HYPERBOLIC REGULARIZATION EFFECTS FOR DEGENERATE ELLIPTIC EQUATIONS

XAVIER LAMY AND RICCARDO TIONE

Abstract. This paper investigates the regularity of Lipschitz solutions u to the general two-dimensional equation $\operatorname{div}(G(Du)) = 0$ with highly degenerate ellipticity. Just assuming strict monotonicity of the field G and heavily relying on the differential inclusions point of view, we establish a pointwise gradient localization theorem and we show that the singular set of nondifferentiability points of u is $\mathcal{H}^1$-negligible. As a consequence, we derive new sharp partial C^1 regularity results under the assumption that G is degenerate only on curves. This is done by exploiting the hyperbolic structure of the equation along these curves, where the loss of regularity is compensated using tools from the theories of Hamilton-Jacobi equations and scalar conservation laws. Our analysis recovers and extends all the previously known results, where the degeneracy set was required to be zero-dimensional.

1. Introduction

The focus of this paper is the regularity of two dimensional Lipschitz solutions to the elliptic equation:

$$\operatorname{div}(G(Du)) = 0 \text{ in } \mathcal{D}'(B_1). \tag{1.1}$$

To motivate the following discussion, let us consider, for the moment, its *variational* counterpart:

$$\operatorname{div}(Df(Du)) = 0 \text{ in } \mathcal{D}'(B_1), \tag{1.2}$$

which is the Euler-Lagrange equation satisfied by minima of the functional

$$\int_{B_1} f(Du)\,dx. \tag{1.3}$$

It is classical that solutions u to (1.2) are smooth provided f is convex, $C^2(\mathbb{R}^2)$, and

$$\Lambda^{-1}\,\mathrm{id} \le D^2 f(\xi) \le \Lambda\,\mathrm{id}, \text{ in the sense of quadratic forms.} \tag{1.4}$$

In higher dimensions, this is due to the celebrated theorems of De Giorgi-Nash-Moser [16, 45, 48], while in two dimensions it has been known since the work of Morrey [44]. In fact, in two dimensions, we will see below that one may assume only one of the bounds in (1.4) and still infer C^1 regularity of Lipschitz solutions to (1.2), provided f is strictly convex. In the context of the more general equation (1.1), strict convexity of $f \in C^1(\mathbb{R}^2)$ should be replaced by strict monotonicity of $G \in C^0(\mathbb{R}^2, \mathbb{R}^2)$:

$$(G(b) - G(a), b - a) \ge \sigma(|b - a|), \quad \forall a, b \in \mathbb{R}^2, \tag{1.5}$$

where

$$\sigma : [0, +\infty) \to [0, +\infty) \text{ is a strictly increasing, convex function.} \tag{1.6}$$

In the following, for brevity we will often say that G fulfills (1.5) to say that G fulfills (1.5) for σ fulfilling (1.6). This discussion motivates the following question, which is the starting point of this paper. Assume $G \in C^0(\mathbb{R}^2, \mathbb{R}^2)$ and strictly monotone:

$$\text{are Lipschitz solutions to (1.1) } C^1\text{, or at least partially regular?} \tag{Q}$$

One motivation for this question is that many important and natural energies, such as the p-Laplacian, do not enjoy the strong ellipticity property (1.4) and their regularity theory is more complicated, even in two dimensions. See [42] for an extensive review of problems where (1.4) fails for $|\xi| \to 0$ and $|\xi| \to \infty$. Cases where (1.4) fails in a more general fashion are much less understood, but have seen a lot of progress in recent years, starting with the work of D. De Silva and O. Savin [20]. In order to review the literature, we introduce some (loosely defined) notation. We let $\mathcal{D}_+$ and $\mathcal{D}_-$ be the sets in $B_{\mathrm{Lip}(u)}$ where the symmetric part of $(DG)^{-1}$, respectively of DG, has a zero eigenvalue, see (1.16) for the formal definition. In the variational case (1.2), $\mathcal{D}_+$ and $\mathcal{D}_-$ should be thought as the sets of points in $B_{\mathrm{Lip}(u)}$ at which $D^2 f$ has an infinite eigenvalue and a zero eigenvalue, respectively. Then, the answer to (Q) is *yes*, provided:

- $\mathcal{D}_+ \cap \mathcal{D}_-$ is empty or $\mathcal{D}_-$ is finite, [20];
- $\mathcal{D}_+ \cap \mathcal{D}_-$ is finite, [35];
- G is δ-monotone, see [2, Chapter 16.4.1]. In this case, u needs not be Lipschitz;
- Du lies in a curve and it solves (1.1)-(1.5) for $\sigma(t) \sim t^4$, [37].

Partial regularity is also obtained in [20] for an obstacle problem involving minimizers of (1.3) under the constraint that Du lies in a convex polygon, with a degeneracy set $\mathcal{D}_+ \cap \mathcal{D}_-$ that may contain the polygon's boundary. Further, an example given in [35, Theorem 1.5] shows that, under the mere assumption (1.5), Lipschitz solutions of (1.1) may have point singularities, hence one cannot expect better in general than partial regularity in (Q).

Having asked that $u \in \mathrm{Lip}(B_1)$, (Q) becomes a question about controlling the *oscillations* in the gradient of Du. Of course, a similar question could be asked about *concentration* effects, namely starting from a $W^{1,p}(B_1)$ function. Some results in this direction, and valid in all dimensions, can be found in [14, 15] and references therein. Notice, however, that in the unbounded gradient case, further and careful assumptions need to be placed on u, f or G, as counterexamples to regularity of very weak solutions to elliptic equations show, see [9, 30]. Besides, in two dimensions the Lipschitz property of u can be inferred naturally in many situations, compare [24, §12.4] and [8], valid in all dimensions. Another direction in which to extend this analysis is, of course, higher dimensions, see [40, 41, 43]. In that case, however, the picture is much less clear than in two dimensions: to the best of our knowledge, in that context even partial regularity for u solving (1.2) for f strictly convex, smooth, and fulfilling only the upper bound in (1.4) is unknown.

As said, the aim of this work is to answer Question (Q). Let us give a rough idea of our main results, deferring a more precise description to §1.2-1.3-1.4 below. The starting point is to write, formally, (1.1) as

$$\langle DG^s(Du), D^2 u\rangle = 0.$$

Then, if $Du(x) \notin \mathcal{D}_+ \cap \mathcal{D}_-$, we can expect some sort of ellipticity in a neighborhood of x, while all ellipticity is lost if $Du(x) \in \mathcal{D}_+ \cap \mathcal{D}_-$. However, if $\mathcal{D}_+ \cap \mathcal{D}_-$ has a sufficiently nice structure, we can interpret

$$Du \in \mathcal{D}_+ \cap \mathcal{D}_- \tag{1.7}$$

as an additional equation, coupled with (1.1). Our results show (partial) C^1 regularity under, for instance, the assumption that $\mathcal{D}_+ \cap \mathcal{D}_-$ is a finite union of curves. In that case, (1.7) coupled with (1.1) becomes a hyperbolic problem: to study it, we rely on the theories of viscosity solutions to Hamilton-Jacobi equations and entropy solutions of scalar conservation laws. It is important to note that our results recover and extend all the results contained in the previous list where, essentially, $\mathcal{D}_+ \cap \mathcal{D}_-$ had to be taken of dimension zero.

To implement the strategy outlined above, the crucial point is to obtain a good localization property of the gradient of a solution of (1.1), Theorem A. This property, inspired by the beautiful separation argument of [22], tells us that oscillations of the gradient are small in L^∞ in any small enough neighborhood of a differentiability point (where oscillations would a priori be small only in L^p for $p < \infty$). Through Theorem A, we are able to efficiently split B_1 into regular and singular points, based on the Lipschitz blow-ups of u. Our next result, Theorem B, shows that the singular set is $\mathcal{H}^1$-negligible. Next, in Theorem C, we show that one can associate to a solution to (1.1) a measure that represents the determinant of its Hessian. It is worth noticing that Theorems A-B-C hold purely under assumptions (1.1)-(1.5), hence they may be of independent interest. Next, we exploit Theorems A-B to analyze the inclusion (1.7) coupled with (1.1), and obtain partial C^1 regularity together with a precise description of the singular set in Theorem D (which extends [37] to general σ). Finally, we combine Theorems A-B-C-D with the very recent [34] to deduce partial C^1 regularity for solutions to the general equation (1.1), Theorem E.

In the course of the paper, it will be convenient to take into account three equivalent viewpoints: other than (1.1), it is better sometimes to look at solutions to differential inclusions, and sometimes at solutions to nonlinear Beltrami equations. While we defer the explanation of the latter to §2.2, where we will also thoroughly explain the interplay among the three, let us focus on the theory of differential inclusions in two dimensions in the next section. This will be useful to introduce our results in §1.2-1.3-1.4.

1.1. **The differential inclusions point of view.** Given a solution $u \in \mathrm{Lip}(B_1)$ to (1.1), it is rather easy to construct a solution $w = (u, v) \in \mathrm{Lip}(B_1, \mathbb{R}^2)$ to

$$Dw \in K \text{ a.e. in } B_1, \tag{1.8}$$

where $K \subset \mathbb{R}^{2\times 2}$ fulfills

$$\sigma(|X - Y|) \le \det(X - Y), \quad \forall X, Y \in K, \tag{1.9}$$

as soon as G fulfills the monotonicity condition (1.5)-(1.6). The converse also holds: given a solution $w = (u, v)$ to (1.8) with K enjoying property (1.9), then u solves (1.1) for G satisfying (1.5)-(1.6). The details of this translation is the content of Proposition 2.2 below.

In the theory of differential inclusions, *ellipticity* can be defined as in [54, §5], where a compact set K is called *elliptic* if it fulfills (1.9), it is a manifold and its tangent space at every point has no rank-one connections. Recall that $A, B \in \mathbb{R}^{n\times m}$ are rank one-connected if $\mathrm{rank}(A - B) = 1$. For consistency with the usual terminology in the world of PDEs, in this paper we refer to this requirement as *uniform ellipticity*. For example, the aforementioned p-Laplacian is an *elliptic* equation which does not admit elliptic linearizations at all points (hence is not uniformly elliptic) if $p \neq 2$. Hence, we give a weaker definition of ellipticity than [54], that a priori includes and extends the one carried by the PDE (1.1), where G fulfills (1.5)-(1.6). While this is just a matter of terminology, it is crucial to underline the difference between our meaning of the word *elliptic* and the one of [54]: the fact that our elliptic sets may not admit elliptic tangent spaces at all points is the crux of the matter in Question (Q). In simple terms, our notion of ellipticity is equivalent to *approximate rigidity* of K:

Definition. A compact set $K \subset \mathbb{R}^{n\times m}$ is called *elliptic* if, given any equi-Lipschitz sequence $(u_k)_k$ on B_1,

$$\mathrm{d}(Du_k, K) \to 0 \text{ in } \mathcal{D}'(B_1) \quad \Longrightarrow \quad (u_k)_k \text{ is } W^{1,1}_{\mathrm{loc}} \text{ strongly precompact.} \tag{1.10}$$

Remark. Equivalently, K is elliptic if it only supports trivial homogeneous gradient Young measures. For a thorough introduction on Young measures, see for example [46].

