} \otimes \mathbf{e_r}) \\
&\quad - 8m_1 \odot \mathbf{e_r} + 4m_2 \odot \mathbf{e_r} \\
&= -2A_3 + 18 m_1 \otimes m_1 + 4\partial_\theta m_2 \odot \mathbf{e}_\theta + \frac{4}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi + 4m_2 \odot \mathbf{e_r} \\
&\quad - \Big[9|m_1|^2 + 2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi\Big](I - \mathbf{e_r} \otimes \mathbf{e_r}).
\end{aligned}
$$

Moreover, denoting by

$$\psi(s) := \frac{1}{L}\left(2s\partial_1\varphi(2s^2/3, 2s^3/9) + s^2\partial_2\varphi(2s^2/3, 2s^3/9)\right)$$

the nonlinear term of order 0 arising in the first equation of (2.8), we have

$$\Delta s - 3s|\nabla n|^2 - \psi(s) = 2s_2 + 2s_1 - 6s_0 + \psi(s_0) + O(r-1).$$

We conclude that the following equalities hold:

$$s_2 = -s_1 + 3s_0 + \frac{1}{2}\psi(s_0), \tag{2.28}$$

$$6s_0m_1 + 2\partial_\theta s_1\mathbf{e}_\theta + 2\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi = 0, \tag{2.29}$$

$$\begin{aligned}12s_0m_2 &+ 6s_0m_1 + 12s_1m_1 \\ &+ 2\partial_\theta s_2\mathbf{e}_\theta - 4\partial_\theta s_1\mathbf{e}_\theta + 2\frac{\partial_\varphi s_2}{\sin\theta}\mathbf{e}_\varphi - 4\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi = 0,\end{aligned} \tag{2.30}$$

$$\begin{aligned}&4\partial_\theta m_1 \odot \mathbf{e}_\theta + \frac{4}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi + 4m_1 \odot \mathbf{e_r} = \\ &\qquad \left[2\partial_\theta m_1 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi\right](I - \mathbf{e_r}\otimes\mathbf{e_r}),\end{aligned} \tag{2.31}$$

$$\begin{aligned}&18m_1\otimes m_1 + 4\partial_\theta m_2 \odot \mathbf{e}_\theta + \frac{4}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi + 4m_2 \odot \mathbf{e_r} \\ &\quad = \left[9|m_1|^2 + 2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi\right](I - \mathbf{e_r}\otimes\mathbf{e_r}).\end{aligned} \tag{2.32}$$

Equation (2.28) comes from the first equation in (2.8), equations (2.29) and (2.30) come from the second equation in (2.8), and equations (2.31) and (2.32) come from the extra equation (2.9).

Since $\psi(s_0)$ is a constant, (2.28) implies that

$$\partial_\theta s_2 = -\partial_\theta s_1, \quad \partial_\varphi s_2 = -\partial_\varphi s_1,$$

so that (2.30) becomes

$$12s_0m_2 + 6s_0m_1 + 12s_1m_1 = 6\partial_\theta s_1\mathbf{e}_\theta + 6\frac{\partial_\varphi s_1}{\sin\theta}\mathbf{e}_\varphi$$

that is, using (2.29),

$$12s_0m_2 + 24s_0m_1 + 12s_1m_1 = 0$$

from which we deduce an expression of m_2 in terms of s_0, s_1 and m_1:

$$m_2 = -\frac{2s_0 + s_1}{s_0}m_1. \tag{2.33}$$

Thus we compute, using also (2.31),

$$\begin{aligned}
&4\partial_\theta m_2 \odot \mathbf{e}_\theta + \frac{4}{\sin\theta}\partial_\varphi m_2 \odot \mathbf{e}_\varphi \\
&\quad = -\frac{2s_0+s_1}{s_0}\left(4\partial_\theta m_1 \odot \mathbf{e}_\theta + \frac{4}{\sin\theta}\partial_\varphi m_1 \odot \mathbf{e}_\varphi\right) \\
&\quad - \frac{1}{s_0}\left(\partial_\theta s_1 m_1 \odot \mathbf{e}_\theta + \frac{1}{\sin\theta}\partial_\varphi s_1 m_1 \odot \mathbf{e}_\varphi\right) \\
&\quad = -\frac{2s_0+s_1}{s_0}\left[2\partial_\theta m_1 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_1 \cdot \mathbf{e}_\varphi\right](I - \mathbf{e_r}\otimes\mathbf{e_r}) \\
&\quad + 4\frac{2s_0+s_1}{s_0} m_1 \odot \mathbf{e_r} \\
&\quad - \frac{1}{s_0}\left(\partial_\theta s_1 m_1 \odot \mathbf{e}_\theta + \frac{1}{\sin\theta}\partial_\varphi s_1 m_1 \odot \mathbf{e}_\varphi\right) \\
&\quad = \left[2\partial_\theta m_2 \cdot \mathbf{e}_\theta + \frac{2}{\sin\theta}\partial_\varphi m_2 \cdot \mathbf{e}_\varphi\right](I - \mathbf{e_r}\otimes\mathbf{e_r}) \\
&\quad + \frac{1}{2s_0}\left[\partial_\theta s_1 m_1 \cdot \mathbf{e}_\theta + \frac{1}{\sin\theta}\partial_\varphi s_1 m_1 \cdot \mathbf{e}_\varphi\right](I - \mathbf{e_r}\otimes\mathbf{e_r}) \\
&\quad - 4m_2 \odot \mathbf{e_r} \\
&\quad - \frac{1}{s_0}\left(\partial_\theta s_1 m_1 \odot \mathbf{e}_\theta + \frac{1}{\sin\theta}\partial_\varphi s_1 m_1 \odot \mathbf{e}_\varphi\right).
\end{aligned}$$

We plug this last computation into (2.32), which gives:

$$\begin{aligned}
&18m_1 \otimes m_1 - 9|m_1|^2(I - \mathbf{e_r}\otimes\mathbf{e_r}) = \\
&\quad \frac{1}{s_0}\left(\partial_\theta s_1 m_1 \odot \mathbf{e}_\theta + \frac{1}{\sin\theta}\partial_\varphi s_1 m_1 \odot \mathbf{e}_\varphi\right) \\
&\quad - \frac{1}{2s_0}\left[\partial_\theta s_1 m_1 \cdot \mathbf{e}_\theta + \frac{1}{\sin\theta}\partial_\varphi s_1 m_1 \cdot \mathbf{e}_\varphi\right](I - \mathbf{e_r}\otimes\mathbf{e_r}).
\end{aligned} \tag{2.34}$$

The identity (2.34) is an equality of symmetric (traceless) matrices, so it amounts to 5 scalar equalities. Actually only two of them are interesting (see Remark 2.8). In the sequel we are going to make use of (2.34) applied – as an equality of bilinear forms – to $(\mathbf{e}_\theta, \mathbf{e}_\theta)$ and $(\mathbf{e}_\theta, \mathbf{e}_\varphi)$, which gives the two following equations:

$$\begin{aligned}
18(m_1\cdot\mathbf{e}_\theta)^2 - 9|m_1|^2 &= \frac{1}{2s_0}\partial_\theta s_1 m_1\cdot\mathbf{e}_\theta - \frac{1}{2s_0\sin\theta}\partial_\varphi s_1 m_1\cdot\mathbf{e}_\varphi \\
18(m_1\cdot\mathbf{e}_\theta)(m_1\cdot\mathbf{e}_\varphi) &= \frac{1}{2s_0}\partial_\theta s_1 m_1\cdot\mathbf{e}_\varphi + \frac{1}{2s_0\sin\theta}\partial_\varphi s_1 m_1\cdot\mathbf{e}_\theta
\end{aligned} \tag{2.35}$$

Eventually we make use of (2.29) to transform (2.34) into equations involving only the derivatives of s_1.

Equation (2.29) may indeed be rewritten as

$$m_1 = -\frac{1}{3s_0}\partial_\theta s_1 \mathbf{e}_\theta - \frac{1}{3s_0\sin\theta}\partial_\varphi s_1 \mathbf{e}_\varphi.$$

Hence we have the following identities:

$$m_1 \cdot \mathbf{e}_\theta = -\frac{1}{3s_0}\partial_\theta s_1, \quad m_1 \cdot \mathbf{e}_\varphi = -\frac{1}{3s_0 \sin\theta}\partial_\varphi s_1,$$
$$|m_1|^2 = \frac{1}{9s_0^2}\left((\partial_\theta s_1)^2 + \frac{(\partial_\varphi s_1)^2}{\sin^2\theta}\right)$$

which we plug into (2.34) to obtain:

$$\frac{2}{s_0^2}(\partial_\theta s_1)^2 - \frac{1}{s_0^2}\left((\partial_\theta s_1)^2 + \frac{(\partial_\varphi s_1)^2}{\sin^2\theta}\right) = -\frac{1}{6s_0^2}(\partial_\theta s_1)^2 + \frac{1}{6s_0^2\sin^2\theta}(\partial_\varphi s_1)^2$$
$$\frac{2}{s_0^2\sin\theta}(\partial_\theta s_1)(\partial_\varphi s_1) = -\frac{1}{3s_0^2\sin\theta}(\partial_\theta s_1)(\partial_\varphi s_1)$$

i.e.

$$(\partial_\theta s_1)^2 - \frac{1}{\sin^2\theta}(\partial_\varphi s_1)^2 = 0$$
$$\frac{1}{\sin\theta}(\partial_\theta s_1)(\partial_\varphi s_1) = 0$$

Clearly, the last equations imply that

$$\partial_\theta s_1 = \partial_\varphi s_1 = 0,$$

which proves *Step 3.* □

2.6 Conclusions and perspectives

2.6.1 Conclusions

We have studied nematic equilibrium configurations under the constraint of uniaxial symmetry. The results we have obtained show that the constraint of uniaxial symmetry is very restrictive and should in general not be satisfied by equilibrium configurations, except in the presence of other strong symmetries.

We have shown that, for a nematic equilibrium configuration presenting translational invariance in one direction, there are only two options: either it does not have any regions with uniaxial symmetry, or it has uniform director field. In particular, when the boundary conditions prevent the director field from being uniform, as it is the case in hybrid cells or in capillaries with radial anchoring, then at equilibrium uniaxial order is destroyed spontaneously within the whole system. In other words, for translationally invariant configurations, biaxial escape has to occur.

Biaxiality had in fact been predicted in such geometries [130, 109, 26], but it was supposed to stay confined to small regions, and to occur only in some parameter range. Here we have provided a rigorous proof that biaxiality must occur everywhere, and for any values of the parameter: the configurations interpreted as uniaxial just correspond to a small degree of biaxiality. Our proof does not rely on free energy minimization, but only on the equilibrium equations – in particular it affects all metastable configurations. It is also remarkable that our results do not depend on the form of the bulk energy density, whereas all the previously cited workers used a four-terms approximation.

For general three-dimensional configurations we have not obtained a complete description of uniaxial equilibrium configurations, but we have studied the model case of the hedgehog defect, and obtained a strong symmetry result: a uniaxial equilibrium must be spherically symmetric. We believe in fact that, in general, the only non trivial uniaxial solutions of the equilibrium equation are spherically symmetric.

2.6.2 Perspectives

Many interesting problems concerning uniaxial equilibrium and biaxial escape remain open. We mention here three directions of further research.

The first one is the complete description of three-dimensional uniaxial solutions of (2.4). Techniques similar to the proof of Theorem 2.12 should allow to prove that, in a smooth bounded domain with normal anchoring, uniaxial solutions exist only if the domain has spherical symmetry. Such a result would constitute a first step towards the conjectured fact that the only non trivial uniaxial solution of (2.4) – whatever the form of the domain and the boundary conditions – are spherically symmetric. For more general boundary conditions however, other techniques would likely be needed.

Another open problem is to consider more general (and more physically relevant) elastic terms (see Remark 2.2). The equation (2.9) corresponding to equilibrium with respect to symmetry-breaking perturbations is more complicated in that case (in particular it is of second order).

A third problem, which is of even greater physical relevance, is to investigate “approximately uniaxial” equilibrium configurations. Hopefully, equation extra could play an interesting role in such a study.

Chapitre 3

Fuite biaxe à basse température

(avec Andres Contreras)

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3.1 Introduction

Nematic liquid crystals are composed of rigid rod-like molecules which tend to align in a preferred direction. As a result of this orientational order, nematics present electromagnetic properties similar to those of crystals. A striking feature of nematics is the appearance of particular optical textures called *defects*. From the mathematical point of view, the study of these defects is carried out using a tensorial order parameter Q (introduced by P.G. de Gennes [45]). The Q-tensor takes values in the five-dimensional space

$$\mathcal{S} = \left\{ Q \in \mathbb{R}^{3\times 3} \colon Q_{ij} = Q_{ji},\ \operatorname{tr} Q = 0 \right\}, \tag{3.1}$$

of symmetric traceless 3×3 matrices. As a symmetric matrix, a Q-tensor has an orthonormal frame of eigenvectors: the eigendirections are the locally preferred mean directions of alignment of the molecules, and the eigenvalues measure the degrees of alignment along those directions. In this context, *uniaxial* states are described by Q-tensors with two equal eigenvalues, and *biaxial* states correspond to Q-tensors with three distinct eigenvalues.

The configuration of a nematic material contained in a domain $\Omega \subset \mathbb{R}^3$ is given by a map $Q\colon \Omega \to \mathcal{S}$. At equilibrium, Q should minimize the Landau-de Gennes free energy given by

$$F_T(Q) = \int_\Omega \left(\frac{L}{2}|\nabla Q|^2 + f_T(Q) \right) dx. \tag{3.2}$$

Here L is an elastic constant and $f_T(Q)$ is the bulk free energy density, usually considered to be of the form

$$f_T(Q) = \frac{\alpha(T - T_*)}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4. \tag{3.3}$$

Above α, b and c are material-dependent positive constants, T is the absolute temperature and T_* a critical temperature. For $T < T_*$, the bulk free energy density $f_T(Q)$ attains its minimum exactly on the vacuum manifold $\mathcal{N}_T \subset \mathcal{S}$ composed of uniaxial Q-tensors with a certain fixed norm:

$$\begin{aligned} \mathcal{N}_T &= \left\{ Q \in \mathcal{S}\colon Q = s_* \left(n \otimes n - \frac{1}{3}I \right),\ n \in \mathbb{S}^2 \right\}, \\ s_* = s_*(T) &= \frac{b + \sqrt{b^2 - 24\alpha(T - T_*)c}}{4c}. \end{aligned} \tag{3.4}$$

Above, the notation $n \otimes n$ denotes the matrix $(n_i n_j)$. Note that $\mathcal{N}_T$ is diffeomorphic to the projective plane $\mathbb{RP}^2$. In this work we consider minimizers of $F_T(Q)$ subject to Dirichlet boundary conditions $Q_{b,T}\colon \partial\Omega \to \mathcal{N}_T$ minimizing the potential $f_T(Q)$:

$$Q_{b,T}(x) = s_* \left(n_b(x) \otimes n_b(x) - \frac{1}{3}I \right), \quad n_b\colon \partial\Omega \to \mathbb{S}^2. \tag{3.5}$$

In the London limit $L \to 0$, a minimizing Q-tensor must be close to an $\mathcal{N}_T$-valued harmonic map Q_*, that is a minimizer of the Dirichlet energy among maps with values in the manifold $\mathcal{N}_T$. This is analogous to the case of the simplified Ginzburg-Landau energy with prescribed topologically nontrivial boundary conditions studied in [25]; in this setting it is proved that minimizers of the corresponding energy converge to harmonic maps with values in $\mathbb{S}^1$, which are then forced to have singularities, known in that context as vortices.

The singularities of the director field n_* associated to the limit of minimizers of $F_T(Q)$ correspond to the optical defects observed in experiments. In the core of a defect, two possible behaviors are considered in the physics literature. The notion of *isotropic melting* refers to a Q-tensor vanishing in the core of the defect. This is comparable to the behaviour observed in the core of Ginzburg-Landau vortices, and can be achieved by remaining in a uniaxial state. Alternatively, Q-tensors may take advantage of the additional degrees of freedom offered by biaxiality: instead of vanishing in the core of the defect, the Q-tensor order parameter may become strongly biaxial. This last behaviour is referred to as *biaxial escape* [130].

Biaxial escape has been first proposed as a way to avoid singularities of the director field by Lyuksyutov [92]. The corresponding mechanism has been studied further by Penzenstadler and Trebin [111], followed by a number of numerical studies of this phenomenon (see for instance [130, 101]). They conclude that biaxial escape is energetically favorable when the bulk free energy (3.3) degenerates to a Ginzburg-Landau-like potential, which occurs for instance at low temperature.

Our main result states that, at low temperatures, isotropic melting is indeed avoided: the minimizing configurations do not vanish.

Theorem 3.1. *Let $\Omega \subset \mathbb{R}^3$ be a smooth bounded simply connected domain. Let $n_b\colon \partial\Omega \to \mathbb{S}^2$ be a smooth director field and $Q_{b,T}\colon \partial\Omega \to \mathcal{N}_T$ the associated boundary datum* (3.5)*. Let Q_T be a solution of the variational problem*

$$\min \left\{ F_T(Q)\colon Q \in H^1(\Omega;\mathcal{S}),\ Q = Q_{b,T} \text{ on } \partial\Omega \right\},$$

where F_T is the Landau-de Gennes free energy (3.2)*. Then, there exists $T_0 = T_0(\Omega, L, \alpha, b, c) \in \mathbb{R}$ such that if $T < T_0$,*

$$\inf_{\Omega} |Q_T| > 0,$$

i.e. Q_T does not vanish in Ω.

To prove Theorem 3.1, we use the fact that any zero x_T of Q_T must converge, as $T \to -\infty$, to a point $x_0 \in \Omega$; this follows from the analysis in [95]. After this, we take advantage of the degeneracy of the bulk potential to a Ginzburg-Landau potential in the low temperature limit. The Ginzburg-Landau potential being minimized by $\mathbb{S}^4$-valued maps, we are able to relate Q_T to an $\mathbb{S}^4$-valued harmonic map. This is done through a blow-up analysis of Q_T at x_T which in turn leads to a local minimization problem in $\mathbb{R}^3$ for a limiting map Q_∞. Next, thanks to the study in [97] based on the work of Lin and Wang [86], a blow-down analysis of the limiting map using the minimality of Q_∞ yields strong convergence to a harmonic map with values in $\mathbb{S}^4$. The conclusion follows with the help of a regularity result for minimizing harmonic maps by Schoen and Uhlenbeck [123].

Next we explain how Theorem 3.1 is related to the phenomenon of biaxial escape. Of course, Theorem 3.1 is only interesting if the boundary condition n_b is topologically non-trivial. In that case, a recent remark of Canevari [33, Lemma 3.10] shows that the only way for Q_T to avoid isotropic melting is to be *strongly* biaxial. To give a precise meaning to this statement, we recall the definition of the biaxiality parameter for a Q-tensor,

$$\beta(Q) = 1 - 6\frac{(\mathrm{tr}(Q^3))^2}{|Q|^6}, \tag{3.6}$$

introduced in [75]. It holds that $0 \le \beta(Q) \le 1$, and Q is uniaxial for $\beta = 0$, biaxial for $\beta > 0$ and is said to be *maximally biaxial* for $\beta = 1$. Canevari's lemma implies the following corollary to our main result:

Corollary 3.2. *If the boundary datum $n_b\colon \partial\Omega \to \mathbb{S}^2$ is topologically non-trivial, then for low enough temperatures $T < T_0$, any minimizing configuration Q_T must be strongly biaxial:*

$$\beta(Q_T(x_0)) = 1$$

for some $x_0 \in \Omega$.

In fact, in [33] Canevari uses the aforementioned lemma to prove a theorem similar to Corollary 3.2, in the case of a two-dimensional domain. Our result is a three-dimensional analog of [33, Theorem 1.1], and could probably be adapted to provide a simpler proof of [33, Theorem 1.1].

Corollary 3.2 generalizes a recent result by Henao, Majumdar and Pisante [94]. In [94], the authors show that for low enough temperature, minimizers can not be purely uniaxial (that is, can not satisfy $\beta = 0$ everywhere). Note that such a result does not exclude the existence of approximately uniaxial minimizers, while Corollary 3.2 does. Moreover the results of the second author in [83] indicate that the uniaxiality constraint is very rigid: non-existence of purely uniaxial solutions may not be specific to low temperature or energy minimization. In contrast, Corollary 3.2 is really specific to the low temperature limit.

The article is organized as follows. In Section 3.2 we reformulate the problem and recall some basic convergence properties of minimizers of F_T. In Section 3.3 we study the blown-up problem, obtain a limiting map and derive its minimal character. In Section 3.4 we conclude the proof of Theorem 3.1 with the aid of a blow-down analysis. Finally, in Section 3.5 we prove Corollary 3.2 and make some final remarks.

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3.2 Properties of minimizing Q-tensors

3.2.1 Rescaling

Introducing the reduced temperature t and rescaled maps $\widetilde{Q}$:

$$t := \frac{-\alpha(T - T_*)c}{b^2}, \quad \widetilde{Q} := \frac{1}{s_*}\sqrt{\frac{3}{2}}Q,$$

we see that, for some constant $K = K(\alpha, b, c, T)$ which plays no role in the sequel,

$$F_T(Q) = \frac{s_*^2 b^2}{3c} \int_\Omega \left(\frac{\widetilde{L}}{2}|\nabla \widetilde{Q}|^2 + \frac{t}{2}(|\widetilde{Q}|^2 - 1)^2 + \lambda(t)\, h(\widetilde{Q}) \right) dx \ + K,$$

where $\widetilde{L} = 3cL/b^2$,

$$\lambda(t) = \frac{\sqrt{24t+1}+1}{12} \underset{t\to+\infty}{\sim} \sqrt{\frac{t}{6}}, \tag{3.7}$$

and

$$h(\widetilde{Q}) = \frac{1}{6} - \frac{2\sqrt{2}}{\sqrt{3}}\mathrm{tr}(\widetilde{Q}^3) + \frac{1}{2}|\widetilde{Q}|^4. \tag{3.8}$$

It holds that $h(Q) \geq 0$ for every $Q \in \mathcal{S}$, and the potential h vanishes exactly at

$$\widetilde{\mathcal{N}} = \left\{ \sqrt{\frac{3}{2}} \left(n \otimes n - \frac{1}{3} I \right) : n \in \mathbb{S}^2 \right\}. \tag{3.9}$$

The limit $T \to -\infty$ corresponds to $t \to +\infty$. Therefore we may reformulate the problem: show that minimizers Q_t of the energy functional

$$\widetilde{F}_t(Q) = \int_\Omega \left(\frac{\widetilde{L}}{2} |\nabla Q|^2 + \frac{t}{2}(|Q|^2 - 1)^2 + \lambda(t)\, h(Q) \right) dx \tag{3.10}$$

subject to the boundary condition

$$Q_t = \widetilde{Q}_b = \sqrt{\frac{3}{2}} \left(n_b \otimes n_b - \frac{1}{3} I \right) \quad \text{on } \partial\Omega, \tag{3.11}$$

do not vanish for large enough t.

We prove Theorem 3.1 by contradiction: we assume the existence of sequences $t_j \to +\infty$ and $(x_j) \subset \Omega$ such that Q_{t_j} minimizes (3.10)-(3.11) and $Q_{t_j}(x_j) = 0$. Note that any minimizer of $\widetilde{F}_t$ is smooth thanks to standard elliptic estimates (see e.g. [95, Proposition 13]), so that evaluation at x_j makes sense. Up to extracting a subsequence, we may assume in addition that $x_j \to x_* \in \overline{\Omega}$.

In the sequel we study the behaviour of the sequence (Q_{t_j}) and obtain a contradiction. To simplify the notations, we drop the subscript j: we write (Q_t) and (x_t) and it is always implied that a subsequence is considered.

3.2.2 Convergence

Since the set $H^1_{n_b}(\Omega; \mathbb{S}^2) = \{n \in H^1(\Omega; \mathbb{S}^2) \colon n_{|\partial\Omega} = n_b\}$ is not empty, we may use an $\widetilde{\mathcal{N}}$-valued comparison map and obtain the bound

$$\widetilde{F}_t(Q_t) = \int_\Omega \left(\frac{\widetilde{L}}{2} |\nabla Q_t|^2 + \frac{t}{2}(|Q_t|^2 - 1)^2 + \lambda(t)\, h(Q_t) \right) dx \leq C. \tag{3.12}$$

In particular, we see that the sequence (Q_t) is bounded in $H^1(\Omega; \mathcal{S})$. Up to extracting a subsequence, we may therefore assume that Q_t converges weakly to a limiting map $Q_* \in H^1(\Omega; \mathcal{S})$. Moreover, since the bound (3.12) implies

$$\int_\Omega h(Q_t) \leq C\lambda(t)^{-1} \sim C\sqrt{\frac{6}{t}},$$

we deduce that $h(Q_*) = 0$ a.e., so that Q_* is $\widetilde{\mathcal{N}}$-valued. From this point on, we can proceed exactly as in [95, Lemma 3]. We conclude that Q_t converges to Q_* strongly in H^1, and that

$$Q_* = \sqrt{\frac{3}{2}} \left(n_* \otimes n_* - \frac{1}{3} I \right),$$

where $n_* \in H^1(\Omega;\mathbb{S}^2)$ is a minimizing harmonic map. In particular, Q_* is smooth in $\Omega \setminus \Sigma$, where $\Sigma \subseteq \Omega$ is a finite set of interior point singularities [121, 122].

As in the Ginzburg-Landau case [23], the convergence of Q_t towards Q_* can be improved away from the singularities Σ. The arguments in [23] have been adapted to the liquid crystal case in [95]. The asymptotic regime $L \to 0$ in [95] corresponds to the limit $t \to +\infty$ in the present work. The arguments in [95, Proposition 4] and [95, Proposition 6] are straightforward to adapt, and we obtain the convergence

$$\frac{1}{2}(|Q_t|^2-1)^2 + \frac{\lambda(t)}{t}\,h(Q_t) \longrightarrow 0, \quad \text{locally uniformly in } \overline{\Omega}\setminus\Sigma.$$

Since we have in addition, thanks to the maximum principle, $|Q_t| \leq 1$ (cf e.g. [95, Proposition 3]), we deduce – using also (3.7) – that

$$|Q_t| \longrightarrow 1 \quad \text{locally uniformly in } \overline{\Omega}\setminus\Sigma. \tag{3.13}$$

Recall that by assumption, $Q_t(x_t) = 0$ for a sequence $x_t \to x_* \in \overline{\Omega}$. The uniform convergence (3.13) away from Σ implies that $x_* \in \Sigma$. In particular x_* lies well inside Ω. Our next step will consist in "blowing up" around x_t.

3.3 Blowing up

We fix $\delta > 0$ such that $B(x_t,\delta) \subset \Omega$ for all j. We consider the blown-up maps

$$\overline{Q}_t(x) = Q_t\left(x_t + \frac{x}{\sqrt{t}}\right), \quad x \in B_{\delta\sqrt{t}}.$$

The map $\overline{Q}_t$ minimizes the energy functional

$$E_t(Q;B_{\delta\sqrt{t}}) = \int_{B_{\delta\sqrt{t}}}\left(\frac{\widetilde{L}}{2}|\nabla Q|^2 + \frac{1}{2}(|Q|^2-1)^2\right)dx + \frac{\lambda(t)}{t}\int_{B_{\delta\sqrt{t}}} h(Q)\,dx, \tag{3.14}$$

with respect to its own boundary conditions. Fix any $R > 0$. For large enough t, $\overline{Q}_t$ is defined in B_R and solves the Euler-Lagrange equation

$$\widetilde{L}\Delta\overline{Q}_t = 2(|\overline{Q}_t|^2-1)\overline{Q}_t + \frac{\lambda(t)}{t}\nabla h(\overline{Q}_t).$$

The uniform bound $|\overline{Q}_t| \leq 1$ and standard elliptic estimates thus imply

$$|\nabla\overline{Q}_t| \leq C_R \quad \text{in } B_R,$$

where C_R is a constant that may depend on R but not on t. Therefore, up to extracting a subsequence, we may assume that $\overline{Q}_t$ converges locally uniformly, and weakly in H^1_{loc}, to a map $Q_\infty \in H^1_{\mathrm{loc}}(\mathbb{R}^3;\mathcal{S})$. Moreover, since the convergence is locally uniform, Q_∞ is continuous and satisfies

$$Q_\infty(0) = 0. \tag{3.15}$$

We claim that Q_∞ locally minimizes a Ginzburg-Landau energy; this is a very important simplification.

Lemma 3.3. *For all $R > 0$, the limiting profile Q_∞ minimizes the energy functional*

$$E(Q;B_R) = \int_{B_R} \left(\frac{\widetilde{L}}{2}|\nabla Q|^2 + \frac{1}{2}(|Q|^2-1)^2 \right) dx, \tag{3.16}$$

with respect to its own boundary condition.

Proof. Let $P \in H_0^1(B_R;\mathcal{S})$. Since $\overline{Q}_t$ is minimizing, it holds

$$\begin{aligned} 0 &\le E_t(\overline{Q}_t + P; B_R) - E_t(\overline{Q}_t; B_R) \\ &= \widetilde{L}\int_{B_R} \nabla\overline{Q}_t \cdot \nabla P + \frac{\widetilde{L}}{2}\int_{B_R} |\nabla P|^2 \\ &\quad + \frac{1}{2}\int_{B_R} (|\overline{Q}_t + P|^2 - 1)^2 - \frac{1}{2}\int_{B_R} (|\overline{Q}_t|^2-1)^2 \\ &\quad + \frac{\lambda(t)}{t}\int_{B_R} \left[h(\overline{Q}_t + P) - h(\overline{Q}_t)\right] dx. \end{aligned}$$

Using the weak H^1 convergence of $\overline{Q}_t$ (which implies also strong L^6 convergence), we obtain in the limit $t \to +\infty$

$$\begin{aligned} 0 &\le \widetilde{L}\int_{B_R} \nabla Q_\infty \cdot \nabla P \, dx + \frac{\widetilde{L}}{2}\int_{B_R} |\nabla P|^2 \\ &\quad + \frac{1}{2}\int_{B_R} (|Q_\infty + P|^2 - 1)^2 - \frac{1}{2}\int_{B_R} (|Q_\infty|^2-1)^2 \\ &= E(Q_\infty + P; B_R) - E(Q_\infty; B_R). \end{aligned}$$

Therefore Q_∞ minimizes (3.16), as claimed. □

Moreover, proceeding exactly as in the proof of [94, Theorem 1.(v)], we obtain the energy bound

$$E(Q_\infty; B_R) \le CR. \tag{3.17}$$

The bound (3.17) follows from two main ingredients: an energy monotonicity inequality for minimizers of (3.14) [95, Lemma 2], and an energy bound for $\mathbb{S}^2$-valued minimizing harmonic maps near their singularities (following from the energy monotonicity for minimizing harmonic maps, see e.g. [87, Lemma 2.2.5]).

3.4 Blowing down

Our last step consists in "blowing down" Q_∞ around the origin, and eventually reaching a contradiction with (3.15). Let B_1 be the unit ball in $\mathbb{R}^3$. We consider the blown-down maps

$$\underline{Q}_R(x) = Q_\infty(Rx), \quad x \in B_1.$$

Note that (3.15) implies that

$$\underline{Q}_R(0) = 0, \quad \forall R > 0. \tag{3.18}$$

By definition, $\underline{Q}_R \in H^1(B_1)$ for all $R > 0$. We have:

Lemma 3.4. *Up to a subsequence,*

$$\underline{Q}_R \longrightarrow \underline{Q} \quad \text{in } H^1(B_1;\mathcal{S}),$$

for some $\mathbb{S}^4$*-valued harmonic map* $\underline{Q}$*. Moreover,* $|\underline{Q}_R|$ *stays bounded away from zero uniformly in* B_1*.*

Proof. Since Q_∞ minimizes (3.16), the map $\underline{Q}_R$ minimizes the energy functional

$$G_R(Q) = \int_{B_1} \left(\frac{\widetilde{L}}{2}|\nabla Q|^2 + \frac{R^2}{2}(|Q|^2-1)^2 \right) dx. \tag{3.19}$$

Moreover, the energy bound (3.17) implies the bound

$$G_R(\underline{Q}_R) \le C, \tag{3.20}$$

so that we may extract a subsequence $R \to +\infty$ (indices are implicit), such that

$$\underline{Q}_R \longrightarrow \underline{Q} \quad \text{weakly in } H^1(B_1;\mathcal{S}). \tag{3.21}$$

The energy bound (3.20) also implies that $\underline{Q}$ is $\mathbb{S}^4$-valued. Now, thanks to Lemma 3.3, we can appeal to Proposition 4.2 in [97] to conclude that the convergence of $\underline{Q}_R$ to $\underline{Q}$ can be improved to strong convergence in H^1. In [97], the proof relies on [86, Theorem C] in the case of $\mathbb{R}^3$-valued maps converging to $\mathbb{S}^2$-valued maps. However, [86, Theorem C] is valid in greater generality and applies to our case. Moreover, the analysis in [97] does not make use of the dimension of the target space other than to provide an explicit constant in their computations.

Next, the minimizing character of $\underline{Q}$ follows from Step 1 in [97, Corollary 4.1], which also applies to our case without modifications. From this we conclude that $\underline{Q}$ is an $\mathbb{S}^4$-valued minimizing harmonic map. As a consequence, Schoen and Uhlenbeck's regularity result [123, Theorem 2.7] ensures that $\underline{Q}$ is smooth in B_1.

Since the proof of [97, Proposition 4.2] also shows that the convergence of $\underline{Q}_R$ towards $\underline{Q}$ is actually uniform away from the singularities of $\underline{Q}$, we obtain in particular that

$$|\underline{Q}_R| \longrightarrow 1 \quad \text{uniformly in } B_1, \tag{3.22}$$

which is the desired conclusion. □

We note that (3.22) contradicts (3.18) and thus the proof of Theorem 3.1 is complete. □

3.5 Proof of Corollary 3.2

In [33], Canevari makes the crucial observation that if Q is *almost uniaxial*, i.e.

$$\max_{\overline{\Omega}} \beta(Q) < 1,$$

then the Q-tensor must vanish. More precisely, in our case the following result holds.

Lemma 3.5. *[33, Lemma 3.10] Let $Q \in C^1(\overline{\Omega}; \mathcal{S})$ with uniaxial boundary condition of the form* (3.5). *If $n_b \colon \partial\Omega \to \mathbb{S}^2$ is topologically non trivial, and Q is almost uniaxial, then*

$$\min_{\overline{\Omega}} |Q| = 0.$$

In [33] the proof is carried out in the two-dimensional case but a careful reading shows that the argument still holds in the three-dimensional setting, the result depending only on topological considerations in the target space $\mathcal{S}$. Indeed, the crucial observation leading to [33, Lemma 3.10] is the fact that, for any $C \geq 1$ and $1 > \delta > 0$, the set

$$\{Q \in \mathcal{S} \colon \delta \leq |Q| \leq C,\ \beta(Q) \leq 1 - \delta\} \subset \mathcal{S}$$

is topologically equivalent to $\mathcal{N} \simeq \mathbb{RP}^2$.

As a consequence of Theorem 3.1 we see that, in light of Lemma 3.5, Q_T must be maximally biaxial at some point for sufficiently low temperature. The proof of Corollary 3.2 is complete. □

We finish with a few remarks. Theorem 3.1 implies the existence of a point where maximal biaxiality is achieved, however it does not provide a characterization of the location of this (or these) point(s) in terms of the domain or the boundary datum. Also the number of these points of biaxial escape cannot be deduced from the topological conclusion in [33, Lemma 3.10]. To finish, a more detailed description of the defect core is also an interesting matter worthy of pursuit. In this last direction, we mention the stability study of the radial hedgehog defect performed in [72].

Chapitre 4

Structure nématique autour d'une particule colloïdale

(avec Stan Alama et Lia Bronsard)

Sommaire

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4.1 Introduction

Liquid crystals are well-known for their many applications in optical devices. The rod-like molecules in a nematic liquid crystal tend to align in a common direction: the resulting orientational order produces an anisotropic fluid with remarkable optical features. This anisotropy also makes it highly interesting to use nematic liquid crystals in colloidal suspensions. Immersion of colloid particles into a nematic system disturbs the orientational order and creates topological defects, which enforce fascinating self-assembly phenomena [113, 104], with many potential applications [125, 112]. This sensitivity to inclusion of small foreign bodies also has promising biomedical applications [141]: for instance, new biological sensors could detect very quickly the presence of microbes, based on the induced change in nematic order [129, 66, 71].

In the present paper we investigate the structure of the nematic order around one spherical particle, with homeotropic (*i.e.* normal) anchoring at the particle surface, and uniform alignment far away from it. The homeotropic anchoring creates a topological charge. This charge has to be balanced in order to match the uniform alignment at infinity, which is topologically trivial. Therefore one expects to observe singularities.

This particular problem is a crucial step in understanding more complex situations, and it has received a lot of attention in the past two decades [136, 80, 91, 131, 115]. These works rely on heuristically supported approximations, and numerical computations. They point out two possible types of configurations, with « dipolar » or « quadrupolar » symmetry (related to their far-field behavior [132, § 4.1]). In a dipolar configuration the topological charge created by the particle is balanced by a point defect, while in a quadrupolar configuration it is balanced by a « Saturn ring » defect around the particle.

The aforementioned works use either Oseen-Frank theory [136, 80, 91, 131] or Landau-de Gennes theory [115] to describe nematic alignment. In Oseen-Frank theory, the order parameter is a director field $n(x) \in \mathbb{S}^2$, which minimizes an elastic energy. One drawback of that model is that line defects have infinite energy. In particular, the energy of a quadrupolar configuration with Saturn ring defect has to be renormalized. Moreover, Oseen-Frank theory only accounts for uniaxial nematic states: it assumes local axial symmetry of the alignment around the average director. On the other hand Landau-de Gennes theory involves a tensorial order parameter that can also describe biaxial states, in which the local axial symmetry is broken. This is the model that we will be using here.

The order parameter in Landau-de Gennes theory is the so-called Q-tensor, which belongs to the space

$$\mathcal{S}_0 := \{Q \in M_3(\mathbb{R})\colon Q_{ij} = Q_{ji},\ \mathrm{tr}(Q) = 0\}\,, \tag{4.1}$$

of symmetric traceless 3×3 matrices. The eigenvectors of Q represent the average directions of alignment of the molecules, and the associated eigenvalues measure the degree of alignment along these directions. Uniaxial states are described by Q-tensors with two equal eigenvalues, which can be put in the form

$$Q = s\left(n \otimes n - \frac{1}{3}I\right), \quad s \in \mathbb{R},\ n \in \mathbb{S}^2.$$

Biaxial states correspond to the generic case of a Q-tensor with three distinct eigenvalues.

The configuration of a nematic material contained in a domain $\Omega \subset \mathbb{R}^3$ is described by a map $Q\colon \Omega \to \mathcal{S}_0$. At equilibrium it should minimize the free energy functional

$$\mathcal{F}(Q) = \int_\Omega \left[\frac{L}{2}\left|\nabla Q\right|^2 + f(Q)\right] dx + \mathcal{F}_s(Q). \tag{4.2}$$

Here $\mathcal{F}_s(Q)$ is a surface energy term which depends on the type of anchoring (see (4.6) below), and the bulk potential $f(Q) \geq 0$ is given by

$$f(Q) = -\frac{a}{2}\,\mathrm{tr}(Q^2) - \frac{b}{3}\,\mathrm{tr}(Q^3) + \frac{c}{4}\,\mathrm{tr}(Q^2)^2 + C_0, \tag{4.3}$$

for some material-dependent constants $a \geq 0,\ b, c > 0$. The constant C_0 is chosen to ensure that $\min f = 0$. The set of $Q \in \mathcal{S}_0$ minimizing the potential (4.3) obviously plays a crucial role. It consists exactly of those Q-tensors which are uniaxial, with fixed eigenvalues:

$$\mathcal{U}_* := \{f = 0\} = \left\{s_*\left(n \otimes n - \frac{1}{3}I\right) : n \in \mathbb{S}^2\right\}, \tag{4.4}$$

where $s_* = (b + \sqrt{b^2 + 24ac})/4c > 0$.

We are interested here in the nematic configuration around a spherical particle:

$$\Omega = \Omega_{r_0} := \mathbb{R}^3 \setminus \overline{B}_{r_0},$$

where $r_0 > 0$ is the particle radius. We impose uniform $\mathcal{U}_*$-valued conditions at infinity

$$\lim_{|x|\to\infty} Q(x) = Q_\infty := s_* \left(e_z \otimes e_z - \frac{1}{3} I \right), \qquad e_z = (0,0,1). \tag{4.5}$$

At the particle surface, weak radial anchoring is enforced through the surface term $\mathcal{F}_s$ in the free energy functional (4.2). This surface contribution is given by

$$\mathcal{F}_s(Q) = \frac{W}{2} \int_{\partial B_{r_0}} |Q_s - Q|^2 \, dA, \tag{4.6}$$

where $W > 0$ is the anchoring strength, and Q_s is the $\mathcal{U}_*$-valued radial map

$$Q_s := s_* \left(e_r \otimes e_r - \frac{1}{3} I \right), \qquad e_r = \frac{x}{|x|}. \tag{4.7}$$

Denoting by ν the exterior normal to Ω_{r_0}, the corresponding boundary conditions are

$$\frac{L}{W} \frac{\partial Q}{\partial \nu} = Q_s - Q \quad \text{for } |x| = r_0. \tag{4.8}$$

We also include in this description the case of strong anchoring, corresponding to $W = +\infty$ and Dirichlet boundary conditions

$$Q = Q_s \quad \text{for } |x| = r_0. \tag{4.9}$$

In every case, the Euler-Lagrange equations

$$L \Delta Q = \nabla f(Q) = -aQ - b \left(Q^2 - \frac{1}{3} |Q|^2 I \right) + c\,|Q|^2 Q, \tag{4.10}$$

are satisfied in $\mathcal{D}'(\Omega_{r_0}; \mathcal{S}_0)$ by any equilibrium configuration.

The existence of minimizers of the free energy functional (4.2) with the uniform far-field condition (4.5) can be obtained if we replace the pointwise condition (4.5) with the integrability condition

$$\int_{\Omega_{r_0}} \frac{|Q_\infty - Q|^2}{|x|^2} \, dx < \infty. \tag{4.11}$$

The Euler-Lagrange equations (4.10) can then be used to see that the strong condition (4.5) is in fact also satisfied. After establishing this existence result in Section 4.2, we turn to studying the two asymptotics regimes of « small » or « large » colloid particle.

According to the numerical computations in [131, 115], small particles favor quadrupolar configurations with a defect ring, while large particles favor dipolar configurations with a point defect. In the present paper we obtain rigorous justifications of these observations. In the small particle regime we also provide exact information on the radius of the defect ring, for which the values computed in [136, 80, 91, 131, 115] did not agree.

More precisely, whether the particle is small or large depends on the ratio r_0^2/L. In addition we have to pay attention to the ratio r_0W/L which affects the anchoring at the particle surface. In Section 4.3 we investigate the small particle regime, and prove:

Theorem 4.1. *Consider (for any r_0, W or L) a map Q, finite-energy solution of* (4.10)-(4.8)-(4.11). *Let $w \in (0, +\infty]$ be the effective « limiting anchoring strength ». Then, as*

$$\left(\frac{r_0^2}{L}, \frac{r_0 W}{L}\right) \longrightarrow (0, w),$$

the rescaled maps $x \mapsto Q(r_0 x)$ converge to

$$Q_0 = s_* \frac{w}{3+w}\frac{1}{r^3}\left(e_r \otimes e_r - \frac{1}{3}I\right) + s_*(1 - \frac{w}{1+w}\frac{1}{r})\left(e_z \otimes e_z - \frac{1}{3}I\right),$$

locally uniformly in $\overline{\Omega}_1$.

While the proof of the convergence is quite standard, an interesting feature of Theorem 4.1 is the explicit form of the limit: it provides a very precise description of the quadrupolar configurations. The Saturn ring defect can be interpreted as the locus of uniaxiality of the limiting map Q_0. This is consistent with the « eigenvalue exchange » mechanism taking place in the biaxial core of the line defect [115]. The ratio of the ring radius to the particle size is thus found to be, for $w > \sqrt{3}$, the solution $r > 1$ of

$$r^3 - \frac{w}{1+w}r^2 - \frac{w}{3+w} = 0.$$

For the strong anchoring $w = \infty$, its value is $r \approx 1.47$. As the anchoring strength decreases, the ring shrinks until it becomes a surface ring for $w = \sqrt{3}$. At very weak anchoring $w < \sqrt{3}$ there is no defect ring anymore. This description is consistent with [136, 80, 91, 131, 115], with the significant improvement of providing exact values for the relevant quantities.

The large particle regime is more delicate to analyse. We restrict ourselves to minimizers of the free energy, and to strong anchoring – that is, $W = \infty$. For the rescaled maps $x \mapsto Q(r_0 x)$, the regime $r_0^2/L \gg 1$ corresponds to the vanishing elastic constant limit studied in [95, 105]. There, the authors prove convergence to a $\mathcal{U}_*$-valued map whose director is an $\mathbb{S}^2$-valued minimizing harmonic map. It is well-known that such a map has a discrete set of singularities [123], and that these defects carry topological degrees ± 1 [32]. Since the strong radial anchoring imposes a degree $+1$ near the particle, while the uniform far-field condition imposes a zero degree at infinity, there must be at least one defect of degree -1. However the number of defects of a minimizing map does not necessarily correspond to the minimal number of defects required by the topology [61].

There are very few cases in which the number of defects is actually known to match the topological degree. This is true in a ball with radial Dirichlet boundary conditions, because then the energy of the radial map can be explicitly computed and seen to coincide with a general lower bound [32], and this is true also for geometries close enough to the radial one [62].

In our case we would like to show that there is exactly one defect, as predicted by [91, 131, 115]. Since determining the exact number of defects is a very difficult question in general, we restrict ourselves to axially symmetric configurations: we impose invariance under any rotation of vertical axis, and that e_θ (horizontal unit vector orthogonal to the radial direction) be everywhere an eigenvector of the Q-tensor. This natural symmetry assumption seems to be supported by the numerical pictures in [115].

In the limit we will therefore obtain an axially symmetric $\mathbb{S}^2$-valued harmonic map. Such maps have been studied in [60, 64]. They are analytic away from a discrete set of defects on the z-axis. For very particular symmetric boundary data, it can be deduced from rearrangement inequalities that the number of defects matches the topological degree [60, Theorem 5.1]. This result does not apply to our case, but – using different arguments – we nevertheless manage to show that there is exactly one defect, thus justifying the dipole configuration predicted by [91, 131, 115]. More precisely, in Section 4.4 we prove:

Theorem 4.2. *Let Q minimize the free energy* (4.2) *among axially symmetric maps satisfying the boundary conditions* (4.9)-(4.11). *Then, as r_0^2/L goes to $+\infty$, a subsequence of the rescaled maps $x \mapsto Q(r_0 x)$ converges to a map*

$$Q_*(x) = s_*(n(x) \otimes n(x) - I/3),$$

locally uniformly in $\overline{\Omega}_1 \setminus \{p_0\}$. Here n minimizes the Dirichlet energy in Ω_1, among axially symmetric $\mathbb{S}^2$-valued maps satisfying the boundary conditions

$$n = e_r \text{ on } \partial B_1, \qquad \text{and} \int_{\Omega_1} \frac{(n_1)^2 + (n_2)^2}{|x|^2}\, dx < \infty,$$

and n is analytic away from ***exactly one point defect*** *p_0.*

The core of Theorem 4.2 is the fact that the minimizing harmonic map n admits at most one defect. We achieve this by investigating the topology of the sets $\{n_3 > 0\}$ and $\{n_3 < 0\}$ where n points « more upward » or « more downward ». Using basic energy comparison arguments and the analyticity of minimizers away from the z-axis, we show that these sets are connected. Merging this with the observation that defects correspond to « jumps » between upward- and downward-pointing n, we conclude that there cannot be more than one defect.

The plan of the paper is as follows. In Section 4.2 we prove the existence of minimizers and some basic properties. In Section 4.3 we investigate the small particle regime and the quadrupolar « Saturn ring » configurations. In Section 4.4 we study the large particle regime and the associated axially symmetric harmonic map problem.

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Notations: We will use cylindrical coordinates (ρ, θ, z) defined by

$$x_1 = \rho \cos\theta, \quad x_2 = \rho \sin\theta, \quad x_3 = z,$$

and the associated orthonormal frame (e_ρ, e_θ, e_z), where

$$e_\rho = (\cos\theta, \sin\theta, 0), \quad e_\theta = (-\sin\theta, \cos\theta, 0).$$

We will also use spherical coordinates (r, θ, φ) with (r, φ) defined by

$$\rho = r \sin\varphi, \quad z = r\cos\varphi.$$

4.2 Existence and first properties of minimizers

We start by remarking that, with $\widetilde{Q}(x) := Q(r_0 x)$, it holds

$$\frac{1}{r_0^3}\mathcal{F}(Q) = \int_{\Omega_1} \left[\frac{L}{2r_0^2} |\nabla \widetilde{Q}|^2 + f(\widetilde{Q}) \right] dx + \frac{W}{2r_0} \int_{\partial B_1} |Q_s - \widetilde{Q}|^2 dA.$$

We may always work in this rescaled setting, and **we assume from now on that $\boldsymbol{r_0 = 1}$**: we consider the domain

$$\Omega := \Omega_1 = \mathbb{R}^3 \setminus \overline{B},$$

where $B = B_1$ is the ball of radius 1. The size of the particle is then encoded in the elastic constant L.

As mentioned in the introduction, an appropriate functional setting to establish the existence of minimizers is the affine Hilbert space

$$\begin{aligned} \mathcal{H}_\infty &:= Q_\infty + \mathcal{H}, \\ \mathcal{H} &:= \left\{ Q \in H^1_{loc}(\Omega; \mathcal{S}_0) \colon \int_\Omega |\nabla Q|^2 + \int_\Omega \frac{|Q|^2}{|x|^2} < \infty \right\}. \end{aligned} \tag{4.12}$$

Note that the free energy functional (4.2) is not everywhere finite on the space $\mathcal{H}_\infty$, since the potential term $f(Q) \geq 0$ may very well not be integrable in Ω. However, since $f(Q_\infty) = 0$, we do know that

$$\inf_{\mathcal{H}_\infty} \mathcal{F} < \infty.$$

In fact, we show that the infimum is attained, and that the minimizer has a limit at infinity:

Proposition 4.3. *Let $L > 0$ and $W \in [0, +\infty]$. Then there exists $Q \in \mathcal{H}_\infty$ such that*

$$\mathcal{F}(Q) = \inf_{\mathcal{H}_\infty} \mathcal{F}.$$

Moreover, the far-field condition holds in the strong sense (4.5)*, and this is true for any solution of the Euler-Lagrange equation* (4.10) *in $\mathcal{H}_\infty$.*

Remark 4.4. The case $W = +\infty$ has to be understood as the strong anchoring case: it amounts to considering only maps $Q \in \mathcal{H}_\infty$ which satisfy the Dirichlet boundary condition (4.9) in the sense of traces.

Proof of Proposition 4.3: Existence follows from the direct method of the calculus of variations. Thanks to Hardy's inequality, any minimizing sequence (Q_n) is bounded in $\mathcal{H}_\infty$ and admits (up to taking a subsequence) a weak limit $Q \in \mathcal{H}_\infty$. We may also assume that the convergence $Q_n \to Q$ holds almost everywhere. Convexity and Fatou's lemma allow to conclude that $\mathcal{F}(Q) \leq \liminf \mathcal{F}(Q_n) = \inf \mathcal{F}$.

The limit at infinity follows from estimates for solutions of the Euler-Lagrange equation (4.10). From Lemma 4.5 below we know that $Q \in L^\infty(\Omega)$, and (4.10) readily implies that $\Delta Q \in L^\infty(\Omega)$. Using standard elliptic estimates, we deduce that $\nabla Q \in L^\infty(\Omega)$, so that Q is uniformly continuous. Since on the other hand Sobolev inequality implies that $|Q - Q_\infty|$ belongs to $L^6(\Omega)$, we conclude that $|Q(x) - Q_\infty|$ converges to zero as $|x|$ goes to $+\infty$. □

In the proof of Proposition 4.3 we used the following L^∞ bound for solutions of (4.10), related to the growth of the potential $f(Q)$.

Lemma 4.5. *If $Q \in \mathcal{H}_\infty$ solves* (4.10)*, it holds*

$$\|Q\|_{L^\infty(\Omega)} \leq q_0,$$

for some $q_0 > 0$ that depends only on a, b and c (but not on L and W).

Proof. Let $\widetilde{Q} := Q - Q_\infty$, so that $\widetilde{Q} \in \mathcal{H}$ solves

$$L\Delta\widetilde{Q} = \nabla f(Q_\infty + \widetilde{Q}). \tag{4.13}$$

We may use as a test function in (4.13) any function $\Psi \in \mathcal{H}$ with compact support in $\overline{\Omega}$. Let us consider a test function

$$\Psi = V\widetilde{Q}, \quad \text{for some } V \geq 0 \text{ with compact support in } \overline{\Omega}.$$

Multiplying (4.13) by Ψ we find

$$L\int_\Omega \nabla\widetilde{Q} \cdot \nabla\Psi = -\int_\Omega V\nabla f(Q_\infty + \widetilde{Q}) \cdot \widetilde{Q} + \int_{\partial\Omega} V\widetilde{Q} \cdot \frac{\partial\widetilde{Q}}{\partial\nu}. \tag{4.14}$$

Clearly there exists $\tilde{q}_0 = \tilde{q}_0(a,b,c) > 0$ such that

$$\nabla f(Q_\infty + \widetilde{Q}) \cdot \widetilde{Q} \geq 0 \quad \text{for } |\widetilde{Q}| \geq \tilde{q}_0, \qquad \text{and} \quad |Q_s - Q_\infty| \leq \tilde{q}_0.$$

Now let us take V of the form

$$V = U\varphi, \quad U = \min\Big((|\widetilde{Q}|^2 - \tilde{q}_0^2)_+, M\Big), \quad 0 \leq \varphi \in C_c^\infty(\overline{\Omega}). \tag{4.15}$$

Note that V is non-negative and supported inside the set $\{|\widetilde{Q}| \geq \tilde{q}_0\}$.

Thanks to the choice of $\tilde{q}_0$, both terms in the right-hand side of (4.14) are non-positive: for the first term this is clear, and the second term is zero in the case of strong anchoring (4.9) and non-positive in the case of weak anchoring (4.8) because $|\widetilde{Q}|^2 - \widetilde{Q} \cdot (Q_s - Q_\infty) \geq 0$ for $|\widetilde{Q}| \geq \tilde{q}_0$. Thus we obtain

$$L\int_\Omega \nabla\widetilde{Q} \cdot \nabla\Psi \leq 0,$$

i.e.

$$\int_\Omega \varphi \left(U|\nabla\widetilde{Q}|^2 + \frac{1}{2}\,|\nabla U|^2 \right) \leq -\int_\Omega U\widetilde{Q}\cdot\nabla\widetilde{Q}\cdot\nabla\varphi.$$

Next we take $\varphi = \varphi_R$ such that

$$\varphi_R(x) = \begin{cases} 1 & \text{for } |x| \leq R, \\ 0 & \text{for } |x| \geq 2R, \end{cases} \qquad \text{and } |\nabla\varphi_R(x)| \leq \frac{C}{|x|},$$

for some constant $C > 0$ independent of R. We obtain

$$\int_\Omega \varphi_R \left(U|\nabla\widetilde{Q}|^2 + \frac{1}{2}\,|\nabla U|^2 \right) \leq MC\|\nabla\widetilde{Q}\|_{L^2(|x|\geq R)}\|\widetilde{Q}/r\|_{L^2(|x|\geq R)}.$$

Since $\widetilde{Q} \in \mathcal{H}$, the right hand side converges to zero as R goes to $+\infty$ and it holds

$$\int_\Omega \left(U|\nabla\widetilde{Q}|^2 + \frac{1}{2}\,|\nabla U|^2 \right) = 0$$

Recalling the definition (4.15) of U, we conclude that $|\widetilde{Q}| \leq \tilde{q}_0$ a.e., and therefore $\|Q\|_{L^\infty} \leq \tilde{q}_0 + s_*\sqrt{2/3}$. □

Remark 4.6. It would be interesting to prove an explicit convergence rate for the far-field behavior (4.5). The small particle limit (cf. Section 4.3) suggests a bound of the form $|Q(x) - Q_\infty| \leq C/\,|x|$, for some $C = C(a, b, c, L) > 0$. An indication that this bound could indeed be true is given by the estimate

$$\operatorname{dist}(Q(x), \mathcal{U}_*) \leq \frac{C}{|x|}, \quad C = C(a, b, c, L) > 0, \tag{4.16}$$

which holds for any solution $Q \in \mathcal{H}^\infty$ with finite potential energy $\int_\Omega f(Q) < \infty$. It can be proven using the ideas in [127] and estimates in [95]. More precisely, consider the map $Q_R(x) = Q(Rx)$ for $x \in A := B_3 \setminus B_2$. Then, as $R \to \infty$, Q_R converges towards Q_∞ in $H^1(A)$ and $R^2 \int_A f(Q_R)$ converges to zero. Following [95, § 4], this implies the estimate $f(Q_R) \leq C\,R^{-2}$. Since on the other hand it is not hard to see that $\operatorname{dist}(Q, \mathcal{U}_*)^2 \leq f(Q)$ for all $Q \in \mathcal{S}_0$, we deduce the bound (4.16). Note that, as in [95, Proposition 7], explicit bounds on $|Q(x)|$ and the eigenvalues $\lambda_j(Q(x))$ can be deduced from (4.16).

4.3 Small particle: the Saturn ring

This section is dedicated to proving Theorem 4.1, and then studying the limiting configuration Q_0, whose expression we recall here:

$$Q_0 = s_* \frac{w}{3+w}\frac{1}{r^3}\left(e_r \otimes e_r - \frac{1}{3}I \right) + s_*(1 - \frac{w}{1+w}\frac{1}{r})\left(e_z \otimes e_z - \frac{1}{3}I \right). \tag{4.17}$$

Proof of Theorem 4.1. Recall that we assume $r_0 = 1$, so that we are in fact taking the limit $(1/L, W/L) \to (0, w)$. It is straightforward to check that the map Q_0 (4.17) belongs to $\mathcal{H}_\infty$ and solves

$$\begin{cases} \Delta Q_0 = 0 \quad \text{in } \Omega, \\ \dfrac{1}{w}\dfrac{\partial Q_0}{\partial \nu} = Q_s - Q_0 \quad \text{on } \partial\Omega. \end{cases} \tag{4.18}$$

In what follows we emphasize the dependence of Q_0 on the parameter $w > 0$ by writing $Q_0 = Q_{0,w}$. Consider the map $\overline{Q} := Q - Q_{0,W/L}$, which solves

$$\begin{cases} \Delta\overline{Q} = \dfrac{1}{L}\nabla f(Q) & \text{in } \Omega, \\ \dfrac{L}{W}\dfrac{\partial\overline{Q}}{\partial\nu} + \overline{Q} = 0 & \text{on } \partial\Omega. \end{cases} \tag{4.19}$$

Next we apply an analog of the interpolated estimate of [23, Lemma A.2] for the oblique derivative problem (boundary conditions $\partial u/\partial\nu + \alpha u = 0$) in Ω: it holds

$$\left\|\nabla\overline{Q}\right\|_{L^\infty}^2 \leq C \left\|\Delta\overline{Q}\right\|_{L^\infty} \left\|\overline{Q}\right\|_{L^\infty}. \tag{4.20}$$

This estimate can be proven exactly as in [23]: apply [23, Lemma A.1] which does not need to be adapted, and then follow the proof of [23, Lemma A.2] using elliptic L^p estimates for the oblique derivative problem near $\partial\Omega$ (see e.g. [2, § 15] or [37]) instead of the homogeneous Dirichlet problem. The constant $C > 0$ depends only on $\delta > 0$ such that $w \geq \delta$.

Since we can infer from Lemma 4.5 that $\|\nabla f(Q)\|_{L^\infty} \leq C$ (where $C > 0$ depends only on a, b and c), the estimate (4.20) and the equation (4.19) imply that

$$\left\|\nabla\overline{Q}\right\|_{L^\infty} \leq \frac{C}{\sqrt{L}}, \tag{4.21}$$

from which we deduce that for any compact $K \subset \overline{\Omega}$, it holds

$$\left\|\overline{Q}\right\|_{L^\infty(K)} \leq C\,\frac{\operatorname{diam}(K)}{\sqrt{L}},$$

with the constant $C > 0$ depending on a, b, c and $\delta > 0$ such that $w \geq \delta$. On the other hand it can be easily checked that, at any order $k \in \mathbb{N}$,

$$\left\|Q_{0,W/L} - Q_{0,w}\right\|_{C^k} \leq c_k \left|1/w - L/W\right|,$$

so that $Q - Q_0 = \overline{Q} + Q_{0,W/L} - Q_{0,w}$ converges to zero locally uniformly in $\overline{\Omega}$, as $(1/L, W/L) \to (0, w)$. □

The main interest of Theorem 4.1 lies in the explicit expression for the limit Q_0. Next we investigate its most important features. We start by interpreting the Saturn ring defect as a locus of uniaxiality. More precisely, let U_w be the uniaxial locus of $Q_0 = Q_{0,w}$ (4.17) deprived from the z-axis (on which $e_r = \pm e_z$ and Q_0 is trivially uniaxial):

$$U_w := \{x \in \Omega \setminus \mathbb{R}e_z \colon Q_0(x) \text{ is uniaxial}\}. \tag{4.22}$$

Then we have:

Proposition 4.7. *For $w > \sqrt{3}$ it holds*

$$U_w = \left\{(x_1, x_2, 0) \colon x_1^2 + x_2^2 = r_w^2\right\}, \tag{4.23}$$

where r_w is the unique solution $r > 1$ of

$$r^3 - \frac{w}{1+w}r^2 - \frac{w}{3+w} = 0. \tag{4.24}$$

The function $w \mapsto r_w$ increases from $r_{\sqrt{3}} = 1$ to a finite value $r_\infty \approx 1.47$, as w increases from $\sqrt{3}$ to ∞.

For $w \leq \sqrt{3}$, U_w is empty.

Proof. Using spherical coordinates (r, θ, φ), the map Q_0 is of the form

$$Q_0 = \alpha(r)(e_r \otimes e_r - I/3) + \beta(r)(e_z \otimes e_z - I/3), \qquad (r > 1),$$

where the functions $\alpha(r), \beta(r) > 0$ are given by

$$\alpha(r) = s_* \frac{w}{3+w} \frac{1}{r^3}, \quad \beta(r) = s_*(1 - \frac{w}{1+w} \frac{1}{r}). \tag{4.25}$$

Using the fact that $e_r = \cos\varphi \; e_z + \sin\varphi \; e_\rho$, and defining

$$\sigma := \alpha + \beta, \quad \nu := \alpha\beta,$$

elementary computations show that the characteristic polynomial of Q_0 is

$$\begin{aligned} P(X) &= \left(X + \frac{\sigma}{3}\right)\left(X^2 - \frac{\sigma}{3}X - \frac{2}{9}\sigma^2 + \nu \sin^2\varphi\right) \\ &= \left(X + \frac{\sigma}{3}\right)\left(X - \frac{\sigma}{6} - \sqrt{\frac{\sigma^2}{4} - \nu\sin^2\varphi}\right)\left(X - \frac{\sigma}{6} + \sqrt{\frac{\sigma^2}{4} - \nu\sin^2\varphi}\right). \end{aligned}$$

Note that

$$\frac{\sigma^2}{4} - \nu \sin^2\varphi = \frac{1}{4}(\alpha - \beta)^2 + \nu\cos^2\varphi \geq 0,$$

so that the above square root is well defined. The eigenvalues $\lambda_1^0 \geq \lambda_2^0 \geq \lambda_3^0$ of Q_0 are given by

$$\begin{aligned} \lambda_1^0 &= \frac{\sigma}{6} + \sqrt{\frac{\sigma^2}{4} - \nu\sin^2\varphi}, \\ \lambda_2^0 &= \frac{\sigma}{6} - \sqrt{\frac{\sigma^2}{4} - \nu\sin^2\varphi}, \\ \lambda_3^0 &= -\frac{\sigma}{3}. \end{aligned}$$

We are looking for the points where Q_0 is uniaxial, *i.e.* either $\lambda_1^0 = \lambda_2^0$ or $\lambda_2^0 = \lambda_3^0$. For $0 < \varphi < \pi$, it holds

$$\begin{aligned} \lambda_1^0 = \lambda_2^0 \quad &\Longleftrightarrow \quad \varphi = \frac{\pi}{2} \text{ and } \alpha = \beta, \\ \lambda_2^0 = \lambda_3^0 \quad &\Longleftrightarrow \quad \alpha = 0 \text{ or } \beta = 0. \end{aligned}$$

Since $\alpha, \beta > 0$, only the first case ($\lambda_1^0 = \lambda_2^0$) can occur. Given the expressions (4.25) of α and β, we deduce that

$$\begin{aligned} &U_w = \left\{(x_1, x_2, 0) \colon 1 < x_1^2 + x_2^2 = r^2, \; p(w, r) = 0\right\}, \\ &\text{where} \quad p(w, r) := r^3 - \frac{w}{1+w}r^2 - \frac{w}{3+w}. \end{aligned}$$

It is straightforward to check that $r \mapsto p(w,r)$ is increasing on $(1,\infty)$ and therefore $p(w,\cdot)=0$ has at most one solution in this interval. Since on the other hand

$$(1+w)(3+w)\cdot p(w,1) = 3-w^2,$$

there is a solution $r_w > 1$ if and only if $w > \sqrt{3}$. The function $w \mapsto r_w$ is easily seen to be smooth, and its derivative has the same sign as

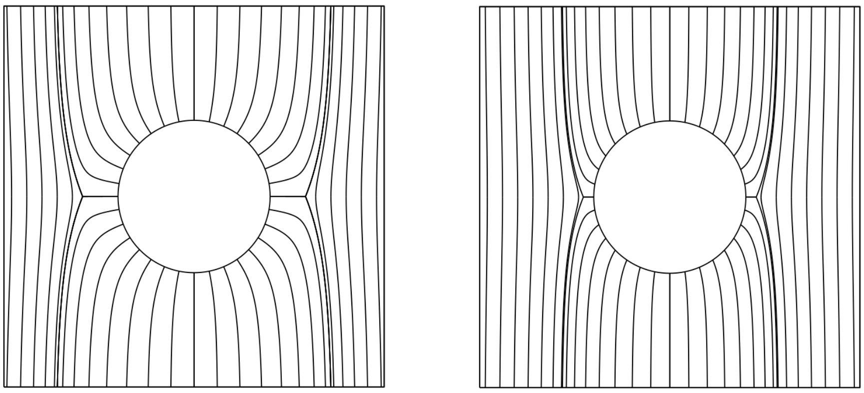
$$-\partial_w p(w,r) = \frac{r^2}{(1+w)^2} + \frac{3}{(3+w)^2} > 0,$$

so that r_w is an increasing function of w. As w increases to ∞, r_w increases to $r_\infty > 1$ such that $r_\infty^3 - r_\infty^2 - 1 = 0$. □

Remark 4.8. Let us define the **director** n_0 of Q_0 as a unit vector associated to the largest eigenvalue λ_1^0. This is well-defined up to a sign, at every point where λ_1^0 is a simple eigenvalue: that is, everywhere but at U_w. Then one may compute, for $0<\varphi\leq\pi/2$,

$$n_0 = \sqrt{\frac{1-\mu}{2}}e_\rho + \sqrt{\frac{1+\mu}{2}}e_z, \qquad \mu := \frac{\alpha(1-2\sin^2\varphi)+\beta}{\sqrt{\alpha^2+\beta^2+2\alpha\beta(1-2\sin^2\varphi)}},$$

with α,β as in (4.25). For $\pi>\varphi>\pi/2$ the director field is obtained by reflecting with respect to the horizontal plane : $n_0(\varphi) = \sqrt{(1-\mu)/2}\,e_\rho - \sqrt{(1+\mu)/2}\,e_z$. Note that this way n_0 is not continuous in $\Omega\setminus U_w$, but it is continuous as an $\mathbb{RP}^2$-valued map : the map $n_0\otimes n_0$ is continuous in $\Omega\setminus U_w$.



(a) $w=\infty$. (b) $w=3$.

Figure 4.1 – The director field n_0 (integral curves).

Corollary 4.9. *Let $w>\sqrt{3}$. There exists $\delta_0>0$ such that, if*

$$\delta := \frac{1}{L} + \left|\frac{L}{W} - \frac{1}{w}\right| < \delta_0,$$

then any solution $Q \in \mathcal{H}_\infty$ *of* (4.10)-(4.8) *admits a uniaxial ring near* U_w. *More precisely,* $Q(x)$ *is uniaxial for all* x *belonging to*

$$U = \{(\rho_u(\theta)\cos\theta, \rho_u(\theta)\sin\theta, z_u(\theta)) \colon \theta \in \mathbb{R}\},$$

where the functions $\rho_u(\theta)$ *and* $z_u(\theta)$ *satisfy*

$$|\rho_u(\theta) - r_w| + |z_u(\theta)| \leq \varepsilon(\delta) \to 0,$$

as $\delta \to 0$.

Proof. Biaxiality can be quantified through the biaxiality parameter [75]

$$\beta(Q) = 1 - 6\frac{(\mathrm{tr}(Q^3))^2}{|Q|^6},$$

which is such that : Q is uniaxial if and only if $\beta(Q) = 0$. Let us fix $R > r_w$. From Theorem 4.1 and Proposition 4.7 we infer that there exists δ_0 such that for $\delta < \delta_0$ it holds

$$\beta(Q) > 0 \quad \text{in } A := \{x \in B_R \colon \mathrm{dist}(x, U_w \cup \mathbb{R}e_z) \geq \varepsilon(\delta)\},$$

for some $\varepsilon(\delta) \to 0$ as $\delta \to 0$. Let us write χ_0, the characteristic polynomial of Q_0, as

$$\chi_0 = (X - \lambda_1^0)P_0,$$

where $P_0 = (X - \lambda_2^0)(X - \lambda_3^0)$. By continuity of the roots of a polynomial, it is clear that the characteristic polynomial χ of Q satisfies

$$\chi = (X - \lambda_1)P, \quad \left|\lambda_1 - \lambda_1^0\right| + |P - P_0| \leq c_1(\delta) \quad \text{in } A,$$

for some $c_1(\delta) \to 0$ as $\delta \to 0$. The eigenvalue λ_1 and the coefficients of P depend continuously on $x \in A$. Note that $P_0(Q_0)n_0 \neq 0$ in A, and define

$$u := \frac{1}{|P_0(Q_0)n_0|}P(Q)n_0.$$

Then $Q\,u = \lambda_1 u$, and $|u - n_0| \leq c_2(\delta) \to 0$ in A, so that $u \neq 0$ and we may define $n = u/|u|$. It holds $Q\,n = \lambda_1 n$ and

$$|n(x) - n_0(x)| \leq c_3(\delta) \to 0.$$

Moreover the map $n \otimes n$ is continuous in A.

