ON THE STABILITY OF RADIAL SOLUTIONS TO AN ANISOTROPIC GINZBURG-LANDAU EQUATION

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Abstract. We study the linear stability of entire radial solutions $u(re^{i\theta}) = f(r)e^{i\theta}$, with positive increasing profile $f(r)$, to the anisotropic Ginzburg-Landau equation

$$-\Delta u - \delta(\partial_x + i\partial_y)^2\bar{u} = (1-|u|^2)u, \quad -1 < \delta < 1,$$

which arises in various liquid crystal models. In the isotropic case $\delta = 0$, Mironescu showed that such solution is nondegenerately stable. We prove stability of this radial solution in the range $\delta \in (\delta_1, 0]$ for some $-1 < \delta_1 < 0$, and instability outside this range. In strong contrast with the isotropic case, stability with respect to higher Fourier modes is *not* a direct consequence of stability with respect to lower Fourier modes. In particular, in the case where $\delta \approx -1$, lower modes are stable and yet higher modes are unstable.

1. Introduction

Given $\delta \in (-1,1)$ and $u : \mathbb{R}^2 \to \mathbb{C}$, we consider the anisotropic energy

$$\mathfrak{E}[u] = \int_{\mathbb{R}^2} \frac{1}{2}|\nabla u|^2 + \frac{\delta}{2}\operatorname{Re}\left\{(\partial_\eta \bar{u})^2\right\} + \frac{1}{4}(1-|u|^2)^2\,\mathrm{d}x, \qquad \text{where } \partial_\eta = \partial_x + i\partial_y. \tag{1}$$

Minimizers and stable critical points of $\mathfrak{E}$ are relevant in describing 2D point defects (or 3D straight-line defects) in some liquid crystal configurations (e.g. smectic-C^* thin films [4] and nematics close to the Fréedericksz transition [2]). This energy can also be viewed as a toy model to understand intricate phenomena triggered by elastic anisotropy in the more complex Landau-de Gennes energy [11].

Remark 1.1. The anisotropic term $\operatorname{Re}\left\{(\partial_\eta \bar{u})^2\right\}$ can be rewritten as

$$\operatorname{Re}\left\{(\partial_\eta \bar{u})^2\right\} = (\nabla\cdot u)^2 - (\nabla\times u)^2,$$

so that, in view of the identity $|\nabla u|^2 = (\nabla\cdot u)^2 + (\nabla\times u)^2 - 2\det(\nabla u)$, energy (1) differs from

$$\widetilde{\mathfrak{E}}[u] = \int \frac{k_s}{2}(\nabla\cdot u)^2 + \frac{k_b}{2}(\nabla\times u)^2 + \frac{1}{4}(1-|u|^2)^2, \qquad k_s = 1+\delta,\ k_b = 1-\delta,$$

only by the integral of the null Lagrangian $\det(\nabla u)$. This is precisely the form that appears in [4] where minimizers of

$$\widetilde{\mathfrak{E}}_\varepsilon[u] = \int_\Omega \frac{k_s}{2}(\nabla\cdot u)^2 + \frac{k_b}{2}(\nabla\times u)^2 + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \tag{2}$$

are investigated in the limit as $\varepsilon \to 0^+$ in a bounded planar domain Ω.

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Critical points of $\mathfrak{E}$ are solutions of the Euler-Lagrange equation

$$\begin{aligned} &\mathfrak{L}_\delta u = (|u|^2 - 1)u \qquad \text{in } \mathbb{R}^2 \\ &\mathfrak{L}_\delta u := \Delta u + \delta\, \partial_{\eta\eta}\bar{u}. \end{aligned} \tag{3}$$

We are interested in symmetric solutions of the form

$$u(re^{i\theta}) = f(r)e^{i\alpha}e^{i\theta} \qquad \text{for some } \alpha \in \mathbb{R}, \tag{4}$$

with a radial profile $f(r)$ satisfying

$$f(0) = 0, \quad \lim_{r\to+\infty} f(r) = 1, \quad |f(r)| > 0 \quad \forall r \in (0,\infty). \tag{5}$$

Formally, one can always look for solutions of (3) in the form (4) (as a consequence of the $O(2)$-invariance of $\mathfrak{E}$), and f must solve

$$Tf + \delta e^{-2i\alpha} T\bar{f} = \left(|f|^2 - 1\right) f, \qquad T = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}.$$

At this point we see a fundamental difference with respect to the isotropic case $\delta = 0$. If $\delta = 0$, one can find solutions as above for a real-valued function f, which moreover does not depend on α. In the anisotropic case $\delta \neq 0$, as remarked in [2], the function f can be real-valued only if $\alpha \equiv 0$ modulo $\pi/2$. In that case, the existence and uniqueness of a solution satisfying (5) follows from the case $\delta = 0$ (see [1,6]). Otherwise, the function f must be complex valued.

Remark 1.2. Another difference with respect to the isotropic case is that for $\delta \neq 0$ the Ansatz $u(re^{i\theta}) = f(r)e^{im\theta}$ cannot provide a solution when the winding number m is $\neq 1$.

In [2], the core energies of the two symmetric solutions corresponding to $\alpha = 0, \pi/2$ are compared, to find that the lowest energy corresponds to $\alpha = 0$ for $\delta < 0$ and $\alpha = \pi/2$ for $\delta > 0$. In view of Remark 1.1 this is consistent with the fact that $\nabla \times e^{i\theta} = 0$, while $\nabla \cdot ie^{i\theta} = 0$; indeed, for $\delta < 0$ the energy $\widetilde{\mathfrak{E}}[u]$ in Remark 1.1 penalizes more strongly the term $(\nabla \times u)^2$ than the term $(\nabla \cdot u)^2$, since in this case $k_b = 1 - \delta > k_s = 1 + \delta$. In [4, Proposition 3.1] the authors use this to show that minimizers of (2) behave like $e^{i\alpha}e^{i\theta}$ around point defects, with $\alpha \equiv 0$ (resp. $\pi/2$) modulo π if $\delta < 0$ (resp. $\delta > 0$). These results tell us, for $\delta \neq 0$, which one is the minimizing behavior at infinity.