The primary examples of elliptic sets are precisely those with property (1.9), compare [54, Theorem 1]:

Theorem 1.1. *If $K \subset \mathbb{R}^{2\times 2}$ is a compact set fulfilling* (1.9)*, then K is elliptic.*

Showing that a set in $\mathbb{R}^{n\times m}$ is elliptic is, in general, very hard. In many situations, for instance in some convex integration schemes [47, 52], it is actually sufficient to show that the set is *not* elliptic by finding special subsets of matrices inside K: common choices of such special sets are rank-one connections and T_N configurations, see [53, Definition 1]. If $K \subset \mathbb{R}^{2\times 2}$, we have a much clearer picture. In particular, from the deep works [22, 53] we have the following very simple way of deciding whether a given set in $\mathbb{R}^{2\times 2}$ is elliptic.

Theorem 1.2. *Let $K \subset \mathbb{R}^{2\times 2}$ be a compact set. Then K is elliptic if and only if K does not contain rank-one connections and T_4 configurations.*

This result sheds a much clearer light on the notion of ellipticity for (1.8), since it provides an algorithmic way of checking it. We can then reformulate question (Q) in the language of differential inclusions:

Let K be compact and elliptic. Are solutions to (1.8) C^1, or at least partially regular? (Q:DI)

A deep fact is that Question (Q:DI) can be completely reduced to its, a priori, simpler counterpart Question (Q). This is done in two steps:

Theorem 1.3 ([32])**.** *Let $K \subset \mathbb{R}^{2\times 2}$ be a compact elliptic set. Then, the gradient of any solution $w \in \mathrm{Lip}(B_1,\mathbb{R}^2)$ to (1.8) has connected essential range.*

Here, if $\Omega \subset \mathbb{R}^m$ is an open set, for a map $f \in L^1(\Omega,\mathbb{R}^N)$ we denote by $[f](\Omega')$ its *essential range in* $\Omega' \subset \Omega$, namely the smallest closed set with the property that

$$f(x) \in [f](\Omega') \text{ for a.e. } x \in \Omega'. \tag{1.11}$$

We will write $[f]$ for $[f](\Omega)$ (and call it simply the *essential range* of f) and we also set:

$$[f](x_0) \doteq [f]\left(\bigcap_{r>0} B_r(x_0)\right). \tag{1.12}$$

We will add a few words on the essential range of the gradient of a map and its relation with Clarke's generalized differential in §2.1.1.

Remark 1.4. The statement of [32, Theorem 1] is more general than Theorem 1.3. It asserts that, for any $w \in \mathrm{Lip}(B_1,\mathbb{R}^2)$, the *rank-one convex hull* $[Dw]^{\mathrm{rc}}$ is connected. This implies Theorem 1.3 because $[Dw]^{\mathrm{rc}} = [Dw]$ if $[Dw] \subset K$ for an elliptic set K. Indeed, the ellipticity assumption ensures that the quasiconvex hull of $[Dw]$, $[Dw]^{\mathrm{qc}}$, which fulfills $[Dw] \subset [Dw]^{\mathrm{rc}} \subset [Dw]^{\mathrm{qc}}$, is equal to $[Dw]$, see [46, §4.4].

To show how to reduce Question (Q:DI) to Question (Q) we need a second, and last, ingredient:

Lemma ([54])**.** *Let $K \subset \mathbb{R}^{2\times 2}$ be a compact, connected set without rank-one connections. Then there exists $c \in \mathbb{R}$ such that $c\det(X - Y) > 0$, $\forall X, Y \in K, X \neq Y$.*

In other words, up to a global multiplication by a matrix with determinant -1, a compact, connected, elliptic set fulfills (1.9). This, combined with Theorem 1.3, allows us to say that, if w is a Lipschitz solution to (1.8) for an elliptic set K, then actually (up to passing to a connected component and up to a global change of sign of determinants) we can assume that K fulfills (1.9), hence there is no real difference between questions (Q) and (Q:DI). This represents a weak converse to Theorem 1.1.

As we see from the above results, the theory of 2×2 elliptic differential inclusions is extremely rich, and this wealth of results and techniques will be essential in our paper. Conversely, our results, especially Theorems A-B-C, represent a step towards a complete understanding of Question (Q:DI).

1.2. **General results.** Our first main theorem asserts that local $W^{1,\infty}$ bounds are stable with respect to local L^∞ convergence for equi-Lipschitz solutions of (1.8).

Theorem A. *Let $K \subset \mathbb{R}^{2\times 2}$ fulfill (1.9), $(w_j)_j \subset \mathrm{Lip}(B_1,\mathbb{R}^2)$ be equi-Lipschitz solutions to (1.8), $x_j \in B_1$, and assume w_j converges locally uniformly in B_1 to w_∞, $x_j \to x_0 \in B_1$. Then we have*

$$\limsup_j \operatorname{diam}([Dw_j](x_j)) \le 2\operatorname{diam}([Dw_\infty](x_0)),$$

where the essential range at a point is defined in (1.12).

It is particularly interesting to apply this result to sequences w_j obtained from rescalings of a single solution w. To that end, it is convenient to set some notation. Given a Lipschitz map $w \in \mathrm{Lip}(B_1,\mathbb{R}^2)$ and any point $x_0 \in B_1$, we can consider the rescalings

$$w_{r,x_0}(h) \doteq \frac{w(x_0 + rh) - w(x_0)}{r}. \tag{1.13}$$

As w is Lipschitz, this sequence is precompact in $C^0(B)$, for any ball $B \subset \mathbb{R}^2$. Therefore, we can introduce the set $\mathcal{B}(w)(x_0)$ of blow-ups of w at x_0 to be the collection of maps obtained as locally uniform limits of any subsequence extracted from w_{r,x_0}. By Rademacher's theorem w is differentiable at a.e. $x_0 \in B_1$. For any such x_0, $\mathcal{B}(w)(x_0) = \{L_{Dw(x_0)}\}$. In general we observe that, for K compact and fulfilling (1.9),

$$\text{any map } w_\infty \in \mathcal{B}(w)(x_0) \text{ still solves (1.8)}, \tag{1.14}$$

due to the strong $W^{1,1}_{\mathrm{loc}}$ convergence of the gradients of rescalings provided by Theorem 1.1.

We may apply Theorem A to a sequence of rescalings as in (1.13) of a given solution w, and $x_j = x_0$ for all j. In that case we infer, in particular, that if *some* blowup $w_\infty \in \mathcal{B}(w)(x_0)$ is linear, say $w_\infty(h) = Ah$, then $\operatorname{diam}([Dw_\infty](x_0)) = 0$, and just by using the definitions, see (1.12), we obtain that

$$\forall \varepsilon > 0, \exists \delta > 0 : \ |Dw(x) - A| \le \varepsilon, \quad \text{for a.e. } x \in B_\delta(x_0). \tag{1.15}$$

We can then say that Dw *localizes near* the matrix A in sufficiently small neighborhoods of x_0. Furthermore, (1.15) implies that w_∞ is the only element of $\mathcal{B}(w)(x_0)$. Using Theorem A, we see that to obtain the localization property (1.15) we do not need w_∞ to be linear in $\mathbb{R}^2$, but we only need it to be differentiable at 0, with differential A. Indeed, since $Dw_\infty \in K$ a.e. in $\mathbb{R}^2$, compare (1.14), then we can employ Theorem A on w_∞ itself to deduce that (1.15) holds for w_∞, which in turn implies that (1.15) holds for w. These considerations allow us to define the regular and the singular set of w. To keep the definitions as general as possible, we assume $\Omega \subset \mathbb{R}^2$ is open and $w \in \operatorname{Lip}(\Omega, \mathbb{R}^2)$ solves $Dw \in K$ a.e. in Ω, for K fulfilling (1.9).

Definition 1.5. We let

$$\operatorname{Reg}(w) \doteq \{x \in \Omega : \mathcal{B}(w)(x) \text{ contains a map which is differentiable at } 0\} \quad \text{and} \quad \operatorname{Sing}(w) \doteq \Omega \setminus \operatorname{Reg}(w).$$

The above discussion shows that, thanks to Theorem A, $x \in \operatorname{Reg}(w)$ if and only if $\mathcal{B}(w)(x)$ consists of a single linear map, that is, w is differentiable at x. If $\operatorname{Reg}(w)$ contains an open ball B, then we infer that w is C^1 on B, see Lemma 2.1, whence the name *regular points*. We cannot show that $\operatorname{Reg}(w)$ is open in this generality, and we will need to restrict ourselves to cases where K has more structure. Nonetheless, our second result still holds for any elliptic inclusion set K, and concerns the size of $\operatorname{Sing}(w)$. We denote by $\mathcal{H}^\alpha$ the α-dimensional Hausdorff measure.

Theorem B. *Let $K \subset \mathbb{R}^{2\times 2}$ fulfill (1.9) and $w \in \operatorname{Lip}(B_1, \mathbb{R}^2)$ solve (1.8). Then $\mathcal{H}^1(\operatorname{Sing}(w)) = 0$.*

Finally, we will show another remarkable property of solutions to general elliptic differential inclusions. Recall that the Hessian determinant of a smooth function u of two variables can be written as $\det(D^2u) = \operatorname{div}\operatorname{div}(Du^\perp \otimes Du^\perp)$, see (5.1), and the latter expression makes sense as a distribution if u is merely H^1.

Theorem C. *Let $K \subset \mathbb{R}^{2\times 2}$ fulfill (1.9) and $w = (u, v) \in \operatorname{Lip}(B_1, \mathbb{R}^2)$ solve (1.8). Then, $\det(D^2u)$ is a non-positive measure μ. Moreover, if $w_\infty = (u_\infty, v_\infty) \in \mathcal{B}(w)(x_0)$, then $\det(D^2u_\infty) = \mu(\{x_0\})\delta_0$.*

Similar properties hold for infinity harmonic functions [33, Theorem 1.5]. Let us add a few words on Theorem C. Ideally, one could hope to deduce from it that $w_\infty \in \mathcal{B}(w)(x_0)$ is linear at points where $\mu(\{x_0\}) = 0$ and that w_∞ is one-homogeneous at points where $\mu(\{x_0\}) \neq 0$. This would tell us that $\operatorname{Sing}(w)$ is actually countable, since a finite measure can have at most countably many atoms. The reason for this hope is that the equation $\det(D^2u) = 0$ in an open set Ω implies a very rigid structure of Du: essentially, that Du is constant along segments connecting the boundary. This, in turn, yields that Du is constant if u is defined in $\mathbb{R}^2$. Such rigidity property has been shown assuming $u \in C^2$ [26], $u \in W^{2,\infty}$ [31, §2.6], $u \in W^{2,2}$ [49] and u in the class *MA* [29]. In all of these cases, Du is (at least weakly) differentiable. Here, however, the only way we can interpret the distribution $\det(D^2u)$ is in the *very weak sense*, see the distribution $\mathcal{D}(u, u)$ in (5.1). The equation $\mathcal{D}(u, u) = 0$ is extremely flexible, even if $u \in C^{1,\alpha}$ for $\alpha > 0$, see [6, 7, 38]. In a sense, a function u which satisfies $\mathcal{D}(u, u) = 0$ and also (1.1) for a strictly monotone G, lies in a realm between the known rigidity and flexibility statements. The fact that some rigidity can be expected can be seen in Proposition 7.5, where Theorem C will be instrumental to exclude some types of blowups.