Now fix $\theta \in \mathbb{R}$ and denote by H_θ the half plane corresponding to the azimuthal angle θ

$$H_\theta = \{(\rho\cos\theta, \rho\sin\theta, z) \colon \rho \geq 0,\ z \in \mathbb{R}\},$$

and by D_θ the disc in H_θ of radius $\varepsilon(\delta)$ and of center at the point where the ring U_w intersects H_θ:

$$D_\theta = \left\{(\rho\cos\theta, \rho\sin\theta, z) \colon (\rho - r_w)^2 + z^2 \leq \varepsilon(\delta)^2\right\}.$$

We claim that for low enough δ there exists $x \in D_\theta$ such that $Q(x)$ is uniaxial (which obviously proves Corollary (4.9)).

To prove the claim, note that when restricted to H_θ, the director field n_0 may be viewed as an $\mathbb{S}^1$-valued map since it takes values in $\mathbb{S}^2 \cap H_\theta$. Then the restriction of $n_0 \otimes n_0$ to ∂D_θ is topologically non trivial : it corresponds to a non trival class of $\pi_1(\mathbb{RP}^1)$, as can be seen by explicitly computing its degree. Since for δ low enough $n(x)$ is arbitrarily close to $n_0(x)$, this implies that the map $n \otimes n$, which is continuous in $A \cap H_\theta$, admits no continuous extension inside D_θ. Therefore Q can not be biaxial everywhere in D_θ : if it were the case, Q would have only simple eigenvalues and admit a differentiable eigenframe [106]. In particular there would be a differentiable vector field $\tilde{n}$ defined in D_θ, such that $\tilde{n} \otimes \tilde{n}$ extends $n \otimes n$. □

4.4 Large particle: the dipole structure

As mentioned in the introduction, the core of Theorem 4.2 is the fact that the axially symmetric harmonic map obtained in the limit of a large particle has exactly one singularity. In this section, we prove this result (Theorem 4.10 below) and then complete the proof of Theorem 4.2.

We define the axially symmetric $\mathbb{S}^2$-valued maps to be exactly the maps $n \in H^1_{loc}(\Omega; \mathbb{S}^2)$ which can be written in cylindrical coordinates (ρ, θ, z) as

$$n = \sin\psi(\rho, z)\, e_\rho + \cos\psi(\rho, z)\, e_z, \tag{4.26}$$

for some real-valued function $\psi \in H^1_{loc}(\Omega_{cyl})$ defined in the domain

$$\Omega_{cyl} := \left\{(\rho, z) \in \mathbb{R}_+ \times \mathbb{R} \colon \rho^2 + z^2 > 1\right\}. \tag{4.27}$$

We consider here strong anchoring conditions given by

$$n = e_r \quad \text{for } |x| = 1, \tag{4.28}$$

and the far-field conditions in integral form

$$\int_\Omega \frac{(n_1)^2 + (n_2)^2}{|x|^2} dx. \tag{4.29}$$

As in Section 4.2, the existence of an axially symmetric $\mathbb{S}^2$ valued map n minimizing the Dirichlet functional

$$E(n) = \int_\Omega |\nabla n|^2 ,$$

under the conditions (4.28)-(4.29) follows from the direct method of the calculus of variations and Hardy's inequality. Such a map is analytic away from a discrete set of singularities on the z-axis [60]. Here we prove:

Theorem 4.10. *Let $n \in H^1_{loc}(\Omega; \mathbb{S}^2)$ be a minimizer of the Dirichlet functional E among all axially symmetric $\mathbb{S}^2$-valued maps satisfying* (4.28)-(4.29). *Then n is analytic away from exactly one point defect* $(0, 0, z_0)$.

Proof of Theorem 4.10. **Preliminaries.** Let $\psi \in H^1_{loc}(\Omega_{cyl})$ be the function associated to n through (4.26). Then ψ minimizes the energy

$$E(\psi) = \int_{\Omega_{cyl}} \left[|\partial_\rho \psi|^2 + |\partial_z \psi|^2 + \frac{1}{\rho^2} \sin^2 \psi \right] \rho d\rho dz,$$

among all functions $\psi \in H^1_{loc}(\Omega_{cyl})$ such that the corresponding n satisfies (4.28)-(4.29). Next we express these boundary conditions in terms of ψ.

The strong anchoring condition (4.28) is more conveniently expressed using spherical coordinates (r, θ, φ):

$$\psi = \varphi \quad \text{for } r = 1. \tag{4.30}$$

It holds in the sense of traces, which makes sense as soon as $E(\psi) < \infty$. (In fact we should have written $\psi \equiv \varphi \bmod 2\pi$, but since any $\mathbb{Z}$-valued function of regularity $H^{1/2}$ is constant [28] we may reduce to the above.)

The far-field condition (4.29) becomes

$$\int_{\Omega_{cyl}} \frac{\sin^2 \psi}{\rho^2 + z^2} \rho d\rho dz < \infty. \tag{4.31}$$

Hence the class of admissible functions consists exactly of the $\psi \in H^1_{loc}(\Omega_{cyl})$ satisfying $E(\psi) < \infty$ and (4.30)-(4.31).

The Euler-Lagrange equation satisfied by ψ is

$$\partial_z^2 \psi + \partial_\rho^2 \psi + \frac{1}{\rho} \partial_\rho \psi = \frac{1}{2\rho^2} \sin(2\psi) \quad \text{in } \Omega_{cyl}. \tag{4.32}$$

Note that, by elliptic regularity, any solution of (4.32) is real-analytic away from the z-axis $\{\rho = 0\}$. Also note that, since replacing ψ by $\max(\psi, 0)$ or $\min(\psi, \pi)$ does not change the boundary conditions and decreases the energy, it holds

$$0 \leq \psi \leq \pi.$$

The rest of the proof is divided into 3 steps.

Step 1. We claim that the open subsets of $\Omega_{cyl} \cap \{\rho > 0\}$,

$$X_+ = \{\psi > \pi/2\} \text{ and } X_- = \{\psi < \pi/2\},$$

are connected.

We split the half-circle $\partial\Omega_{cyl} \cap \{\rho > 0\}$ into two arcs:

$$A_\pm = \{r = 1,\ \varphi = \pi/2 \pm t \colon t \in (0, \pi/2)\}.$$

The boundary conditions ensure that $A_\pm \subset X_\pm$. We denote by $\omega_\pm$ the connected component of $A_\pm$ in $X_\pm$.

Let us show first that $X_+ = \omega_+$. Consider the function

$$\widetilde{\psi} = \begin{cases} \psi & \text{in } \omega_+, \\ \min(\psi, \pi - \psi) & \text{in } \Omega_{cyl} \setminus \omega_+. \end{cases}$$

Then it can be checked that $\widetilde{\psi} \in H^1_{loc}(\Omega_{cyl})$. Moreover, $E(\widetilde{\psi}) = E(\psi)$ and $\widetilde{\psi}$ clearly satisfies the boundary conditions (4.30)-(4.31). Therefore $\widetilde{\psi}$ minimizes

E, and is analytic away from the z-axis. In particular, $\min(\psi, \pi-\psi)$ is analytic in $\Omega_{cyl} \setminus \overline{\omega}_+$.

On the other hand, since X_- is open and non-void, there is an open subset of $\Omega_{cyl} \setminus \overline{\omega}_+$ in which $\psi < \pi/2$. In that open subset, the two analytic functions ψ and $\min(\psi, \pi-\psi)$ coincide, so they must coincide in the whole $\Omega_{cyl} \setminus \overline{\omega}_+$. We deduce that

$$\psi \leq \pi - \psi \quad \text{i.e.} \quad \psi \leq \pi/2 \qquad \text{in } \Omega_{cyl} \setminus \overline{\omega}_+,$$

and therefore $X_+ = \omega_+$ is connected.

To show that X_- is connected, consider

$$\widetilde{\psi} = \begin{cases} \psi & \text{in } \omega_-, \\ \max(\psi, \pi-\psi) & \text{in } \Omega_{cyl} \setminus \omega_-. \end{cases}$$

As above, $E(\widetilde{\psi}) = E(\psi)$ and we conclude that $X_- = \omega_-$ is connected.

Step 2. There is at most one singularity.

We know [60] that n is analytic in Ω away from a set of isolated points

$$Z \subset \{\rho = 0, \ |z| > 1\}.$$

In particular ψ is continuous in $\overline{\Omega}_{cyl} \setminus Z$, and since

$$\int_{\Omega_{cyl}} \frac{\sin^2\psi}{\rho} d\rho dz \leq E(\psi) < \infty,$$

and $0 \leq \psi \leq \pi$, it follows that

$$\psi \in \{0, \pi\} \quad \text{on } (\partial\Omega_{cyl} \cap \{\rho = 0\}) \setminus Z.$$

At every point $(0, z_0) \in Z$, ψ must be discontinuous, because otherwise n would be continuous around that point (and then real analytic). Therefore it must hold

$$\text{either } \psi = \begin{cases} 0 & \text{in } (z_0 - \delta, z_0), \\ \pi & \text{in } (z_0, z_0 + \delta), \end{cases} \qquad \text{or } \psi = \begin{cases} \pi & \text{in } (z_0 - \delta, z_0), \\ 0 & \text{in } (z_0, z_0 + \delta), \end{cases}$$

for some $\delta > 0$.

Let us argue by contradiction and assume that there exist two distinct points

$$(0, z_1), \, (0, z_2) \in Z, \quad z_1 < z_2, \quad [z_1, z_2] \cap Z = \{z_1, z_2\}.$$

There are three cases: either $z_1 < z_2 < -1$, or $z_1 < -1 < 1 < z_2$, or $1 < z_1 < z_2$. Note that the boundary conditions (together with the boundary regularity of n [60]) ensure that $\psi = \pi$ in $(-1-\delta, -1)$ and $\psi = 0$ in $(1, 1+\delta)$ for some $\delta > 0$. In all three cases, it is easy to see that there must exist four distinct points

$$(0, z_j') \notin Z, \quad z_1' < z_2' < z_3' < z_4',$$

such that

$$\begin{aligned} \text{either} \quad & \psi(z_1') = \psi(z_3') = 0, \ \psi(z_2') = \psi(z_4') = \pi, \\ \text{or} \quad & \psi(z_1') = \psi(z_3') = \pi, \ \psi(z_2') = \psi(z_4') = 0. \end{aligned}$$

We may assume that we are in the first case (the second case can be dealt with similarly). Then by continuity there exists $\delta > 0$ such that

$$\begin{aligned}&\left[B((0,z_1'),\delta)\cap\overline{\Omega}_{cyl}\right]\cup\left[B((0,z_3'),\delta)\cap\overline{\Omega}_{cyl}\right]\subset X_-,\\&\left[B((0,z_2'),\delta)\cap\overline{\Omega}_{cyl}\right]\cup\left[B((0,z_4'),\delta)\cap\overline{\Omega}_{cyl}\right]\subset X_+.\end{aligned}$$

Since the sets $X_\pm$ are path-connected we deduce from the above that there is a continuous path γ_- from z_1' to z_3' inside X_-, and a continuous path γ_+ from z_2' to z_4' inside X_+. Since $z_1' < z_2' < z_3' < z_4'$, these paths must intersect, but then there intersection would belong to $X_- \cap X_+ = \emptyset$. This contradiction shows that Z contains at most one point.

Step 3. There is at least one singularity.

Assume that Z is empty. Then n is continuous in $\overline{\Omega}$, and therefore

$$\deg(n,\partial B_r) = -\frac{1}{2}\int_0^\pi \partial_\varphi\psi(r,\varphi)\sin\psi(r,\varphi)\,d\varphi,$$

is independent of $r \geq 1$. We deduce that

$$\begin{aligned}1 &= \deg(n,\partial B_1) = \deg(n,\partial B_r)\\&\leq \frac{1}{4}\int_0^\pi \frac{\sin^2\psi(r,\varphi)}{\sin^2\varphi}\sin\varphi d\varphi + \frac{1}{4}\int_0^\pi \partial_\varphi\psi(r,\varphi)^2\,\sin\varphi d\varphi,\end{aligned}$$

which implies $E(\psi) \geq 4\int_1^\infty dr = \infty$, a contradiction. □

Before turning to the proof of Theorem 4.2, we define rigorously the axially symmetric Q-tensor maps. They are the maps $Q \in \mathcal{H}_\infty$ which satisfy the two following natural symmetry constraints:

— the map Q is invariant by rotation around the vertical axis:

$$Q(Rx) = {}^tRQ(x)R \quad \text{for all rotations } R \text{ of axis } e_z, \tag{4.33}$$

— the vector e_θ is everywhere an eigenvector of Q:

$$Q(x)e_\theta\cdot e_\rho = Q(x)e_\theta\cdot e_z \equiv 0. \tag{4.34}$$

These constraints are natural, in the sense that a minimizer of the free energy (4.2) under those restrictions is still a solution of the complete (unconstrained) Euler-Lagrange system (4.10) – as can be easily checked. We denote this class of symmetric maps by $\mathcal{H}_\infty^{sym} \subset \mathcal{H}_\infty$.

Proof of Theorem 4.2. Recall that we assume $r_0 = 1$ and we are therefore considering the limit $1/L \to \infty$. Since axial symmetry (4.33)-(4.34) is clearly preserved by pointwise convergence, we may proceed exactly as in [23, Proposition 1] to obtain a subsequence (Q_k) converging in $\mathcal{H}_\infty$ to an axially symmetric $\mathcal{U}_*$-valued map Q_* minimizing the Dirichlet energy $\int_\Omega |\nabla Q|^2$. The estimates in [95, 105] show that the convergence is in fact locally uniform in $\overline{\Omega}$, away from the singular set of Q_*.

Because Ω is simply connected, $\mathcal{U}_*$-valued H^1_{loc} maps can be lifted to $\mathbb{S}^2$-valued H^1_{loc} maps [14]: for any $\mathcal{U}_*$-valued $Q \in \mathcal{H}_\infty$, there exists $n \in H^1_{loc}(\Omega;\mathbb{S}^2)$ such that

$$Q = Q_n := s_*(n\otimes n - I/3).$$

Therefore, the limiting map Q_* can be written as $Q_* = Q_n$, and the map n minimizes the Dirichlet energy in the class

$$\mathcal{H}_*^{sym} := \left\{ n \in H^1_{loc}(\Omega;\mathbb{S}^2)\colon Q_n \in \mathcal{H}_\infty^{sym} \text{ with strong anchoring (4.9)} \right\}.$$

To conclude the proof, it remains to show that the class $\mathcal{H}_*^{sym}$ actually corresponds to the class considered in Theorem 4.10.

Using the fact that $|n|^2 = 1$, we calculate

$$s_*^{-2}\,|Q_n - Q_\infty| = 2(n_1^2 + n_2^2),$$

so that Q_n satisfying (4.11) translates into n satisfying (4.29).

The strong anchoring condition (4.9) for Q_n is equivalent to

$$n_{|\partial B} = \tau\, e_r,$$

for some $\{\pm 1\}$-valued function τ, which must be of regularity $H^{1/2}$ and therefore constant [28]. Therefore, up to multiplying the map n by a sign, the strong anchoring condition becomes (4.28).

The fact that Q_n admits e_θ as an eigenvector (4.34) is equivalent to

$$(n\cdot e_\theta)(n\cdot e_\rho) = (n\cdot e_\theta)(n\cdot e_z) = 0 \quad \Leftrightarrow \quad n\cdot e_\theta \in \{0,\pm 1\}.$$

Since the function $n\cdot e_\theta$ is H^1_{loc}, it must therefore be constant. The boundary conditions prevent it to be equal to ± 1, so that $n\cdot e_\theta \equiv 0$.

The invariance by rotation (4.33) for Q_n is equivalent to $n(Rx) = \pm Rn(x)$ for any rotation R of axis e_z. The sign ± 1 may depend on x and on R, but the H^1_{loc} regularity implies that it does not depend on x. Therefore it holds, using cylindrical coordinates,

$$n(\rho,\theta,z) = \tau(\theta)\, R_\theta\, n(\rho,0,z), \quad \tau(\theta) = \pm 1,$$

where $R_\theta \in SO(3)$ is the rotation of axis e_z and angle θ. The function τ is easily seen to belong to $H^{1/2}(\mathbb{R}/2\pi\mathbb{Z})$: we conclude that $\tau \equiv 1$.

Therefore the axial symmetry (4.33)-(4.34) of Q_n is equivalent to

$$n(\rho,\theta,z) = R_\theta\, n(\rho,0,z), \quad \text{and} \quad n\cdot e_\theta \equiv 0,$$

which implies that

$$n(\rho,\theta,z) = u_1(\rho,z)\, e_\rho + u_2(\rho,z) e_z, \quad u \in H^1_{loc}(\Omega_{cyl};\mathbb{S}^1).$$

Since Ω_{cyl} is simply connected, u can be lifted to a real-valued function $\psi \in H^1_{loc}(\Omega_{cyl})$: $u_1 = \sin\psi$, $u_2 = \cos\psi$. Therefore n is of the form (4.26), and $\mathcal{H}_*^{sym}$ corresponds indeed (up to a sign) to the class of axially symmetric $\mathbb{S}^2$-valued maps satisfying (4.28)-(4.29). □

Chapter 5

Analyse de bifurcation dans une cellule nématique frustrée

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5.1 Introduction

In a nematic liquid crystal, rigid rod-like molecules tend to align in a common preferred direction. To describe this orientational order, de Gennes [45] introduced the so called Q-tensor: a 3×3 traceless symmetric matrix. The eigenframe of the Q-tensor describes the principal mean directions of alignment, while the corresponding eigenvalues describe the degrees of alignment along those directions. A null Q-tensor corresponds to the *isotropic* liquid state. A Q-tensor with two equal eigenvalues corresponds to the *uniaxial* state, which is axially symmetric around one eigenvector, called the director. The generic case of a Q-tensor with three distinct eigenvalues corresponds to the *biaxial* state. There exist other continuum theories for the description of nematic order. The Oseen-Frank theory and the Ericksen theory make use of simpler order parameters but do not account for biaxiality. The Q-tensor description allows for a much finer understanding of some phenomena, as defects (see e.g. [130]) and material frustration [109].

In the present paper we focus on a hybrid cell consisting of nematic material confined between two parallel bounding plates, and subject to competing strong anchoring conditions on each plate. Hybrid-aligned nematic cells are interesting for technological applications [70, 96], as is, in general, the effect of confining geometries on liquid crystals [38]. Nematic systems similar to the one considered here have been studied numerically in [26, 109] as a model for material frustration. On each bounding plate, the prescribed boundary condition is uniaxial and parallel to the plate, with director orthogonal to the one prescribed on the opposite plate. The numerics presented in [26, 109] bring to light two different families of solutions: *eigenvalue exchange* (EE) and *bent director* (BD) configurations. In an eigenvalue exchange solution, the Q-tensor eigenframe remains constant through the whole cell, and only the eigenvalues vary to match the boundary conditions. Therefore, inside the cell the material is strongly biaxial. In the bent director configuration however, the eigenframe rotates to connect the two orthogonal uniaxial states on the plates. Hence the tensor remains approximately uniaxial, with director bending from one plate to the other. Those two kinds of configurations are depicted in Figure 5.1.

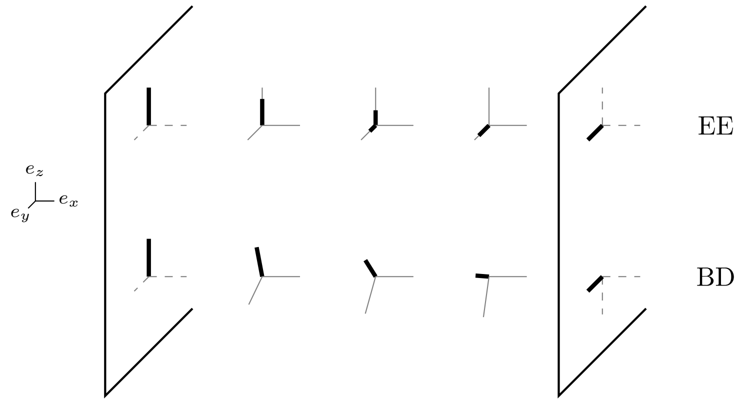

Figure 5.1 – Schematic representation of EE and BD configurations: variations of the eigenframe of Q through the cell. Eigenvectors corresponding to the largest eigenvalues are emphasized. In an EE configuration, as we move through the cell from the left to the right, the eigenvalue associated to $\mathbf{e_z}$ (the prescribed director on the left plate) decreases until, in the middle of the cell, it becomes equal to the eigenvalue associated to $\mathbf{e_y}$ (the prescribed director on the right plate), which in turn increases to match the boundary condition. In a BD configuration, as we move through the cell from the left to the right, the eigenframe rotates, so that the eigenvector corresponding to the largest eigenvalue rotates from $\mathbf{e_z}$ to $\mathbf{e_y}$. Note that a similar eigenframe rotation could occur in the opposite way as the one pictured here: there are two possible types of BD configurations.

When working with dimensionless variables, two parameters influence the behaviour of the system: a reduced temperature θ, and a typical length λ proportional to the thickness of the cell. In [26, 109], a bifurcation analysis is performed numerically as the cell thickness varies, at fixed temperatures. In both studies, a symmetric pitchfork bifurcation diagram is obtained [26, Fig. 8], which can be described as follows (see Figure 5.2). When the parameter λ (and thus the cell thickness) is small, the only equilibrium is an eigenvalue exchange configuration, which is stable. Letting the cell thickness grow, a critical value is attained, at which this eigenvalue exchange solution loses stability. At this point, bifurcation occurs and two new stable branches of solutions appear, corresponding to bent director configurations, with their eigenframe rotating in one way or the other. The results pictured in [26, Fig. 8] were obtained for a special value of the reduced temperature, $\theta = -8$, at which computations are simplified.

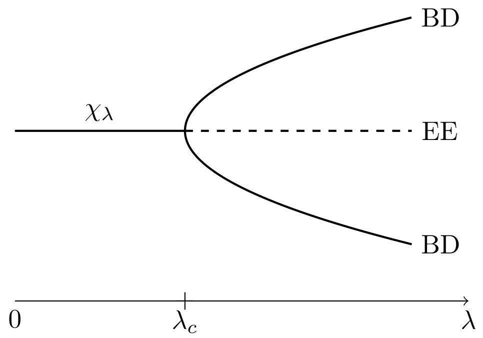

Figure 5.2 – Shape of the pitchfork bifurcation described in [26, 109].

In the present paper, we aim at providing rigorous mathematical arguments justifying the shape of the bifurcation diagram pictured in Figure 5.2. Thus we fix the temperature and let the cell thickness vary.

In a first step, we study the limits of small and large cell thickness. For a very large cell thickness, we check that energy minimizers converge towards two possible limiting uniaxial configurations, corresponding to a rotation of the director in one way or the other. On the other hand, when the cell is sufficiently narrow, we prove indeed that the energy admits a unique critical point. Symmetry considerations imply that this unique solution is an eigenvalue exchange configuration – thus showing that Figure 5.2 is valid for small λ. The method used to prove uniqueness applies to a quite wide class of problems, and we prove a general uniqueness result for a class of perturbed quasilinear elliptic systems in Appendix 5.B.

In a second step, we perform a bifurcation analysis and show that there is indeed a symmetric pitchfork bifurcation, at least when the reduced temperature θ is close to $\theta = -8$ (the special value at which [26, Fig. 8] was obtained). More specifically, we prove the following result.

Theorem. *Let $\theta \approx -8$. Consider, for small λ, the unique solution χ_λ. The branch of eigenvalue exchange solutions $\lambda \mapsto \chi_\lambda$ may be extended smoothly to larger λ, and loses stability at a critical value λ_c. At this point, a symmetric pitchfork bifurcation occurs.*

More precisely, we prove first the above Theorem in the case $\theta = -8$. Then we identify the properties that make this special case work, which leads to an abstract result of the form: if θ satisfies some properties, then bifurcation occurs. And eventually we check that those properties are stable: if a θ_0 satisfies them, they extend to nearby $\theta \approx \theta_0$. In particular we obtain the above Theorem.

Similar nematic systems have also been studied using Oseen-Frank theory [15] and Ericksen theory [11]. Oseen-Frank theory describes nematic order with help of a sole director field, while in Ericksen theory the degree of alignment along the director is also accounted for. In both cases, only uniaxial states can be described, so that there is no equivalent to the eigenvalue exchange solution considered in the present paper. However the aforementioned works [11, 15] do bring into light the existence of a critical cell width – as in our case – below which the bent director solution is no longer valid. Within Oseen-Frank theory [15] the boundary anchoring is relaxed and a uniform configuration is preferred below the critical width. Within Ericksen theory [11] the strong boundary anchoring is preserved, forcing the director field to have a discontinuity in the cell center. In our case such a discontinuity is avoided by the eigenvalue exchange mechanism.

The plan of the paper is the following. In Section 5.2 we present the precise model used to describe the cell. In Section 5.3 we discuss the existence and some properties of eigenvalue exchange configurations. In Section 5.4 we study the limits of large and small cell thickness. Then we concentrate on the unique branch of eigenvalue exchange solutions starting from small λ, and show that a symmetric pitchfork bifurcation occurs. We treat the case $\theta = -8$ in Section 5.5, and the perturbed case $\theta \approx -8$ in Section 5.6.

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5.2 Model

The cell consists of nematic material confined between two parallel bounding plates (of infinite size), with competing strong anchoring conditions on each plate. In an orthonormal basis $(\overrightarrow{e_x}, \overrightarrow{e_y}, \overrightarrow{e_z})$, the bounding plates are perpendicular to $\overrightarrow{e_x}$ and parallel to the (y, z) plane. The width of the cell is $2d$: one plate at $x = -d$, the other at $x = d$. On the left plate ($x = -d$) the boundary condition is uniaxial with director $\overrightarrow{e_z}$, and on the right plate ($x = d$) the boundary condition is uniaxial with director $\overrightarrow{e_y}$ (see Figure 5.1).

Nematic order is described by means of de Gennes' Q-tensor – a traceless symmetric 3×3 matrix –, and Landau-de Gennes free energy density

$$e(Q) = \frac{L}{2}|\nabla Q|^2 + f_b(Q),$$

where the bulk energy density f_b is given by

$$f_b(Q) = \frac{a(T)}{2}|Q|^2 - \frac{b}{3}\mathrm{tr}(Q^3) + \frac{c}{4}|Q|^4.$$

Here and in the sequel, the notation $|\cdot|$ refers to the usual euclidean norm $|Q|^2 = \mathrm{tr}(Q^tQ) = \sum Q_{ij}Q_{ij}$.

In [26, 109], the numerical simulations are performed under two symmetry restrictions : the Q-tensor depends only on x, and $\overrightarrow{e_x}$ is always an eigenvector. These restrictions are natural, since the system is invariant in the x and y directions, and since $\overrightarrow{e_x}$ is an eigenvector of the boundary conditions. It is not our goal here to justify rigorously the validity of these symmetry assumptions : we will, from the beginning, consider Q-tensors depending only on x, with $\overrightarrow{e_x}$ as an eigenvector. The assumption on the eigenvector implies that the Q-tensors considered here will always have four zero entries, as in (5.1) below.

More precisely, we will study maps

$$Q(x) = \begin{pmatrix} -2q_1(x) & 0 & 0 \\ 0 & q_1(x) - q_2(x) & q_3(x) \\ 0 & q_3(x) & q_1(x) + q_2(x) \end{pmatrix}, \quad x \in [-d, d] \quad (5.1)$$

minimizing the energy functional

$$E(Q) = \int_{-d}^{d} \left(\frac{L}{2} |Q'|^2 + f_b(Q) \right) dx,$$

when subject to boundary conditions

$$Q(-d) = \begin{pmatrix} -2q_+ & 0 & 0 \\ 0 & -2q_+ & 0 \\ 0 & 0 & 4q_+ \end{pmatrix}, \quad Q(-d) = \begin{pmatrix} -2q_+ & 0 & 0 \\ 0 & 4q_+ & 0 \\ 0 & 0 & -2q_+ \end{pmatrix}.$$

Here, q_+ is such that the boundary conditions minimize f_b.

After an appropriate rescaling [26], we may actually consider a dimensionless version of the problem, where we are left with only two parameters: a reduced temperature $\theta \in (-\infty, 1)$, and a reduced elastic constant $1/\lambda^2$. The parameter $\lambda > 0$ is proportional to $d/\sqrt{L}$: it accounts for the effects of the elastic constant L, and of the distance between the plates d. From now on we will work with the reduced free energy

$$E_\lambda(Q) = \int_{-1}^{1} \left(\frac{1}{2\lambda^2} |Q'|^2 + f(Q) \right) dx, \quad (5.2)$$

where

$$\begin{aligned} f(Q) &= \frac{\theta}{6}|Q|^2 - \frac{2}{3}\mathrm{tr}(Q^3) + \frac{1}{8}|Q|^4 + c(\theta) \\ &= \frac{\theta}{3}(3q_1^2 + q_2^2 + q_3^2) + 4q_1(q_1^2 - q_2^2 - q_3^2) + \frac{1}{2}(3q_1^2 + q_2^2 + q_3^2)^2 + c(\theta). \end{aligned} \quad (5.3)$$

Here the constant $c(\theta)$ is choosen in such a way that $\min f = 0$. Note that this minimum is attained exactly [95] at uniaxial Q-tensors of the form

$$Q = 6q_+ \left(n \otimes n - \frac{1}{3} I \right), \quad n \in \mathbb{S}^2, \ 6q_+ = 1 + \sqrt{1 - \theta}.$$

Although we do not emphasize it in the notation, the free energy obviously depends on θ.

The direct method of the calculus of variations applies to the energy functional (5.2) in the natural space $H^1(-1,1)^3$. Hence minimizers always exist. They are critical points of the energy, and as such they satisfy the Euler-Lagrange equation

$$\frac{1}{\lambda^2} Q'' = \frac{\theta}{3} Q - 2\left(Q^2 - \frac{|Q|^2}{3} I \right) + \frac{1}{2}|Q|^2 Q. \quad (5.4)$$

Solutions of (5.4) are analytic, and they satisfy the maximum principle [95]

$$|Q| \leq 2\sqrt{6}q_+. \tag{5.5}$$

In terms of q_1, q_2 and q_3 defined by (5.1), the Euler-Lagrange equation (5.4) becomes the system

$$\begin{cases} \dfrac{1}{\lambda^2}q_1'' = \dfrac{\theta}{3}q_1 - \dfrac{2}{3}(q_2^2+q_3^2-3q_1^2) + (3q_1^2+q_2^2+q_3^2)q_1 \\ \dfrac{1}{\lambda^2}q_2'' = \dfrac{\theta}{3}q_2 - 4q_1q_2 + (3q_1^2+q_2^2+q_3^2)q_2 \\ \dfrac{1}{\lambda^2}q_3'' = \dfrac{\theta}{3}q_3 - 4q_1q_3 + (3q_1^2+q_2^2+q_3^2)q_3 \end{cases} \tag{5.6}$$

and the boundary conditions read

$$\begin{array}{ll} q_1(-1) = q_+, & q_1(1) = q_+, \\ q_2(-1) = 3q_+, & q_2(1) = -3q_+, \\ q_3(-1) = 0, & q_3(1) = 0. \end{array} \tag{5.7}$$

In the sequel we will denote by $\mathcal{H}$ the space of all admissible configurations, i.e. the space of H^1 configurations satisfying the boundary conditions. Thus $\mathcal{H}$ is an affine subspace of $H^1(-1,1)^3$, consisting of all Q-tensors of the form (5.1), which satisfy the boundary conditions (5.7).

5.3 Eigenvalue exchange configurations

Consider the group G defined as the subgroup of $O(3)$ generated by the matrices S_y and S_z of the orthogonal reflections with respect to the axes $\mathbb{R}\overrightarrow{e_y}$ and $\mathbb{R}\overrightarrow{e_z}$. Explicitly,

$$S_y = \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix} \quad \text{and} \quad S_z = \begin{pmatrix} -1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & 1 \end{pmatrix}.$$

As a subgroup of $O(3)$, G acts naturally on symmetric traceless matrices, and thus on $H^1(-1,1)^3$, via the following formula:

$$(R \cdot Q)(x) = RQ(x)^t R, \qquad R \in G.$$

One easily sees that the affine subspace $\mathcal{H} \subset H^1(-1,1)^3$ of admissible configurations is stable under this action: if Q satisfies the boundary conditions (5.7) then $R \cdot Q$ satisfies them also, for $R \in G$. Thus G acts on $\mathcal{H}$.

Moreover, the free energy functional E_λ is invariant under this action:

$$E_\lambda(R \cdot Q) = E_\lambda(Q) \qquad \forall R \in G,\ Q \in \mathcal{H}.$$

Therefore the principle of symmetric criticality [108] ensures that critical points among G-invariant configurations are critical points of E_λ, that is solutions of the Euler-Lagrange system (5.6).

More precisely, we denote by $\mathcal{H}^{ee}$ the affine subspace of $\mathcal{H}$ consisting of all invariant configurations, and by $E_\lambda^{ee} = E_\lambda|_{\mathcal{H}^{ee}}$ the free energy functional restricted to invariant configurations. It is straightforward to check that

$$\mathcal{H}^{ee} = \{Q \in \mathcal{H}\,;\, q_3 \equiv 0\},$$

and the principle of symmetric criticality simply asserts that critical points of E_λ^{ee} correspond to solutions of (5.6) with $q_3 \equiv 0$. Of course this fact could also be checked by a direct computation.

The elements of $\mathcal{H}^{ee}$ are the eigenvalue exchange configurations, since $\chi \in \mathcal{H}^{ee}$ corresponds to (q_1, q_2) via

$$\chi(x) = \begin{pmatrix} -2q_1(x) & 0 & 0 \\ 0 & q_1(x) - q_2(x) & 0 \\ 0 & 0 & q_1(x) + q_2(x) \end{pmatrix}, \quad x \in [-1,1].$$

The free energy of such a $\chi \in \mathcal{H}^{ee}$ is given by

$$\begin{aligned} E_\lambda^{ee}(\chi) &= \int_{-1}^{1} \left(\frac{1}{2\lambda^2}|\chi'|^2 + f(\chi) \right) dx \\ &= \int_{-1}^{1} \Bigg(\frac{3(q_1')^2 + (q_2')^2}{\lambda^2} + \frac{\theta}{3}(3q_1^2 + q_2^2) \\ &\qquad + 4q_1(q_1^2 - q_2^2) + \frac{1}{2}(3q_1^2 + q_2^2)^2 + c(\theta) \Bigg) dx, \end{aligned} \tag{5.8}$$

and critical points of E_λ^{ee} solve the boundary value problem

$$\begin{cases} \dfrac{1}{\lambda^2} q_1'' = \dfrac{\theta}{3} q_1 - \dfrac{2}{3}(q_2^2 - 3q_1^2) + (3q_1^2 + q_2^2) q_1, \\ \dfrac{1}{\lambda^2} q_2'' = \dfrac{\theta}{3} q_2 - 4q_1 q_2 + (3q_1^2 + q_2^2) q_2, \\ q_1(\pm 1) = q_+, \\ q_2(\pm 1) = \mp 3q_+. \end{cases} \tag{5.9}$$

Since the direct method of the calculus of variations applies to E_λ^{ee}, there always exists an eigenvalue exchange minimizer, which is an equilibrium configuration in $\mathcal{H}$. This eigenvalue exchange equilibrium is stable in $\mathcal{H}^{ee}$, but need not be stable as an equilibrium among all admissible configurations: in principle, symmetry-breaking perturbations may induce a negative second variation of the total free energy E_λ. To study this phenomenon we need to understand the structure of that second variation.

Consider a family $\chi_\lambda = (q_{1,\lambda}, q_{2,\lambda})$ of eigenvalue exchange configurations. That is, χ_λ is a critical point of E_λ^{ee}, and hence also of E_λ. The principle of symmetric criticality (see Appendix 5.A) ensures that the orthogonal decomposition

$$H_0^1(-1,1)^3 = H_{sp} \oplus H_{sb} = \{(h_1, h_2, 0)\} \oplus \{(0,0,h_3)\},$$

corresponding to the decomposition into 'symmetry-preserving' perturbations and 'symmetry-breaking' perturbations, is also orthogonal for the bilinear form

$D^2E_\lambda(\chi_\lambda)$. Namely, for $H \in H_0^1(-1,1)^3$,

$$\begin{aligned} D^2E_\lambda(\chi_\lambda)[H] &= D^2E_\lambda(\chi_\lambda)[(h_1,h_2,0)] + D^2E_\lambda(\chi_\lambda)[(0,0,h_3)] \\ &= \Phi_\lambda[h_1,h_2] + \Psi_\lambda[h_3]. \end{aligned}$$

Here $\Phi = \Phi_\lambda$ and $\Psi = \Psi_\lambda$ are quadratic forms defined on $H_0^1(-1,1)^2$, respectively $H_0^1(-1,1)$, by the above equality. Note that Φ_λ is nothing else than $D^2E_\lambda^{ee}(\chi_\lambda)$, the second variation of restricted free energy. From the computations in Appendix 5.C we obtain

$$\begin{aligned} \Phi[h_1,h_2] = \int_{-1}^{1} \Bigg\{ & \frac{6(h_1')^2 + 2(h_2')^2}{\lambda^2} + 6\left(\frac{\theta}{3} + 2q_1 + 9q_1^2 + q_2^2\right) h_1^2 \\ & + 2\left(\frac{\theta}{3} - 4q_1 + 3q_1^2 + 3q_2^2\right) h_2^2 + 8q_2(3q_1-2)h_1h_2 \Bigg\} dx \end{aligned} \tag{5.10}$$

and

$$\Psi[h_3] = \int_{-1}^{1} \left\{ \frac{2(h_3')^2}{\lambda^2} + 2\left(\frac{\theta}{3} - 4q_1 + 3q_1^2 + q_2^2\right) h_3^2 \right\} dx. \tag{5.11}$$

To the quadratic forms Φ_λ and Ψ_λ, we may associate bounded linear operators $\mathcal{M}_\lambda : H_0^1(-1,1)^2 \to H^{-1}(-1,1)^2$ and $\mathcal{L}_\lambda : H_0^1(-1,1) \to H^{-1}(-1,1)$ such that

$$\langle \mathcal{M}_\lambda(h_1,h_2),(h_1,h_2)\rangle = \Phi_\lambda[h_1,h_2] \quad \text{and} \quad \langle \mathcal{L}_\lambda h_3, h_3\rangle = \Psi_\lambda[h_3]. \tag{5.12}$$

Of particular interest to us will be the first eigenvalues of these operators, since they measure the local stability of the eigenvalue exchange equilibrium. We will denote the first eigenvalue of $\mathcal{M}_\lambda$ (respectively $\mathcal{L}_\lambda$) by $\nu(\lambda)$ (respectively $\mu(\lambda)$). They are given by the following formulas:

$$\nu(\lambda) = \inf \frac{\Phi_\lambda[h_1,h_2]}{\int (h_1^2 + h_2^2)}, \qquad \mu(\lambda) = \inf \frac{\Psi_\lambda[h]}{\int h^2}. \tag{5.13}$$

5.4 The limits of very large and very small cell thickness

So far, we know that there always exists an eigenvalue exchange solution. However, as the cell thickness grows larger, the numerics in [109, 26] predict the existence of a bent director solution, that is, a solution of (5.6) with $q_3 \neq 0$. In addition this solution should be approximately uniaxial. In Proposition 5.1 below we study the limiting behaviour of minimizers as λ grows to infinity, and obtain in fact a convergence towards a uniaxial tensor. In particular the minimizer can not stay in $\mathcal{H}^{ee}$, thus for large λ there do exist solutions other than the eigenvalue exchange minimizer.

Before stating the result, we should remark that, due to the symmetry of the energy functional, any solution with $q_3 \neq 0$ automatically gives rise to another, distinct solution. Recall indeed from Section 5.3 that E_λ is G-invariant, where G is the subgroup of $O(3)$ generated by the orthogonal reflections S_y and S_z (with respect to the y-axis and to the z-axis). For a Q-tensor associated to (q_1, q_2, q_3) via (5.1), it holds

$$S_y \cdot Q = \begin{pmatrix} -2q_1 & 0 & 0 \\ 0 & q_1 - q_2 & -q_3 \\ 0 & -q_3 & q_1 + q_2 \end{pmatrix}.$$

Therefore, if Q is a solution of (5.6), then the Q-tensor with opposite q_3 is also solution of (5.6). Moreover, those two solutions Q and $S_y \cdot Q$ have same energy. That is why, when studying the limit of minimizers of E_λ in Proposition 5.1 below, we will restrict ourselves to Q-tensors satisfying, say, $q_3(0) \geq 0$, to ensure the uniqueness of the limit.

The limit of a small elastic constant – which corresponds to a large λ – has already been studied in [95] in the three dimensional case, and in [16, 33, 56] in the two dimensional case. The one dimensional case considered in the present article is particularly simple and we obtain the following result.

Proposition 5.1. *Let Q_λ be a minimizer of E_λ, with $q_3(0) \geq 0$. It holds*

$$Q_\lambda \to Q_* \quad \text{in } H^1(-1,1)^3$$

as λ tends to $+\infty$, where

$$Q_*(x) = 6q_+ \left(n_*(x) \otimes n_*(x) - \frac{1}{3} I \right), \quad n_*(x) = \begin{pmatrix} 0 \\ \cos\left(\frac{\pi}{4} - \frac{\pi}{4}x\right) \\ \sin\left(\frac{\pi}{4} - \frac{\pi}{4}x\right) \end{pmatrix}$$

Proof. One proves, exactly as in [95, Lemma 3], that there exists a subsequence

$$Q_{\lambda_k} \longrightarrow Q_* = 6q_+ \left(n \otimes n - \frac{1}{3} I \right) \qquad \text{in } H^1,$$

where Q_* minimizes $\int |Q'|^2$ among maps in $\mathcal{H}$ which are everywhere of the form $Q = 6q_+(n \otimes n - I/3)$ – that is, maps Q in $\mathcal{H}$ which satisfy $f(Q) = 0$ everywhere.

Since Q_* is continuous on $(-1,1)$ it follows from [14, Lemma 3] that there exists a unique continuous map $n_* : (-1,1) \to \mathbb{S}^2$ such that

$$Q_*(x) = 6q_+ \left(n_*(x) \otimes n_*(x) - \frac{1}{3} I \right), \quad n_*(-1) = \overrightarrow{e_z}.$$

Moreover, by [14, Lemma 1], the map n_* lies in $H^1(-1,1;\mathbb{S}^2)$.

Since Q_* minimizes $\int |Q'|^2$, we deduce that n_* minimizes $\int |n'|^2$ among maps $n \in H^1(-1,1;\mathbb{S}^2)$ satisfying the same boundary conditions as n_*. It holds $n_*(-1) = \overrightarrow{e_z}$, and the boundary conditions on Q_* imply that $n_*(1) = \alpha \overrightarrow{e_y}$ for some $\alpha = \pm 1$. Using the fact that the geodesics on the sphere $\mathbb{S}^2$ are arcs of large circles, we obtain

$$n_*(x) = \left(0, \alpha \cos(\frac{\pi}{4} - \frac{\pi}{4}x), \sin(\frac{\pi}{4} - \frac{\pi}{4}x) \right).$$

On the other hand, since the maps Q_λ satisfy $q_{3,\lambda}(0) \geq 0$, the limiting map Q_* must satisfy also $q_{3,*}(0) \geq 0$. Since the above formula for n_* implies that

$$q_{3,*}(0) = 6\alpha q_+ \cos(\pi/4)\sin(\pi/4) = 3\alpha q_+,$$

we conclude that $\alpha \geq 0$, and thus $\alpha = 1$. In particular, we obtain the formula for n_* in the statement of the theorem. Moreover we have shown that the limit

$$Q_* = \lim Q_{\lambda_k}$$

is uniquely determined, independently of the converging subsequence. Therefore we do actually have

$$Q_\lambda \longrightarrow Q_* \quad \text{in } H^1,$$

as λ tends to $+\infty$. □

Now we turn to studying the case of a very narrow cell. That is, we investigate the limit $\lambda \to 0$. The numerics in [109, 26] predict that for small λ, there is only one solution, which is an eigenvalue exchange configuration. This is indeed the content of the next result, which is in the same spirit as the similar uniqueness result [25, Theorem VIII.7] in a Ginzburg-Landau setting.

Proposition 5.2. *There exists $\lambda_0 > 0$, such that for any $\lambda \in (0, \lambda_0)$, E_λ admits a unique critical point $\chi_\lambda \in \mathcal{H}^{ee}$.*

Proof. The uniqueness is a consequence of a more general result, stated as Theorem 5.17 in Appendix 5.B. The fact that the unique solution belongs to $\mathcal{H}^{ee}$ is immediate from the considerations in Section 5.3, since there always exists a solution $\chi \in \mathcal{H}^{ee}$. □

Proposition 5.2 provides us with a family of solutions

$$(0, \lambda_0) \ni \lambda \mapsto \chi_\lambda \in \mathcal{H}^{ee}.$$

The next result gives further properties of this branch of solutions. Recall from (5.13) the definitions of $\nu(\lambda)$ and $\mu(\lambda)$: ν is the first eigenvalue of $D^2 E_\lambda^{ee}(\chi_\lambda)$, and μ is the first eigenvalue of $D^2 E_\lambda(\chi_\lambda)$ restricted to symmetry-breaking perturbations.

Proposition 5.3. *The map $\lambda \mapsto \chi_\lambda$ is smooth on $(0, \lambda_0)$ and can be extended uniquely to a smooth map of eigenvalue exchange solutions*

$$(0, \lambda_*) \to \mathcal{H}^{ee}, \ \lambda \mapsto \chi_\lambda,$$

where $\lambda_* \in [\lambda_0, +\infty]$ is determined by the following property:

$$\big(\,\nu(\lambda) > 0 \quad \forall \lambda \in (0, \lambda_*)\,\big) \quad \textit{and} \quad \big(\,\lambda_* = +\infty \ \textit{or}\ \nu(\lambda_*) = 0\,\big) \tag{5.14}$$

Moreover, the map $\lambda \mapsto \mu(\lambda)$ is smooth on $(0, \lambda_*)$.

Remark 5.4. It is not clear wether both alternatives in (5.14) can actually occur. In Theorem 5.5 below we show that $\lambda_* = +\infty$ provided $\theta = -8$. In general we do not know if λ_* could be finite.

Proof of Proposition 5.3: In the proof of Theorem 5.17, λ_0 is chosen in such a way that E_λ is strictly convex around χ_λ, and in particular $D^2 E_\lambda(\chi_\lambda)$ is positive for $\lambda \in (0, \lambda_0)$. In fact it is straightforward to check (using Poincaré's inequality) that the choice of λ_0 in the proof of Theorem 5.17 ensures that $D^2 E_\lambda(\chi_\lambda)$ is positive definite for $\lambda \in (0,\lambda_0)$. In particular, $D^2E^{ee}_\lambda(\chi_\lambda)$ is positive definite, or equivalently, $\nu(\lambda) > 0$ for $\lambda \in (0,\lambda_0)$.

Therefore χ_λ is a non degenerate critical point, and we may apply the implicit function theorem to the smooth map

$$\mathcal{F}\colon (0,+\infty)\times \mathcal{H}^{ee} \to H^{-1}(-1,1)^2,\ (\lambda,\chi)\mapsto D^2E^{ee}_\lambda(\chi),$$

around a solution (λ,χ_λ) of $\mathcal{F}=0$, for $\lambda\in(0,\lambda_0)$. Since this solution is unique, we deduce that $\lambda\mapsto\chi_\lambda$ is given by the implicit function theorem and as such, is smooth.

As long as $D^2E_\lambda(\chi_\lambda)$ stays positive definite, i.e. $\nu(\lambda)>0$, we may apply the implicit function theorem to smoothly extend the map $\lambda\mapsto\chi_\lambda$, until we reach a λ_* satisfying (5.14). Note that the extension is unique since for each λ, χ_λ is a non degenerate – an thus isolated – critical point of E^{ee}_λ.

It remains to prove that $\lambda\mapsto\mu(\lambda)$ is a smooth map. Recall that $\mu(\lambda)$ is the first eigenvalue of the bounded linear operator

$$\begin{aligned}\mathcal{L}_\lambda\colon\ & H^1_0(-1,1)\to H^{-1}(-1,1)\\ & h\mapsto -\frac{2}{\lambda^2}h'' + 2\left(\frac{\theta}{3} - 4q_{1,\lambda} + 3q_{1,\lambda}^2 + q_{2,\lambda}^2\right)h,\end{aligned} \tag{5.15}$$

where $(q_{1,\lambda},q_{2,\lambda}) = \chi_\lambda$. From the smoothness of $\lambda\mapsto\chi_\lambda$ we deduce easily that $\mathcal{L}_\lambda$ depends smoothly on λ.

Let us fix $\lambda_0\in(0,\lambda_*)$. From the theory of Sturm-Liouville operators, we know that $\mu(\lambda)$ is a simple eigenvalue of $\mathcal{L}_\lambda$. In fact, in the terminology of [43, Definition 1.2], μ_0 is an i-simple eigenvalue of $\mathcal{L}_{\lambda_0}$, where $i: H^1_0\to H^{-1}$ is the injection operator. Indeed, since $\mathcal{L}_{\lambda_0}$ is Fredholm of index 0 and symmetric, if we fix an eigenfunction $h_0\in H^1_0$, $\int h_0^2 = 1$ associated to μ_0, then it holds

$$\mathrm{Ran}(\mathcal{L}_{\lambda_0}-\mu_0 i) = \{f\in H^{-1};\ <f,h_0>=0\},$$

so that $ih_0\notin \mathrm{Ran}(\mathcal{L}_{\lambda_0}-\mu_0 i)$ and the i-simplicity of μ_0 follows easily.

Therefore we may invoke [43, Lemma 1.3] to obtain the existence of smooth maps $\lambda\mapsto\tilde\mu(\lambda)$, $\lambda\mapsto h_\lambda$ defined for $\lambda\approx\lambda_0$, such that $\tilde\mu(\lambda)$ is the unique eigenvalue of $\mathcal{L}_{\lambda_0}$ close enough to μ_0, and h_λ a corresponding eigenfunction.

On the other hand, it can be easily checked that $\lambda\mapsto\mu(\lambda)$ is continuous: upper semi-continuity is obvious since μ is an infimum of continuous functions, and lower semi-continuity follows from the inequalities

$$\mu(\lambda_0)\le\mu(\lambda) + \|\mathcal{L}_{\lambda_0}-\mathcal{L}_\lambda\|\|h_\lambda\|_{H^1} \le \mu(\lambda) + C\|\mathcal{L}_{\lambda_0}-\mathcal{L}_\lambda\|,$$

where $h_\lambda\in H^1_0$ is a L^2-normalized eigenfunction associated to $\mu(\lambda)$, and λ is close to λ_0. (Note that $\|h_\lambda\|_{H^1}$ is bounded since $\langle\mathcal{L}_\lambda h_\lambda, h_\lambda\rangle$ is bounded.)

Therefore, for λ close enough to λ_0, $\mu(\lambda)$ is close enough to μ_0. Hence by the uniqueness in [43, Lemma 1.3], $\mu(\lambda)$ must coincide with $\tilde\mu(\lambda)$. In particular, μ is smooth. □

Although we did not emphasize this dependence in the notations, everything we have done so far depends on the fixed parameter $\theta\in(-\infty,1)$. In the next section, we choose a special value for this parameter, $\theta=-8$, at which computations are simplified.

5.5 The special temperature $\theta = -8$

Throughout the present section, we assume that $\theta = -8$. As already noticed in [26], in that case the equations (5.9) admit a particularly simple solution, and we are able to say a lot more about the branch of solutions $\lambda \mapsto \chi_\lambda$ obtained in Proposition 5.3.

First of all, we obtain more information about the maximal value λ_* of definition of χ_λ, and about the eigenvalue $\mu(\lambda)$ measuring the stability with respect to symmetry-breaking perturbations. In fact we are going to prove the following theorem, which is the first of two main results in the present section.

Theorem 5.5. *Assume that $\theta = -8$. Then $\lambda_* = +\infty$. That is, the unique eigenvalue exchange solution χ_λ for small λ, can be extended to a smooth branch of eigenvalue exchange solutions*

$$(0, +\infty) \to \mathcal{H}^{ee},\ \lambda \mapsto \chi_\lambda.$$

with $\nu(\lambda) > 0$ for all $\lambda > 0$. Moreover, there exists $\lambda_c > 0$ such that

$$\mu(\lambda) > 0 \quad \forall \lambda \in (0, \lambda_c), \qquad \mu(\lambda_c) = 0, \quad \text{and } \mu'(\lambda_c) < 0. \tag{5.16}$$

In fact it holds $\mu'(\lambda) < 0$ for all λ, and $\lim_{+\infty} \mu < 0$.

In particular, Theorem 5.5 provides a rigorous justification for part of the bifurcation diagram pictured in Figure 5.2. Namely, there is a smooth branch of eigenvalue exchange solutions defined for all λ and loosing stability at some critical value of λ.

The next natural step is to investigate what happens at the critical value λ_c, where the branch of eigenvalue exchange solutions looses stability. This is the content of the second main result of the present section. Let $h_c \in \ker \mathcal{L}_{\lambda_c}$ (a perturbation responsible for the loss of stability at λ_c), and denote by $h_c^\perp \subset H_0^1(-1,1)^3$ the space of perturbations orthogonal to $(0, 0, h_c) \in H_0^1(-1,1)^3$.

Theorem 5.6. *Assume $\theta = -8$. There exist $\delta, \varepsilon > 0$ and a neighborhood $\mathfrak{A}$ of χ_{λ_c} in $\mathcal{H}$, such that the solutions of*

$$DE_\lambda(Q) = 0, \qquad (\lambda, Q) \in (\lambda_c - \delta, \lambda_c + \delta) \times \mathfrak{A}, \tag{5.17}$$

are exactly

$$Q = \chi_\lambda \quad \text{or} \quad \begin{cases} \lambda = \lambda(t) \\ Q = \chi_{\lambda_c} + t(0,0,h_c) + t^2 H_t, \end{cases} \quad \text{for some } t \in (-\varepsilon, \varepsilon) \tag{5.18}$$

where $\lambda(t) \in (\lambda_c - \delta, \lambda_c + \delta)$ and $H_t \in h_c^\perp$ are smooth functions of $t \in (-\varepsilon, \varepsilon)$. Moreover, the following symmetry properties are satisfied:

$$\lambda(-t) = \lambda(t), \quad \text{and} \quad h_{1,-t} = h_{1,t},\ h_{2,-t} = h_{2,t},\ h_{3,-t} = -h_{3,t}, \tag{5.19}$$

where H_t is identified with $(h_{1,t}, h_{2,t}, h_{3,t})$ via (5.1).

The rest of the section will be devoted to the proofs of Theorems 5.5 and 5.6, which we decompose into several intermediate results.

5.5.1 The proof of Theorem 5.5

We start by proving that the eigenvalue exchange solution branch χ_λ obtained in Proposition 5.3 has constant q_1, and can be extended to all $\lambda > 0$. In particular we obtain the first part of Theorem 5.5.

Proposition 5.7. *Assume $\theta = -8$. Then $\lambda_* = +\infty$, and for every $\lambda \in (0, +\infty)$, $\chi_\lambda = (2/3, q_{2,\lambda})$, where $q_2 = q_{2,\lambda}$ solves*

$$\begin{cases} \dfrac{1}{\lambda^2} q_2'' = \left(q_2^2 - 4\right) q_2, \\ q_2(-1) = 2, \; q_2(1) = -2. \end{cases} \tag{5.20}$$

Proof. When the value of the reduced temperature θ is set to $\theta = -8$, then $q_+ = 2/3$. Let us define $\tilde{q}_1 = q_1 - 2/3$. For $\tilde{q}_1$, the boundary conditions become $\tilde{q}_1(\pm 1) = 0$. The boundary conditions for q_2 are $q_2(\pm 1) = \mp 2$.

In terms of $\tilde{q}_1$, the bulk energy density – for the eigenvalue exchange solution (that is, with $q_3 = 0$) – reads

$$f(q_1, q_2) = 16\tilde{q}_1^2 \left(\frac{3}{2} + \tilde{q}_1\right) + \frac{1}{2}\left(q_2^2 - 4 + 3\tilde{q}_1^2\right)^2, \tag{5.21}$$

and the Euler-Lagrange equations become

$$\begin{aligned} \frac{1}{\lambda^2}\tilde{q}_1'' &= \left(4 + 8\tilde{q}_1 + 3\tilde{q}_1^2 + q_2^2\right)\tilde{q}_1, \\ \frac{1}{\lambda^2} q_2'' &= \left(q_2^2 - 4 + 3\tilde{q}_1^2\right) q_2 \end{aligned} \tag{5.22}$$

Therefore, there exists a solution with $\tilde{q}_1 \equiv 0$, i.e. $q_1 \equiv 2/3$. Indeed, a constant $\tilde{q}_1$ solves the first equation (for any q_2), and the corresponding q_2 is obtained by minimizing the energy E_λ^{ee} in which q_1 is taken to be constant. That is, q_2 minimizes

$$I_\lambda(q_2) = \int_{-1}^{1} \left(\frac{1}{\lambda^2}(q_2')^2 + \frac{1}{2}(q_2^2 - 4)^2\right) dx. \tag{5.23}$$

Hence q_2 solves (5.20). From Lemma 5.8 below we know that (5.20) actually admits a unique solution. Hence we may define for all $\lambda > 0$, without ambiguity, the eigenvalue exchange solution

$$\tilde{\chi}_\lambda := (2/3, q_{2,\lambda}), \quad \text{where } q_{2,\lambda} \text{ solves (5.20)}.$$

The uniqueness proven in Proposition 5.2 ensures that $\chi_\lambda = \tilde{\chi}_\lambda$ for $\lambda \in (0, \lambda_0)$. On the other hand, Lemma 5.9 below ensures that $\tilde{\chi}_\lambda$ is a smooth extension of χ_λ satisfying $\nu(\lambda) > 0$ for all $\lambda > 0$. Therefore we conclude, by the uniqueness in Proposition 5.3, that $\lambda_* = +\infty$ and $\chi_\lambda = (2/3, q_{2,\lambda})$. □

In the proof of Proposition 5.7, we made use of two lemmas, Lemma 5.8 and Lemma 5.9, that we are going to prove next. The first one gives properties of the boundary value problem (5.20) satisfied by $q_{2,\lambda}$.

Lemma 5.8. *The boundary value problem* (5.20) *has a unique solution, which is odd and decreasing.*

Proof. Very similar results are classical in the study of reaction-diffusion equations (see for instance [50, Section 4.3.]). Since the present case is particularly simple, we nevertheless give a complete proof here. Recall that the existence of a solution follows directly from minimizing the energy I_λ defined in (5.23).

We start by proving the bounds

$$-2 \leq q_2 \leq 2. \tag{5.24}$$

Assume that q_2^2 attains its maximum in $(-1,1)$. Then, at a point where the maximum is attained, it holds

$$0 \geq \frac{1}{2\lambda^2}(q_2^2)'' \geq \frac{1}{\lambda^2} q_2'' q_2 = (q_2^2 - 4) q_2^2,$$

so that $q_2^2 \leq 4$. Since this bound is satisfied (with equality) on the boundary, (5.24) is proved.

Multiplying (5.20) by q_2', we obtain the first integral

$$\left[\frac{1}{2\lambda^2}(q_2')^2\right]' = \left[\frac{1}{4}(q_2^2 - 4)^2\right]'. \tag{5.25}$$

Integrating (5.25), we obtain

$$\frac{1}{2\lambda^2}(q_2')^2 = \frac{1}{4}(q_2^2 - 4)^2 + \frac{1}{2\lambda^2} q_2'(-1)^2. \tag{5.26}$$

Since $q_2'(-1) \neq 0$ (otherwise q_2 would satisfy the same Cauchy problem at -1 as the constant solution), it follows in particular that q_2' does not vanish. On the other hand, the bounds (5.24) ensure that $q_2'(-1)$ is negative. Therefore q_2' must stay negative:

$$q_2' < 0, \tag{5.27}$$

hence every solution of (5.20) is decreasing.

Now we prove that (5.20) has a unique solution. Assume $\underline{q}_2$ and $\overline{q}_2$ are distinct solution. Then they must have distinct derivatives at -1 (otherwise they would satisfy the same Cauchy problem). Say

$$\underline{q}_2'(-1) < \overline{q}_2'(-1) < 0. \tag{5.28}$$

Since $\underline{q}_2$ and $\overline{q}_2$ take the same value at 1, we may consider

$$x_0 = \min\left\{x > -1;\ \underline{q}_2(x) = \overline{q}_2(x)\right\} \in (-1, 1].$$

At this point x_0, $\underline{q}_2$ and $\overline{q}_2$ must have distinct derivatives, and since $\underline{q}_2 < \overline{q}_2$ in $(-1, x_0)$, it holds

$$\overline{q}_2'(x_0) < \underline{q}_2'(x_0) < 0 \tag{5.29}$$

From (5.28) and (5.29) we deduce that

$$\overline{q}_2'(-1)^2 - \overline{q}_2'(x_0)^2 < \underline{q}_2'(-1)^2 - \underline{q}_2'(x_0)^2,$$

which is obviously incompatible with the facts that $\underline{q}_2$ and $\overline{q}_2$ satisfy (5.25) and coincide at -1 and x_0. Therefore (5.20) has a unique solution.

Next we prove that q_2 satisfying (5.20) must be odd. This is a direct consequence of uniqueness: the functions $q_2(x)$ and $-q_2(-x)$ are both solutions of the boundary problem (5.20), therefore they must coincide. ☐

Now we turn to the proof of the second lemma used in the proof of Proposition 5.7, in which we show that the eigenvalue exchange solution with constant q_1 is non degenerately stable in $\mathcal{H}^{ee}$.

Lemma 5.9. *Assume $\theta = -8$. Let $q_{2,\lambda}$ be the unique solution of* (5.20), *and $\chi_\lambda := (2/3, q_{2,\lambda}) \in \mathcal{H}^{ee}$. Then $\nu(\lambda)$, defined as in* (5.13), *satisfies*

$$\nu(\lambda) > 0 \quad \forall \lambda > 0. \tag{5.30}$$

As a consequence, $\lambda \mapsto q_{2,\lambda}$ is smooth.

Proof. First note that the smoothness of $\lambda \mapsto q_{2,\lambda}$ follows from (5.30). Indeed, (5.30) implies that $D^2 I_\lambda(q_{2,\lambda})$ is invertible, so that near $q_{2,\lambda}$, a solution q_2 of $DI_\lambda(q_2) = 0$ depending smoothly on λ may be obtained by the implicit function theorem. On the other hand, the uniqueness proven in Lemma 5.8 implies that $q_{2,\lambda}$ coincide with this smooth solution.

Now we turn to the proof of (5.30). Recall that $\nu(\lambda)$ is the first eigenvalue of the quadratic form $\Phi_\lambda = D^2 E^{ee}_\lambda(\chi_\lambda)$. Since $\theta = -8$ and $q_1 \equiv 2/3$, it holds

$$\begin{aligned}\Phi_\lambda[h_1, h_2] = &\int_{-1}^{1} \left\{ \frac{6}{\lambda^2}(h_1')^2 + 6\left(\frac{8}{3} + q_{2,\lambda}^2\right) h_1^2 \right\} dx \\ &+ \int_{-1}^{1} \left\{ \frac{2}{\lambda^2}(h_2')^2 + 2\left(3q_{2,\lambda}^2 - 4\right) h_2^2 \right\} dx.\end{aligned}$$

That is, Φ_λ decomposes into a quadratic form in h_1, which is obviously positive definite, and a quadratic form in h_2, which is nothing else than $D^2 I_\lambda(q_{2,\lambda})$. Therefore, to prove (5.30) we only need to show that $D^2 I_\lambda(q_{2,\lambda})$ is positive definite.

Let us define

$$\begin{aligned}\eta(\lambda) :=& \inf_{h \in H_0^1(-1,1),\, \int h^2 = 1} D^2 I_\lambda(q_{2,\lambda})[h] \\ =& \inf_{h \in H_0^1(-1,1)\ \int h^2 = 1} \int \left\{ \frac{2}{\lambda^2}(h')^2 + 2\left(3q_{2,\lambda}^2 - 4\right) h^2 \right\} dx.\end{aligned} \tag{5.31}$$

We need to prove that $\eta(\lambda) > 0$ for every $\lambda > 0$. Since $q_{2,\lambda}$ minimizes I_λ, it clearly holds $\eta(\lambda) \geq 0$. To prove that $\eta(\lambda)$ can not vanish, we are going to establish that $\lambda \mapsto \eta(\lambda)$ is decreasing.

To this end, we remark that after a rescaling it holds

$$\eta(\lambda) = \inf_{h \in H_0^1(-\lambda,\lambda),\, \int h^2 = 1} \int_{-\lambda}^{\lambda} \left\{ 2(h')^2 + 2\left(3\bar{q}_{2,\lambda}^2 - 4\right) h^2 \right\} dy,$$

where $\bar{q}_{2,\lambda}$ is the rescaled map defined by

$$\bar{q}_{2,\lambda}(y) = q_{2,\lambda}(y/\lambda), \tag{5.32}$$

and extended to the whole real line by putting $\bar{q}_{2,\lambda} = 2$ in $(-\infty, -\lambda)$ and $\bar{q}_{2,\lambda} = -2$ in $(\lambda, +\infty)$. In Lemma 5.10 below we show that $\bar{q}_{2,\lambda}(y)$ is a monotone function of λ.

Using Lemma 5.10, we prove that $\eta(\lambda)$ is decreasing: let $\lambda' > \lambda$ and consider a map $h_\lambda \in H_0^1(-\lambda, \lambda)$ at which the infimum defining $\eta(\lambda)$ is attained. Then h_λ

is admissible in the infimum defining $\eta(\lambda')$, and we obtain $\eta(\lambda') < \eta(\lambda)$, since $\bar{q}_{2,\lambda'}^2 \leq \bar{q}_{2,\lambda}^2$, with strict inequality on $(-\lambda, 0) \cup (0, \lambda)$. The latter fact follows from Lemma 5.10 and the fact that $q_{2,\lambda}$ is odd.

We may now complete the proof of Lemma 5.9: since $\eta(\lambda)$ decreases, and in addition $\eta(\lambda) \geq 0$ for all λ, we must have $\eta(\lambda) > 0$ for any λ. □

In the following lemma, we prove the monotonicity of $\lambda \mapsto \bar{q}_{2,\lambda}$.

Lemma 5.10. *For any $y > 0$, $(0, y) \ni \lambda \mapsto q_{2,\lambda}(y/\lambda) = \bar{q}_{2,\lambda}(y)$ is increasing.*

Proof. The rescaled map $\bar{q}_{2,\lambda}$ minimizes the energy functional

$$\tilde{I}_\lambda(\bar{q}_2) = \int_0^\lambda \left[(\bar{q}_2')^2 + \frac{1}{2}(\bar{q}_2^2 - 4)^2 \right] dy,$$

subject to the boundary conditions $\bar{q}_2(0) = 0$, $\bar{q}_2(\lambda) = -2$. Note that we were able to restrict the integral to the positive half-line since q_2 is odd.

Let $\lambda' > \lambda > 0$. Consider the respective minimizers $\bar{q}_{2,\lambda'}$ and $\bar{q}_{2,\lambda}$, and assume that it does not hold

$$\bar{q}_{2,\lambda'}(y) > \bar{q}_{2,\lambda}(y) \quad \forall y \in (0, \lambda').$$

Then $\bar{q}_{2,\lambda'}(y_0) = \bar{q}_{2,\lambda}(y_0)$ for some $y_0 \in (0, \lambda)$, since in (λ, λ') it does hold $\bar{q}_{2,\lambda'} > \bar{q}_{2,\lambda}$.

Thus, the maps

$$\tilde{q}_{2,\lambda} = \bar{q}_{2,\lambda'} \mathbf{1}_{y \leq y_0} + \bar{q}_{2,\lambda} \mathbf{1}_{y \geq y_0}, \quad \tilde{q}_{2,\lambda'} = \bar{q}_{2,\lambda} \mathbf{1}_{y \leq y_0} + \bar{q}_{2,\lambda'} \mathbf{1}_{y \geq y_0}$$

belong to $H^1(0, \lambda)$, respectively $H^1(0, \lambda')$. We claim that $\tilde{q}_{2,\lambda}$ minimizes $\tilde{I}_\lambda$. Assume indeed that $\tilde{q}_{2,\lambda}$ has strictly higher energy than $\bar{q}_{2,\lambda}$: then $\tilde{q}_{2,\lambda'}$ would have strictly lower energy than $\bar{q}_{2,\lambda'}$, which is absurd. In particular, $\tilde{q}_{2,\lambda}$ is analytical (as a minimizer of $\tilde{I}_\lambda$), which is possible only if $q_{2,\lambda'}$ and $q_{2,\lambda}$ coincide. But then the analytical function $q_{2,\lambda'}$ would be constant on (λ, λ'), and we obtain a contradiction. □

So far we have proven the first part of Theorem 5.5, about the extension of χ_λ until $\lambda = +\infty$. Now we turn to proving the second part, about the behaviour of $\mu(\lambda)$. We split this second part into Propositions 5.11 and 5.12 below. We start by showing that $\mu(\lambda)$ decreases, with non vanishing derivative.

Proposition 5.11. *Assume $\theta = -8$. Then it holds*

$$\mu'(\lambda) < 0,$$

for all $\lambda > 0$.

Proof. The fact that $\mu(\lambda)$ decreases can be obtained quite easily as a consequence of Lemma 5.10. The fact that its derivative does not vanish, however, is not immediate. Our proof is very similar to the proof of [8, Proposition 5.18].

First we show that

$$\frac{\partial}{\partial \lambda}[\bar{q}_{2,\lambda}(x)] > 0 \quad \text{for } x \in (0, \lambda].$$

Consider the smooth map

$$\phi : [0, +\infty) \times \mathbb{R} \to \mathbb{R}, \; (x, \alpha) \mapsto \phi(x, \alpha),$$

defined as the solution of the Cauchy problem

$$\begin{cases} \phi_{xx} = (\phi^2 - 4)\phi, \\ \phi(0, \alpha) = 0, \; \phi_x(0, \alpha) = \alpha. \end{cases}$$

Clearly, for any $\lambda > 0$, and for $x \in (0, \lambda]$,

$$\bar{q}_{2,\lambda}(x) = \phi(x, \alpha_\lambda) \quad \text{with } \alpha_\lambda = \bar{q}'_{2,\lambda}(0).$$

Notice that α_λ solves

$$\phi(\lambda, \alpha_\lambda) = -2.$$

We claim that, for any $x \in (0, \lambda]$, $\partial_\alpha \phi(x, \alpha_\lambda) > 0$. In fact, let $h(x) = \partial_\alpha \phi(x, \alpha_\lambda)$. The function h solves

$$\begin{cases} \quad h'' = (3\bar{q}_{2,\lambda}^2 - 4)h, \\ h(0) = 0, \quad h'(0) = 1. \end{cases}$$

Assume that $h(x_0) = 0$ for some $x_0 \in (0, \lambda]$. Then $h\mathbf{1}_{(0,x_0)}$ would be an admissible test function in the variational problem defining $\eta(\lambda)$, and we would obtain $\eta(\lambda) = 0$, which is not true. Recall that $\eta(\lambda)$ was defined in (5.31), as the first eigenvalue of the second variation of I_λ, and that we have shown in Lemma 5.9 that $\eta(\lambda) > 0$ for every λ.