Here, in contrast, we fix the far-field behavior and investigate the local stability of radial solutions with respect to compactly supported perturbations. For the isotropic case $\delta = 0$, this study has been performed in [12] (see also [5]), and the radial solution is stable. In the anisotropic situation $\delta \neq 0$ we find that the corresponding symmetric solution stays stable for negative δ close to zero and it loses stability for δ either positive or close to minus one (see Theorem 1.3 for precise statements).

It can be readily seen that the case $\alpha = \pi/2$ corresponds to $\alpha = 0$, after changing the sign of δ. Accordingly, we only treat the case where $\alpha = 0$. That is, we investigate the linear stability of solutions u of the form

$$u^\delta_{\mathrm{rad}}(r,\theta) = f(r)e^{i\theta}, \quad f\colon (0,+\infty) \to (0,+\infty) \quad \text{with} \quad f(0) = 0, \ \lim_{r\to+\infty} f(r) = 1. \tag{6}$$

Let us note that the equation satisfied by u^δ_{rad}, (3), reduces to the following ODE for f

$$(1+\delta)Tf = (f^2-1)f, \qquad T = \frac{d^2}{dr^2} + \frac{1}{r}\frac{d}{dr} - \frac{1}{r^2}. \tag{7}$$

As pointed out in [2], the rescaling of the variable by $(1+\delta)^{\frac{1}{2}}$ simplifies (7) to the standard ODE corresponding to the isotropic case $\delta = 0$. Whence, existence and uniqueness of f follow from [1,6]. Moreover, it is known that f takes values in $(0,1)$ and is strictly increasing.

The second variation of the energy $\mathfrak{E}$ around u^δ_{rad} is the quadratic form

$$\begin{aligned}\mathfrak{Q}^\delta_{\text{rad}}[v] &= \int_{\mathbb{R}^2} |\nabla v|^2 + \delta \operatorname{Re}\left\{(\partial_\eta \bar v)^2\right\} - (1-|u^\delta_{\text{rad}}|^2)|v|^2 + 2\,(u^\delta_{\text{rad}}\cdot v)^2 \,\mathrm{d}x \\ &= \int_{\mathbb{R}^2} |\nabla v|^2 + \delta \operatorname{Re}\left\{(\partial_\eta \bar v)^2\right\} - (1-f^2)|v|^2 + 2f^2\,(e^{i\theta}\cdot v)^2 \,\mathrm{d}x \end{aligned} \tag{8}$$

associated to the linear operator obtained by linearizing (3) around u^δ_{rad}:

$$\mathcal{L}(u^\delta_{\text{rad}})[v] = -\mathfrak{L}_\delta v - (1-|u^\delta_{\text{rad}}|^2)v + 2\,(u^\delta_{\text{rad}}\cdot v)\,u^\delta_{\text{rad}},$$

where $u\cdot v := \operatorname{Re}\{u\bar v\}$ denotes the standard inner product of complex-valued functions.

Taking into account the asymptotic expansion $f(r) = 1 + O(r^{-2})$ as $r\to\infty$ (see [1,6]), it follows that the energy space of $\mathfrak{Q}^\delta_{\text{rad}}$ naturally corresponds to

$$\mathcal{H} := \left\{ v\in H^1_{loc}(\mathbb{R}^2)\colon \int_{\mathbb{R}^2} |\nabla v|^2 + \frac{1}{r^2}|v|^2 + (e^{i\theta}\cdot v)^2 \,\mathrm{d}x < +\infty \right\}.$$

Also, the translational invariance of $\mathfrak{E}$ readily provides two elements of $\mathcal{H}$ at which $\mathfrak{Q}^\delta_{\text{rad}}$ vanishes, namely

$$\partial_x u^\delta_{\text{rad}} = e^{i\theta}\left(f'\cos\theta - i\frac{f}{r}\sin\theta\right), \qquad \partial_y u^\delta_{\text{rad}} = e^{i\theta}\left(f'\sin\theta + i\frac{f}{r}\cos\theta\right),$$

and the linear space they generate is denoted by

$$K_0 = \operatorname{span}\{\partial_x u^\delta_{\text{rad}}, \partial_y u^\delta_{\text{rad}}\}.$$

Our main result shows that the symmetric solution u^δ_{rad} is stable when $\delta \le 0$ is small, and unstable otherwise:

Theorem 1.3. *Let u^δ_{rad} denote the radial solution (6) of the anisotropic Ginzburg-Landau equation (3), and let $\mathfrak{Q}^\delta_{\text{rad}}$ denote the quadratic form (8) associated to the energy $\mathfrak{E}$ around u^δ_{rad}. Then, there exists a unique number $\delta_1\in(-1,0)$ such that*

- *for every $\delta\in(\delta_1,0]$, u^δ_{rad} is nondegenerately stable: namely,*

  $\mathfrak{Q}^\delta_{\text{rad}}[v] > 0$ *for all $v\in H\setminus K_0$,*

- *for every $\delta\in(-1,\delta_1)\cup(0,1)$, u^δ_{rad} is linearly unstable: namely,*

  $\mathfrak{Q}^\delta_{\text{rad}}[v] < 0$ *for some $v\in H$.*

Remark 1.4. The most relevant range from the stand point of physics is $\delta\in(-1,0]$ since for $\delta>0$ the far-field behavior corresponding to $\alpha=0$ is non-minimizing, and this translates here into instability of the radial solution.

Remark 1.5. In the stability range $\delta \in (\delta_1, 0]$, a contradiction argument as in [5, Lemma 3.1] provides a coercivity estimate of the form

$$\mathfrak{Q}^{\delta}_{\mathrm{rad}}[v] \geq C(\delta) \int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x \qquad \forall v \in K_0^{\perp} : \int_{\mathbb{S}^1} (ie^{i\theta}) \cdot v(re^{i\theta})\, d\theta = 0 \ \ \forall r > 0,$$

where $\perp$ denotes orthogonality in $\mathcal{H}$. Using this coercivity for $\delta = 0$, one can deduce stability for small negative δ via a relatively simple perturbation argument, combined with properties of the lower modes in § 3. Instead, we will give a more quantitative proof, which provides an explicit range for stability: we deduce that $\delta_1 \leq -1/\sqrt{5}$.