1.3. **Regularity for inclusions in elliptic curves.** Assume now that $K = \Gamma \subset \mathbb{R}^{2\times 2}$ satisfies (1.9) and is a C^1 curve, that is, a connected, compact C^1 submanifold of $\mathbb{R}^{2\times 2}$ of dimension one, which may have a nonempty boundary. Hence it is given by $\Gamma = \gamma(I)$ for some $I = [a, b]$ or $\mathbb{R}/L\mathbb{Z}$ and $\gamma \in C^1(I, \mathbb{R}^{2\times 2})$ a homeomorphism onto Γ with $|\gamma'| > 0$ on I. As previously mentioned, the next Theorem generalizes the main result of [37], where the same conclusion was obtained under the assumption that (1.9) is satisfied with $\sigma(t) = ct^4$ for some $c > 0$.

Theorem D. *Let $\Omega \subset \mathbb{R}^2$ be open. If $w \in \mathrm{Lip}(\Omega, \mathbb{R}^2)$ satisfies $Dw \in \Gamma$ a.e. in Ω, then $\mathrm{Sing}(w)$ is locally finite. If Ω is convex, then $\mathrm{Sing}(w)$ contains at most two points. Moreover, $\mathrm{Sing}(w)$ is empty if one of the following sufficient conditions is satisfied.*

- *Γ is simply connected, that is, $I = [a, b]$;*
- *Γ is not fully degenerate, that is, it admits a tangent line generated by a rank-2 matrix;*
- *there exists $a \in \mathbb{S}^1$ such that the projection $a\Gamma \subset \mathbb{R}^2$ is the boundary of a strictly convex open set.*

In fact, under the last assumption, the conclusion holds even if Γ is not assumed C^1, see Corollary 6.3. We also show that the map Dw is locally constant along characteristic lines outside its singular set, see Proposition 6.9, and that the singularities have a rigid structure, see Proposition 6.18.

1.4. **Regularity for solutions to degenerate equations.** Following [20, 35], for a strictly monotone $G \in C^0(\mathbb{R}^2, \mathbb{R}^2)$ we define its degeneracy set $\mathcal{D} \doteq \mathcal{D}_- \cap \mathcal{D}_+$ which can be interpreted as the set of points where the symmetric parts of DG and $(DG)^{-1}$ both have zero eigenvalues. More precisely, the sets $\mathcal{D}_\pm$ are:

$$\begin{aligned} \mathcal{D}_- = \mathcal{D}_-(G) &= \bigcap_{\lambda>0} \overline{\left\{ X \in \mathbb{R}^2 \colon \liminf_{H \to 0} \frac{(G(X+H) - G(X), H)}{|H|^2} \le \lambda \right\}}, \\ \mathcal{D}_+ = \mathcal{D}_+(G) &= \bigcap_{\lambda>0} \overline{\left\{ X \in \mathbb{R}^2 \colon \liminf_{H \to 0} \frac{(G(X+H) - G(X), H)}{|G(X+H) - G(X)|^2} \le \lambda \right\}}. \end{aligned} \tag{1.16}$$

They correspond to the smallest closed sets outside which G is locally elliptic from below or above:

$$\begin{aligned} X \in \mathcal{D}_-^c \quad &\iff \quad \exists \lambda, \delta > 0 \,:\, (G(X_2) - G(X_1), X_2 - X_1) \ge \lambda |X_2 - X_1|^2 \quad \forall X_1, X_2 \in B_\delta(X)\,, \\ X \in \mathcal{D}_+^c \quad &\iff \quad \exists \lambda, \delta > 0 \,:\, (G(X_2) - G(X_1), X_2 - X_1) \ge \lambda |G(X_2) - G(X_1)|^2 \quad \forall X_1, X_2 \in B_\delta(X)\,. \end{aligned}$$

The latter is also equivalent to $G(X) \in \mathcal{D}_-^c(G^{-1})$, justifying its interpretation as local ellipticity from above. In [34], it is shown that blow-up limits of solutions to (1.1) are either affine or take values into $\mathcal{D}$. Combining this remarkable fact with the previous Theorems, we obtain a new partial regularity result under structural assumptions on the connected components of $\mathcal{D}$.

Theorem E. *Assume that each connected component of $\mathcal{D} = \mathcal{D}_- \cap \mathcal{D}_+$ has image through the graph map*

$$\mathbb{R}^2 \ni x \mapsto \begin{pmatrix} x \\ G(x) \end{pmatrix} \in \mathbb{R}^{2\times 2}\,,$$

contained in a C^1 curve. If all but a finite number of these components are

$$\text{either simply connected or boundaries of strictly convex open sets,} \tag{1.17}$$

then any Lipschitz solution u of (1.1) is C^1 outside a locally finite singular set. Moreover, that singular set is empty if all components satisfy (1.17).

This result is sharp, namely solutions can indeed develop singularities, as shown in [35, Theorem 1.5]. Note that Theorem E applies to the field G constructed in [35, Theorem 1.5], for which $\mathcal{D}$ is the first-row projection of a smooth curve, hence all solutions of (1.1) have a discrete singular set. It is also interesting to apply Theorem E in the variational case $G = Df$ with f strictly convex and radial, that is, $f(x) = g(|x|)$ for some increasing and strictly convex C^1 function $g \colon [0, \infty) \to [0, \infty)$ such that $g(0) = g'(0) = 0$. Then we have $X \in \mathcal{D}$ if and only if $|X| \in D_g$, where

$$D_g = \bigcap_{\lambda>0} \overline{\left\{ r \ge 0 \colon \liminf_{h \to 0} \frac{g'(|r+h|) - g'(r)}{h} \le \lambda \right\}} \cap \bigcap_{\lambda>0} \overline{\left\{ r \ge 0 \colon \limsup_{h \to 0} \frac{g'(|r+h|) - g'(r)}{h} \ge \frac{1}{\lambda} \right\}}.$$

If we assume that D_g is totally disconnected, then all connected components of $\mathcal{D}$ are (possibly degenerate) circles, whose image through the graph map is either the zero matrix or a C^1 curve, and we deduce that any Lipschitz minimizer of (1.3) is C^1.

Structure of the paper. In §2 we will introduce the notation, and recall some well-known facts about the essential range that we will need in the sequel. Importantly, we will also explain how to translate elliptic equations into differential inclusions and nonlinear Beltrami systems, and viceversa. Next, §3-4-5-6-7 contain the proofs of Theorems A-B-C-D-E, respectively. Some sections are lengthier than others: in that case we will explain the structure of the section at the start of it.

Acknowledgments. XL is supported by the ANR project ANR-22-CE40-0006.

2. Preliminaries

Let us first recall the notation and some results we will use in the course of our paper.

2.1. General notation and some elementary facts.

Topology. Let $E \subset \mathbb{R}^n$ be any set. Then, $\overline{E}$ denotes its closure, ∂E its topological boundary, E^c its complement in $\mathbb{R}^n$ and $\operatorname{diam}(E)$ its diameter. For two sets A, B, we denote by $\mathrm{d}(A, B)$ the distance between them. Moreover, $A \Subset B$ means that $\overline{A} \subset B$. The open ball of radius r centered at X in $\mathbb{R}^n$ or in $\mathbb{R}^{n\times m}$ is denoted by $B_r(X)$. If $X = 0$, we will simply write B_r. Finally, given a Lipschitz function $f : E \subset \mathbb{R}^m \to \mathbb{R}^n$, we write $\operatorname{Lip}(f)$ for its Lipschitz constant.

Measure theory. $|E|$ denotes the Lebesgue measure of a measurable set $E \subset \mathbb{R}^n$. We write $\mathbf{1}_E$ for the indicator function of E, namely $\mathbf{1}_E(x) = 1$ if $x \in E$ and $\mathbf{1}_E(x) = 0$ otherwise. In the case of super-level sets of a function $u : F \subset \mathbb{R}^n \to \mathbb{R}$ (and analogously for sub-level sets and for preimages of intervals) we will use expressions such as $\mathbf{1}_{\{u>\alpha\}}$ to denote the function that, at $x \in F$, returns 1 if $u(x) > \alpha$ and 0 otherwise. Let $|E| < +\infty$, then $⨍_E f(x)dx$ denotes the average of f over E. In most cases we will consider only L^p spaces with respect to the Lebesgue measure. If we need to say that $\Phi \in L^p$ with respect to another finite measure, we will write $L^p(\Omega, X; \mu)$, the space of L^p functions with respect to the measure μ on Ω and with values in the set X.

Linear Algebra. We use the notation e_i for the vectors of the canonical basis of $\mathbb{R}^2$, $e_1 = (1, 0)$ and $e_2 = (0, 1)$. For a matrix $A \in \mathbb{R}^{n\times m}$, we denote by L_A the linear map associated to A, $L_A(x) \doteq Ax$. The image of L_A is the *range* of A, $\operatorname{ran} A$. $\det(A)$, A^T and $|A|$ denote the determinant, the transpose and the Euclidean norm of the matrix A, respectively. The (standard) scalar product between matrices is denoted by $\langle A, B\rangle$, while for vectors a, b we use (a, b) or, if confusion may arise, $a \cdot b$. The cofactor matrix is

$$\operatorname{cof}(A) \doteq \begin{pmatrix} d & -c \\ -b & a \end{pmatrix}, \quad \text{if } A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}, \quad \text{so that } \operatorname{cof}^T(A)A = A\operatorname{cof}^T(A) = \det(A)\,\mathrm{id}\,. \tag{2.1}$$

Complex derivatives. It will be convenient to use *complex* notation to represent matrices $A \in \mathbb{R}^{2\times 2}$. Let A be as in (2.1). Its conformal and anti-conformal parts are defined, respectively, as follows:

$$[A]_{\mathcal{H}} \doteq \frac{1}{2}[(a + d) + i(c - b)] \quad \text{and} \quad [A]_{\overline{\mathcal{H}}} \doteq \frac{1}{2}[(a - d) + i(c + b)]$$

In particular, we have the fundamental identity:

$$Az = [A]_{\mathcal{H}} z + [A]_{\overline{\mathcal{H}}}\overline{z}, \quad \forall z \in \mathbb{R}^2, \tag{2.2}$$

where the product on the left-hand side is the classical one between a matrix and a vector and the one on the right-hand side is the complex multiplication. We have the following relations:

$$\det(A) = |[A]_{\mathcal{H}}|^2 - |[A]_{\overline{\mathcal{H}}}|^2, \quad |A|^2 = 2|[A]_{\mathcal{H}}|^2 + 2|[A]_{\overline{\mathcal{H}}}|^2. \tag{2.3}$$

As a short-hand notation, we will write expressions such as:

$$A = (a, b), \tag{2.4}$$

which mean that a is the conformal part of A and b is the anti-conformal part of A.