In particular, $\partial_\alpha \phi(\lambda, \alpha_\lambda) > 0$ and we can apply the implicit function theorem to obtain a smooth map $\lambda \mapsto \alpha(\lambda)$ such that

$$\phi(\lambda, \alpha(\lambda)) = -2.$$

Then $x \mapsto \phi(x, \alpha(\lambda))$ solves the same boundary problem as $\bar{q}_{2,\lambda}$. By the uniqueness proven in Lemma 5.8, those two functions must be equal, and in particular we have $\alpha_\lambda = \alpha(\lambda)$. Moreover, differentiating the equation satisfied by $\alpha(\lambda)$, we obtain

$$\alpha'(\lambda) = -\frac{\partial_x \phi(\lambda, \alpha_\lambda)}{\partial_\alpha \phi(\lambda, \alpha_\lambda)} = -\frac{\bar{q}'_{2,\lambda}(\lambda)}{\partial_\alpha \phi(\lambda, \alpha_\lambda)} > 0.$$

In fact, the bounds (5.24) ensure that $\bar{q}'_{2,\lambda}(\lambda) \leq 0$, and equality can not occur, else $\bar{q}_{2,\lambda}$ would satisfy the same Cauchy problem as the constant map $q \equiv -2$.

Thus we have

$$\frac{\partial}{\partial \lambda}[\bar{q}_{2,\lambda}(x)] = \alpha'(\lambda)\partial_\alpha \phi(x, \alpha_\lambda) > 0 \quad \text{for } x \in (0, \lambda].$$

Let $\lambda_1 > \lambda_0 > 0$. Using the facts that $(x, \lambda) \mapsto \bar{q}_{2,\lambda}(x)$ is smooth, that $\bar{q}_{2,\lambda} < 0$ on $(0, +\infty)$, and that $\bar{q}'_{2,\lambda}(0) < 0$ (else $\bar{q}_{2,\lambda}$ would coincide with the constant solution $q \equiv 0$), we obtain

$$\bar{q}_{2,\lambda}(x) \leq -cx \quad \forall x \in [0, \lambda], \; \lambda \in [\lambda_0, \lambda_1],$$

for some constant $c > 0$. Similarly, we have

$$\partial_\lambda[\bar{q}_{2,\lambda}(x)] \geq c'x \quad \forall x \in [0,\lambda],\ \lambda \in [\lambda_0,\lambda_1].$$

Therefore we deduce from the mean value theorem the existence of a constant $C > 0$ such that

$$\bar{q}_{2,\lambda}^2(x) - \bar{q}_{2,\lambda_0}^2(x) \leq -C(\lambda-\lambda_0)x^2 \quad \forall x \in (0,\lambda_0),\ \lambda \in [\lambda_0,\lambda_1]. \tag{5.33}$$

Note that, since $\bar{q}_{2,\lambda}$ is odd, estimate (5.33) holds also for all $x \in (-\lambda_0,\lambda_0)$.

We remark that, since $\theta = -8$ and $q_1 \equiv 2/3$, formula (5.13) for $\mu(\lambda)$ simplifies to

$$\begin{aligned}\mu(\lambda) &= 2 \inf_{h\in H_0^1(-1,1),\ \int h^2=1} \int_{-1}^{1} \left(\frac{1}{\lambda^2}(h')^2 + (q_{2,\lambda}^2 - 4)h^2\right) dx \\ &= 2 \inf_{h\in H_0^1(-\lambda,\lambda)\ \int h^2=1} \int_{-\lambda}^{\lambda} \left((h')^2 + (\bar{q}_{2,\lambda}^2 - 4)h^2\right) dy.\end{aligned}$$

Let $h_0 \in H_0^1(-\lambda,\lambda)$, $\int h_0^2 = 1$, be a function at which the infimum defining $\mu(\lambda_0)$ is attained. Using the estimate (5.33), we compute, for $\lambda \in [\lambda_0,\lambda_1]$:

$$\begin{aligned}\mu(\lambda) &= 2 \inf_{h\in H_0^1(-\lambda,\lambda),\ \int h^2=1} \int_{-\lambda}^{\lambda} \left[(h')^2 + (\bar{q}_{2,\lambda}^2 - 4)h^2\right] dx \\ &\leq 2\int_{-\lambda_0}^{\lambda_0} \left[(h_0')^2 + (\bar{q}_{2,\lambda}^2 - 4)h_0^2\right] dx \\ &\leq \mu(\lambda_0) - 2C(\lambda-\lambda_0)\int h_0^2 x^2 dx,\end{aligned}$$

so that $\mu'(\lambda_0) < 0$. □

To complete the proof of Theorem 5.5, it remains to show that, for large λ, the eigenvalue exchange solution is unstable with respect to symmetry breaking perturbations. This is the content of the next result.

Proposition 5.12. *Assume $\theta = -8$. Then it holds*

$$\lim_{\lambda\to+\infty} \mu(\lambda) < 0.$$

Proof. We start by studying the limit of the rescaled map $\bar{q}_{2,\lambda}(y) = q_{2,\lambda}(y/\lambda)$ (extended to $(-\infty,+\infty)$ by $\bar{q}_{2,\lambda} \equiv \mp 2$ near $\pm\infty$). This rescaled map $\bar{q}_{2,\lambda}$ minimizes the integral

$$J_\lambda(\bar{q}_2) = \int_{-\lambda}^{\lambda} \left((\bar{q}_2')^2 + \frac{1}{2}(\bar{q}_2^2 - 4)^2\right) dy$$

subject to the boundary conditions $\bar{q}_2(\pm\lambda) = \mp 2$. For $\lambda' > \lambda$, $\bar{q}_{2,\lambda}$ is admissible in $J_{\lambda'}$. Therefore we deduce that

$$\lambda \mapsto J_\lambda(\bar{q}_{2,\lambda}) \quad \text{is non increasing.}$$

In particular, it holds

$$\int_{-\infty}^{+\infty} \left((\bar{q}_{2,\lambda}')^2 + \frac{1}{2}(\bar{q}_{2,\lambda}^2 - 4)^2\right) dx \leq C,$$

and $(\bar{q}_{2,\lambda})_{\lambda>0}$ is bounded in $H^1_{loc}(\mathbb{R})$, so that we may extract a weakly converging subsequence. On the other hand, we know from Lemma 5.10 that $\bar{q}_{2,\lambda}(y)$ is a monotonic function of λ, so that the whole sequence converges pointwise. Therefore the weak H^1_{loc} limit is unique and we do not need to take a subsequence: there exists $\bar{q}_{2,*} \in H^1_{loc}(\mathbb{R})$ such that $\bar{q}_{2,\lambda}$ converges to $\bar{q}_{2,*}$ as $\lambda \to +\infty$, on every compact interval, H^1-weakly and uniformly. Using the differential equation satisfied by $\bar{q}_{2,\lambda}$, we see that the second derivatives converge uniformly on every compact interval, so that we actually obtain convergence in $C^2_{loc}(\mathbb{R})$. In particular, the rescaled limiting map $\bar{q}_{2,*} \in C^2(\mathbb{R})$ solves the equation

$$\bar{q}''_{2,*} = (\bar{q}^2_{2,*} - 4)\bar{q}_{2,*}. \tag{5.34}$$

Moreover, using Fatou's lemma, we find that the map $\bar{q}_{2,*}$ has finite energy:

$$\int_{-\infty}^{+\infty} \left((\bar{q}'_{2,*})^2 + \frac{1}{2}(\bar{q}^2_{2,*} - 4)^2 \right) dy < +\infty. \tag{5.35}$$

Since $\bar{q}_{2,*}$ is obviously odd and non increasing, the finite energy property implies that it satisfies the boundary conditions

$$\bar{q}_{2,*}(-\infty) = 2, \quad \bar{q}_{2,*}(+\infty) = -2.$$

Recall that, since $\theta = -8$ and $q_1 \equiv 2/3$,

$$\mu(\lambda) = 2 \inf_{h \in H^1_0(-\lambda,\lambda),\, \int h^2 = 1} \int_{-\lambda}^{\lambda} \left((h')^2 + (\bar{q}^2_{2,\lambda} - 4)h^2 \right) dx.$$

We claim that the convergence of $\bar{q}_{2,\lambda}$ towards $\bar{q}_{2,*}$ implies that

$$\lim_{\lambda \to +\infty} \mu(\lambda) = 2 \inf_{h \in H^1(\mathbb{R}),\, \int h^2 = 1} \int_{-\infty}^{+\infty} \left((h')^2 + (\bar{q}^2_{2,*} - 4)h^2 \right) dx. \tag{5.36}$$

Indeed, for any $\varepsilon > 0$, we may find $h_0 \in C^\infty_c(\mathbb{R})$ such that

$$\int_{-\infty}^{+\infty} \left((h'_0)^2 + (\bar{q}^2_{2,*} - 4)h_0^2 \right) dx \leq m + \varepsilon,$$

where m denotes the infimum in the right hand side of (5.36). Choose $\Lambda > 0$ such that $\operatorname{supp} h_0 \subset [-\Lambda, \Lambda]$. Then, for any $\lambda \geq \Lambda$, it holds

$$\mu(\lambda) \leq 2 \int_{-\infty}^{+\infty} \left((h'_0)^2 + (\bar{q}^2_{2,\lambda} - 4)h_0^2 \right) dx.$$

Since $\bar{q}_{2,\lambda}$ converges uniformly to $\bar{q}_{2,*}$ on $\operatorname{supp} h_0$, we may pass to the limit in the last inequality, and deduce that

$$\lim_{\lambda \to +\infty} \mu(\lambda) \leq 2m + 2\varepsilon,$$

which proves (5.36) since ε is arbitrary.

In view of (5.36), to conclude the proof we need to find a function $h \in H^1(\mathbb{R})$, $h \neq 0$, such that

$$\int_{-\infty}^{+\infty} \left((h')^2 + (\bar{q}^2_{2,*} - 4)h^2 \right) dx < 0.$$

We claim that $h = \bar{q}_{2,*}'$ is a suitable choice. The fact that $h \neq 0$ is clear in view of the boundary conditions satisfied by $\bar{q}_{2,*}$. The fact that $h \in H_0^1(\mathbb{R})$ follows from the finite energy property (5.35). Indeed, (5.35) clearly implies that $h \in L^2(\mathbb{R})$, and also that $(\bar{q}_{2,*}^2 - 4) \in L^2$, so that

$$h' = (\bar{q}_{2,*}^2 - 4)\bar{q}_{2,*} \in L^2(\mathbb{R})$$

since $\bar{q}_{2,*} \in L^\infty$.

Moreover, differentiating the equation satisfied by $\bar{q}_{2,*}$, we obtain

$$h'' = (3\bar{q}_{2,*}^2 - 4)h,$$

so that

$$\int_{-\infty}^{+\infty} \left((h')^2 + (\bar{q}_{2,*}^2 - 4)h^2\right) dx = -2\int_{-\infty}^{+\infty} \bar{q}_{2,*}^2 h^2 \, dx < 0,$$

and the proof is complete. □

Now Theorem 5.5 is obtained directly by putting together the propositions 5.7, 5.11 and 5.12 above.

5.5.2 The proof of Theorem 5.6

We define the map

$$\mathcal{G} : (0, +\infty) \times H_0^1(-1,1)^3 \longrightarrow H^{-1}(-1,1)^3,$$

defined by

$$G(\lambda, Q) = DE_\lambda(\chi_\lambda + Q), \quad \text{where } \chi_\lambda = \left(\frac{2}{3}, q_{2,\lambda}, 0\right).$$

By definition of the eigenvalue exchange solution, it holds

$$\mathcal{G}(\lambda, 0) = 0.$$

From Section 5.3 we know that

$$D_Q\mathcal{G}(\lambda, 0)[h_1, h_2, h_3] = (\mathcal{M}_\lambda(h_1, h_2), \mathcal{L}_\lambda h_3),$$

and $\mathcal{M}_\lambda$ is invertible since $\nu(\lambda) > 0$. Recall indeed from Proposition 5.3 that the branch χ_λ is defined only when $\nu(\lambda) > 0$.

As for $\mathcal{L}_{\lambda_c}$, its first eigenvalue is $\mu(\lambda_c) = 0$ and it is simple. Therefore we obtain

$$\dim \operatorname{Ker} D_Q\mathcal{G}(\lambda_c, 0) = 1 = \operatorname{codim} \operatorname{Ran} D_Q\mathcal{G}(\lambda_c, 0),$$

since $D_Q\mathcal{G}(\lambda_c, 0)$ is obviously Fredholm of index 0.

Next we show that, for all $H \in \operatorname{Ker} D_Q\mathcal{G}(\lambda_c, 0)$, $H \neq 0$, it holds

$$\partial_\lambda D_Q\mathcal{G}(\lambda_c, 0) \cdot H \notin \operatorname{Ran} D_Q\mathcal{G}(\lambda_c, 0).$$

To this end, we use an argument similar to one in the proof of [8, Theorem 5.24]. Recall that

$$\text{Ker } D_Q\mathcal{G}(\lambda_c, 0) = \text{Span }(0, 0, h_{\lambda_c}),$$

where h_λ is an eigenfunction associated with the first eigenvalue of $\mathcal{L}_\lambda$ and can be chosen to depend smoothly on λ (see the proof of Proposition 5.3).

Hence it suffices to show that

$$\partial_\lambda \mathcal{L}_\lambda|_{\lambda=\lambda_c} h_{\lambda_c} \notin \text{Ran } \mathcal{L}_{\lambda_c}.$$

We obtain this latter fact as a consequence of $\mu'(\lambda_c) < 0$. Indeed, assume that there exists $h \in H_0^1$ such that

$$\partial_\lambda \mathcal{L}_\lambda|_{\lambda=\lambda_c} h_{\lambda_c} = \mathcal{L}_{\lambda_c} h.$$

Then we compute, using the facts that $\mathcal{L}_{\lambda_c} h_{\lambda_c} = 0$ and that $\mathcal{L}_{\lambda_c}$ is symmetric,

$$\begin{aligned} 0 > \mu'(\lambda_c) &= \frac{d}{d\lambda}\left[< \mathcal{L}_\lambda h_\lambda, h_\lambda >\right]_{\lambda=\lambda_c} \\ &=< \partial_\lambda \mathcal{L}_\lambda|_{\lambda=\lambda_c} h_{\lambda_c}, h_{\lambda_c} > \\ &=< \mathcal{L}_{\lambda_c} h, h_{\lambda_c} >= 0, \end{aligned}$$

and we obtain a contradiction.

Thus, all the assumptions needed to apply Crandall-Rabinowitz' bifurcation theorem [42, Theorem 1.7] are satisfied: there exists a smooth function $\lambda(t)$ defined for small t, with $\lambda(0) = \lambda_c$, and a regular family $H_t = (h_{1,t}, h_{2,t}, h_{3,t})$ taking values in $(0, 0, h_{\lambda_c})^\perp \subset H_0^1(-1, 1)^3$ with $H_0 = 0$, such that, for any Q close enough to χ_{λ_c},

$$DE_\lambda(Q) = 0 \quad \Leftrightarrow \quad \begin{cases} \quad Q = \chi_\lambda \\ \text{or} \quad \lambda = \lambda(t) \text{ and } Q = \chi_{\lambda(t)} + t(0, 0, h_{\lambda_c}) + t^2 H_t. \end{cases}$$

One can say a little bit more about the new branch of solutions thus obtained. Indeed, changing q_3 to $-q_3$ leaves the equations (5.6) invariant. More precisely, given $Q = (q_1, q_2, q_3)$ a solution of (5.6), the map $\widetilde{Q} = (q_1, q_2, -q_3)$ is automatically a solution of (5.6). In particular, to a solution

$$Q = \chi_{\lambda(t)} + t(0, 0, h_{\lambda_c}) + t^2 H_t,$$

corresponds a solution

$$\widetilde{Q} = \chi_{\lambda(t)} - t(0, 0, h_{\lambda_c}) + t^2 \widetilde{H}_t.$$

Since both Q and $\widetilde{Q}$ are close to χ_{λ_c}, we deduce that

$$\lambda(t) = \lambda(-t) \quad \text{and } h_{1,-t} = h_{1,t},\ h_{2,-t} = h_{2,t},\ h_{3,-t} = -h_{3,t}.$$

This ends the proof of Theorem 5.6.

5.6 The perturbed case $\theta \approx -8$

Now we turn back to the case of a general temperature $\theta \in (-\infty, 1]$. A closer look at the proof in subsection 5.5.2 will convince us that a result similar to Theorem 5.6 holds for any θ satisfying some nondegeneracy assumptions. After having checked that these non degeneracy assumptions are stable under small perturbations of θ, we will obtain as a corollary a result similar to Theorem 5.6 in the perturbed case $\theta \approx -8$.

Theorem 5.13. *Assume that θ is such that the branch of eigenvalue exchange solutions $\lambda \mapsto \chi_\lambda$ given by Proposition 5.3 has the following two properties:*
(i) there exists $\lambda \in (0, \lambda_*)$ such that $\mu(\lambda) < 0$.
(ii) denoting by $\lambda_c > 0$ the infimum of all such λ:

$$\lambda_c = \inf \{\lambda \in (0, \lambda_*) \colon \mu(\lambda) < 0\},$$

it holds

$$\mu'(\lambda_c) < 0.$$

Then there exist $\delta, \varepsilon > 0$ and a neighborhood $\mathfrak{A}$ of χ_{λ_c} in $\mathcal{H}$, such that the solutions of

$$DE_\lambda(Q) = 0, \qquad (\lambda, Q) \in (\lambda_c - \delta, \lambda_c + \delta) \times \mathfrak{A},$$

are exactly

$$Q = \chi_\lambda \quad or \quad \begin{cases} \lambda = \lambda(t) \\ Q = \chi_{\lambda_c} + t(0,0,h_c) + t^2 H_t, \end{cases} \qquad for\ some\ t \in (-\varepsilon, \varepsilon)$$

where $\lambda(t) \in (\lambda_c - \delta, \lambda_c + \delta)$ and $H_t \in h_c^\perp$ are smooth functions of $t \in (-\varepsilon, \varepsilon)$. Moreover, the following symmetry properties are satisfied:

$$\lambda(-t) = \lambda(t), \quad and \quad h_{1,-t} = h_{1,t},\ h_{2,-t} = h_{2,t},\ h_{3,-t} = -h_{3,t}, \tag{5.37}$$

where H_t is identified with $(h_{1,t}, h_{2,t}, h_{3,t})$ via (5.1).

Proof. Looking at the proof of Theorem 5.6 in subsection 5.5.2, we see that we have really only used the facts that for $\theta = -8$, *(i)* and *(ii)* are satisfied. Hence the proof of Theorem 5.6 may be reproduced word for word to prove Theorem 5.13. □

Theorem 5.13 is an abstract theorem: if θ satisfies some conditions, then we have a concrete result. But it does not tell us anything about the validity of such conditions in general.

Let us say a few words about these conditions. In view of Proposition 5.3, λ_* can be interpreted as the point where the eigenvalue exchange solution looses its stability with respect to symmetry-preserving perturbations. Condition *(i)* asks for $\mu(\lambda)$ to become negative before this point is reached. That is, condition *(i)* could be rephrased as: as λ grows, starting from the unique solution for small λ, a symmetry-breaking loss of stability occurs *before* a possible symmetry-preserving loss of stability. And condition *(ii)* asks for the symmetry-breaking loss of stability to be non degenerate. Hence condition *(ii)* is typically a generic condition.

Remark that in the special case $\theta = -8$, we have shown (Theorem 5.5) that symmetry-preserving loss of stability does not occur at all, and that symmetry-breaking loss of stability does occur, in a non degenerate way. Now we are going to show that these conditions propagate to nearby θ. This is the content of the next result.

Proposition 5.14. *If $\theta_0 < 1$ satisfies conditions (i)-(ii) of Theorem 5.13, then there exists $\varepsilon > 0$ such that every $\theta < 1$ with $|\theta - \theta_0| < \varepsilon$ also satisfies (i)-(ii).*

Proof. During this proof we will emphasize the dependence on θ of the objects we have been working with. For instance we will write $\mathcal{H}_\theta$, $E^{ee}_{\theta,\lambda}$, $\lambda_*(\theta)$, and so on.

Let us start by remarking that a value λ_0 (provided by Proposition 5.2), under which there is uniqueness of the solution, may be chosen independently of θ in a neighborhood of θ_0. Indeed, the proof of Lemma 5.17 shows that this value of λ_0 depends on the $W^{2,\infty}$ norm of the bulk energy density f restricted to values of Q satisfying the maximum principle (5.5). It is clear from the expression of f and (5.5) that this $W^{2,\infty}$ norm depends at least continuously on θ. We may thus choose a λ_0 that works for all θ in a fixed neighborhood of θ_0.

The idea of the proof is to use the implicit function theorem to define eigenvalue exchange solutions $\chi_{\lambda,\theta}$ depending smoothly on λ and θ. For θ close enough to θ_0, this branch will look very much like the branch χ_{λ,θ_0}, and thus will satisfy *(i)-(ii)*.

To apply the implicit function theorem we need a fixed space, but $\mathcal{H}^{ee}_\theta$ depends on θ. Thus we fix $\chi_\theta \in \mathcal{H}^{ee}_\theta$ depending smoothly on θ (for instance take χ_θ to be affine), and we will work instead in the space $H^1_0(-1,1)^2$ after having translated by χ_θ.

Since we will apply the implicit function theorem near each λ, but need to obtain for each θ a whole branch $\lambda \mapsto \chi_{\lambda,\theta}$, we will have to restrict λ to a compact interval. That is why we choose $\lambda_1 \in (\lambda_c, \lambda_*)$, where $\lambda_c = \lambda_c(\theta_0)$ (defined by *(ii)*) and $\lambda_* = \lambda_*(\theta_0)$.

We consider the smooth function $\mathcal{F}$ defined by

$$\mathcal{F}(\theta,\lambda,\chi) = DE^{ee}_{\lambda,\theta}(\chi_\theta + \chi) \in H^{-1}(-1,1)^2,$$

for $\theta < 1$, $\lambda > 0$ and $\chi \in H^1_0(-1,1)^2$. For all $\lambda \in (0,\lambda_*)$, it holds

$$\mathcal{F}(\theta_0,\lambda,\chi_{\lambda,\theta_0} - \chi_{\theta_0}) = 0,$$

and, since $\nu(\lambda) > 0$, the partial differential

$$D_\chi\mathcal{F}(\theta_0,\lambda,\chi_{\lambda,\theta_0} - \chi_{\theta_0}) \text{ is invertible.}$$

Hence the implicit function theorems provides us with $\varepsilon_\lambda > 0$ and $\mathfrak{A}_\lambda$ a neighborhood of $\chi_{\lambda,\theta_0} - \chi_{\theta_0}$ such that the equation

$$\mathcal{F}(\theta,\lambda',\chi) = 0, \quad |\theta - \theta_0| < \varepsilon_\lambda,\ |\lambda' - \lambda| < \varepsilon_\lambda,\ \chi \in \mathfrak{A}_\lambda,$$

has a unique solution $\chi_{\lambda',\theta} - \chi_\theta$ depending smoothly on (λ',θ).

Using the compactness of $[\lambda_0/2, \lambda_1]$, we deduce the existence of $\varepsilon > 0$ and $\mathfrak{A}$ a neighborhood of 0 in $H_0^1(-1,1)^2$, such that the equation

$$\mathcal{F}(\theta, \lambda, \chi) = 0, \quad |\theta - \theta_0| < \varepsilon, \ \frac{\lambda_0}{2} \leq \lambda \leq \lambda_1, \chi \in \chi_{\lambda,\theta_0} - \chi_{\theta_0} + \mathfrak{A},$$

has a unique solution $\chi_{\theta,\lambda} - \chi_\theta$ which depends smoothly on (θ, λ). Hence, for every $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$, the unique smooth branch of eigenvalue exchange solutions given by Proposition 5.3 is $\lambda \mapsto \chi_{\theta,\lambda}$, defined at least up to λ_1, and it depends smoothly on θ.

More precisely, we have just proven that $\chi_{\theta,\lambda}$ depends smoothly on $(\theta, \lambda) \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \times (\lambda_0/2, \lambda_1)$. On the other hand, since λ_0 is choosen in such a way that the unique solution $\chi_{\theta,\lambda}$ is non degenerate for $\lambda < \lambda_0$ (see the proof of Theorem 5.17), we may apply the implicit function theorem to obtain that $(\theta, \lambda) \mapsto \chi_{\theta,\lambda}$ is smooth also for small λ. Hence $\chi_{\theta,\lambda}$ depends smoothly on $(\theta, \lambda) \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \times (0, \lambda_1)$.

Recall that, for fixed θ, given a branch of eigenvalue exchange solutions χ_λ, we have defined $\mu(\lambda)$ in (5.13), as the first eigenvalue of the free energy second variation around χ_λ with respect to symmetry-breaking perturbations. Here we emphasize the dependence on θ by writing $\mu(\theta, \lambda)$. That is, $\mu(\theta, \lambda)$ is the first eigenvalue of $\mathcal{L}_{\theta,\lambda}$, which is the linear operator associated to the quadratic form $D^2 E_{\theta,\lambda}$ restricted to the space H_{sb} of symmetry-breaking perturbations (see Section 5.3).

Since $(\theta, \lambda) \mapsto \chi_{\theta,\lambda}$ is smooth, we prove, exactly as in Proposition 5.3 for $\lambda \mapsto \mu(\lambda)$, that

$$(\theta_0 - \varepsilon, \theta_0 + \varepsilon) \times (0, \lambda_1) \ni (\theta, \lambda) \mapsto \mu(\theta, \lambda)$$

is smooth.

In particular, since – by (i) – there exists $\lambda_2 \in (\lambda_c(\theta_0), \lambda_1)$ such that $\mu(\theta_0, \lambda_2) < 0$, it follows that we may chose ε small enough, so that

$$\mu(\theta, \lambda_2) < 0 \quad \forall \theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon),$$

i.e. (i) is satisfied for θ close enough to θ_0.

By definition of $\lambda_c = \lambda_c(\theta_0)$, and since θ_0 satisfies (ii), it holds

$$\mu(\theta_0, \lambda_c) = 0, \quad \frac{\partial \mu}{\partial \lambda}(\theta_0, \lambda_c) > 0.$$

Therefore the implicit function theorem ensures the existence of a smooth map $\lambda(\theta)$ defined – up to choosing ε small enough – for $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$, such that

$$\mu(\theta_0, \lambda(\theta)) = 0, \quad \frac{\partial \mu}{\partial \lambda}(\theta, \lambda(\theta)) > 0.$$

In order to complete the proof of Proposition 5.14, we need to show that this $\lambda(\theta)$ is really the critical value $\lambda_c(\theta)$ that appears in (ii).

That is, we need to prove that (for ε small enough),

$$\mu(\theta, \lambda) > 0 \quad \text{for } \theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon), \ \lambda \in (0, \lambda(\theta)). \tag{5.38}$$

We start by noting that the choice of λ_0 in the proof of the uniqueness result Theorem 5.17 can be such that

$$\mu(\theta, \lambda) \geq c_0 \quad \forall (\theta, \lambda) \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \times (0, \lambda_0), \tag{5.39}$$

for some $c_0 > 0$. On the other hand, ε may be choosen in such a way that it holds

$$\frac{\partial \mu}{\partial \lambda}(\theta, \lambda) > 0 \quad \text{for } (\theta, \lambda) \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \times (\lambda_c - \delta, \lambda_c + \delta). \tag{5.40}$$

Using the compactness of $[\lambda_0, \lambda_c - \delta]$ and the fact that $\mu(\theta_0, \lambda) > 0$ for all $\lambda \in (0, \lambda_c)$, we may also choose ε such that we have

$$\mu(\theta, \lambda) > 0 \quad \forall (\theta, \lambda) \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon) \times [\lambda_0, \lambda_c - \delta]. \tag{5.41}$$

Putting together (5.39), (5.6) and (5.41), we obtain (5.38). Therefore, $\lambda(\theta)$ is really the infimum of those λ for which $\mu(\theta, \lambda) < 0$, and $\theta \in (\theta_0 - \varepsilon, \theta_0 + \varepsilon)$ satisfies (ii). □

As we pointed out at the beginning of the present section, a corollary of Theorem 5.5 and Proposition 5.14 is that the bifurcation result Theorem 5.13 applies to all θ close enough to the special value $\theta = -8$.

Corollary 5.15. *There exists $\varepsilon > 0$ such that, for any $\theta < 1$ with $|\theta + 8| < \varepsilon$, a symmetric pitchfork bifurcation occurs from the branch of eigenvalue exchange solutions starting at small λ, in the sense that Theorem 5.13 applies.*

Appendix 5.A Principle of symmetric criticality

Proposition 5.16. *Let H be a Hilbert space, G a group acting linearly and isometrically on H and $\Sigma = H^G$ the subspace of symmetric elements (that is, $x \in \Sigma$ iff $gx = x$ $\forall g \in G$). Let $f : H \to \mathbb{R}$ be a G-invariant C^1 function. It holds:*

(i) If $x \in \Sigma$ is a critical point of $f_{|\Sigma}$, then x is a critical point of f.

(ii) If in addition f is C^2, it further holds

$$D^2 f(x) \cdot h \cdot k = 0 \quad \text{for } h \in \Sigma,\ k \in \Sigma^\perp,$$

i.e. the orthogonal decomposition $H = \Sigma \oplus \Sigma^\perp$ is also orthogonal for the bilinear form $D^2 f(x)$.

Item *(i)* of the above proposition is only a particularly simple case of Palais' Principle of symmetric criticality [108]. Item *(ii)* however does not seem to be explicitly stated in the literature – as far as we know. Using the same tools as in Section 2 of [108], it is not hard to see that an equivalent of *(ii)* is actually valid if H is replaced by a Riemannian manifold $\mathcal{M}$ on which the group G acts isometrically. In this case, Σ is a submanifold of $\mathcal{M}$ and, at a symmetric critical point x, the orthogonal decomposition

$$T_x\mathcal{M} = T_x\Sigma \oplus (T_x\Sigma)^\perp$$

is also orthogonal for the bilinear form $D^2 f(x)$.

Proof of Proposition 5.16. As already pointed out, item *(i)* is a particular case of [108, Section 2]. We nevertheless present a complete proof of Proposition 5.16 here, since in the simple framework we consider, the proof of *(i)* is really straightforward.

The fact that f is G-invariant means that it holds

$$f(gx) = f(x) \quad \forall g \in G,\ x \in H. \tag{5.42}$$

Since the action of G on H is linear, differentiating (5.42) we obtain

$$Df(gx) \cdot gh = Df(x) \cdot h \quad \forall h \in H. \tag{5.43}$$

Applying (5.43) for a symmetric x, i.e. $x \in \Sigma$, we have

$$< \nabla f(x), gh >=< \nabla f(x), h > \quad \forall h \in H.$$

Note that here we distinguish between the differential $Df(x) \in H^*$ and the gradient $\nabla f(x) \in H$. Similarly, below we will distinguish between the second order differential $D^2 f(x) \in \mathcal{L}(H, H^*)$ and the hessian $\nabla^2 f(x) \in \mathcal{L}(H)$. Since g is a linear isometry, we conclude that

$$g^{-1}\nabla f(x) = \nabla f(x) \quad \forall g \in G, \qquad \text{i.e. } \nabla f(x) \in \Sigma. \tag{5.44}$$

Therefore, if we know in addition that x is a critical point of $f_{|\Sigma}$, which means that $\nabla f(x) \in \Sigma^\perp$, it must hold $\nabla f(x) = 0$. This proves *(i)*.

Now assume that f is C^2 and differentiate (5.43) to obtain

$$D^2 f(gx) \cdot gh \cdot gk = D^2 f(x) \cdot h \cdot k \quad \forall h, k \in H. \tag{5.45}$$

In particular, if x and h are symmetric (i.e. belong to Σ), and if we denote by $\nabla^2 f(x)$ the Hessian of f at x, (5.45) becomes

$$< g^{-1}(\nabla^2 f(x)h), k >=< \nabla^2 f(x)h, k > \quad \forall x \in \Sigma,\ h \in \Sigma,\ k \in H,$$

so that $\nabla^2 f(x)h$ is symmetric. Hence it is orthogonal to any $k \in \Sigma^\perp$, which proves *(ii)*. □

Appendix 5.B Uniqueness of critical points for small λ

Let $\Omega \subset \mathbb{R}^N$ be a smooth bounded domain, and $f : \mathbb{R}^d \to \mathbb{R}$ a $W^{2,\infty}_{loc}$ map. We are interested in critical points of functionals of the form

$$E_\lambda(u) = \int_\Omega \frac{1}{2\lambda^2}|\nabla u|^2 + \int_\Omega f(u), \tag{5.46}$$

i.e. solutions $u \in H^1(\Omega)^d$ of the equation

$$\Delta u = \lambda^2 \nabla f(u) \quad \text{in } \mathcal{D}'(\Omega). \tag{5.47}$$

Note that (5.47) implies in particular that $\nabla f(u) \in L^1_{loc}$.

We prove the following:

Theorem 5.17. *Assume that there exists $C > 0$ such that $\nabla f(x) \cdot x \geq 0$ for any $x \in \mathbb{R}^d$ with $|x| \geq C$.*

Let $g \in L^\infty \cap H^{1/2}(\partial\Omega)^d$. There exists $\lambda_0 = \lambda_0(\Omega, f, g)$ such that, for any $\lambda \in (0, \lambda_0)$, E_λ admits at most one critical point with $\operatorname{tr} u = g$ *on $\partial\Omega$.*

Theorem 5.17 is a direct consequence of Lemmas 5.18 and 5.19 below. Indeed, Lemma 5.18 ensures that, for sufficiently small λ, E_λ admits at most one critical point satisfying a given L^∞ bound (independent of λ). And in Lemma 5.19 we prove that the assumption on f implies such a bound for critical points of E_λ.

Lemma 5.18. *Let $C > 0$. There exists $\lambda_0 = \lambda_0(C, f, \Omega)$ such that, for any $\lambda \in (0, \lambda_0)$ and any $g \in H^{1/2}(\partial\Omega)^d$, E_λ admits at most one critical point u satisfying $|u| \leq C$ a.e. and $\operatorname{tr} u = g$.*

Proof. Let

$$X := \left\{u \in H^1(\Omega) : \ |u| \leq C \text{ a.e.}\right\}.$$

We show that, for λ small enough, E_λ is strictly convex on X.

Let $u, v \in X$. Then $u - v \in H^1_0(\Omega)^d$. Using Poincaré's inequality, we obtain

$$\begin{aligned} E_\lambda\left(\frac{u+v}{2}\right) &= \frac{1}{8\lambda^2}\int |\nabla u + \nabla v|^2 + \int f\left(\frac{u+v}{2}\right) \\ &= \frac{1}{4\lambda^2}\int |\nabla u|^2 + \frac{1}{4\lambda^2}\int |\nabla v|^2 - \frac{1}{8\lambda^2}\int |\nabla(u-v)|^2 \\ &\quad + \int f\left(\frac{u+v}{2}\right) \\ &= \frac{1}{2}E_\lambda(u) + \frac{1}{2}E_\lambda(v) - \frac{1}{8\lambda^2}\int |\nabla(u-v)|^2 \\ &\quad + \int \left[f\left(\frac{u+v}{2}\right) - \frac{1}{2}f(u) - \frac{1}{2}f(v)\right] \\ &\leq \frac{1}{2}E_\lambda(u) + \frac{1}{2}E_\lambda(v) - \frac{c_1(\Omega)}{\lambda^2}\|u-v\|^2_{L^2} \\ &\quad + \int \left[f\left(\frac{u+v}{2}\right) - \frac{1}{2}f(u) - \frac{1}{2}f(v)\right]. \end{aligned} \tag{5.48}$$

On the other hand, for any $x, y \in \mathbb{R}^d$ satisfying $|x|, |y| \leq C$, it holds

$$f\left(\frac{x+y}{2}\right) - \frac{1}{2}f(x) - \frac{1}{2}f(y) \leq \|f\|_{W^{2,\infty}(B_C)}|x-y|^2. \tag{5.49}$$

Plugging (5.49) into (5.48) we obtain, for some $c_2 = c_2(\Omega, f, C) > 0$,

$$\begin{aligned} E_\lambda\left(\frac{u+v}{2}\right) &\leq \frac{1}{2}E_\lambda(u) + \frac{1}{2}E_\lambda(v) - \frac{c_1}{\lambda^2}\|u-v\|^2_{L^2} + c_2\|u-v\|^2_{L^2} \\ &= \frac{1}{2}E_\lambda(u) + \frac{1}{2}E_\lambda(v) - \frac{c_1}{2\lambda^2}\|u-v\|^2_{L^2} - c_2\left(\frac{c_1}{2c_2\lambda^2} - 1\right)\|u-v\|^2_{L^2}. \end{aligned}$$

Hence, for $\lambda \leq \lambda_0 := \sqrt{c_1/(2c_2)}$, it holds

$$E_\lambda\left(\frac{u+v}{2}\right) < \frac{1}{2}E_\lambda(u) + \frac{1}{2}E_\lambda(v) \quad \forall u, v \in X, \ u \neq v.$$

Thus E_λ is strictly convex on X.

To conclude the proof, assume that for a $\lambda \in (0, \lambda_0)$, there exist two solutions u_1 and u_2 of (5.47), belonging to X. Then one easily shows that $[0,1] \ni t \mapsto E_\lambda(tu_1 + (1-t)u_2)$ is C^1 and that its derivative vanishes at 0 and 1, which is incompatible with the strict convexity of E_λ. ∎

Lemma 5.19. *Assume that there exists $C > 0$ such that*

$$|x| \geq C \quad \Rightarrow \quad \nabla f(x) \cdot x \geq 0.$$

Let $g \in L^\infty \cap H^{1/2}(\partial\Omega)^d$. If $u \in H^1_g(\Omega)^d$ is a critical point of E_λ, then it holds

$$|u| \leq \max(C, \|g\|_\infty) \quad a.e.$$

Proof. We may assume $C = \max(C, \|g\|_\infty) > 0$.

Let $\varphi \in C^\infty(\mathbb{R})$ be such that:

$$\begin{cases} \varphi \geq 0, \\ \varphi' \geq 0, \\ \varphi(t) = 0 \quad \text{for } t \leq C^2, \\ \varphi(t) = 1 \quad \text{for } t \geq T, \text{ for some } T > C^2. \end{cases} \tag{5.50}$$

Let $w = \varphi(|u|^2)$. The assumptions on φ ensure that $w \geq 0$, and $w = 0$ in $\{|u| \leq C\}$.

Therefore, taking the scalar product of (5.47) with wu and using the assumption that $\nabla f(u) \cdot u \geq 0$ outside of $\{|u| \leq C\}$, we obtain

$$\frac{1}{\lambda^2} wu \cdot \Delta u = w \nabla f(u) \cdot u \geq 0 \quad \text{a.e.} \tag{5.51}$$

Since $wu \in H^1_0(\Omega)^d$, we may apply Lemma 5.20 below, to deduce

$$\int_\Omega \nabla u \cdot \nabla(wu) \leq 0. \tag{5.52}$$

On the other hand, it holds

$$\int_\Omega \nabla u \cdot \nabla(wu) = \int_\Omega w|\nabla u|^2 + \int_\Omega 2 \sum_k (u \cdot \partial_k u)^2 \varphi'(|u|^2),$$

so that we have in fact

$$\int_\Omega w|\nabla u|^2 \leq 0. \tag{5.53}$$

Finally we may choose an increasing sequence φ_k of smooth maps satisfying (5.50) and converging to $\mathbf{1}_{t>C^2}$. Then $w_k = \varphi_k(|u|^2)$ is increasing and converges a.e. to $\mathbf{1}_{|u|>C}$, and we conclude that

$$\int_{|u|>C} |\nabla u|^2 = 0,$$

so that $|u| \leq C$ a.e. □

The following result, which we used in the proof of Lemma 5.19, is due to Pierre Bousquet.

Lemma 5.20. *Let $u \in H^1(\Omega)^d$ and assume that $\Delta u = g \in L^1_{loc}(\Omega)^d$. Then, for any $\zeta \in H^1_0(\Omega)^d$,*

$$\zeta \cdot g \geq 0 \text{ a.e.} \implies \int \nabla\zeta \cdot \nabla u \leq 0. \tag{5.54}$$

Proof. We proceed in three steps: first we show that (5.54) is valid for $\zeta \in H^1 \cap L^\infty(\Omega)^d$ with compact support in Ω, then for $\zeta \in H^1_0 \cap L^\infty(\Omega)^d$, and eventually for $\zeta \in H^1_0(\Omega)^d$.

Step 1: $\zeta \in H^1_c \cap L^\infty$.

Since ζ is bounded and compactly supported, there exists a sequence ζ_k of C_c^∞ functions, a constant $C > 0$, and a compact $K \subset \Omega$, such that

$$\operatorname{supp} \zeta_k \subset K, \quad \|\zeta_k\|_\infty \leq C, \text{ and } \zeta_k \longrightarrow \zeta \text{ in } H^1 \text{ and a.e..}$$

Since $\zeta_k \in C_c^\infty(\Omega)^d$, it holds, by definition of the weak laplacian,

$$\int \zeta_k \cdot g = -\int \nabla \zeta_k \cdot \nabla u,$$

and we may pass to the limit (using dominated convergence on the compact K for the left hand side) to obtain

$$\int_\Omega \zeta \cdot g = -\int \nabla \zeta \cdot \nabla u,$$

which implies (5.54).

Step 2: $\zeta \in H^1_0 \cap L^\infty$.

Let $\theta_k \in C_c^\infty(\Omega)$ be such that

$$0 \leq \theta_k \leq 1,\ \theta_k(x) = 1 \text{ if } d(x, \partial\Omega) > \frac{1}{k}, \text{ and } |\nabla \theta_k(x)| \leq \frac{c}{d(x, \partial\Omega)},$$

and define $\zeta_k = \theta_k \zeta \in H^1_c \cap L^\infty(\Omega)^d$.

Assuming that $\zeta \cdot g \geq 0$ a.e., we deduce that $\zeta_k \cdot g \geq 0$ a.e., and thus we may apply *Step 1* to ζ_k: it holds

$$0 \geq \int \nabla \zeta_k \cdot \nabla u = \int \theta_k \nabla \zeta \cdot \nabla u + \int \nabla \theta_k \cdot \nabla u \cdot \zeta. \tag{5.55}$$

The first term in the right hand side of (5.55) converges to $\int \nabla \zeta \cdot \nabla u$, by dominated convergence. Therefore we only need to prove that the second term in the right hand side of (5.55) converges to zero. To this end we use the following Hardy-type inequality:

$$\int \frac{|\zeta|^2}{d(x, \partial\Omega)^2} \leq C \int |\nabla \zeta|^2, \qquad \forall \zeta \in H^1_0(\Omega). \tag{5.56}$$

Using (5.56) and the Hölder inequality, we obtain

$$\left| \int \nabla \theta_k \cdot \nabla u \cdot \zeta \right|^2 \leq C \|\nabla \zeta\|^2_{L^2} \int_{d(x,\partial\Omega) > 1/k} |\nabla u|^2 \longrightarrow 0,$$

which concludes the proof of *Step 2*.

Step 3: $\zeta \in H^1_0$.

We define $\zeta_k = P_k(\zeta)$, where $P_k : \mathbb{R}^d \to \mathbb{R}^d$ is given by

$$P_k(x) = \begin{cases} x & \text{if } |x| \leq k, \\ \frac{k}{|x|} x & \text{if } |x| > k. \end{cases}$$

Then $\zeta_k \in H_0^1 \cap L^\infty(\Omega)$ and $\zeta_k \to \zeta$ in H^1.

If $\zeta \cdot g \geq 0$, then it obviously holds $\zeta_k \cdot g \geq 0$, so that we may apply *Step 2* to ζ_k and obtain

$$\int \nabla \zeta_k \cdot \nabla u \leq 0.$$

Letting k go to ∞ in this last inequality provides the desired conclusion. □

Appendix 5.C Second variation of the energy

At a map $Q \in H^1(-1,1)^3$, the second variation of the energy reads

$$D^2 E(Q)[H] = \int \left(\frac{1}{\lambda^2}(H')^2 + D^2 f(Q)[H] \right) dx,$$

where

$$D^2 f(Q)[H] = \frac{\theta}{3}|H|^2 - 4Q \cdot H^2 + (Q \cdot H)^2 + \frac{1}{2}|Q|^2|H|^2.$$

If we take $Q = \chi = (q_1, q_2, 0)$, and consider separately perturbations $H_{sp} = (h_1, h_2, 0)$ and $H_{sb} = (0, 0, h_3)$, we have

$$\begin{aligned}
|H_{sp}|^2 &= 6h_1^2 + 2h_2^2 & |H_{sb}|^2 &= 2h_3^2 \\
\chi \cdot H_{sp}^2 &= 2q_1(h_2^2 - 3h_1^2) + 4q_2 h_1 h_2 & \chi \cdot H_{sb}^2 &= 2q_1 h_3^2 \\
\chi \cdot H_{sp} &= 6q_1 h_1 + 2q_2 h_2 & \chi \cdot H_{sb} &= 0,
\end{aligned}$$

so that we can compute

$$\begin{aligned}
D^2 f(\chi)[H_{sp}] &= 6\left(\frac{\theta}{3} + 2q_1 + 9q_1^2 + q_2^2\right) h_1^2 \\
&\quad + 2\left(\frac{\theta}{3} - 4q_1 + 3q_1^2 + 3q_2^2\right) h_2^2 \\
&\quad + 8q_2(3q_1 - 2)h_1 h_2 \\
D^2 f(\chi)[H_{sb}] &= 2\left(\frac{\theta}{3} - 4q_1 + 3q_1^2 + q_2^2\right) h_3^2.
\end{aligned}$$

Deuxième partie

Supraconductivité

Chapitre 6

Existence de points critiques avec conditions semi-rigides dans des domaines simplement connexes du plan

(avec Petru Mironescu)

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6.1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a smooth bounded simply connected domain. Let a map u belong to the space

$$\mathcal{E} := \left\{ u \in H^1(\Omega, \mathbb{C});\ |\operatorname{tr} u| = 1 \right\},$$

where $\operatorname{tr} u$ denotes the trace of u on the boundary $\partial\Omega$. Then the trace $\operatorname{tr} u$ of u on $\partial\Omega$ belongs to the space $H^{1/2}(\partial\Omega; \mathbb{S}^1)$, and therefore we can define its winding number or degree, which we denote by $\deg(u, \partial\Omega)$ (see [30, Appendix]; see also [20, Section 2] for more details). This allows us to define the class

$$\mathcal{E}_d = \left\{ u \in H^1(\Omega;\mathbb{C});\ |\operatorname{tr} u| = 1,\ \deg(u, \partial\Omega) = d \right\}.$$

In this paper we study the existence of critical points of the Ginzburg-Landau energy functional

$$E_\varepsilon(u) = \frac{1}{2}\int_\Omega |\nabla u|^2 + \frac{1}{4\varepsilon^2}\int_\Omega (1-|u|^2)^2$$

in the space $\mathcal{E}_d$, i.e., of critical points with prescribed degree d. More specifically, we are interested in non trivial critical points, that is critical points which are not constants of modulus one.

The prescribed degree boundary condition is an intermediate model between the Dirichlet and the Neumann boundary conditions. The asymptotic of minimizers of the Ginzburg-Landau energy E_ε with Dirichlet boundary condition was first studied by Bethuel, Brezis and Hélein in their classical work [25]. In particular, it was shown in [25] that minimizers u_ε have zeros « well-inside » Ω, and that these zeros approach the singularities (vortices) of the limit u_* of the u_ε's as $\varepsilon \to 0$. In contrast, the only minimizers of E_ε with no boundary condition are constants. The same holds even for stable critical points of E_ε with Neumann boundary conditions [126]. The analysis of the prescribed degrees boundary condition (in domains which may be multiply connected) leads to a richer global picture [17], [55], [19], [18], [21], [49], [20]. More specifically, in multiply connected domains minimizers of E_ε may exist [55], [19] or not [18]. However, in such domains critical points of E_ε always exist [21], [49]. In simply connected domains, minimizers never exist [19]. More involved is the study of the existence of critical points in simply connected domains; this is our purpose. Typical methods in absence of absolute minimizers consist in constructing local minimizers, or in constructing critical points by minimax methods. Construction of local minimizers proved to be successful in multiply connected domains [21], but the arguments there do not adapt to our case. Minimax techniques led in [20] to the proof of the existence of critical points in simply connected domains for *large* ε, but again these techniques do not seem to work for *small* ε.

The present paper is devoted to the existence of critical points for *small* ε and thus complements [20]. Our approach relies on singular perturbations techniques, in the spirit of Pacard and Rivière [107]. We explain this approach in the special case where the prescribed degree is $d = 1$. We first recall the main result in [25]. Consider the minimization of E_ε with Dirichlet boundary condition:

$$\min\{E_\varepsilon(u);\ \operatorname{tr} u = g \text{ on } \partial\Omega\}.$$

Here, $g : \partial\Omega \to \mathbb{S}^1$ is smooth, and we assume that $\deg(g, \partial\Omega) = 1$. Then there exists some $a \in \Omega$ such that, possibly up to a subsequence, minimizers u_ε satisfy $u_\varepsilon \to u_*$, with

$$u_*(z) = u_{*,a,g}(z) = \frac{z-a}{|z-a|} e^{\imath H}, \text{ with } H = H_{a,g} \text{ harmonic.} \tag{6.1}$$

In (6.1), the function H is uniquely determined (mod 2π) by the condition

$$u_* = g \text{ on } \partial\Omega. \tag{6.2}$$

The point a is not arbitrary: it has to be a critical point (actually, a point of minimum) of the « renormalized energy » $W(\cdot, g)$ associated with g.

In order to explain our main results in the case of prescribed degree boundary condition, we perform a handwaving analysis of our problem when $d = 1$. Assume that u_ε is a critical point of E_ε in $\mathcal{E}_1$. Then u_ε has to vanish at some point a_ε, and up to a subsequence we have either

(i) $a_\varepsilon \to a \in \Omega$

or

(ii) $a_\varepsilon \to a \in \partial\Omega$.

Assume that (i) holds. Assume further, for the purpose of our discussion, that a_ε is the only zero of u_ε. Then the analysis in [25] suggests that the limit u_* of the u_ε's should be again of the form $u_*(z) = \dfrac{z-a}{|z-a|} e^{\imath\psi}$. Formally, the fact that u_ε is a critical point of E_ε leads, as in [25], to the conclusion that the limiting point a is a critical point of a suitable renormalized energy $\widehat{W}(\cdot)$. Some basic properties of the energy $\widehat{W}$ are studied in [85]; we will come back to this in Section 6.2. Of interest to us is the fact that $\widehat{W}$ is smooth and does have critical points.

Let a be a critical point of $\widehat{W}$, and let u_* be as in (6.1)-(6.2). We plan to construct critical points u_ε of E_ε in $\mathcal{E}_1$ such that $u_\varepsilon \to u_*$ as $\varepsilon \to 0$. Our approach is inspired by the one of Pacard and Rivière [107]. In [107], critical points of E_ε with Dirichlet boundary condition g are constructed under a nondegeneracy assumption for the corresponding renormalized energy $W(\cdot, g)$. We encounter a similar situation in our problem: we are able to construct critical points of E_ε under some nondegeneracy assumptions that we explain below.

To start with, we will see in Section 6.2 that we may associate with each point $a \in \Omega$ a natural boundary datum g^a, solution of the minimization problem

$$\min\{W(a, g);\, g : \partial\Omega \to \mathbb{S}^1,\ \deg(g, \partial\Omega) = 1\}.$$

It turns out that, if a is a critical point of $\widehat{W}$, then a is also a critical point of $W(\cdot, g^a)$ (Section 6.2). Since $\widehat{W}$ has a global maximum (Section 6.11), $\widehat{W}$ has critical points, and thus there exists some $a \in \Omega$ critical point of $W(\cdot, g^a)$. Our first nondegeneracy assumption is

$(ND1)$ there exists some $a \in \Omega$ nondegenerate critical point of $W(\cdot, g^a)$.

Assuming that *(ND1)* holds, set $g_0 := g^a$. Then we may prove that, for each g « close » to g_0 in a suitable sense, $W(\cdot, g)$ has a critical point $a(g)$ close to a (Section 6.5). Thus, to such $g \in C^{1,\beta}(\partial\Omega; \mathbb{S}^1)$ we may associate the function

$$T_*(g) \in \dot{C}^\beta(\partial\Omega; \mathbb{R}),\ T_*(g) := u_* \wedge \frac{\partial u_*}{\partial \nu},$$

where $u_* = u_{*,a(g),g}$ is given by (6.1)-(6.2). One may prove that the map $g \mapsto T_*(g)$ is C^1 near g_0, and that its differential L at g_0 is a Fredholm operator of index one (Section 6.10). Our second nondegeneracy assumption is

$(ND2)$ L is onto.

We may now state our first result.

Theorem 6.1. *Assume that (ND1) and (ND2) hold. Then, for small ε, E_ε has critical points u_ε with prescribed degree one.*

A similar result holds for an arbitrary prescribed degree d.

The conditions *(ND1)* and *(ND2)* seem to be « generic ».[1] However, it is not clear whether the assumptions *(ND1)* and *(ND2)* are ever satisfied. Therefore, our next task is to exhibit nondegeneracy situations.

Loose Theorem. *Assume that $d = 1$ and that Ω is « close » to a disc. Then (ND1) and (ND2) hold. In particular, for small ε, E_ε has critical points of prescribed degree* 1.

The above theorem applies to the unit disc $\mathbb{D}$. However, no sophisticated argument is needed for a disc. Indeed, when $\Omega = \mathbb{D}$ it is possible to construct explicit hedgehog type critical points of E_ε by minimizing E_ε in the class of the maps of the form $f(|z|)\dfrac{z}{|z|}$.

Concerning the existence of critical points of E_ε in arbitrary domains, we do not know whether *(ND1)* and *(ND2)* do always hold. However, we have the following result.

Loose Theorem. *Assume that $d = 1$. Then every Ω can be approximated with domains satisfying (ND1)-(ND2).*

Our paper contains the proof of the three above theorems, as well as generalizations to higher degrees d and a discussion about the « generic » nature of our results. The plan of the paper is the following. In Section 6.2, we recall the definition and the main properties of the renormalized energies corresponding to either Dirichlet or prescribed degree boundary condition, and establish few new properties. In Sections 6.3 and 6.4, we derive new useful formulas for the renormalized energies. In Section 6.5, we prove that nondegeneracy of critical points of $W(\cdot, g)$ is stable with respect to small perturbations of g. Section 6.6 is devoted to the proof of a variant of the Pacard-Rivière [107] construction of critical points with Dirichlet condition; this is a key step in our proof. We prove Theorem 6.1 (for arbitrary degrees d) in Section 6.8. The proof relies on a Leray-Schauder degree argument, and the corresponding key estimate is obtained in Section 6.7. In Section 6.9, we prove that the couple of conditions *(ND1)*-*(ND2)* is stable with respect to small perturbations of the domain. This and the fact that $\Omega = \mathbb{D}$ satisfies *(ND1)*-*(ND2)* (Section 6.10) implies (a rigorous form of) the first Loose Theorem. We finally discuss in Section 6.11 the « generic » nature of our results, and establish (a rigorous form of) the second Loose Theorem.

Notation

1. Points in $\mathbb{R}^2$ are denoted z in the Sections 6.3 and 6.4 relying on complex analysis techniques, and x or y elsewhere.
2. $\mathbb{D}(z, r)$, $\overline{\mathbb{D}}(z, r)$ and $C(z, r)$ denote respectively the open disc, the closed disc and the circle of center z and radius r. We let $\mathbb{D} = \mathbb{D}(0, 1)$ denote the unit disc and set $\mathbb{D}_r = \mathbb{D}(0, r)$. $\mathbb{S}^1$ is the unit circle.

1. Critical points of smooth functionals are « generically » nondegenerate, and Fredholm operators of index one are « generically » onto.

3. $\wedge$ stands for the vector product of complex numbers or vectors. Examples: $(u_1+\imath u_2)\wedge(v_1+\imath v_2)=u_1v_2-u_2v_1$, $(u_1+\imath u_2)\wedge(\nabla v_1+\imath\nabla v_2)=u_1\nabla v_2-u_2\nabla v_1$, $(\nabla u_1+\imath\nabla u_2)\wedge(\nabla v_1+\imath\nabla v_2)=\nabla u_1\cdot\nabla v_2-\nabla u_2\cdot\nabla v_1$.
4. If A is a set and k an integer, then we let
$$A_*^k=\{a=(a_1,\dots,a_k)\in A^k;\ a_j\neq a_l,\forall\, j\neq l\}.$$
5. When $k=1$, we identify a collection $a=(a_1)$ with (the point or number) a_1.
6. Additional indices emphasize the dependence of objects on variables. E.g.: $\psi_a=\psi_{a,g}$ recalls that ψ depends not only on a, but also on g.

6.2 Renormalized energies and canonical maps

In the first part of this section, we follow [25] and [85].

We fix $k\in\mathbb{N}$ and a collection $\overline{d}=(d_1,\dots,d_k)\in\mathbb{Z}^k$, and we let $d:=d_1+\cdots+d_k$. The bounded domain $\Omega\subset\mathbb{R}^2$ is assumed to be simply connected and $C^{1,\beta}$.

We consider a collection of mutually distinct points in Ω, $a=(a_1,\dots,a_k)\in\Omega_*^k$ (the prescribed singularities), and also a boundary datum $g\in H^{1/2}(\partial\Omega;\mathbb{S}^1)$, of degree d. We denote by $\mathcal{B}_d$ the space of all such boundary data. Thus
$$\mathcal{B}_d:=\{g\in H^{1/2}(\partial\Omega;\mathbb{S}^1);\ \deg(g,\partial\Omega)=d\}.$$
For small $\rho>0$, we define the open set $\Omega_\rho=\Omega\setminus\bigcup_{j=1}^k\overline{\mathbb{D}}(a_j,\rho)$, and the classes of functions
$$\mathcal{F}_{\rho,g}=\left\{v\in H^1(\Omega_\rho;\mathbb{S}^1);\ \operatorname{tr} v=g,\ \deg(v,C(a_j,\rho))=d_j\right\},\tag{6.3}$$
$$\widehat{\mathcal{F}}_\rho=\left\{v\in H^1(\Omega_\rho;\mathbb{S}^1);\ \deg(v,\partial\Omega)=d,\ \deg(v,C(a_j,\rho))=d_j\right\}.\tag{6.4}$$
The functions in these classes have prescribed winding number d_j around each a_j, and prescribed boundary condition g (respectively prescribed degree d) on $\partial\Omega$. Of course, although we do not make this dependence explicit, the above classes depend not only on ρ and g, but also on a.

In [25] and [85], minimization of the Dirichlet energy $1/2\int|\nabla v|^2$ over these spaces is studied, and the following asymptotic expansions are obtained as $\rho\to 0$:
$$\inf\left\{\frac12\int_{\Omega_\rho}|\nabla v|^2;\ v\in\mathcal{F}_{\rho,g}\right\}=\pi\left(\sum_{j=1}^k d_j^2\right)\log\frac1\rho+W(a,g)+O(\rho),\tag{6.5}$$
$$\inf\left\{\frac12\int_{\Omega_\rho}|\nabla v|^2;\ v\in\widehat{\mathcal{F}}_\rho\right\}=\pi\left(\sum_{j=1}^k d_j^2\right)\log\frac1\rho+\widehat{W}(a)+O(\rho).\tag{6.6}$$
In the above expressions, $W(a,g)$ and $\widehat{W}(a)$ are the so-called renormalized energies. These quantities depend not only on a and g, but also on $\overline{d}$ and Ω.

Explicit formulae for the above renormalized energies can be found in [25] and [85], and involve the functions $\Phi_{a,g}$ and $\widehat{\Phi}_a$ defined as follows. $\Phi_{a,g}$ is the unique solution of

$$\begin{cases} \Delta\Phi_{a,g} = 2\pi \sum_{j=1}^k d_j\delta_{a_j} & \text{in } \Omega \\ \dfrac{\partial\Phi_{a,g}}{\partial\nu} = g \wedge \dfrac{\partial g}{\partial\tau} & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega} \Phi_{a,g} = 0 & \end{cases}, \tag{6.7}$$

while $\widehat{\Phi}_a$ is the unique solution of

$$\begin{cases} \Delta\widehat{\Phi}_a = 2\pi \sum_{j=1}^k d_j\delta_{a_j} & \text{in } \Omega \\ \widehat{\Phi}_a = 0 & \text{on } \partial\Omega \end{cases}. \tag{6.8}$$

For further use, let us note that, if $\alpha \in \mathbb{S}^1$, then $\Phi_{a,g} = \Phi_{a,\alpha g}$. Therefore, we may naturally define $\Phi_{a,g}$ when g is an equivalence class in $H^{1/2}(\partial\Omega;\mathbb{S}^1)/\mathbb{S}^1$.

We also define the regular parts $R_{a,g}$ and $\widehat{R}_a$ of $\Phi_{a,g}$ and $\widehat{\Phi}_a$ as follows:

$$R_{a,g}(x) = \Phi_{a,g}(x) - \sum_{j=1}^k d_j \log|x-a_j|, \quad \forall\, x \in \Omega, \tag{6.9}$$

respectively

$$\widehat{R}_a(x) = \widehat{\Phi}_a(x) - \sum_{j=1}^k d_j \log|x-a_j|, \quad \forall\, x \in \Omega. \tag{6.10}$$

The expressions of W and $\widehat{W}$ are

$$W(a,g) = -\pi \sum_{j\neq l} d_j d_l \log|a_j - a_l| + \frac{1}{2}\int_{\partial\Omega} \Phi_{a,g}\left(g \wedge \frac{\partial g}{\partial\tau}\right) - \pi\sum_{j=1}^k d_j R_{a,g}(a_j), \tag{6.11}$$

respectively

$$\widehat{W}(a) = -\pi \sum_{j\neq l} d_j d_l \log|a_j - a_l| - \pi\sum_{j=1}^k d_j \widehat{R}_a(a_j). \tag{6.12}$$

The next result was proved in [85].

Proposition 6.2. *We have*

$$\widehat{W}(a) = \inf\ \{W(a,g);\ g \in \mathcal{B}_d\}, \tag{6.13}$$

and the infimum is attained in (6.13).

Recall that $\mathcal{B}_d := \{g \in H^{1/2}(\partial\Omega;\mathbb{S}^1);\ \deg(g,\partial\Omega) = d\}$.

We present here an alternative proof of Proposition 6.2, in the course of which we exhibit a formula of the form

$$W(a,g) = \widehat{W}(a) + \text{ non negative terms },$$

which will be useful in the sequel.

Proof of Proposition 6.2. We identify a map $\psi \in H^{1/2}(\partial\Omega;\mathbb{R})$ with its harmonic extension to Ω, still denoted ψ. Given $\psi \in H^{1/2}(\partial\Omega;\mathbb{R})$, we define its (normalized) harmonic conjugate $\psi^* \in H^{1/2}(\partial\Omega;\mathbb{R})$ as follows. The harmonic extension of ψ^* (still denoted ψ^*) is the unique solution of

$$\begin{cases} \psi + \imath\psi^* \text{ is holomorphic in } \Omega, \\ \displaystyle\int_{\partial\Omega} \psi^* = 0. \end{cases} \tag{6.14}$$

Note that the Cauchy-Riemann equations imply

$$\frac{\partial\psi^*}{\partial\nu} = -\frac{\partial\psi}{\partial\tau} \quad \text{and} \quad \frac{\partial\psi^*}{\partial\tau} = \frac{\partial\psi}{\partial\nu}, \tag{6.15}$$

at least when ψ is smooth. When ψ is merely $H^{1/2}$, the distributions $\dfrac{\partial\psi}{\partial\nu}, \dfrac{\partial\psi^*}{\partial\nu} \in H^{-1/2}$ are respectively defined as the trace on $\partial\Omega$ of the normal derivatives of ψ and ψ^*, and the equalities in (6.15) are to be understood as equalities of distributions in $H^{-1/2}$.

We consider the space

$$H^{1/2}(\partial\Omega;\mathbb{R})/\mathbb{R} \simeq \dot{H}^{1/2}(\partial\Omega;\mathbb{R}) := \left\{\psi \in H^{1/2}(\partial\Omega;\mathbb{R});\ \int_{\partial\Omega} \psi = 0\right\}, \tag{6.16}$$

which is endowed with the natural norm

$$|\psi|^2_{H^{1/2}} = \int_\Omega |\nabla\psi|^2 = \int_\Omega |\nabla\psi^*|^2 = \int_{\partial\Omega} \frac{\partial\psi^*}{\partial\nu}\psi^* = -\int_{\partial\Omega} \frac{\partial\psi}{\partial\tau}\psi^*. \tag{6.17}$$

If ψ not smooth, then the two last integrals are to be understood as $H^{-1/2}$–$H^{1/2}$ duality brackets.

Given $a \in \Omega^k_*$, we define the canonical boundary datum associated with a as the unique element $g = g^a \in H^{1/2}(\partial\Omega;\mathbb{S}^1)/\mathbb{S}^1$ such that $\deg(g,\partial\Omega) = d$ and

$$g^a \wedge \frac{\partial g^a}{\partial\tau} = \frac{\partial\widehat{\Phi}_a}{\partial\nu}. \tag{6.18}$$

Our first observation is that g^a is well-defined and smooth. (It would be more accurate to assert that every map in the equivalence class defining g^a is smooth.) Indeed, existence of a smooth $g : \partial\Omega \to \mathbb{S}^1$ satisfying $g \wedge \dfrac{\partial g}{\partial\tau} = h$ (with given h) is equivalent to h smooth and

$$\int_{\partial\Omega} h = 2\pi d. \tag{6.19}$$

In addition, g (if it exists) is unique modulo $\mathbb{S}^1$. In our case, we have $h = \dfrac{\partial\widehat{\Phi}_a}{\partial\nu}$, which is smooth (since $\widehat{\Phi}_a$ is smooth near $\partial\Omega$). In addition, using the equation (6.8), we see that (6.19) holds. If we compare the definition of g^a to the one of $\Phi_{a,g}$, we see that the canonical datum g^a is the unique g (modulo multiplication by a constant in $\mathbb{S}^1$) such that

$$\widehat{\Phi}_a = \Phi_{a,g}. \tag{6.20}$$

Given $g \in H^{1/2}(\partial\Omega;\mathbb{S}^1)$ with $\deg(g,\partial\Omega) = d$, we have $\deg(g/g^a,\partial\Omega) = 0$. Therefore, we may find $\psi = \psi_{a,g} \in H^{1/2}(\partial\Omega;\mathbb{R})$, unique modulo a constant, such that [29]

$$g = g^a e^{\imath\psi} = g^a e^{\imath\psi_{a,g}}. \tag{6.21}$$

Thus we have

$$\begin{cases} \Delta\left[\Phi_{a,g} - \widehat{\Phi}_a\right] = 0 & \text{in } \Omega \\ \dfrac{\partial}{\partial\nu}\left[\Phi_{a,g} - \widehat{\Phi}_a\right] = \dfrac{\partial\psi}{\partial\tau} & \text{on } \partial\Omega \\ \displaystyle\int_{\partial\Omega}\left(\Phi_{a,g} - \widehat{\Phi}_a\right) = 0 \end{cases}. \tag{6.22}$$

Combining the above with the definition of the harmonic conjugate, we find that

$$\Phi_{a,g} = \widehat{\Phi}_a - \psi^* = \widehat{\Phi}_a - \psi^*_{a,g}. \tag{6.23}$$

Plugging (6.23) into the expression of $W(a,g)$ given by formula (6.11), we find

$$\begin{aligned} W(a,g) =& -\pi\sum_{j\neq l} d_j d_l \log|a_j - a_l| + \frac{1}{2}\int_{\partial\Omega}\left(\widehat{\Phi}_a - \psi^*\right)\left(g^a\wedge\frac{\partial g^a}{\partial\tau} + \frac{\partial\psi}{\partial\tau}\right) \\ & - \pi\sum_{j=1}^k d_j(\widehat{R}_a(a_j) - \psi^*(a_j)) \\ =& \widehat{W}(a) - \frac{1}{2}\int_{\partial\Omega}\psi^*\left(g^a\wedge\frac{\partial g^a}{\partial\tau}\right) - \frac{1}{2}\int_{\partial\Omega}\psi^*\frac{\partial\psi}{\partial\tau} + \pi\sum_{j=1}^k d_j\psi^*(a_j). \end{aligned} \tag{6.24}$$

In the last equality we used the fact that $\widehat{\Phi}_a = 0$ on $\partial\Omega$. Furthermore, using the definition of g^a and the fact that ψ^* is harmonic, we obtain

$$\begin{aligned} \int_{\partial\Omega}\psi^*\left(g^a\wedge\frac{\partial g^a}{\partial\tau}\right) &= \int_{\partial\Omega}\psi^*\frac{\partial\widehat{\Phi}_a}{\partial\nu} \\ &= \int_{\partial\Omega}\frac{\partial\psi^*}{\partial\nu}\widehat{\Phi}_a + \int_{\Omega}\psi^*\Delta\widehat{\Phi}_a = 2\pi\sum_{j=1}^k d_j\psi^*(a_j). \end{aligned} \tag{6.25}$$

Using (6.17), (6.24) and (6.25), we finally obtain

$$W(a,g) = \widehat{W}(a) + \frac{1}{2}\left|\psi_{a,g}\right|^2_{H^{1/2}}. \tag{6.26}$$

In particular, we recover the conclusion of Proposition 6.2 in the following stronger form: the minimum of $W(a,\cdot)$ is attained (exactly) when $g = g^a$ (modulo $\mathbb{S}^1$). □

Remark 6.3. The canonical boundary datum g^a will play a crucial role in our subsequent analysis. We emphasize here the fact that g^a is the (unique modulo $\mathbb{S}^1$) solution of

$$g^a\wedge\frac{\partial g^a}{\partial\tau} = \frac{\partial\widehat{\Phi}_a}{\partial\nu} \quad \text{on } \partial\Omega. \tag{6.27}$$

The limit (as $\rho \to 0$) of the variational problem (6.5) is also connected to the so-called canonical harmonic map $u_{*,a,g}$ associated to prescribed singularities $a \in \Omega_*^k$ and to the Dirichlet condition $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$. In fact, in [25, Chapter I] it is proved that the unique solution $u_{\rho,g}$ of the minimization problem $\inf \left\{\int |\nabla u|^2 \,;\, u \in \mathcal{F}_{\rho,g}\right\}$ tends to $u_{*,a,g}$, in $C^k_{loc}(\Omega \setminus \{a_j\})$ as $\rho \to 0$. [2]

The canonical harmonic map is defined by the formula

$$\begin{cases} u = u_{*,a,g} = e^{\imath H} \displaystyle\prod_{j=1}^{k} \left(\frac{z-a_j}{|z-a_j|}\right)^{d_j} & \text{in } \Omega \\ \Delta H = 0 & \text{in } \Omega \\ u = g & \text{on } \partial\Omega \end{cases}. \tag{6.28}$$

The fact that $\deg(g, \partial\Omega) = d = \sum d_j$ guarantees that $H = H_g$ is well defined. Indeed, there exists $\psi \in H^{1/2}(\partial\Omega; \mathbb{R})$ such that

$$g \prod_{j=1}^{k} \left(\frac{z-a_j}{|z-a_j|}\right)^{-d_j} = e^{\imath\psi},$$

and then we can simply let H be the harmonic extension of ψ. On the other hand, we note that H is uniquely defined up to a multiple of 2π.