Our proof of Theorem 1.3 follows the general strategy of [12]: we decompose v into Fourier modes

$$v = e^{i\theta} \sum_{n \in \mathbb{Z}} w_n(r) e^{in\theta}.$$

and we are led to studying the sign of $\mathfrak{Q}^{\delta}_{\mathrm{rad}}$, separately, for each mode

$$e^{i\theta} \left(w_n(r) e^{in\theta} + w_{-n}(r) e^{-in\theta} \right).$$

As in [12], the lower modes $n = 0$ and $n = 1$ play a special role. They can be studied via an appropriate decomposition already used in [12] (see also [5]). For any $\delta \in (-1, 0]$ we find that these lower modes are stable, while for $\delta > 0$ the mode corresponding to $n = 0$ is unstable.

A major difference of the present work compared to [12] (or similar results in [8–10]) pertains to the higher modes $n \geq 2$. In contrast with the cited works, stability for the higher modes is not an obvious consequence of stability for the lower modes. More precisely in the isotropic case we have

$$\mathfrak{Q}^{0}_{\mathrm{rad}} \left[e^{i\theta} \left(w_+(r) e^{in\theta} + w_-(r) e^{-in\theta} \right) \right] \geq \mathfrak{Q}^{0}_{\mathrm{rad}} \left[e^{i\theta} \left(w_+(r) e^{i\theta} + w_-(r) e^{-i\theta} \right) \right] \quad \forall n \geq 1,$$

but for $\delta \neq 0$ this is not valid anymore, see (16). This feature is new and specific to the anisotropic case $\delta \neq 0$. Our strategy to study the sign of these higher modes is based on the same decomposition used for $n = 1$, and a careful balance of the contributions of additional terms, which end up causing instability for $\delta \approx -1$.

The article is organized as follows. In Section 2 we recall the splitting property of the quadratic form $\mathfrak{Q}^{\delta}_{\mathrm{rad}}$ with respect to Fourier expansion. In Section 3 we study the stability of lower modes, and in Section 4 the instability of higher modes. In Section 5 we give the proof of Theorem 1.3. In addition, we included Appendix A to recall the details of the decomposition used to study the lower modes, adapted to our notations.

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2. Fourier splitting

Recall that $f(r) = f_0((1+\delta)^{-\frac{1}{2}} r)$ where f_0 is the classical Ginzburg-Landau vortex profile corresponding to the case $\delta = 0$. That is, the unique solution of

$$f_0'' + \frac{1}{r} f_0' - \frac{1}{r^2} f_0 = -(1 - f_0^2) f_0, \quad f_0 > 0 \text{ on } (0, +\infty), \quad f_0(0) = 0, \quad \lim_{r \to +\infty} f_0(r) = 1. \tag{9}$$

We rescale variables and consider $\mathcal{Q}^\delta[v] = \mathfrak{Q}^\delta_{\mathrm{rad}}[\tilde{v}]$ where $\tilde{v}(\tilde{x}) = v((1+\delta)^{-\frac{1}{2}} \tilde{x})$, so that

$$\mathcal{Q}^\delta[v] = \int_{\mathbb{R}^2} |\nabla v|^2 + \delta \operatorname{Re}\left\{(\partial_\eta \bar{v})^2\right\} + (1+\delta)\left\{2 f_0^2 \,(e^{i\theta} \cdot v)^2 - (1 - f_0^2)|v|^2\right\} \mathrm{d}x, \tag{10}$$

which corresponds to the second variation of the appropriately rescaled energy around u^0_{rad}. Following [12] we decompose v using Fourier series, as

$$v = e^{i\theta} w = e^{i\theta} \sum_{n \in \mathbb{Z}} w_n(r) e^{in\theta}, \tag{11}$$

where we have conveniently shifted the index $n - 1 \mapsto n$.

This decomposition provides a "diagonalization" of the linearized operator:

Lemma 2.1. *The quadratic form* (10) *splits as*

$$\mathcal{Q}^\delta[v] = \mathcal{Q}^\delta\left[w_0(r) e^{i\theta}\right] + \sum_{n \geq 1} \mathcal{Q}^\delta\left[e^{i\theta}\left(w_n(r) e^{in\theta} + w_{-n}(r) e^{-in\theta}\right)\right].$$

Proof of Lemma 2.1. Lemma 2.1 essentially asserts that the family of functions

$$w_0(r) e^{i\theta}, \quad \{e^{i\theta}\left(w_n(r) e^{in\theta} + w_{-n}(r) e^{-in\theta}\right) : \; n \geq 1\}, \tag{12}$$

is orthogonal for the quadratic form $\mathcal{Q}$. This quadratic form (10) is composed of three terms. For the first term,

$$\int_{\mathbb{R}^2} |\nabla v|^2 \,\mathrm{d}x,$$

the orthogonality of (12) is a standard fact (recall e.g. in [12]). For the third term,

$$\int_{\mathbb{R}^2} \left\{f_0^2 \,(e^{i\theta} \cdot v)^2 - (1 - f_0^2)|v|^2\right\} \mathrm{d}x,$$

the orthogonality of (12) is proved in [12]. The novelty here, with respect to [12], concerns the anisotropic term

$$\int_{\mathbb{R}^2} \operatorname{Re}\left\{(\partial_\eta \bar{v})^2\right\} \mathrm{d}x.$$

The orthogonality of (12) for this anisotropic term, as a matter of fact, follows from the calculations in [3, § 3.2]. As our notations are different, we sketch a proof here for the reader's convenience.

We compute

$$\partial_\eta \bar{v} = e^{i\theta} \partial_r \bar{v} + \frac{i e^{i\theta}}{r} \partial_\theta \bar{v} = \sum_{n \in \mathbb{Z}} \left(\bar{w}_n' + \frac{1+n}{r} \bar{w}_n\right) e^{-in\theta},$$

and deduce, using the orthogonality of $\{e^{in\theta}\}$ in $L^2(\mathbb{S}^1)$,

$$\begin{aligned}
&\fint_{\mathbb{S}^1} \mathrm{Re}\left\{(\partial_\eta \bar{v})^2\right\}\, d\theta \\
&= \mathrm{Re}\left\{\sum_{n,m\in\mathbb{Z}} \left(\bar{w}_n' + \frac{1+n}{r}\bar{w}_n\right)\left(\bar{w}_m' + \frac{1+m}{r}\bar{w}_m\right) \fint_{\mathbb{S}^1} e^{-i(n+m)\theta} d\theta\right\} \\
&= \mathrm{Re}\left\{\sum_{n\in\mathbb{Z}} \left(\bar{w}_n' + \frac{1+n}{r}\bar{w}_n\right)\left(\bar{w}_{-n}' + \frac{1-n}{r}\bar{w}_{-n}\right)\right\} \\
&= \sum_{n\in\mathbb{Z}} \mathrm{Re}\left\{\left(\bar{w}_n' + \frac{1+n}{r}\bar{w}_n\right)\left(\bar{w}_{-n}' + \frac{1-n}{r}\bar{w}_{-n}\right)\right\}.
\end{aligned}$$