Finally, if $\Omega \subset \mathbb{C}$ is an open set and $f : \Omega \to \mathbb{C}$, whenever its differential makes sense we can set

$$\begin{aligned} \partial_z f = f_z &\doteq [Df]_{\mathcal{H}} = \frac{1}{2}[(\partial_1 f_1 + \partial_2 f_2) + i(\partial_1 f_2 - \partial_2 f_1)], \\ \partial_{\bar z} f = f_{\bar z} &\doteq [Df]_{\overline{\mathcal{H}}} = \frac{1}{2}[(\partial_1 f_1 - \partial_2 f_2) + i(\partial_1 f_2 + \partial_2 f_1)]. \end{aligned} \tag{2.5}$$

2.1.1. *Essential range and generalized differentials.* As defined in (1.11), given any $w \in \mathrm{Lip}(\Omega, \mathbb{R}^n)$, $\Omega \subset \mathbb{R}^m$ open, we let $[Dw]$ be its essential range over Ω, defined as the smallest closed set K' such that $Dw(x) \in K'$ for a.e. $x \in \Omega$. It is immediate to see that if $E \subset \Omega$ is the set of Lebesgue points for Dw, then,

$$[Dw] = \overline{\{Dw(x) \in \mathbb{R}^{n\times m} : x \in E\}}.$$

This, together with (1.12), is closely related to the set of *reachable gradients of* w, see for instance [5, Definition 3.1.10], which is typically defined by considering the set E of differentiability points of w. In the case we are interested in, namely for solutions w to (1.8), there is no difference between the two sets: if K fulfills (1.9) and w solves (1.8), then x_0 is a differentiabilty point for w if and only if x_0 is a Lebesgue point for Dw. Finally, let us show a simple but crucial property of the set $\mathrm{Reg}(w)$, which justifies its name.

Lemma 2.1. *Let $K \subset \mathbb{R}^{2\times 2}$ fulfill (1.9). Let w solve (1.8) and let $\mathrm{Reg}(w)$ be as in Definition 1.5. If $\mathrm{Reg}(w)$ contains a ball B, then $w \in C^1(B, \mathbb{R}^2)$.*

Proof. By Theorem A and the discussion preceding Definition 1.5, for all $x_0 \in \mathrm{Reg}(w)$ we have $[Dw](x_0) = \{A(x_0)\}$, for some $A(x_0) \in \mathbb{R}^{2\times 2}$. But then the blow-ups at x_0, w_{r,x_0}, actually converge in $W^{1,\infty}_{\mathrm{loc}}$ to $L_{A(x_0)}$, as can be seen by using (1.12). We infer that w is differentiable everywhere in B. Then, continuity follows, again, from (1.12), as $[Dw](x_0) = \{A(x_0)\}$ implies that for all $\varepsilon > 0$, there exists $r > 0$ such that $[Dw](B_r(x_0)) \subset B_\varepsilon(A(x_0)) = B_\varepsilon(Dw(x_0))$. □

2.2. **Three equivalent viewpoints.** We have the following equivalence:

Proposition 2.2. *Let $u \in \mathrm{Lip}(B_1)$ be a solution to (1.1), where G is defined on $F \supseteq [Du]$. Then, there exists $v \in \mathrm{Lip}(B_1)$ such that $w \doteq (u, v)$ solves*

$$Dw \in K = \left\{ \begin{pmatrix} x & y \\ -G_2(x,y) & G_1(x,y) \end{pmatrix} : (x,y) \in F \right\}, \qquad \text{a.e. in } B_1.$$

Moreover, G satisfies (1.5) for all $(x, y) \in F \supseteq [Du]$ with a function σ if and only if K fulfills (1.9) with the same σ. Conversely, if $w = (u, v) \in \mathrm{Lip}(B_1, \mathbb{R}^2)$ satisfies (1.8) and K fulfills (1.9), then there exists a field $G \in C^0(\pi_1(K), \mathbb{R}^2)$ fulfilling (1.5) such that u solves (1.1). Here, π_1 is the projection onto the first row.

Proof. To show the first statement we simply use (1.1) and the convexity of B_1 to find $v \in \mathrm{Lip}(B_1)$ such that $Dv = -JG(Du)$ a.e. in B_1. The rest of the properties are straightforward. Here and in what follows,

$$J \doteq \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix}. \tag{2.6}$$

For the converse statement, we notice that inequality (1.9) reads as

$$(JX_2 - JY_2, X_1 - Y_1) \geq \sigma(|X - Y|), \quad \forall X, Y \in K, \tag{2.7}$$

if X_i, Y_i are the i-th row of X and Y respectively. We thus see that the map $Z \in K \mapsto Z_1 \in \mathbb{R}^2$ is injective. Therefore, it must be invertible when the target is restricted to $\pi_1(K)$. Let $I(Z_1)$ be such inverse, and define also $\tilde G(Z_1) \doteq \pi_2(I(Z_1))$. Finally, set $G(x, y) \doteq J\tilde G(x, y)$, for all $(x, y) \in \pi_1(K)$. By definition,

$$G(Du) = J\tilde G(Du) = JDv,$$

so that u solves (1.1). From (2.7), (1.5) also follows, and the proof is concluded. □

In the same spirit, as observed in [55, Theorem 3.2], we have:

Proposition 2.3. *Let $w \in \mathrm{Lip}(B_1, \mathbb{R}^2)$ be a solution to*

$$\partial_{\bar z} w = h(\partial_z w) \text{ a.e. in } B_1, \tag{2.8}$$

for $h : F \supseteq [\partial_z w] \to \mathbb{C}$. Using notation (2.4), if $K \doteq \{(a, h(a)) : a \in F\}$, then w solves (1.8) for such K. Moreover, if h is strictly contractive, in the sense that there exists

$$\text{an increasing function } \tilde\sigma\colon [0,\infty) \to [0,1] \tag{2.9}$$

such that

$$|h(a) - h(b)| \le (1 - \tilde\sigma(|a-b|))|a-b|, \qquad \forall a, b \in F\,, \tag{2.10}$$

then K fulfills (1.9) for some function σ with properties (1.6). Conversely, let $w \in \mathrm{Lip}(B_1, \mathbb{R}^2)$ solve (1.8), and assume K is a compact set fulfilling (1.9). Then, there exists a strictly contractive $h : [K]_{\mathcal{H}} \to \mathbb{C}$ such that w solves (2.8). We denoted by $[K]_{\mathcal{H}} \doteq \{[X]_{\mathcal{H}} : X \in K\}$.

Proof. The fact that w solves (1.8) for K given by the graph of h is immediate. Now (1.9) follows from (2.9)-(2.10) by direct computations that exploit (2.3). The converse statement is less direct, but it follows from the same arguments as the previous proposition: we rewrite the ellipticity (1.9) using (2.3) as

$$\sigma(|X-Y|) + \left|[X]_{\overline{\mathcal{H}}} - [Y]_{\overline{\mathcal{H}}}\right|^2 \le \left|[X]_{\mathcal{H}} - [Y]_{\mathcal{H}}\right|^2 \qquad \forall X, Y \in K\,.$$

Then, this inequality implies that the projection map $g\colon K \to \mathbb{C}$, $X \mapsto [X]_{\mathcal{H}}$ is injective and that the map $h\colon [K]_{\mathcal{H}} \to \mathbb{C}$, $z \mapsto [g^{-1}(z)]_{\overline{\mathcal{H}}}$, is strictly contractive. □

The last viewpoint we introduced in Proposition 2.3, the one of nonlinear Beltrami systems, is another rather useful one, see [2, 22] and references therein. We exploit it now to show how all of our problems (1.1)-(1.8) and (2.8) can be *extended* in a suitable way so that, for instance, instead of working with G defined on $[Du]$ in (1.1) we can work with $G \in C^0(\mathbb{R}^2, \mathbb{R}^2)$ fulfilling some special properties *at infinity*. Thanks to Propositions 2.2-2.3, it is enough to present this extension for nonlinear Beltrami systems.

Lemma 2.4. *Let $F \subset \mathbb{C}$ be compact and $h\colon F \to \mathbb{C}$ be a strictly contractive mapping, that is, h satisfies (2.9)-(2.10). Then h admits a strictly contractive extension $H\colon \mathbb{C} \to \mathbb{C}$, for a possibly different $\tilde\sigma$ than h still fulfilling (2.9). Moreover, H can be chosen to be constant outside a large ball.*

Proof. The existence of a contractive extension H_1 of h follows from [13, Theorem 3.1]. Then it suffices to set $H = H_1 \circ \Phi$, where $\Phi\colon \mathbb{C} \to \mathbb{C}$ is 1-Lipschitz, is the identity on B_R for some $R > 0$ such that $F \subset B_R$, and has compact support. To obtain such a map Φ, one can set for instance $\Phi(z) = \chi(|z|)z$ with

$$\chi(r) = \begin{cases} 1 & \text{for } 0 \le r \le R\,, \\ 1 - \ln \frac{r}{R} & \text{for } R < r < eR\,, \\ 0 & \text{for } R \ge eR\,. \end{cases}$$

The differential of Φ at $z = re^{i\theta}$ is symmetric with eigenvalues $\chi(r) \in [0,1]$ and $\chi(r) + r\chi'(r) \in [-1,1]$. It has therefore operator norm at most 1, hence Φ is 1-Lipschitz. □

3. Theorem A: elliptic Curves and separation

3.1. **Setup and reduction.** This section is devoted to the proof of Theorem A. To show it, we need to make a few simplifications. Firstly, we reduce it to the following, cleaner, upper semicontinuity property:

Theorem 3.1. *Let $K \subset \mathbb{R}^{2\times 2}$ be compact and fulfill (1.9), $(w_j)_j \subset \mathrm{Lip}(B_1, \mathbb{R}^2)$ be equi-Lipschitz solutions to (1.8), $x_j \in B_1$, and assume w_j converges locally uniformly in B_1 to w_∞, $x_j \to x_0 \in B_1$. Then,*

$$\limsup_j \operatorname{diam}([\partial_z w_j](x_j)) \le \operatorname{diam}([\partial_z w_\infty](x_0)).$$

To pass from Theorem 3.1 to Theorem A, we can simply exploit Proposition 2.3 and write K as the graph of a 1-Lipschitz map h. Hence it suffices to show Theorem 3.1. Lemma 2.4 tells us that we can, without loss of generality, assume that h is defined in $\mathbb{C}$ and:

$$|h(a)-h(b)| \le (1-\tilde\sigma(|a-b|))|a-b|, \quad \text{for all } a,b\in\mathbb{C}, \tag{3.1}$$

$$\|h\|_{L^\infty(\mathbb{C},\mathbb{C})} < +\infty, \tag{3.2}$$

for some $\tilde\sigma$ fulfilling (2.9). Now the property of solving (1.8) for a Lipschitz map w is equivalent to solving

$$w_{\bar z} = h(w_z) \text{ a.e. in } \mathbb{C}. \tag{3.3}$$

Notice that, by Theorem 1.1, w_∞ also solves (1.8)-(3.3). From now on we will consider K to be the graph of h. Furthermore, by considering $W_j(x) \doteq w_j(x+x_j) - w_j(x_j)$ and $W_\infty(x) = w_\infty(x+x_0) - w_\infty(x_0)$ instead of w_j and w_∞, we can assume in addition that

$$x_j = x_0 = 0 \text{ and } w_j(0) = w_\infty(0) = 0, \quad \forall j\in\mathbb{N}. \tag{3.4}$$

By rescaling the domain, we can then still assume these maps are defined in B_1.