Equivalently, u in (6.28) is characterized by [25, Chapter I]

$$\begin{cases} |u| = 1 \\ u \wedge \dfrac{\partial u}{\partial x_1} = -\dfrac{\partial \Phi_{a,g}}{\partial x_2} \\ u \wedge \dfrac{\partial u}{\partial x_2} = \dfrac{\partial \Phi_{a,g}}{\partial x_1} \\ u = g \text{ on } \partial\Omega \end{cases}. \tag{6.29}$$

In particular, we have

$$u_{*,a,g} \wedge \frac{\partial u_{*,a,g}}{\partial \nu} = -\frac{\partial \Phi_{a,g}}{\partial \tau} \quad \text{on } \partial\Omega \tag{6.30}$$

and

$$\int_{\partial\Omega} u_{*,a,g} \wedge \frac{\partial u_{*,a,g}}{\partial \nu} = 0. \tag{6.31}$$

Remark 6.4. For the minimization problem (6.6), the situation is similar. As established in [85], the solution v_ρ to $\inf \left\{\int |\nabla v|^2 \,;\, v \in \widehat{\mathcal{F}}_\rho\right\}$ converges (in an appropriate sense) as $\rho \to 0$, to $v_{*,a} := u_{*,a,g^a}$. Since g^a is defined modulo $\mathbb{S}^1$, $v_{*,a}$ is also defined modulo $\mathbb{S}^1$. Therefore, in this context the convergence actually means that subsequences of (v_ρ) converge to representatives (modulo $\mathbb{S}^1$) of $v_{*,a}$.

We end this section with the definition of the following quantity, which will play a very important role in what follows. For $a \in \Omega_*^k$ and $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$, we set

$$N(a, g) := u_{*,a,g} \wedge \frac{\partial u_{*,a,g}}{\partial \nu} = -\frac{\partial \Phi_{a,g}}{\partial \tau} \in H^{-1/2}(\partial\Omega; \mathbb{R}). \tag{6.32}$$

2. Actually, in [25, Chapter I] the map g is supposed smooth, but the argument adapts to a general $g \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$.

6.3 Transport of formulas onto the unit disc

Let $f : \mathbb{D} \to \Omega$ be a conformal representation. The assumption $\Omega \in C^{1,\beta}$ ensures that f and its inverse $\varphi := f^{-1} : \Omega \to \mathbb{D}$ are $C^{1,\beta}$ up to the boundary.

The goal of this section is to understand how the objects defined in Section 6.2 are transported by φ and f.

We will stress the dependence on the domain by using superscripts (e.g. $W = W^\Omega$). For $\alpha \in \mathbb{D}^k_*$, the notation $a = f(\alpha)$ stands for $a := (f(\alpha_1), \ldots, f(\alpha_k)) \in \Omega^k_*$.

First of all, for $a \in \Omega^k_*$, we have

$$\Phi^\Omega_{a,g} = \Phi^\mathbb{D}_{\varphi(a),g\circ f} \circ \varphi + C, \tag{6.33}$$

where $C = C(a, g, f) = -\int_{\mathbb{S}^1} \Phi^\mathbb{D}_{\varphi(a),g\circ f}|f'|$. Indeed, (6.33) is justified as follows. By a direct calculation, both sides of (6.33) satisfy the same Poisson equation, with the same Neumann boundary condition. The constant C comes from the normalization condition $\int_{\partial\Omega} \Phi_{a,g} = 0$. The same argument applies to show that

$$\widehat{\Phi}^\Omega_a = \widehat{\Phi}^\mathbb{D}_{\varphi(a)} \circ \varphi. \tag{6.34}$$

Here there is no renormalization constant since $\widehat{\Phi}_a$ satisfies a Dirichlet boundary condition.

Normal and tangential derivatives transform in the following way. If $v : \mathbb{D} \to \mathbb{C}$, then

$$\frac{\partial}{\partial\tau}\left[v \circ \varphi\right](z) = |\varphi'(z)| \frac{\partial v}{\partial\tau}\left(\varphi(z)\right), \quad z \in \partial\Omega, \tag{6.35}$$

$$\frac{\partial}{\partial\nu}\left[v \circ \varphi\right](z) = |\varphi'(z)| \frac{\partial v}{\partial\nu}\left(\varphi(z)\right), \quad z \in \partial\Omega. \tag{6.36}$$

Using (6.34), (6.35), (6.36) together with formula (6.18) characterizing g^a, we find, for $a \in \Omega^k_*$,

$$g^a \circ f = g^{\varphi(a)}. \tag{6.37}$$

On the other hand, we claim that

$$u^\Omega_{*,a,g} = u^\mathbb{D}_{*,\varphi(a),g\circ f} \circ \varphi. \tag{6.38}$$

Indeed, this follows from the observation that the two sides of (6.38) agree on $\partial\Omega$, combined with (6.28) and with the fact, when H is harmonic in $\mathbb{D}$, we may write

$$\frac{\varphi(z) - \varphi(a)}{|\varphi(z) - \varphi(a)|} e^{\imath H\circ\varphi(z)} = \frac{z-a}{|z-a|} e^{\imath K(z)}, \quad \text{with } K \text{ harmonic in } \Omega.$$

As a consequence of (6.38) and (6.36), we obtain, recalling the definition (6.32) of N,

$$N^\Omega(a, g) = |\varphi'|\, N^\mathbb{D}(\varphi(a), g \circ f) \circ \varphi. \tag{6.39}$$

The formulas of the renormalized energies $\widehat{W}$ and W transport in a more complicated way.

Lemma 6.5. *Let* $\alpha \in \mathbb{D}^k_*$, $a := f(\alpha)$ *and* $g : \partial\Omega \to \mathbb{S}^1$. *Then*

$$W^\Omega(a,g) = W^{\mathbb{D}}(\alpha, g\circ f) + \pi \sum_j d_j^2 \log|f'(\alpha_j)|, \tag{6.40}$$

$$\widehat{W}^\Omega(a) = \widehat{W}^{\mathbb{D}}(\alpha) + \pi \sum_j d_j^2 \log|f'(\alpha_j)|. \tag{6.41}$$

Proof. Using definition of $R_{a,g}$ (6.9), together with (6.33), we compute, for $z \in \mathbb{D}$,

$$\begin{aligned} R^\Omega_{a,g}(f(z)) &= \Phi^{\mathbb{D}}_{\alpha,g\circ f}(z) - \sum_{l=1}^k d_l \log|f(z) - f(\alpha_l)| + C \\ &= R^{\mathbb{D}}_{\alpha,g\circ f}(z) - \sum_{l=1}^k d_l \log\left|\frac{f(z)-f(\alpha_l)}{z-\alpha_l}\right| + C. \end{aligned} \tag{6.42}$$

The above is well-defined when $z \neq \alpha_j$, and extends by continuity at $z = \alpha_j$. In particular,

$$R^\Omega_{a,g}(f(\alpha_j)) = R^{\mathbb{D}}_{\alpha,g\circ f}(\alpha_j) - \sum_{l\neq j} d_l \log\left|\frac{f(\alpha_j)-f(\alpha_l)}{\alpha_j-\alpha_l}\right| - d_j \log|f'(\alpha_j)| + C. \tag{6.43}$$

Finally, we plug (6.33) and (6.43) into formula (6.11) expressing W in terms of $\Phi_{a,g}$ and $R_{a,g}$. We obtain, using also the fact that $\deg(g,\partial\Omega) = d = \sum d_j$,

$$\begin{aligned} W^\Omega(a,g) =& -\pi \sum_{j\neq l} d_j d_l \log|\alpha_j - \alpha_l| + \frac{1}{2}\int_{\partial\Omega} \Phi^{\mathbb{D}}_{\alpha,g\circ f}\circ\varphi \left(g \wedge \frac{\partial g}{\partial \tau}\right) \\ & + \frac{1}{2} C \int_{\partial\Omega} g \wedge \frac{\partial g}{\partial\tau} - \pi \sum_j d_j C - \pi \sum_{j=1}^k d_j R^{\mathbb{D}}_{\alpha,g\circ f}(\alpha_j) + \pi \sum_{j=1}^k d_j^2 \log|f'(\alpha_j)| \\ =& -\pi \sum_{j\neq l} d_j d_l \log|\alpha_j - \alpha_l| + \frac{1}{2}\int_{\partial\mathbb{D}} \Phi^{\mathbb{D}}_{\alpha,g\circ f} (g\circ f) \wedge \frac{\partial}{\partial\tau}(g\circ f) \\ & - \pi \sum_{j=1}^k d_j R^{\mathbb{D}}_{\alpha,g\circ f}(\alpha_j) + \pi \sum_{j=1}^k d_j^2 \log|f'(\alpha_j)| \\ =& \ W^{\mathbb{D}}(\alpha, g\circ f) + \pi \sum_j d_j^2 \log|f'(\alpha_j)|. \end{aligned}$$

Formula (6.41) can be proved following the same lines (the calculations are even simpler than for (6.40)). Alternatively, we can obtain (6.41) via the relation $\widehat{W}(a) = W(a, g^a)$. □

6.4 Explicit formulas in the unit disc

In this section we derive explicit formulas for $\widehat{W}^{\mathbb{D}}$, $W^{\mathbb{D}}$ and $N^{\mathbb{D}}$.

We start by recalling the explicit formulas for $\widehat{\Phi}^{\mathbb{D}}_\alpha$ and $\widehat{W}^{\mathbb{D}}$ [85]: for $\alpha \in \mathbb{D}^k_*$, we have

$$\widehat{\Phi}^{\mathbb{D}}_\alpha(z) = \sum_{j=1}^k d_j \left(\log|z-\alpha_j| - \log|1-\overline{\alpha}_j z|\right), \quad \forall\, z \in \mathbb{D}, \tag{6.44}$$

$$\widehat{W}^{\mathbb{D}}(\alpha) = -\pi \sum_{j\neq l} d_j d_l \log|\alpha_j - \alpha_l| + \pi \sum_{j\neq l} d_j d_l \log|1-\overline{\alpha_j}\alpha_l| \tag{6.45}$$
$$+ \pi \sum_j d_j^2 \log(1-|\alpha_j|^2).$$

The formulas for W and N are more involved.

Lemma 6.6. *Let $\alpha^0 \in \mathbb{D}^k_*$ be fixed, and $g^0 := g^{\alpha^0} : \mathbb{S}^1 \to \mathbb{S}^1$ be an associated canonical boundary map. (Recall that g^0 is defined up to a multiplicative constant.) Then it holds:*
(i) For $\alpha \in \mathbb{D}^k_*$ and for $\psi \in H^{1/2}(\mathbb{S}^1;\mathbb{R})$,

$$W^{\mathbb{D}}(\alpha, g^0 e^{\imath\psi}) = \widehat{W}^{\mathbb{D}}(\alpha) + \frac{1}{2}\int_{\mathbb{D}} \left|\nabla\left(\psi^*_{\alpha,g^0} + \psi^*\right)\right|^2, \tag{6.46}$$

and, for $z \in \mathbb{D}$,

$$\nabla\psi^*_{\alpha,g^0}(z) = 2\sum_{j=1}^k d_j \left(\frac{\alpha_j(1-\overline{\alpha}_j z)}{|1-\overline{\alpha}_j z|^2} - \frac{\alpha_j^0(1-\overline{\alpha}_j^0 z)}{|1-\overline{\alpha}_j^0 z|^2}\right) \in \mathbb{C} \simeq \mathbb{R}^2. \tag{6.47}$$

(ii) For $\alpha \in \mathbb{D}^k_*$ and for $\psi \in H^{1/2}(\mathbb{S}^1;\mathbb{R})$,

$$N^{\mathbb{D}}(\alpha, g^0 e^{\imath\psi}) = \frac{\partial\psi^*}{\partial\tau} + 2\sum_j d_j \frac{\alpha_j^0 \wedge z}{|z-\alpha_j^0|^2} - 2\sum_j d_j \frac{\alpha_j \wedge z}{|z-\alpha_j|^2}. \tag{6.48}$$

Proof of (i). Since we will always work in the unit disc, we drop the superscript $\mathbb{D}$.

We know from (6.26) that for $g \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$,

$$W(\alpha, g) = \widehat{W}(\alpha) + \frac{1}{2}\left|\psi_{\alpha,g}\right|^2_{H^{1/2}}, \tag{6.49}$$

where $\psi_{\alpha,g}$ is defined (modulo a constant) in (6.21) by

$$g = g^\alpha e^{\imath\psi_{\alpha,g}}. \tag{6.50}$$

Taking $g = g^0 e^{\imath\psi}$, and using $g^0 = g^\alpha e^{\imath\psi_{\alpha,g^0}}$, we find

$$g = g^\alpha e^{\imath\psi_{\alpha,g^0}} e^{\imath\psi} = g^\alpha e^{\imath(\psi_{\alpha,g^0}+\psi)}, \tag{6.51}$$

so that it holds

$$\psi_{\alpha,g} = \psi_{\alpha,g^0} + \psi. \tag{6.52}$$

This leads to

$$W(\alpha, g^0 e^{\imath\psi}) = \widehat{W}(\alpha) + \frac{1}{2}\left|\psi_{\alpha,g^0} + \psi\right|^2_{H^{1/2}} = \widehat{W}(\alpha) + \frac{1}{2}\int_{\mathbb{D}} \left|\nabla\left(\psi^*_{\alpha,g^0} + \psi^*\right)\right|^2,$$

(6.53)

i.e., (6.46) holds. In order to complete the proof of *(i)*, it remains to compute $\nabla\psi^*_{\alpha,g^0}$.

Recall that ψ^*_{α,g^0} is characterized by

$$\begin{cases} \Delta\psi^*_{\alpha,g^0} = 0 & \text{in } \mathbb{D} \\ \dfrac{\partial}{\partial\nu}\psi^*_{\alpha,g^0} = -\dfrac{\partial}{\partial\tau}\psi_{\alpha,g^0} & \text{on } \mathbb{S}^1 \\ \displaystyle\int_{\mathbb{S}^1}\psi^*_{\alpha,g^0} = 0 \end{cases}. \tag{6.54}$$

Since $e^{\imath\psi_{\alpha,g^0}} = g^0/g^\alpha$, we have

$$\frac{\partial\psi_{\alpha,g^0}}{\partial\tau} = g^0 \wedge \frac{\partial g^0}{\partial\tau} - g^\alpha \wedge \frac{\partial g^\alpha}{\partial\tau}. \tag{6.55}$$

By definition of g^α and $g^0 = g^{\alpha^0}$, and using (6.44), we obtain

$$\begin{aligned} g^\alpha \wedge \frac{\partial g^\alpha}{\partial\tau} &= \frac{\partial\widehat{\Phi}_\alpha}{\partial\nu} = \sum_j d_j \frac{\partial}{\partial\nu}\left[\log|z-\alpha_j| - \log|1-\overline{\alpha_j}z|\right], \\ g^0 \wedge \frac{\partial g^0}{\partial\tau} &= \frac{\partial\widehat{\Phi}_{\alpha^0}}{\partial\nu} = \sum_j d_j \frac{\partial}{\partial\nu}\left[\log|z-\alpha_j^0| - \log|1-\overline{\alpha_j^0}z|\right]. \end{aligned} \tag{6.56}$$

We also note the identity

$$1 = \frac{\partial}{\partial\nu}\left[\log|1-\overline{\alpha}z| + \log|z-\alpha|\right], \quad \forall\, \alpha\in\mathbb{D}. \tag{6.57}$$

Combining (6.55))-(6.57), we obtain

$$\frac{\partial\psi^*_{\alpha,g^0}}{\partial\nu} = -\frac{\partial\psi_{\alpha,g^0}}{\partial\tau} = \frac{\partial}{\partial\nu}\left[2\sum_j d_j\left(\log|1-\overline{\alpha_j^0}z| - \log|1-\overline{\alpha_j}z|\right)\right]. \tag{6.58}$$

Therefore, there exists a constant $c(\alpha)\in\mathbb{R}$ such that

$$\psi^*_{\alpha,g^0}(z) = 2\sum_j d_j\left(\log|1-\overline{\alpha}_j^0 z| - \log|1-\overline{\alpha}_j z|\right) + c(\alpha), \qquad \forall\, x\in\mathbb{D}. \tag{6.59}$$

Indeed, the right-hand side of (6.59) satisfies (6.54), and so does ψ^*_{α,g^0}. The constant $c(\alpha)$ is determined by the normalization condition $\int\psi^*_{\alpha,g^0} = 0$. From (6.59) we immediately obtain (6.47). □

Proof of (ii). In view of formula (6.28), we have

$$N(\alpha, g^0 e^{\imath\psi}) = \frac{\partial H}{\partial\nu} + \frac{\partial}{\partial\nu}\left[\sum_j d_j\theta(z-\alpha_j)\right] = \frac{\partial H^*}{\partial\tau} - \sum_j d_j\frac{\partial}{\partial\tau}\left[\log|z-\alpha_j|\right], \tag{6.60}$$

where H^* is the harmonic conjugate of H, characterized (up to a constant) by

$$\begin{cases} \Delta H^* = 0 & \text{in } \mathbb{D} \\ \dfrac{\partial H^*}{\partial \nu} = -\dfrac{\partial H}{\partial \tau} & \text{on } \mathbb{S}^1. \end{cases} \tag{6.61}$$

On the boundary $\mathbb{S}^1$, we have

$$e^{\imath H} = \prod_j \left(\frac{z-\alpha_j}{|z-\alpha_j|} \right)^{-d_j} g = \prod_j \left(\frac{z-\alpha_j}{|z-\alpha_j|} \right)^{-d_j} g^0 e^{\imath\psi}, \tag{6.62}$$

so that

$$\begin{aligned} \frac{\partial H}{\partial \tau} &= \frac{\partial \psi}{\partial \tau} + g^0 \wedge \frac{\partial g^0}{\partial \tau} - \sum_j d_j \frac{\partial}{\partial \tau} \left[\theta(z-\alpha_j)\right] \\ &= -\frac{\partial \psi^*}{\partial \nu} + \frac{\partial \widehat{\Phi}_{\alpha^0}}{\partial \nu} - \sum_j d_j \frac{\partial}{\partial \nu} \left[\log|z-\alpha_j|\right] \\ &= -\frac{\partial \psi^*}{\partial \nu} + \sum_j d_j \frac{\partial}{\partial \nu} \left[\log|z-\alpha_j^0| - \log|1-\overline{\alpha}_j^0 z|\right] - \sum_j d_j \frac{\partial}{\partial \nu} \left[\log|z-\alpha_j|.\right] \end{aligned} \tag{6.63}$$

Here we have used the definition of $g^0 = g^{\alpha^0}$ and the explicit formula (6.44) for $\widehat{\Phi}_\alpha$. Using (6.57), we obtain

$$\frac{\partial H}{\partial \tau} = -\frac{\partial}{\partial \nu} \left[\psi^* + \sum_j d_j (2\log|1-\overline{\alpha}_j^0 z| - \log|1-\overline{\alpha}_j z|) \right]. \tag{6.64}$$

We deduce that there exists a constant $c = c(\psi, \alpha)$ such that

$$H^* = \psi^* + \sum_j d_j (2\log|1-\overline{\alpha}_j^0 z| - \log|1-\overline{\alpha}_j z|) + c. \tag{6.65}$$

From (6.65) and (6.60) we obtain

$$N(\alpha, g^0 e^{\imath\psi}) = \frac{\partial \psi^*}{\partial \tau} + \sum_j d_j \frac{\partial}{\partial \tau} \left[2\log|1-\overline{\alpha}_j^0 z| - \log|1-\overline{\alpha}_j z| - \log|z-\alpha_j|\right]. \tag{6.66}$$

Using the fact that for every $\alpha \in \mathbb{D}$ we have

$$\frac{\partial}{\partial \tau} \left[\log|z-\alpha|\right] = \frac{\partial}{\partial \tau} \left[\log|1-\overline{\alpha} z|\right] = \frac{\alpha \wedge z}{|z-\alpha|^2}, \quad \forall\, z \in \mathbb{S}^1, \tag{6.67}$$

we finally obtain

$$N(\alpha, g^0 e^{\imath\psi}) = \frac{\partial \psi^*}{\partial \tau} + 2\sum_j d_j \frac{\alpha_j^0 \wedge z}{|z-\alpha_j^0|^2} - 2\sum_j d_j \frac{\alpha_j \wedge z}{|z-\alpha_j|^2}, \tag{6.68}$$

as claimed. □

6.5 Nondegeneracy of W is stable

In this section we show that, if $a^0 \in (\Omega_0)^k_*$ is a nondegenerate critical point of $W^{\Omega_0}(\cdot, g_0)$, with $g_0 : \partial\Omega_0 \to \mathbb{S}^1$, then for Ω « close to » Ω_0, and for $g : \partial\Omega \to \mathbb{S}^1$ « close to » g_0, there exists a unique nondegenerate critical point a of $W^{\Omega}(\cdot, g)$ « close to » a^0. Unlike the analysis we perform in subsequent sections, smoothness (of the domain or of the boundary datum) is not crucial here. In order to emphasize this fact, we first state and prove a result concerning rough boundary datum (Proposition 6.7). We next present a « smoother » variant of the stability result (Proposition 6.9).

The notion of closeness will be expressed in terms of conformal representations. Let us first introduce some definitions. Let X be the space

$$X := \left\{ f \in C^1\left(\overline{\mathbb{D}}; \mathbb{C}\right);\ f \text{ is holomorphic in } \mathbb{D} \right\}, \tag{6.69}$$

which is a Banach space with the $\|\cdot\|_{C^1}$ norm. In X we will consider the open set

$$V := \left\{ f \in X;\ f \text{ is bijective and } f^{-1} \in X \right\}. \tag{6.70}$$

Every $f \in V$ induces a conformal representation $f : \mathbb{D} \to \Omega := f(\mathbb{D})$, which is C^1 up to the boundary. In what follows, we denote by f^{-1} both the inverse of $f : \overline{\mathbb{D}} \to \overline{\Omega}$ and of $f_{|\mathbb{S}^1} : \mathbb{S}^1 \to \partial\Omega$.

Similar considerations apply to the space

$$X_\beta := \left\{ f \in C^{1,\beta}\left(\overline{\mathbb{D}}; \mathbb{C}\right);\ f \text{ is holomorphic in } \mathbb{D} \right\}, \tag{6.71}$$

and to the open set

$$V_\beta := \left\{ f \in X;\ f \text{ is bijective and } f^{-1} \in X_\beta \right\}. \tag{6.72}$$

Here, $0 < \beta < 1$.

Proposition 6.7. *Let Ω_0 be a smooth bounded simply connected $C^{1,\beta}$ domain and $f_0 : \mathbb{D} \to \Omega_0$ be a conformal representation. Assume that there exists $\alpha^0 \in \mathbb{D}^k_*$ and $g_0 \in H^{1/2}(\mathbb{S}^1; \mathbb{S}^1)$ such that $a^0 := f_0(\alpha^0)$ is a nondegenerate critical point of $W^{\Omega_0}(\cdot, g_0 \circ f_0^{-1})$.*

Then there exist a neighborhood $\mathcal{V}$ of $(f_0, 0)$ in $V \times H^{1/2}(\mathbb{S}^1; \mathbb{R})$, a smooth map $\alpha : \mathcal{V} \to \mathbb{D}^k_*$, and some $\delta > 0$, such that the following holds.

Let $(f, \psi) \in \mathcal{V}$ and consider the domain $\Omega := f(\mathbb{D})$ together with the boundary datum $g := (g_0 e^{\imath\psi}) \circ f^{-1} \in H^{1/2}(\partial\Omega; \mathbb{S}^1)$. Then $W^{\Omega}(\cdot, g)$ admits a unique critical point $a \in \Omega^k_*$ satisfying $|f^{-1}(a) - \alpha^0| < \delta$. This a is given by the map $a(f, \psi) = f\left(\alpha(f, \psi)\right)$. Furthermore, a is a nondegenerate critical point of $W^{\Omega}(\cdot, g)$.

Before proving Proposition 6.7 we state as a lemma the following smoothness result.

Lemma 6.8. *The map $\widetilde{W} : \mathbb{D}^k_* \times V \times H^{1/2}(\mathbb{S}^1; \mathbb{R}) \to \mathbb{R}$, defined by*

$$\widetilde{W}(\alpha, f, \psi) = W^{f(\mathbb{D})}\left(f(\alpha), (g_0 e^{\imath\psi}) \circ f^{-1}\right), \tag{6.73}$$

is smooth.

Similarly, the map $\widetilde{W}^{\beta} : \mathbb{D}_*^k \times V_\beta \times C^{1,\beta}(\mathbb{S}^1;\mathbb{R}) \to \mathbb{R}$*, defined by*

$$\widetilde{W}^{\beta}(\alpha, f, \psi) = W^{f(\mathbb{D})}\left(f(\alpha), (g_0 e^{\imath\psi}) \circ f^{-1}\right), \tag{6.74}$$

is smooth.

Proof of Lemma 6.8. The idea is to rely on the formulas derived in Sections 6.3 and 6.4 in order to obtain an explicit formula for $\widetilde{W}$, from which it will be clear that $\widetilde{W}$ is smooth.

To start with, formula (6.40) gives

$$\widetilde{W}(\alpha, f, \psi) = W^{\mathbb{D}}(\alpha, g_0 e^{\imath\psi}) + \pi \sum_j d_j^2 \log|f'(\alpha_j)|. \tag{6.75}$$

Using the fact that for holomorphic f all derivatives can be estimated locally using only $\|f\|_\infty$, it can be easily shown that the maps

$$\mathbb{D}_*^k \times V \ni (\alpha, f) \mapsto \log|f'(\alpha_j)| \tag{6.76}$$

are smooth.

Therefore the second term in the right-hand side of (6.75) is smooth, and in order to complete the proof of Lemma 6.8 it suffices to prove that

$$\mathbb{D}_*^k \times H^{1/2}(\mathbb{S}^1;\mathbb{R}) \ni (\alpha, \psi) \mapsto W^{\mathbb{D}}(\alpha, g_0 e^{\imath\psi}) := P_{g_0}(\alpha, \psi) \tag{6.77}$$

is smooth. Clearly, if $g \in H^{1/2}(\mathbb{S}^1;\mathbb{S}^1)$ is such that $\deg(g, \mathbb{S}^1) = \deg(g_0, \mathbb{S}^1)$, then we may write $g = g_0 e^{\imath\psi_0}$ for some $\psi_0 \in H^{1/2}(\mathbb{S}^1;\mathbb{R})$, and then we have $P_g(\alpha, \psi) = P_{g_0}(\alpha, \psi + \psi_0)$. This implies that the smoothness of P_{g_0} does not depend on the choice of g_0. Therefore, we may assume that $g_0 = g^{\alpha^0}$ for some $\alpha^0 \in \mathbb{D}_*^k$. This assumption allows us to use Lemma 6.6. Using (6.46), we obtain

$$\begin{aligned} W^{\mathbb{D}}(\alpha, g_0 e^{\imath\psi}) = \widehat{W}^{\mathbb{D}}(\alpha) &+ \frac{1}{2}\int_{\mathbb{D}} \left|\nabla\psi_{\alpha,g_0}^*\right|^2 + \frac{1}{2}\int_{\mathbb{D}} |\nabla\psi^*|^2 \\ &+ \int_{\mathbb{D}} \nabla\psi_{\alpha,g_0}^* \cdot \nabla\psi^*. \end{aligned} \tag{6.78}$$

We examine the smoothness of the four terms on the right-hand side of (6.78). The first term depends only on α and is smooth thanks to formula (6.45). The second term depends only on α and is smooth thanks to formula (6.47). The third term depends only on ψ and is a continuous quadratic form, hence it is smooth. The fourth and last term depends linearly on ψ and is smooth thanks to formula (6.47) again.

Hence the map (6.77) is smooth, and the proof of the $H^{1/2}$ part of the lemma is complete.

The proof of the $C^{1,\beta}$ part of the follows the same lines and is left to the reader. □

Proof of Proposition 6.7. Let us first remark the following fact. Fix $f \in V$ and $\psi \in H^{1/2}(\partial\mathbb{D};\mathbb{R})$ and consider the domain $\Omega = f(\mathbb{D})$ together with the boundary datum $g = (g_0 e^{\imath\psi}) \circ f^{-1}$. Then, for any $\alpha \in D_*^k$, $f(\alpha)$ is a nondegenerate critical point of $W^{\Omega}(\cdot, g)$ if and only if α is a nondegenerate critical point of $\widetilde{W}(\cdot, f, \psi)$. This is a simple consequence of the fact that f induces a diffeomorphism from $\mathbb{D}_*^k$ into Ω_*^k.

We consider the map $F : \mathbb{D}_*^k \times V \times H^{1/2}(\mathbb{S}^1;\mathbb{R}) \to \mathbb{R}^{2k}$,

$$F : (\alpha, f, \psi) \mapsto \nabla_\alpha \widetilde{W}(\alpha, f, \psi). \tag{6.79}$$

Lemma 6.8 ensures that F is smooth. Moreover, the assumption that a^0 is a nondegenerate critical point of $W^{\Omega_0}(\cdot, g_0 \circ f^{-1})$ ensures that α^0 is a nondegenerate critical point of $\widetilde{W}(\cdot, f_0, 0)$. Therefore $F(\alpha^0, f_0, 0) = 0$, and $D_\alpha F(\alpha^0, f_0, 0)$ is invertible.

This enables us to apply the implicit function theorem: there exist of a neighborhood $\mathcal{V}$ of $(f_0, 0)$ in $V \times H^{1/2}(\mathbb{S}^1;\mathbb{R})$, a smooth map $\alpha : \mathcal{V} \to \mathbb{D}_*^k$, and $\delta > 0$, such that, for $(f, \psi) \in \mathcal{V}$ and $|\alpha - \alpha^0| < \delta$,

$$F(\alpha, f, \psi) = 0 \Longleftrightarrow \alpha = \alpha(f, \psi). \tag{6.80}$$

We may also assume that $D_\alpha F(\alpha(f,\psi), f, \psi)$ is invertible, so that $\alpha(f, \psi)$ is a nondegenerate critical point of $\widetilde{W}(\cdot, f, \psi)$. This implies that $a := f(\alpha(f, \psi))$ is a nondegenerate critical point of $W^\Omega(\cdot, g)$, where $\Omega = f(\mathbb{D})$ and $g = (g_0 e^{\imath\psi}) \circ f^{-1}$. In view of (6.80), a is the unique critical point of $W^\Omega(\cdot, g)$ satisfying $|f^{-1}(a) - \alpha^0| < \delta$.

The proof of Proposition 6.7 is complete. □

In what follows, we will use the following smoother version of Proposition 6.7.

Proposition 6.9. *Let Ω_0 be a smooth bounded simply connected $C^{1,\beta}$ domain and $f_0 : \mathbb{D} \to \Omega_0$ be a conformal representation. Assume that there exists $\alpha^0 \in \mathbb{D}_*^k$ and $g_0 \in C^{1,\beta}(\mathbb{S}^1;\mathbb{S}^1)$ such that $a^0 := f_0(\alpha^0)$ is a nondegenerate critical point of $W^{\Omega_0}(\cdot, g_0 \circ f_0^{-1})$.*

Then there exist a neighborhood $\mathcal{V}$ of $(f_0, 0)$ in $V_\beta \times C^{1,\beta}(\mathbb{S}^1;\mathbb{R})$, a smooth map $\alpha : \mathcal{V} \to \mathbb{D}_*^k$, and some $\delta > 0$, such that the following holds.

Let $(f, \psi) \in \mathcal{V}$ and consider the domain $\Omega := f(\mathbb{D})$ together with the boundary datum $g := (g_0 e^{\imath\psi}) \circ f^{-1} \in C^{1,\beta}(\partial\Omega;\mathbb{S}^1)$. Then $W^\Omega(\cdot, g)$ admits a unique critical point $a \in \Omega_*^k$ satisfying $|f^{-1}(a) - \alpha^0| < \delta$, given by the map $a(f, \psi) = f(\alpha(f, \psi))$. Furthermore, a is a nondegenerate critical point of $W^\Omega(\cdot, g)$.

Here, V_β is given by (6.72). The proof of Proposition 6.9 is identical to the one of Proposition 6.7 and is left to the reader.

We will need later the following special case of Proposition 6.9, where Ω is fixed.

Corollary 6.10. *Let $a^0 \in \Omega_*^k$ be a nondegenerate critical point of $W(\cdot, g_0)$, for some $g_0 \in C^{1,\beta}(\partial\Omega; \S^1)$. Then, for g in a small $C^{1,\beta}$-neighborhood $\mathfrak{A}$ of g_0, $W(\cdot, g)$ has, near a^0, a unique nondegenerate critical point $a(g)$. In addition, the map $\psi \mapsto a(g_0 e^{\imath\psi})$, defined for ψ in a sufficiently small neighborhood of the origin in $C^{1,\beta}(\partial\Omega;\mathbb{R})$, is smooth.*

We note here that Corollary 6.10 allows us to define a map

$$T_* = T^\Omega_{*,a^0,g_0} : \mathfrak{A} \to C^\beta(\partial\Omega;\mathbb{R}),\ T_*(g) := N^\Omega(a(g), g) = u_{*,a(g),g} \wedge \frac{\partial u_{*,a(g),g}}{\partial \nu}. \tag{6.81}$$

Since $W^\Omega(\cdot,g)$ does not depend on the class of g modulo $\mathbb{S}^1$, neither do $a(g)$ and T_*. Moreover, in view of (6.30) and (6.31) we have

$$\int_{\partial\Omega} u_{*,a,g}\wedge\frac{\partial u_{*,a,g}}{\partial\nu}=\int_{\partial\Omega}\frac{\partial\Phi_{a,g}}{\partial\tau}=0.$$

We find that the map T_* induces a map, still denoted T_*, from $\mathfrak{A}/\mathbb{S}^1$ into $\dot{C}^\beta(\partial\Omega;\mathbb{R})$. Here, we define

$$\dot{C}^\beta(\partial\Omega;\mathbb{R}):=\left\{\psi\in C^\beta(\partial\Omega;\mathbb{R});\ \int_{\partial\Omega}\psi=0\right\}.$$

It is also convenient to consider, in a sufficiently small neighborhood $\mathfrak{B}$ of the origin in $C^{1,\beta}(\partial\Omega;\mathbb{R})$, the maps (both denoted U_*)

$$U_*=U^\Omega_{*,a^0,g_0}:\mathfrak{B}\to\dot{C}^\beta(\partial\Omega;\mathbb{R}),\ U_*(\psi)=T_*(g_0e^{\imath\psi}) \tag{6.82}$$

and

$$U_*=U^\Omega_{*,a^0,g_0}:\mathfrak{B}/\mathbb{R}\to\dot{C}^\beta(\partial\Omega;\mathbb{R}),\ U_*(\psi)=T_*(g_0e^{\imath\psi}). \tag{6.83}$$

The above U_*'s are smooth. Indeed, this is obtained by combining (6.39) with (6.48) and with the fact that $\psi\mapsto a(g_0e^{\imath\psi})$ is smooth.

6.6 A uniform version of the Pacard-Rivière construction of critical points

We start by explaining how the results established in this section compare to the existent literature.

Let us first briefly recall the Bethuel-Brezis-Hélein analysis of critical points of the Ginzburg-Landau energy E_ε with prescribed Dirichlet boundary condition $g\in C^\infty(\partial\Omega;\mathbb{S}^1)$ [25, Chapter X]. Consider a fixed boundary condition $g\in C^\infty(\partial\Omega;\mathbb{S}^1)$, with $\deg(g,\partial\Omega)=d=\sum_{j=1}^k d_j$. Given a critical point $a\in\Omega_*^k$ of $W(\cdot,d_1,\ldots,d_k,g)$, consider the canonical harmonic map given by (6.28). The fact that a is a critical point of $W(\cdot,d_1,\ldots,d_k,g)$ is equivalent to the fact that the harmonic function H_j, defined near a_j by

$$u_*=e^{\imath H_j}\left(\frac{z-a_j}{|z-a_j|}\right)^{d_j}, \tag{6.84}$$

satisfies $\nabla H_j(a_j)=0$ [25, Chapter VII].

The main result in [25, Chapter X] asserts that, when Ω is starshaped critical points of E_ε converge, as $\varepsilon\to 0$ (up to subsequences and in appropriate function spaces), to a canonical harmonic map $u_*=u_{*,a,g}$ associated with a critical point $a\in\Omega_*^k$ of $W(\cdot,d_1,\ldots,d_k,g)$.

Granted this result, one can address the converse: given a critical point $a\in\Omega_*^k$ of $W(\cdot,d_1,\ldots,d_k,g)$, does there exist critical points u_ε of E_ε with prescribed boundary condition g, such that $u_\varepsilon\longrightarrow u_{*,a,g}$ as $\varepsilon\to 0$? Here we will be interested in the answer provided by Pacard and Rivière [107].

Theorem 6.11 ([107, Theorem 1.4]). *Let* $0 < \beta, \gamma < 1$. *Assume that* $g \in C^{2,\beta}(\partial\Omega; \mathbb{S}^1)$ *and* $d_j \in \{\pm 1\}$. *Let* $a \in \Omega_*^k$ *be a nondegenerate critical point of* $W(\cdot, d_1, \ldots, d_k, g)$.

Then there exists $\varepsilon_0 > 0$ *such that for every* $\varepsilon \in (0, \varepsilon_0)$, *there exists* u_ε *a critical point of* E_ε *with* $u_\varepsilon = g$ *on* $\partial\Omega$, *and*

$$u_\varepsilon \longrightarrow u_{*,a(g),g} \quad \text{as } \varepsilon \to 0 \tag{6.85}$$

in $C^{2,\gamma}_{loc}(\Omega \setminus \{a_1, \ldots, a_k\})$.

The purpose of this section is to establish a variant of Theorem 6.11, in which g is assumed to be merely $C^{1,\beta}$ and is not fixed anymore. In addition, we will obtain a uniform existence theorem, and uniform convergence rate. More specifically, we fix integers $d_1, \ldots, d_k$. Since these integers do not depend on the boundary datum g we consider, we will omit the dependence of W with respect to $d_1, \ldots, d_k$: we write $W(\cdot, g)$ instead of $W(\cdot, d_1, \ldots, d_k, g)$. We consider $a_0 \in \Omega_*^k$ a nondegenerate critical point of the renormalized energy $W(\cdot, g_0)$ associated with $g_0 \in C^{1,\beta}(\partial\Omega; \mathbb{S}^1)$. By Corollary 6.10, we know that, for g in a small $C^{1,\beta}$-neighborhood $\mathfrak{A}$ of g_0, $W(\cdot, g)$ has, near a_0, a unique nondegenerate critical point $a(g)$.

In this section, we establish the following variant of Theorem 6.11.

Theorem 6.12. *Let* $0 < \beta, \gamma < 1$. *Let* $g_0 \in C^{1,\beta}(\partial\Omega; \mathbb{S}^1)$. *Let* $d_1, \ldots, d_k \in \{-1, 1\}$. *Let* a_0 *be a nondegenerate critical point of* $W(\cdot, g_0)$. *Then there exist* $\delta > 0$ *and* $\varepsilon_0 > 0$ *such that the following holds. For every* $g \in C^{1,\beta}(\partial\Omega; \mathbb{S}^1)$ *satisfying* $\|g - g_0\|_{C^{1,\beta}} \leq \delta$, *and for every* $\varepsilon \in (0, \varepsilon_0)$, *there exists* $u_\varepsilon = u_{\varepsilon,g}$ *a critical point of* E_ε *with prescribed boundary condition* g, *such that*

$$u_{\varepsilon,g} \longrightarrow u_{*,a(g),g} \quad \text{as } \varepsilon \to 0 \tag{6.86}$$

in $C^{2,\gamma}_{loc}(\Omega \setminus \{a_1, \ldots, a_k\})$.

As announced, the difference with Theorem 1.4 in [107] is that we merely assume that $g \in C^{1,\beta}$; in addition, we prove that ε_0 can be chosen independent of g. Theorem 6.12 allows us to define a map $F_\varepsilon : g \mapsto u_{\varepsilon,g}$ for every $\varepsilon \in (0, \varepsilon_0)$.

Theorem 6.12 is obtained by following the proof of Theorem 6.11 in [107]. All we have to check (and we will do in what follows) is that the estimates in [107] are uniform in g; we also have to modify some arguments relying on the regularity assumption $g \in C^{2,\beta}$.

Proof of Theorem 6.12. For the convenience of the reader, we recall the main steps of the proof of Theorem 6.11 in [107], and examine the crucial points where the estimates depend on g, respectively where the regularity of g plays a role.

The general strategy in [107] is to construct an « approximate solution » $\widetilde{u}_\varepsilon$ of the Ginzburg Landau equation

$$\mathcal{N}_\varepsilon(u) = 0, \quad \text{where } \mathcal{N}_\varepsilon(u) := \Delta u + \frac{u}{\varepsilon^2}(1 - |u|^2), \tag{6.87}$$

using the fairly precise knowledge we have of the form of solutions for small ε. Then, using a fixed point argument, one can prove that some perturbation of $\widetilde{u}_\varepsilon$ is in fact an exact solution of (6.87). The main difficulty lies in finding the good functional setting that makes the linearized operator $L_\varepsilon = D\mathcal{N}_\varepsilon$ around

$\widetilde{u}_\varepsilon$ invertible, uniformly with respect to ε. This is achieved in [107] in the frame of appropriate weighted Hölder spaces.

In [107] the proof of Theorem 6.11 is divided into five chapters: Chapters 3 through 7. In what follows, we detail the content of these chapters and explain how to adapt the arguments for the need of Theorem 6.12.

Chapters 3 and 4 in [107]

[107, Chapter 3] is devoted to the study of the radially symmetric solution $u(re^{\imath\theta}) = f(r)e^{\imath\theta}$ of the Ginzburg-Landau equation in $\mathbb{C}$ satisfying $\lim_{r\to\infty} f(r) = 1$. In particular, [107, Chapter 3] characterizes the bounded solutions of the linearized equation about this radial solution. This characterization is used in [107, Chapter 4] in the study of the mapping properties of the linearization of the Ginzburg-Landau operator (at the radial solution) in the punctured unit disc $\mathbb{D} \setminus \{0\}$. In particular, it is shown that the linearized operator is invertible between appropriate weighted Hölder spaces.

These two chapters (3 and 4) are independent of the boundary condition g, so that they can be used with no changes in the proof of Theorem 6.12.

Chapter 5 in [107]

The next step, in [107, Chapter 5], consists in constructing and estimating the approximate solution $\widetilde{u}_\varepsilon$. This approximate solution looks like $u_* = u_{*,g,a(g)}$ away from its zeros (which are close to the singularities of u_*), and like the radial solution studied in [107, Chapter 3] near its zeros. Since $\widetilde{u}_\varepsilon$ is built upon u_*, the estimates satisfied by $\widetilde{u}_\varepsilon$ involve u_*, and thus g.

More specifically, in [107, Chapter 5], various quantities are estimated in terms of constants $c(u_*)$ depending on u_* and its derivatives. An inspection of the proofs there combined with (6.28) shows that these constants depend only on $a(g)$, on the harmonic function $H = H_g$ and on the derivatives of H_g.

We claim that the constants $c(u_*)$ can be chosen independent of g satisfying

$$\|g - g_0\|_{C^{1,\beta}(\partial\Omega)} \leq \delta. \tag{6.88}$$

Here, δ is sufficiently small in order to have the conclusion of Corollary 6.10. Indeed, the key observation is that there exists a constant $C > 0$ independent of g such that

$$\|H\|_{C^{1,\beta}(\overline{\Omega})} \leq C; \tag{6.89}$$

this follows from the fact that H is harmonic and $\|H\|_{C^{1,\beta}(\partial\Omega)} \leq C$.

In particular, we have

$$\|H\|_{C^k(\omega)} \leq C(k,\omega) \quad \text{for } k \in \mathbb{N} \text{ and } \overline{\omega} \subset \Omega. \tag{6.90}$$

Estimate (6.90) implies that all the interior estimates in [107, Chapter 5] are satisfied uniformly in $g \in C^{1,\beta}$ satisfying (6.88). This settles the case of estimates (5.8), (5.9), (5.33), (5.42) and (5.43) in [107, Chapter 5].

It remains to consider the global and boundary estimates (5.29), (5.32) and (5.41) in [107]. These estimates rely on bounds on the solution ξ of the problem

$$\begin{cases} \Delta\xi - |\nabla u_*|^2\xi + \dfrac{1-\xi^2}{\varepsilon^2}\xi & = 0 & \text{in } \Omega_{\delta/2} \\ \xi & = S_\varepsilon + w_{j,r} & \text{on } \partial\mathbb{D}_{\delta/2}(a_j) \\ \xi & = 1 & \text{on } \partial\Omega \end{cases} . \tag{6.91}$$

Here, $\Omega_\sigma := \Omega \setminus \bigcup_j \mathbb{D}_\sigma(a_j)$ (for sufficiently small $\sigma > 0$), and $\delta := \varepsilon^2$. The auxiliary function S_ε is independent of g and is defined in [107, Section 3.6]. Finally, $w_{j,r}$, defined in [107, (5.7)], depends only $a(g)$ and on the restriction of H_j to compacts of Ω; therefore, the estimates involving $w_{j,r}$ are uniform in g.

In [107, Lemma 5.1] the following estimates (numbered as (5.29) in [107]) are shown to hold:

$$1 - c\varepsilon^2 \le \xi \le 1 \qquad \text{in } \Omega_\sigma, \tag{6.92}$$

$$1 - c\varepsilon^2 r_j^{-2} \le \xi \le 1 \qquad \text{in } \mathbb{D}_\sigma(a_j) \setminus \mathbb{D}_{\delta/2}(a_j), \tag{6.93}$$

$$|\nabla^k \xi| \le c_k \varepsilon^2 r_j^{-2-k} \qquad \text{in } \mathbb{D}_{2\sigma}(a_j) \setminus \mathbb{D}_\delta(a_j) \quad (k \ge 1). \tag{6.94}$$

Here $\sigma > 0$ is fixed, and $r_j = r_j(x)$ denotes the distance from x to a_j. Estimates (6.93) and (6.94) are interior estimates, and therefore they hold uniformly in $g \in C^{1,\beta}$ satisfying (6.88), as explained above. We claim that the same conclusion applies to (6.92). Indeed, an inspection of the proof in [107] shows that the constant c in (6.92) is controlled by $\sup_{\Omega_\sigma} |\nabla u_*|$. The latter quantity is uniformly bounded, thanks to (6.89), whence the conclusion. This settles the case of the estimate (5.29) in [107].

We next turn to the estimate (5.32) in [107, Lemma 5.2]. Under the assumption that $g \in C^{2,\beta}$, this lemma asserts that

$$\sup_{\Omega_\sigma} |\nabla^k \xi| \le c\varepsilon^{2-k}, \quad k = 1, 2. \tag{6.95}$$

In our case, we only assume $g \in C^{1,\beta}$. The corresponding estimates are given by our next result.

Lemma 6.13. *Assume that* (6.88) *holds. Then we have*

$$\sup_{\Omega_\sigma} |\nabla\xi| \le c\varepsilon \text{ and } |\nabla\xi|_{\beta,\Omega_\sigma} \le c\varepsilon^{1-\beta}. \tag{6.96}$$

Here, $|\cdot|_{\beta,\Omega_\sigma}$ denotes the C^α semi-norm in Ω_σ:

$$|u|_{\alpha,\Omega_\sigma} := \sup \left\{ \frac{|u(x) - u(y)|}{|x-y|^\beta}; \, x, y \in \Omega_\sigma \right\}.$$

Proof. We apply Lemma 6.27 in the Appendix with $w = \xi - 1$ in $G := \Omega_{\sigma/2}$, and find that

$$\sup_{\Omega_\sigma} |\nabla\xi| \le C \left(\|w\|^{1/2}_{L^\infty(\Omega_\sigma)} \|\Delta w\|^{1/2}_{L^\infty(\Omega_\sigma)} + \|w\|_{C^{1,\beta}(\partial\Omega_\sigma)} \right), \tag{6.97}$$

$$|\nabla\xi|_{\beta,\Omega_\sigma} \le C \left(\|w\|^{1/2-\beta/2}_{L^\infty(\Omega_\sigma)} \|\Delta w\|^{1/2+\beta/2}_{L^\infty(\Omega_\sigma)} + \|w\|_{C^{1,\beta}(\partial\Omega_\sigma)} \right). \tag{6.98}$$

The conclusion then follows by combining (6.97)-(6.98) with the equation (6.91) and with with estimates (6.92) and (6.94). □

Finally, we examine estimate (5.41) in the last section of [107, Chapter 5]; this is a global estimate for $\mathcal{N}_\varepsilon(\widetilde{u}_\varepsilon)$. Recall here that $\mathcal{N}_\varepsilon$ is the Ginzburg-Landau operator, and that $\widetilde{u}_\varepsilon$ is the approximate solution of (6.87) constructed in [107, Chapter 5]. The estimate [107, (5.41)] involves an interior estimate and a boundary estimate. As above, the interior estimate is settled with the help of (6.90). We now turn to the boundary estimate, which is the following:

$$\|\mathcal{N}_\varepsilon(\widetilde{u}_\varepsilon)\|_{C^\beta(\Omega_\sigma)} \leq c\varepsilon^{1-\beta}. \tag{6.99}$$

The proof of (6.99) in [107] relies on the estimates (6.95) above (see[107, Proof of Lemma 5.2]). In our case, (6.95) need not hold, since we only assume that $g \in C^{1,\beta}$. However, we still obtain (6.99) as follows. We note that

$$\mathcal{N}_\varepsilon(\widetilde{u}_\varepsilon) = 2\nabla u_* \cdot \nabla\xi \quad \text{in } \Omega_\sigma \tag{6.100}$$

(this is formula (5.46) in [107]). By (6.100), we have

$$\|\mathcal{N}_\varepsilon(\widetilde{u}_\varepsilon)\|_{C^\beta(\Omega_\sigma)} \leq c\|\nabla u_*\|_{C^\beta(\Omega_\sigma)}\|\nabla\xi\|_{C^\beta(\Omega_\sigma)}. \tag{6.101}$$

We obtain (6.99) as a consequence of (6.89) and of Lemma 6.13.

As a conclusion of this inspection, we find that all the estimates in [107, Chapter 5] are uniform in g satisfying (6.88); the arguments there need only minor changes. The most relevant change is that [107, Lemma 5.2] has to be replaced by Lemma 6.13.

Chapter 6 in [107]

We now turn to [107, Chapter 6], which deals with the conjugate linearized operator $\widetilde{\mathcal{L}}_\varepsilon$ around the approximate solution. The main result of this chapter is [107, Theorem 6.1], which states that $\widetilde{\mathcal{L}}_\varepsilon$ is invertible for $\varepsilon \in (0, \varepsilon_0)$, with the norm of its inverse bounded independently of ε. In order to adapt this theorem to our situation, we need to check that this ε_0, and the bound on $\widetilde{\mathcal{L}}_\varepsilon^{-1}$, can be chosen independently of g satisfying (6.88).

The proof of [107, Theorem 6.1] is divided into three parts:

(a) The « interior » problem, consisting in the study of the linearized operator $\widetilde{\mathcal{L}}_\varepsilon$ near the zeros of $\widetilde{u}_\varepsilon$ [107, Section 6.2].

(b) The « exterior » problem, requiring the study of the linearized operator $\widetilde{\mathcal{L}}_\varepsilon$ away from the zeros of $\widetilde{u}_\varepsilon$ [107, Section 6.3].

(c) The study of the Dirichlet-to-Neumann mappings [107, Section 6.4]. (These mappings are used later in order to « glue » the two first steps together.)

The interior and the exterior problem rely on the estimates obtained in [107, Chapter 5]. An inspection of the proofs shows that all the estimates obtained there are uniform in g, with one possible exception: the estimates in [107, Proposition 6.2]. Indeed, these estimates rely on [107, Lemma 5.2], and more specifically on (6.95) (which does not hold in our setting). However, a closer look to [107, Proof of Proposition 6.2] shows that the conclusion of [107, Proposition 6.2] still holds if we replace (6.95) by Lemma 6.13. In conclusion, the first two steps can be carried out with uniform estimates, provided (6.88) holds.

The third step (Dirichlet-to-Neumann mappings) requires more care. In [107, Section 6.4], the following two operators are defined, for fixed small $\zeta > 0$ and for sufficiently small ε:

$$DN_{int,\varepsilon},\, DN_{ext,\varepsilon} : \prod_{j=1}^{k} C^{2,\beta}\left(C(\zeta, a_j)\right) \longrightarrow \prod_{j=1}^{k} C^{1,\beta}\left(C(\zeta, a_j)\right). \qquad (6.102)$$

(These are the interior and exterior Dirichlet-to-Neumann mappings.) The crucial result in part (c) is [107, Proposition 6.5], which states the existence of some ε_0 such that $DN_{int,\varepsilon} - DN_{ext,\varepsilon}$ is an isomorphism for $\varepsilon \in (0, \varepsilon_0)$. The proof of this fact goes as follows. First the convergence

$$DN_{int,\varepsilon} - DN_{ext,\varepsilon} \longrightarrow DN_{int,0} - DN_{ext,0} \quad \text{as } \varepsilon \to 0 \qquad (6.103)$$

is shown to hold in operator norm. The proof of (6.103) relies on the interior estimate (6.90). Therefore, the convergence in (6.103) is uniform in g satisfying (6.88).

We now return to the proof in [107, Chapter 6]. Once (6.103) is established, it remains to prove that the limiting operator $DN_{int,0} - DN_{ext,0}$ is invertible. This is done in [107, Proposition 6.5]; this is where the nondegeneracy of a as a critical point of $W(\cdot, g)$ comes into the picture. In order to extend the conclusion of [107, Proposition 6.5] to our setting, and to obtain a uniform bound for the inverse of $DN_{int,\varepsilon} - DN_{ext,\varepsilon}$, it suffices to check that $DN_{int,0} - DN_{ext,0}$ depends continuously on g. Indeed, this will lead to a uniform bound for the inverse of $DN_{int,\varepsilon} - DN_{ext,\varepsilon}$ provided ε is sufficiently small, uniformly in g satisfying (6.88) (possibly with a smaller δ). For this purpose, we examine the formulas of $DN_{ext,0}$ and $DN_{int,0}$. The definition of $DN_{ext,0}$ is given in [107, Proposition 6.4], and it turns out that that $DN_{ext,0}$ does not depend on g. As for $DN_{int,0}$, it is a diagonal operator of the form

$$DN_{int,0}(\phi_1, \ldots, \phi_k) = \left(DN^1_{int,0}(\phi_1), \ldots, DN^k_{int,0}(\phi_k)\right), \qquad (6.104)$$

with $DN^j_{int,0} : C^{2,\beta}\left(C(\zeta, a_j)\right) \to C^{1,\beta}\left(C(\zeta, a_j)\right)$, $\forall\, j \in [\![1, k]\!]$.

Furthermore, from [107, Proposition 6.3] we know that $DN^j_{int,0}$ further splits as

$$DN^j_{int,0} = T_1 \oplus T_2, \quad \text{with } \begin{cases} T_1 : \operatorname{span}\left\{e^{\pm\imath n\theta}\right\}_{n\geq 2} \to \operatorname{span}\left\{e^{\pm\imath n\theta}\right\}_{n\geq 2} \\ T_2 : \operatorname{span}\left\{e^{\pm\imath\theta}\right\} \to \operatorname{span}\left\{e^{\pm\imath\theta}\right\} \end{cases}. \qquad (6.105)$$

Here, the operator T_1 does not depend on g. Therefore, we only need to check that T_2 depends continuously on g. As a linear operator on a two-dimensional space, T_2 is represented by a 2×2 matrix. It is clear from [107, Proposition 6.3] that the coefficients of this matrix are smooth functions of $\nabla^2 H(a_j)$. In turn, $\nabla^2 H(a_j)$ depends smoothly on g, by Corollary 6.10.

Hence $DN_{int,0} - DN_{ext,0}$ depends continuously on g, as claimed.

This allows us to choose ε_0 independent of g satisfying (6.88) in [107, Proposition 6.5] and in [107, Theorem 6.1], and to obtain a uniform estimate for the inverse of $\widetilde{\mathcal{L}}_\varepsilon$.

Chapter 7 in [107]

Finally, in [107, Chapter 7] the results and estimates in [107, Chapters 3-6] are combined in order to prove Theorem 6.11. Our above analysis shows that these estimates are uniform, and therefore lead to the uniform version Theorem 6.12 of Theorem 6.11.

Conclusion

As a conclusion of our analysis, Theorem 6.12 holds. □

For further use, we record two additional properties of the maps $u_{\varepsilon,g}$; these properties are immediate consequences of the construction in [107]. Let δ be as in Theorem 6.12. We consider the set

$$\mathfrak{A} := \left\{g \in C^{1,\beta}(\partial\Omega;\mathbb{S}^1);\ \|g-g_0\|_{C^{1,\beta}} < \delta\right\}. \tag{6.106}$$

Lemma 6.14. *Let $K \Subset \overline{\Omega}\setminus\{a_j\}$. Then we have $|u_{\varepsilon,g}| \to 1$ as $\varepsilon \to 0$, uniformly in K and in $g \in \mathfrak{A}$.*

Proof. This follows by an inspection of the construction in [107]. Formulas (5.36) and (5.37) in [107] ensure that, for small ε, the approximate solution $\widetilde{u}_\varepsilon$ satisfies $|\widetilde{u}_\varepsilon| = |\xi|$ in K. The convergence then follows from the estimates on ξ, and from formula (7.1) in [107] connecting the approximate solution to the exact solution. □

For the next result, it may be necessary to replace δ by a smaller value.

Lemma 6.15. *Let $g \in \mathfrak{A}$ and $\omega \in \mathbb{S}^1$. If $\omega g \in \mathfrak{A}$, then $u_{\varepsilon,\omega g} = \omega u_{\varepsilon,g}$.*

Proof. We have $W(\cdot,g) = W(\cdot,\omega g)$. Therefore, if a is a nondegenerate critical point of $W(\cdot,g)$, then a is also a nondegenerate critical point of $W(\cdot,\omega g)$. By Corollary 6.10, we find that $a(\omega g) = a(g)$. Using this equality, an inspection of the construction in [107] shows that

$$\widetilde{u}_{\varepsilon,\omega g} = \omega \widetilde{u}_{\varepsilon,g}. \tag{6.107}$$

Thanks to (6.107), we obtain that $\omega u_{\varepsilon,g}$ has all the properties satisfied by the solution $u_{\varepsilon,\omega g}$ constructed from $\widetilde{u}_{\varepsilon,\omega g}$ via the inverse function theorem. Since the solution provided by the inverse function theorem is unique, we find that $u_{\varepsilon,\omega g} = \omega u_{\varepsilon,g}$, as claimed. □

6.7 Convergence of the normal differentiation operators

In this section, we fix integers $d_1, \ldots, d_k \in \{-1, 1\}$ as in Section 6.6. We assume that a^0 is a nondegenerate critical point of $W(\cdot, g_0)$. Let $g \in \mathfrak{A}$, where $\mathfrak{A}$ is given by (6.106), and let $0 < \varepsilon < \varepsilon_0$. For such g and ε, we define $u_\varepsilon = u_{\varepsilon,g}$ as in Section 6.6. We also define $u_{*,g} := u_{*,a(g),g}$, where $a(g)$ is the unique critical point of $W(\cdot,g)$ close to a_0 (see Corollary 6.10). We consider the operators

$$T_\varepsilon, T_* : \mathfrak{A} \to C^\beta(\partial\Omega;\mathbb{R}),\ T_\varepsilon(g) := u_{\varepsilon,g} \wedge \frac{\partial u_{\varepsilon,g}}{\partial\nu} \text{ and } T_*(g) := u_{*,g} \wedge \frac{\partial u_{*,g}}{\partial\nu}.$$

The main result of this section is the following

Proposition 6.16. *Let $0 < \gamma < 1$. Then (possibly after replacing δ by a smaller number) we have*

$$\lim_{\varepsilon\to 0}\sup_{g\in\mathfrak{A}}\|T_\varepsilon(g) - T_*(g)\|_{C^\gamma(\partial\Omega)} = 0. \tag{6.108}$$

In particular, given $\mu > 0$ there exists some $\varepsilon_0 > 0$ such that, for $0 < \varepsilon < \varepsilon_0$, $T_\varepsilon - T_* : \mathfrak{A} \to C^\beta(\partial\Omega;\mathbb{R})$ is compact and satisfies

$$\|T_\varepsilon(g) - T_*(g)\|_{C^\beta(\partial\Omega)} \le \mu, \quad \forall\, \varepsilon < \varepsilon_0,\ \forall\, g \in \mathfrak{A}.$$

Proof. The last part of the proposition follows from the fact that the embedding $C^\gamma(\partial\Omega;\mathbb{R}) \hookrightarrow C^\beta(\partial\Omega;\mathbb{R})$ is compact when $\gamma > \beta$.

Whenever needed in the proof, we will replace δ by a smaller number. Let $a = a(g)$, $g \in \mathfrak{A}$, be such that $\nabla_a W(a,g) = 0$ and a is close to $a^0 = (a_1^0, \ldots a_k^0)$. Let $t > 0$ be a small number and set

$$\omega := \left\{x \in \Omega;\ |x - a_j^0| > t,\ \forall\, j \in [\![1,k]\!]\right\}.$$

We may assume that $|a(g) - a^0| < t/2$, $\forall\, g \in \mathfrak{A}$. In view of Theorem 6.12, we have $u_{\varepsilon,g} \to u_{*,g}$ in $C^{2,\gamma}(K)$ as $\varepsilon \to 0$, for every $g \in \mathfrak{A}$ and for every K compact set such that $K \subset \overline{\omega} \setminus \partial\Omega$. In addition, by Lemma 6.14 we have $|u_{\varepsilon,g}| \to 1$ as $\varepsilon \to 0$ uniformly in $\overline{\omega}$ and in $g \in \mathfrak{A}$.

Let $\theta = \theta_g$ be the multi-valued argument of

$$z \mapsto \prod_{j=1}^{k}\frac{(z - a_j(g))^{d_j}}{|z - a_j(g)|^{d_j}}.$$

We note that $\nabla\theta_g$ is single-valued and that we have

$$\|\nabla\theta_g\|_{C^{1,\beta}(\omega)} \le C, \quad \forall\, g \in \mathfrak{A}. \tag{6.109}$$

For small ε (independent of g), we have $\deg(u_{\varepsilon,g}) = \deg(u_{*,g}) = d_j$ on $C(a_j^0, t)$, and thus we may write, locally in $\overline{\omega}$,

$$u_{\varepsilon,g} = \rho e^{\imath\varphi} = \rho_{\varepsilon,g}e^{\imath\varphi_{\varepsilon,g}} = \rho e^{\imath(\theta+\psi)} = \rho_{\varepsilon,g}e^{\imath(\theta_g+\psi_{\varepsilon,g})},$$

and similarly

$$u_{*,g} = e^{\imath(\theta+\psi_*)} = e^{\imath(\theta_g+\psi_{*,g})}.$$

We may choose $\psi_{*,g}$ in order to have

$$\|\psi_{*,g}\|_{C^{1,\beta}(\omega)} \le C, \quad \forall\, g \in \mathfrak{A}, \tag{6.110}$$

and we normalize $\psi_{\varepsilon,g}$ by the condition

$$\psi_{\varepsilon,g} = \psi_{*,g} \quad \text{on } \partial\Omega. \tag{6.111}$$

In terms of ρ, φ and ψ, the Ginzburg-Landau equation reads

$$\begin{cases}\operatorname{div}(\rho^2\nabla\varphi) = \operatorname{div}(\rho^2(\theta+\psi)) = 0\\ -\Delta\rho = \dfrac{1}{\varepsilon^2}\rho(1-\rho^2) - \rho|\nabla\varphi|^2\end{cases}.$$

Step 1. We have

$$\|\nabla\varphi_{\varepsilon,g}\|_{L^p(\omega)} \le C_p, \quad ,\forall\,\varepsilon<\varepsilon_0\ \forall\,g\in\mathfrak{A},\ \forall\,1<p<\infty.$$

Indeed, we start by noting that we have $\|\nabla\theta_g\|_{L^p(\omega)} \le C_p$; therefore, it suffices to prove that $\|\nabla\psi_{\varepsilon,g}\|_{L^p(\omega)} \le C_p$. Using the equation $\operatorname{div}(\rho^2\nabla\varphi)=0$, we see that $\psi_{\varepsilon,g}$ satisfies

$$\Delta\psi_{\varepsilon,g} = \operatorname{div}\left(\left(1-\rho_{\varepsilon,g}^2\right)\nabla\theta_g + \left(1-\rho_{\varepsilon,g}^2\right)\nabla\psi_{\varepsilon,g}\right) \quad \text{in } \omega. \tag{6.112}$$

We obtain

$$\begin{aligned}\|\nabla\psi_{\varepsilon,g}\|_{L^p(\omega)} &\le C\left(\|\psi_{\varepsilon,g}\|_{W^{1-1/p,p}(\partial\omega)} + \|(1-\rho_{\varepsilon,g}^2)\nabla\theta_g\|_{L^p(\omega)} + \|(1-\rho_{\varepsilon,g}^2)\nabla\psi_{\varepsilon,g}\|_{L^p(\omega)}\right)\\ &\le C_p + C\|1-\rho_{\varepsilon,g}^2\|_{L^\infty(\omega)}\|\nabla\psi_{\varepsilon,g}\|_{L^p(\omega)}.\end{aligned}$$

Since $\rho_{\varepsilon,g}\to 1$ as $\varepsilon\to 0$ uniformly in $\overline{\omega}$ and in $g\in\mathfrak{A}$, the second term in the right-hand side of the above inequality can be absorbed in the left-hand side and we obtain the announced result.

Step 2. For $1<p<\infty$ we have $\nabla\rho_{\varepsilon,g}\to 0$ in $L^p(\omega)$ as $\varepsilon\to 0$, uniformly in $g\in\mathfrak{A}$.
This is obtained as follows. Let $\eta := \eta_{\varepsilon,g} := 1-\rho_{\varepsilon,g}\in[0,1]$, which satisfies

$$\begin{cases} -\Delta\eta + \dfrac{1}{\varepsilon^2}\rho(1+\rho)\eta = \rho|\nabla\varphi|^2 & \text{in } \omega\\ \eta = 0 & \text{on } \partial\Omega.\end{cases} \tag{6.113}$$

Moreover, we have

$$\frac{1}{4\varepsilon^2}\eta \le \frac{1}{\varepsilon^2}\rho(1+\rho)\eta = \rho|\nabla\varphi|^2 + \Delta\eta \le C \quad \text{on } \partial\omega\setminus\partial\Omega, \tag{6.114}$$

since $u_{\varepsilon,g}\to u_{*,g}$ in $C^{2,\gamma}(K)$ for any compact $K\subset\overline{\omega}\setminus\partial\Omega$, uniformly in $g\in\mathfrak{A}$.

We may assume that $p\ge 2$. Multiplying (6.113) by η^{p-1} and using Step 1, Hölder's inequality and (6.114) we find that, for small ε, we have

$$\begin{aligned}\frac{1}{4\varepsilon^2}\int_\omega \eta^p &\le \frac{1}{\varepsilon^2}\int_\omega \rho(1+\rho)\eta^p\\ &= \int_\omega \rho|\nabla\varphi|^2\eta^{p-1} + \int_{\partial\omega\setminus\partial\Omega}\eta^{p-1}\frac{\partial\eta}{\partial\nu} - (p-1)\int_\omega \eta^{p-2}|\nabla\eta|^2\\ &\le \int_\omega \rho|\nabla\varphi|^2\eta^{p-1} + \int_{\partial\omega\setminus\partial\Omega}\eta^{p-1}\frac{\partial\eta}{\partial\nu} \le C\left(\int_\omega\eta^p\right)^{(p-1)/p} + C\varepsilon^{2(p-1)},\end{aligned}$$

and thus

$$\|\eta_{\varepsilon,g}\|_{L^p(\omega)} \le C_p\varepsilon^2, \quad \forall\,\varepsilon<\varepsilon_0,\ \forall\,g\in\mathfrak{A},\ \forall\,p<\infty. \tag{6.115}$$

Inserting (6.115) into (6.113), we find that $\Delta\eta$ is bounded in $L^p(\omega)$, $\forall\,p<\infty$. By standard elliptic estimates, we find that η (and thus ρ) is bounded in $W^{2,p}(\omega)$, $\forall\,p<\infty$. We conclude via the compact embedding $W^{2,p}\hookrightarrow W^{1,p}$ and the fact that, by Lemma 6.14, we have $\rho\to 1$ uniformly in $\overline{\omega}$.

Step 3. For every $\gamma < 1$, we have $\psi_{\varepsilon,g} \to \psi_{*,g}$ in $C^{1,\gamma}(\overline{\omega})$ as $\varepsilon \to 0$, uniformly in $g \in \mathfrak{A}$.
Indeed, $\psi_{\varepsilon,g} - \psi_{*,g}$ satisfies

$$\begin{cases} \Delta(\psi_{\varepsilon,g} - \psi_{*,g}) = -\dfrac{2}{\rho_{\varepsilon,g}} \nabla\rho_{\varepsilon,g} \cdot \nabla(\theta_g + \psi_{\varepsilon,g}) & \text{in } \omega \\ \psi_{\varepsilon,g} - \psi_{*,g} = 0 & \text{on } \partial\Omega \\ \psi_{\varepsilon,g} - \psi_{*,g} \to 0 \text{ in } C^2 & \text{on } \partial\omega \setminus \partial\Omega \end{cases}, \tag{6.116}$$

the latter convergence being uniform in g. By Steps 1 and 2, we have

$$\|\Delta(\psi_{\varepsilon,g} - \psi_{*,g})\|_{L^p(\omega)} \to 0 \quad \text{as } \varepsilon \to 0, \text{ uniformly in } g.$$

Using (6.116), we find that $\psi_{\varepsilon,g} - \psi_{*,g} \to 0$ in $W^{2,p}(\omega)$. We conclude via the embedding $W^{2,p}(\omega) \hookrightarrow C^{1,\gamma}(\overline{\omega})$, valid when $p > 2$ and $\gamma = 1 - 2/p$.