This implies the announced orthogonality and completes the proof of Lemma 2.1. □

According to the decomposition of Lemma (2.1), we define the quadratic forms

$$\begin{aligned}
Q_0^\delta[\varphi] &= \frac{1}{2\pi}\mathcal{Q}^\delta\left[\varphi(r)e^{i\theta}\right] && \text{for } \varphi\in\mathcal{H}_0, \\
Q_n^\delta[\varphi,\psi] &= \frac{1}{2\pi}\mathcal{Q}^\delta\left[e^{i\theta}\left(\varphi(r)e^{in\theta}+\psi(r)e^{-in\theta}\right)\right] && \text{for } (\varphi,\psi)\in\mathcal{H}_1,
\end{aligned}$$

where $\mathcal{H}_0$ and $\mathcal{H}_1$ are the natural spaces corresponding to the conditions $\varphi(r)e^{i\theta}\in\mathcal{H}$ and $e^{i\theta}\left(\varphi(r)e^{in\theta}+\psi(r)e^{-in\theta}\right)\in\mathcal{H}$ for $n\geq 1$, respectively.

$$\begin{aligned}
\mathcal{H}_0 &= \left\{\varphi\in H^1_{loc}(0,\infty)\colon \int_0^{+\infty}\left(\left|\varphi'\right|^2 + \frac{|\varphi|^2}{r^2} + \mathrm{Re}\left\{\varphi\right\}^2\right) r\,dr < +\infty\right\}, \\
\mathcal{H}_1 &= \left\{(\varphi,\psi)\in (H^1_{loc}(0,\infty))^2\colon \int_0^{+\infty}\left(\left|\varphi'\right|^2 + \left|\psi'\right|^2 + \frac{|\varphi|^2+|\psi|^2}{r^2} + \left|\varphi+\bar{\psi}\right|^2\right) r\,dr < +\infty\right\}
\end{aligned}$$

Remark 2.2. Using the density of smooth functions in H^1_{loc} and cut-off functions χ_ε such that $\mathbf{1}_{2\varepsilon<r<\varepsilon^{-1}} \leq \chi_\varepsilon(r) \leq \mathbf{1}_{\varepsilon<r<2\varepsilon^{-1}}$ and $|\chi_\varepsilon'(r)|\leq C/r$, we see that smooth test functions with compact support in $(0,\infty)$ are dense in $\mathcal{H}_0$ and $\mathcal{H}_1$. Hence, in the sequel, we will always be able to perform calculations assuming, without loss of generality, that φ and ψ are such test functions.

The quadratic forms Q_0^δ and Q_n^δ are explicitly given by

$$\begin{aligned} Q_0^\delta[\varphi] = \int_0^\infty \Bigg[|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 + \delta \operatorname{Re}\left\{ \left(\bar{\varphi}' + \frac{1}{r}\bar{\varphi} \right)^2 \right\} \\ + (1+\delta)\left\{ 2f_0^2 (\operatorname{Re}\{\varphi\})^2 - (1-f_0^2)|\varphi|^2 \right\} \Bigg] r dr, \end{aligned} \tag{13}$$

$$\begin{aligned} Q_n^\delta[\varphi,\psi] = \int_0^\infty \Bigg[|\varphi'|^2 + |\psi'|^2 + \frac{(1+n)^2}{r^2}|\varphi|^2 + \frac{(1-n)^2}{r^2}|\psi|^2 \\ + 2\delta \operatorname{Re}\left\{ \left(\bar{\varphi}' + \frac{1+n}{r}\bar{\varphi} \right) \left(\bar{\psi}' + \frac{1-n}{r}\bar{\psi} \right) \right\} \\ + (1+\delta)\left\{ f_0^2 |\varphi + \bar{\psi}|^2 - (1-f_0^2)\left(|\varphi|^2 + |\psi|^2 \right) \right\} \Bigg] r dr. \end{aligned} \tag{14}$$

Remark 2.3. For every $n \geq 1$ there is a further splitting, namely

$$Q_n^\delta[\varphi,\psi] = Q_n^\delta\left[\operatorname{Re}\{\varphi\}, \operatorname{Re}\{\psi\}\right] + Q_n^\delta\left[\operatorname{Im}\{\varphi\}, -\operatorname{Im}\{\psi\}\right].$$

Consequently, it will be sufficient to consider real-valued test functions φ, ψ.

3. Study of the lower modes Q_0^δ and Q_1^δ

We show that Q_0^δ is positive for $\delta \leq 0$, but it can become negative for $\delta > 0$. In addition, we prove that Q_1^δ is nonnegative for all $\delta \in (-1, 0]$.

3.1. Positivity of Q_0^δ for $\delta \in (-1, 0]$.

Let us recall from (13) that Q_0^δ is given by

$$\begin{aligned} Q_0^\delta[\varphi] = \int_0^\infty \Bigg[|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 + \delta \operatorname{Re}\left\{ \left(\bar{\varphi}' + \frac{1}{r}\bar{\varphi} \right)^2 \right\} \\ + (1+\delta)\left\{ 2f_0^2 (\operatorname{Re}\{\varphi\})^2 - (1-f_0^2)|\varphi|^2 \right\} \Bigg] r dr \end{aligned}$$

We now introduce the quadratic form

$$\begin{aligned} A_0[\varphi] &:= Q_0^0[\varphi] \\ &= \int_0^\infty \left[|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 + 2f_0^2(\operatorname{Re}\{\varphi\})^2 - (1-f_0^2)|\varphi|^2 \right] r dr. \end{aligned}$$

It is known that $A_0[\varphi] > 0$, unless $\varphi = 0$ (see Appendix A for more details). Moreover, we have the identity

$$\begin{aligned} Q_0^\delta[\varphi] &= (1+\delta)A_0[\mathrm{Re}\{\varphi\}] + (1-\delta)A_0[i\,\mathrm{Im}\{\varphi\}] - 2\delta\int(1-f_0^2)(\mathrm{Im}\{\varphi\})^2\, r\,dr \\ &\quad + \delta\int_0^\infty \frac{d}{dr}\left[(\mathrm{Re}\{\varphi\})^2 - (\mathrm{Im}\{\varphi\})^2\right]\,dr \\ &= (1+\delta)A_0[\mathrm{Re}\{\varphi\}] + (1-\delta)A_0[i\,\mathrm{Im}\{\varphi\}] - 2\delta\int(1-f_0^2)(\mathrm{Im}\{\varphi\})^2\, r\,dr, \end{aligned}$$

which is valid for any $\varphi \in C_c^\infty(0,\infty)$, hence for $\varphi \in \mathcal{H}_0$ thanks to Remark 2.2. Since $1-f_0^2 \geq 0$, we deduce the positivity of Q_0^δ for every $\delta \in (-1, 0]$.