Finally, we reduce the proof of Theorem 3.1 to the following:

Theorem 3.2. *Let w_j, w_∞ be as in Theorem 3.1. Let $C\subset\mathbb{R}^2$ be an open, bounded convex set with $[\partial_z w_\infty](B_r)\subset C$ for some $r>0$. Then, given $a\in(\overline{C})^c$, there exist δ and J depending on r and a such that*

$$[\partial_z w_j](B_\delta)\subset B^c_{\frac{\mathrm{d}(a,C)}{2}}(a), \quad \forall j\ge J.$$

Lemma 3.3. *Under the simplifications (3.4), Theorem 3.2 implies Theorem 3.1.*

Proof. Let $C'\Subset C''\subset\mathbb{R}^2$ be open convex sets containing $[\partial_z w_\infty](0)$ with $\operatorname{diam}(C'')\le\operatorname{diam}([\partial_z w_\infty](0))+\alpha$, $\alpha>0$ to be chosen. Let $L = 1+\sup_j\{\|Dw_j\|_{L^\infty}\}$. We can cover the compact set $(C'')^c\cap\overline{B_L}$ with finitely many balls taken from the open cover given by the balls $B_{\frac{\mathrm{d}(a,C')}{2}}(a)$, centered at $a\in(C'')^c\cap\overline{B_L}$. Let a_i be the centers of these balls, for $i=1,\dots,N$. By (1.12), we find $r>0$ such that $[\partial_z w_\infty](B_r)\subset C'$. Hence, by Theorem 3.2 for each i we find δ_i and J_i for which

$$[\partial_z w_j](B_{\delta_i})\subset B^c_{\frac{\mathrm{d}(a_i,C')}{2}}(a_i), \quad \forall j\ge J_i.$$

Let $\delta = \min_i\delta_i>0$ and $J=\max_i J_i<+\infty$. Then,

$$[\partial_z w_j](B_\delta)\subset\bigcap_i B^c_{\frac{\mathrm{d}(a_i,C')}{2}}(a_i) = \Big(\bigcup B_{\frac{\mathrm{d}(a_i,C')}{2}}(a_i)\Big)^c \subset \big((C'')^c\cap\overline{B_L}\big)^c = C''\cup(\overline{B_L})^c, \quad \forall j\ge J.$$

We also have, for all j, $[\partial_z w_j](0)\subset[\partial_z w_j](B_\delta)\subset\overline{B_L}$, and hence

$$\operatorname{diam}([\partial_z w_j](0))\le\operatorname{diam}(C'')\le\operatorname{diam}([\partial_z w_\infty](0))+\alpha, \quad \forall j\ge J.$$

The arbitrariness of α concludes the proof. □

We will show Theorem 3.2 in §3.3, and we start by collecting a few preliminary results.

3.2. Preliminary results.

3.2.1. *Topological degree.* In this section we recall some well-known results on the topological degree $\deg(u,\Omega,p)$. We refer the reader to [23] and references therein for the definition and a detailed introduction.

Proposition 3.4. *Let $\Omega\subset\mathbb{R}^n$ be an open, bounded and connected set and let $u\in W^{1,p}(\Omega,\mathbb{R}^n)$ for $p>n$. Then for all open sets $U\Subset\Omega$ with $|\partial U|=0$:*

$$\int_U v(u(x))\det(Du)(x)dx = \int_{\mathbb{R}^n} v(y)\deg(u,U,y)dy, \quad \forall v\in L^\infty(\mathbb{R}^n), \tag{3.5}$$

and, if $\det(Du)>0$ a.e. in Ω, then

$$\deg(u,U,y) = N(u,U,y), \quad \text{for a.e. } y\in\mathbb{R}^n, \tag{3.6}$$

where $N(u, U, y) \doteq \#\{x \in U : u(x) = y\}$.

Proof. (3.5) is shown in [23, Theorem 5.31]. Through (3.5) and [23, Theorem 5.30], we also get (3.6). □

Proposition 3.5. *Let* $\Omega \subset \mathbb{R}^n$ *be open and bounded,* $v \in C^0(\overline{\Omega}, \mathbb{R}^n)$*, and let* $p \in \mathbb{R}^n \setminus v(\partial\Omega)$*. Then:*

$$\deg(v, \Omega, p) = \deg(u, \Omega, p) \text{ if } u \in C(\overline{\Omega}, \mathbb{R}^n) \text{ and } \|u - v\|_\infty < \mathrm{d}(p, v(\partial\Omega)); \tag{3.7}$$

Moreover, the degree is invariant under homotopies, namely

$$\deg(H(\cdot, t), \Omega, p) = \deg(H(\cdot, 0), \Omega, p), \tag{3.8}$$

for every homotopy $H \in C^0(\overline{\Omega} \times [0, 1], \mathbb{R}^n)$ *such that* $p \notin H(\partial\Omega, t)$ *for all* $t \in [0, 1]$.

Proof. This is classical and can be found e.g. in [23, Theorem 2.3]. □

Remark 3.6. The degree $\deg(u, \Omega, p)$ is, in general, not defined if $p \in u(\partial\Omega)$. Therefore, every time we write expressions involving $\deg(u, \Omega, p)$ and $p \in B$, for some set B, this will implicitly entail that $u(\partial\Omega) \cap B = \emptyset$.

3.2.2. *Quasiregular mappings.* We start by recalling the definition.

Definition 3.7. Let $\Omega \subset \mathbb{R}^2$ be open. $\varphi \in W^{1,2}(\Omega, \mathbb{R}^2)$ is (K-)quasiregular if there exists $K \geq 1$ such that

$$|D\varphi|^2(x) \leq K \det(D\varphi(x)), \quad \text{for a.e. } x \in \Omega. \tag{3.9}$$

Equivalently, $f \in W^{1,2}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$ is quasiregular if and only if there exists $\kappa \in [0, 1)$ such that

$$|f_{\bar{z}}|(z) \leq \kappa |f_z|(z) \text{ for a.e. } z \in \Omega. \tag{3.10}$$

Remark 3.8. For instance, if we consider equation (2.8) for some k-Lipschitz map h for some $0 \leq k < 1$, then the difference $u = v - w$ of any two solutions v, w to (2.8) is quasiregular:

$$|u_{\bar{z}}|^2 = |v_{\bar{z}} - w_{\bar{z}}|^2 = |h(v_z) - h(w_z)|^2 \leq k^2 |v_z - w_z|^2 = k^2 |u_z|^2 .$$

Let us recall a few properties of quasiregular maps, referring the reader to [2, 3, 28] for more details.

Proposition 3.9. *Let* $\varphi \in W^{1,2}(\Omega, \mathbb{C})$ *be quasiregular. Then:*

1. *there exists* $p = p(K) > 2$ *such that* $\varphi \in W^{1,p}_{\mathrm{loc}}(\Omega, \mathbb{R}^2)$*. In particular,* φ *is continuous;*
2. φ *is either constant, or open and discrete;*
3. φ *is either constant, or* $\det(D\varphi) > 0$ *a.e. in* Ω*;*
4. *if* $U, W \subset \mathbb{R}^2$ *are open sets and* $N(\varphi, U, y) = 1$ *for a.e.* $y \in W$*, then* φ *is injective on* $U \cap \varphi^{-1}(W)$.

Proof. (1) can be found in [3, Theorem 5.1] (see also [1] for the precise expression of p). (2)-(3) can be found in [2, Corollary 5.5.2]. Let us show (4), following the same argument of [19, Lemma 4.3]. Observe that φ is open by (2). Suppose by contradiction that there exist distinct $x_1, x_2 \in U \cap \varphi^{-1}(W)$ such that $\varphi(x_1) = \varphi(x_2) = y$. Then, taking a sufficiently small $r > 0$ such that

$$B_r(x_1) \cap B_r(x_2) = \emptyset \tag{3.11}$$

and $B_r(x_i) \subset U \cap \varphi^{-1}(W)$ for all i, define the open set $V = \varphi(B_r(x_1)) \cap \varphi(B_r(x_2)) \subset W$. Observe that $y \in V$. Since V is open and nonempty, our assumption implies that we can find $p \in V$ for which $N(\varphi, U, p) = 1$. This yields a contradiction with the definition of V and (3.11). □

We state the *stability* result for quasiregular maps established in [32, Proposition 1]. In that reference, the statement is shown in $\mathbb{R}^n$, but we will only need it for planar maps. We derive from it Corollary 3.12, which is essentially contained in [32, Proposition 2].

Proposition 3.10. *Let* $u : \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ *be a* K*-quasiregular mapping such that* $|Du(x)| \geq \varepsilon$ *a.e. in* Ω*. Let* $G \Subset \Omega$ *and assume that* $M \doteq \sup_{y \in \mathbb{R}^2} N(u, G, y) < +\infty$*. Then there exist a constant* $\delta = \delta(\varepsilon, K, M) > 0$ *and for any* $x_0 \in G$ *a radius* $r(x_0) > 0$ *such that for any Lipschitz mapping* $\phi : \Omega \to \mathbb{R}^2$ *with* $\|D\phi\|_\infty < \delta$,

$$\min_{|x - x_0| = r} |u^t(x) - u^t(x_0)| \geq \delta r, \qquad \text{for all } r < r(x_0), t \in [0, 1], \text{ if } u^t = u + t\phi.$$

Remark 3.11. There are several ways to see that the assumption $M < +\infty$ in the previous proposition is automatically satisfied. One can use, for instance, Stoilow Factorization [2, Theorem 5.5.1] or more purely topological methods valid in all dimensions [51, Proposition 4.10(3)].