Step 4. Conclusion.
We have

$$T_\varepsilon(g) = u_{\varepsilon,g} \wedge \frac{\partial u_{\varepsilon,g}}{\partial\nu} = \frac{\partial\varphi_{\varepsilon,g}}{\partial\nu} = \frac{\partial\theta_g}{\partial\nu} + \frac{\partial\psi_{\varepsilon,g}}{\partial\nu},$$

and similarly $T_*(g) = \dfrac{\partial\theta_g}{\partial\nu} + \dfrac{\partial\psi_{*,g}}{\partial\nu}$. Using Step 3, we find that

$$T_\varepsilon(g) - T_*(g) = \frac{\partial(\psi_{\varepsilon,g} - \psi_{*,g})}{\partial\nu} \to 0 \text{ in } C^\gamma(\partial\Omega) \quad \text{as } \varepsilon \to 0, \text{ uniformly in } g \in \mathfrak{A}. \square$$

6.8 Existence of critical points in nondegenerate domains

Before stating the main result of this section, let us recall the definition (6.83) of the operator U_* in Section 6.5. Given a^0 a nondegenerate critical point of $W(\cdot, g)$, we first define, in a $C^{1,\beta}$ neighborhood of g, the operator $T_* = T_{*,a^0,g}$. Then U_* is defined in a neighborhood $\mathfrak{B}$ of the origin in $C^{1,\beta}(\partial\Omega;\mathbb{R})$ by

$$U_*(\psi) = U_{*,a^0,g}(\psi) = T_*(ge^{\imath\psi}) = T_{*,a^0,g}(ge^{\imath\psi}).$$

We still denote by U_* the induced map $U_* : \mathfrak{B}/\mathbb{R} \to \dot{C}^\beta(\partial\Omega;\mathbb{R})$, and recall that U_* is smooth.

Theorem 6.17. *Let $d_1, \dots, d_k \in \{-1, 1\}$ and set $d := d_1 + \dots + d_k$.*

Let Ω be a bounded simply connected $C^{1,\beta}$ domain satisfying the two following nondegeneracy conditions:

(ND1) There exists $a^0 \in \Omega^k_*$ such that a^0 is a nondegenerate critical point of $W(\cdot, g^0) = W^\Omega(\cdot, d_1, \dots, d_k, g^0)$, with $g^0 = g^{a^0}$ the canonical boundary data associated with a^0 and $d_1, \dots, d_k$.

(ND2) The corresponding operator $U_{*,a^0,g^0} : \mathfrak{B}/\mathbb{R} \to \dot{C}^\beta(\partial\Omega;\mathbb{R})$ is a local diffeomorphism at the origin, i.e., the differential

$$DU_*(0) : C^{1,\beta}(\partial\Omega;\mathbb{R})/\mathbb{R} \longrightarrow \dot{C}^\beta(\partial\Omega;\mathbb{R})$$

is invertible.

Then there exists $\varepsilon_0 > 0$ such that, for $\varepsilon \in (0, \varepsilon_0)$, there exists $u_\varepsilon \in \mathcal{E}_d$ a critical point of E_ε with prescribed degree d.

Remark 6.18. It will be clear from the proof of Theorem 6.17 that the non-degeneracy condition *(ND2)* can actually be replaced by the following weaker condition:
(ND2') U_* is a local homeomorphism near the origin.

However in what follows it will be more convenient for us to consider the condition *(ND2)*. The main reason for this is that *(ND2)* is stable under small perturbation of the domain, while it is not clear that *(ND2')* is stable.

Remark 6.19. We connect here the hypothesis *(ND2)* in Theorem 6.17 to the hypothesis *(ND2)* presented in the introduction. As we will see in Section 6.11, [3] $DU_*(0)$ is a Fredholm operator of index zero. Thus the above hypothesis *(ND2)* is equivalent to the fact that $DU_*(0)$ is onto. It is not difficult to see (but will not be needed in what follows) that the surjectivity of $DU_*(0)$ is equivalent to the hypothesis *(ND2)* in the introduction, and that the index of the operator L that appears in the introduction is $\operatorname{ind} L = \operatorname{ind} DU_*(0) + 1 = 1$.

Proof of Theorem 6.17. Since Ω satisfies *(ND1)*, the results of Section 6.6 and 6.7 apply. We consider, as in Section 6.7, the operators

$$T_\varepsilon, T_* : \mathfrak{A} \to \dot{C}^\beta(\partial\Omega; \mathbb{R})$$

and

$$U_* : \mathfrak{B}/\mathbb{R} \to \dot{C}^\beta(\partial\Omega; \mathbb{R}),$$

where $\mathfrak{A} = \{g \in C^{1,\beta}(\Omega, \mathbb{S}^1);\ \|g - g^0\| < \delta\}$ and $\mathfrak{B} = \{\psi \in C^{1,\beta}(\partial\Omega; \mathbb{R});\ \|\psi\| < \delta\}$. Here, δ and ε are sufficiently small. We note that T_ε takes its values in $\dot{C}^\beta(\partial\Omega; \mathbb{R})$. Indeed, $u = u_{\varepsilon,g}$ satisfies

$$\int_{\partial\Omega} u \wedge \frac{\partial u}{\partial \nu} = \int_\Omega \operatorname{div}(u \wedge \nabla u) = \int_\Omega u \wedge \Delta u = \int_\Omega \frac{|u|^2 - 1}{\varepsilon^2} u \wedge u = 0.$$

By Lemma 6.15, we may also consider the induced operators

$$U_\varepsilon : \mathfrak{B}/\mathbb{R} \to \dot{C}^\beta(\partial\Omega; \mathbb{R}),\ U_\varepsilon(\psi) = T_\varepsilon(g^0 e^{\imath\psi}).$$

By definition of the canonical boundary datum, it holds

$$U_*(0) = u_{*,a^0,g^0} \wedge \frac{\partial u_{*,a^0,g^0}}{\partial \nu} = \frac{\partial \Phi_{a^0,g^0}}{\partial \tau} = \frac{\partial \widehat{\Phi}_{a^0}}{\partial \tau} = 0. \tag{6.117}$$

Thanks to *(ND2)*, by considering a smaller δ if necessary, we may assume that U_* is a homeomorphism onto its image. By (6.117), there exists some $\eta > 0$ such that

$$U_*(\mathfrak{B}/\mathbb{R}) \supset \left\{\psi \in \dot{C}^\beta(\partial\Omega; \mathbb{R});\ \|\psi\|_{C^\beta(\partial\Omega)} < \eta\right\} := B_\eta. \tag{6.118}$$

3. In the special case where $d = 1$ and $k = 1$, but the arguments there adapt to the general case.

Recall the result of Proposition 6.16: for sufficiently small ε, $U_\varepsilon - U_*$ is compact and we have

$$\lim_{\varepsilon\to 0} \sup \left\{ \|U_\varepsilon(\psi) - U_*(\psi)\|_{C^\beta(\partial\Omega)};\ \psi \in \mathfrak{B} \right\} = 0. \tag{6.119}$$

Using (6.118), (6.119) and standard properties of the Leray-Schauder degree, we find that

$$U_\varepsilon(\mathfrak{B}/\mathbb{R}) \supset \left\{ \psi \in \dot{C}^\beta(\partial\Omega;\mathbb{R});\ \|\psi\|_{C^\beta(\partial\Omega)} < \frac{\eta}{2} \right\} = B_{\eta/2}, \tag{6.120}$$

for sufficiently small ε. Indeed, the argument goes as follows. We start from

$$U_\varepsilon(\mathfrak{B}/\mathbb{R}) = \left(\mathrm{Id} + (U_\varepsilon - U_*)\circ U_*^{-1}\right)(U_*(\mathfrak{B}/\mathbb{R})) \supset \left(\mathrm{Id} + (U_\varepsilon - U_*)\circ U_*^{-1}\right)(B_\eta). \tag{6.121}$$

Here, Id denotes the identity map in $\dot{C}^\beta(\partial\Omega;\mathbb{R})$.

Let $L_\varepsilon := (U_\varepsilon - U_*)\circ U_*^{-1}$. Then $L_\varepsilon : B_\eta \to \dot{C}^\beta(\partial\Omega;\mathbb{R})$ is compact and, by (6.119), there exists $\varepsilon_0 > 0$ such that

$$\sup \left\{ \|L_\varepsilon(\psi)\|_{C^\beta(\partial\Omega)};\ \psi \in B_\eta \right\} < \eta/2, \quad \forall\, \varepsilon \in (0,\varepsilon_0). \tag{6.122}$$

We complete the proof of (6.120) by showing that $B_{\eta/2} \subset (Id + L_\varepsilon)(B_\eta)$ for $\varepsilon \in (0,\varepsilon_0)$. For this purpose, we let $\psi_0 \in B_{\eta/2}$ and consider the compact operator $T : B_\eta \to \dot{C}^\beta(\partial\Omega;\mathbb{R})$, $T(\psi) := L_\varepsilon(\psi) - \psi_0$. We claim that

$$(\mathrm{Id} + sT)(\psi) \neq 0, \quad \forall\, s \in [0,1],\ \forall\, \psi \in \partial B_\eta. \tag{6.123}$$

Indeed, (6.123) is obtained by contradiction. Otherwise, using (6.122), we obtain, for some ψ such that $\|\psi\|_{C^\beta(\partial\Omega)} = \eta$:

$$\eta/2 < \eta - s\|L_\varepsilon(\psi)\|_{C^\beta(\partial\Omega)} \le \|\psi + sL_\varepsilon(\psi)\|_{C^\beta(\partial\Omega)} = \|s\psi_0\|_{C^\beta(\partial\Omega)} < \eta/2.$$

By (6.123), the Leray-Schauder degree $\deg(\mathrm{Id} + sT, B_\eta, 0)$ is well defined. By homotopy invariance, we find that

$$\deg(\mathrm{Id} + sT, B_\eta, 0) = \deg(\mathrm{Id}, B_\eta, 0) = 1.$$

As a consequence, the equation $(\mathrm{Id} + T)(\psi) = 0$ admits at least a solution $\psi \in B_\eta$. This ψ satisfies $(\mathrm{Id} + L_\varepsilon)(\psi) = \psi_0$. The proof of (6.120) is complete.

Let $\varepsilon \in (0,\varepsilon_0)$. Then, by (6.120), there exists some $\psi \in \mathfrak{B}$ such that $U_\varepsilon(\psi) = 0$. Let $g = g^0 e^{\imath\psi}$. Then $u_\varepsilon = u_{\varepsilon,g} \in \mathcal{E}_d$ is a solution of the Ginzburg-Landau equation, and it satisfies the semi-stiff boundary condition

$$u_\varepsilon \wedge \frac{\partial u_\varepsilon}{\partial \nu} = T_\varepsilon(g) = U_\varepsilon(\psi) = 0 \quad \text{on } \partial\Omega.$$

Therefore, u_ε is a critical point of E_ε with prescribed degree d. □

6.9 Nondegeneracy of domains is stable

In this section we show that, if a domain Ω_0 satisfies the nondegeneracy conditions *(ND1)-(ND2)* required in Theorem 6.17, then a slightly perturbed domain $\Omega \approx \Omega_0$ still satisfies these nondegeneracy conditions.

Theorem 6.20. *Assume that Ω_0 satisfies (ND1)-(ND2). Fix a conformal representation $f_0 : \mathbb{D} \to \Omega_0$. There exists $\delta > 0$ such that, for every holomorphic map $f \in C^{1,\beta}(\overline{\mathbb{D}})$ satisfying $\|f - f_0\|_{C^{1,\beta}} < \delta$, the domain $\Omega := f(\mathbb{D})$ satisfies (ND1)-(ND2).*

Proof. Let V_β be as in (6.72). We let $\widetilde{g}^0 \in C^{1,\beta}(\mathbb{S}^1; \mathbb{S}^1)/\mathbb{S}^1$ denote the canonical boundary datum associated with $\alpha^0 := f_0^{-1}(a^0)$, so that $\widetilde{g}^0 = g^0 \circ f_0$.

Since Ω_0 satisfies *(ND1)*, we know from Proposition 6.9 that there exist: a neighborhood $\mathcal{V}_1$ of f_0 in V_β, a neighborhood $\mathcal{V}_2$ of the origin in $C^{1,\beta}(\mathbb{S}^1; \mathbb{R})$, and a smooth map $\alpha : \mathcal{V}_1 \times \mathcal{V}_2 \to \mathbb{D}^k_*$, such that the following holds. For $f \in \mathcal{V}_1$ and $\psi \in \mathcal{V}_2$, let $\Omega = f(\mathbb{D})$ and $g = (\widetilde{g}^0 e^{\imath\psi}) \circ f^{-1}$. Then $a(f, \psi) := f(\alpha(f, \psi))$ is a nondegenerate critical point of $W^\Omega(\cdot, g)$.

By the above, we may define, as in (6.81), the smooth operator $U_{*,f} = U_{*,a(f,0),\widetilde{g}^0 \circ f^{-1}}$,

$$U_{*,f}(\psi) = N^\Omega\left(a(f, \psi \circ f), (\widetilde{g}^0 \circ f^{-1})e^{\imath\psi}\right) \quad \text{for small } \psi \in C^{1,\beta}(\partial\Omega; \mathbb{R})/\mathbb{R}. \tag{6.124}$$

The spaces between which $U_{*,f}$ is defined vary with f. In order to deal with fixed spaces, we consider the linear isomorphisms

$$\Theta_f : \dot{C}^\beta(\partial\Omega; \mathbb{R}) \to \dot{C}^\beta(\mathbb{S}^1; \mathbb{R}),\ \psi \mapsto |f'|\psi \circ f, \tag{6.125}$$

$$\Xi_f : C^{1,\beta}(\partial\Omega; \mathbb{R})/\mathbb{R} \to C^{1,\beta}(\mathbb{S}^1; \mathbb{R})/\mathbb{R},\ \psi \mapsto \psi \circ f, \tag{6.126}$$

and we let

$$U(f, \psi) = \Theta_f \circ U_{*,f} \circ \Xi_f^{-1}(\psi) \quad \text{for } (f, \psi) \in \mathcal{V}_1 \times (\mathcal{V}_2/\mathbb{R}), \tag{6.127}$$

so that $U_{*,f}$ is a local diffeomorphism if and only if $U(f, \cdot)$ is a local diffeomorphism.

Moreover, if we express N^Ω using (6.39), then we obtain

$$U(f, \psi) = N^{\mathbb{D}}(\alpha(f, \psi), \widetilde{g}_0 e^{\imath\psi}). \tag{6.128}$$

By combining (6.128) with the explicit formula (6.48) for $N^{\mathbb{D}}$, we find that $U : \mathcal{V}_1 \times (\mathcal{V}_2/\mathbb{R}) \to \dot{C}^\beta(\mathbb{S}^1; \mathbb{R})$ is smooth.

On the other hand, using the definition (6.18) of the canonical boundary datum, we have

$$u_{*,a,g^a} \wedge \frac{\partial u_{*,a,g^a}}{\partial \nu} = \frac{\partial \Phi_{a,g^a}}{\partial \tau} = \frac{\partial \widehat{\Phi_a}}{\partial \tau} = 0,$$

so that $U(f_0, 0) = 0$.

Moreover, since Ω_0 satisfies *(ND2)*, $U(f_0, \cdot)$ is a local diffeomorphism near the origin, i.e., $D_\psi U(f_0, 0)$ is invertible. By the implicit function theorem, possibly after shrinking $\mathcal{V}_1$, for every $f \in \mathcal{V}_1$ there exists $\psi(f) \in \mathcal{V}_2$ such that

$$U(f, \psi(f)) = 0. \tag{6.129}$$

In addition, the map $f \mapsto \psi(f)$ is smooth and we can assume that $D_\psi U(f, \psi(f))$ is invertible.

Let $f \in \mathcal{V}_1$ and set $\Omega := f(\mathbb{D})$. We claim that Ω satisfies *(ND1)-(ND2)*. Assuming the claim proved for the moment, we complete the proof of Theorem 6.20 by taking any $\delta > 0$ such that

$$\{f \in X_\beta;\ \|f - f_0\|_{C^{1,\beta}} < \delta\} \subset \mathcal{V}_1.$$

We next turn to the proof of the claim. Let $g_\Omega := (\widetilde{g}_0 e^{\imath\psi(f)})\circ f^{-1} \in C^{1,\beta}(\partial\Omega; \mathbb{S}^1)$, and $a_\Omega := a(f, \psi(f)) \in \Omega_*^k$. By the definition (6.129) of $\psi(f)$ and the definition (6.127) of U, we obtain

$$U_{*,f}(\psi(f) \circ f^{-1}) = 0. \tag{6.130}$$

By combining (6.130) with the definition (6.124) of $U_{*,f}$, we find that

$$N^\Omega(a_\Omega, g_\Omega) = u_{*,a_\Omega,g_\Omega} \wedge \frac{\partial u_{*,a_\Omega,g_\Omega}}{\partial \nu} = \frac{\partial \Phi_{a_\Omega,g_\Omega}}{\partial \tau} = 0. \tag{6.131}$$

The normalization condition in (6.7) combined with (6.131) implies that $\Phi_{a_\Omega,g_\Omega} = 0$ on $\partial\Omega$, and thus

$$\Phi_{a_\Omega,g_\Omega} = \widehat{\Phi}_{a_\Omega}. \tag{6.132}$$

In turn, (6.132) implies that $g_\Omega = g^{a_\Omega}$ is the canonical boundary data associated with a_Ω. Since, by definition of the map $(f, \psi) \mapsto a(f, \psi)$, the configuration a_Ω is a nondegenerate critical point of $W(\cdot, g_\Omega)$, we find that the nondegeneracy condition *(ND1)* is satisfied by Ω.

On the other hand, since $D_\psi U(f, \psi(f))$ is invertible, $U_{*,f}$ is a local diffeomorphism near $\psi(f) \circ f^{-1}$, which means that U_{*,a_Ω,g_Ω} is a local diffeomorphism near the origin. We find that Ω satisfies *(ND2)*.

The proof of Theorem 6.20 is complete. □

6.10 The radial configuration is nondegenerate

In this section we let $d = 1$, $k = 1$, $d_1 = 1$, and prove that the unit disc $\mathbb{D}$ satisfies *(ND1)-(ND2)*. As a consequence, domains close to the unit disc satisfy the nondegeneracy conditions when $d = 1$, $k = 1$, $d_1 = 1$.

Proposition 6.21. *Assume $\Omega = \mathbb{D}$, $k = 1$, $d = 1$. Then $a = 0$ is a nondegenerate critical point of $W(\cdot, g^0)$, and $DU_{*,0,g^0}(0)$ is invertible.*

Proof. Step 1. 0 is a nondegenerate critical point of $W(\cdot, g^0)$.
Indeed, by combining (6.27) and (6.44), we easily obtain that the canonical boundary datum $g^0 : \mathbb{S}^1 \to \mathbb{S}^1$ corresponding to $a = 0$ is given by $g^0(z) = z$. From (6.53) we know that

$$W(a, g^0) = \widehat{W}(a) + \frac{1}{2}\int_{\mathbb{D}} \left|\nabla\psi^*_{a,g^0}\right|^2. \tag{6.133}$$

On the other hand, (6.59) leads to

$$\nabla\psi^*_{a,g^0}(x) = -2\frac{a(\overline{a}x - 1)}{|1 - \overline{a}x|^2}, \quad \forall\, x \in \mathbb{D}, \tag{6.134}$$

and therefore

$$\frac{1}{2}\int_{\mathbb{D}}\left|\nabla\psi^*_{a,g^0}\right|^2 = 2|a|^2\int_{\mathbb{D}}\frac{dx}{|1-\overline{a}x|^2}. \tag{6.135}$$

Thanks to the $|a|^2$ factor, if we differentiate (6.135) with respect to a, and next let $a=0$, we obtain

$$\nabla_a\left[\frac{1}{2}\int_{\mathbb{D}}\left|\nabla\psi^*_{a,g^0}\right|^2\right]\Big|_{a=0} = 0. \tag{6.136}$$

If we differentiate twice (6.135) with respect to a, and next let $a=0$, then we are left with only one non zero term (thanks to the $|a|^2$ factor again). More specifically, we obtain

$$\nabla_a^2\left[\frac{1}{2}\int_{\mathbb{D}}\left|\nabla\psi^*_{a,g^0}\right|^2\right]\Big|_{a=0} = 4\int_{\mathbb{D}}\frac{dx}{|1-\overline{a}x|^2}\Big|_{a=0} I_2 = 4\int_{\mathbb{D}} dx\, I_2 = 4\pi I_2. \tag{6.137}$$

By combining (6.133) with (6.136) and (6.137), we find that

$$\nabla_a W(0,g^0) = \nabla\widehat{W}(0), \quad \nabla_a^2 W(0,g^0) = \nabla^2\widehat{W}(0) + 4\pi I_2. \tag{6.138}$$

We next compute $\nabla\widehat{W}(0)$ and $\nabla^2\widehat{W}(0)$. When $k=1$ and $d=1$, formula (6.45) reads

$$\widehat{W}(a) = \pi\log\left(1-|a|^2\right), \quad \forall\, a\in\mathbb{D}. \tag{6.139}$$

Identifying the complex number a with a vector in $\mathbb{R}^2$, the two first derivatives of $\widehat{W}$ are respectively given by:

$$\nabla\widehat{W}(a) = \frac{2\pi}{|a|^2-1}a \in \mathbb{R}^2 \tag{6.140}$$

$$\nabla^2\widehat{W}(a) = \frac{2\pi}{|a|^2-1}I_2 - \frac{4\pi}{\left(|a|^2-1\right)^2}a\otimes a \in M_2(\mathbb{R}). \tag{6.141}$$

In particular, we obtain $\nabla\widehat{W}(0)=0$ and $\nabla^2\widehat{W}(0) = -2\pi I_2$. Plugging this into (6.138) yields

$$\nabla_a W(0,g^0) = 0, \quad \nabla_a^2 W(0,g^0) = 2\pi I_2, \tag{6.142}$$

so that $a=0$ is indeed a nondegenerate critical point of $W(\cdot,g^0)$.

Step 2. $DU_*(0)$ is invertible.
In our case, formula (6.48) becomes

$$N(a,g^0e^{\imath\psi}) = \frac{\partial\psi^*}{\partial\tau} - 2\frac{a\wedge z}{|z-a|^2}. \tag{6.143}$$

Therefore

$$U_*(\psi) = \frac{\partial\psi^*}{\partial\tau} - 2\frac{a(\psi)\wedge z}{|z-a(\psi)|^2}, \tag{6.144}$$

where $\psi \mapsto a(\psi)$ is smooth, $a(0) = 0$ and $a(\psi)$ is a nondegenerate critical point of $W(\cdot, g^0 e^{\imath\psi})$.

Using (6.144) together with the fact that $a(0) = 0$, we obtain that

$$DU_*(0)\psi = \frac{\partial \psi^*}{\partial \tau} - 2(Da(0)\psi) \wedge z. \tag{6.145}$$

In (6.145), ψ is either a function in $C^{1,\beta}(\mathbb{S}^1;\mathbb{R})$, or a class in $C^{1,\beta}(\mathbb{S}^1;\mathbb{R})/\mathbb{R}$. Thus the linear operator $DU_*(0) : C^{1,\beta}(\mathbb{S}^1;\mathbb{R})/\mathbb{R} \to \dot{C}^{\beta}(\mathbb{S}^1;\mathbb{R})$ can be written $DU_*(0) = L - K$, where

$$L(\psi) := \frac{\partial \psi^*}{\partial \tau} \text{ and } K(\psi) := 2(Da(0)\psi) \wedge z, \quad \forall\, \psi \in C^{1,\beta}(\mathbb{S}^1;\mathbb{R})/\mathbb{R}.$$

The operator L is an isomorphism, and K is compact since it has finite range. As a consequence, $DU_*(0)$ is Fredholm of index zero and, in order to complete Step *2*, it suffices to prove that $DU_*(0)$ is injective. For this purpose, we compute $Da(0)$ using the implicit equation

$$F(a(\psi), \psi) := \nabla_a W(a(\psi), g^0 e^{i\psi}) = 0 \tag{6.146}$$

satisfied by a. By differentiating (6.146) with respect to ψ we obtain (via (6.142))

$$D_\psi F(0,0)\psi = -\nabla_a^2 W(0, g^0) Da(0)\psi = -2\pi Da(0)\psi. \tag{6.147}$$

Let us compute $D_\psi F(0,0)$. Recalling (6.46), we find that

$$\begin{aligned} F(a,\psi) =& \nabla \widehat{W}(a) + \nabla_a \left[\frac{1}{2}\int_{\mathbb{D}} \left|\nabla \psi^*_{a,g^0} + \nabla \psi^*\right|^2\right] \\ =& \nabla \widehat{W}(a) + \nabla_a \left[\frac{1}{2}\int_{\mathbb{D}} \left|\nabla \psi^*_{a,g^0}\right|^2\right] + \nabla_a \left[\int_{\mathbb{D}} \nabla \psi^*_{a,g^0} \cdot \nabla \psi^*\right]. \end{aligned} \tag{6.148}$$

The two first terms do not depend on ψ, and the last term depends linearly on ψ. Hence we obtain

$$D_\psi F(a,0)\psi = \nabla_a \left[\int_{\mathbb{D}} \nabla \psi^*_{a,g^0} \cdot \nabla \psi^*\right]. \tag{6.149}$$

Integrating by parts, using the explicit formula (6.59) for ψ^*_{a,g^0}, and the fact that

$$\int_{\mathbb{S}^1} \frac{\partial \psi^*}{\partial \nu} = \int_{\mathbb{S}^1} \frac{\partial \psi}{\partial \tau} = 0,$$

we find that

$$\int_{\mathbb{D}} \nabla \psi^*_{a,g^0} \cdot \nabla \psi^* = -2\int_{\mathbb{S}^1} \log|1 - \overline{a}z| \frac{\partial \psi^*}{\partial \nu}. \tag{6.150}$$

If we first plug (6.150) into (6.149) and next let $a = 0$, then we obtain

$$D_\psi F(0,0)\psi = 2\int_{\mathbb{S}^1} z \frac{\partial \psi^*}{\partial \nu}, \tag{6.151}$$

and finally, using (6.147),

$$Da(0)\psi = -\frac{1}{\pi}\int_{\mathbb{S}^1} z\frac{\partial\psi^*}{\partial\nu}. \tag{6.152}$$

We are now in position to prove that $DU_*(0)$ is injective. Let $\psi \in \ker DU_*(0)$. Then, recalling (6.145), we have

$$\frac{\partial\psi^*}{\partial\tau} = 2(Da(0)\psi)\wedge z = \alpha\wedge z, \tag{6.153}$$

where

$$\alpha = -\frac{2}{\pi}\int_{\mathbb{S}^1} z\frac{\partial\psi^*}{\partial\nu} \in \mathbb{C}. \tag{6.154}$$

Since ψ^* is harmonic and has zero average on $\mathbb{S}^1$, we may write

$$\psi^*(re^{\imath\theta}) = \sum_{n\neq 0} a_n r^n e^{\imath n\theta}. \tag{6.155}$$

Hence (6.153) yields

$$\frac{\overline{\alpha}}{2\imath}e^{\imath\theta} - \frac{\alpha}{2\imath}e^{-\imath\theta} = \frac{\partial\psi^*}{\partial\tau}(e^{\imath\theta}) = \sum_{n\neq 0}\imath n a_n e^{\imath n\theta}. \tag{6.156}$$

Identifying the Fourier coefficients, we obtain

$$a_n = 0 \text{ for } |n| > 1, \quad a_1 = -\frac{\overline{\alpha}}{2}, \quad a_{-1} = -\frac{\alpha}{2}, \tag{6.157}$$

so that (6.155) becomes

$$\psi^*(re^{\imath\theta}) = -\frac{\overline{\alpha}}{2}re^{\imath\theta} - \frac{\alpha}{2}re^{-\imath\theta}. \tag{6.158}$$

By (6.158), we have

$$\int_{\mathbb{S}^1} z\frac{\partial\psi^*}{\partial\nu} = -\frac{1}{2}\int_0^{2\pi} e^{i\theta}(\overline{\alpha}e^{\imath\theta} + \alpha e^{-\imath\theta})d\theta = -\pi\alpha. \tag{6.159}$$

Plugging (6.159) into (6.154) we obtain $\alpha = 2\alpha$, so that $\alpha = 0$ and consequently $\psi^* = 0$. Therefore, we have $\psi = 0$ modulo $\mathbb{R}$, and thus $DU_*(0)$ is invertible. □

Corollary 6.22. *If a domain Ω is sufficiently close to the unit disc, in the sense that there exists a conformal representation $f : \mathbb{D} \to \Omega$ such that $\|f - \mathrm{Id}\|_{C^{1,\beta}} < \delta$ for sufficiently small δ, then, for small ε, E_ε admits critical points with prescribed degree one.*

6.11 In degree one, « most » of the domains are non degenerate

In this section, we assume that $k = 1$ and $d = 1$, and we prove that every domain can be approximated with domains satisfying the nondegeneracy conditions *(ND1)-(ND2)*. More specifically, we establish the following result.

Theorem 6.23. *Assume that $k = 1$ and $d = 1$. Let $\Omega_0 \subset \mathbb{R}^2$ be a simply connected bounded domain with $C^{1,\beta}$ boundary, and fix a conformal representation $f_0 : \mathbb{D} \to \Omega_0$.*

Then, for every $\eta > 0$, there exists a conformal representation $f : \mathbb{D} \to \Omega := f(\mathbb{D})$ such that $\|f_0 - f\|_{C^{1,\beta}} < \eta$ and such that the corresponding domain Ω satisfies (ND1)-(ND2).

The main idea of the proof of Theorem 6.23 is to use transversality. Among other ingredients, we will rely on the following abstract transversality result [114, Theorem 3].

Theorem 6.24. *Let X, Λ, Y be smooth separable Banach manifolds. Let $\Phi : X \times \Lambda \to Y$ be a smooth map.*

Assume that:
1. for every $\lambda \in \Lambda$, $\Phi_\lambda := \Phi(\cdot, \lambda) : X \to Y$ is Fredholm. [4]
2. Φ is transverse to $\{0\}$, i.e., for every (x, λ) such that $\Phi(x, \lambda) = 0$, the differential $D\Phi(x, \lambda)$ is onto.

Then the set $\{\lambda;\ \Phi_\lambda$ is transverse to $\{0\}\}$ is dense in Λ.

Note that, if X and Y are finite dimensional, then condition *1.* is automatically satisfied.

Another ingredient of the proof is the following fact, which relates non degenerate critical points of $\widehat{W}$ to non degenerate critical points of $W(\cdot, g^a)$.

Proposition 6.25. *Assume that $k = 1$ and $d = 1$. Let $a_0 \in \Omega$ be a non degenerate critical point of $\widehat{W}^\Omega$. Then a_0 is a non degenerate critical point of $W(\cdot, g^{a_0})$.*

Proof. Let us first remark that a_0 is automatically a critical point of $W^{\Omega_0}(\cdot, g^{a_0})$. [5] Indeed, using (6.26), in which each term is smooth thanks to the formulas in Sections 6.3 and 6.4, and the fact that (by definition) we have $\psi_{a,g^a} = 0$, we find that

$$\nabla_a W(a, g)|_{g=g^a} = \nabla \widehat{W}(a). \tag{6.160}$$

It remains to prove that a_0 is non degenerate as a critical point of $W^\Omega(\cdot, g^{a_0})$.

Let $f : \mathbb{D} \to \Omega$ be a conformal representation and set $\alpha_0 := f^{-1}(a_0)$. Then $\widetilde{f}(0) = a_0$, where

$$\widetilde{f}(z) = f\left(\frac{z + \alpha_0}{1 + \overline{\alpha_0} z}\right).$$

Therefore, by replacing f with $\widetilde{f}$, we may actually assume that $f(0) = a_0$. In view of (6.37) and of the fact that, in the unit disc, we have $g^0 = \mathrm{Id}$, we obtain

$$g^{a_0} \circ f = g^0 = \mathrm{Id}. \tag{6.161}$$

Recall that, by Lemmas 6.5 and 6.6 and by (6.45) we have

$$\widehat{W}^\Omega(f(\alpha)) = \widehat{W}^{\mathbb{D}}(\alpha) + P(\alpha, f), \tag{6.162}$$

$$W^\Omega(f(\alpha), g^{a_0}) = W^{\mathbb{D}}(\alpha, g^0) + P(\alpha, f), \tag{6.163}$$

4. That is, the linearized operator $D_x\Phi(x, \lambda)$ is Fredholm for every x and every λ.
5. This is not specific to the case where $k = 1$ and $d = 1$, but holds for arbitrary k and degrees d_j, $j \in [\![1, k]\!]$.

where

$$\widehat{W}^{\mathbb{D}}(\alpha) = \pi \log(1 - |\alpha|^2), \; P(\alpha, f) := \pi \log |f'(\alpha)| \tag{6.164}$$

and

$$W^{\mathbb{D}}(\alpha, g^0) \text{ is given by (6.46) with } \psi = 0. \tag{6.165}$$

By (6.162)-(6.165) and the discussion at the beginning of the proof of Proposition 6.7, the assumption that a_0 is a non degenerate critical point of $\widehat{W}^{\Omega}$ is equivalent to the fact that 0 is a non degenerate critical point of $\widehat{W}^{\mathbb{D}} + P(\cdot, f)$. Similarly, the desired conclusion (that a_0 is a nondegenerate critical point of $W^{\Omega}(\cdot, g^{a_0})$) is equivalent to the fact that 0 is non degenerate as a critical point of $W^{\mathbb{D}}(\cdot, g^0) + P(\cdot, f)$.

Since

$$\nabla \left[\widehat{W}^{\mathbb{D}} + P(\cdot, f)\right](\alpha) = \frac{-2\pi\alpha}{1 - |\alpha|^2} + \pi \overline{\frac{f''(\alpha)}{f'(\alpha)}} \in \mathbb{C} \simeq \mathbb{R}^2, \tag{6.166}$$

and since 0 is a critical point of $\widehat{W}^{\mathbb{D}} + P(\cdot, f)$, we have $f''(0) = 0$.

In order to calculate the Hessian of $P(\cdot, f)$ at the origin, we find the second order Taylor expansion of $P(\cdot, f)$:

$$\begin{aligned} P(\alpha, f) &= \pi \log \left| f'(0) + f^{(3)}(0)\alpha^2 + o(|\alpha|^2) \right| \\ &= \pi \log |f'(0)| + \frac{\pi}{2} \log \left(\left| 1 + \frac{f^{(3)}(0)}{f'(0)}\alpha^2 + o(|\alpha|^2) \right|^2 \right) \\ &= P(0, f) + \frac{\pi}{2} \log \left(1 + 2 \operatorname{Re} \left(\frac{f^{(3)}(0)}{f'(0)}\alpha^2 \right) + o(|\alpha|^2) \right) \\ &= P(0, f) + \frac{\pi}{2} \left(2 \operatorname{Re} \left(\frac{f^{(3)}(0)}{f'(0)}\alpha^2 \right) + o(|\alpha|^2) \right) \\ &= P(0, f) + \pi \left(\overline{\frac{f^{(3)}(0)}{f'(0)}}\, \overline{\alpha} \right) \cdot \alpha + o(|\alpha|^2). \end{aligned} \tag{6.167}$$

In the last equality, $z \cdot w$ stands for the real scalar product of the complex numbers z and w (identified with vectors in $\mathbb{R}^2$). As a consequence, we have

$$\nabla^2_{\alpha} P(0, f) = \pi M_{f^{(3)}(0)/f'(0)}, \tag{6.168}$$

where, for a complex number $z \in \mathbb{C}$, M_z denotes the matrix corresponding to the $\mathbb{R}$-linear map

$$T : \mathbb{C} \to \mathbb{C}, \; \xi \overset{T}{\longmapsto} \overline{z\xi},$$

i.e.,

$$M_z = \begin{pmatrix} \operatorname{Re} z & -\operatorname{Im} z \\ -\operatorname{Im} z & -\operatorname{Re} z \end{pmatrix}.$$

Recall that, from (6.141) and (6.142), it holds

$$\nabla^2 \widehat{W}^{\mathbb{D}}(0) = -2\pi I_2 \quad \text{and} \quad \nabla^2_{\alpha} W(0, g^0) = 2\pi I_2. \tag{6.169}$$

By combining (6.168) with (6.169), we obtain

$$\nabla^2 \left[\widehat{W}^{\mathbb{D}} + P(\cdot, f)\right](0) = \pi M_{f^{(3)}(0)/f'(0)} - 2\pi I_2, \tag{6.170}$$

$$\nabla^2 \left[W^{\mathbb{D}}(\cdot, g^0) + P(\cdot, f)\right](0) = \pi M_{f^{(3)}(0)/f'(0)} + 2\pi I_2. \tag{6.171}$$

We claim that the two Hessian matrices (6.170) and (6.171) have the same determinant. In fact, for every $z \in \mathbb{C}$, we have

$$\det(M_z - 2I_2) = \begin{vmatrix} \operatorname{Re} z - 2 & -\operatorname{Im} z \\ -\operatorname{Im} z & -\operatorname{Re} z - 2 \end{vmatrix} = (2 - \operatorname{Re} z)(\operatorname{Re} z + 2) - (\operatorname{Im} z)^2$$
$$= \begin{vmatrix} 2 + \operatorname{Re} z & -\operatorname{Im} z \\ -\operatorname{Im} z & 2 - \operatorname{Re} z \end{vmatrix} = \det(2I_2 + M_z).$$

The Hessian matrix in (6.170) being non degenerate by assumption, so is the Hessian in (6.171). Therefore 0 is a non degenerate critical point of $W^{\mathbb{D}}(\cdot, g^0) + P(\cdot, f)$, which means that a_0 is a non degenerate critical point of $W^{\Omega}(\cdot, g^{a_0})$. □

Before proceeding to the proof of Theorem 6.23, we introduce some notation. For $\alpha \in \mathbb{D}$ and $f \in V_\beta$, let

$$\widehat{F}(\alpha, f) = \nabla_\alpha \left[\widehat{W}^{\mathbb{D}}(\alpha) + \pi \log|f'(\alpha)|\right], \tag{6.172}$$

so that $\widehat{F} : \mathbb{D} \times V_\beta \to \mathbb{R}^2$ is smooth (thanks to the computations in Lemma 6.8), and, by Lemma 6.5, a point $a = f(\alpha) \in \Omega = f(\mathbb{D})$ is a non degenerate critical point of $\widehat{W}^{\Omega}$ if and only if α is a non degenerate zero of $\widehat{F}(\cdot, f)$.

Similarly, $g_0 \in C^{1,\beta}(\mathbb{S}^1; \mathbb{S}^1)$ being fixed, we define, for $\alpha \in \mathbb{D}$, $f \in V_\beta$ and $\psi \in C^{1,\beta}(\mathbb{S}^1; \mathbb{R})$,

$$F(\alpha, \psi, f) = \nabla_\alpha \left[W^{\mathbb{D}}(\alpha, g_0 e^{\imath\psi}) + \pi \log|f'(\alpha)|\right], \tag{6.173}$$

so that $F : \mathbb{D} \times C^{1,\beta}(\mathbb{S}^1; \mathbb{R}) \times V_\beta \to \mathbb{R}^2$ is smooth. By Lemma 6.5 and the discussion at the beginning of the proof of Proposition 6.7, a point $a = f(\alpha) \in \Omega = f(\mathbb{D})$ is a non degenerate critical point of $W^{\Omega}(\cdot, (g_0 e^{\imath\psi}) \circ f^{-1})$ if and only if α is a non degenerate zero of $F(\cdot, \psi, f)$.

Using (6.46), we may split

$$\widehat{F}(\alpha, f) = F_1(\alpha) + F_2(\alpha, f) \tag{6.174}$$

and

$$F(\alpha, \psi, f) = F_1(\alpha) + F_2(\alpha, f) + F_3(\alpha, \psi), \tag{6.175}$$

where the smooth maps F_1, F_2 and F_3 are respectively given by

$$F_1(\alpha) = \nabla \widehat{W}^{\mathbb{D}}(\alpha), \tag{6.176}$$

$$F_2(\alpha, f) = \nabla_\alpha [\pi \log|f'(\alpha)|] = \pi \frac{\overline{f''(\alpha)}}{\overline{f'(\alpha)}} \in \mathbb{C} \simeq \mathbb{R}^2, \tag{6.177}$$

$$F_3(\alpha, \psi) = \nabla_\alpha \left[\frac{1}{2} \int_{\mathbb{D}} \left|\nabla(\psi^*_{\alpha, g_0} + \psi^*)\right|^2\right]. \tag{6.178}$$

Proof of Theorem 6.23. The proof is divided into two steps. In each step we apply the abstract transversality result (Theorem 6.24) in order to prove that a certain nondegeneracy is generic.

Step 1. We may assume that $\widehat{W}^{\Omega_0}$ has a non degenerate critical point $a_0 \in \Omega_0$. Indeed, we claim that $\widehat{F}$ is transverse to $\{0\}$. This will follow if we prove that $D_f\widehat{F}(\alpha, f)$ is surjective for every (α, f). In turn, surjectivity is established as follows. For every $h \in X_\beta$ we have

$$D_f\widehat{F}(\alpha, f)\cdot h = D_fF_2(\alpha, f)\cdot h = \pi \frac{\overline{f'(\alpha)h''(\alpha) - f''(\alpha)h'(\alpha)}}{\overline{f'(\alpha)}^2} \in \mathbb{C} \simeq \mathbb{R}^2. \quad (6.179)$$

If $f''(\alpha) \neq 0$, then the choice $h(z) = -\lambda z$ (with $\lambda \in \mathbb{C}$ arbitrary constant) leads to

$$\pi \frac{\overline{f''(\alpha)}}{\overline{f'(\alpha)}^2}\overline{\lambda} \in \text{range } D_f\widehat{F}(\alpha, f),$$

so $D_f\widehat{F}(\alpha, f)$ is surjective. If $f''(\alpha) = 0$, then we take $h(z) = \lambda z^2$ and obtain

$$\frac{2\pi}{\overline{f'(\alpha)}}\overline{\lambda} \in \text{range } D_f\widehat{F}(\alpha, f),$$

and thus the claim is proved.

Therefore we can apply the transversality theorem: we can choose f arbitrarily close to f_0, such that $\widehat{F}(\cdot, f)$ is transverse to $\{0\}$. Thus, by slightly perturbing f_0, we may actually assume that $\widehat{F}(\cdot, f_0)$ is transverse to $\{0\}$.

Since

$$\begin{aligned} \widehat{W}^{\Omega_0}(f_0(\alpha)) &= \widehat{W}^{\mathbb{D}}(\alpha) + \pi \log |f_0'(\alpha)| \\ &= \pi \log(1 - |\alpha|^2) + \pi \log |f_0'(\alpha)| \longrightarrow -\infty \quad \text{as } |\alpha| \to 1, \end{aligned} \quad (6.180)$$

there exists some $a_0 \in \Omega$, such that $\widehat{W}^{\Omega}(a_0) = \max_{\Omega_0} \widehat{W}^{\Omega}$. Hence a_0 is a critical point of $\widehat{W}^{\Omega_0}$, which is equivalent to the fact that $\alpha_0 := f_0^{-1}(a_0)$ is a zero of $\widehat{F}(\cdot, f_0)$. Since the map $\widehat{F}(\cdot, f_0)$ is transverse to $\{0\}$, its differential is surjective at α_0. Therefore, α_0 is a non degenerate zero of $\widehat{F}(\cdot, f_0)$, which means that a_0 is a non degenerate critical point of $\widehat{W}^{\Omega_0}$. The proof of Step *1* is complete.

Step 2. There exists f arbitrarily close to f_0, such that $\Omega = f(\mathbb{D})$ satisfies *(ND1)-(ND2).*

Thanks to Step *1* and Proposition 6.25, possibly after slightly perturbing f_0, we may assume that there exists some $a_0 = f_0(\alpha_0) \in \Omega_0$, which is a non degenerate critical point of both $\widehat{W}^{\Omega_0}$ and $W^{\Omega_0}(\cdot, g_0)$ (with $g_0 = g^{a_0}$).

Since $\widehat{F}(\alpha_0, f_0) = 0$, and since $D_\alpha\widehat{F}(\alpha_0, f_0)$ is invertible, we can apply the implicit function theorem to $\widehat{F}$. There exists an open neighborhood $\mathcal{V}_1$ of f_0 in V_β, and a smooth function $\alpha : \mathcal{V}_1 \to \mathbb{D}$, such that, for every $f \in \mathcal{V}_1$ and for every α sufficiently close to α_0, we have

$$\widehat{F}(\alpha, f) = 0 \Longleftrightarrow \alpha = \alpha(f). \quad (6.181)$$

By Proposition 6.9 and by the invertibility of $D_\alpha\widehat{F}(\alpha_0, f_0)$, we may choose the open neighborhood $\mathcal{V}_1$ such that, for every $f \in \mathcal{V}_1$, the point $a = a(f) =$

$f(\alpha(f)) \in \Omega = f(\mathbb{D})$ is doubly non degenerate, that is: non degenerate as a critical point of $\widehat{W}^\Omega$ and non degenerate as a critical point of $W^\Omega(\cdot, g^a)$. In particular, every domain $\Omega = f(\mathbb{D})$, with $f \in \mathcal{V}_1$, satisfies *(ND1)*.

Again by the second nondegeneracy property of every $f \in \mathcal{V}_1$, we may consider the map U_{*,a,g^a}, defined as in (6.83), and corresponding to $a = a(f)$. In order to complete Step *2*, we have to find some f arbitrarily close to f_0, such that the map U_{*,a,g^a} is a local diffeomorphism at the origin. To this end we will again rely on the transversality theorem. More specifically, we define, exactly as in formula (6.127) in the proof of Theorem 6.20, the smooth map

$$U : \mathcal{V}_1 \times \mathcal{V}_2/\mathbb{R} \longrightarrow \dot{C}^\beta(\mathbb{S}^1;\mathbb{R}). \tag{6.182}$$

Recall that $\mathcal{V}_1$ is an open neighborhood of f_0 in V_β, that $\mathcal{V}_2$ is an open neighborhood of the origin in $C^{1,\beta}(\mathbb{S}^1;\mathbb{R})$, and that

$$U(f,\psi) = N^{\mathbb{D}}(\widetilde{\alpha}(\psi, f), g_0 e^{\imath\psi}) \quad \forall\, (f,\psi) \in \mathcal{V}_1 \times \mathcal{V}_2/\mathbb{R}. \tag{6.183}$$

Here, $\widetilde{\alpha}$ is the smooth implicit solution of

$$F(\widetilde{\alpha}(\psi, f), \psi, f) = 0 \tag{6.184}$$

obtained in Proposition 6.7. We recall the following fact established in the proof of Theorem 6.20: the map $U_{*,a(f),g^{a(f)}}$ is a local diffeomorphism at the origin if and only if $U(f,\cdot)$ is a local diffeomorphism at $-\psi_{\alpha(f),g_0}$.

Recalling the formula (6.48) for $N^{\mathbb{D}}$, we obtain the following explicit formula for U:

$$U(f,\psi) = \frac{\partial \psi^*}{\partial \tau} + 2\frac{\alpha_0 \wedge z}{|z-\alpha_0|^2} - 2\frac{\alpha(\psi, f) \wedge z}{|z - \alpha(\psi, f)|^2}. \tag{6.185}$$

Hence, for every $(f,\psi) \in \mathcal{V}_1 \times \mathcal{V}_2/\mathbb{R}$, we have

$$\begin{aligned} D_\psi U(f,\psi)\cdot\zeta =& \frac{\partial \zeta^*}{\partial \tau} - 2\frac{(D_\psi \widetilde{\alpha}(\psi, f)\cdot\zeta)\wedge z}{|z-\widetilde{\alpha}(\psi, f)|^2} \\ & - 4\frac{(z-\widetilde{\alpha}(\psi, f))\cdot(D_\psi\widetilde{\alpha}(\psi, f)\cdot\zeta)}{|z-\widetilde{\alpha}(\psi, f)|^4}\widetilde{\alpha}(\psi, f)\wedge z. \end{aligned}$$

In particular $D_\psi U(f,\psi)$ is a Fredholm operator of index zero, since it can be written as $L - K$, where

$$L : C^{1,\beta}(\mathbb{S}^1;\mathbb{R})/\mathbb{R} \to \dot{C}^\beta(\mathbb{S}^1;\mathbb{R}),\ \zeta \stackrel{L}{\longmapsto} \frac{\partial \zeta^*}{\partial \tau}$$

is invertible and K has finite range. Hence $U(f,\cdot)$ is a smooth Fredholm map for every $f \in \mathcal{V}_1$.

We want to apply the transversality theorem to U. We already know that assumption *1.* of the transverality theorem is satisfied. It remains to check that U is transverse to 0. To this end we compute the differential of U at some point (f,ψ), using (6.185):

$$\begin{aligned} DU(f,\psi)\cdot(h,\zeta) = & \frac{\partial \zeta^*}{\partial \tau} - 2\frac{(D\widetilde{\alpha}(\psi, f)\cdot(h,\zeta))\wedge z}{|z-\widetilde{\alpha}(\psi, f)|^2} \\ & - 4\frac{(z-\widetilde{\alpha}(\psi, f))\cdot(D\widetilde{\alpha}(\psi, f)\cdot(h,\zeta))}{|z-\widetilde{\alpha}(\psi, f)|^4}\widetilde{\alpha}(\psi, f)\wedge z. \end{aligned} \tag{6.186}$$

Let us show that $DU(f,\psi)$ is onto. Let $\Psi \in \dot{C}^{\beta}(\mathbb{S}^1;\mathbb{R})$. Then there exists some $\zeta \in C^{1,\beta}(\mathbb{S}^1;\mathbb{R})/\mathbb{R}$ such that

$$\frac{\partial \zeta^*}{\partial \tau} = \Psi. \tag{6.187}$$

We claim that there exists $h = h_\zeta \in X_\beta$ such that

$$D\widetilde{\alpha}(\psi, f)\cdot(h_\zeta,\zeta) = 0. \tag{6.188}$$

Then, plugging (6.188) and (6.187) into (6.186), we obtain

$$DU(f,\psi)\cdot(h_\zeta,\zeta) = \Psi,$$

and thus $DU(f,\psi)$ is onto.

In order to complete Step *2*, it remains to prove the existence of h_ζ. From the implicit equation (6.184) satisfied by $\widetilde{\alpha}$, we obtain

$$\begin{aligned} &D\widetilde{\alpha}(\psi, f)\cdot(h,\zeta) \\ &= -D_\alpha F(\widetilde{\alpha}(\psi,f),\psi,f)^{-1}\left[D_f F(\widetilde{\alpha}(\psi,f),\psi,f)\cdot h + D_\psi F(\widetilde{\alpha}(\psi,f),\psi,f)\cdot\zeta\right]. \end{aligned} \tag{6.189}$$

Since $D_f F(\alpha,\psi,f) = D_f\widetilde{F}(\alpha,f)$ is surjective (by Step *1*), we may clearly choose h_ζ such that (6.189) holds.

Therefore we can apply the transversality theorem to U: the set of f such that $U(f,\cdot)$ is transverse to $\{0\}$ is dense.

Let $\eta > 0$. We can choose $f \in \mathcal{V}_1$, such that $\|f - f_0\|_{C^{1,\beta}} < \eta$, and $U(f,\cdot)$ is transverse to $\{0\}$. In particular, the differential of $U(f,\cdot)$ at $-\psi_{\alpha(f),g_0}$ is onto, which implies that the differential is invertible (since it is a zero index Fredholm operator). Hence $U(\cdot,f)$ is a local diffeomorphism at $-\psi_{\alpha(f),g_0}$, which is equivalent to $U_{*,a(f),g^{a(f)}}$ being a local diffeomorphism at the origin, i.e. $\Omega = f(\mathbb{D})$ satisfies *(ND2)*.

Step *2* and the proof of Theorem 6.23 are complete. □

Remark 6.26. In Theorem 6.23 we have established that nondegeneracy of the domain is generic in the case of prescribed degree $d = 1$. Some, but not all, of the ingredients of our proof can be generalized to arbitrary d. For example, it is possible to adapt our arguments and obtain the transversality of $\widehat{F}$ to 0 when d is arbitrary. However, this does not lead to the conclusion that *(ND1)* is generically true. The reason is that when $d \neq \pm 1$, we cannot rely on (6.180) anymore, and we actually do not know whether $\widehat{W}^{\Omega}$ does have critical points. A similar difficulty occurs in Step *2*. Indeed, the first ingredient in Step *2* is Proposition 6.25, yielding the existence of a non degenerate critical point a_0 of $W(\cdot, g^{a_0})$. Clearly, our proof of Proposition 6.25 is specific to the case $d = 1$.

However, it is plausible the the transversality arguments extend to an arbitrary degree d, and thus the main difficulty arises in the existence of critical points of $\widehat{W}^{\Omega}$. It would be interesting to investigate, e.g. by topological methods in the spirit of [10], whether such points do exist.

Appendix

The following is a $C^{1,\beta}$ variant of [23, Lemmas A1, A2].

Lemma 6.27. *Let $G \subset \mathbb{R}^n$ be a bounded open set of class $C^{1,\beta}$. Assume that*

$$\begin{cases} \Delta w &= f \quad \text{in } G \\ \quad w &= \varphi \quad \text{on } \partial G \end{cases}. \tag{6.190}$$

Then

$$\sup_G |\nabla w| \le C\left(\|f\|_{L^\infty}^{1/2}\left(\|w\|_{L^\infty}^{1/2} + \|\varphi\|_{L^\infty(\partial G)}^{1/2}\right) + \|\varphi\|_{C^{1,\beta}(\partial G)}\right), \tag{6.191}$$

$$|\nabla w|_{0,\beta,G} \le C\left(\|f\|_{L^\infty}^{1/2+\beta/2}\left(\|w\|_{L^\infty}^{1/2-\beta/2} + \|\varphi\|_{L^\infty(\partial G)}^{1/2-\beta/2}\right) + \|\varphi\|_{C^{1,\beta}(\partial G)}\right), \tag{6.192}$$

for a constant C depending only on G. In addition, when $G = \Omega_\sigma$, where $\sigma_1 \le \sigma \le \sigma_2$ and σ_1, σ_2 are fixed small numbers, we may take C independent of σ.

Proof. We write $w = u + v$, where

$$\begin{cases} \Delta u &= 0 \quad \text{in } G \\ \quad u &= \varphi \quad \text{on } \partial G \end{cases}, \tag{6.193}$$

and

$$\begin{cases} \Delta v &= f \quad \text{in } G \\ \quad v &= 0 \quad \text{on } \partial G \end{cases}. \tag{6.194}$$

By standard elliptic estimates [54, Theorem 8.33] we have

$$\|u\|_{C^{1,\beta}} \le c\|\varphi\|_{C^{1,\beta}}. \tag{6.195}$$

Therefore we only need to prove that v satisfies the estimates

$$\sup_G |\nabla v| \le C\|f\|_{L^\infty}^{1/2}\|v\|_{L^\infty}^{1/2}, \tag{6.196}$$

$$|\nabla v|_{0,\beta,G} \le C\|f\|_{L^\infty}^{1/2+\beta/2}\|v\|_{L^\infty}^{1/2-\beta/2}. \tag{6.197}$$

Estimate (6.196) is proved in [23, Lemma A.2] by combining an interior estimate with a boundary estimate. Estimate (6.197) can be obtained following exactly the same lines. In order to see this, we detail for example the proof of the interior estimate corresponding to (6.197). Proceeding as in [23, Lemma A.1], we first show that

$$|\nabla v|_{0,\beta,G_d} \le C\left(\|f\|_{L^\infty}^{1/2+\beta/2}\|v\|_{L^\infty}^{1/2-\beta/2} + \frac{1}{d^{1+\beta}}\|v\|_{L^\infty}\right), \tag{6.198}$$

where, for $d > 0$, we let $G_d := \{x \in G; \operatorname{dist}(x, \partial G) > d\}$. In order to prove (6.198), we let $x_0 \in G_d$ and $\lambda \in (0, d]$, and define

$$v_\lambda(y) := v(x_0 + \lambda y), \quad y \in B_1(0). \tag{6.199}$$

Then the function v_λ satisfies the equation

$$\Delta v_\lambda = f_\lambda \text{ in } B_1(0), \quad \text{with } f_\lambda(y) := \lambda^2 f(x_0 + \lambda y). \tag{6.200}$$

Standard elliptic estimates [54, Theorem 8.33] yield

$$\begin{aligned} \lambda^{1+\beta} |\nabla v|_{0,\beta,B_{\lambda/2}(x_0)} = |\nabla v_\lambda|_{0,\beta,B_{1/2}(0)} &\leq C \left(\|v_\lambda\|_{L^\infty} + \|f_\lambda\|_{L^\infty}\right) \\ &\leq C \left(\|v\|_{L^\infty} + \lambda^2 \|f\|_{L^\infty}\right). \end{aligned} \tag{6.201}$$

We next discuss the two following cases.
Case 1. $\dfrac{\|v\|_{L^\infty}}{\|f\|_{L^\infty}} \leq d^2$.
In this case, we apply (6.201) with $\lambda = (\|v\|_{L^\infty}/\|f\|_{L^\infty})^{1/2}$. We find that

$$|\nabla v|_{0,\beta,B_{\lambda/2}(x_0)} \leq 2C \|v\|_{L^\infty}^{1/2-\beta/2} \|f\|_{L^\infty}^{1/2+\beta/2}, \tag{6.202}$$

so that (6.198) is satisfied.
Case 2. $\dfrac{\|v\|_{L^\infty}}{\|f\|_{L^\infty}} > d^2$.
In this case, we apply (6.201) with $\lambda = d$. We obtain

$$\begin{aligned} |\nabla v|_{0,\beta,B_{\lambda/2}(x_0)} &\leq C \left(d^{-1-\beta} \|v\|_{L^\infty} + d^{1-\beta} \|f\|_{L^\infty}\right) \\ &\leq C \left(d^{-1-\beta} \|v\|_{L^\infty} + \|v\|_{L^\infty}^{1/2-\beta/2} \|f\|_{L^\infty}^{1/2+\beta/2}\right), \end{aligned} \tag{6.203}$$

so that in both cases (6.198) is satisfied.

Once (6.198) is established, we easily obtain the interior estimate corresponding to (6.197). Indeed, standard elliptic estimates [54, Theorem 3.7] imply $\|v\|_{L^\infty} \leq C \|f\|_{L^\infty}$, so that from (6.198) we obtain

$$|\nabla v|_{0,\beta,K} \leq C \|f\|_{L^\infty}^{1/2+\beta/2} \|v\|_{L^\infty}^{1/2-\beta/2}, \tag{6.204}$$

for every compact set $K \subset G$.

The proof of the boundary version of (6.204) is also a straightforward adaptation of the corresponding estimate established in [23, proof of Lemma A.2], and we omit it here. □

Chapitre 7

Persistance de la supraconductivité dans de fines coques

(avec Andres Contreras)

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7.1 Introduction

Let $\mathcal{M}$ be a compact surface homeomorphic to $\mathbb{S}^2$, embedded in $\mathbb{R}^3$. For $\kappa, h > 0$ and $\mathbf{A}$ a vector field on $\mathcal{M}$, we consider the Ginzburg-Landau functional $\mathcal{G}_{\mathcal{M},\kappa} : H^1(\mathcal{M};\mathbb{C}) \to \mathbb{R}_+$,

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_{\mathcal{M}} \left(|\nabla_{\mathcal{M}} - ih\mathbf{A}\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x). \tag{7.1}$$

The functional $\mathcal{G}_{\mathcal{M},\kappa}$ arises as the Γ-limit (see [41]) of the full 3d Ginzburg-Landau energy

$$\begin{aligned} G_{\varepsilon,\kappa}(\psi, A) = \frac{1}{\varepsilon}\Bigg[\int_{\Omega_\varepsilon} \left(|(\nabla - iA)\psi|^2 + \frac{\kappa^2}{2}\left(|\psi|^2 - 1\right)^2 \right) dx \\ + \int_{\mathbb{R}^3} |\nabla \times A - \mathbf{H}_{ext}|^2 \, dx \Bigg]. \end{aligned} \tag{7.2}$$

where for all $\varepsilon > 0$ sufficiently small, Ω_ε corresponds to a uniform tubular neighborhood of $\mathcal{M}$. In (7.2) $\mathbf{H}_{ext}$ is the external magnetic field. As $\varepsilon \to 0$, the field completely penetrates the sample which then implies that in the Γ-limit A is prescribed to be equal to $\mathbf{A}$, the tangential component of a divergence free vector field $\mathbf{A}^e$ such that $\nabla \times h\mathbf{A}^e = \mathbf{H}_{ext}$.

A central question in Ginzburg-Landau theory is the determination of the so-called *critical fields.* The first critical field corresponds to the appearance of zeros of ψ carrying non-trivial degree – called vortices in this context – in minimizers of the energy.

The analysis in [41] includes the computation of the first critical field of a thin shell of a surface of revolution subject to a constant vertical field which turns out to be surprisingly simple and depending only on an intrinsic quantity, in the $\kappa \to \infty$ limit:

$$H_{c_1} \sim \left(\frac{4\pi}{\text{Area of } \mathcal{M}} \right) \ln \kappa.$$

This result is extended in [40], to general surfaces and magnetic fields. For a fixed field $\mathbf{H}^e$, an external magnetic field of the form $\mathbf{H}_{ext} = h(\kappa)\mathbf{H}^e = h(\kappa)\nabla \times \mathbf{A}^e$ is considered. Then the first critical field is

$$H_{c_1} \sim \frac{1}{\max_{\mathcal{M}} *F - \min_{\mathcal{M}} *F} \ln \kappa,$$

where $d^*F = *d * F = \mathbf{A}$ and $*$ denotes the Hodge star-operator. In fact, the study shows also that, somewhat remarkably, not all fields $\mathbf{H}^e$ give rise to a first critical field. This phenomenon is related to the geometry and relative location of $\mathcal{M}$ with respect to $\mathbf{H}^e$. For $\mathbf{H}^e$ that yield a finite H_{c_1}, the topological obstruction imposed by $\mathcal{M}$ implying that the total degree of $\frac{\psi}{|\psi|}$ is zero is used in [40] to show that there is an even number of vortices in minimizers of $\mathcal{G}_{\mathcal{M},\kappa}$, half with positive degree, half with negative degree concentrating respectively on the set where $*F$ achieves its minimum and maximum. The optimal number $2n$ and location of vortices and anti-vortices in $\mathcal{M}$ is established in [40] for values of $h(k)$ slightly above H_{c_1} and in addition it is shown that if the minimum and maximum of $*F$ is attained at finitely many points then the two sets of vortices minimize, independently, a renormalized energy.

The results in [40] and [41] cover only a moderate regime; in these works the intensity of the applied field is $H_{c_1} + \mathcal{O}(\ln\ln\kappa)$ and thus the number of vortices remains bounded as κ goes to infinity.

Once the value of h becomes much larger than H_{c_1}, that is there is a constant $C > 0$ such that $h - H_{c_1} \geq C \ln \kappa$, then the number of vortices in minimizers diverges as $\kappa \to \infty$. For even larger h, superconductivity persists only in a narrow region in the sample.

In the case of an infinite cylinder whose cross section is a domain $\Omega \subseteq \mathbb{R}^2$ and for constant applied fields parallel to the axis of the cylinder a reduction to a two-dimensional problem is possible. In this case it is known that as the intensity increases superconductivity is lost in the bulk and only a thin superconductivity region near $\partial\Omega$ persists (see Chapter 7 in [119]). For much higher values still, superconductivity is completely lost: this value is known as H_{c_3} and is estimated by a delicate spectral analysis of the magnetic Laplacian operator as in the monograph [51].

In our setting, corresponding to the above functional $\mathcal{G}_{\mathcal{M},\kappa}$ (7.1) on the compact surface $\mathcal{M}$, there is no boundary, so what happens to the superconductivity region is not obvious. Another crucial difference lies in the behaviour of the (normalized) magnetic field H induced on $\mathcal{M}$, which is the normal component of $\mathbf{H}^e$, or equivalently $H\, d\mathcal{H}^2_{\mathcal{M}} = d\mathbf{A}$ (viewing $\mathbf{A}$ as a 1-form). Namely, in our case, H vanishes and changes sign. The spectral analysis in [102] therefore suggests that superconductivity should persist near the set $\{H = 0\}$, where the external magnetic field is tangent to the surface $\mathcal{M}$. In [110] the authors study the case of a vanishing magnetic field in the infinite cylinder model, and observe indeed nucleation of superconductivity near the zero locus of the magnetic field, for very high values of the applied field (near the putative H_{c_3}) under the condition that the gradient of the magnetic field does not vanish on its zero locus. The problem of the determination of the upper critical field for vanishing fields remains largely open otherwise. Here, we are concerned with much lower values of the applied field: a main motivation of this work is to understand the transition from the vortexless to normal state regimes.

Another interesting difference is the fact that in the infinite cylinder model only positive vortices exist and so the location and growth of the vortex region is always ruled by the competing effects of mutual repulsion, and confinement provided by the external field. In the present setting, this is no longer the case. Vortices of positive and negative degree must coexist and so repulsion and attraction are common features of the relative placement of vortices in $\mathcal{M}$, this without taking into account the external field.

In this way, the shrinking of the superconductivity region is a multifaceted phenomenon. Moreover, the problems mentioned in the characterization of this region are present even in the most emblematic case of a constant external field $\mathbf{H}^e$: the region of persistence of superconductivity does not only depend on the field and on the topology of $\mathcal{M}$, but also on extrinsic geometric properties of the surface; the relative position of $\mathcal{M}$ with respect to $\mathbf{H}^e$ affects H and therefore the zero locus of the induced field.

In the present work we address the question of identifying the region where superconductivity persists in the $\kappa \to \infty$ limit, when

$$\frac{H_{c1}}{h}$$

is small; we show that as this quantity gets small superconductivity persists in a small neighborhood of the place where the applied field is tangential to the sample, provided the field satisfies a generic non-degeneracy condition(see (7.14) below). Another thrust of this work is aimed at uncovering some new intermediate regimes only present in this setting, when the normal component of the external field changes sign multiples times. In the model problem of a surface of revolution and constant vertical field, we identify several structural transitions undergone by the superconductivity region. Furthermore, we observe a new phenomenon which we refer to as *freezing of the boundary*, where a component of the vortex region stops growing even after increasing the intensity of the external field. This phenomenon holds in great generality (not only in the surface of revolution case), as is shown at the end of section 7.4.

To carry out our analysis we start by using a reduction to a mean field model, first derived rigorously in [118]. More precisely, if we write a critical point ψ of $\mathcal{G}_{\mathcal{M},\kappa}$ in polar form $\psi = \rho e^{i\phi}$, variations of the phase yield $d(\rho^2(d\phi - hd^*F)) = 0$, and because $H^1_{dr}(\mathcal{M}) = 0$ this implies there is a V such that $*dV = \rho^2(d\phi - hd^*F)$. Taking $V = hW$, the function W is expected to minimize

$$\int_{\mathcal{M}} |\nabla_{\mathcal{M}} W|^2 \, d\mathcal{H}^2_{\mathcal{M}} + \frac{\ln\kappa}{h} \int_{\mathcal{M}} |-\Delta_{\mathcal{M}} W + \Delta_{\mathcal{M}} * F| \, d\mathcal{H}^2_{\mathcal{M}}. \tag{7.3}$$

The details of this mean field reduction can be found in [118] in the case of a positive external field applied in a bounded planar domain. However, the analysis in [118] does not handle the additional restriction of total zero mass which affects the construction of an upper bound in this setting. The steps needed to extend the proof to the present case are included in Appendix 7.A.

The measure $-\Delta_{\mathcal{M}} V + \Delta_{\mathcal{M}} * F$ can be interpreted as the normalized measure generated by the vortices. On the other hand, we observe that

$$\Delta_{\mathcal{M}} * F \, d\mathcal{H}^2_{\mathcal{M}} = d * d * F = d\mathbf{A} = H d\mathcal{H}^2_{\mathcal{M}},$$

where the function H is the normal component of the external magnetic field $\mathbf{H}^e$ relative to $\mathcal{M}$. In what follows we refer to H simply as *the* magnetic field, and we assume that $H \in C^1(\mathcal{M})$. Moreover, we drop the explicit dependence on $\mathcal{M}$ in expressions like $\Delta_{\mathcal{M}}$, $\nabla_{\mathcal{M}}$.

Before we state our main result we make the following assumption: there exists $\beta > 0$ such that

$$\lim_{\kappa \to \infty} \frac{\ln \kappa}{h} = \beta. \tag{7.4}$$

Once the connection to the mean field problem (7.3) is established we proceed to locate very precisely the region of persistence of superconductivity, that is, the region SC_β where the vorticity measure $-\Delta V + H$ vanishes. We find that this region corresponds to a $\beta^{\frac{1}{3}}$ neighborhood of the set where H vanishes, in the $\beta \to 0$ limit. More precisely,

Theorem 7.1. *Under the nondegeneracy assumption that ∇H is nowhere vanishing on $\{H = 0\}$, there exists $C > 0$ independent of β such that the superconductivity region SC_β is contained in $\{x \in \mathcal{M} : d(x, \{H = 0\}) < C\beta^{\frac{1}{3}}\}$, and contains $\{x \in \mathcal{M} : d(x, \{H = 0\}) < C^{-1}\beta^{\frac{1}{3}}\}$, for β sufficiently small.*

The nondegeneracy assumption on H implies that the set $\{H = 0\}$ is a finite union of smooth closed curves. It is the same assumption as the one made in [102, 110] for the study of the third critical field H_{c_3}.

To prove Theorem 7.1 we reformulate the mean field approximation as an obstacle problem, and construct comparison functions. We note that a construction in the same spirit was carried out in [120, Appendix A] for the planar Ginzburg-Landau model in a different context. In our case however the construction is not immediate, because our obstacle problem is two-sided and our magnetic field H changes sign. Indeed, our proof makes use of a comparison principle for two sided obstacle problems proved in [44] which allows to compare solutions to obstacle problems corresponding to different data H. Hence the comparison functions will not be merely "super- or sub-solutions" of our problem, but actual solutions of modified problems. In particular they have to be quite regular. As a consequence, we cannot use functions of the distance to $\{H = 0\}$ as comparison functions. We have to use a particular coordinate system near each component of $\{H = 0\}$ and explicitly build local functions satisfying local obstacle problems with appropriate modifications of H. Pasting these constructions we are able to appeal to [44] to obtain the desired estimates. In so doing we note a key feature of the proof, related to the fact that the obstacle problem is two-sided: the barriers thus obtained cannot be used independently to get neither the inner nor the outer bound separately, but together they yield the conclusion of the theorem. This is explained in more detail in section 7.3.

Thanks to Theorem 7.1, we have a clear picture of the superconductivity region for $\beta \to 0$: it is a union of tubular neigborhoods of the connected components of $\{H = 0\}$. In particular, the superconductivity region has at least as many connected components as $\{H = 0\}$. On the other hand, we also have a clear picture of the superconductivity region as $\beta \to \beta_c$, where positive (resp. negative) vortices are concentrated near the points where $*F$ achieves its maximum (resp. minimum). In particular, the superconductivity region has, generically, one connected component. In the last part of this work, we investigate the intermediate regimes. If $\{H = 0\}$ has more than one connected component, transitions *have* to occur: when β crosses some critical value, the number of connected components of SC_β changes.