3.2. **Instability for $\delta > 0$.** Using the formula (18) obtained for A_0 in Appendix A, we see that for any compactly supported real-valued test function χ we have

$$Q_0^\delta[if_0\chi] = (1-\delta)\int f_0^2(\chi')^2\, r\,dr - 2\delta\int(1-f_0^2)f_0^2\chi^2\, r\,dr.$$

Applying this to $\chi_n(r) = \chi_1(r/n)$, for a fixed test function χ_1, and using the asymptotic expansion [1,6]:

$$f_0(r) = 1 - \frac{1}{2r^2} + O(r^{-4}) \qquad \text{as } r \to \infty,$$

we see that

$$\lim_{n\to\infty} Q_0^\delta[if_0\chi_n] = (1-\delta)\int(\chi_1')^2\, r\,dr - 2\delta\int\frac{\chi_1^2}{r^2}\, r\,dr.$$

When $\delta > 0$, this expression must be negative for some χ_1, since Hardy's inequality is known to fail in two dimensions. Explicitly, by choosing

$$\chi_1(r) = \sin(\sqrt{\lambda}\ln r)\mathbf{1}_{(1,e^{\pi/\sqrt{\lambda}})}(r) \qquad \text{for } \lambda = \frac{\delta}{1-\delta} > 0,$$

we have that $\chi_1 \in H^1(0,\infty)$ is compactly supported, and

$$\lim_{n\to\infty} Q_0^\delta[if_0\chi_n] = -\delta\int\frac{\chi_1^2}{r^2}\, r\,dr < 0.$$

Whence, for $\delta > 0$, the mode of order 0 already brings instability. This comes as no surprise as this mode corresponds to infinitesimal rotations (see Appendix A), and we know that the far-field behavior $e^{i\theta}$ is unstable: rotating this far-field behavior decreases the energy.

3.3. **Positivity of Q_1^δ for $\delta \le 0$.** Recall, according to (14), that Q_1^δ is given by

$$Q_1^\delta[\varphi,\psi] = \int_0^\infty \Bigg[|\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 + 2\delta \operatorname{Re}\left\{ \left(\bar\varphi' + \frac{2}{r}\bar\varphi \right) \bar\psi' \right\} + (1+\delta)\left\{ f_0^2 |\varphi + \bar\psi|^2 - (1-f_0^2)\left(|\varphi|^2 + |\psi|^2 \right) \right\} \Bigg] rdr.$$

We introduce the quadratic form $A_1 := Q_1^0$, namely

$$A_1[\varphi,\psi] = \int_0^\infty \Bigg[|\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 + f_0^2 |\varphi + \bar\psi|^2 - (1-f_0^2)\left(|\varphi|^2 + |\psi|^2 \right) \Bigg] rdr.$$

It is a known fact that A_1 is nonnegative on $\mathcal{H}_1$, and vanishes exactly at pairs (φ,ψ) corresponding to maps v which are linear combinations of $\partial_x u^0_{\mathrm{rad}}$ and $\partial_y u^0_{\mathrm{rad}}$ (see Appendix A for more details). Moreover, we have

$$\begin{aligned} Q_1^\delta[\varphi,\psi] - (1+\delta)A_1[\varphi,\psi] &= -\delta \int_0^\infty \left[|\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 \right] rdr \\ &\quad + 2\delta \int_0^\infty \operatorname{Re}\left\{ \left(\bar\varphi' + \frac{2}{r}\bar\varphi \right) \bar\psi' \right\} rdr \\ &= -\delta \int_0^\infty \left| \varphi' + \frac{2}{r}\varphi - \bar\psi' \right|^2 rdr - 2\delta \int_0^\infty \frac{d}{dr}\left[|\varphi|^2 \right] dr \\ &= -\delta \int_0^\infty \left| \varphi' + \frac{2}{r}\varphi - \bar\psi' \right|^2 rdr, \end{aligned} \tag{15}$$

for $(\varphi,\psi) \in (C_c^\infty(0,\infty))^2$, hence for all $(\varphi,\psi) \in \mathcal{H}_1$. From this identity we infer that $Q_1^\delta \ge 0$ for every $\delta \in (-1,0]$, and equality can only occur when v is a linear combination of $\partial_x u^0_{\mathrm{rad}}$ and $\partial_y u^0_{\mathrm{rad}}$.

4. Study of the higher modes Q_n^δ for $n \ge 2$

4.1. **Positivity of Q_n^δ for $n \ge 2$ and $\delta \in [-1/\sqrt{5}, 0]$.** Let us recall: in the isotropic case, the positivity of Q_n^δ (any $n \ge 2$) is a consequence of the fact that $Q_n^0 \ge Q_1^0$. Here, from the definition (14) of Q_n^δ, we have

$$Q_n^\delta[\varphi,\psi]-Q_1^\delta[\varphi,\psi] = (n-1)\int_0^\infty \bigg[\frac{n+3}{r^2}|\varphi|^2+\frac{n-1}{r^2}|\psi|^2-2\delta\frac{n+1}{r^2}\operatorname{Re}\left\{\bar\varphi\bar\psi\right\} + 2\frac{\delta}{r}\operatorname{Re}\left\{\bar\varphi\bar\psi'-\bar\varphi'\bar\psi\right\}\bigg]\,rdr. \tag{16}$$

Unlike what happens in the isotropic case, this does not obviously have a sign (because of the last term which contains derivatives).