Corollary 3.12. *Let $R, \varepsilon > 0$, $u \in \mathrm{Lip}(B_R, \mathbb{R}^2)$ such that $u(0) = 0$, and $\Gamma \subset \mathbb{R}^{2\times 2}$ be a compact and path-connected set of matrices such that*

$$\det(Du - A) \geq \varepsilon \quad a.e. \text{ in } B_R\,, \quad \forall A \in \Gamma\,. \tag{3.12}$$

Then there exists $r_0 = r_0(u, \Gamma) \in (0, R)$ and $\delta = \delta(\varepsilon, \Gamma) > 0$ such that

$$A \mapsto \deg(u - L_A, B_r, y) \text{ is defined and constant on } \Gamma \text{ for all } r \in (0, r_0) \text{ and } y \in \overline{B_{\delta r}}\,.$$

Proof. Fix $\Lambda > 0$ such that $|Du| + |A| \leq \Lambda$ for all $A \in \Gamma$ and a.e. $x \in B_R$. The finiteness of such Λ and (3.12) imply that the map $u^A \doteq u - L_A$ is a nonconstant quasiregular mapping in B_R for all $A \in \Gamma$. Thanks to Remark 3.11 we may apply Proposition 3.10 to u^A at $x_0 = 0$. Recalling that $u^{A'}(0) = 0$ for all $A' \in K$, this yields the existence of $r_A = r_A(u) \in (0, R)$ and $\delta_A = \delta_A(\varepsilon, \Lambda) > 0$ such that

$$\mathrm{d}(0, u^{A'}(\partial B_r)) > \delta_A r \quad \forall r \in (0, r_A),\ \forall A' \in B_{\delta_A}(A)\,.$$

Covering Γ with a finite number of open balls $B_{\delta_A}(A)$, we infer the existence of $r_0 > 0$ and $\delta > 0$ such that

$$\mathrm{d}(0, u^A(\partial B_r)) > \delta r \quad \forall r \in (0, r_0),\ \forall A \in \Gamma\,.$$

Thus for all $A \in \Gamma$ we have $u^A(\partial B_r) \cap \overline{B_{\delta r}} = \emptyset$, hence the degree $\deg(u^A, B_r, y)$ is well-defined for all $y \in \overline{B_{\delta r}}$. Moreover it is independent of $A \in \Gamma$ by path-connectedness of Γ and homotopy invariance of the degree, see Proposition 3.5. □

3.2.3. *Approximation of the problem.* Let w be a solution to (3.3) and h satisfy (3.1)-(3.2). We want to obtain w as limit of solutions to strongly elliptic problems.

Lemma 3.13. *For every $\varepsilon \in (0, 1)$, the nonlinear Beltrami system*

$$\begin{cases} \partial_{\bar z} w^\varepsilon = (1 - \varepsilon) h(\partial_z w^\varepsilon) \text{ in } B_1, \\ \mathrm{Re}(w^\varepsilon) = \mathrm{Re}(w) \text{ on } \partial B_1. \end{cases} \tag{3.13}$$

admits a unique solution $w^\varepsilon \in W^{1,2}(B_1, \mathbb{C})$ with w_2^ε of zero average. Furthermore, $w_\varepsilon \to w$ locally uniformly and locally strongly in $W^{1,p}$ for all $p \in [1, \infty)$.

Proof. The solvability of this system can be found in [22, Proposition 2], where it is shown that if $H : B_1 \times \mathbb{C} \to \mathbb{C}$ is a measurable function satisfying $H(z, 0) = 0$ for a.e. $z \in B_1$ and

$$|H(z, w_1) - H(z, w_2)| \leq k|w_1 - w_2|, \text{ for a.e. } z \in B_1 \text{ and all } w_1, w_2 \in \mathbb{C} \text{ for some } k < 1,$$

then for any $\sigma \in L^2(B_1, \mathbb{C})$ there exists a $W^{1,2}(B_1, \mathbb{C})$ solution v (which is unique up to the addition of a constant to the second component) of

$$\begin{cases} \partial_{\bar z} v = H(z, \partial_z v) + \sigma(z) \text{ in } B_1, \\ \mathrm{Re}(v) = 0 \text{ on } \partial B_1. \end{cases} \tag{3.14}$$

To solve (3.13) we can just consider a solution v^ε to (3.14) with

$$H_\varepsilon(z, \xi) \doteq (1 - \varepsilon)(h(\partial_z w + \xi) - h(\partial_z w)) \text{ and } \sigma(z) = -\varepsilon h(\partial_z w).$$

and then set $w^\varepsilon \doteq v^\varepsilon + w$. We will now show the required estimates. We start by obtaining $W^{1,2}$ bounds uniform in $\varepsilon > 0$. Using complex notation, see (2.3):

$$\int_{B_1} |Dw^\varepsilon|^2 dx = 2\int_{B_1} |w_z^\varepsilon|^2 + |w_{\bar z}^\varepsilon|^2 dx = 2\int_{B_1} |w_z^\varepsilon|^2 - |w_{\bar z}^\varepsilon|^2 dx + 4\int_{B_1} |w_{\bar z}^\varepsilon|^2 dx.$$

The last addendum is bounded, since our assumptions on h, see (3.2), and (3.13) imply that actually $w^\varepsilon_{\bar z}$ is equibounded in L^∞. It then suffices to estimate the first addendum using the boundary conditions:

$$\int_{B_1}|w^\varepsilon_z|^2-|w^\varepsilon_{\bar z}|^2dx \overset{(2.3)}{=} \int_{B_1}\det(Dw^\varepsilon)dx=\int_{B_1}\det(D\psi^\varepsilon)dx,$$

where $\psi^\varepsilon=(\mathrm{Re}(w),w^\varepsilon_2)$. Now by Young's inequality, for any $\delta>0$ we find:

$$\int_{B_1}\det(D\psi^\varepsilon)dx\le C(\delta)\int_{B_1}|Dw|^2dx+\delta\int_{B_1}|Dw^\varepsilon|^2dx.$$

Combining all these estimates, we get a universal constant C such that

$$\int_{B_1}|Dw^\varepsilon|^2dx\le C\left(C(\delta)\int_{B_1}|Dw|^2dx+\delta\int_{B_1}|Dw^\varepsilon|^2dx+C(h)\right).$$

By choosing δ small, we deduce the required uniform bound. Now we can use the boundedness of the Beurling transform [2, Theorem 4.5.3] to obtain interior $W^{1,p}$ bounds for all $p\ge1$, uniformly in $\varepsilon>0$: indeed, for any test function $\eta\in C^\infty_c(B_1)$, the derivative $(\eta w^\varepsilon)_{\bar z}=\eta_{\bar z}w^\varepsilon+(1-\varepsilon)\eta h(w^\varepsilon_z)$ is uniformly bounded in L^p since $W^{1,2}(B_1)\subset L^p(B_1)$ and $h\in L^\infty(\mathbb{C})$, and thus $D(\eta w_\varepsilon)$ is uniformly bounded in L^p. We are only left to show the convergence of w^ε to w. Using the boundary conditions,

$$\begin{aligned}0=\int_{B_1}\det(Dw^\varepsilon-Dw)dx&=\int_{B_1}|\partial_z w^\varepsilon-\partial_z w|^2-|\partial_{\bar z}w^\varepsilon-\partial_{\bar z}w|^2dx\\&=\int_{B_1}|\partial_z w^\varepsilon-\partial_z w|^2-|(1-\varepsilon)h(\partial_z w^\varepsilon)-h(\partial_z w)|^2dx\end{aligned}$$

Thanks to the uniform bounds shown above, this gives us for all $\varepsilon>0$:

$$\int_{B_1}|\partial_z w^\varepsilon-\partial_z w|^2-|h(\partial_z w^\varepsilon)-h(\partial_z w)|^2dx\le C\varepsilon.$$

From (3.1) and (2.9), this readily yields the convergence in measure of $\partial_z w^\varepsilon$ to $\partial_z w$ as $\varepsilon\to0$. In turn (3.13) tells us that $\partial_{\bar z}w^\varepsilon$ does the same and this concludes the proof. □

3.3. **Proof of Theorem 3.2.** We write K as the graph of h, as in (3.1)-(3.2)-(3.3). For simplicity, we let

$$g:\mathbb{R}^2\to K \text{ be defined as } g(z)\doteq(z,h(z)), \text{ and } U\doteq g\left((\overline{C})^c\right),$$

and we also let $\tau\doteq\frac{\mathrm{d}(a,C)}{2}$. We also recall (3.4).

Step 1: *degree of w_j-L_A for $A\in U$.* We show the existence of $J\in\mathbb{N}$, $R\in(0,r)$ and $\alpha>0$ such that

$$\deg(w_j-L_A,B_R,y)=1\,,\quad\forall y\in\overline{B_{2\alpha}}\,,\ \forall A\in\overline{U}\,,\ \forall j\ge J\,. \tag{3.15}$$

Recall from Remark 3.6 that this assertion implicitly entails $(w_j-L_A)(\partial B_R)\cap\overline{B_{2\alpha}}=\emptyset$. Thanks to the uniform convergence of w_j to w_∞ and Proposition 3.5, it suffices to show

$$\deg(w_\infty-L_A,B_R,y)=1\,,\quad\forall y\in\overline{B_{3\alpha}}\,,\ \forall A\in\overline{U}\,. \tag{3.16}$$

From (1.9) and the facts that $Dw_\infty\in K$ a.e., $\overline{U}\subset K$ and $\overline{U}\cap[Dw_\infty](B_r)=\emptyset$, we see that

$$\det(Dw_\infty-A)\ge\varepsilon\quad\text{a.e. in } B_r\,,\quad\forall A\in\overline{U}\,,$$

where $\varepsilon=\sigma(\mathrm{d}(\overline{U},[Dw_\infty](B_r)))>0$. Noting that the set $\overline{U}\cap g(\overline{B_M})$ is compact and path-connected for any $M>0$ such that $\overline{C}\subset B_M$, we apply Corollary 3.12 and deduce the existence of $R=R(M)\in(0,r)$, $\delta=\delta(M)>0$ such that

$$A\mapsto\deg(w_\infty-L_A,B_R,y)\text{ is defined and constant on }\overline{U}\cap g(\overline{B_M})\,,\text{ for all }y\in\overline{B}_\delta\,.$$

To conclude the proof of (3.16), it suffices therefore to find $M>0$ such that

$$\deg(w_\infty-L_A,B_\rho,y)=1\quad\forall y\in\overline{B_\rho}\,,\ \forall\rho\in(0,r)\,,\ \forall A\in g(B^c_M)\,, \tag{3.17}$$

apply this to $\rho = R$ and choose $3\alpha = \min\{\delta, R\}$, for R, δ given by this choice of M. To establish (3.17), let $M \geq 2\|h\|_\infty$, $z \in B_M^c$, so that $|z| \geq M \geq 2\|h\|_\infty$, and $A = g(z) = (z, h(z))$. Recalling (2.3), we have

$$|z|^2 \leq |A|^2 = |z|^2 + |h(z)|^2 \leq 2|z|^2 , \quad \frac{1}{2}|z|^2 \leq \det(A) = |z|^2 - |h(z)|^2 \leq |z|^2 .$$

This implies that A is invertible and $|A^{-1}| = (\det A)^{-1}|A| < 4/M$, so $M|x| < 8|Ax|$ for all $x \in \mathbb{R}^2 \setminus \{0\}$. Hence we have $L_A(\partial B_\rho) \cap \overline{B_{M\rho/8}} = \emptyset$ for all $\rho > 0$, and

$$\deg(-L_A, B_\rho, y) = 1 , \quad \forall y \in \overline{B_{M\rho/8}} .$$

Assume moreover $M > 16 \,\mathrm{Lip}(w_\infty)$. Then for $0 < \rho < r$ we have $\|w_\infty\|_{L^\infty(B_\rho)} < M\rho/16$ and we can invoke Proposition 3.5 to deduce

$$\deg(w_\infty - L_A, B_\rho, y) = 1 , \quad \forall y \in \overline{B_{M\rho/16}} , \ \forall \rho \in (0, r) ,$$

which gives (3.17) if we impose in addition $M \geq 16$.