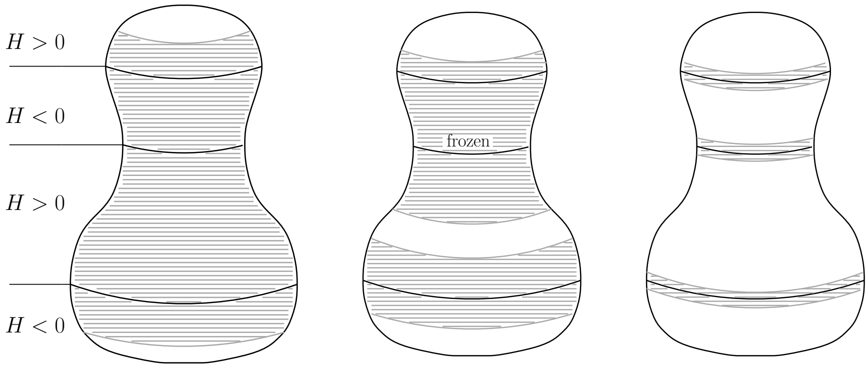
Studying such transitions, and determining the values of β at which they occur, seems out of our reach in all generality. That is why we concentrate first on a simple model problem. We consider a surface of revolution around the vertical axis $\mathbf{e_z}$, and assume that the external magnetic field $\mathbf{H}^e = \mathbf{e_z}$ is vertical and constant. (In fact in Section 7.4.1, more general magnetic fields are considered.) In that case, the induced field H on $\mathcal{M}$ is just $H = \mathbf{e_z} \cdot \nu$, where ν is an outward normal vector on $\mathcal{M}$. The set $\{H = 0\}$ consists exactly of the points where $\mathbf{e_z}$ is tangent to $\mathcal{M}$, and it is a union of circles. Note that H has to change sign an odd number of times, since $H = -1$ at the 'south pole' and $+1$ at the 'north pole', thus there are an odd number of those circles. As explained above, interesting transitions happen when $\{H = 0\}$ has more than one connected component. Therefore we focus on the simplest nontrivial situation, which corresponds to $\{H = 0\}$ consisting of three circles. We state loosely here the result that we obtain for that simple model problem in Section 7.4.1 (see Figure 7.1).

Proposition 7.2. *Let $\mathcal{M}$ be a surface of revolution of the form* (7.39) *with constraints specified in 7.4.1 below. Assume the induced magnetic potential is rotationally symmetric. Then there exist $\beta_c > \beta_1^* \geq \beta_2^* > 0$ such that*

— *for $\beta \in (\beta_1^*, \beta_c)$, SC_β has one connected component,*
— *for $\beta \in (\beta_2^*, \beta_1^*)$, SC_β has two connected components,*
— *for $\beta \in (0, \beta_2^*)$, SC_β has three connected components.*

Moreover, for $\beta \in (\beta_2^*, \beta_1^*)$, one connected component of SC_β remains constant.

The most striking part of Proposition 7.2 is the appearance of an intermediate regime in which one connected component of SC_β remains constant: one



(a) $\beta \in (\beta_1^*, \beta_c)$ (b) $\beta \in (\beta_2^*, \beta_1^*)$ (c) $\beta \in (0, \beta_2^*)$

Figure 7.1 – The region SC_β in the three regimes of Proposition 7.2

part of the free boundary is *frozen.* In [6] a similar ocurrence is observed in an explicit solution to a two-sided obstacle problem arising in the study of almost planar thin films in the presence of strong parallel fields. In Section 7.4.2 we identify the features responsible for such 'freezing of the boundary' phenomenon depicted in Proposition 7.2 and prove a similar 'freezing property' in a general (non-symmetric) setting (see Proposition 7.17). We note that since our proof relies on a general comparison principle, it is likely that it could be adapted to include the setting in [6].

An other interesting outcome of the precise version of Proposition 7.2 (Proposition 7.15 in Section 7.4.1) are the expressions of the critical values β_1^* and β_2^*, in terms of integral quantities involving $\mathbf{A}$ and the parametrization of $\mathcal{M}$. Transfering these conditions to a general non-symmetric setting seems far from obvious and constitutes an interesting challenge.

The plan of the paper is as follows. In the next section we collect some basic properties of solutions to an obstacle problem that serves as the starting point in our analysis. In section 7.3 we identify the thin region of superconductivity when β is small. In section 7.4 we turn to the symmetric situation and identify in Proposition 7.15 the further transitions as β decreases to zero from $\beta_c = \max(*F) - \min(*F)$. We also prove the 'freezing of the boundary' property at the end of section 7.4.

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7.2 The obstacle problem

This preamble is devoted to the derivation of the obstacle problem dual to the mean field approximation. We also prove some basic results we will need later on. We think it is worthwhile recording these properties because in our setting, even in the by now classical application of the duality theorem which allows for the obstacle problem formulation, there is an inherent degeneracy we have to account for which is not present in other similar results in the literature.

In the first part of this section we show that – as in [119, Chapter 7] – the minimizer of

$$E_\beta(V) = \int_{\mathcal{M}} |\nabla V|^2 + \beta \int_{\mathcal{M}} |-\Delta V + H| \tag{7.5}$$

is the solution of an obstacle problem, and then we study general properties of the contact set. There are two main differences with the obstacle problem arising in [119, Chapter 7].

— In our case there are no boundary conditions and the minimizer is well-defined only up to a constant. We need to deal with this degeneracy.

— While in [119, Chapter 7] the obstacle problem is one-sided, we have to consider a two-sided obstacle problem. This is due to the fact that, in our case, the magnetic field H changes sign.

The functional E_β is, under assumption (7.4), the limit of the sequence of energies considered in (7.3). The link between E_β and the superconductivity region is, as mentioned in the introduction, proved in appendix A.

7.2.1 Derivation of the obstacle problem

Proposition 7.3. *Let $\beta > 0$. A function $V_0 \in H^1(\mathcal{M})$ minimizes E_β (7.5) if and only if V_0 minimizes*

$$\mathcal{F}(V) = \int_{\mathcal{M}} \Big(|\nabla V|^2 + 2HV \Big) \tag{7.6}$$

among all $V \in H^1(\mathcal{M})$ such that $(\operatorname{ess\,sup} V - \operatorname{ess\,inf} V) \leq \beta$.

Remark 7.4. Since the functional $\mathcal{F}(V)$ is translation invariant, V_0 coincides, up to a constant, with any minimizer of the two-sided obstacle problem

$$\min \left\{ \int_{\mathcal{M}} \Big(|\nabla V|^2 + 2HV \Big) : V \in H^1(\mathcal{M}),\ |V| \leq \beta/2 \right\}.$$

Moreover, recalling that $H = \Delta * F$, this obstacle problem can also be rephrased as

$$\min \left\{ \int_{\mathcal{M}} |\nabla (V - *F)|^2 : V \in H^1(\mathcal{M}),\ |V| \leq \beta/2 \right\}. \tag{7.7}$$

The fact that minimizers coincide only up to a constant does not matter, since the physically relevant object is the vorticity measure $-\Delta V + H$. Moreover, it is easy to check that, if the obstacle problem (7.7) admits a solution V that 'touches' the obstacles, i.e. satisfies $\max V - \min V = \beta$, then this solution is unique because any other solution differs from it by a constant, which has to be zero. On the other hand, a solution satisfying $\max V - \min V < \beta$ would have to be $V = *F + \alpha$ for some constant α. Therefore, for $\beta \leq \max *F - \min *F$ the solution is unique.

The proof of Proposition 7.3 relies on the following classical result of convex analysis (easily deduced from [116] or [31, Theorem 1.12]).

Lemma 7.5. *Let $\mathcal{H}$ be a Hilbert space and $\varphi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semi-continuous function. Then the minimizers of the problems*

$$\min_{x \in \mathcal{H}} \left(\frac{1}{2} \|x\|_{\mathcal{H}}^2 + \varphi(x) \right) \quad \text{and} \quad \min_{y \in \mathcal{H}} \left(\frac{1}{2} \|y\|_{\mathcal{H}}^2 + \varphi^*(-y) \right)$$

coincide, where φ^* denotes the Fenchel conjugate of φ,

$$\varphi^*(y) := \sup_{z \in \mathcal{H}} \langle y, z \rangle_{\mathcal{H}} - \varphi(z).$$

Proof of Proposition 7.3: We apply Lemma 7.5 in the Hilbert space

$$\mathcal{H} := \dot{H}^1(\mathcal{M}) = \left\{ V \in H^1(\mathcal{M}) \colon \int_{\mathcal{M}} V = 0 \right\},$$

endowed with the norm $\|V\|^2 = \int |\nabla V|^2$, to the function

$$\varphi(V) = \varphi_\beta(V) = \frac{\beta}{2} \int_{\mathcal{M}} |-\Delta V + H| \,. \tag{7.8}$$

In formula (7.8), it is implicit that $\varphi(V) = +\infty$ if $\mu = -\Delta V + H$ is not a Radon measure. Note that, when μ *is* a Radon measure, it must have zero average $\int \mu = 0$, since $\mu = \Delta(*F - V)$.

We compute the Fenchel conjugate of φ. It holds

$$\begin{aligned}
\varphi^*(V) &= \sup_{U \in \mathcal{H}} \left\{ \int_{\mathcal{M}} \nabla V \cdot \nabla U - \frac{\beta}{2} \int_{\mathcal{M}} |-\Delta U + H| \right\} \\
&= - \int_{\mathcal{M}} HV + \sup_{U \in \mathcal{H}} \left\{ \int_{\mathcal{M}} (-\Delta U + H) V - \frac{\beta}{2} \int_{\mathcal{M}} |-\Delta U + H| \right\} \\
&= - \int_{\mathcal{M}} HV + \sup_{\int P = 0} \left\{ \int_{\mathcal{M}} \left(PV - \frac{\beta}{2} |P| \right) \right\}.
\end{aligned}$$

In the last equality, the supremum may – by a density argument – be taken over all L^2 functions P with zero average.

If $(\operatorname{ess\,sup} V - \operatorname{ess\,inf} V) \leq \beta$, then $|V + \alpha| \leq \beta/2$ for some $\alpha \in \mathbb{R}$, so that

$$\int_{\mathcal{M}} \left(PV - \frac{\beta}{2} |P| \right) = \int_{\mathcal{M}} \left((V + \alpha) P - \frac{\beta}{2} |P| \right) \leq 0,$$

and in that case

$$\varphi^*(V) = - \int_{\mathcal{M}} HV.$$

On the other hand, if $(\operatorname{ess\,sup} V - \operatorname{ess\,inf} V) > \beta$, then up to translating V we may assume that $\{V > \beta/2\}$ and $\{V < -\beta/2\}$ have positive measures. It is then easy to construct a function P supported in those sets, such that $\int P = 0$, $\int |P| = 1$, and $\int PV > \beta/2$. Using λP as a test function for arbitrary $\lambda > 0$, we deduce that $\varphi^*(V) = +\infty$.

From Lemma 7.5 it follows that $V_0 \in \dot{H}^1(\mathcal{M})$ minimizes E_β if and only if V_0 minimizes

$$\frac{1}{2}\int_{\mathcal{M}} |\nabla V|^2 + \int_{\mathcal{M}} HV$$

among $V \in \dot{H}^1(\mathcal{M})$ such that $\operatorname{ess\,sup} V - \operatorname{ess\,inf} V \leq \beta$. Since both problems are invariant under addition of a constant, the restriction to the space $\dot{H}^1(\mathcal{M})$ can be relaxed to obtain Proposition 7.3. $\square$

7.2.2 Basic properties

In this section we concentrate on the obstacle problem

$$\min\left\{\int_{\mathcal{M}} \left(|\nabla V|^2 + 2HV\right) : V \in H^1(\mathcal{M}),\ |V| \leq \beta/2\right\}. \tag{7.9}$$

We recall the classical interpretation of (7.9) as a free boundary problem, and establish a monotonicity property of the free boundary.

The first step to these basic properties is the reformulation of the obstacle problem (7.9) as a variational inequality: a function $V \in H^1(\mathcal{M})$ solves (7.9) if and only if $|V| \leq \beta/2$ and

$$\int_{\mathcal{M}} \nabla V \cdot \nabla (W - V) \geq -\int_{\mathcal{M}} H(W - V) \qquad \forall W \in H^1(\mathcal{M}),\ |W| \leq \beta/2. \tag{7.10}$$

The proof of this weak formulation is elementary and can be found in many textbooks on convex analysis. See for instance [117].

Next we recall the standard reformulation of (7.10) as a free boundary problem.

Lemma 7.6. *A function $V \in H^1(\mathcal{M})$ with $|V| \leq \beta/2$ solves* (7.9) *or equivalently* (7.10) *if and only if*

$$\begin{cases} V \in W^{2,p}(\mathcal{M}), \quad 1 < p < \infty, \\ \Delta V = H \quad \text{in } \{|V| < \beta/2\}, \\ 0 \geq H \quad \text{in } \{V = \beta/2\}, \\ 0 \leq H \quad \text{in } \{V = -\beta/2\}. \end{cases} \tag{7.11}$$

In particular $V \in C^{1,\alpha}(\mathcal{M})$, so that at every regular point of the free boundaries $\partial\{V = \pm\beta/2\}$, the function V satisfies the overdetermining boundary conditions $V = \pm\beta/2$ and $\partial V/\partial\nu = 0$.

The only non-elementary part of Lemma 7.6 is the $W^{2,p}$ regularity of the solution. For the one-sided obstacle problem, it is proven for instance in [52, Theorem 3.2]. The proof adapts easily to our two-sided obstacle problem: see e.g. [52, Problem 2, p.29].

Recall that in our case, $\mu = -\Delta V + H$ represents the vorticity measure. In light of Lemma 7.6, this measure is supported in $\{V = \pm\beta/2\}$. In that region, vortices are distributed with density H.

For $\beta > \beta_c$, where

$$\beta_c := \max(*F) - \min(*F), \tag{7.12}$$

the function $*F + \alpha$ solves the obstacle problem (7.9), as long as the constant α satisfies $\max(*F) - \beta/2 \leq \alpha \leq \min(*F) + \beta/2$, and the vorticity measure $-\Delta V + H$ is identically zero.

For $\beta \leq \beta_c$, the solution $V = V_\beta$ of the obstacle problem (7.9) must satisfy

$$\max V_\beta - \min V_\beta = \beta,$$

and therefore is unique (see Remark 7.4). Recall that the superconductivity region SC_β is defined as the set where the vorticity measure $-\Delta V + H$ vanishes. According to Lemma 7.6, that region is exactly

$$SC_\beta = \{|V_\beta| < \beta/2\}. \tag{7.13}$$

A first basic property of the superconductivity region SC_β is its monotonicity.

Proposition 7.7. *For any $0 < \beta_1 < \beta_2 \leq \beta_c$, it holds*

$$SC_{\beta_1} \subset SC_{\beta_2}.$$

In other words, increasing the intensity of the applied magnetic field shrinks the region of persisting superconductivity, which consistant with physical intuition. Since we have to deal with a two-sided obstacle problem, this monotonicity property is not as obvious as in [119, Chapter 7]. To prove it, we use a comparison principle for two-sided obstacle problems [44, Lemma 2.1]. We state and prove here a particular form that will also be useful later on.

Lemma 7.8. *Let $H_1 \geq H_2$ be bounded, real-valued functions on $\mathcal{M}$. Let also $\alpha_1 \leq \alpha_2$ and $\beta_1 \leq \beta_2$ be real numbers. For $j = 1, 2$, let $V_j \in H^1(\mathcal{M})$ solve respectively the obstacle problems*

$$\min \left\{ \int_{\mathcal{M}} \left(|\nabla V|^2 + 2H_j V \right) : \alpha_j \leq V \leq \beta_j \right\}.$$

Then either $V_1 - V_2$ is constant, or $V_1 \leq V_2$.

Proof. For the convenience of the reader, we provide here the elementary proof, which consists in remarking that

$$W_1 = \min(V_1, V_2) \quad \text{and } W_2 = \max(V_1, V_2)$$

are admissible test functions in the variational inequalities

$$\int_{\mathcal{M}} \nabla V_j \cdot \nabla (W_j - V_j) \geq -\int_{\mathcal{M}} H_j (W_j - V_j), \qquad \forall W_j \in H^1,\, \alpha_j \leq W_j \leq \beta_j.$$

Substracting the resulting inequalities, we obtain

$$\int_{\mathcal{M}} |\nabla (V_1 - V_2)_+|^2 \leq \int_{\mathcal{M}} (H_2 - H_1)(V_1 - V_2)_+ \leq 0,$$

where $(V_1 - V_2)_+ = \max(V_1 - V_2, 0)$. We conclude that $(V_1 - V_2)_+$ is a constant function. □

With Lemma 7.8 at hand, we may prove the monotonicity of the superconductivity region.

Proof of Proposition 7.7: Let V_1 and V_2 denote the solution of the obstacle problem (7.9) corresponding respectively to $\beta = \beta_1$ and $\beta = \beta_2$. Let

$$\widetilde{V}_1 = V_1 + \beta_1/2, \quad \text{and} \quad \widetilde{V}_2 = V_2 + \beta_2/2,$$

so that for $j = 1, 2$, $\widetilde{V}_j$ solves the obstacle problem

$$\min \left\{ \int_{\mathcal{M}} \left(|\nabla V|^2 + 2HV \right) : 0 \le V \le \beta_j \right\}.$$

Therefore, applying Lemma 7.8 with $H_1 = H_2 = H$, $\alpha_1 = \alpha_2 = 0$ and $\beta_1 \le \beta_2$, we deduce that

$$V_1 + \beta_1/2 \le V_2 + \beta_2/2.$$

(If $\widetilde{V}_1 - \widetilde{V}_2$ is constant, then $\beta_2 = \max V_1 - \min V_1 = \beta_1$.) In particular, we obtain that

$$\{V_1 > -\beta_1/2\} \subset \{V_2 > -\beta_2/2\}.$$

By a similar argument, we show that

$$\{V_1 < \beta_1/2\} \subset \{V_2 < \beta_2/2\},$$

and conclude that $SC_{\beta_1} \subset SC_{\beta_2}$. □

Remark 7.9. It follows from the above proof that

$$|V_1 - V_2| \le (\beta_2 - \beta_1)/2,$$

thus proving the continuity of $\beta \mapsto V_\beta$ for $0 \le \beta \le \beta_c$.

7.3 The small β limit

In this section we study what happens to the superconductivity set when the intensity of the field is high enough to confine it in a narrow region. We make the (generic) non degeneracy assumption that

$$|H| + |\nabla H| > 0 \quad \text{in } \mathcal{M}. \tag{7.14}$$

In other words, $\nabla H \ne 0$ in $\{H = 0\}$. This implies in particular that the set $\Sigma := \{H = 0\}$ where the magnetic field vanishes is a finite disjoint union of smooth closed curves. We also note that condition (7.14) also implies that we are not in the situation where not even the first critical field is defined(see [40], Theorem 3.1).

Let us say a few words here about the nondegeneracy assumption (7.14). This is the same nondegeneracy assumption that has been considered in works on the spectral analysis of the magnetic Laplacian [102] and on higher applied magnetic fields in Ginzburg-Landau [110, 13]. Moreover, we emphasize that (7.14) is a generic assumption, in the following sense.

Lemma 7.10. *Let* $X = \left\{H \in C^1(\mathcal{M}) \colon \int_{\mathcal{M}} H = 0\right\}$. *The functions satisfying* (7.14) *form and open and dense subset of* X.

Proof. The fact that (7.14) is an open condition is clear. For the density, it suffices to show that any $H \in X \cap C^\infty(\mathcal{M})$ can be approached by functions satisfying (7.14). This follows from a simple transversality argument : recall (see e.g. [48, § 3.7]) that a smooth function Φ is transverse to $\{0\}$ if and only if Φ is a submersion on $\{\Phi = 0\}$. In particular (7.14) is equivalent to H being transverse to $\{0\}$. Fix $H_1, H_2 \in X \cap C^\infty(\mathcal{M})$ such that $\{H_1 = H_2 = 0\}$ is void. Then the smooth function

$$\Phi \colon \mathbb{R}^2 \times \mathcal{M} \to \mathbb{R}, \ (\lambda, x) \mapsto H(x) + \lambda_1 H_1(x) + \lambda_2 H_2(x),$$

is transverse to $\{0\}$, and therefore $\Phi(\lambda, \cdot) = H + \lambda_1 H_1 + \lambda_2 H_2$ is transverse to $\{0\}$ for λ arbitrarily small [48, Theorem 3.7.4]. ☐

We are interested in the behavior, as $\beta \to 0$, of the superconductivity region SC_β (7.13). We let $d \colon \mathcal{M} \to \mathbb{R}_+$ denote the distance function to the set $\Sigma = \{H = 0\}$, that is

$$d(x) = \mathrm{dist}(x, \{H = 0\}). \tag{7.15}$$

In this context we characterize the behavior of SC_β in terms of the function d, as follows (this is a more explicit version of Theorem 7.1).

Theorem 7.11. *Under the non-degeneracy assumption* (7.14) *on the magnetic field, there exists* $\beta_0 > 0$ *and* $C > 0$ *such that, for* $\beta \in (0, \beta_0)$,

$$\left\{d \leq \frac{1}{C}\beta^{1/3}\right\} \subset SC_\beta \subset \left\{d \leq C\beta^{1/3}\right\}, \tag{7.16}$$

where SC_β *is the superconductivity region* (7.13)*, and* d *denotes the distance to the zero locus of the magnetic field* (7.15)*.*

In the proof we construct explicit solutions to modified obstacle problems, in order to apply the comparison principle Lemma 7.8. The comparison functions are constructed locally near each component Γ of $\{H = 0\}$, and then we need to extend and paste these functions and the associated modified obstacle problem data. Although the construction looks local, it is worth noting that we really need to make it near *every* component Γ of $\{H = 0\}$. Otherwise the pasting would not provide us with obstacle problems comparable to the original one, because a solution has to change sign near *every* curve Γ.

Remark 7.12. Another natural approach to proving Theorem 7.11 would be to construct separate comparison functions in $\{H > 0\}$ and $\{H < 0\}$. In those regions, the obstacle problem becomes one-sided, so that more standard constructions with a classical comparison principle can be made. On the other hand, there is no boundary conditions in those regions, so that such a construction would only provide us with the outer bound

$$SC_\beta \subset \{d \leq C\beta^{1/3}\}. \tag{7.17}$$

To obtain the bounds (7.16) which show that the superconductivity set extends to <u>both</u> sides of the zero locus of H by a $\beta^{\frac{1}{3}}$ margin, it seems that we really have to appeal to the comparison principle for two-sided obstacle problems. However, if we would just content ourselves with showing that the superconductivity set had 'thickness' proportional to $\beta^{\frac{1}{3}}$, namely

$$\operatorname{dist}(\{V = \beta/2\}, \{V = -\beta/2\}) \geq c\beta^{1/3}, \tag{7.18}$$

there would be a simpler and elegant way. In fact (7.18) can be directly inferred from (7.17). This is a simple consequence of the interpolated elliptic estimate (see [24, Appendix A])

$$\|\nabla V\|_{\infty}^{2} \leq C \|\Delta V\|_{\infty} \|V\|_{\infty}, \tag{7.19}$$

which implies, since $|V| \leq \beta$ and $|\Delta V| = |H \mathbb{1}_{SC_\beta}| \leq C\beta^{1/3}$, that

$$|\nabla V| \leq C\beta^{2/3} \quad \text{in } \mathcal{M}. \tag{7.20}$$

Hence, for any $x_\pm \in \{V = \pm\beta/2\}$ and any arc-length parametrized curve $\gamma(s)$, $(0 \leq s \leq \ell)$ going from x_- to x_+, it holds

$$\beta = V(x_+) - V(x_-) = \int_0^\ell \nabla V(\gamma(s)) \cdot \gamma'(s)\, ds \leq C\beta^{2/3}\ell,$$

so that the length of γ satisfies $\ell \geq c\beta^{1/3}$, which proves (7.18). Let us emphasize again that (7.17)-(7.18) really is weaker than (7.16), since (7.18) does not prevent vortices from coming arbitrarily close to one side of $\{H = 0\}$.

Next we turn to the proof of Theorem 7.11.

Proof of Theorem 7.11: We will construct, for small enough β, bounded functions $H_1 \leq H \leq H_2$, and comparison functions V_1 and V_2 of regularity $W^{2,\infty}$, satisfying for $j = 1, 2$,

$$\begin{gathered} \Delta V_j = H_j \mathbb{1}_{|V_j| < \beta/2}, \\ |V_j| \leq \beta/2, \quad H_j \geq 0 \text{ in } \{V_j = -\beta/2\}, \quad H_j \leq 0 \text{ in } \{V_j = \beta/2\}, \end{gathered} \tag{7.21}$$

and the bounds

$$\left\{d \leq \frac{1}{C}\beta^{1/3}\right\} \subset \{|V_j| < \beta/2\} \subset \left\{d \leq C\beta^{1/3}\right\}. \tag{7.22}$$

By Lemma 7.6, (7.21) implies that V_j solves the obstacle problem (7.9) with $H = H_j$. Therefore we may apply the comparison principle for two-sided obstacle problems (Lemma 7.8) to conclude that $V_1 \geq V \geq V_2$. In view of the bounds (7.22) satisfied by V_1 and V_2, this obviously implies that the superconductivity region satisfies the bounds (7.16).

The rest of the proof is devoted to constructing V_1 and V_2. To this end we introduce good local coordinates in a neighborhood of $\Sigma = \{H = 0\}$. Recall that, thanks to the nondegeneracy assumption (7.14), Σ is a finite union of closed smooth curves. Let us fix one of them, Γ, together with an arc-length parametrization of it:

$$\Gamma = \{\gamma(x) \colon x \in \mathbb{R}/\ell\mathbb{Z}\}, \quad |\gamma'(x)| = 1.$$

Let us also fix a smooth normal vector $\nu(x)$ to Γ on $\mathcal{M}$, that is

$$\nu(x) \in T_{\gamma(x)}\mathcal{M}, \quad |\nu| = 1, \quad \nu \cdot \gamma' = 0,$$

and impose that $\nu(x)$ points in the direction of $\{H > 0\}$ (since $H < 0$ on one side of Γ and $H < 0$ on the other side). We introduce Fermi coordinates along Γ: for small enough δ, the map

$$\mathbb{R}/\ell\mathbb{Z} \times (-\delta, \delta) \to \mathcal{M}, \quad (x, y) \mapsto \exp_{\gamma(x)}(y\nu(x)),$$

is a diffeomorphism. It defines local coordinates (x, y) on $\mathcal{M}$ in a neighborhood of Γ, in which the Laplace operator has the form

$$\Delta = \frac{1}{f}\left(\partial_y f \partial_y + \partial_x f^{-1}\partial_x\right), \tag{7.23}$$

where $f(x, y) = 1 - y\kappa(x, y)$ for some smooth function κ. Note that y is nothing else than the signed distance to Γ, and in particular $|y| = d$ in a neighborhood of Γ. While this is a coordinate system that follows well the geometry of a neighborhood of γ, we actually need one where the Laplacian allows us to reduce our construction to a $1d$ problem. To that end let (x, z) be the local coordinates where

$$z = y + \frac{1}{2}y^2\kappa(x, y). \tag{7.24}$$

Clearly the map $(x, y) \mapsto (x, z)$ is a diffeomorphism for small enough y, so that (x, z) define indeed local coordinates on $\mathcal{M}$. The reason for using the coordinates (x, z) is that the Laplace operator is then approximately

$$\Delta \approx \partial_x^2 + \partial_z^2,$$

which will allow us to obtain nice bounds for functions depending only on z.

Note that, since we choose the normal vector ν to point in the direction of $\{H > 0\}$, and since $|\nabla H| \geq c > 0$ in a neighborhood of Γ thanks to the nondegeneracy assumption (7.14), it holds

$$\partial_z H \geq c > 0, \quad |z| < \delta.$$

On the other hand, ∇H is bounded, so that there exist $C \geq c > 0$ such that

$$Cz\mathbb{1}_{z<0} + cz\mathbb{1}_{z>0} \leq H \leq cz\mathbb{1}_{z<0} + Cz\mathbb{1}_{z>0}, \qquad |z| < \delta. \tag{7.25}$$

Next we concentrate on the construction of V_1 (H_1 will be defined accordingly). Away from the set Σ, we simply define

$$V_1 = -\mathrm{sign}(H)\beta/2 \quad \text{in } \{d > \delta/2\}. \tag{7.26}$$

The interesting part is of course what happens near Σ. Near each of the smooth curves $\Gamma \subset \Sigma$, we will look for V_1 in the form $V_1 = v(z)$, where v is a $W^{2,\infty}$ function satisfying

$$v(z) = \begin{cases} \beta/2 & \text{for } z < -\eta_-, \\ -\beta/2 & \text{for } z > \eta_+, \end{cases} \tag{7.27}$$

for some parameters $\eta_\pm > 0$ that will depend on β. A straightforward computation using (7.23) and (7.24) shows that

$$\Delta V_1 = v''(z) + z\left(g_1(x,z)v''(z) + g_2(x,z)v'(z)\right), \tag{7.28}$$

where g_1 and g_2 are bounded functions. We are going to define in $(-\eta_-, \eta_+)$ the function v so that

$$v'' \le 2Cz\mathbb{1}_{z<0} + \frac{c}{2}z\mathbb{1}_{z>0}, \quad |v'| = o(\beta),\ |v''| = o(\beta). \tag{7.29}$$

We then define H_1 in $(-\eta_-, \eta_+)$ simply as ΔV_1. Thus, recalling (7.25), we will have, for small enough $\beta > 0$,

$$\Delta V_1 = H_1\mathbb{1}_{|V_1|<\beta/2} \qquad \text{with } H_1 \le H \text{ in } \{-\eta_- < z < \eta_+\}. \tag{7.30}$$

It is then straightforward to extend H_1 to a function defined on $\mathcal{M}$, such that $H_1 \le H$, and having the same sign as H outside of $\{-\eta_- < z < \eta_+\}$. The resulting H_1 and V_1 satisfy (7.21).

Thus it remains to show that we can indeed define $v(z)$ in $\{-\eta_- < z < \eta_+\}$, satisfying the bounds (7.29). We look for v in the form

$$v(z) = \begin{cases} v_-(z) & \text{for } -\eta_- < z < 0, \\ v_+(z) & \text{for } 0 < z < \eta_+, \end{cases} \quad \text{with } v_\pm(z) \text{ polynomial.} \tag{7.31}$$

First of all, for v to be of class $W^{2,\infty}$ around the points $\pm\eta_\pm$, we should impose

$$v_-(-\eta_-) = \beta/2, \quad v_+(\eta_+) = -\beta/2, \quad v'_-(-\eta_-) = v'_+(\eta_+) = 0. \tag{7.32}$$

Thus we take $v_\pm$ to be of the form

$$\begin{aligned} v_-(z) &= (z+\eta_-)^2(A_-z + B_-) + \frac{\beta}{2} \\ &= A_-z^3 + (B_- + 2\eta_-A_-)z^2 + (2\eta_-B_- + \eta_-^2A_-)z + \eta_-^2B_- + \frac{\beta}{2}, \\ v_+(z) &= (z-\eta_+)^2(A_+z + B_+) - \frac{\beta}{2} \\ &= A_+z^3 + (B_+ - 2\eta_+A_+)z^2 + (-2\eta_+B_+ + \eta_+^2A_+)z + \eta_+^2B_+ - \frac{\beta}{2}. \end{aligned} \tag{7.33}$$

For v to be of class $W^{2,\infty}$ around $z = 0$, we have to impose

$$\begin{aligned} \eta_-^2B_- + \frac{\beta}{2} &= \eta_+^2B_+ - \frac{\beta}{2}, \\ 2\eta_-B_- + \eta_-^2A_- &= -2\eta_+B_+ + \eta_+^2A_+. \end{aligned} \tag{7.34}$$

We also need to ensure that

$$v'' \le 2Cz\mathbb{1}_{z<0} + \frac{c}{2}z\mathbb{1}_{z>0}, \tag{7.35}$$

so we impose

$$6A_- = 2C, \quad 6A_+ = \frac{c}{2}, \quad B_- + 2\eta_-A_- = B_+ - 2\eta_+A_+ = 0, \tag{7.36}$$

so that we even have an equality in (7.35). Plugging (7.36) into (7.34), we find

$$\frac{c}{6}\eta_+^3 + \frac{2C}{3}\eta_-^3 = \beta, \quad 4C\eta_-^2 = c\eta_+^2, \tag{7.37}$$

which leads us to choose

$$\eta_\pm = \alpha_\pm \beta^{1/3}, \tag{7.38}$$

where $\alpha_\pm > 0$ are the solutions of

$$4C\alpha_-^2 = c\alpha_+^2, \quad \frac{c}{6}\alpha_+^3 + \frac{2C}{3}\alpha_-^3 = 1.$$

With $A_\pm$, $B_\pm$ and $\eta_\pm$ chosen as in (7.36)-(7.38), the function v is of class $W^{2,\infty}$ and satisfies (7.35). Moreover, it is straightforward to check that

$$|v'| + |v''| \leq C\beta^{1/3} \quad \text{in } (-\eta_-, \eta_+),$$

so that (7.29) is satisfied, which concludes the construction of V_1 satisfying (7.21). On the other hand V_1 obviously satisfies (7.22) since

$$\{|V_1| < \beta/2\} = \{-\eta_- < z < \eta_+\}.$$

We omit the construction of V_2, which is completely similar to the one just performed. □

7.4 Intermediate regimes

As discussed in the introduction (Section 7.1), in the present section we want to understand the transitions occurring as β decreases from β_c to 0, when the set $\{H = 0\}$ has more than one connected component.

In Section 7.4.1 we study in detail a special case with rotational symmetry along a vertical axis, to provide some insight into the transition from the vortexless state to the zero solution. The reason to restrict to this setting is that it encapsules, what we believe are, the most interesting changes in the superconducting set that can occur.

On the one hand, once we drop the assumption of rotational symmetry, changes in H inside the sample could lead to arbitrarily intricate solutions to the obstacle problem for different values of β, so a general theorem is not available. On the other hand the symmetries we consider highlight many model situations with remarkable properties. One of these is the striking phenomenon that some parts of the free boundary may freeze: that is, remain constant with respect to β, for β in some interval. In Section 7.4.2 we generalize this observation to the general, non-symmetric case.

As mentioned earlier, a generalization of the other properties is precluded due to the wide variety of solutions one could construct, having the freedom to choose both H and $\mathcal{M}$. Nevertheless, we believe that under some more restrictive assumptions, in particular fixing the topology of the level sets of H, one could extend the result on existence of the transitions observed in Proposition 7.15, however the role of the integral conditions on $I_\pm, J$ is not so easily transferable or even identifiable anymore.

7.4.1 Detailed study of a symmetric case

Here we consider a surface of revolution of the form

$$\mathcal{M} = \{(\rho(\phi)\cos\theta, \rho(\phi)\sin\theta, z(\phi)) \colon \phi \in [0,\pi],\, \theta \in [0,2\pi]\}, \tag{7.39}$$

where ρ and z are smooth functions linked by the relation

$$z(\phi)\tan\phi = \rho(\phi),$$

and satisfying $\rho(0) = \rho(\pi) = 0$, $\rho > 0$ in $(0,\pi)$, $z'(0) = z'(\pi) = 0$, and

$$\gamma := \sqrt{(\rho')^2 + (z')^2} \geq c > 0.$$

The volume form on such $\mathcal{M}$ is $d\mathcal{H}^2_{\mathcal{M}} = \rho\gamma d\theta d\phi$.

The induced magnetic potential $\mathbf{A}$ on $\mathcal{M}$ is also assumed to be symmetric, of the form

$$\mathbf{A} = a(\phi)d\theta = \frac{a(\phi)}{\rho(\phi)}\hat{e}_\theta,$$

and we make the following assumptions on the functions a:

(a1): $a(0) = a(\pi) = 0$, and $a > 0$ in $(0,\pi)$.

(a2): $a' > 0$ in $(0,\phi_1)$ and (ϕ_2,ϕ_3) and $a' < 0$ in (ϕ_1,ϕ_2) and (ϕ_3,π), for some $0 < \phi_1 < \phi_2 < \phi_3 < \pi$.

The function $a(\phi)$ hast two local maxima $a_1 = a(\phi_1)$ and $a_3 = a(\phi_3)$, and one local minimum $a_2 = a(\phi_2)$. To simplify notations to come, we assume in addition that $a_1 < a_3$. See Figure 7.2.

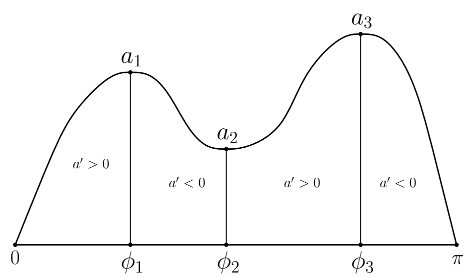


Figure 7.2 – The shape of $a(\phi)$.

Remark 7.13. The case, presented in the introduction (Section 7.1), of a uniform external magnetic field $\mathbf{H}^e = \mathbf{e_z}$ corresponds to $a = \rho^2/2$.

In that setting, the functions H and $*F$ are also axially symmetric: they depend only on ϕ , and are given by

$$H = \frac{a'}{\rho\gamma}, \quad (*F)' = a\frac{\gamma}{\rho}.$$

By uniqueness (up to a possible additive constant), the solution of the obstacle problem (7.9) is also rotationally symmetric: it holds $V = v(\phi)$. Since $V \in C^1(\mathcal{M})$, the function v should satisfy

$$v \in C^1([0,\pi]), \quad v'(0) = v'(\pi) = 0.$$

Moreover, the free boundary problem (7.11) becomes

$$\begin{cases} |v| \leq \beta/2 & \text{in } [0,\pi], \\ \left(\rho\gamma^{-1}v' - a\right)' = 0 & \text{in } \{|u| < \beta/2\}, \\ a' \geq 0 & \text{in } \{v = -\beta/2\}, \\ a' \leq 0 & \text{in } \{v = \beta/2\}. \end{cases} \tag{7.40}$$

We investigate, for $\beta < \beta_c$, the changes in the shape of the superconducting set $SC_\beta = \{|v| < \beta/2\}$. The critical values at which that shape changes depend on the values of integrals $\int a\,\gamma\rho^{-1}d\phi$ on some intervals related to the level sets of $a(\phi)$. That is why we start by fixing some notations concerning the level sets of $a(\phi)$. There are three different cases, depicted in Figure 7.3:

— For $\alpha \in (0, a_2)$, $\{a = \alpha\} = \{\phi_- < \phi_+\}$.

— For $\alpha \in (a_2, a_1)$, $\{a = \alpha\} = \{\phi_- < \psi_+ < \psi_- < \phi_+\}$.

— For $\alpha \in (a_1, a_3)$, $\{a = \alpha\} = \{\psi_- < \phi_+\}$.

The functions $\phi_\pm(\alpha)$, $\psi_\pm(\alpha)$ are continuous on their intervals of definition.

For $\alpha \in (a_2, a_1)$, we define

$$\begin{gathered} I_-(\alpha) = \int_{\phi_-}^{\psi_+} (a-\alpha)\frac{\gamma}{\rho}\,d\phi, \quad I_+(\alpha) = \int_{\psi_-}^{\phi_+} (a-\alpha)\frac{\gamma}{\rho}\,d\phi, \\ J(\alpha) = -\int_{\psi_+}^{\psi_-} (a-\alpha)\frac{\gamma}{\rho}\,d\phi. \end{gathered} \tag{7.41}$$

Those integrals corresponds to “weighted” areas of the regions depicted in Figure 7.4, with respect to the measure $\gamma\rho^{-1}d\phi$. Note that both the integrands and the intervals of integration depend on α.

We identify a critical value of α with respect to these integrals.

Lemma 7.14. *There exists $\alpha_* \in (a_2, a_1)$ such that:*

— for $a_2 < \alpha < \alpha_*$, $J < \min(I_\pm)$.

— for $\alpha_* < \alpha < a_1$, $\min(I_\pm) < J$.

Proof. It follows from the obvious facts that J is increasing, $I_\pm$ are decreasing, $J(a_2) = 0$, $I_-(a_1) = 0$, and the functions are continuous. □

Now we may give the precise version of Proposition 7.2.

Proposition 7.15. *Let $\beta_c > \beta_1^* \geq \beta_2^* > 0$ be defined by*

$$\beta_1^* := \max(I_\pm(\alpha^*)), \quad \beta_2^* := \min(I_\pm(\alpha^*)).$$

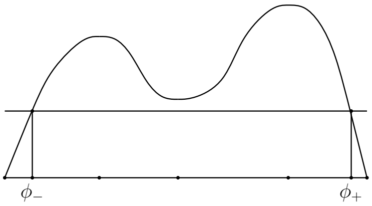
Then the conclusion of Propostion 7.2 holds:

— For $\beta_c > \beta > \beta_1^*$, SC_β is an interval.

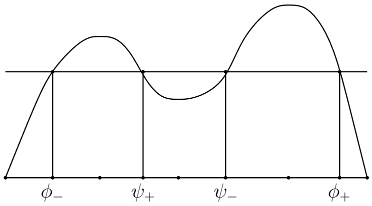
— For $\beta_1^* > \beta > \beta_2^*$, SC_β is the union of two disjoint intervals, one of them independent of β.

— For $\beta_2^* > \beta > 0$, SC_β is the union of three disjoint intervals.

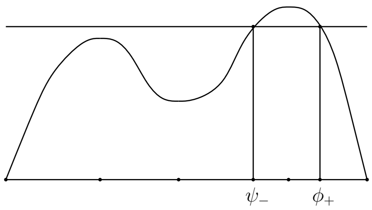
Remark 7.16. It may happen that $I_-(\alpha^*) = I_+(\alpha^*)$. In that case, $\beta_1^* = \beta_2^*$ and the second regime predicted by Proposition 7.15 never happens.



(a) $\alpha \in (0, a_2)$



(b) $\alpha \in (a_2, a_1)$



(c) $\alpha \in (a_1, a_3)$

Figure 7.3 – Level sets $\{a = \alpha\}$

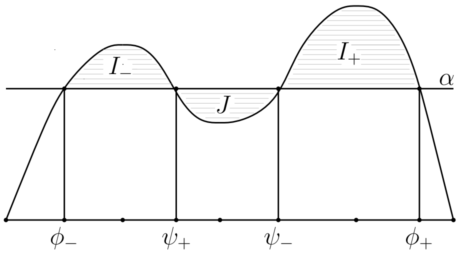


Figure 7.4 – The integrals $I_\pm$ and J.

Proof of Proposition 7.15: By uniqueness (see Remark 7.4), it suffices to exhibit, for each regime listed in Proposition 7.15, a solution of (7.40) satisfying the desired properties.

Case 1: $\beta \in (\beta_1^*, \beta_c)$. The function

$$I(\alpha) := \int_{\phi_-}^{\phi_+} (a - \alpha)\frac{\gamma}{\rho}\, d\phi, \quad \alpha \in (0, a_1),$$

is continuous, decreasing and satisfies $I(0) = \beta_c$ and $I(\alpha^*) = \beta_1^*$. Therefore there exists a unique $\alpha \in (0, \alpha^*)$ such that $I(\alpha) = \beta$. We define

$$v(\phi) = \begin{cases} -\beta/2 & \text{for } \phi \in (0, \phi_-), \\ -\beta/2 + \int_{\phi_-}^{\phi} (a - \alpha)\frac{\gamma}{\rho}\, d\tilde{\phi} & \text{for } \phi \in (\phi_-, \phi_+), \\ \beta/2 & \text{for } \phi \in (\phi_+, \pi). \end{cases}$$

The shape of the function v is sketched in Figure 7.5.

The function v is clearly continuous since β has been chosen accordingly.

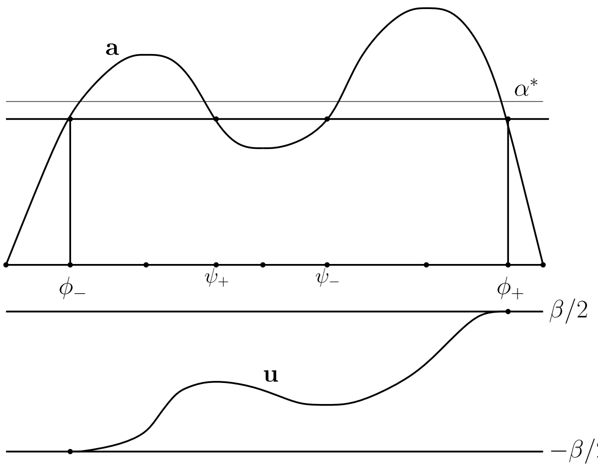


Figure 7.5 – The shape of v for $\beta \in (\beta_1^*, \beta_c)$

Moreover, it holds

$$v'(\phi_+) = (a(\phi_+) - \alpha)\frac{\gamma}{\rho} = v'(\phi_-) = 0,$$

since by definition $a(\phi_+) = a(\phi_-) = \alpha$. Hence v is in fact C^1 in $[0, \pi]$. Also by definition, $a' \geq 0$ in $(0, \phi_-)$ and $a' \leq 0$ in (ϕ_+, π). In addition, we clearly have $(\rho\gamma^{-1}v' - a)' = 0$ in (ϕ_-, ϕ_+). To prove that v solves (7.40), it only remains to show that $|v| < \beta/2$ in (ϕ_+, ϕ_-). We consider two different cases, depending on whether $\alpha \in (0, a_2]$ or $\alpha \in (a_2, \alpha^*)$.

If $\alpha \in (0, a_2)$, then (see Figure 7.3a)

$$v' = (a - \alpha)\frac{\gamma}{\rho} > 0 \quad \text{in } (\phi_-, \phi_+),$$

so that v is increasing on (ϕ_-, ϕ_+) and it clearly holds $|v| < \beta/2$. For $\alpha = a_2$ the derivative v' only vanishes at one point and the same conclusion is valid.

If, on the other hand $\alpha \in (a_2, \alpha^*)$, then (see Figure 7.5)

$$v' = (a - \alpha)\frac{\gamma}{\rho}\begin{cases} > 0 & \text{in } (\phi_-, \psi_+), \\ < 0 & \text{in } (\psi_+, \psi_-), \\ > 0 & \text{in } (\psi_-, \phi_+). \end{cases}$$

Therefore it suffices to check that $v(\psi_+) < \beta/2$ and $v(\psi_-) > -\beta/2$. We have, since $I(\alpha) = \beta$ and by definition of $I_\pm$ and J (see Figure 7.4),

$$\begin{aligned} v(\psi_+) - \beta/2 &= I_-(\alpha) - \beta = I_-(\alpha) - I(\alpha) = J(\alpha) - I_+(\alpha), \\ v(\psi_-) + \beta/2 &= I_-(\alpha) - J(\alpha). \end{aligned}$$

Since $\alpha < \alpha^*$ we find indeed (by definition of α^*) that $v(\psi_+) < \beta/2$ and $v(\psi_-) > -\beta/2$, and in that case also we conclude that v solves the free boundary problem (7.40).

Case 2: $\beta \in (\beta_2^*, \beta_1^*)$. We treat the case where $\min(I_\pm(\alpha^*)) = I_-(\alpha^*)$. Thus $\beta_1^* = I_+(\alpha^*)$ and $\beta_2^* = I_-(\alpha^*)$. The other case can be dealt with similarly.

The function $I_+(\alpha)$ is continuous and decreasing on (a_2, a_3) and satisfies $I_+(\alpha^*) = \beta_1^*$ and $I_+(a_3) = 0 < \beta_2^*$ (see Figure 7.4). Therefore there exists $\alpha > \alpha^*$ such that $I_+(\beta) = \alpha$. We denote by ψ_- and ϕ_+ the two points of $\{a = \alpha\} \cap (\phi_2, \pi)$, and by $\phi_-^* < \psi_+^* < \psi_-^*$ the three points of $\{a = \alpha^*\} \cap (0, \phi_3)$ (as in Figure 7.6 below). Note that, since $\alpha > \alpha^*$, $\psi_-^* < \psi_-$. Next we define

$$v(\phi) = \begin{cases} -\beta/2 & \text{for } \phi \in (0, \phi_-^*), \\ -\beta/2 + \int_{\phi_-^*}^{\phi} (a - \alpha^*)\frac{\gamma}{\rho}\, d\tilde{\phi} & \text{for } \phi \in (\phi_-^*, \psi_-^*), \\ -\beta/2 & \text{for } \phi \in (\psi_-^*, \psi_-), \\ -\beta/2 + \int_{\psi_-}^{\phi} (a - \alpha)\frac{\gamma}{\rho}\, d\tilde{\phi} & \text{for } \phi \in (\psi_-, \phi_+), \\ \beta/2 & \text{for } \phi \in (\phi_+, \pi). \end{cases}$$

The shape of the function v is sketched in Figure 7.6.

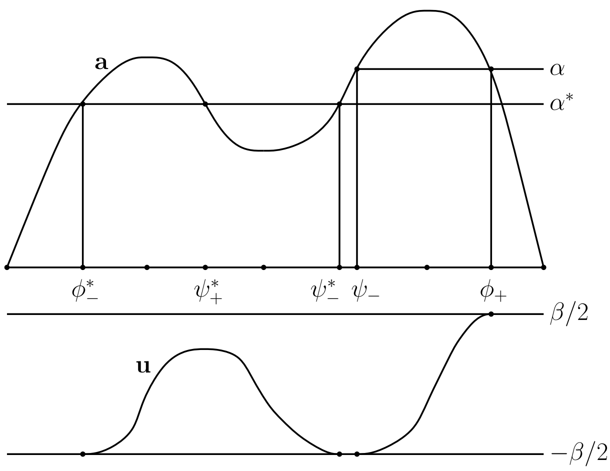


Figure 7.6 – The shape of v for $\beta \in (\beta_2^*, \beta_1^*)$

Continuity of v at ψ_-^* is ensured by the fact that $I_-(\alpha^*) = J(\alpha^*)$. Continuity at ϕ_+ by $I_+(\alpha) = \beta$. The function v is C^1 because the facts that $a(\phi_-^*) = a(\psi_-^*) = \alpha^*$ and $a(\psi_-) = a(\phi_+) = \alpha$ guarantee that $v'(\phi_-^*) = v'(\psi_-^*) = v'(\psi_-) = v'(\phi_+) = 0$. The sign of a' is positive in $(0, \phi_-^*)$ and (ψ_-^*, ψ_-) and negative in (ϕ_+, π). In the two intervals (ϕ_-^*, ψ_-^*) and (ψ_-, ϕ_+), the equation $(\rho\gamma^{-1}v' - a)' = 0$ is obviously satisfied, and it remains to check that $|v| < \beta/2$ in those intervals.

Since $v' = (a-\alpha)\gamma\rho^{-1} > 0$ in (ψ_-, ϕ_+), it clearly holds $|v| < \beta/2$ in (ψ_-, ϕ_+).

In the interval (ϕ_-^*, ψ_-^*), the sign of v' shows that v attains its minimum at the boundary and its maximum at ψ_+^*, and it holds

$$v(\psi_+^*) - \beta/2 = -\beta + I_-(\alpha^*) = -\beta + \beta_2^* < 0.$$

We conclude that v solves the free boundary problem (7.40). Moreover, the interval (ϕ_-^*, ψ_-^*) clearly does not depend on β.

Case 3: $\beta \in (0, \beta_2^*)$. Since I_- is continuous and decreasing, $I_-(\alpha^*) > \beta_2^*$ and $I_-(a_1) = 0$, there exists $\alpha_1 > \alpha^*$ such that $I_-(\alpha_1) = \beta$. Similarly, there exist $\alpha_2 < \alpha^*$ and $\alpha_3 > \alpha^*$ such that $J(\alpha_2) = I_+(\alpha_3) = \beta$. We denote by

$$0 < \phi_-^1 < \psi_+^1 < \psi_+^2 < \psi_-^2 < \psi_-^3 < \phi_+^3 < \pi$$

the points such that (see Figure 7.7)

$$\begin{aligned} \{a = \alpha_1\} \cap (0, \phi_2) &= \{\phi_-^1, \psi_+^1\}, \\ \{a = \alpha_2\} \cap (\phi_1, \phi_3) &= \{\psi_+^2, \psi_-^2\}, \\ \{a = \alpha_3\} \cap (\phi_2, \pi) &= \{\psi_-^3, \phi_+^3\}. \end{aligned}$$

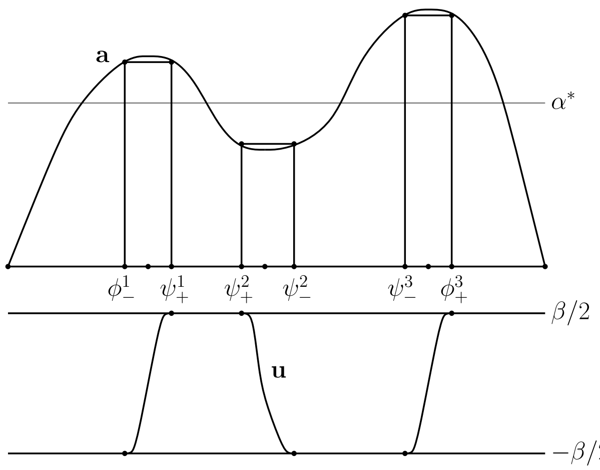


Figure 7.7 – The shape of v for $\beta \in (0, \beta_2^*)$

Then we define

$$v(\phi) = \begin{cases} -\beta/2 & \text{for } \phi \in (0, \phi_-^1) \text{ or } \phi \in (\psi_-^2, \psi_-^3) \\ -\beta/2 + \int_{\phi_-^1}^{\phi} (a - \alpha_1)\frac{\gamma}{\rho}\, d\tilde{\phi} & \text{for } \phi \in (\phi_-^1, \psi_+^1), \\ \beta/2 & \text{for } \phi \in (\psi_+^1, \psi_+^2) \text{ or } \phi \in (\phi_+^3, \pi), \\ \beta/2 + \int_{\psi_+^2}^{\phi} (a - \alpha_2)\frac{\gamma}{\rho}\, d\tilde{\phi} & \text{for } \phi \in (\psi_+^2, \psi_-^2) \\ -\beta/2 + \int_{\psi_-^3}^{\phi} (a - \alpha_3)\frac{\gamma}{\rho}\, d\tilde{\phi} & \text{for } \phi \in (\psi_-^3, \phi_+^3). \end{cases}$$

The shape of the function v is sketched in Figure 7.7.

As above the C^1 regularity of v follows from the definitions of α_1, α_2 and α_3. The sign of a' is positive in $(0, \phi_-^1) \cup (\psi_-^2, \psi_-^3)$ and negative in $(\psi_+^1, \psi_+^2) \cup (\phi_+^3, \pi)$. The equation $(\rho\gamma^{-1}v' - a)' = 0$ is satisfied in the three intervals (ϕ_-^1, ψ_+^1), (ψ_+^2, ψ_-^2) and (ψ_-^3, ϕ_+^3). Moreover in those intervals, the function v is monotone, hence $|v| < \beta/2$. Therefore v solves the free boundary problem (7.40). □

7.4.2 'Freezing' of the free boundary

Proposition 7.17. *Assume that, for some $\beta_0 \in (0, \beta_c)$, one connected component ω of the superconductivity set SC_{β_0} is such that V_{β_0} takes the same value on each connected component of $\partial\omega$. Then there exists $\delta > 0$ such that*

$$SC_\beta \cap \overline{\omega} = SC_{\beta_0} \cap \overline{\omega} = \omega, \tag{7.42}$$

for all $\beta \in (\beta_0 - \delta, \beta_0]$.

In Figure 7.8 we show a situation corresponding to Proposition 7.17, with $V = -\beta/2$ on every connected component of $\partial\omega$.

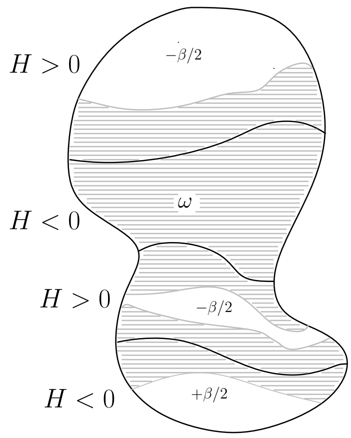


Figure 7.8 – An example of the situation of Proposition 7.17

Remark 7.18. The assumption on β_0 in Proposition 7.17 corresponds exactly to what happens in the symmetric case (Proposition 7.15) in the regime $\beta_1^* > \beta > \beta_2^*$, where $v(\phi_-^*) = v(\psi_-^*) = -\beta/2$ (Figure 7.6).

Proof of Proposition 7.17: We present the proof in the case where $V = -\beta_0/2$ on every connected component of $\partial\omega$. The case $V = \beta_0/2$ on $\partial\omega$ can be dealt with similarly.

Since $V < \beta_0/2$ in ω and $V = -\beta_0/2$ on $\partial\omega$, it holds

$$m := \max_{\overline{\omega}} V < \beta_0/2,$$

and we define

$$\delta := \frac{1}{2}\beta_0 - m > 0.$$

Let $\beta \in (\beta_0 - \delta, \beta_0]$, and define

$$\widetilde{V}_0 := V_{\beta_0} + \frac{1}{2}(\beta_0 - \beta). \tag{7.43}$$

The definitions of m and δ ensure that it holds

$$-\beta/2 \leq \widetilde{V}_0 \leq \frac{1}{2}\beta_0 - \delta + \frac{1}{2}(\beta_0 - \beta) < \beta/2 \quad \text{in } \overline{\omega}. \tag{7.44}$$

We claim that

$$V_\beta = \widetilde{V}_0 \quad \text{in } \overline{\omega}, \tag{7.45}$$

which obviously implies (7.42).

Note that the proof of Proposition 7.7 implies that it always holds

$$V_\beta \leq \widetilde{V}_0 \quad \text{in } \mathcal{M}. \tag{7.46}$$

Let $\omega_\beta = SC_\beta \cap \omega$, and

$$U := \widetilde{V}_0 - V_\beta \geq 0. \tag{7.47}$$

Note that $U \in C^{1,\alpha}(\overline{\omega})$, and $U = 0$ on $\partial\omega$ (since, by definition of ω, $\widetilde{V}_0 = 0$ on $\partial\omega$).

Let $\omega' := \omega \cap SC_\beta$. It holds

$$\Delta U = H \mathbb{1}_{\omega \setminus \omega'} \quad \text{in } \omega. \tag{7.48}$$

From (7.44) and (7.46) it follows that

$$V_\beta < \beta/2 \quad \text{in } \overline{\omega}.$$

Therefore, recalling the free boundary formulation (7.11), we have $H \geq 0$ in $\omega \setminus \omega'$. In particular (7.48) implies that

$$\Delta U \geq 0 \quad \text{in } \omega.$$

Let $\varepsilon > 0$ and consider

$$\varphi := \max(U - \varepsilon, 0) \in H^1(\omega).$$

Recalling that $U \in C(\overline{\omega})$ and $U = 0$ on $\partial\omega$, we know that φ has compact support inside ω. Thus we may integrate by part (without knowing anything about the regularity of $\partial\omega$) to obtain

$$\int_\omega |\nabla\varphi|^2 = \int_\omega \nabla\varphi \cdot \nabla U = -\int_\omega \varphi \Delta U \leq 0,$$

and we deduce that $\varphi \equiv 0$ in ω, which implies that $U \leq \varepsilon$ in ω. Letting $\varepsilon \to 0$, we conclude that $U \leq 0$ in ω, which, together with (7.47), shows that (7.45) holds. □

Appendix 7.A The mean field approximation

Recall we assume $\mathcal{M} \subset \mathbb{R}^3$ is a closed compact surface homeomorphic to a sphere, $\mathbf{A}$ a 1-form on $\mathcal{M}$ such that $\mathbf{A} = d^*F = *d*F$ for some smooth non constant 2-form F, and $\mathcal{G}_{\mathcal{M},\kappa}$ the Ginzburg-Landau energy

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_\mathcal{M} |(\nabla - ih\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}\int_\mathcal{M} (|\psi|^2 - 1)^2.$$

The parameter $\kappa > 0$ is going to tend to $+\infty$, as is the strength of the applied field $h(\kappa) > 0$.

If ψ is a critical point of $\mathcal{G}_{\mathcal{M},\kappa}$, written locally as $\psi = \rho e^{i\varphi}$, then it holds

$$d(\rho^2(d\varphi - hd^*F)) = 0.$$

We deduce that there exists a function V such that

$$*dV = \rho^2(hd^*F - d\varphi).$$

The function V is uniquely defined up to an additive constant, which we may fix by imposing $\int_{\mathcal{M}} V = 0$. The function

$$\mu = -\Delta(V - h * F) = -\Delta V + H$$

is the vortex density.

In this paper we appeal to a mean field approximation result proved by Sandier and Serfaty in [118]. In our case we also have to handle positive and negative measures μ_+, μ_- with total zero mass $\mu_+(\mathcal{M}) - \mu_-(\mathcal{M}) = 0$. In this appendix we verify that under the additional constraints present in our context, we still have such a reduction. For an intensity $h(\kappa)$ comparable to $\ln\kappa$, the mean field approximation consists in approximating the problem of minimizing $\mathcal{G}_{\mathcal{M},\kappa}$ by a limiting problem on the vorticity measure. The result also relates the

Proposition 7.19. *Assume that* $\beta := \lim_{\kappa\to\infty} \frac{\ln\kappa}{h(\kappa)} \geq 0$ *and* $h(\kappa) = o(\kappa^2)$. *Let* ψ_κ *be a minimizer of* $\mathcal{G}_{\mathcal{M},\kappa}$, *and the corresponding* V_κ *be defined as above. Then, up to a subsequence, as* $\kappa \to \infty$,

$$\frac{V_\kappa}{h(\kappa)} \text{ converges to } W_*,$$

weakly in H^1 *(and strongly in* $W^{1,q}$ *for* $q < 2$*) where* W_* *minimizes the energy*

$$E_\beta(W) = \frac{1}{2}\int_{\mathcal{M}} |\nabla W|^2 \, d\mathcal{H}^2 + \frac{\beta}{2}\| - \Delta W + H\|_{TV},$$

over the set of all $W \in H^1(\mathcal{M})$ *such that* $(-\Delta W + H)$ *is a Radon measure. Here* $\|\mu\|_{TV} = |\mu|(\mathcal{M})$ *denotes the total variation norm of the Radon measure* μ.

Moreover, it holds $\mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa) = h(\kappa)^2 E_\beta(W_*) + o(h(\kappa)^2)$.

We impose the normalization conditions $\int_{\mathcal{M}} W \, d\mathcal{H}^2 = \int_{\mathcal{M}} *F \, d\mathcal{H}^2 = 0$. Then with some slight abuse of notation $E_\beta(W)$ can be expressed in terms of $\mu = -\Delta W + H$, as

$$E_\beta(W) = E_\beta(\mu) = \frac{\beta}{2}\|\mu\|_{TV} + \frac{1}{2}\int_{\mathcal{M}} G(x,y)d(\mu - H)(x)d(\mu - H)(y),$$

where $G(x, y)$ is the Green's function satisfying

$$-\Delta_{\mathcal{M}} G(\cdot, y) = \delta_y - \frac{1}{\mathcal{H}^2(\mathcal{M})}.$$

Here μ has to be a Radon measure of zero average since it comes from $\mu = -\Delta(W - *F)$, hence $\int_{\mathcal{M}} d\mu = 0$.

Note that $E_\beta(\mu)$ may not be well-defined for every measure μ, but at the end we will only need it to be well-defined for the particular μ_* associated to W_* solving the obstacle problem (7.9), and this follows from the regularity theory for the obstacle problem (see Lemma 7.6).

Sketch of the proof of the upper bound in Proposition 7.19

The proof of the lower bound and compactness for minimizers follows directly from Theorems 7.1 and 7.2 in [118]. We note that a by product of the analysis in [118] is that $\frac{2\pi\sum_{i\in I} d_i\delta_{a_i}}{h}$ converges to $-\Delta\left(\frac{V_\kappa}{h} - *F\right)$ in the sense of measures and in $W^{1,p}$, for $p < 2$.

The upper bound on the other hand is a little more delicate to adapt. Next we provide the details. The main tool to derive the upper bound in [118] is a construction of measures μ_κ which approximate the measure μ_* minimizing I_β, and which are concentrated in balls of size κ^{-1} each carrying a weight 2π. Before stating the precise result, we introduce the functional $J = J_\beta$

$$J(\mu) := \beta\|\mu\|_{TV} + \int_{\mathcal{M}\times\mathcal{M}} G(x,y)d\mu(x)d\mu(y). \tag{7.49}$$

The following result then corresponds to Proposition 2.2 in [118].

Proposition 7.20. *Let $\mu = \mu_+ - \mu_-$ be the minimizer of I_β. Then, for κ large enough, there exist points $a^\kappa_{j,\pm}$, $1 \le j \le n_\pm(\kappa)$, such that*

$$n_\pm(\kappa) \sim \frac{h(\kappa)\mu_\pm(\mathcal{M})}{2\pi}, \qquad d(a^\kappa_{j,\pm}, a^\kappa_{\ell,\pm}) > 4\kappa^{-1},$$

and, letting $\mu_\kappa^{j,\pm}$ be the uniform measure on $\partial B(a_{j,\pm}, \kappa^{-1})$ of mass 2π, the measure

$$\mu_\kappa := \frac{1}{h(\kappa)}\sum_{j=1}^{n_+(\kappa)} \mu_\kappa^{j,+} - \frac{1}{h(\kappa)}\sum_{j=1}^{n_-(\kappa)} \mu_\kappa^{j,-} \quad \textit{converges to } \mu,$$

in the sense of measures as $\kappa \to +\infty$. Moreover it holds $\int_{\mathcal{M}} d\mu_\kappa = 0$, and

$$\limsup_{\kappa\to\infty} \int_{\mathcal{M}\times\mathcal{M}} G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) \le J(\mu), \tag{7.50}$$

where $J = J_\beta$ is defined in (7.49).

Above, d denotes geodesic distance and ∂B denotes a geodesic circle accordingly.The zero average property $\int d\mu_\kappa = 0$ is needed later to solve $-\Delta(V - *F) = \mu_\kappa$. It actually amounts to asking $n_+(\kappa) = n_-(\kappa)$. The upper bound (7.50) is crucial to estimate the energy of the testing configuration constructed with help of the measures μ_κ, and requires great care in the way the points a^κ_j are distributed.

In [118], the authors consider non-negative measures defined on a domain in the plane, with no average constraint. Here we are dealing with measures on a surface having positive and negative parts, and, more importantly, satisfying the zero average constraint.

Next we state a lemma that can be directly adapted from [118, Proposition 2.2], which deals only with positive measures with support inside a coordinate neighborhood. Then we will explain how to use this lemma to obtain Proposition 7.20 above.

Lemma 7.21. *[118, Proposition 2.2] Assume that μ is a non-negative Radon measure on $\mathcal{M}$, absolutely continuous with respect to the 2-dimensional measure on $\mathcal{M}$, and with support contained inside a coordinate neighborhood. Then, there exist points a_j^κ, $1 \le j \le n(\kappa)$, with*

$$n(\kappa) \sim \frac{h(\kappa)\mu(\mathcal{M})}{2\pi} \quad \textit{and } d(a_j^\kappa, a_\ell^\kappa) > 4\kappa^{-1},$$

such that, with μ_κ^j the uniform measure of mass 2π on $\partial B(a_j^\kappa, \kappa^{-1})$, it holds

$$\mu_\kappa = \frac{1}{h(\kappa)} \sum_{j=1}^{n(\kappa)} \mu_\kappa^j \quad \textit{converges to } \mu,$$

and the upper bound (7.50) *is satisfied.*

The proof of Lemma 7.21 is just a straightforward adaptation of [118, Proposition 2.2], using the coordinate chart to transport their construction from the plane to our surface and general properties of the Green's function of the Laplacian on a compact surface.

Next we explain how to deal with non-negative measures whose support does not lie inside a coordinate neighborhood.

Lemma 7.22. *Assume that μ is a non-negative Radon measure on $\mathcal{M}$, absolutely continuous with respect to the* 2*-dimensional measure on $\mathcal{M}$. Then the conclusion of Lemma 7.21 holds.*

Proof. Step 1: We reduce to the case where the support of μ is a finite disjoint union of compact coordinate neighborhoods. Assume indeed that the conclusion of Lemma 7.22 holds for such measures. It is possible to construct a sequence μ_n of such measures, such that $0 \le \mu_n \le \mu$ and μ_n converges to μ. Indeed, just define $\mu_n = \mathbb{1}_{K_n}\mu$, where K_n is a finite disjoint union of compact subsets of coordinate neighborhoods, and $\mu(\mathcal{M} \setminus K_n) \to 0$. Such a sequence K_n exists because $\mathcal{M}$ is compact and the measure μ is inner regular. For each μ_n we obtain a sequence μ_n^κ tending to μ_n with the good properties. After a diagonal process, we obtain a sequence μ_κ converging to μ, such that

$$\limsup_{\kappa\to\infty} \int_{\mathcal{M}\times\mathcal{M}} G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) \le \liminf J(\mu_n).$$

It remains to show that the right-hand side is less than $J(\mu)$, which follows from $0 \le \mu_n \le \mu$ and $G \ge 0$.

Step 2: We prove Lemma 7.22 for μ that can be decomposed in the form

$$\mu = \mu_1 + \cdots + \mu_N,$$

where the supports of the μ_j are inside disjoint compact coordinate neighborhoods, and each μ_j is non-negative and absolutely continuous with respect to $\mathcal{H}^2_{\mathcal{M}}$. Then one can apply Lemma 7.21 to each μ_j to obtain sequences $\mu_{j,\kappa}$ with the good properties. Then, defining $\mu_\kappa = \mu_{1,\kappa} + \cdots + \mu_{N,\kappa}$, one obtains

$$\limsup \int G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) \le \sum_j J(\mu_j) + \limsup \sum_{j\neq\ell} \int G(x,y)d\mu_{j,\kappa}(x)d\mu_{\ell,\kappa}(y).$$

Since the supports of distinct μ_j are disjoint and $G(x,y)$ is continuous outside the diagonal $\{x=y\}$, it holds

$$\int G(x,y)d\mu_{j,\kappa}(x)d\mu_{\ell,\kappa}(y) \to \int G(x,y)d\mu_j(x)d\mu_\ell(y) \qquad \text{for } j\neq \ell,$$

and we conclude that

$$\limsup \int G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) \le \sum_j J(\mu_j)+\sum_{j\neq\ell}\int G(x,y)d\mu_j(x)d\mu_\ell(y) = J(\mu).$$

The proof is complete. □

Finally we deal with measures having positive and negative parts, and satisfying the zero average constraint.