It seems reasonable to use a decomposition for φ,ψ adapted to Q_1^δ, as in Appendix A. Accordingly, we define for any real-valued test functions ζ,η, the adapted quadratic form

$$B_n^\delta[\zeta,\eta]=\frac12 Q_n^\delta\left[f_0'\zeta-r^{-1}f_0\eta, f_0'\zeta+r^{-1}f_0\eta\right]$$

Decomposing

$$Q_n^\delta=(1+\delta)A_1+Q_1^\delta-(1+\delta)A_1+Q_n^\delta-Q_1^\delta$$

and using the above expressions of $Q_n^\delta-Q_1^\delta$ (16) and $Q_1^\delta-(1+\delta)A_1$ (15), we have, for real-valued $(\varphi,\psi)\in\mathcal{H}_1$:

$$\begin{aligned}Q_n^\delta[\varphi,\psi]=(1+\delta)&A_1[\varphi,\psi]\\ &-\delta\int_0^\infty\left(\varphi'+\frac2r\varphi-\psi'\right)^2 rdr\\ &+(n-1)\int_0^\infty\left[\frac{n+3}{r^2}\varphi^2+\frac{n-1}{r^2}\psi^2-2\delta\frac{n+1}{r^2}\varphi\psi\right]\\ &+2\delta(n-1)\int_0^\infty\frac1r\left(\varphi\psi'-\varphi'\psi\right) rdr.\end{aligned}$$

When plugging in $\varphi=f_0'\zeta-r^{-1}f_0\eta$, $\psi=f_0'\zeta+r^{-1}f_0\eta$, the first term significantly simplifies thanks to the formula (19) for A_1 in Appendix A. For the other terms we directly expand

$$\begin{aligned}&\varphi'+\frac2r\varphi-\psi'=2f_0'\frac{\zeta-\eta}{r}-2\frac{f_0}{r}\eta',\\ &\frac{n+3}{r^2}\varphi^2+\frac{n-1}{r^2}\psi^2-2\delta\frac{n+1}{r^2}\varphi\psi\\ &=2(1-\delta)\frac{n+1}{r^2}(f_0'\zeta)^2+2(1+\delta)\frac{n+1}{r^2}\left(\frac{f_0}{r}\eta\right)^2-\frac{8}{r^2}f_0'\zeta\frac{f_0}{r}\eta\\ &\varphi\psi'-\varphi'\psi=2\left(\frac{f_0}{r}\eta\right)'f_0'\zeta-2(f_0'\zeta)'\frac{f_0}{r}\eta,\end{aligned}$$

from which it follows that $B_n^\delta[\zeta,\eta]=(1/2)Q_n^\delta[f_0'\zeta - r^{-1}f_0\eta, f_0'\zeta + r^{-1}f_0\eta]$ can be rewritten as

$$\begin{aligned}
B_n^\delta[\zeta,\eta] &= (1+\delta)\int_0^\infty \left[\frac{f_0^2}{r^2}(\eta')^2 + (f_0')^2(\zeta')^2 + \frac{2}{r^3}f_0 f_0'(\eta-\zeta)^2\right] rdr \\
&\quad - 2\delta\int_0^\infty \left[\frac{f_0'}{r}(\eta-\zeta) + \frac{f_0}{r}\eta'\right]^2 rdr \\
&\quad + (n-1)\int_0^\infty \left[(1-\delta)\frac{n+1}{r^2}(f_0'\zeta)^2 + (1+\delta)\frac{n+1}{r^2}\left(\frac{f_0}{r}\eta\right)^2 - \frac{4}{r^2}(f_0'\zeta)\left(\frac{f_0}{r}\eta\right)\right] rdr \\
&\quad + 2\delta(n-1)\int_0^\infty \frac{1}{r}\left[\left(\frac{f_0}{r}\eta\right)' f_0'\zeta - (f_0'\zeta)'\frac{f_0}{r}\eta\right] rdr.
\end{aligned} \tag{17}$$

Integrating by parts, the last integral becomes

$$\begin{aligned}
\int_0^\infty \frac{1}{r}\left[\left(\frac{f_0}{r}\eta\right)' f_0'\zeta - (f_0'\zeta)'\frac{f_0}{r}\eta\right] rdr &= 2\int_0^\infty \left(\frac{f_0}{r}\eta\right)' f_0'\frac{\zeta}{r}\, rdr \\
&= 2\int_0^\infty \left[\left(f_0' - \frac{f_0}{r}\right) f_0'\frac{\eta}{r}\frac{\zeta}{r} + \frac{f_0}{r}\eta' f_0'\frac{\zeta}{r}\right] rdr.
\end{aligned}$$

We use the first positive term in (17) in order to absorb this latter term: thanks to the identity

$$\begin{aligned}
(1+\delta)\frac{f_0^2}{r^2}(\eta')^2 + 4\delta(n-1)\frac{f_0}{r}\eta' f_0'\frac{\zeta}{r} &= (1+\delta)\left(\frac{f_0}{r}\eta' + \frac{2\delta}{1+\delta}(n-1)f_0'\frac{\zeta}{r}\right)^2 \\
&\quad - 4\frac{\delta^2}{1+\delta}(n-1)^2\left(f_0'\right)^2\left(\frac{\zeta}{r}\right)^2,
\end{aligned}$$

we rewrite (17) as

$$\begin{aligned}
B_n^\delta[\zeta,\eta] &= B_n^{\delta,1}[\zeta,\eta] + (n-1)B_n^{\delta,2}[\zeta,\eta], \\
B_n^{\delta,1}[\zeta,\eta] &= (1+\delta)\int_0^\infty \left[\left(\frac{f_0}{r}\eta' + \frac{2\delta}{1+\delta}(n-1)f_0'\frac{\zeta}{r}\right)^2 + (f_0')^2(\zeta')^2\right] rdr \\
&\quad + 2\int_0^\infty \left\{(1+\delta)f_0'\frac{f_0}{r}\frac{(\eta-\zeta)^2}{r^2} - \delta\left[\frac{f_0'}{r}(\eta-\zeta) + \frac{f_0}{r}\eta'\right]^2\right\} rdr, \\
B_n^{\delta,2}[\zeta,\eta] &= \int_0^\infty q_n^\delta(r)\left[f_0'\frac{\zeta}{r}, \frac{f_0}{r}\frac{\eta}{r}\right] rdr,
\end{aligned}$$

and $q_n^\delta(r)$ is the quadratic form on $\mathbb{R}^2$ given by

$$\begin{aligned}
q_n^\delta(r)[X,Y] &= a_n X^2 + b_n Y^2 + 2c(r)XY, \\
a_n &= (1-\delta)(n+1) - 4\frac{\delta^2}{1+\delta}(n-1) \\
b_n &= (1+\delta)(n+1) \\
c(r) &= -2 - 2\delta\left(1 - r\frac{f_0'}{f_0}\right)
\end{aligned}$$