Step 2: *Jordan curves and the degree of the approximating maps.* For $t \in [0,1]$ and $j \in \mathbb{N}$ we define

$$\gamma(t) \doteq \tau e^{2\pi i t} + a, \quad \Gamma(t) \doteq g(\gamma(t)) , \quad f_{j,t} = w_j - L_{\Gamma(t)} . \tag{3.18}$$

Since $\Gamma(t) \in U$ for all $t \in [0,1]$, from (3.15) we infer

$$\deg(f_{j,t}, B_R, y) = 1 \quad \text{ for all } y \in \overline{B_{2\alpha}} , \ j \geq J, \ t \in [0,1] . \tag{3.19}$$

Now we fix $j \geq J$ and let w_j^ε be the family parametrized by $\varepsilon \in (0,1)$ provided by Lemma 3.13 applied to $w = w_j$. Then we define the maps

$$w_{j,\varepsilon,t} \doteq w_j^\varepsilon - L_{\Gamma_\varepsilon(t)}, \qquad \Gamma_\varepsilon \doteq (\gamma, (1-\varepsilon)h(\gamma)) ,$$

which are quasiregular for all $j \in \mathbb{N}$, $\varepsilon \in (0,1)$ and for all $t \in [0,1]$, compare Remark 3.8. For this fixed $j \geq J$, we see from Lemma 3.13 that $w_{j,\varepsilon,t}$ converges locally uniformly in B_1 to $f_{j,t}$ as $\varepsilon \to 0$, uniformly with respect to $t \in [0,1]$. Recall from Remark 3.6 that (3.19) contains the information $f_{j,t}(\partial B_R) \cap \overline{B_{2\alpha}} = \emptyset$. Choosing $\varepsilon_0 = \varepsilon_0(j) > 0$ such that $\|w_{j,\varepsilon,t} - f_{j,t}\|_{L^\infty(B_R)} < \alpha$ for all $\varepsilon \in (0, \varepsilon_0)$, by Proposition 3.5 we infer

$$1 \overset{(3.19)}{=} \deg(f_{j,t}, B_R, y) = \deg(w_{j,\varepsilon,t}, B_R, y) \text{ for all } t \in [0,1] \text{ and } y \in \overline{B_\alpha}. \tag{3.20}$$

Further, recalling that $f_{j,t}(0) = 0$ and $\mathrm{Lip}(f_{j,t}) \leq C$, we may fix $\delta = \delta(C, R) > 0$ such that

$$B_{2\delta} \subset B_R \cap (f_{j,t})^{-1}(B_{\alpha/2}), \quad \forall t \in [0,1], \forall j \in \mathbb{N} ,$$

hence, for a possibly smaller $\varepsilon_0 = \varepsilon_0(j)$,

$$B_{2\delta} \subset B_R \cap (w_{j,\varepsilon,t})^{-1}(B_\alpha), \quad \text{ for all } t \in [0,1], 0 < \varepsilon \leq \varepsilon_0. \tag{3.21}$$

We now claim that

$$w_{j,\varepsilon,t} \text{ is injective on } B_{2\delta}, \quad \text{ for all } t \in [0,1], 0 < \varepsilon \leq \varepsilon_0. \tag{3.22}$$

To see this, note first that $w_{j,\varepsilon,t}$ cannot be constant on B_R by (3.20), so it has positive Jacobian a.e. in B_R by Proposition 3.9(3). Recalling also from Lemma 3.13 that $w_{j,\varepsilon,t} \in W^{1,p}(B_R)$ for all $p \geq 1$, we can apply (3.6) and deduce that, for $0 < \varepsilon \leq \varepsilon_0$ and $t \in [0,1]$:

$$N(w_{j,\varepsilon,t}, B_R, y) = \deg(w_{j,\varepsilon,t}, B_R, y) \overset{(3.20)}{=} 1, \quad \text{ for a.e. } y \in B_\alpha, \tag{3.23}$$

Through Proposition 3.9(4), we infer that $w_{j,\varepsilon,t}$ is injective on $w_{j,\varepsilon,t}^{-1}(B_\alpha) \cap B_R$, hence on $B_{2\delta}$ by (3.21), thus proving (3.22).

Step 3: *Separation and conclusion.* We can finally conclude the argument. Fix for the moment $j \geq J$, where J was found in Step 1. By (3.22), we can write:

$$w_{j,\varepsilon,t}(x + re_1) \neq w_{j,\varepsilon,t}(x), \quad \forall \varepsilon \leq \varepsilon_0, r < \delta, x \in B_\delta.$$

The definitions yield

$$w_j^\varepsilon(x+re_1)-w_j^\varepsilon(x)\neq r\Gamma_\varepsilon(t)e_1,\quad \forall\varepsilon\leq\varepsilon_0, r<\delta, x\in B_\delta. \tag{3.24}$$

Now we need to observe that $t\mapsto\Gamma_\varepsilon(t)e_1$ is a Jordan curve. Hence, it divides $\mathbb{R}^2$ into two connected components, the bounded one, ω_ε, and the unbounded one, $\mathbb{R}^2\setminus\omega_\varepsilon$. From (3.24), we deduce that

$$x\mapsto\frac{w_j^\varepsilon(x+re_1)-w_j^\varepsilon(x)}{r}$$

does not intersect this Jordan curve. Hence, for fixed $j\geq J, \varepsilon<\varepsilon_0$, for all $r<\delta$ and for every $x\in B_\delta$,

$$\text{either }\frac{w_j^\varepsilon(x+re_1)-w_j^\varepsilon(x)}{r}\in\omega_\varepsilon\quad\text{or}\quad\frac{w_j^\varepsilon(x+re_1)-w_j^\varepsilon(x)}{r}\in\mathbb{R}^2\setminus\omega_\varepsilon. \tag{3.25}$$

Let us characterize ω_ε. If we consider, for any $\varepsilon\in[0,1]$, the continuous map defined in complex notation by:

$$\Phi_\varepsilon(z)=ze_1+(1-\varepsilon)h(z)\bar{e}_1=z+(1-\varepsilon)h(z),$$

then Φ_ε is injective from $\mathbb{R}^2$ to $\mathbb{R}^2$, thanks to (3.1). Thus, it is a homeomorphism onto its open image. The image is also closed thanks to the boundedness of h (3.2), and hence $\Phi_\varepsilon(\mathbb{R}^2)=\mathbb{R}^2$, $\forall\varepsilon\in[0,1]$. Noticing that $\Phi_\varepsilon(\gamma(t))=\Gamma_\varepsilon(t)e_1$ for all $t\in[0,1]$ by (2.2) and recalling (3.18), we deduce:

$$\omega_\varepsilon=\Phi_\varepsilon(B_\tau(a)). \tag{3.26}$$

Letting $\varepsilon\to0$ in (3.25), since Φ_ε converges uniformly to Φ_0 on $\mathbb{R}^2$ we find that for all $r<\delta$:

$$\text{either }\frac{w_j(x+re_1)-w_j(x)}{r}\in\overline{\omega_0}\quad\text{or}\quad\frac{w_j(x+re_1)-w_j(x)}{r}\in\mathbb{R}^2\setminus\omega_0,\quad\text{for all }x\in B_\delta. \tag{3.27}$$

If, by contradiction, the first alternative occurred for infinitely many j, then we would find

$$Dw_j(x)e_1=\Phi_0(\partial_z w(x))\in\overline{\omega_0}=\Phi_0(\overline{B_\tau(a)}),\text{ for a.e. }x\in B_\delta.$$

In turn, since Φ_0 is a homeomorphism, we deduce that $\partial_z w_j(x)\in\overline{B_\tau(a)}$, for a.e. $x\in B_\delta$ and infinitely many $j\geq J$. By Theorem 1.1, this would also imply that

$$\partial_z w_\infty(h)\in\overline{B_\tau(a)}\subset(\overline{C})^c\text{ a.e. in }B_\delta, \tag{3.28}$$

which is against our assumption $[\partial_z w_\infty](B_r)\subset C$. Therefore, the second alternative in (3.27) must occur. Following the same reasoning of the previous contradiction argument, this leads to

$$|\partial_z w_j(x)-a|\geq\tau,\qquad\text{for a.e. }x\in B_\delta,\forall j\geq J,$$

which is precisely the conclusion of Theorem 3.2. □

4. Theorem B: the size of non-differentiability points

In this section we give the proof of Theorem B, that we deduce from the following result valid in $\mathbb{R}^n$.

Proposition 4.1. *Let $u\in\mathrm{Lip}(B_1)$, $B_1\subset\mathbb{R}^n$, be a solution to (1.1), where $G\in C^0([Du],\mathbb{R}^n)$ fulfills*

$$(G(b)-G(a),b-a)\geq\sigma(|b-a|)\qquad\forall a,b\in[Du], \tag{4.1}$$

for some σ as in (1.6). Then, $\mathcal{H}^{n-1}(S)=0$, where S is the set

$$S\doteq\left\{x\in B_1:\liminf_{r\to0}\fint_{B_r(x)}\sigma\left(\left|Du(y)-\fint_{B_r(x)}Du(z)dz\right|\right)dy>0\right\}. \tag{4.2}$$

Proof. As the statement is local, we can assume that u is defined in B_2. We start with a preliminary estimate. Let $|h| < \frac{1}{2}$ and φ be a smooth cut-off function of $B_{\frac{3}{2}}$ inside B_2. Then, (1.1) implies:

$$\int_{B_2} (G(Du(x+h)) - G(Du(x)), D[\varphi(x)(u(x+h) - u(x))])dx = 0.$$

Routine calculations and (4.1) then yield

$$\begin{aligned}\int_{B_{\frac{3}{2}}} \sigma(|Du(x+h) - Du(x)|)dx &\le \int_{B_2} \varphi\sigma(|Du(x+h) - Du(x)|)dx \\ &\le \int_{B_2} |D\varphi||u(x+h) - u(x)||G(Du(x+h)) - G(Du(x))|dx \\ &\le C|h| \int_{B_2} |D\varphi||G(Du(x+h)) - G(Du(x))|dx \le C|h|D(|h|),\end{aligned} \tag{4.3}$$

for $C = \mathrm{Lip}(u)$ and

$$D(t) \doteq \sup_{h:|h|\le t} \int_{B_2} |D\varphi||G(Du(x+h)) - G(Du(x))|dx.$$