Lemma 7.23. *Let μ be a zero-average Radon measure on $\mathcal{M}$, absolutely continuous with respect to $\mathcal{H}^2_{\mathcal{M}}$. Then the conclusions of Proposition 7.20 hold.*

Proof. Step 1: It suffices to construct measures μ_κ satisfying all the conclusions of Proposition 7.20, except for the zero average constraint. Assume indeed that we have such a sequence. Since μ satisfies the zero average constraint, it holds $\mu_+(\mathcal{M}) = \mu_-(\mathcal{M})$ and we deduce that $n_+(\kappa) - n_-(\kappa) = o(h(\kappa))$. Up to considering a subsequence, we may assume that either $n_+(\kappa) \ge n_-(\kappa)$ for every κ (or the opposite, but this is completely symmetric). We fix a compact K such that $\mu_+(K) > 0$ and K is disjoint from the support of μ_-. Since $\mu_\kappa^+(K)$ converges to $\mu_+(K)$, the number of points $a^\kappa_{j,+}$ that are contained in K for large κ is larger than $c\cdot h(\kappa)$ for $c>0$. In particular it is larger that $n_+ - n_-$, and we may define a measure $\tilde\mu^+_\kappa$ obtained from μ^+_κ by removing $(n_+ - n_-)$ points $a^\kappa_{j,+}$ that lie inside K. The measure $\tilde\mu_\kappa = \tilde\mu^+_\kappa - \mu^-_\kappa$ now satisfies the zero average condition, and since $n_+ - n_- = o(h)$ the convergence $\tilde\mu_\kappa \to \mu$ still holds. It remains to prove that the upper bound (7.50) is satisfied also by $\tilde\mu_\kappa$. Since $G \ge 0$ and $0 \le \tilde\mu^+_\kappa \le \mu^+_\kappa$, it holds

$$\int G(x,y)d\tilde\mu_\kappa(x)d\tilde\mu_\kappa(y) \le \int G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) + 2\int G(x,y)d(\mu^+_\kappa - \tilde\mu^+_\kappa)(x)d\mu^-_\kappa(y).$$

The last term converges to zero since G is continuous outside the diagonal and $\mu^+_\kappa - \tilde\mu^+_\kappa$ converges to zero and has support inside K which is disjoint from the support of μ_-. Hence we conclude that (7.50) holds.

Step 2: As in Step 1 of Lemma 7.22, we reduce to the case of a measure μ such that μ_+ and μ_- have disjoint compact supports. Assume indeed that Lemma 7.23 holds for such measures, and consider, by truncating, monotone approximations $\mu^\pm_n$ of $\mu_\pm$, with disjoints compact supports and such that $0 \le \mu^\pm_n \le \mu_\pm$. For each n there exist measures μ^n_κ with the good properties, converging to $\mu_n := \mu^+_n - \mu^-_n$. After a diagonal process, one obtains a sequence μ_κ such that

$$\limsup_{\kappa\to\infty} \int_{\mathcal{M}\times\mathcal{M}} G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) \le \liminf J(\mu_n).$$

Since $G \geq 0$, by monotone convergence (or dominated convergence) terms of the form $\int G d\mu_n^{\pm} d\mu_n^{\pm}$ converge to $\int G d\mu^{\pm} d\mu^{\pm}$, so that

$$\int G(x,y) d\mu_n(x) d\mu_n(y) \longrightarrow \int G(x,y) d\mu(x) d\mu(y),$$

and we also have $\|\mu_n\| \to \|\mu\|$, so that $J(\mu_n) \to J(\mu)$ and we conclude that (7.50) holds.

Step 3: We assume now that μ_+ and μ_- have disjoint compact supports. Applying Lemma 7.22 to each of these non-negative measures, we can proceed exactly as in Step 2 of Lemma 7.22 to obtain the conclusion. □

With Lemma 7.23 at hand, the proof of Proposition 7.20 simply follows from the regularity theory for the obstacle problem (see Lemma 7.6), which ensures in particular that the minimizing measure μ_* is absolutely continuous with respect to $\mathcal{H}^2_{\mathcal{M}}$.

Then the upper bound is obtained by constructing test configurations with vortices at the $a^{\kappa}_{j,\pm}$ as in the proof of [118, Proposition 2.1]. Those test configurations are obtained by solving $-\Delta(V_\kappa - h * F) = h\mu_\kappa$ and constructing the corresponding ψ_κ which has modulus 1 outside the balls $B(a^{\kappa}_{j,\pm}, 2\kappa^{-1})$'s, and phase given by $d\varphi_\kappa = hd^* F - * dV_\kappa$. □

Chapitre 8

Structure d'un vortex dans les supraconducteurs à symétrie 'p'

(avec Stan Alama & Lia Bronsard)

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8.1 Introduction

With the discovery of high temperature superconductors physicists have investigated many new and unusual families of superconducting materials, many with properties which are quite different from the metal superconductors which were originally studied a century ago. Among these is $Sr_2\,Ru\,O_4$, which (although it is not a high temperature superconductor) has a layered perovskite crystalline structure which is very similar to the cuprate high T_C materials. This material is special, however, in that it has a different electronic structure from conventional "s-wave" superconductors described by the microscopic BCS model, but instead exhibits a "p-wave" electron pairing symmetry (see [3]). Superconductors with p-wave pairing develop such unconventional properties as spontaneous magnetization and surface currents[65, 84], and square vortex lattices in certain parameter regimes [3].

In this paper we consider a Ginzburg–Landau model for p-wave superconductors in two dimensions. The state of the superconductor is described by a pair of complex wave functions, $\eta = (\eta_-, \eta_+) : \Omega \subset \mathbb{R}^2 \to \mathbb{C}^2$ and the magnetic vector potential, $A : \Omega \to \mathbb{R}^2$. The p-wave symmetry is encoded in the kinetic energy by means of an anisotropic gradient term,

$$E(\eta, A) = \int \left(e_{kin}(\eta, A) + \kappa^2 e_{pot}(\eta) + |\operatorname{curl} A|^2\right),$$

where

$$\begin{aligned} e_{kin}(\eta, A) = |D\eta_+|^2 + |D\eta_-|^2 &+ (1+\nu)\left[D_x\eta_+ \cdot D_x\eta_- - D_y\eta_+ \cdot D_y\eta_-\right] \\ &+ (1-\nu)\left[D_x\eta_- \wedge D_y\eta_+ - D_x\eta_+ \wedge D_y\eta_-\right] \end{aligned}$$

and

$$e_{pot}(\eta) = \frac{1}{2}(|\eta_+|^2 - 1)^2 + \frac{1}{2}(|\eta_-|^2 - 1)^2 + 2|\eta_+|^2|\eta_-|^2 + \nu(\eta_+^2) \cdot (\eta_-^2). \quad (8.1)$$

Here κ is the Ginzburg–Landau parameter, $\nu \in (-1, 1)$ is an anisotropy parameter, and the operator $D = \nabla - iA$. The dot and wedge product on $\mathbb{C}$ are calculated by treating $z = x + iy \in \mathbb{C}$ as a real vector $(x, y) \in \mathbb{R}^2$, and applying the usual definitions.

By writing the potential energy in the form,

$$\begin{aligned} e_{pot} = \frac{1}{2} &+ \frac{1}{2}(|\eta_+|^2 + |\eta_-|^2 - 1)^2 + (1 - |\nu|)|\eta_+|^2|\eta_-|^2 \\ &+ |\nu|\left[|\eta_+|^2|\eta_-|^2 + \operatorname{sign}(\nu)(\eta_+^2) \cdot (\eta_-^2)\right], \end{aligned}$$

we note that for $-1 < \nu < 1$, the minimum of the potential e_{pot} is attained exactly at

$$(\eta_-, \eta_+) = (1, 0) \text{ or } (0, 1).$$

Thus, we expect that energy minimizers will have this form away from any vortices, with one "dominant" component, which we take to be η_-, $|\eta_-| \simeq 1$, and one "admixed" component [65] η_+ which is small in the bulk of the sample.

Also note that E is gauge invariant: for smooth enough φ,

$$E(\eta_\pm, A) = E(e^{i\varphi}\eta_\pm, A + \nabla\varphi).$$

The goal of this paper is to study isolated vortices in this p-wave Ginzburg–Landau model, and thus we concentrate on energy minimizing solutions with given degrees imposed on the boundary of a disk or at infinity, in the case of entire solutions (defined on $\Omega = \mathbb{R}^2$.) As in the classical Ginzburg–Landau functional, in questions concerning isolated vortices the role of the magnetic field $h = \operatorname{curl} A$ is secondary, and so we neglect the vector potential A in this paper. We expect that our results should extend to the full system with vector potential with some minor technical adjustments. With this simplification, the energy functional takes the form:

$$E(\eta) = \int_\Omega \left[e_{kin}(\eta) + \kappa^2 e_{pot}(\eta)\right] dx,$$

with e_{pot} as before, and

$$\begin{aligned} e_{kin}(\eta) &= |\nabla\eta_+|^2 + |\nabla\eta_-|^2 + (1+\nu)\left[\partial_x\eta_+\cdot\partial_x\eta_- - \partial_y\eta_+\cdot\partial_y\eta_-\right] \\ &\quad + (1-\nu)\left[\partial_x\eta_-\wedge\partial_y\eta_+ - \partial_x\eta_+\wedge\partial_y\eta_-\right] \\ &= |\nabla\eta_+|^2 + |\nabla\eta_-|^2 + (\Pi_-\eta_+)\cdot(\Pi_+\eta_-) + \nu(\Pi_+\eta_+)\cdot(\Pi_-\eta_-), \end{aligned} \tag{8.2}$$

with operators $\Pi_+ = \Pi = \partial_x + i\partial_y$, $\Pi_- = -\Pi^* = \partial_x - i\partial_y$. As we will see shortly, the kinetic energy is nonnegative, but not coercive: it vanishes along a nontrivial linear subspace of functions η. This is an early indication of the difficulties involved in the analysis of the p-wave functional. Energy minimizers solve a system of Euler–Lagrange equations, which are coupled in the second derivative terms:

$$\left.\begin{aligned} 2\Delta\eta_- + [\Pi_-^2 + \nu\Pi_+^2]\eta_+ &= \kappa^2\left(2\eta_-(|\eta_-|^2-1) + 4\eta_-|\eta_+|^2 + 2\nu\overline{\eta}_-\eta_+^2\right) \\ 2\Delta\eta_+ + [\Pi_+^2 + \nu\Pi_-^2]\eta_- &= \kappa^2\left(2\eta_+(|\eta_+|^2-1) + 4\eta_+|\eta_-|^2 + 2\nu\overline{\eta}_+\eta_-^2\right) \end{aligned}\right\} \tag{8.3}$$

Our first result concerns the existence of energy minimizing solutions in any smooth bounded simply connected domain $\Omega \subset \mathbb{R}^2$. Consider the Dirichlet boundary condition

$$\eta_\pm|_{\partial\Omega} = g_\pm, \tag{8.4}$$

where $g_\pm:\ \partial\Omega \to \mathbb{C}$ are given smooth functions.

Theorem 8.1. *Let $g_\pm \in H^{1/2}(\partial\Omega)$ and define*

$$W = \{\eta \in H^1(\Omega;\mathbb{C}^2):\ (8.4)\ \textit{is satisfied}\}.$$

Assume that $(c_+ + \alpha z, c_- - \alpha\bar{z}) \notin W$ for any constants $\alpha, c_\pm \in \mathbb{C}$. Then, there exist a minimizer of $E(\eta)$ in W.

In particular, there exists a minimizer in $\Omega = B_R$ for $g_\pm = \alpha_\pm e^{in_\pm\theta}$ provided that one of $n_\pm \neq \pm 1$ or $\alpha_+ \neq -\alpha_-$.

We recall that the potential energy is minimized with $|\eta_-| = 1$, $|\eta_+| = 0$ (or vice-versa,) and hence a natural choice of boundary condition is

$$\eta_-|_{\partial\Omega} = e^{in\theta}, \qquad \eta_+|_{\partial\Omega} = 0, \tag{8.5}$$

with $n \in \mathbb{N}$, in analogy with Ginzburg–Landau vortices but recognizing the bulk states preferred by e_{pot}. Theorem 8.1 is proved in section 8.2. There we show that the restriction on the boundary data can compensate for the general lack of coercivity in the whole space $H^1(\Omega)$.

As in the classical Ginzburg–Landau model, it is to be expected that the symmetric (equivariant) vortex solutions, $\eta_\pm = f(r)e^{in_\pm\theta}$, play a special role. Here we already see the effect of the p-wave symmetry, as radial solutions do not exist in general, but only for certain choices of the parameters. Indeed, in section 8.3 we show that equivariant solutions cannot exist for anisotropy $\nu \neq 0$, and that for $\nu = 0$ there is a restriction on the degrees, $n_+ = n_- + 2$.

Assuming $\nu = 0$ and $n_+ = n_- + 2$, the equivariant ansatz reduces the problem to finding real-valued functions $(f_-(r), f_+(r))$, $r \in (0,\infty)$, which solve the Euler-Lagrange equations, a system of two coupled second-order ordinary differential equations (see (8.10) below.) As with the classical Ginzburg–Landau model, entire solutions (in all $\mathbb{R}^2$) with nontrivial degree at infinity must have infinite energy. We thus adopt the strategy of passing to the limit in balls B_R of increasing radius, in which we minimize the energy subject to the boundary condition (8.5) on $\partial\Omega = \partial B_R$. Even in this simpler context, there are significant obstacles to overcome. Although the existence of solutions in the balls B_R is guaranteed by Theorem 8.1, for general $n \in \mathbb{N}$ the coupling of the system at highest order prevents us from obtaining the necessary *a priori* estimates to pass to the limit $R \to \infty$, except when $n = -1$. For $n = -1$, which is the most physically relevant case [65], we prove:

Theorem 8.2. *There exists a smooth entire equivariant solution* $\eta = (\eta_-, \eta_+) = (f_-(r)e^{-i\theta}, f_+(r)e^{+i\theta})$ *to the Ginzburg–Landau system* (8.3)*, with* $f_-(r) \to 1$ *and* $f_+(r) \to 0$ *as* $r \to \infty$*. Moreover it holds*

$$f_- = 1 - \frac{1}{2r^2} - \frac{7}{4r^4} + O(r^{-6}), \qquad f_+ = -\frac{1}{2r^2} - \frac{13}{4r^4} + O(r^{-6}), \tag{8.6}$$

as $r \to +\infty$.

The existence of entire equivariant solutions with degrees $(n, n+2)$, $n \neq -1$, is an open problem, as is uniqueness.

Given the usual interpretation of $f_\pm(r)$ as a local density of superconducting electrons, we would expect that these solutions have fixed sign. This is a nontrivial question, as the coupling of the two components in the kinetic energy term precludes the usual arguments used in Ginzburg–Landau vortices, and even the methods developed for semilinear Ginzburg–Landau systems [7] fail in this context. To obtain a result in this direction we introduce an additional parameter into the model, and employ perturbative methods. For $t \in [0, 1]$, we consider the family of functionals,

$$E_t(\eta; R) = \int_{B_R} (|\nabla\eta_+|^2 + |\nabla\eta_-|^2 + t(\Pi_+\eta_-)\cdot(\Pi_-\eta_+) + e_{pot}). \tag{8.7}$$

When $t = 0$ the system couples only through the potential energy term. Vortices in a two-component model with similar potential energy were studied by Lin & Lin [89], and with an applied magnetic field by Alama & Bronsard [5, 4]. With the equivariant ansatz $\eta = (\eta_-, \eta_+) = (f_-(r)e^{-i\theta}, f_+(r)e^{+i\theta})$, the Euler–Lagrange equations take the form

$$\begin{aligned} \Delta_r f_- - \frac{1}{r^2}f_- + \frac{t}{2}(\Delta_r f_+ - \frac{1}{r^2}f_+) &= f_-(f_-^2 - 1) + 2f_-f_+^2, \\ \Delta_r f_+ - \frac{1}{r^2}f_+ + \frac{t}{2}(\Delta_r f_- - \frac{1}{r^2}f_-) &= f_+(f_+^2 - 1) + 2f_+f_-^2. \end{aligned} \tag{8.8}$$

When $t = 1$, this is exactly the system satisfied by the physical p-wave functions with the equivariant ansatz and $n = -1$. On the other hand, when $t = 0$ the system (8.8) partially decouples, and admits a solution of the form $f^0 = (f_-^0, f_+^0) = (f, 0)$, with $f(r)$ the radial degree-one Ginzburg–Landau vortex profile. We verify that f^0 gives a nondegenerate locally minimizing solution to the system (8.8) at $t = 0$, and the solutions for $t > 0$ are obtained via the Implicit Function Theorem. In section 8.4 we prove:

Theorem 8.3. *There exists t_0 such that for all $t \in (0, t_0)$ there exist smooth bounded solutions (f_-^t, f_+^t) of (8.8) such that:*

(a) $f_-^t(0) = 0 = f_+^t(0)$;

(b) $f_-^t(r) \to 1$, $f_+^t(r) \to 0$ *as* $r \to \infty$;

(c) $0 < f_-^t(r) < 1$, $f_+^t(r) < 0$ *for all* $r \in (0, \infty)$;

(d) As $r \to \infty$,

$$f_-^t = 1 - \frac{1}{2r^2} - \frac{5t^2+9}{8r^4} + O(r^{-6}), \qquad f_+^t = t\left[-\frac{1}{2r^2} - \frac{13}{4r^4} + O(r^{-6})\right].$$

Note that $0 > f_+(r) = -|\eta_+|$, and so the components of the equivariant solution incorporate a relative phase shift of π, in addition to having conjugate phases. The asymptotic estimate in (d) may be made uniform for $r \geq R$ and $t \in (0, t_0)$; see Theorem 8.10 for a more precise statement. We note that it is thanks to the uniform bounds on the asymptotic error that we may obtain the global control of the signs of the components in (c). Our result does not preclude the possiblity that one or both of $f_\pm^t$ vanishes or changes sign at some value of $t \in (0, 1]$. If this were to occur at some t, the solution $\eta_\pm = f_\pm^t e^{\pm i\theta}$ would still be a valid solution to the system of equations, but with a very unconventional profile for vortices. We conjecture that in fact (c) remains valid for all $t \in (0, 1]$, but again this question is open.

The methods employed in this paper extend various techniques used to study vortices in Ginzburg–Landau systems. In particular, the perturbation arguments rely on the extensive analysis of the linearization of the classical Ginzburg–Landau functional by Mironescu [98]. The asymptotic expansion follows the basic strategy followed in [9], based on [35]. The use of perturbative methods to study entire vortex solutions to the d-wave symmetric coupled Ginzburg–Landau system were also introduced by Kim & Phillips [76] and Han & Lin [58], although their approach was different from ours.

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8.2 Existence of minimizers

We begin with the existence of minimizers for the general functional

$$E(\eta) = \int_\Omega \left(e_{kin}(\eta) + \kappa^2 e_{pot}(\eta)\right)\, dx$$

with e_{kin} as in (8.2), e_{pot} as in (8.1), and with Dirichlet boundary condition (8.4). The existence of minimizers, even in a bounded domain $\Omega \subset \mathbb{R}^2$, is not obvious, since the kinetic energy is not coercive:

Proposition 8.4. *For any given $\eta_\pm \in H^1(\Omega)$, it holds that $e_{kin}(\eta) \geq 0$, with equality if and only if*

$$\eta_+ = c_+ + \alpha z, \quad \eta_- = c_- - \alpha \bar{z},$$

for some $c_+, c_-, \alpha \in \mathbb{C}$.

Proof. The kinetic energy may be rewritten as

$$\begin{aligned} e_{kin} = &\frac{1+\nu}{2}|\partial_x\eta_+ + \partial_x\eta_-|^2 + \frac{1+\nu}{2}|\partial_y\eta_+ - \partial_y\eta_-|^2 \\ &+ \frac{1-\nu}{2}|\partial_y\eta_+ + i\partial_x\eta_-|^2 + \frac{1-\nu}{2}|\partial_x\eta_+ + i\partial_y\eta_-|^2. \end{aligned}$$

In particular it is non-negative, and $e_{kin} = 0$ implies

$$\partial_x[\eta_+ + \eta_-] = 0, \quad \partial_y[\eta_+ - \eta_-] = 0, \quad \text{and } (\partial_x + i\partial_y)\eta_+ = 0.$$

Thus there exist one-dimensional distributions $u, v \in \mathcal{D}'(\mathbb{R})$ such that

$$\eta_+ = u(y) + v(x), \quad \eta_- = u(y) - v(x), \quad iu'(y) + v'(x) = 0.$$

Differentiating the last equation, we deduce that $u'' = v'' = 0$. Therefore u and v are affine functions with $u' = iv'$:

$$u = u_0 + i\alpha y, \quad v = v_0 + \alpha x, \quad \text{for some } \alpha \in \mathbb{C},$$

and we obtain the desired conclusion with $c_+ = u_0 + v_0$ and $c_- = u_0 - v_0$. □

As a consequence of Proposition 8.4, there is no hope for a general inequality of the form $\int e_{kin} \geq c\|\nabla\eta\|_{L^2}^2$ to be valid. However, we have the following:

Lemma 8.5. *Let Ω be an open subset of $\mathbb{R}^2$. Let $W \subset H^1(\Omega)^2$ be a closed affine subspace such that*

$$W \cap \{(c_+ + \alpha z, c_- - \alpha \bar{z}) \colon c_\pm, \alpha \in \mathbb{C}\} = \emptyset.$$

Then there exists $c > 0$ (depending on Ω and W) such that

$$\int_\Omega e_{kin}(\eta) \geq c\|\eta\|_{H^1}^2$$

for every $\eta \in W$.

Proof. We argue by contradiction. If the conclusion does not hold, then (using the homogeneity of the involved quantities) there exists a sequence $(\eta^k) \subset W$ such that

$$\|\eta^k\|_{H^1} = 1, \quad \int e_{kin}(\eta^k) \longrightarrow 0.$$

Up to considering a subsequence, and since W is weakly closed, we may assume that η^k converges H^1-weakly to $\eta \in W$. On the other hand, since the kinetic energy is convex (as a non-negative quadratic form), it holds

$$\int e_{kin}(\eta) \leq \liminf \int e_{kin}(\eta^k) = 0,$$

so that by Lemma 8.4, $\eta_\pm = c_\pm + \alpha(y \pm ix)$, thus contradicting the assumption on W. □

In particular, we may impose Dirichlet boundary conditions ensuring that the assumption of Lemma 8.5 is satisfied. For instance, the following result will allow us to construct – in Section 8.3 below – physically relevant 'radial vortex' solutions.

Proof of Theorem 8.1. The first assertion follows from Proposition 8.4 and Lemma 8.5. In the case $\Omega = B_R$, $g_\pm = \alpha_\pm e^{in_\pm\theta}$, it suffices to show that for any $c_\pm, \alpha \in \mathbb{C}$,

$$\eta_\pm = c_\pm \pm \alpha r e^{\pm i\theta} \notin W,$$

which follows from the uniqueness of Fourier decomposition on ∂B_R. □

8.3 Entire vortex solutions

In this section we study symmetric vortices, that is, solutions of the form

$$\eta_\pm(re^{i\theta}) = f_\pm(r)e^{in_\pm\theta}, \quad n_\pm \in \mathbb{Z},$$

where $f_\pm$ are real-valued functions. However, because of the coupling term in the kinetic energy, and in contrast with other coupled systems of Ginzburg-Landau equations [9], not all values of $n_\pm \in \mathbb{Z}$ are natural.

Indeed, the existence of such symmetric solutions is related to invariance properties of the energy. More specifically, for any $n_\pm \in \mathbb{Z}$, one may define an action of $\mathbb{S}^1$ on functions $\eta_\pm(z)$:

$$(\omega \cdot \eta_\pm)(z) = \omega^{n_\pm}\eta(\omega^{-1}z), \quad \omega \in \mathbb{S}^1.$$

A straightforward computation shows that

$$\begin{aligned} E(\eta) - E(\omega\cdot\eta) = &\int([1-\omega^{n_+-n_--2}]\Pi_-\eta_+)\cdot(\Pi_+\eta_-) \\ &+ \nu\int([1-\omega^{n_+-n_-+2}]\Pi_+\eta_+)\cdot(\Pi_-\eta_-) \\ &+ \kappa^2\nu\int([1-\omega^{2(n_+-n_-)}]\eta_+^2)\cdot(\eta_-^2). \end{aligned}$$

Hence we see that, in the case $\nu = 0$, the energy is invariant if and only if

$$n_+ = n_- + 2.$$

In the case $\nu \neq 0$, the energy can not be invariant, and the only invariance that can be expected is for the subgroup $\mathbb{U}_4 \subset \mathbb{S}^1$, which explains why vortices with square symmetry are predicted [65, 128].

In view of the above discussion, we consider from now on the case $\nu = 0$. Moreover, since we will be interested in solutions defined in the whole plane $\mathbb{R}^2$, the parameter κ can be scaled out, and we assume also $\kappa = 1$. In that case the Euler-Lagrange equations read

$$\begin{aligned} \Delta\eta_- + \frac{1}{2}\Pi_-^2\eta_+ &= \eta_-(|\eta_-|^2-1) + 2\eta_-|\eta_+|^2, \\ \Delta\eta_+ + \frac{1}{2}\Pi_+^2\eta_- &= \eta_+(|\eta_+|^2-1) + 2\eta_+|\eta_-|^2. \end{aligned} \tag{8.9}$$

in terms of $f_\pm$ defined by (8.11), and using the notation $\Delta_r f = r^{-1}(rf')' = f'' + r^{-1}f'$, the system (8.9) takes the form,

$$\begin{aligned}
&\Delta_r f_- - \frac{n^2}{r^2} f_- + \frac{1}{2}\left(\Delta_r f_+ + 2\frac{n+1}{r} f'_+ + \frac{n(n+2)}{r^2} f_+\right) \\
&\qquad = f_-(|f_-|^2 - 1) + 2f_- f_+^2, \\
&\Delta_r f_+ - \frac{(n+2)^2}{r^2} f_+ + \frac{1}{2}\left(\Delta_r f_- - 2\frac{n+1}{r} f'_- + \frac{n(n+2)}{r^2} f_-\right) \\
&\qquad = f_+(|f_+|^2 - 1) + 2f_+ f_-^2.
\end{aligned} \tag{8.10}$$

In the following we will show the existence of entire solutions of (8.10) with $n = -1$, that is equivariant solutions of the form

$$\eta_-(re^{i\theta}) = f_-(r)e^{-i\theta}, \qquad \eta_+(re^{i\theta}) = f_+(r)e^{+i\theta}, \tag{8.11}$$

where $f_\pm$ are real-valued functions. This is the choice of degrees made in [65], in the expectation that these solutions are the "most stable". In fact, the choice $n = -1$ simplifies the equations by eliminating a troublesome first order cross term in each equation. Existence of entire equivariant solutions for $n \neq -1$ remains an open problem.

With the choice $n = -1$, the kinetic energy becomes

$$e_{kin} = |f'|^2 + \frac{1}{r^2}|f|^2 + \left(f'_- + \frac{1}{r} f_-\right)\left(f'_+ + \frac{1}{r} f_+\right), \tag{8.12}$$

where $|f'|^2 = (f'_-)^2 + (f'_+)^2$ and $|f|^2 = f_-^2 + f_+^2$. Moreover, the system (8.9) reads

$$\begin{aligned}
&\Delta_r f_- - \frac{1}{r^2} f_- + \frac{1}{2}\left(\Delta_r f_+ - \frac{1}{r^2} f_+\right) = f_-(|f_-|^2 - 1) + 2f_- f_+^2, \\
&\Delta_r f_+ - \frac{1}{r^2} f_+ + \frac{1}{2}\left(\Delta_r f_- - \frac{1}{r^2} f_-\right) = f_+(|f_+|^2 - 1) + 2f_+ f_-^2.
\end{aligned} \tag{8.13}$$

Note that the continuity of $\eta_\pm$ forces $f_\pm$ to satisfy homogeneous boundary conditions at the origin:

$$f_-(0) = f_+(0) = 0. \tag{8.14}$$

In fact these conditions (8.14) are automatically satisfied by any bounded solutions of (8.13). As for boundary conditions at ∞ we impose, in agreement with (8.5),

$$\lim_{r\to\infty} (f_-, f_+) = (1, 0). \tag{8.15}$$

The strategy to obtain entire solutions of (8.13)-(8.15) is standard: we first obtain solutions in balls B_R by direct minimization, and then let $R \to \infty$. We denote by $\mathcal{H}_R$ the admissible energy space for vortex configurations in B_R:

$$\begin{aligned}
\mathcal{H}_R &= \left\{ \text{real-valued } (f_-, f_+) \colon \eta_\pm = f(r)e^{\pm i\theta} \in H^1(B_R) \right\} \\
&= \left\{ \text{real-valued } (f_-, f_+) \colon \int_0^R \left(|f'|^2 + \frac{1}{r^2}|f|^2\right) r dr < \infty \right\}.
\end{aligned} \tag{8.16}$$

We also denote by $\mathcal{H}_R^{bc}$ the vortex configurations in B_R, having the right boundary conditions at R, and by H_R^0 the admissible perturbations, i.e. with zero boundary conditions at R:

$$\mathcal{H}_R^{bc} = \{(f_-, f_+) \in \mathcal{H}_R \colon f_-(R) = 1,\ f_+(R) = 0\}\,, \tag{8.17}$$

$$\mathcal{H}_R^{0} = \{(\varphi_-, \varphi_+) \in \mathcal{H}_R \colon \varphi_-(R) = \varphi_+(R) = 0\}\,. \tag{8.18}$$

To obtain entire solutions of (8.13)-(8.15), we will need two kinds of *a priori* estimates on solutions in $\mathcal{H}_R$: an L^∞ bound, and a bound on the potential energy.

Lemma 8.6. *Let $f_\pm \in \mathcal{H}_R^{bc}$ solve* (8.13) *in* $(0, R)$*, with $f_-(R) = 1$, $f_+(R) = 0$. Then it holds*

$$2\int_0^R e_{pot}\, rdr \le 1. \tag{8.19}$$

and

$$f_-^2 + f_+^2 \le 3 \quad \text{in } (0, R). \tag{8.20}$$

If in addition we know that $f_- \ge 0$ and $f_+ \le 0$ in $(0, R)$, then we have

$$f_-^2 + f_+^2 \le 1 \quad \text{in } (0, R). \tag{8.21}$$

Proof of the L^∞ estimate (8.20) *and* (8.21)*:* We use the weak formulation of the system (8.13). That is, for any test functions $\varphi_\pm \in \mathcal{H}_R^0$, it holds

$$\begin{aligned}\int_0^R \Bigg\{ & f_-'\varphi_-' + \frac{1}{r^2} f_-\varphi_- + f_+'\varphi_+' + \frac{1}{r^2} f_+\varphi_+ \\ & + \frac{1}{2}(f_-' + \frac{1}{r} f_-)(\varphi_+' + \frac{1}{r}\varphi_+) + \frac{1}{2}(f_+' + \frac{1}{r} f_+)(\varphi_-' + \frac{1}{r}\varphi_-)\Bigg\} rdr \\ = -\frac{1}{2}\int_0^R & [De_{pot}(f)\cdot\varphi]\, rdr,\end{aligned} \tag{8.22}$$

where

$$\frac{1}{2} De_{pot}(f)\cdot\varphi = (2f_+^2 + f_-^2 - 1) f_-\varphi_- + (2f_-^2 + f_+^2 - 1) f_+\varphi_+. \tag{8.23}$$

We apply that weak formulation to test functions of the form $\varphi_\pm = f_\pm V$, where $V \ge 0$ will be chosen appropriately later on. We find

$$\begin{aligned}\int_0^R & \left\{ e_{kin}(f)V + \frac{1}{2}(f_-^2 + f_+^2 + f_+f_-)'V' + \frac{1}{r} f_+f_-V' \right\} rdr \\ & = -\frac{1}{2}\int_0^R V\,(De_{pot}(f)\cdot f)\, rdr.\end{aligned}$$

Integrating by parts, we rewrite that last equation as

$$\begin{aligned}\int_0^R & \left\{ \left(e_{kin} - \frac{1}{r}(f_-f_+)'\right) V + \frac{1}{2}(f_-^2 + f_+^2 + f_+f_-)'V' \right\} rdr \\ & = -\frac{1}{2}\int_0^R V\,(De_{pot}(f)\cdot f)\, rdr.\end{aligned} \tag{8.24}$$

Next we notice that

$$\begin{aligned} De_{pot}(f)\cdot f &= (f_-^2+f_+^2-1)(f_-^2+f_+^2)+2f_-^2f_+^2 \\ &\geq 0 \qquad \text{if } f_-^2+f_+^2\geq 1, \end{aligned} \tag{8.25}$$

and

$$\begin{aligned} e_{kin}-\frac{1}{r}(f_-f_+)' &= (f_-')^2+(f_+')^2+f_-'f_+'+\frac{1}{r^2}\left(f_-^2+f_+^2+f_-f_+\right) \\ &\geq 0. \end{aligned} \tag{8.26}$$

Now we can choose the function V. We define, for an arbitrary $M>0$,

$$U=\max(f_-^2+f_+^2+f_+f_--3/2,0) \quad \text{and } V=\min(U,M). \tag{8.27}$$

It is easy to check that $\varphi_\pm = f_\pm V \in \mathcal{H}_R$ are indeed admissible test functions (8.18). Plugging (8.27) into (8.24), and using the inequalities (8.25) and (8.26), we obtain

$$\int_0^R (V')^2 r dr \leq 0, \tag{8.28}$$

and therefore $V=0$ a.e. We deduce that

$$f_+^2+f_-^2+f_+f_-\leq 3/2,$$

which obviously implies the L^∞ estimate (8.20).

In case that $f_-\geq 0$ and $f_+\leq 0$ in $(0,R)$, let $W=f_-^2+f_+^2-1$. If W attains a positive maximum at $r\in(0,R)$, we easily compute

$$\begin{aligned} 0\geq \Delta_r W(r) &\geq 2f_-\Delta_r f_- + 2f_+\Delta_r f_+ \\ &= 2W(W+1)+4f_-^2f_+^2-f_-f_+(3f_-^2+3f_+^2-2)+\frac{2}{r^2}(f_-^2+f_+^2) \\ &\geq 2W(W+1)>0, \end{aligned}$$

thus proving (8.21). □

Proof of the potential energy estimate (8.19)*:* The potential energy estimate is classically proven using a Pohozaev identiy. The Pohozaev identity is obtained by multiplying the first line of (8.13) by $r^2 f_-'$ and the second line by $r^2 f_+'$, and adding them. The resulting equality can be rewritten as

$$\begin{aligned} &\left[r^2(f_-')^2+r^2(f_+')^2+r^2f_+'f_-'-f_-^2-f_+^2-f_-f_+\right]' \\ &\qquad = r^2\left[e_{pot}\right]' = \left[r^2 e_{pot}\right]' - 2r(e_{pot}). \end{aligned} \tag{8.29}$$

Integrating (8.29) from 0 to R and using the boundary conditions $f_\pm(0)=0$, $f_\pm(R)=(0,1)$, we obtain

$$2\int_0^R (e_{pot})\, rdr = 1-R^2\left[f_-'(R)^2+f_+'(R)^2+f_-(R)f_+(R)\right]\leq 1, \tag{8.30}$$

thus proving (8.19). □

With the *a priori* estimates of Lemma 8.6 at hand, we are ready to prove Theorem 8.2.

Proof of Theorem 8.2. We prove here the existence part of Theorem 8.2. The asymptotic expansion (8.6) is then a consequence of Theorem 8.10, which is proven in Section 8.5.

We proceed in three steps: first we show the existence of solutions in finite balls, then let the radii tend to $+\infty$ and obtain entire solutions of (8.13) , and eventually we show that those solutions satisfy the boundary conditions (8.15). The first two steps are fairly standard after the preliminary work in Section 8.2 and the uniform bound of Lemma 8.6. The last step classically relies on the potential energy bound of Lemma 8.6, but requires an extra argument that was not needed in previous related works (as e.g. [9]).

Step 1: Existence of solutions $f_\pm^R$ in $(0, R)$ with $f_\pm^R(R) = (0, 1)$.

By Lemma 8.5, the kinetic energy functional is coercive on the closed affine (real) subspace

$$\left\{\eta_\pm = f_\pm(r)e^{\pm i\theta} \colon f \in \mathcal{H}_R^{bc}\right\} \subset H^1(B_R)^2. \tag{8.31}$$

Therefore the direct method of the calculus of variation ensures the existence of a minimizer $\eta_\pm = f_\pm^R(r)e^{\pm i\theta}$. The functions $f_\pm$ solve (8.13) in (0,R). Moreover, $f_\pm \in \mathcal{H}_R^{bc}$ and Lemma 8.6 applies: it holds

$$|f|^2 \le 3, \quad 2\int_0^R e_{pot}(f)\, rdr \le 1.$$

Note that the L^∞ bound (8.20) ensures that $\Delta_r f_\pm \in L^\infty_{loc}$, and therefore by elliptic regularity $f_\pm$ are smooth.

Step 2: Taking the limit as $R \to \infty$.

We regard $f_\pm^R$ as being defined on $(0, \infty)$ by setting $f_\pm^R \equiv (0, 1)$ in (R, ∞). Thanks to the L^∞ bound $|f|^2 \le 3$, elliptic estimates ensure that $(f_\pm^R)'$ is uniformly bounded in any compact interval of $(0, \infty)$. Hence we may extract a converging subsequence

$$f_\pm^{R_n} \longrightarrow f_\pm \quad \text{locally uniformly in } (0, R).$$

It follows that $f_\pm$ are smooth bounded solutions of (8.13).

Step 3: Boundary conditions (8.15).

From the bound on the potential energy (8.19) and Fatou's lemma, we obtain that

$$\int_0^\infty e_{pot}\, rdr < \infty.$$

We claim that this finite energy property implies that $\lim_{r\to\infty} e_{pot} = 0$. To this end, remark that it holds $|f_\pm'(r)| \le C(1 + r)$, which is easily established using the uniform bound $|f_\pm| \le 3$ together with the differential system (8.13) satisfied by $f_\pm$. Now assume that there exists a subsequence $r_n \to \infty$ such that $e_{pot}(r_n) \ge \varepsilon > 0$. We may assume in addition that $r_{n+1} - r_n \ge 1$. From $|f_\pm| \le 3$ and $|f_\pm'| \le C(1+r)$ we obtain that $|e_{pot}'| \le C(1+r)$, and we deduce that there exists $\delta > 0$ such that $e_{pot} \geq \varepsilon/2$ on $(r_n - \delta/r_n, r_n + \delta/r_n)$. But this would imply

$$\int_0^\infty e_{pot}\, rdr \geq \frac{\varepsilon}{2}\sum_n \int_{r_n-\delta/r_n}^{r^n+\delta/r_n} rdr = \frac{\varepsilon}{2}\sum_n 2\delta = \infty,$$

which contradicts the finite energy property. Therefore it holds

$$\lim_{r\to\infty} e_{pot} = 0.$$

On the other hand, recall that $e_{pot} = 0$ exactly at the points $(0,1)$ and $(1,0)$. As a consequence, any converging subsequence $f_\pm(r_n)$ must converge to either $(0,1)$ or $(1,0)$.

In fact only one of these two points can be such a limit: if there exists sequences $f_\pm(r_n^1) \to (0,1)$ and $f_\pm(r_n^2) \to (1,0)$, then using the continuity of $f_\pm$ one easily constructs a sequence $r_n^3 \to \infty$ such that

$$\mathrm{dist}(f_\pm(r_n^3), \{(0,1),(1,0)\}) \geq 1/2.$$

But then one could extract a subsequence $f_\pm(r_{n'}^3) \to \ell_\pm \notin \{(0,1),(1,0)\}$, contradicting the fact that $\lim e_{pot} = 0$.

Therefore there is a unique possible limit for converging subsequences $f_\pm(r_n)$, and we conclude that the limit $f_\pm(\infty)$ exists and is either $(0,1)$ or $(1,0)$. Up to exchanging f_+ with f_- (the equations are symmetric), we have the right boundary conditions at ∞. □

Now that we have the existence of entire vortices, we would like to investigate qualitative properties of the radial profiles $f_\pm$. The first natural question is whether or not they have a sign. In the classical one-component Ginzburg-Landau setting [25], as in other two-component models [9], existence of the radial profile components with a sign follow from a simple energy argument: replacing f with $|f|$ or $-|f|$ does not increase the energy. In the present case however, this argument does not work, because of the coupling term in the kinetic energy.

If there do exist radial profiles with a sign, it is clear that f_- should be positive since $f_-(\infty) = 1$. On the other hand, due to the asymptotic expansion (8.6), f_+ should be negative. This is in agreement with numerical computations performed in [128]. In the next section we give arguments supporting the conjecture that $f_- \geq 0$ and $f_+ \leq 0$. We consider a perturbed model and prove the existence of vortices with such signs.

8.4 Vortex structure for a perturbed model

This section is devoted to proving Theorem 8.3. We start by presenting and proving the main tools needed in the proof.

8.4.1 Main ingredients

Recall that we consider the family of perturbed functionals (8.7) and we look for radial vortex solutions of the form

$$\eta_+ = f_+(r)e^{i\theta}, \quad \eta_- = f_-(r)e^{-i\theta}.$$

Then the energy (8.7) becomes

$$I_t(f;R) := \int_0^R \left(|f'|^2 + \frac{1}{r^2}|f|^2 + t(f'_- + \frac{1}{r}f_-)(f'_+ + \frac{1}{r}f_+) + e_{pot} \right) rdr, \quad (8.32)$$

where $|f'|^2 = (f'_-)^2 + (f'_+)^2$ and $|f|^2 = f_-^2 + f_+^2$, and the corresponding Euler-Lagrange equations are (8.8).

The solutions f^t are obtained by perturbation of a solution f^0 of (8.8) for $t = 0$, given by

$$f_-^0 = f, \quad f_+^0 \equiv 0,$$

where f is the classical Ginzburg-Landau radial vortex profile solving

$$\Delta_r f - \frac{1}{r^2} f = f(f^2 - 1), \qquad f(0) = 0, \quad f(\infty) = 1. \quad (8.33)$$

More specifically, the solution f^t will be of the form

$$f^t = f^0 + g^t, \qquad g_\pm^t(\infty) = 0.$$

Perturbed solutions will be obtained through the implicit function theorem, and to this end we need a stability result. The space of admissible perturbation is

$$\mathcal{H} = \left\{ \varphi_\pm \in H^1_{loc}(0,\infty) \colon \eta = \varphi_\pm(r) e^{\pm i\theta} \in H^1(\mathbb{R}^2) \right\}. \quad (8.34)$$

Although the entire solution f^0 does not have finite energy I_0 in $(0, \infty)$, it makes sense to consider variations with respect to compact perturbations: for $\varphi_\pm \in C_c^\infty(0,\infty)$, such that $\operatorname{supp} \varphi_\pm \subset (0, R_0)$, it holds

$$I_0(f^0 + \varphi; R_0) - I_0(f^0; R_0) = Q_0[\varphi] + o(\|\varphi\|^2_{\mathcal{H}}),$$

where

$$\begin{aligned} Q_0[\varphi] = &\int_0^\infty \left\{ (\varphi'_-)^2 + \frac{1}{r^2}\varphi_-^2 + (3f^2 - 1)\varphi_-^2 \right\} rdr \\ &+ \int_0^\infty \left\{ (\varphi'_+)^2 + \frac{1}{r^2}\varphi_+^2 + (2f^2 - 1)\varphi_+^2 \right\} rdr. \end{aligned} \quad (8.35)$$

Note that $Q_0[\varphi]$ is well-defined for any $\varphi \in \mathcal{H}$.

Lemma 8.7. *There exists $\delta > 0$ such that*

$$Q_0[\varphi] \geq \delta \|\varphi\|^2_{\mathcal{H}}, \quad (8.36)$$

for all $\varphi \in \mathcal{H}$.

Part of Lemma 8.7, namely the fact that Q_0 is non-negative, will be otained as a consequence of Mironescu's stability result [98] for the classical one-component Ginzburg-Landau equation. To obtain the positive definiteness we will need an extra argument.

With Lemma 8.7 at hand, we will be able to construct the map $t \mapsto f^t$ as in Theorem 8.3. The next step will be to obtain information on the sign of f_+^t for $t > 0$. This will be done mostly by examining the equation (8.38) satisfied by

$$h := \frac{d}{dt}\left[f_+^t\right]_{t=0}. \quad (8.37)$$

We will prove the following crucial result:

Lemma 8.8. *Let h be a smooth function in $[0,\infty)$, satisfying the boundary value problem*

$$\Delta_r h - \frac{1}{r^2}h = (2f^2-1)h + \frac{1}{2}f(1-f^2), \tag{8.38}$$

$$h(0)=0, \quad \lim_{r\to\infty} h(r) = 0. \tag{8.39}$$

Then $h<0$ in $(0,\infty)$. In addition, $h'(0)<0$.

We now present the proofs of Lemma 8.7 and Lemma 8.8.

Proof of Lemma 8.7: We first remark that it suffices to establish the weaker estimate

$$Q_0[\varphi] \geq \delta \|\varphi\|^2_{L^2(rdr)} \qquad \forall \varphi \in \mathcal{H}. \tag{8.40}$$

Assume indeed that (8.40) holds, but that (8.36) does not. Then there is a sequence φ_k such that $\|\varphi_k\|_{\mathcal{H}} = 1$ and $Q_0[\varphi_k] \to 0$. Using (8.40), it follows that $\|\varphi_k\|_{L^2(rdr)} \to 0$. Since f is uniformly bounded, this clearly implies that $Q_0[\varphi_k] = \|\varphi_k\|_{\mathcal{H}} + o(1)$, which is absurd.

In view of the decoupled expression of Q_0 (8.35), it is enough to show that, for every $\varphi \in H^1_{loc}(0,\infty;\mathbb{R})$ s.t. $\eta = \varphi(r)e^{i\theta} \in H^1(\mathbb{R}^2)$, it holds

$$\widetilde{Q}[\varphi] := \int_0^\infty \left\{ (\varphi')^2 + \frac{1}{r^2}\varphi^2 + (2f^2-1)\varphi^2 \right\} rdr \geq \delta \|\varphi\|^2_{L^2(rdr)}\,. \tag{8.41}$$

We appeal to Mironescu's stability result [98], which implies that, for any $\psi \in H^1(\mathbb{R}^2;\mathbb{C})$,

$$P[\psi] = \int_{\mathbb{R}^2} \left\{ |\nabla\psi|^2 + (f^2-1)|\psi|^2 + 2f^2(e^{i\theta}\cdot\psi)^2 \right\} \geq 0. \tag{8.42}$$

On the other hand, $\widetilde{Q}$ can be rewritten as

$$\widetilde{Q}[\varphi] = P[i\varphi(r)e^{i\theta}] + \int_0^\infty f^2\varphi^2\, rdr. \tag{8.43}$$

Of course the second term in the right-hand side of (8.43) is, by itself, not enough to make $\widetilde{Q}$ positive definite, since there exist sequences φ_k with $\|\varphi_k\|_{L^2} = 1$ and

$$\int_0^R f^2\varphi_k^2\, rdr \longrightarrow 0.$$

However, such sequences have their mass concentrated near zero, which makes the first term in the right-hand side of (8.43) large. In other words, the competition between the two terms in the right-hand side of (8.43) will ensure the positive definiteness of $\widetilde{Q}$.

Let us assume that (8.41) does not hold: there is a sequence φ_k such that

$$\|\varphi_k\|_{L^2(rdr)} = 1, \quad \widetilde{Q}[\varphi_k] \longrightarrow 0.$$

Since $\widetilde{Q}[\varphi_k]$ is bounded, the sequence $\eta_k = \varphi_k(r)e^{i\theta}$ is bounded in $H^1(\mathbb{R}^2)$ and therefore weakly compact: up to extracting a subsequence, η_k converges a.e., and strongly in L^2_{loc}. Hence there is a function $\varphi \in L^2(0,\infty)$ such that $\varphi_k \to \varphi$ a.e., and strongly in $L^2(0,1)$. Since, by (8.43) and (8.42),

$$\int f^2\varphi_k^2 rdr \le \widetilde{Q}[\varphi_k],$$

we deduce, using Fatou's lemma, that $\int f^2\varphi^2 rdr = 0$, and therefore $\varphi \equiv 0$. In particular, it holds

$$\int_0^1 \varphi_k^2\, rdr \longrightarrow 0,$$

from which we infer that

$$\begin{aligned}\widetilde{Q}[\varphi_k] &\ge \int_0^\infty f^2\varphi_k^2\, rdr = \int_1^\infty f^2\varphi^2\, rdr + o(1)\\ &\ge f(1)^2\int_1^\infty \varphi_k^2\, rdr + o(1) = f(1)^2 + o(1),\end{aligned}$$

contradicting the fact that $\widetilde{Q}[\varphi_k] \to 0$. □

Proof of Lemma 8.8: It is well known [69] that $f > 0$ in $(0,\infty)$. Hence we may write

$$h = fg$$

for some function g which is smooth in $(0,\infty)$ and continuous up to 0. In fact g is smooth up to 0, since $f(r) = r\tilde{f}(r)$ and $h(r) = r\tilde{h}(r)$ for some functions $\tilde{f}$ and $\tilde{h}$ which are smooth on $[0,\infty)$ and $\tilde{f}$ does not vanish on $[0,\infty)$.

The idea of decomposing h as $h = fg$ is reminiscent of Mironescu's method [99] to show the radial symmetry of entire vortices of degree one in the classical one-component Ginzburg-Landau framework.

Let us compute the differential equation satisfied by g. It holds

$$\begin{aligned}g' &= \left(\frac{h}{f}\right)' = \frac{h'}{f} - \frac{hf'}{f^2},\\ g'' &= \frac{h''}{f} - 2\frac{f'h'}{f^2} - \frac{hf''}{f^2} + 2\frac{h(f')^2}{f^3}.\end{aligned}$$

Therefore we find

$$\begin{aligned}f^2g'' &= h''f - hf'' - 2f'h' + 2g(f')^2\\ &= \left[(2f^2-1)h + \frac{1}{2}f(1-f^2) + \frac{1}{r^2}h - \frac{1}{r}h'\right]f\\ &\quad - \left[f(f^2-1) + \frac{1}{r^2}f - \frac{1}{r}f'\right]h - 2f'h' + 2g(f')^2\\ &= f^3h + \frac{1}{2}f^2(1-f^2) - \frac{1}{r}h'f + \frac{1}{r}f'h - 2f'h' + 2g(f')^2\\ &= f^4g + \frac{1}{2}f^2(1-f^2) - \frac{1}{r}f^2g' - 2f'(f'g + g'f) + 2(f')^2g\\ &= -\left(\frac{1}{r}f^2 + 2f'f\right)g' + f^4g + \frac{1}{2}f^2(1-f^2).\end{aligned}$$

Hence g satisfies the differential equation

$$g'' + \left(\frac{1}{r} + 2\frac{f'}{f}\right) g' = f^2 g + \frac{1-f^2}{2},$$

and the boundary condition

$$g(R) = 0.$$

Recall that it holds $0 < f < 1$ in $(0, \infty)$. Therefore the equation implies that g can not admit a positive maximum in $(0, \infty)$, and it holds

$$g \leq \max(0, g(0)).$$

Next we prove that $g(0) < 0$. To this end we show that $g'(0) = 0$ and $g''(0) > 0$. Therefore g is initially increasing. In particular, if we assume that $g(0) \geq 0$, then to match the boundary condition $g(\infty) = 0$, g would have to attain a positive maximum inside $(0, \infty)$ which is impossible.

To show that $g'(0) = 0$ and $g''(0) > 0$, we perform a Taylor expansion near zero: write

$$g = g_0 + g_1 r + \frac{g_2}{2} r^2 + O(r^3), \quad f = f_1 r + \frac{f_2}{2} r^2 + O(r^3),$$

so that

$$\begin{aligned} g' &= g_1 + g_2 r + O(r^2), \quad g'' = g_2 + O(r), \quad \frac{f'}{f} = \frac{1}{r} + \frac{f_2}{2f_1} + O(r), \\ \left(\frac{1}{r} + 2\frac{f'}{f}\right) g' &= \left(\frac{3}{r} + \frac{f_2}{f_1} + O(r)\right)(g_1 + g_2 r + O(r^2)) = \frac{3g_1}{r} + 3g_2 + g_1\frac{f_2}{f_1} + O(r) \\ g'' + \left(\frac{1}{r} + 2\frac{f'}{f}\right) g' - f^2 g - \frac{1-f^2}{2} &= \frac{3g_1}{r} + 4g_2 + g_1\frac{f_2}{f_1} - \frac{1}{2} + O(r). \end{aligned}$$

Hence it holds $g_1 = 0$ and $g_2 = 1/8 > 0$.

As explained above, it follows that $g(0) < 0$. In particular, $\max(0, g(0)) = 0$ and $g \leq 0$ in $[0, \infty)$. We claim that in fact this inequality is strict: it holds

$$g < 0 \quad \text{in } [0, \infty).$$

Assume indeed that $g(r_0) = 0$ for some $r_0 \in (0, \infty)$. Then r_0 is a point of maximum of g, so that $g''(r_0) \leq 0$. But on the other hand it holds $2g''(r_0) = 1 - f(r_0)^2 > 0$, so that we obtain a contradiction. We conclude that $g < 0$ in $[0, \infty)$ and therefore $h < 0$ in $(0, \infty)$. Moreover, $h'(0) = f'(0)g(0) < 0$. □

Also of use will be the fact that the space $\mathcal{H}$ is embedded into the space of continuous maps vanishing at zero and infinity.

Lemma 8.9. *It holds*

$$\mathcal{H} \subset \left\{\varphi \in [C(0,\infty)]^2 : \varphi(0) = 0, \lim_{r\to\infty} \varphi(r) = 0\right\},$$

and $\|\varphi\|_{L^\infty} \leq \|\varphi\|_{\mathcal{H}}$ *for all* $\varphi \in \mathcal{H}$.

Proof. Let $\varphi \in \mathcal{H}$. Then φ is absolutely continuous in $(0,\infty)$. So are $\varphi_\pm^2$, and $(\varphi_\pm^2)' = 2\varphi_\pm\varphi_\pm'$. For any $r_1 \geq r_2$ it holds

$$\begin{aligned}\left|\varphi_\pm(r_1)^2 - \varphi_\pm(r_2)^2\right| &\leq 2\int_{r_1}^{r_2} |\varphi_\pm|\left|\varphi_\pm'\right| dr \\ &\leq \int_{r_1}^{r_2}\left[\frac{\varphi_\pm^2}{r^2} + (\varphi_\pm')^2\right] rdr,\end{aligned}$$

so that $\varphi_\pm^2$ is Cauchy at 0 and ∞. Obviously the corresponding limits must be zero. The estimate on the supremum norm follows by choosing $r_1 = 0$ in the inequality above. □

Finally, we require an asymptotic expansion of solutions which is uniform in the parameter t. The following result is proven in section 8.5:

Theorem 8.10. *Let $[f_{t,-}, f_{t,+}]$ be solutions of (8.8), and assume that for every $\delta > 0$ there exists $R_0 > 0$ and $0 \leq T_1 \leq T_2 \leq 1$ such that for every $R > R_0$ and $t \in [T_1, T_2]$,*

$$|f_+^t(r)| \leq t\delta, \qquad |f_-^t(r) - 1| \leq \delta, \tag{8.44}$$

for all $r \geq R$. Then we have

$$f_{t,-} = 1 - \frac{1}{2r^2} - \frac{5t^2+9}{8r^4} + O(r^{-6}), \qquad f_{t,+} = t\left[-\frac{1}{2r^2} - \frac{13}{4r^4} + O(r^{-6})\right], \tag{8.45}$$

as $r \to \infty$. More precisely, there exist positive constants $C_\pm, C_\pm', R > 0$ such that

$$\left|f_{t,-}(r) - \left(1 - \frac{1}{2r^2} - \frac{5t^2+9}{8r^4}\right)\right| \leq \frac{C_-}{r^6}, \tag{8.46}$$

$$\left|f_{t,+}(r) + t\left[\frac{1}{2r^2} + \frac{13}{4r^4}\right]\right| \leq t\frac{C_+}{r^6} \tag{8.47}$$

$$\left|f_{t,-}'(r) + \frac{1}{r^3}\right| \leq \frac{C_-'}{r^5}, \tag{8.48}$$

$$\left|f_{t,+}'(r) + \frac{t}{r^3}\right| \leq t\frac{C_+'}{r^5}, \tag{8.49}$$

hold for all $r \geq R$ and all $t \in [T_1, T_2]$.

8.4.2 Proof of Theorem 8.3

Step 1: Construction of the family $t \mapsto f^t$.
We denote by $\mathcal{N}_t(f)$ the quasilinear differential operator such that

$$\langle DI_t(f;R), \varphi\rangle_{(H_R^0)^*, \mathcal{H}_R^0} = \langle \mathcal{N}_t(f), \varphi\rangle_{L^2(rdr)}.$$

for $\varphi \in C_c^\infty(0,R)$. In other words, the system (8.8) is exactly $\mathcal{N}_t(f) = 0$.
Using the fact that $\mathcal{N}_0(f^0) = 0$, one may check that

$$\mathcal{N}_t(f^0 + g) \in \mathcal{H}^* \qquad \forall g \in \mathcal{H}.$$

Moreover, the map

$$\mathcal{F}\colon (-1,1)\times\mathcal{H}\to\mathcal{H}^*,\quad (t,g)\mapsto \mathcal{N}_t(f^0+g),$$

is smooth. Since

$$\langle D_g\mathcal{F}(0,0)\varphi,\varphi\rangle_{\mathcal{H}^*,\mathcal{H}} = Q_0[\varphi],$$

Lemma 8.7 and Lax-Milgram theorem imply that $D_g\mathcal{F}(0,f^0)$ is invertible. Applying the implicit function theorem, we find that there exists $t_0>0$, $\delta_0>0$ and a smooth map

$$(-t_0,t_0)\ni t\mapsto g^t\in\mathcal{H},\quad g^0=0,$$

such that, for $|t|<t_0$ and $\|g\|_{\mathcal{H}}<\delta_0$,

$$\mathcal{F}(t,g)=0 \quad\Longleftrightarrow\quad g=g^t. \tag{8.50}$$

In particular, $f^t=f^0+g^t$ solves (8.8). Elliptic regularity ensures that for every t, f^t is a smooth function.

Step 2: The map $t\mapsto f^t\in C^k([0,R])$ is smooth, for any integer k and $R>0$.

In fact we consider spaces of differentiable functions which are more appropriate to our problem: let

$$\widetilde{C}^k(0,R)=\left\{f_\pm\in C^{k,\alpha}(0,R)\colon \eta_\pm=f_\pm(r)e^{\pm i\theta}\in C^{k,\alpha}(\overline{B}_R)\right\}.$$

Let $t_1\in(-t_0,t_0)$. Since (by Lemma 8.9) $\mathcal{H}$ is embedded in a space of continuous functions, the map $t\mapsto g^t(R)$ is smooth, and we may fix a smooth map $t\mapsto\psi^t\in\widetilde{C}^{k+2}(0,R)$ such that

$$\psi^0\equiv 0,\quad (\psi^t+g^{t_1})(R)=g^{t_1+t}(R).$$

Next we consider the smooth map

$$\widetilde{\mathcal{F}}\colon(-\varepsilon,\varepsilon)\times\widetilde{C}^{k+2}(0,R)\to\widetilde{C}^k(0,R),\quad (t,g)\mapsto\mathcal{N}_{t_1+t}(f^{t_1}+\psi^t+g).$$

The small constants t_0 and δ_0 in Step 1 may be chosen so that

$$\langle D_g\mathcal{F}(t,g)\varphi,\varphi\rangle_{\mathcal{H}^*,\mathcal{H}}\geq c\,\|\varphi\|^2_{\mathcal{H}}\,,\qquad |t|<t_0,\ \|g\|<\delta_0,$$

for some $c>0$. It is then easy to check, using elliptic regularity, that $D_g\widetilde{\mathcal{F}}(0,0)$ is invertible. Therefore the implicit function theorem provides us with a smooth family $t\mapsto\tilde{g}^t\in\widetilde{C}^{k+2}(0,R)$ defined for small t and solving

$$\mathcal{N}_{t_1+t}(f^{t_1}+\psi^t+\tilde{g}^t)=0.$$

For small enough t, the function

$$\hat{g}^t=\begin{cases} g^{t_1}+\psi^t+\tilde{g}^t & \text{in } (0,R),\\ g^{t_1+t} & \text{in } (R,\infty),\end{cases}$$

satisfies $\|\hat{g}^t\|_{\mathcal{H}}<\delta_0$. Moreover, it holds $\mathcal{F}(t_1+t,\hat{g}^t)=0$, so that by (8.50) we deduce that $\hat{g}^t=g^{t_1+t}$. In particular, the map $t\mapsto g^t\in\widetilde{C}^{k+2}(0,R)$ is smooth.

Step 3: It holds $f_+^t < 0$ and $0 < f_-^t < 1$ in $(0,\infty)$ for small enough t.

Let $\phi^t = \frac{\partial}{\partial t} f^t$. By Step 2, the map $t \mapsto \phi^t$ is smooth in $\mathcal{H} \cap C^k_{loc}$ for each k, and hence ϕ^t solves the system obtained by differentiating the equations (8.53) with respect to t. As ϕ^t is continuous at $t = 0$, a computation reveals that $\phi^0 = (\phi^0_-, \phi^0_+) \in \mathcal{H} \cap C^k_{loc}$, with $\phi^0_+ = h$, the solution of (8.38) and ϕ^0_- solving the linearized radial Ginzburg-Landau equation, $\Delta_r \phi^0_- - \frac{1}{r^2}\phi^0_- = \phi^0_-(3f^2 - 1)$, and thus, $\phi^0_- = 0$. As the map $t \mapsto f^t$ is smooth in $\mathcal{H} \cap C^k_{loc}$, it follows that $f^t = f^0 + t\phi^0 + O(t^2)$, with error term uniform in supremum norm on $[0,\infty)$, by Lemma 8.9. Since $\phi^0_\pm(r) \to 0$ as $r \to \infty$, for any $\delta > 0$ we may find $R_0 > 0$ such that $|\phi^0_\pm(r)| < \frac{\delta}{2}$ for all $r \geq R_0$. By the Taylor expansion of f^t we may then conclude that for any $R \geq R_0$, there exists $T > 0$ for which

$$|f_+^t(r)| \leq t\delta, \text{ and } |f_-^t(r) - 1| \leq \delta,$$

for all $r \geq R$ and $t \in [0,T]$.

The solutions f^t thus satisfy the hypotheses of Theorem 8.10, therefore we may choose $R > 0$ such that for all $t \in (0,T]$,

$$f_+^t < 0 \quad \text{and} \quad 0 < f_-^t < 1 \qquad \text{in } [R,\infty).$$

Thus it only remains to show that $f_+^t < 0$ and $0 < f_-^t < 1$ in $(0,R)$ for small enough t.

It is well-known [69] that $(f_-^0)' = f' \geq c > 0$ in $(0,R)$, so that Step 2 ensures that $(f_-^t)' > 0$ in $(0,R)$ for small enough t, and we deduce that $0 < f_-^t < 1$ in $(0,R)$.

Next we show that $f_+^t < 0$. We recall that $h = \frac{\partial}{\partial t}\left[f_+^t\right]_{t=0}$ solves (8.38). In view of Lemma 8.9, h is bounded and satisfies $h(0) = h(\infty) = 0$. Elliptic regularity ensures that h is smooth in $[0,\infty)$, and we may apply Lemma 8.8. Thus it holds $h < 0$ in $(0,\infty)$, and $h'(0) < 0$. There exists $r_0 > 0$ and $\eta > 0$ such that

$$h'(r) \leq -\eta \text{ in } [0,r_0], \qquad h(r) \leq -\eta \text{ in } [r_0,R].$$

Using Step 2, we infer that for all small enough t,

$$\frac{\partial}{\partial t}[(f_+^t)'] \leq -\eta/2 < 0 \text{ in } [0,r_0], \qquad \frac{\partial}{\partial t}[f_+^t] \leq -\eta/2 \text{ in } [r_0,R],$$

which obviously implies, since $f_+^0 \equiv 0$, that $f_+^t < 0$ in $(0,R]$. □

8.5 Asymptotics

We derive the asymptotic behavior of solutions $f_\pm(r)$ as $r \to \infty$ by means of the sub- and super-solutions method. We recall the notation for the Laplacian of radial functions in $\mathbb{R}^2$, $\Delta_r u(r) := \frac{1}{r}(r\, u'(r))'$. This we accomplish thanks to the following comparison lemma, which is an adaptation of Lemma 3.1 in [9]:

Lemma 8.11. *Let $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ be bounded functions on $[R,\infty)$, with $\mathbb{A}, \mathbb{D} > 0$, $\mathbb{B}, \mathbb{C} \leq 0$, and such that the quadratic form defined by $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ satisfies the bound*

$$\mathbb{A}(r)x^2 + (\mathbb{B}(r) + \mathbb{C}(r))xy + \mathbb{D}(r)y^2 \geq \delta(x^2 + y^2), \tag{8.51}$$

for all $r \in [R, \infty)$, $(x, y) \in \mathbb{R}^2$, and constant $\delta > \frac{1}{2R^2}$. Then, if u, v satisfy:

$$\begin{cases} -\Delta_r u + \dfrac{1}{r^2} u + \mathbb{A} u + \mathbb{B} v \le 0, \\ -\Delta_r v + \dfrac{1}{r^2} v + \mathbb{C} u + \mathbb{D} v \le 0, \end{cases}$$

for $r \in (R, \infty)$, with

$$u(R) \le 0, \quad v(R) \le 0, \qquad u(r), v(r) \to 0 \quad \text{as } r \to \infty.$$

we have that $u \le 0$ and $v \le 0$ in $[R, \infty)$.

Proof. Let $u^\pm = \max(\pm u, 0)$ and $v^\pm = \max(\pm v, 0)$, the positive and negative parts of each component. We set $\eta_R(r) = e^{-(r-R)/R}$, $r \in [R, \infty)$, multiply the first equation by $u^+ \eta_R$ and the second equation by $v^+ \eta_R$, integrate over $[R, \infty)$, and add the two resulting inequalities, to obtain:

$$\int_R^\infty \left\{ -u^+ \Delta_r u - v^+ \Delta_r v + \frac{(u^+)^2 + (v^+)^2}{r^2} + \right. \\ \left. \mathbb{A}(u^+)^2 + \mathbb{B} u^+ v + \mathbb{C} v^+ u + \mathbb{D}(v^+)^2 \right\} \eta_R \, r \, dr \le 0. \quad (8.52)$$

Applying (8.51), all but the first two terms in (8.52) may be bounded as follows:

$$\begin{aligned} &\int_R^\infty \left[\frac{(u^+)^2 + (v^+)^2}{r^2} + \mathbb{A}(u^+)^2 + \mathbb{B} u^+ v + \mathbb{C} v^+ u + \mathbb{D}(v^+)^2 \right] \eta_R \, r \, dr \\ &\quad \ge \int_R^\infty \left[\mathbb{A}(u^+)^2 + \mathbb{B}(u^+ v^+ - u^+ v^-) + \mathbb{C}(v^+ u^+ - v^+ u^-) + \mathbb{D}(v^+)^2 \right] \eta_R \, r \, dr \\ &\quad \ge \int_R^\infty \left[\mathbb{A}(u^+)^2 + (\mathbb{B} + \mathbb{C}) v^+ u^+ + \mathbb{D}(v^+)^2 \right] \eta_R \, r \, dr \\ &\quad \ge \delta \int_R^\infty \left[(u^+)^2 + (v^+)^2 \right] \eta_R \, r \, dr. \end{aligned}$$

Integrating the first term by parts, using the hypothesis $u(R) \le 0$ and the explicit form of η_R, we obtain:

$$\begin{aligned} -\int_R^\infty \eta_R \, u^+ \Delta_r u \, r \, dr &= u^+(R) \eta_R(R) R u'(R) + \int_R^\infty \left\{ \eta_R [(u^+)']^2 + \frac{1}{R} \eta_R u^+ \, [u^+]' \right\} r \, dr \\ &\ge \frac{1}{2} \int_R^\infty \eta_R [(u^+)']^2 \, r \, dr - \frac{1}{2R^2} \int_R^\infty \eta_R \, [u^+]^2 \, r \, dr. \end{aligned}$$

An analogous computation may be made for the second term (involving v^+), and inserting in (8.52) we conclude that

$$0 \ge \int_R^\infty \left\{ ([u^+]')^2 + ([v^+]')^2 + \left(\delta - \frac{1}{2R^2} \right) ([u^+]^2 + [v^+]^2) \right\} \eta_R \, r \, dr.$$

As $\eta_R > 0$ on $[R, \infty)$, we conclude that $u^+, v^+ \equiv 0$ on $[R, \infty)$, and the lemma is proven. $\square$

The proof of the asymptotic formulae in Theorem 8.10 thus relies on the construction of appropriate sub- and super-solutions for the system (8.8). By taking linear combinations of the equations (8.8) we can rewrite the system in 'diagonalized' form:

$$\begin{aligned}(1-\frac{t^2}{4})\left(\Delta_r f_- - \frac{1}{r^2}f_-\right) &= f_-(2f_+^2+f_-^2-1) - \frac{t}{2}f_+(2f_-^2+f_+^2-1),\\ (1-\frac{t^2}{4})\left(\Delta_r f_+ - \frac{1}{r^2}f_+\right) &= f_+(2f_-^2+f_+^2-1) - \frac{t}{2}f_-(2f_+^2+f_-^2-1),\end{aligned} \tag{8.53}$$

which will be more convenient to work with during the proof of Theorem 8.10.

Proof of Theorem 8.10. For simplicity of notation, we denote $f_\pm^t = f_\pm$ in the proof, suppressing the dependence on t. We also denote by $\tau := 1 - \frac{t^2}{4} \in [\frac{3}{4}, 1)$.

Step 1: Construction of subsolution/supersolution pairs. We begin with supersolutions. Let

$$w_+ = t\left[\frac{a_+}{r^2} + \frac{b_+}{r^4} + c_+\frac{R^6}{r^6}\right], \tag{8.54}$$

$$w_- = 1 + \frac{a_-}{r^2} + \frac{b_-}{r^4} + c_-\frac{R^6}{r^6}, \tag{8.55}$$

where $a_\pm, b_\pm, c_\pm$ and R are to be chosen so that

$$E_- := [-\tau\Delta_R w_- + \frac{w_-}{r^2}] + w_-(2w_+^2 + w_-^2 - 1) - \frac{t}{2}w_+(2w_-^2 + w_+^2 - 1) \geq 0, \tag{8.56}$$

$$E_+ := [-\tau\Delta_R w_+ + \frac{w_+}{r^2}] + w_+(2w_-^2 + w_+^2 - 1) - \frac{t}{2}w_-(2w_+^2 + w_-^2 - 1) \geq 0, \tag{8.57}$$

for all $r \geq R$, and

$$w_-(R) \geq f_-(R), \qquad w_+(R) \geq f_+(R). \tag{8.58}$$

Expanding (8.57) and (8.56) yields terms which are polynomials in even powers of r^{-1}, of the form:

$$E_+ = t\sum_{k=1}^{9} M_{2k}^+\frac{1}{r^{2k}}, \qquad E_- = \sum_{k=1}^{9} M_{2k}^-\frac{1}{r^{2k}},$$

where $M_{2k}^\pm = M_{2k}^\pm(t, R, a_\pm, b_\pm, c_\pm)$ is a polynomial in each of its arguments. The expansion is quite horrific, but may be explicitly evaluated with the help of a symbolic algebra program such as Maple. First, we choose $a_\pm$ in order to force the lowest order coefficients $M_2^\pm$ to vanish: indeed, the expansion yields

$$M_2^- = 2a_- - \frac{t^2}{2}a_+ + \tau = 0, \qquad M_2^+ = -a_- + a_+ = 0,$$

which gives the coefficients of r^{-2}, $a_- = -\frac{1}{2} = a_+$, as in (8.45).