We readily see that $B_n^{\delta,1}$ is nonnegative for $\delta \leq 0$. Moreover, since $1 > rf_0'/f_0 > 0$ [7, Proposition 2.2], for $\delta \leq 0$, it follows that

$$|c(r)| \leq 2.$$

As $b_n > 0$, a sufficient condition for $q_n^\delta(r)$ to be positive definite for all $r > 0$ is

$$4 < a_n b_n = (1-\delta^2)(n+1)^2 - 4\delta^2(n^2-1).$$

This amounts to the condition

$$0 < \alpha(\delta)n^2 + \beta(\delta)n + \gamma(\delta),$$

where

$$\begin{aligned} \alpha(\delta) &= 1 - 5\delta^2, \\ \beta(\delta) &= 2(1-\delta^2), \\ \gamma(\delta) &= -3(1-\delta^2). \end{aligned}$$

For $\delta \in [-1/\sqrt{5}, 0]$ we have $\alpha(\delta), \beta(\delta) \geq 0$ so that the above polynomial in n is nondecreasing on $[0, +\infty)$. Hence, it is positive for all values of $n \geq 2$ if and only if it is positive for $n = 2$. That is,

$$0 < 4\alpha(\delta) + 2\beta(\delta) + \gamma(\delta) = 5 - 21\,\delta^2.$$

We deduce that q_n^δ is a positive definite quadratic form for all $n \geq 2$ whenever $\delta \in [-1/\sqrt{5}, 0]$. In particular, $B_n^{\delta,2} \geq 0$ and therefore $Q_n^\delta \geq 0$ for $\delta \in [-1/\sqrt{5}, 0]$, with equality only at $(0,0)$.

4.2. **Instability for $\delta \approx -1$.** In this section we show that Q_n^δ can take negative values for $\delta \approx -1$ and $n \geq 1$ large enough. To this end, we choose $\eta = \zeta$ in (17), to obtain

$$\begin{aligned} \hat{B}_n^\delta[\zeta] &= B_n^\delta[\zeta, \zeta] \\ &= (1-\delta)\int_0^\infty \frac{f_0^2}{r^2}(\zeta')^2\, rdr + (1+\delta)\int_0^\infty (f_0')^2(\zeta')^2\, rdr + (n-1)\int_0^\infty \frac{\zeta^2}{r^2}\alpha_n^\delta(r)\, rdr \end{aligned}$$

$$\alpha_n^\delta(r) = (1-\delta)(n+1)(f_0')^2 + (1+\delta)(n+1)\left(\frac{f_0}{r}\right)^2 - 2(2+\delta)f_0'\frac{f_0}{r} + 2\delta(f_0')^2 - 2\delta f_0 f_0''.$$

Using the asymptotics of f_0 ([1,6])

$$f_0(r) = 1 - \frac{1}{2}r^{-2} + O(r^{-4}), \quad f_0'(r) = r^{-3} + O(r^{-5}), \quad f_0''(r) = -3r^{-4} + O(r^{-6}),$$

we find, for $r \to +\infty$,

$$\alpha_n^\delta(r) = \frac{(1+\delta)(n+1)}{r^2}\left(1 - \frac{1}{r^2}\right) - 4\frac{1-\delta}{r^4} + O(r^{-6}).$$

For $\delta = -1$ the leading order is negative. Hence, there exists $\varepsilon > 0$ and a compact interval $[r_0, r_0+1]$ on which $\alpha_n^{-1} \leq -2\varepsilon$. Thus, we deduce that for all $n \geq 2$ there exists $\delta_n > -1$ such that for all $\delta \in (-1, \delta_n]$,

$$-\varepsilon \geq \alpha_n^\delta(r), \qquad \forall r \in [r_0, r_0+1].$$

Choosing a nonzero test function ζ_0 with support in $[r_0, r_0+1]$, we obtain

$$\hat{B}_n^\delta[\zeta_0] \le C_1(\zeta_0) - (n-1)\varepsilon C_2(\zeta_0) \qquad \forall \delta \in (-1, \delta_n],$$

for some $C_1(\zeta_0), C_2(\zeta_0) > 0$. If n is large enough this becomes negative. Compared to the isotropic case this is a really new situation: lower modes are positive but higher modes can bring instability.

5. Proof of Theorem 1.3

In what precedes we have shown that $u_{\rm rad}^\delta$ is nondegenerately stable for small $\delta \le 0$, and unstable for $\delta > 0$ and δ close to -1. In particular, setting

$$\delta_1 = \sup\{\delta \in (-1,0)\colon u_{\rm rad}^\delta \text{ is unstable }\},$$

we know that $-1 < \delta_1 < 0$. It remains to show that $u_{\rm rad}^\delta$ is unstable for all $\delta \in (-1, \delta_1)$, and nondegenerately stable for $\delta \in (\delta_1, 0]$.

Let $\delta' \in (-1, \delta_1)$ be such that $u_{\rm rad}^\delta$ is unstable, that is, $\mathcal{Q}^{\delta'}[v] < 0$ for some choice of $v \in H$. Given that $\delta \mapsto \mathcal{Q}^\delta[v]$ is an affine function which is nonnegative for $\delta = 0$ and negative for $\delta = \delta'$, we deduce that $\mathcal{Q}^\delta[v] < 0$ for all $\delta \le \delta'$. Therefore, $u_{\rm rad}^\delta$ is unstable for all $\delta \in (-1, \delta')$. By arbitrariness of δ' we deduce that $u_{\rm rad}^\delta$ is unstable for all $\delta \in (-1, \delta_1)$.

Let us now fix $\delta \in (\delta_1, 0]$. By definition of δ_1, $u_{\rm rad}^\delta$ is not unstable for all $\delta \in (\delta_1, 0]$. In other words, $\mathcal{Q}^\delta[v]$ is nonnegative for all $v \in \mathcal{H}$. It remains to show that, in fact, $\mathcal{Q}^\delta[v] > 0$ for all $v \in \mathcal{H} \setminus \mathrm{span}(\partial_x u_{\rm rad}^0, \partial_y u_{\rm rad}^0)$. We observe that the function $\delta \mapsto \mathcal{Q}^\delta[v]$ is affine for any given $v \in \mathcal{H} \setminus \mathrm{span}(\partial_x u_{\rm rad}^0, \partial_y u_{\rm rad}^0)$; it is positive for $\delta = 0$ because $u_{\rm rad}^0$ is nondegenerately stable, and it is nonnegative for $\delta \in (\delta_1, 0)$. Thus, it must be strictly positive for $\delta \in (\delta_1, 0)$. This proves the desired nondegenerate stability in the announced range.

Appendix A. Positivity of A_0, A_1

We sketch here the approach in [12], adapted to our notation (see also [5]), based on Hardy-type decompositions to show positivity of the two following quadratic forms

$$\begin{aligned} A_0[\varphi] = \int_0^\infty \Bigg[&|\varphi'|^2 + \frac{1}{r^2}|\varphi|^2 \\ &+ 2f_0^2(\mathrm{Re}\,\{\varphi\})^2 - (1-f_0^2)|\varphi|^2 \Bigg] r dr, \\ A_1[\varphi,\psi] = \int_0^\infty \Bigg[&|\varphi'|^2 + |\psi'|^2 + \frac{4}{r^2}|\varphi|^2 \\ &+ f_0^2|\varphi + \bar{\psi}|^2 - (1-f_0^2)\left(|\varphi|^2 + |\psi|^2\right) \Bigg] r dr. \end{aligned}$$

Testing equation (9), solved by f_0, against $f_0|\tilde\varphi|^2$ for any smooth compactly supported $\tilde\varphi\in C_c^\infty(\mathbb{R};\mathbb{C})$, one obtains

$$\int_0^\infty \left[(f_0')^2|\tilde\varphi|^2+2f_0f_0'\tilde\varphi\cdot\tilde\varphi'+\frac{f_0^2}{r^2}|\tilde\varphi|^2-(1-f_0^2)f_0^2|\tilde\varphi|^2\right] rdr=0,$$

so that

$$A_0[f_0\tilde\varphi]=\int_0^\infty \left[f_0^2\left|\tilde\varphi'\right|^2+2f_0^4(\mathrm{Re}\,\{\tilde\varphi\})^2\right] rdr. \tag{18}$$

By density of test functions, and since $f_0>0$, we deduce that $A_0[\varphi]>0$ for any non-zero $\varphi\in\mathcal{H}_0$. Moreover $A_0[\varphi]\approx 0$ exactly when $\varphi\approx if_0$. This corresponds to the fact that in the isotropic case $\delta=0$,

$$\partial_\alpha[e^{i\alpha}u^\delta_{\mathrm{rad}}]_{\lfloor\alpha=0}=if_0e^{i\theta}$$

solves the linearized equation due to rotational invariance.

For A_1, it is convenient to start by splitting it as

$$A_1[\varphi,\psi]=A_1\left[\mathrm{Re}\,\{\varphi\}\,,\mathrm{Re}\,\{\psi\}\right]+A_1\left[\mathrm{Im}\,\{\varphi\}\,,-\mathrm{Im}\,\{\psi\}\right],$$

so we may just treat the case of real-valued test functions φ,ψ. Guided by the fact that

$$\partial_x u^0_{\mathrm{rad}}=e^{i\theta}(f_0'\cos\theta-i\frac{f_0}{r}\sin\theta),\qquad \partial_y u^0_{\mathrm{rad}}=e^{i\theta}(f_0'\sin\theta+i\frac{f_0}{r}\cos\theta),$$

solve the linearized equation around u^0_{rad}, one uses the ansatz

$$\varphi=f_0'\zeta-\frac{f_0}{r}\eta,\qquad \psi=f_0'\zeta+\frac{f_0}{r}\eta,$$

for some real-valued $\eta,\zeta\in C_c^\infty(0,\infty)$. Testing equation (9), solved by f_0, against $f_0r^{-2}\eta^2$ we obtain

$$\int_0^\infty\left[\left(\left(\frac{f_0}{r}\right)'\right)^2\eta^2+2\left(\frac{f_0}{r}\right)'\frac{f_0}{r}\eta\eta'+\frac{2}{r^4}f_0^2\eta^2-\frac{2}{r^3}f_0f_0'\eta^2-(1-f_0^2)\frac{f_0^2}{r^2}\eta^2\right] rdr=0,$$

and similarly testing (9) against $(f_0'\zeta^2)'$ we find

$$\int_0^\infty\left[(f_0'')^2\zeta^2+2f_0'f_0''\zeta\zeta'+\frac{2}{r^2}(f_0')^2\zeta^2-\frac{2}{r^3}f_0f_0'\zeta^2+(3f_0^2-1)(f_0')^2\zeta^2\right] rdr=0.$$

As a consequence of these two identities, we learn

$$\begin{aligned} &A_1\left[f_0'\zeta-r^{-1}f_0\eta,f_0'\zeta+r^{-1}f_0\eta\right]\\ &\quad=2\int_0^\infty\left[\frac{f_0^2}{r^2}(\eta')^2+(f_0')^2(\zeta')^2+\frac{2}{r^3}f_0f_0'(\eta-\zeta)^2\right] rdr. \end{aligned} \tag{19}$$

Since $f_0,f_0'>0$ one may consider the choice

$$\zeta=\frac{1}{2f_0'}(\varphi+\psi),\quad \eta=\frac{r}{2f_0}(\psi-\varphi),$$

and deduce from the above that $A_1[\varphi,\psi] > 0$ for all non-zero $(\varphi,\psi)\in\mathcal{H}_1$. Moreover $A_1[\varphi,\psi] = 0$ exactly when (φ,ψ) is in the real linear span of

$$\left(f_0' - \frac{f_0}{r}, f_0' + \frac{f_0}{r}\right), \quad \left(i\left(f_0' - \frac{f_0}{r}\right), -i\left(f_0' + \frac{f_0}{r}\right)\right),$$

which corresponds to the fact that $\partial_x u^0_{\mathrm{rad}}$ and $\partial_y u^0_{\mathrm{rad}}$ solve the linearized equation.

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