Similarly, we fix the values of the coefficients $b_\pm$ in order that the r^{-4} terms vanish,

$$M_4^- = 2b_- - \frac{t^2}{2}b_+ - 3\tau a_- + 3a_-^2 - 2t^2 a_+ a_- + 2t^2 a_+^2 = 0,$$
$$M_4^+ = b_+ - b_- - 3\tau a_+ - \frac{3}{2}a_-^2 - t^2 a_+^2 + 4a_+ a_- = 0.$$

Thus, $b_- = -\frac{5t^2+9}{8}$, $b_+ = -\frac{13}{4}$ are the coefficients of r^{-4} given in the expansion (8.45).

The values of $a_\pm, b_\pm$ may then be substituted into the expansions of (8.57) and (8.56), and the expressions for $M_{2k}^\pm$ may be viewed as functions of R. The exact form of the coefficients $M_{2k}^\pm$ is very complex, but they are all polynomials in R, t, and $c_\pm$. As we will choose R large, we are only interested in the leading order of each. We obtain:

$$M_6^+ = (-c_- + c_+)R^6 + O(1), \qquad M_6^- = \left(2c_- - \frac{t^2}{2}c_+\right)R^6 + O(1),$$
$$M_8^\pm = O(R^6), \qquad M_{10}^\pm = O(R^6),$$
$$M_{12}^+ = \left(4c_+c_- - \frac{t^2}{2}(2c_+^2 + 3c_-^2)\right)R^{12} + O(R^6),$$
$$M_{12}^- = (-2t^2 c_+ c_- + 2t^2 c_+^2 + 3c_-^2)R^{12} + O(R^6),$$
$$M_{14}^\pm = O(R^{12}), \qquad M_{16}^\pm = O(R^{12}),$$
$$M_{18}^+ = \left(c_+^3 t^2 - c_- c_+^2 t^2 + 2c_+ c_-^2 - \frac{1}{2}c_-^3\right)R^{18},$$
$$M_{18}^- = \left(c_-^3 + 2c_+^2 c_- t^2 - c_+ c_-^2 t^2 - \frac{1}{2}c_+^3 t^4\right)R^{18}.$$

In each expression, the lower terms are uniformly bounded for $t \in [0,1]$.

Let $c_- = \delta$ and $c_+ = 2\delta$, with $\delta > 0$ to be chosen later. With this definition,

$$M_6^+ = \delta R^6 + O(1), \qquad M_6^- = \delta(2 - t^2)R^6 + O(1),$$

where the remainder terms are uniformly bounded for $t \in [0,1]$. As $M_6^\pm$ are the leading order terms in r, this will ensure that we obtain the correct sign in each equation, and the value of δ will be fixed in order that the r^{-6} terms indeed dominate the others in the expansion. By choosing $R_1 = R_1(\delta)$ sufficiently large, we may then ensure that when $R \geq R_1$,

$$\left|\frac{M_8^\pm}{r^8} + \frac{M_{10}^\pm}{r^{10}} + \frac{M_{14}^\pm}{r^{14}} + \frac{M_{16}^\pm}{r^{16}}\right| \leq C\frac{R^6}{r^8} < \frac{\delta}{4}\frac{R^6}{r^6}, \tag{8.59}$$

for all $r \in [R, \infty)$, with constant C chosen independent of $t \in [0,1]$. Next, with our choice of $c_\pm$, we have

$$|M_{12}^\pm| \leq 7\delta^2 R^{12} + O(R^6) \leq 8\delta^2 R^{12},$$

for all $R \geq R_1$, making R_1 larger if necessary. Hence we may fix δ with $0 < \delta < \frac{1}{32}$, we have:

$$\left|\frac{M_{12}^\pm}{r^{12}}\right| \leq \frac{8\delta^2 R^{12}}{r^{12}} < \frac{\delta}{4}\frac{R^6}{r^6},$$

holds for all $r \in [R,\infty)$ with $R \geq R_1$. Finally, we note that

$$M_{18}^{+} = \left(\frac{7}{2} + 4t^2\right)\delta^3, \qquad M_{18}^{-} = (1 + 6t^2 - 4t^4)\delta^3,$$

and for $t \in [0,1]$ each has the same sign as δ, and thus these terms contribute with the desired sign in the evaluation of (8.57), (8.56), and may be neglected.

Putting these estimates together, it follows that for all $R \geq R_1$,

$$E_{\pm} \geq M_6^{\pm} r^{-6} - \left|\sum_{k=4}^{8} M_{2k}^{\pm} r^{-2k}\right| \geq M_6^{\pm} r^{-6} - \frac{\delta}{2}\frac{R^6}{r^6} > \frac{\delta}{4}\frac{R^6}{r^6} > 0,$$

for all $r \in [R,\infty)$, and uniformly in $t \in (0,1]$. Thus, (w_-, w_+) indeed satisfy the supersolution conditions (8.56) and (8.57) for $R \geq R_1$, as desired.

It remains to consider the behavior at the endpoint, $r = R$. Since

$$w_-(R) = 1 + \delta + O(R^{-2}), \quad w_+(r) = 2t\delta + O(R^{-2}),$$

with $0 < \delta < \frac{1}{32}$, by the hypothesis (8.44) we may fix $R \geq R_1$ such that $f_-(R) \leq w_-(R)$ and $f_+(R) \leq w_+(R)$ holds for all $t \in [T_1, T_2]$. Thus (8.58) holds as well, and we have completed the construction of supersolutions.

We also require a subsolution pair, (z_-, z_+) for which $E_+ \leq 0$ and $E_- \leq 0$ for all $r \in [R,\infty)$ and $z_-(R) \leq f_-(R)$, $z_+(R) \leq f_+(R)$, for R sufficiently large proceeds in exactly the same way as for the supersolution pair above, except the coefficients $\frac{1}{2}c_+ = c_- = -\delta < 0$. This completes Step 1 in the proof.

Step 2: We apply the comparison Lemma 8.11 to the pair $(h_-, h_+) = (f_- - w_-, f_+ - w_+)$. Denote by $Lu := -\Delta_r u + r^{-2}u$ Then, an explicit calculation together with the construction of Step 1 shows that, for any sufficiently large R,

$$\begin{cases} Lh_- + \mathbb{A}h_- + \mathbb{B}h_+ \leq 0 \\ Lh_+ + \mathbb{C}h_- + \mathbb{D}h_+ \leq 0 \end{cases} \tag{8.60}$$

for $r \in [R,\infty)$, with $h_{\pm}(R) \leq 0$. The coefficients are functions of r, but have uniform limits as $r \to \infty$,

$$\begin{aligned}
\mathbb{A} &= f_-^2 + f_- w_- + w_-^2 + 2f_+^2 - 1 - \frac{t}{2}(2w_+(f_- + w_-)) \longrightarrow 2, \\
\mathbb{B} &= 2w_-(f_+ + w_+) - \frac{t}{2}(f_+^2 + f_+ w_+ + w_+^2 + 2f_-^2 - 1) \longrightarrow -\frac{t}{2}, \\
\mathbb{C} &= 2w_+(f_- + w_-) - \frac{t}{2}(f_-^2 + f_- w_- + w_-^2 + 2f_+^2 - 1) \longrightarrow -t, \\
\mathbb{D} &= f_+^2 + f_+ w_+ + w_+^2 + 2f_-^2 - 1 - \frac{t}{2}(2w_-(f_+ + w_+)) \longrightarrow 1.
\end{aligned}$$

Thus (taking R larger if necessary) we may assume the positivity condition (8.51) is satisfied in $[R,\infty)$ with $\delta = \frac{3}{4}$, for example. Lemma 8.11 applies, and we conclude that $h_{\pm}(r) \leq 0$ on $[R,\infty)$, that is $f_{\pm}(r) \leq w_{\pm}(r)$.

Taking $(h_-, h_+) = (z_- - f_-, z_+ - f_+)$, with (z_-, z_+) the subsolution pair, we may repeat the above computations to arrive at the same system (8.60), with coefficients $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ satisfying the same asymptotic conditions as above. Thus, by Lemma 8.11 we may also conclude that $f_\pm(r) \geq z_\pm(r)$ for all $r \in [R, \infty)$, for R sufficiently large. Using the explicit form of $z_\pm, w_\pm$ from Step 1, we may conclude that the estimates (8.46) and (8.47) both hold.

Step 3: The derivative estimates. Here we follow the method of Chen, Elliot, and Qi [35]. Let

$$g_-(r) = f_-(r) - \left(1 + \frac{a_-}{r^2} + \frac{b_-}{r^4}\right) = f_-(r) - w_-(r) + \frac{c_-}{r^6},$$
$$g_+(r) = f_+(r) - \left(\frac{a_+}{r^2} + \frac{b_+}{r^4}\right) = f_+(r) - w_+(r) + \frac{c_+}{r^6},$$

where $a_\pm, b_\pm, c_\pm$ are as in Step 1. By Step 2, we thus know that $g_\pm(r) = O(r^{-6})$. A calculation then yields

$$\Delta_r g_- = \frac{g_-}{r^2} + \mathbb{A} g_- + \mathbb{B} g_+ + O(r^{-6}) = O(r^{-6}),$$
$$\Delta_r g_+ = \frac{g_+}{r^2} + \mathbb{C} g_- + \mathbb{D} g_+ + O(r^{-6}) = O(r^{-6}),$$

uniformly for $t \in [T_1, T_2]$, with $\mathbb{A}, \mathbb{B}, \mathbb{C}, \mathbb{D}$ as in Step 2.

For each $k \in \mathbb{R}$ there exists $r_k \in (k, 2k)$ such that

$$g'_\pm(r_k) = \frac{g_\pm(2k) - g_\pm(k)}{k} = O(k^{-7}) = O(r_k^{-7}).$$

Integrating the estimate on $\Delta_r g_\pm$, we have, for all $r \geq R$,

$$|r g'_\pm(r) - r_k g'_\pm(r_k)| = \left| \int_r^{r_k} \Delta_r g_\pm(r)\, r\, dr \right| \leq C \int_r^{r_k} r^{-5} dr \leq \frac{4C}{r^4},$$

with constant $C > 0$. We now let $k \to \infty$, and use $r_k g'_\pm(r_k) \to 0$, to obtain $|r g'_\pm(r)| \leq \frac{4C}{r^4}$, and hence $|f'_\pm(r) + \frac{2a_\pm}{r^3}| \leq \frac{C'}{r^5}$, which gives (8.48), (8.49). □

Part III

Espaces de Besov et noyaux de convolution peu réguliers

Chapter 9

Caractérisation d'espaces de fonctions via convolution par des noyaux peu réguliers

(avec Petru Mironescu)

9.1 Introduction

The smoothness of a function $f : \mathbb{R}^n \to \mathbb{R}$ can be measured by different decay properties, for example via the decay properties of its harmonic extension, or the ones of its Littlewood-Paley decomposition, or the ones of its coefficients in an appropriate wavelets frame. See [139, Chapter 2] for a thorough discussion on this subject. Another characterization is related to the rate of convergence of $f * \rho_\varepsilon$ to f, where ρ is an appropriate mollifier. For example, for non integer $s > 0$ and $1 \leq p < \infty$ we have

$$\|f\|_{W^{s,p}}^p \sim \|f\|_{L^p}^p + \int_0^1 \frac{1}{\varepsilon^{sp+1}} \|f - f * \rho_\varepsilon\|_{L^p}^p \, d\varepsilon, \text{ where } \rho_\varepsilon(x) = \frac{1}{\varepsilon^n} \rho\left(\frac{x}{\varepsilon}\right), \quad (9.1)$$

provided

$$\rho \in \mathcal{S} \text{ and } \int \rho = 1. \quad (9.2)$$

Here $\mathcal{S}$ denotes the Schwartz class of smooth, rapidly decreasing functions.

We address here the question of the validity of (9.1) under assumptions as weak as possible on ρ. This is a « continuous » (vs « discrete ») counterpart of the analysis of Bourdaud [27] concerning the minimal assumptions required on the (father and mother) wavelets appropriate for the characterization of Besov spaces.

Usually, the assumption $\rho \in \mathcal{S}$ is weakened as follows. First, validity of (9.1) is established for some $\widetilde{\rho} \in \mathcal{S}$. Next, one expresses an arbitrary ρ in the form

$$\rho = \sum_{j \geq 0} \eta^j * \widetilde{\rho}_{2^{-j}} \text{ [134, Lemma 2, p. 93].} \quad (9.3)$$

Then, using (9.3) and the validity of (9.2) for $\widetilde{\rho}$, it follows that property (9.1) holds for ρ provided the η^j's decay sufficiently fast. Finally, decay of η^j is obtained by requiring a sufficient decay of the Fourier transform $\widehat{\rho}$ of ρ. With more work, spatial conditions on ρ (of Fourier multiplier's theorem type) ensure the decay of $\widehat{\rho}$ and thus lead to (usually suboptimal) sufficient conditions for the validity of (9.1). [1] Alternatively, in standard function spaces one can rely on the decomposition of functions in simple building blocks (e.g. atoms) and obtain almost sharp spatial sufficient conditions. For such an approach in the framework of the Hardy spaces, see [135], [133].

In what follows, we will obtain, using very little technology, necessary and sufficient conditions on ρ in order to have (9.1), and simple sufficient spatial conditions on ρ, close to being optimal.

Of special interest to us will be the validity of (9.1) when $f * \rho_\varepsilon$ is particularly simple to compute. A typical example consists in taking ρ the characteristic function of a unit cube, e.g. $Q = (0,1)^n$ or $Q = (-1/2, 1/2)^n$. We will determine the spaces $W^{s,p}$ which can be described via such a ρ.

It turns out that our techniques are adapted not only to the Sobolev spaces with non integer s, but more generally to the Besov spaces $B^s_{p,q}$ with $s > 0$, $1 \le p \le \infty$ and $1 \le q \le \infty$. Recall that this scale of spaces includes the one of fractional Sobolev spaces, since $W^{s,p} = B^s_{p,p}$ for non integer s [139, Chapter 2]. For simplicity, we will write all our formulas and statements only when $q < \infty$. However, our results hold also when $q = \infty$, and the corresponding results are obtained by straightforward adaptations of the formulas and arguments.

Our first result is a one sided estimate, which surprisingly requires no smoothness of ρ.

Theorem 9.1. *Let $\rho \in L^1$ be such that $\int \rho = 1$. Then for every $s > 0$, $1 \le p \le \infty$ and $1 \le q < \infty$ we have*

$$\|f\|^q_{B^s_{p,q}} \lesssim \|f\|^q_{L^p} + \int_0^1 \frac{1}{\varepsilon^{sq+1}} \|f - f * \rho_\varepsilon\|^q_{L^p}\, d\varepsilon. \tag{9.4}$$

Remark 9.2. It is tempting to extend Theorem 9.1 to finite measures, but the example $\rho = \delta$ (the Dirac mass at the origin) shows that Theorem 9.1 need not hold for a measure. We do not know how to characterize the finite measures of total measure 1 satisfying (9.4).

We next discuss what is needed in order to obtain the reverse of (9.4). For this purpose, we fix some $\eta \in \mathcal{S}$. Assuming that the reverse of (9.4) holds, we have

$$\int_0^1 \frac{1}{\varepsilon^{sq+1}} \|\eta - \eta * \rho_\varepsilon\|^q_{L^p}\, d\varepsilon < \infty. \tag{9.5}$$

It turns out that (9.5) with $p = q = 1$ is also sufficient.

Theorem 9.3. *Let $\rho \in L^1$ satisfy $\int \rho = 1$. Let $s > 0$. Then the following are equivalent.*

1. A typical result for which this approach is followed is the fact that the norm on the Besov spaces $B^s_{p,q}$ does not depend on the choice of the rapidly decreasing mollifier; see [138, Section 2.3, p. 168] and the use of the Fourier multipliers theory [138, Section 2.2.4, p. 161].

1. *There exists some $\eta \in \mathcal{S}$ such that $\int \eta \neq 0$ and*

$$\int_0^1 \frac{1}{\varepsilon^{s+1}} \|\eta - \eta * \rho_\varepsilon\|_{L^1}\, d\varepsilon < \infty. \tag{9.6}$$

2. *For every $1 \leq p \leq \infty$ and every $1 \leq q < \infty$ we have*

$$\|f\|^q_{B^s_{p,q}} \sim \|f\|^q_{L^p} + \int_0^1 \frac{1}{\varepsilon^{sq+1}} \|f - f * \rho_\varepsilon\|^q_{L^p}\, d\varepsilon. \tag{9.7}$$

An additional equivalent characterization of ρ satisfying the above properties will be provided in Section 9.4.

We now turn to the case where ρ is the characteristic function of a set A. In that case, the range of values of s for which the equivalent characterizations of Theorem 9.3 are satisfied depends only on whether or not the set A is centered:

Proposition 9.4. *Let $\rho = \frac{1}{|A|}\mathbb{1}_A$, where $A \subset \mathbb{R}^n$ is a bounded measurable set of positive Lebesgue measure. Then ρ characterizes all the spaces $B^s_{p,q}$ for fixed s (that is,* (9.7) *is valid) if and only if:*

1. *Either $\int_A y\, dy = 0$ and $s < 2$.*
2. *Or $\int_A y\, dy \neq 0$ and $s < 1$.*

Finally, we provide sufficient spatial conditions for the validity of (9.7) when $0 < s < 1$.

Proposition 9.5. *Let $\rho \in L^1$ satisfy $\int \rho = 1$, and $0 < s < 1$. If ρ satisfies the moment condition*

$$\int |y|^s |\rho(y)|\, dy < \infty, \tag{9.8}$$

then ρ characterizes all spaces $B^s_{p,q}$. That is, (9.7) *is valid.*

For $s \geq 1$, the exemple of $\rho = \mathbb{1}_A$ with uncentered A shows that there is no such simple sufficient finite moment condition. In order to obtain the validity of (9.7) for higher s, one would need to ask for the vanishing of moments, as in the case of $\rho = \mathbb{1}_A$. For more details see Proposition 9.10 below.

The sufficient spatial condition (9.8) turns out to be optimal, in the sense that for non negative ρ it is also necessary:

Proposition 9.6. *Let $s > 0$. Let $\rho \in L^1$ satisfy $\int \rho = 1$ and $\rho \geq 0$. If* (9.7) *is valid, then ρ necessarily satisfies the moment condition* (9.8)*.*

The plan of the paper is as follows. In Section 9.2 we introduce some preliminary notation, definitions and tools required in the sequel. In Sections 9.3 and 9.4 we prove our two main results, Theorems 9.1 and 9.3. Eventually, Section 9.5 is devoted to proving Propositions 9.4, 9.5 and 9.6.

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9.2 Preliminaries

9.2.1 Littlewood-Paley decomposition and $B^s_{p,q}$

We will make use of the (inhomogeneous) Littlewood-Paley decomposition of a temperate distribution. Let $\zeta, \varphi \in \mathcal{S}(\mathbb{R}^n)$ be as follows:
— $\operatorname{supp} \widehat{\zeta} \subset B(0,2)$ and $\widehat{\zeta} \equiv 1$ in a neighborhood of $\overline{B}(0,1)$,
— $\varphi := \zeta_{1/2} - \zeta$, so that $\widehat{\varphi} = \widehat{\zeta}(\cdot/2) - \widehat{\zeta}$ and $\operatorname{supp} \widehat{\varphi} \subset B(0,4) \setminus \overline{B}(0,1)$.
The (inhomogeneous) Littlewood-Paley decomposition of a temperate distribution $f \in \mathcal{S}'(\mathbb{R}^n)$ is then given by

$$f = \sum_{j \geq 0} f_j, \quad \text{where } f_0 = f * \zeta \text{ and } f_j = f * \varphi_{2^{1-j}} \text{ for } j \geq 1. \tag{9.9}$$

See for instance [134, Section VI.4.1].

The Littlewood-Paley decomposition can be used to characterize the space $B^s_{p,q}$ [139, Section 2.3.2, Proposition 1, p. 46], and this is the definition we adopt here:

$$B^s_{p,q} = \left\{ f \in L^p;\ |f|^q_{B^s_{p,q}} := \sum_{j \geq 0} 2^{sjq} \|f_j\|^q_{L^p} < \infty \right\}. \tag{9.10}$$

The norm on $B^s_{p,q}$ is defined by

$$\|f\|^q_{B^s_{p,q}} = \|f\|^q_{L^p} + |f|^q_{B^s_{p,q}}. \tag{9.11}$$

Different choices of ζ yield equivalent norms [140, Section 2.3]. See also [140, Chapter 3] for other equivalent characterizations of $B^s_{p,q}$.

9.2.2 Schur's criterion

We will also make use of the following Schur-type estimate for kernel operators; see e.g. [57, Appendix I].

Lemma 9.7. *Let (X, μ) and (Y, ν) be two (σ-finite) measure spaces, let $1 \leq p \leq \infty$, and $\kappa\colon X \times Y \to \mathbb{C}$ a measurable kernel. If the quantities*

$$M_1 := \operatorname{ess\,sup}_x \int |\kappa(x,y)|\, d\nu(y) \quad \text{and} \quad M_2 := \operatorname{ess\,sup}_y \int |\kappa(x,y)|\, d\mu(x),$$

are finite, then the formula

$$Tu(x) = \int \kappa(x,y) u(y)\, d\nu(y)$$

defines a bounded linear operator from $L^p(Y)$ to $L^p(X)$, with norm

$$\|T\| \leq M_1^{1/p'} M_2^{1/p}.$$

Here $p' = p/(p-1)$ is the conjugate exponent of p.

9.3 Proof of Theorem 9.1

The proof of Theorem 9.1 relies on the following ingredient.

Lemma 9.8. *Let $\rho \in L^1$, and let $\psi \in L^1$ satisfy $\int \psi = 0$. Then*

$$\lim_{\varepsilon \to 0} \|\rho * \psi_\varepsilon\|_{L^1} = 0.$$

More generally, for ρ and ψ as above we have the following uniform estimate:

$$\lim_{\varepsilon \to 0} \sup_{1/2 \le \delta \le 1} \|\rho_\delta * \psi_\varepsilon\|_{L^1} = 0. \tag{9.12}$$

Proof of Theorem 9.1. We are going to prove a discrete version of (9.4). We start from the inequalities

$$\begin{aligned}\sum_{j\ge 0} 2^{sjq} \int_{1/2}^1 \|f - f * \rho_{2^{-j}\varepsilon}\|_{L^p}^q \, d\varepsilon &\le \int_0^1 \|f - f * \rho_\varepsilon\|_{L^p}^q \frac{d\varepsilon}{\varepsilon^{sq+1}} \\ &\le 2^{sq+1} \sum_{j \ge 0} 2^{sjq} \int_{1/2}^1 \|f - f * \rho_{2^{-j}\varepsilon}\|_{L^p}^q \, d\varepsilon.\end{aligned} \tag{9.13}$$

In view of (9.13), it suffices to establish the estimate

$$\|f\|_{B^s_{p,q}}^q \le C(s,p,q) \left(\|f\|_{L^p}^q + \sum_{j\ge 0} 2^{sjq} \|f - f * \rho_{2^{-j}\varepsilon}\|_{L^p}^q \right), \tag{9.14}$$

uniformly with respect to $\varepsilon \in (1/2, 1)$. Integrating (9.14) and using (9.13), we obtain indeed the desired inequality (9.4).

To simplify the notation, we will establish (9.14) for $\varepsilon = 1$, which amounts to considering $\widetilde{\rho} = \rho_\varepsilon$ instead of ρ. It will be clear at the end of the proof that all estimates are indeed uniform with respect to $\varepsilon \in (1/2, 1)$.

We introduce a function $\psi \in \mathcal{S}$ satisfying the following:

$$\widehat{\psi} \equiv 1 \text{ on } \operatorname{supp} \widehat{\varphi}, \quad \text{and } \widehat{\psi}(0) = 0. \tag{9.15}$$

Recall that φ is the function used in the definition of the Littlewood-Paley decomposition (9.9). Since the support of $\widehat{\varphi}$ is contained in the annulus $\{1 \le |\xi| \le 4\}$, it is indeed possible to choose ψ satisfying (9.15).

We need to estimate the $B^s_{p,q}$ semi-norm of f, hence the sum

$$\sum_{j \ge 0} 2^{sjq} \|f_j\|_{L^p}^q,$$

where $f = \sum_j f_j$ is the Littlewood-Paley decomposition (9.9). We introduce an integer $k > 0$, to be fixed later, and split the sum into two parts:

$$|f|_{B^s_{p,q}}^q \le \sum_{j \le k} 2^{sjq} \|f_j\|_{L^p}^q + \sum_{j > k} 2^{sjq} \|f_j\|_{L^p}^q. \tag{9.16}$$

Using the fact that

$$\|f_j\|_{L^p} = \|f * \varphi_{2^{1-j}}\|_{L^p} \le \|f\|_{L^p}\|\varphi\|_{L^1}, \ \forall\, j \ge 1,$$
$$\text{and } \|f_0\|_{L^p} = \|f * \zeta\|_{L^p} \le \|f\|_{L^p}\|\zeta\|_{L^1},$$

we simply estimate the first sum in the right-hand side of (9.16) by

$$\sum_{j \le k} 2^{sjq}\|f_j\|_{L^p}^q \lesssim \|f\|_{L^p}^q. \tag{9.17}$$

We next turn to estimating the second sum. In the remaining part of the proof, we will use the notation

$$\rho^j := \rho_{2^{-j}}, \ \varphi^j := \varphi_{2^{-j}}, \ \psi^j := \psi_{2^{-j}}.$$

Taking advantage of the fact that $\psi * \varphi = \varphi$ (and thus $\psi^j * \varphi^j = \varphi^j$) we write, for $j > k$,

$$\begin{aligned} f_{j+1} &= (f - f * \rho^{j-k} + f * \rho^{j-k}) * \varphi^j \\ &= (f - f * \rho^{j-k}) * \varphi^j + f * \rho^{j-k} * \psi^j * \varphi^j \\ &= (f - f * \rho^{j-k}) * \varphi^j + f_{j+1} * (\rho * \psi^k)^{j-k}. \end{aligned}$$

We deduce the estimate

$$\|f_{j+1}\|_{L^p} \le \|\varphi\|_{L^1}\|f - f * \rho^{j-k}\|_{L^p} + \|\rho * \psi^k\|_{L^1}\|f_{j+1}\|_{L^p}. \tag{9.18}$$

Since $\widehat{\psi}(0) = 0$, we can apply Lemma 9.8 above: it holds

$$\|\rho * \psi^k\|_{L^1} = \|\rho * \psi_{2^{-k}}\|_{L^1} \to 0, \quad \text{as } k \to \infty. \tag{9.19}$$

Thus for sufficiently large k we may absorb the last term of the right-hand side of (9.18) into the left-hand side. For such k, we have

$$\|f_{j+1}\|_{L^p} \lesssim \|f - f * \rho^{j-k}\|_{L^p} \quad \text{for } j \ge k. \tag{9.20}$$

Plugging (9.20) into (9.16) and recalling (9.17), we obtain

$$\|f\|_{B^s_{p,q}}^q \lesssim \|f\|_{L^p}^q + \sum_{j \ge 0} 2^{sjq}\|f - f * \rho_{2^{-j}}\|_{L^p}^q. \tag{9.21}$$

The latter estimate is exactly the desired estimate (9.14) with $\varepsilon = 1$. The corresponding estimate for $1/2 \le \varepsilon \le 1$ is found by replacing ρ with $\widetilde{\rho} = \rho_\varepsilon$ in the proof of (9.21). The resulting estimate is uniform with respect to $\varepsilon \in (1/2, 1)$ (by formula (9.12) in Lemma 9.8). This concludes the proof of Theorem 9.1. □

Proof of Lemma 9.8. We introduce a parameter $R > 0$. Taking advantage of the fact that $\int \psi = 0$, we may write

$$\begin{aligned} \rho * \psi_\varepsilon(x) &= \frac{1}{\varepsilon^n}\int \Big(\rho(y) - \fint_{B_{R\varepsilon}(x)} \rho\Big)\psi\left(\frac{x-y}{\varepsilon}\right) dy \\ &= \frac{1}{R^n\varepsilon^{2n}\omega_n}\iint_{|z-x|<R\varepsilon} (\rho(y) - \rho(z))\psi\left(\frac{x-y}{\varepsilon}\right) dydz, \end{aligned} \tag{9.22}$$

where $B_{R\varepsilon}(x)$ is the open ball of center x and radius $R\varepsilon$, and ω_n is the Lebesgue measure of the unit ball. We then have

$$\int |\rho * \psi_\varepsilon(x)|\,dx \le \int M_R(x)\,dx + \int N_R(x)\,dx, \tag{9.23}$$

where

$$M_R(x) = \frac{1}{R^n\varepsilon^{2n}\omega_n} \iint_{\substack{|z-x|<R\varepsilon\\ |y-x|<R\varepsilon}} |\rho(y)-\rho(z)| \left|\psi\left(\frac{x-y}{\varepsilon}\right)\right| dydz, \tag{9.24}$$

$$N_R(x) = \frac{1}{R^n\varepsilon^{2n}\omega_n} \iint_{\substack{|z-x|<R\varepsilon\\ |y-x|\ge R\varepsilon}} (|\rho(y)|+|\rho(z)|) \left|\psi\left(\frac{x-y}{\varepsilon}\right)\right| dydz. \tag{9.25}$$

To estimate $\int M_R(x)\,dx$, we perform the change of variable $x \rightsquigarrow w = (x-y)/\varepsilon$ and find

$$\begin{aligned}\int M_R(x)\,dx &\le \frac{1}{R^n\varepsilon^n\omega_n}\int_{|w|<R}|\psi(w)|\,dw \iint_{|z-y|<2R\varepsilon}|\rho(y)-\rho(z)|\,dydz\\ &\le \frac{\|\psi\|_{L^1}}{R^n\varepsilon^n\omega_n}\int_{|h|<2R\varepsilon}\|\rho(\cdot+h)-\rho\|_{L^1}\,dh,\end{aligned}$$

and thus

$$\int M_R(x)\,dx \le 2^n\|\psi\|_{L^1}\sup_{|h|<2R\varepsilon}\|\rho(\cdot+h)-\rho\|_{L^1}. \tag{9.26}$$

Note that, for any fixed R, the right-hand side of (9.26) converges to 0 as $\varepsilon \to 0$.

We next estimate $\int N_R(x)\,dx$. To this end we compute

$$\begin{aligned}&\frac{1}{R^n\varepsilon^{2n}\omega_n}\iiint_{\substack{|z-x|<R\varepsilon\\ |y-x|\ge R\varepsilon}}|\rho(y)|\left|\psi\left(\frac{x-y}{\varepsilon}\right)\right| dxdydz\\ =&\frac{1}{\varepsilon^n}\iint_{|y-x|\ge R\varepsilon}|\rho(y)|\left|\psi\left(\frac{x-y}{\varepsilon}\right)\right| dxdy = \|\rho\|_{L^1}\int_{|w|\ge R}|\psi(w)|\,dw,\end{aligned} \tag{9.27}$$

and

$$\begin{aligned}&\frac{1}{R^n\varepsilon^{2n}\omega_n}\iiint_{\substack{|z-x|<R\varepsilon\\ |y-x|\ge R\varepsilon}}|\rho(z)|\left|\psi\left(\frac{x-y}{\varepsilon}\right)\right| dxdydz\\ =&\frac{1}{R^n\varepsilon^n\omega_n}\int_{|w|\ge R}|\psi(w)|\,dw\iint_{|z-x|<R\varepsilon}|\rho(z)|dxdz = \|\rho\|_{L^1}\int_{|w|\ge R}|\psi(w)|\,dw.\end{aligned} \tag{9.28}$$

Plugging (9.27) and (9.28) into formula (9.25), we obtain

$$\int N_R(x)\,dx \le 2\|\rho\|_{L^1}\int_{|w|\ge R}|\psi(w)|\,dw. \tag{9.29}$$

Combining (9.23), (9.26) and (9.29) we obtain

$$\limsup_{\varepsilon\to 0}\|\rho*\psi_\varepsilon\|_{L^1} \le C\|\rho\|_{L^1}\int_{|w|\ge R}|\psi(w)|\,dw,$$

and complete the proof of the first assertion in Lemma 9.8 by letting $R \to \infty$.

Estimate (9.12) follows from the following calculations:

$$\lim_{\varepsilon\to 0} \sup_{1/2\le\delta\le 1} \|\rho_\delta * \psi_\varepsilon\|_{L^1} = \lim_{\varepsilon\to 0} \sup_{1/2\le\delta\le 1} \|(\rho * \psi_{\varepsilon/\delta})_\delta\|_{L^1} = \lim_{\varepsilon\to 0} \sup_{1/2\le\delta\le 1} \|\rho * \psi_{\varepsilon/\delta}\|_{L^1} = \lim_{\varepsilon\to 0} \|\rho * \psi_\varepsilon\|_{L^1}.$$

□

9.4 Proof of Theorem 9.3

Proof of Theorem 9.3. We clearly have « 2 $\implies$ 1 », and it remains to prove that « 1 $\implies$ 2 ». For the convenience of the reader, we start by establishing a consequence of property 1, and then we proceed to the proof of the desired implication.

Step 1. A discrete-uniform version of 1.
Assume that property 1 holds. Then we claim that for every $\varphi \in \mathcal{S}$ we have

$$\sup_{1/2\le\varepsilon\le 1} \sum_{j\ge 0} 2^{sj} \|\varphi - \varphi * \rho_{2^{-j}\varepsilon}\|_{L^1} \le C < \infty. \tag{9.30}$$

In order to prove (9.30), we start from the following fact. We fix a function $\lambda \in \mathcal{S}$ such that $\int \lambda \neq 0$. Then every function $\psi \in \mathscr{S}(\mathbb{R}^n)$ may be written as

$$\psi = \sum_{k\ge 0} \lambda_\psi^k * \lambda_{2^{-k}}. \tag{9.31}$$

Here $(\lambda_\psi^k)_k \subset \mathcal{S}$ is a sequence that decays rapidly as $k \to \infty$, in the following sense: if ψ belongs to a bounded subset $\mathscr{B} \subset \mathcal{S}$, then for every $M > 0$ there exists a constant C such that

$$\|\lambda_\psi^k\|_{L^1} \le \frac{C}{2^{Mk}},\ \forall\, k \ge 0,\ \forall\, \psi \in \mathscr{B}; \tag{9.32}$$

see [134, Lemma 2, p. 93]. In particular, if we fix $\varphi \in \mathcal{S}$ then we may write

$$\varphi_t = \sum_{k\ge 0} \lambda^{k,t} * \lambda_{2^{-k}},\ \forall\, t \in [1,2], \tag{9.33}$$

with

$$\|\lambda^{k,t}\|_{L^1} \le \frac{C}{2^{Mk}},\ \forall\, k \ge 0,\ \forall\, t \in [1,2]. \tag{9.34}$$

We now choose an appropriate $\lambda \in \mathcal{S}$. In view of (9.13), if property 1 holds then we may find some $\varepsilon \in [1/2, 1]$ such that $\lambda := \eta_{1/\varepsilon}$ satisfies

$$\sum_{k\ge 0} 2^{sk} \|\lambda - \lambda * \rho_{2^{-k}}\|_{L^1} = \sum_{k\ge 0} 2^{sk} \|\eta - \eta * \rho_{2^{-k}\varepsilon}\|_{L^1} < \infty. \tag{9.35}$$

By combining (9.33)-(9.35) we find that, with $\varepsilon \in [1/2, 1]$ and $t := 1/\varepsilon \in [1, 2]$, we have

$$\begin{aligned}
\sum_{j\geq 0} 2^{sj}\|\varphi - \varphi * \rho_{2^{-j}\varepsilon}\|_{L^1} &= \sum_{j\geq 0} 2^{sj}\|\varphi_t - \varphi_t * \rho_{2^{-j}}\|_{L^1} \\
&\leq \sum_{j\geq 0} 2^{sj} \sum_{k\geq 0} \|\lambda^{k,t} * \lambda_{2^{-k}} - \lambda^{k,t} * \lambda_{2^{-k}} * \rho_{2^{-j}}\|_{L^1} \\
&\leq \sum_{j\geq 0}\sum_{k\geq 0} 2^{sj}\|\lambda^{k,t}\|_{L^1}\|\lambda_{2^{-k}} - \lambda_{2^{-k}} * \rho_{2^{-j}}\|_{L^1} \\
&\leq C \sum_{j\geq 0}\sum_{k> j} 2^{sj}\|\lambda^{k,t}\|_{L^1} \\
&\quad + \sum_{j\geq 0}\sum_{k\leq j} 2^{sj}\|\lambda^{k,t}\|_{L^1}\|\lambda - \lambda * \rho_{2^{k-j}}\|_{L^1} \\
&\leq C \sum_{j\geq 0}\sum_{k> j} 2^{sj} 2^{-(s+1)k} \\
&\quad + \sum_{\ell\geq 0}\sum_{j\geq \ell} 2^{sj}\|\lambda^{j-\ell,t}\|_{L^1}\|\lambda - \lambda * \rho_{2^{-\ell}}\|_{L^1} \\
&\leq C + C\sum_{\ell\geq 0}\sum_{j\geq \ell} 2^{sj} 2^{-(s+1)(j-\ell)}\|\lambda - \lambda * \rho_{2^{-\ell}}\|_{L^1} \\
&\leq C + C\sum_{\ell\geq 0} 2^{s\ell}\|\lambda - \lambda * \rho_{2^{-\ell}}\|_{L^1} \leq C,
\end{aligned}$$

with constants independent of t, i.e., (9.30) holds.

Step 2. Proof of « $1 \implies 2$ ».
As we proved in the previous step, we may assume that there exists some $\eta \in \mathcal{S}$ such that

$$\widehat{\eta} \equiv 1 \text{ in } B(0, 4), \tag{9.36}$$

and such that η satisfies the following uniform and discrete version of (9.6):

$$S_\varepsilon := \sum_{j\geq 0} 2^{sj}\|\eta - \eta * \rho_{2^{-j}\varepsilon}\|_{L^1} \leq C, \ \forall\, \varepsilon \in [1/2, 1], \tag{9.37}$$

with C independent of $\varepsilon \in [1/2, 1]$.

Let $f \in L^p$. We will establish the estimate

$$\sum_{j\geq 0} 2^{sjq}\|f - f * \rho_{2^{-j}\varepsilon}\|_{L^p}^q \leq C\ (1 + S_\varepsilon)^q\ |f|_{B^s_{p,q}}^q, \forall\, \varepsilon \in [1/2, 1], \tag{9.38}$$

with C independent of $\varepsilon \in [1/2, 1]$. We obtain (9.7) by integrating (9.38) in ε and using (9.37).

In turn, estimate (9.38) is obtained as follows. Set

$$\alpha_{j,\varepsilon} := 2^{sj}\|\eta - \eta * \rho_{2^{-j}\varepsilon}\|_{L^1}, \text{ which satisfies } \sum_{j\geq 0} \alpha_{j,\varepsilon} \leq C, \forall\, \varepsilon \in [1/2, 1]. \tag{9.39}$$

Let $f = \sum_{\ell\geq 0} f_\ell$ be the (inhomogeneous) Littlewood-Paley decomposition of $f \in L^p$, defined in Section 9.2.1. By (9.36), for every ℓ we have $f_\ell = f_\ell * \eta_{2^{-\ell}}$, and thus

$$\begin{aligned}
f - f * \rho_{2^{-j}\varepsilon} &= \sum_{\ell\geq 0}(f_\ell - f_\ell * \rho_{2^{-j}\varepsilon}) \\
&= \sum_{\ell\geq j}(f_\ell - f_\ell * \rho_{2^{-j}\varepsilon}) + \sum_{\ell<j}(f_\ell - f_\ell * \rho_{2^{-j}\varepsilon}) \\
&= \sum_{\ell\geq j}(f_\ell - f_\ell * \rho_{2^{-j}\varepsilon}) + \sum_{\ell<j} f_\ell * (\eta_{2^{-\ell}} - \eta_{2^{-\ell}} * \rho_{2^{-j}\varepsilon}) \\
&= \sum_{\ell\geq j}(f_\ell - f_\ell * \rho_{2^{-j}\varepsilon}) + \sum_{\ell<j} f_\ell * (\eta - \eta * \rho_{2^{\ell-j}\varepsilon})_{2^{-\ell}}.
\end{aligned} \tag{9.40}$$

Using (9.40), we find that

$$\|f - f * \rho_{2^{-j}\varepsilon}\|_{L^p} \lesssim \sum_{\ell\geq j} \|f_\ell\|_{L^p} + \sum_{\ell<j} 2^{-s(j-\ell)}\alpha_{j-\ell,\varepsilon}\|f_\ell\|_{L^p}, \tag{9.41}$$

i.e.,

$$2^{sj}\|f - f*\rho_{2^{-j}\varepsilon}\|_{L^p} \lesssim \sum_{\ell} \left[2^{s(j-\ell)}\mathbb{1}_{\{\ell\geq j\}}(\ell) + \alpha_{j-\ell,\varepsilon}\mathbb{1}_{\{\ell<j\}}(\ell)\right] 2^{s\ell}\|f_\ell\|_{L^p}. \tag{9.42}$$

We obtain (9.38) by combining (9.39) with (9.42) and with Schur's criterion (Lemma 9.7) applied to:

$$X = Y = \mathbb{Z}_+,\ \mu = \nu = \text{the counting measure on } \mathbb{Z}_+,$$
$$\text{and } k(j,\ell) = 2^{s(j-\ell)}\mathbb{1}_{\{\ell\geq j\}}(\ell) + \alpha_{j-\ell,\varepsilon}\mathbb{1}_{\{\ell<j\}}(\ell),\ \forall\, j,\, \ell \in \mathbb{Z}_+. \quad \square$$

We continue with another characterization of the kernels ρ satisfying the equivalent properties 1 and 2 in Theorem 9.3. For simplicity, the main results of our article were stated for inhomogeneous Besov spaces. It turns out that the homogeneous version of our next result is easier to understand than the inhomogeneous one, so that we start by presenting (without proof) the homogeneous cousin of Theorem 9.9 below.

In order to avoid subtle issues concerning the realization of homogeneous Besov spaces as spaces of distributions, we consider only temperate distributions f such that

$$\widehat{f} \text{ is compactly supported in } \mathbb{R}^n \setminus \{0\}. \tag{9.43}$$

Any such f is smooth, and we have $f = \sum_{j\in\mathbb{Z}} f_j$ in $\mathcal{S}'$, where (in the spirit of (9.9)) $f_j = f * \varphi_{2^{1-j}}$, $\forall\, j \in \mathbb{Z}$. For f satisfying (9.43), we set

$$|f|^q_{\dot{B}^s_{p,q}} = \sum_{j\in\mathbb{Z}} 2^{sjq}\|f_j\|^q_{L^p},$$

with the obvious modification when $q = \infty$. Let us note that, the series $\sum_{j\in\mathbb{Z}} f_j$ containing only a finite number of non zero terms, we actually have

$$\dot{B}^s_{p,q} = \{f \in L^p(\mathbb{R}^n);\ f \text{ satisfies } (9.43)\},$$

but that the norm we consider is not equivalent to the L^p norm.

As in the inhomogeneous case considered in this article, we may try to characterize the L^1 kernels ρ such that

$$|f|^q_{\dot{B}^s_{p,q}} \sim \int_0^\infty \frac{1}{\varepsilon^{sq+1}} \|f - f * \rho_\varepsilon\|^q_{L^p}\, d\varepsilon, \quad \text{for every } f \text{ satisfying (9.43).} \tag{9.44}$$

The homogeneous counterpart of Theorem 9.3 consists of the following equivalence: for a fixed s (not necessarily positive) (9.44) holds if and only if for a function φ as in the Littlewood-Paley decomposition we have

$$\int_0^\infty \frac{1}{\varepsilon^{s+1}} \|\varphi - \varphi * \rho_\varepsilon\|_{L^1}\, d\varepsilon < \infty. \tag{9.45}$$

Necessity of (9.45) comes from the fact that (9.44) holds with $p = q = 1$.

Let us now examine what is required in order to have (9.44) when $p = q = \infty$. If (9.44) holds and if $|f|_{\dot{B}^s_{\infty,\infty}} < \infty$, then the distribution

$$f - f * \rho_\varepsilon = (\delta - \rho)_\varepsilon * f$$

is well-defined (as the convolution of a finite measure with a smooth bounded function).[2] Moreover, $\|f - f * \rho_\varepsilon\|_{L^\infty}$ is controlled by the norm $|f|_{\dot{B}^s_{\infty,\infty}}$ (since (9.44) holds). A moment thought shows that in particular $\delta - \rho$ is an element of the dual of $\dot{B}^s_{\infty,\infty}$. Remarkably, this necessary condition is also sufficient, and is equivalent to the property (9.45).

Theorem 9.9 is the inhomogeneous counterpart of the above fact. In order to state this result, it is convenient to define ad hoc norm and function space. Fix ζ, φ as in the Littlewood-Paley decomposition (9.9). In order to simplify the proof of Theorem 9.9, we make the (unessential) assumption that

$$\varphi \text{ is even.} \tag{9.46}$$

Our appropriate function space is defined starting from the identity

$$f = (f - f * \zeta) + \sum_{j \le -1} f * \varphi_{2^{-j}} := \sum_{j \le 0} f^\sharp_j, \ \forall f \in \mathcal{S}' \text{ satisfying (9.43).} \tag{9.47}$$

We define the appropriate norm

$$[f]^q_{X^s_{p,q}} = \sum_{j \le 0} 2^{sjq} \|f^\sharp_j\|^q_{L^p}, \tag{9.48}$$

with the corresponding modification when $q = \infty$. Let $X^s_{p,q}$ be the space of temperate distributions satisfying (9.43) and such that $[f]_{X^s_{p,q}} < \infty$.[3]

Theorem 9.9. *Let $s > 0$. Then property* (9.6) *is equivalent to*

$$\delta - \rho \in \left(X^s_{\infty,\infty}\right)^*. \tag{9.49}$$

Proof.

2. Here, δ stands for the Dirac mass at the origin.
3. This space is $\{f \in L^p(\mathbb{R}^n);\ f \text{ satisfies (9.43)}\}$, but not with the L^p norm.

« (9.6) $\Longrightarrow$ (9.49) ». Let φ be as in the Littlewood-Paley decomposition and let ψ be as in (9.15). We may assume that ψ is even. If $f \in \mathcal{S}'$ and $\varepsilon > 0$ are such that $f * \varphi_\varepsilon \in L^\infty$, then we have

$$\begin{aligned}(\delta - \rho)(f * \varphi_\varepsilon) &= (\delta - \rho)(f * \varphi_\varepsilon * \psi_\varepsilon) = [(\delta - \rho) * \psi_\varepsilon](f * \varphi_\varepsilon)\\ &= \int [(\delta - \rho) * \psi_\varepsilon(x)]\, [f * \varphi_\varepsilon(x)]\, dx.\end{aligned}$$

In particular, if $j < 0$ and $f \in X^s_{\infty,\infty}$, then

$$\left|(\delta - \rho)(f^\sharp_j)\right| = \left|\int (\delta - \rho) * \psi_{2^{-j}}(x)\, f^\sharp_j(x)\, dx\right| \leq \|(\delta-\rho)*\psi_{2^{-j}}\|_{L^1} \|f^\sharp_j\|_{L^\infty}. \quad (9.50)$$

On the other hand, for $j = 0$ we have $f^\sharp_0 \in C^\infty \cap L^\infty$ (in view of (9.43) and of the definition of $X^s_{\infty,\infty}$) and thus

$$\left|(\delta - \rho)(f^\sharp_0)\right| \leq (1 + \|\rho\|_{L^1}) \|f^\sharp_0\|_{L^\infty}. \quad (9.51)$$

We next note that (9.30) (applied to ψ instead of φ), which is a consequence of (9.6), implies that

$$\begin{aligned}\sum_{j<0} 2^{-sj} \|(\delta - \rho) * \psi_{2^{-j}}\|_{L^1} &= \sum_{j<0} 2^{-sj} \|\psi_{2^{-j}} - \rho * \psi_{2^{-j}}\|_{L^1}\\ &= \sum_{j<0} 2^{-sj} \|\psi - \psi * \rho_{2^j}\|_{L^1}\\ &= \sum_{k>0} 2^{sk} \|\psi - \psi * \rho_{2^{-k}}\|_{L^1} < \infty.\end{aligned} \quad (9.52)$$

By combining (9.50)–(9.52), we obtain

$$\begin{aligned}|(\delta - \rho)(f)| &\lesssim \sum_{j\leq 0} \left|(\delta - \rho)(f^\sharp_j)\right| \lesssim \|f^\sharp_0\|_{L^\infty} + \sum_{j<0} \|(\delta - \rho) * \psi_{2^{-j}}\|_{L^1} \|f^\sharp_j\|_{L^\infty}\\ &\leq \|f^\sharp_0\|_{L^\infty} + \sup_{j<0} 2^{sj} \|f^\sharp_j\|_{L^\infty} \sum_{j<0} 2^{-sj} \|(\delta - \rho) * \psi_{2^{-j}}\|_{L^1} \lesssim \|f\|_{X^s_{\infty,\infty}},\end{aligned}$$

and thus (9.49) holds.

« (9.49) $\Longrightarrow$ (9.6) ». We start by noting that an equivalent formulation of (9.49) is

$$\begin{aligned}&\left[f = \sum_{j\in J} f^\sharp_j, \text{ with } f^\sharp_j \text{ as in (9.47) and } J \subset \mathbb{Z}_- \text{ finite}\right]\\ &\Longrightarrow \left|(\delta - \rho)\left(\sum_{j\in J} f^\sharp_j\right)\right| \lesssim \sup_{j\in J} 2^{sj} \|f^\sharp_j\|_{L^\infty}.\end{aligned} \quad (9.53)$$

Step 1 in the proof of Theorem 9.3 implies that, if we find some $\lambda \in \mathcal{S}$ such that $\int \lambda \neq 0$ and

$$\sum_{j\geq 0} 2^{sj} \|\lambda - \lambda * \rho_{2^{-j}}\|_{L^1} < \infty, \quad (9.54)$$

then (9.6) holds.

Let ζ, φ be as in the Littlewood-Paley decomposition. We will prove that (9.54) holds with $\lambda = \zeta$.

Set

$$\alpha_j := \|\varphi_{2^j} - \varphi_{2^j} * \rho\|_{L^1} = \|(\varphi - \varphi * \rho_{2^{-j}})_{2^j}\|_{L^1} = \|(\varphi - \varphi * \rho_{2^{-j}})\|_{L^1}, \ \forall j > 0.$$

We divide the proof of (9.54) into two steps.

Step 1. It suffices to prove the key estimate

$$\sum_{j>0} 2^{sj} \alpha_j < \infty. \tag{9.55}$$

Granted (9.55), we prove (9.54) for $\lambda = \zeta$. Indeed, using the fact that

$$\lim_{M\to\infty} \|\zeta_M - \zeta_M * \rho\|_{L^1} = \lim_{M\to\infty} \|\zeta - \zeta * \rho_{1/M}\|_{L^1} = 0,$$

we find that, in L^1, we have

$$\begin{aligned} \lim_{\ell\to\infty} \sum_{j=k+1}^{\ell} (\varphi_{2^j} - \varphi_{2^j} * \rho) &= \lim_{\ell\to\infty} [(\zeta_{2^k} - \zeta_{2^k} * \rho) - (\zeta_{2^\ell} - \zeta_{2^\ell} * \rho)] \\ &= \zeta_{2^k} - \zeta_{2^k} * \rho. \end{aligned} \tag{9.56}$$

By (9.56), we have

$$\|\zeta_{2^k} - \zeta_{2^k} * \rho\|_{L^1} \le \sum_{j\ge k+1} \alpha_j. \tag{9.57}$$

By combining (9.55) with (9.57), we obtain

$$\sum_{k\ge 0} 2^{sk} \|\zeta - \zeta * \rho_{2^{-k}}\|_{L^1} = \sum_{k\ge 0} 2^{sk} \|\zeta_{2^k} - \zeta_{2^k} * \rho\|_{L^1} \le \sum_{k\ge 0} \sum_{j\ge k+1} 2^{sk} \alpha_j \lesssim \sum_{j>0} 2^{sj} \alpha_j < \infty,$$

and thus (9.54) holds.

Step 2. Proof of (9.55) completed.

For $\ell < 0$, let $\psi^\ell \in C_c^\infty(\mathbb{R}^n)$ be such that $|\psi^\ell| \le 1$ and

$$\int [(\delta - \rho) * \varphi_{2^{-\ell}}] \psi^\ell \ge \frac{1}{2} \|(\delta - \rho) * \varphi_{2^{-\ell}}\|_{L^1} = \frac{1}{2} \alpha_{-\ell}. \tag{9.58}$$

Let $J \subset \mathbb{Z}_-^*$ be a fixed arbitrary finite set, and set

$$f := \sum_{\ell\in J} 2^{-s\ell} \psi^\ell * \varphi_{2^{-\ell}}.$$

By (9.58), we have (using (9.46))

$$\sum_{\ell\in J} 2^{-s\ell} \alpha_{-\ell} \lesssim \sum_{\ell\in J} 2^{-s\ell} \int [(\delta - \rho) * \varphi_{2^{-\ell}}] \ \psi^\ell = (\delta - \rho) \left(\sum_{\ell\in J} 2^{-s\ell} \psi^\ell * \varphi_{2^{-\ell}} \right). \tag{9.59}$$

By (9.53) and (9.59), we have

$$\sum_{\ell\in J} 2^{-s\ell}\alpha_{-\ell} \lesssim \sup_{j\in M} 2^{sj}\|f_j^\sharp\|_{L^\infty}, \tag{9.60}$$

where $M \subset \mathbb{Z}_-$ is finite and such that $f_j^\sharp = 0$ when $j \notin M$.[4]

We next note that, when $j, \ell < 0$, we have

$$\varphi_{2^{-\ell}} * \varphi_{2^{-j}} = 0 \text{ when } |j-\ell| > 1. \tag{9.61}$$

By (9.61), when $j < 0$ we have

$$\begin{aligned} f_j^\sharp &= \sum_{\ell\in J} 2^{-s\ell}\left(\psi^\ell * \varphi_{2^{-\ell}}\right)_j^\sharp = \sum_{\ell\in J} 2^{-s\ell}\psi^\ell * \varphi_{2^{-\ell}} * \varphi_{2^{-j}} \\ &= \sum_{\substack{\ell\in J\\ |\ell-j|\le 1}} 2^{-s\ell}\psi^\ell * \varphi_{2^{-\ell}} * \varphi_{2^{-j}}, \end{aligned} \tag{9.62}$$

and thus

$$\|f_j^\sharp\|_{L^\infty} \lesssim \sum_{\substack{\ell\in J\\ |\ell-j|\le 1}} 2^{-s\ell}\|\psi^\ell\|_{L^\infty} \lesssim 2^{-sj}. \tag{9.63}$$

By (9.60) and (9.63), we have

$$\sum_{\ell\in J} 2^{-s\ell}\alpha_{-\ell} \le C < \infty, \tag{9.64}$$

with C independent of J.

We obtain (9.55) by taking, in (9.64), the supremum over J. □

9.5 Further results

This section is devoted to the proofs of Propositions 9.4, 9.5 and 9.6.

9.5.1 Proof of Proposition 9.4

Proposition 9.4 is a direct consequence of the following more general result.

Proposition 9.10. *Let $\rho \in L^1$ satisfy $\int \rho = 1$ and let $\eta \in \mathcal{S}$ be such that $\int \eta \neq 0$. Assume that ρ has finite moments of any order:*

$$\int |y|^k |\rho(y)|\, dy < \infty \quad \text{for all } k \in \mathbb{N}.$$

Then

$$\int_0^1 \frac{1}{\varepsilon^{s+1}}\|\eta - \eta * \rho_\varepsilon\|_{L^1}\, d\varepsilon < \infty \quad \text{if and only if} \quad s < k_0, \tag{9.65}$$

where $k_0 \in \mathbb{N}^* \cup \{\infty\}$ is the smallest non-zero moment of ρ:

$$k_0 = \min\left\{k \ge 1 :\ \int y^{\otimes k}\rho(y)\, dy \neq 0\right\}.$$

Here $y^{\otimes k}$ denotes the k-th order tensor $(y_{j_1}\cdots y_{j_k})_{1\le j_1,\dots,j_k\le n}$.

4. Existence of such M follows from the identity (9.62).

Note that Proposition 9.10 implies indeed Proposition 9.4 since for a bounded set A of positive measure the second moment $\int_A y^{\otimes 2}\,dy$ is always non zero.

We now turn to the

Proof of Proposition 9.10. We first treat the case of a finite k_0. Since it holds

$$\eta(x) - \eta * \rho_\varepsilon(x) = \int (\eta(x) - \eta(x - \varepsilon y))\rho(y)\,dy,$$

we find, applying Taylor's formula,

$$\eta(x)-\eta*\rho_\varepsilon(x) = \frac{(-1)^{k_0+1}}{k_0!}\varepsilon^{k_0} \sum_{1\le j_1,\dots,j_{k_0}\le n} \alpha_{j_1,\dots,j_{k_0}}\partial_{j_1}\cdots\partial_{j_{k_0}}\eta(x)+\varepsilon^{k_0+1}R_\varepsilon(x),$$

where

$$\alpha_{j_1,\dots,j_k} := \int y_{j_1}\cdots y_{j_k}\rho(y)\,dy, \tag{9.66}$$

and

$$\|R_\varepsilon\|_{L^1} \le \frac{\|D^{k_0+1}\eta\|_{L^1}}{(k_0+1)!}\int |y|^{k_0+1}|\rho(y)|\,dy.$$

Therefore it holds

$$\|\eta-\eta*\rho_\varepsilon\|_{L^1} = \frac{1}{k_0!}\varepsilon^{k_0}\Big\| \sum_{1\le j_1,\dots,j_{k_0}\le n} \alpha_{j_1,\dots,j_{k_0}}\partial_{j_1}\cdots\partial_{j_{k_0}}\eta\Big\|_{L^1}+O(\varepsilon^{k_0+1}), \tag{9.67}$$

as $\varepsilon \to 0$.

We next claim that

$$c := \Big\|\sum_{1\le j_1,\dots,j_{k_0}\le n} \alpha_{j_1,\dots,j_{k_0}}\partial_{j_1}\cdots\partial_{j_{k_0}}\eta\Big\|_{L^1} \ne 0.$$

Indeed, assume that $c = 0$. Then we have

$$\sum_{1\le j_1,\dots,j_{k_0}\le n} \alpha_{j_1,\dots,j_{k_0}}\xi_{j_1}\cdots\xi_{j_{k_0}}\hat\eta(\xi) = 0 \quad \forall\,\xi \in \mathbb{R}^n.$$

Since $\hat\eta(0) \ne 0$ we deduce that

$$\sum_{1\le j_1,\dots,j_{k_0}\le n} \alpha_{j_1,\dots,j_{k_0}}\xi_{j_1}\cdots\xi_{j_{k_0}} = 0$$

for all sufficiently small ξ, and thus by homogeneity for every ξ. This is absurd since, by assumption, at least one of the coefficients $\alpha_{j_1,\dots,j_{k_0}}$ is non zero.

Therefore $c \ne 0$ and the Taylor expansion (9.67) provides the equivalent

$$\|\eta - \eta * \rho_\varepsilon\|_{L^1} \sim \frac{c}{k_0!}\varepsilon^{k_0}$$

as $\varepsilon \to 0$, which readily implies (9.65). This concludes the proof of Proposition 9.10 when k_0 is finite.

When $k_0 = \infty$, the Taylor expansion shows that

$$\|\eta - \eta * \rho_\varepsilon\|_{L^1} = O(\varepsilon^k) \quad \text{for all } k \in \mathbb{N},$$

so that it holds indeed

$$\int_0^1 \frac{1}{\varepsilon^{s+1}} \|\eta - \eta * \rho_\varepsilon\|_{L^1} \, d\varepsilon < \infty$$

for every $s > 0$. □

9.5.2 Proof of Proposition 9.5

We fix $\rho \in L^1$ with $\int \rho = 1$ and $0 < s < 1$, and assume that ρ satisfies the moment condition (9.8):

$$\int |y|^s |\rho(y)| \, dy < \infty.$$

We consider an arbitrary test function $\eta \in \mathcal{S}$ and are going to show that condition (9.6) is satisfied (so that, by Theorem 9.3, the norm equivalence (9.7) is valid). To this end we compute

$$\begin{aligned}
\int_0^\infty \|\eta - \eta * \rho_\varepsilon\|_{L^1} \frac{d\varepsilon}{\varepsilon^{s+1}} &\leq \int_0^\infty \int \|\eta - \eta(\cdot - \varepsilon y)\|_{L^1} |\rho(y)| \, dy \frac{d\varepsilon}{\varepsilon^{s+1}} \\
&= \int |y|^s \rho(y) \int_0^\infty \frac{\|\eta - \eta(\cdot - \varepsilon y)\|_{L^1}}{|\varepsilon y|^s} \frac{d\varepsilon}{\varepsilon} \, dy \\
&= \int |y|^s \rho(y) \int_0^\infty \|\eta - \eta(\cdot - \delta \frac{y}{|y|})\|_{L^1} \frac{d\delta}{\delta^{s+1}} \, dy.
\end{aligned}$$

On the other hand, for every $\omega \in \mathbb{S}^{n-1}$ we have the estimate

$$\int_0^\infty \|\eta - \eta(\cdot - \delta\omega)\|_{L^1} \frac{d\delta}{\delta^{s+1}} \leq \|D\eta\|_{L^1} \int_0^1 \frac{d\delta}{\delta^s} + 2\|\eta\|_{L^1} \int_1^\infty \frac{d\delta}{\delta^{s+1}} =: C(\eta) < \infty,$$

and therefore we conclude that

$$\int_0^\infty \|\eta - \eta * \rho_\varepsilon\|_{L^1} \frac{d\varepsilon}{\varepsilon^{s+1}} \leq C(\eta) \int |y|^s |\rho(y)| \, dy < \infty,$$

which finishes the proof of Proposition 9.5. □

9.5.3 Proof of Proposition 9.6

Let $s > 0$ and let $\rho \in L^1$ satisfy $\int \rho = 1$ and $\rho \geq 0$. We assume that the norm equivalence (9.7) is valid. Then by Theorem 9.3 (and Step 1 in its proof), it holds

$$\int_0^1 \|\eta - \eta * \rho_\varepsilon\|_{L^1} \frac{d\varepsilon}{\varepsilon^{s+1}} < \infty$$

for every $\eta \in \mathcal{S}$. We fix such a function $\eta \geq 0$, $\eta \not\equiv 0$, with support in the unit ball:

$$\eta(x) = 0 \quad \text{for } |x| \geq 1.$$

We are going to show that

$$\int_0^1 \|\eta-\eta*\rho_\varepsilon\|_{L^1} \frac{d\varepsilon}{\varepsilon^{s+1}} \ge c\|\eta\|_{L^1} \int |y|^s \rho(y)\, dy - C(\|\eta\|_{L^1}+\|\eta\|_{L^\infty}\|\rho\|_{L^1}), \quad (9.68)$$

for some constants $c = c(s), C = C(s) > 0$. Obviously (9.68) implies the conclusion of Proposition 9.6: the function ρ satisfies the finite moment condition

$$\int |y|^s \rho(y)\, dy < \infty.$$

We now turn to the proof of (9.68). Note that

$$\int_1^\infty \frac{1}{\varepsilon^{s+1}} \|\eta - \eta * \rho_\varepsilon\|_{L^1}\, d\varepsilon \le \int_1^\infty \frac{d\varepsilon}{\varepsilon^{s+1}} (\|\eta\|_{L^1} + \|\eta\|_{L^\infty}\|\rho\|_{L^1}).$$

Hence it suffices to show that

$$\int_0^\infty \frac{1}{\varepsilon^{s+1}} \|\eta - \eta * \rho_\varepsilon\|_{L^1}\, d\varepsilon \ge c\|\eta\|_{L^1} \int |y|^s \rho(y)\, dy.$$

Since $\eta(x) = 0$ for $|x| \ge 1$, and since η and ρ are non negative, it holds

$$\|\eta - \eta * \rho_\varepsilon\|_{L^1} \ge \iint_{|x|\ge 1} \eta(x-\varepsilon y)\rho(y)\, dy = \iint_{|z+\varepsilon y|\ge 1} \eta(z)\rho(y)\, dydz.$$

Thus we obtain

$$\begin{aligned} \int_0^\infty \frac{1}{\varepsilon^{s+1}} \|\eta - \eta * \rho_\varepsilon\|_{L^1}\, d\varepsilon &\ge \iiint_{|z+\varepsilon y|\ge 1} \eta(z) \frac{\rho(y)}{\varepsilon^{s+1}}\, dydzd\varepsilon \\ &= \iiint_{|z+\delta y/|y||\ge 1} \eta(z) \frac{\rho(y)|y|^s}{\delta^{s+1}}\, dydzd\delta. \end{aligned} \quad (9.69)$$

Note that it holds

$$[|\delta| \ge 2 \text{ and } |z| < 1] \implies |z + \delta y/|y|| \ge 1.$$

Therefore, the domain of integration in the last integral in (9.69) contains the set

$$\{(y, z, \delta);\ y \ne 0,\ |z| < 1,\ \delta \ge 2\}.$$

We find that

$$\int_0^\infty \frac{1}{\varepsilon^{s+1}} \|\eta - \eta * \rho_\varepsilon\|_{L^1}\, d\varepsilon \ge \|\eta\|_{L^1} \int_2^\infty \frac{d\delta}{\delta^{s+1}} \int \rho(y)|y|^s\, dy,$$

which completes the proof of (9.68). □

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Singularities of vector-valued maps in condensed matter physics

Abstract: The present thesis is devoted mainly to the mathematical analysis of models arising in the physics of liquid crystals and superconductivity. A common feature of these models is that one has to deal with elliptic systems whose solutions have singularities: optical defects in liquid crystals, vorticity defects in superconductivity.

The rod-like molecules in a liquid crystals, while being (as in a liquid) "randomly" distributed, tend to align in a common direction: this "orientational order" enhances crystal-like optical properties, which are responsible for their many industrial applications. The orientational order can be "isotropic" (no preferred optical axis), "uniaxial" or "biaxial". Transitions between these states create singularities: optical defects which can be observed in practice. Some models consider only the uniaxial phase, but near the defects this approximation does not seem to be valid: a "biaxial escape" phenomenon is expected to occur. The results of this thesis bring a new light on the structure of defects and the mechanisms of biaxiality: rigidity of the pure uniaxial constraint, biaxial escape at low temperature (with A. Contreras), dipolar and quadrupolar configurations around a colloidal particle (with S. Alama and L. Bronsard), strongly biaxial solutions in a frustrated hybrid cell.

We also present some results related to the Ginzburg-Landau model for type II superconductivity, and to "vortices": isolated points at which superconductivity is destroyed. Understanding the formation and distribution of vortices is a fundamental question which triggers fascinating nonlinear analysis problems, and a very large literature has been devoted to it. In a joint work with P. Mironescu, we study the existence of solutions to a simplified model introduced by Bethuel, Brezis and Hélein, in which we replace the Dirichlet boundary condition by a mere topological degree constraint (making the variational problem non-compact). In collaboration with A. Contreras, we look into the proliferation of vortices in a thin superconducting shell, a natural sequel to previous works by Contreras and Sternberg. And a joint work with S. Alama and L. Bronsard is devoted to describing the vortex profile in a strongly coupled two-component Ginzburg-Landau model.

The last part of this thesis addresses regularity characterization for a function f through the convergence rate of $f * \rho_\varepsilon$, for some kernel ρ. In a joint work with Petru Mironescu we study the minimal regularity of ρ that allows such characterization. We prove a necessary and sufficient condition to characterize $W^{s,p}$ regularity and draw some consequences, in particular in the case where ρ is a characteristic function.

Keywords: Nonlinear analysis; Calculus of variationsn; Elliptic systems; Defects; Ginzburg-Landau; Liquid crystals.

Autour des singularités d'applications vectorielles en physique de la matière condensée

Résumé : Cette thèse est consacrée principalement à l'analyse mathématique de modèles issus de la physique des cristaux liquides et de la supraconductivité. Ces modèles ont en commun de faire intervenir des systèmes elliptiques dont les solutions présentent des singularités : défauts optiques dans les cristaux liquides, défauts de vorticité en supraconductivité.

Les cristaux liquides se composent de molécules allongées qui, tout en étant distribuées « au hasard » comme dans un liquide, tendent à s'aligner dans une direction commune : cet « ordre d'orientation » leur confère des propriétés optiques similaires à celles d'un cristal, à l'origine de leurs nombreuses applications industrielles. L'ordre d'orientation peut être « isotrope » (pas d'axe optique privilégié), « uniaxe » ou « biaxe », et les transitions entre ces différents états créent des singularités : défauts optiques qui sont observables en pratique. Certains modèles ne considèrent que la phase uniaxe, mais près des défauts cette approximation ne semble plus être valide : on s'attend à observer un phénomène de « fuite biaxe ». Les résultats démontrés dans cette thèse apportent un éclairage nouveau sur la structure des défauts et les mécanismes liés à la biaxie : rigidité de la contrainte de pure uniaxie, fuite biaxe à basse température (avec A. Contreras), configurations dipolaire et quadrupolaire autour d'une particule colloïdale (avec S. Alama et L. Bronsard), solutions fortement biaxes dans une cellule hybride frustrée.

On présente aussi dans cette thèse différents résultats liés au modèle de Ginzburg-Landau pour les supraconducteurs de type II, et aux « défauts de vorticité » : points isolés autour desquels la supraconductivité est détruite. Comprendre l'apparition, la structure et la répartition des ces « vortex » est une question fondamentale liée à de fascinants problèmes d'analyse non linéaire, et une très large littérature lui est consacrée. Dans un travail commun avec P. Mironescu, on étudie l'existence de solutions à un modèle simplifié proposé par Bethuel, Brezis et Hélein, en remplaçant la condition de Dirichlet par une simple contrainte de degré topologique (ce qui rend le problème non compact). Dans un travail en collaboration avec A. Contreras, on s'intéresse à la prolifération des vortex dans une fine coque supraconductrice, suite naturelle à des travaux de Contreras et Sternberg. Enfin, un travail en commun avec S. Alama et L. Bronsard est consacré à l'étude du profil d'un vortex dans un modèle de Ginzburg-Landau à deux composantes fortement couplées.

Une dernière partie de cette thèse traite de la caractérisation de la régularité d'une fonction f à travers la vitesse de convergence de $f * \rho_\varepsilon$ pour un certain noyau ρ. Dans un travail commun avec Petru Mironescu, on s'intéresse à la question de la régularité des noyaux ρ qui permettent une telle caractérisation. On démontre une condition nécessaire et suffisante pour caractériser la régularité $W^{s,p}$, et on étudie certaines conséquences, notamment dans le cas où ρ est une fonction caractéristique.

Mots clés : Analyse non linéaire ; Calcul des variations ; Systèmes elliptiques ; Défauts ; Ginzburg-Landau ; Cristaux liquides.

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