Observe that

$$\lim_{t\to 0} D(t) = 0. \tag{4.4}$$

We now turn to our main goal, i.e. showing that $\mathcal{H}^{n-1}(S) = 0$. We notice that

$$S \subset \bigcup_{p\ge 1} \bigcup_{m\ge 10} S_{p,m} = \bigcup_{p\ge 1} \bigcup_{m\ge 10} \bigcap_{0<r<\frac{1}{m}} E_{p,r}, \tag{4.5}$$

where

$$E_{p,r} \doteq \left\{ x \in B_1 : \fint_{B_r(x)} \sigma\left(\left|Du(y) - \fint_{B_r(x)} Du(z)dz\right|\right) dy \ge \frac{1}{p} \right\}.$$

Let us recall that $\mathcal{H}^{n-1}(E) = \lim_{r\to 0} \mathcal{H}^{n-1}_r(E)$, where, for $r > 0$,

$$\mathcal{H}^{n-1}_r(E) \doteq \inf\left\{ \sum_{i=1}^{\infty} \mathrm{diam}(C_i)^{n-1} : E \subset \bigcup_{i=1}^{\infty} C_i, \mathrm{diam}(C_i) \le r \right\}$$

see [21, Definition 2.1]. Our definition differs from that of [21, Definition 2.1] by a constant factor, which is anyway irrelevant for what we need to show. We claim that:

$$\lim_{r\to 0} \mathcal{H}^{n-1}_{5r}(E_{p,r}) = 0. \tag{4.6}$$

If this holds, then $\mathcal{H}^{n-1}(S) = 0$, since

$$\mathcal{H}^{n-1}_{5r}(S_{p,m}) \le \mathcal{H}^{n-1}_{5r}(E_{p,r}), \text{ for all } r < \frac{1}{m}, \tag{4.7}$$

and again by [21, Definition 2.1]:

$$0 \overset{(4.6)-(4.7)}{=} \lim_{r\to 0} \mathcal{H}^{n-1}_{5r}(S_{p,m}) = \mathcal{H}^{n-1}(S_{p,m}).$$

The σ-subadditivity of $\mathcal{H}^{n-1}$, [21, Theorem 2.1, Claim # 2], and (4.5) would then conclude the proof. We are only left to show (4.6). To this aim, pick any $x \in E_{p,r}$. Then, monotonicity and convexity of σ imply:

$$\frac{1}{p} \le \fint_{B_r(x)} \sigma\left(\left|Du(y) - \fint_{B_r(x)} Du(z)dz\right|\right) dy \le \fint_{B_r(x)} \fint_{B_r(x)} \sigma\left(|Du(y) - Du(z)|\right) dydz.$$

Hence there exists $c = c(n,p) > 0$ such that for all $a \in B_r(x)$:

$$c(n,p) \leq \fint_{B_{2r}(a)} \fint_{B_{2r}(a)} \sigma\left(|Du(y) - Du(z)|\right) dydz.$$

In the next lines, $c(n,p)$ may decrease, but for the sake of brevity we will not denote it differently. Integrating over every such a and changing variables, we see that:

$$r^n c(n,p) \leq \int_{B_r(x)} \fint_{B_{2r}} \fint_{B_{2r}} \sigma\left(|Du(y+a) - Du(z+a)|\right) dydzda \tag{4.8}$$

As r is fixed and $E_{p,r} \subset B_1$, we can cover $E_{p,r}$ with finitely many balls $\overline{B_r(x_i)}$ centered at $\{x_i\}_{i=1}^N \subset E_{p,r}$. From Vitali's Covering Theorem, [21, Theorem 1.24], we find a subset of centers $x_{i_1}, \dots, x_{i_M}$ such that

$$E_{p,r} \subset \bigcup_{i=1}^N \overline{B_r(x_i)} \subset \bigcup_{j=1}^M \overline{B_{5r}(x_{i_j})} \quad \text{and} \quad B_r(x_{i_j}) \cap B_r(x_{i_k}) = \emptyset \quad \text{if } j \neq k. \tag{4.9}$$

Let $E \doteq \bigcup_{j=1}^M B_r(x_{i_j})$. Summing (4.8) over this subset of centers, we obtain

$$Mr^n c(n,p) \leq \int_E \fint_{B_{2r}} \fint_{B_{2r}} \sigma\left(|Du(y+a) - Du(z+a)|\right) dydzda \tag{4.10}$$

For any $y, z \in B_{2r}$, and since $10r \leq 1$, we get

$$\int_E \sigma\left(|Du(y+a) - Du(z+a)|\right) da \leq \int_{B_{\frac{11}{10}}} \sigma\left(|Du(y+a) - Du(z+a)|\right) da.$$

Changing variables and renaming $z - y \doteq h$, we finally obtain

$$\int_{B_{\frac{11}{10}}} \sigma\left(|Du(y+a) - Du(z+a)|\right) da \leq \int_{B_{\frac{3}{2}}} \sigma\left(|Du(a+h) - Du(a)|\right) da \overset{(4.3)}{\leq} C|h|D(|h|) \leq CrD(4r),$$

for a dimensional constant C. Combining this inequality with (4.10), we infer that

$$Mr^{n-1} \leq C(n,p)D(4r). \tag{4.11}$$

Therefore, from (4.9) and (4.11) we infer

$$\mathcal{H}^{n-1}_{5r}(E_{p,r}) \leq \sum_{j=1}^M \operatorname{diam}(B_{5r}(x_{i_j}))^{n-1} = (10)^{n-1} Mr^{n-1} \overset{(4.11)}{\leq} C(n,p)D(4r),$$

for a possibly larger constant $C(n,p)$. This and (4.4) show (4.6) and conclude the proof. □

4.1. **Proof of Theorem B.** Let w solve (1.8). Writing $w = (u,v)$, by Proposition 2.2 we can find a monotone field $G : [Du] \to \mathbb{R}^2$ fulfilling (4.1) such that u solves (1.1). To show Theorem B we prove that $S^c \subset \operatorname{Reg}(w)$. Let then $x_0 \in S^c$. This means that there exists a sequence $r_n \to 0$ such that

$$\lim_n \fint_{B_{r_n}(x_0)} \sigma\left(\left|Du(y) - \fint_{B_{r_n}(x_0)} Du(z)dz\right|\right) dy = 0. \tag{4.12}$$

We may assume, up to a non-relabeled subsequence, that $\fint_{B_{r_n}(x_0)} Du \to a \in \mathbb{R}^2$. Up to considering another non-relabeled subsequence, we can assume that w_{r_n,x_0}, recall (1.13), converges locally uniformly and, thanks to Theorem 1.1, strongly in $W^{1,1}_{\mathrm{loc}}$ to $w_\infty = (u_\infty, v_\infty)$. Now the strong convergence and (4.12) imply that $Du_\infty \equiv a$ on B_1. From (1.9)-(1.6)-(1.14), we infer that $Dw_\infty \equiv A$ in B_1, where $A \in K$ is the only matrix in K whose first row is a. Hence w_∞ is differentiable at 0, and $x_0 \in \operatorname{Reg}(w)$.

5. Theorem C: Monge-Ampère measure associated to solutions of elliptic PDEs

This section is devoted to showing Theorem C. For $u, v \in W^{1,2}(B_1)$, we define the distribution

$$\mathcal{D}(u,v) = -\frac{1}{4}\operatorname{div}(\operatorname{div}(JDu \otimes JDv + JDv \otimes JDu)), \tag{5.1}$$

where J is defined in (2.6). Recalling (2.1), if $u, v \in W^{2,2}$, a direct computation shows that $\mathcal{D}(u,v) = \frac{1}{2}\langle D^2u, \operatorname{cof} D^2v\rangle$, so that $\mathcal{D}(u,u) = \det(D^2u)$. Theorem C follows from Theorem 5.1 and Corollary 5.2.

Theorem 5.1. *Let $K \subset \mathbb{R}^{2\times 2}$ fulfill (1.9), $\Omega \subset \mathbb{R}^2$ be open, and let $w = (u,v) \in \operatorname{Lip}(\Omega, \mathbb{R}^2)$ be a solution to (1.8). Define the symmetric matrix of distributions*

$$\mathcal{D}(w) \doteq \begin{pmatrix} \mathcal{D}(u,u) & \mathcal{D}(u,v) \\ \mathcal{D}(u,v) & \mathcal{D}(u,v) \end{pmatrix}. \tag{5.2}$$

Then, for any $\varphi \in C_c^\infty(\Omega)$ with $\varphi \geq 0$ everywhere,

$$\mathcal{D}(w)(\varphi) \leq 0 \text{ in the sense of quadratic forms.}$$

Thus, $\mathcal{D}(w)$ is a locally finite measure on Ω with values in the set of nonpositive semidefinite symmetric matrices, $\operatorname{Sym}^-(2)$.

Proof. We can assume $\Omega = B_1$. We only need to show that the distribution $\mathcal{D}(u,u)$ is a non-positive measure. Let us show how to conclude the proof assuming this claim. Consider $\tilde{w} = wA$, for $A \in \mathbb{R}^{2\times 2}$ with $\det(A) > 0$. Then, $D\tilde{w} \in \tilde{K} \doteq A^TK$, which fulfills (1.9), and by the claim applied to $\tilde{w}$ we deduce that:

$$\mathcal{D}(au+bv, au+bv) = a^2\mathcal{D}(u,u) + 2ab\mathcal{D}(u,v) + b^2\mathcal{D}(v,v) \leq 0, \quad \forall a, b \in \mathbb{R},$$

which would conclude the proof. Let us show the claim. By Proposition 2.2 we have that u is a Lipschitz solution to (1.1) with $G \in C^0(\mathbb{R}^2, \mathbb{R}^2)$ satisfying (1.5). By [35, Lemma A.4], u is a strong H^1 limit of smooth functions u_ε solving

$$\operatorname{div}(G^\varepsilon(Du^\varepsilon)) = \langle DG^\varepsilon(Du^\varepsilon), D^2u^\varepsilon\rangle = 0\,, \tag{5.3}$$

where G^ε are smooth fields satisfying (1.5) with $\sigma(t) = c_\varepsilon t^2$ for some $c_\varepsilon > 0$, hence $DG^\varepsilon + (DG^\varepsilon)^T \geq 2c_\varepsilon$. This and (5.3) imply that $\mathcal{D}(u^\varepsilon, u^\varepsilon) = \det(D^2u^\varepsilon) \leq 0$ in $\mathcal{D}'(B_1)$. By the strong H^1 convergence provided by [35, Lemma A.4], we deduce the same for $\mathcal{D}(u,u)$. ☐

Corollary 5.2. *Let $K \subset \mathbb{R}^{2\times 2}$ fulfill (1.9), and let $w = (u,v) \in \operatorname{Lip}(B_1, \mathbb{R}^2)$ be a solution to (1.8). Consider the matrix-valued measure $\mathcal{D}(w)$ defined as in (5.2). Factorize it as a $\mathcal{D}(w) = P\mu$, where μ is a finite, positive measure on B_1, and $P \in L^\infty(B_1, \operatorname{Sym}^-(2); \mu)$. Then, for any $w_\infty \in \mathcal{B}(w)(x_0)$, recall (1.14),*

$$\mathcal{D}(w_\infty) = P(x_0)\mu(\{x_0\})\delta_0, \tag{5.4}$$

i.e. $\mathcal{D}(w_\infty) = P(x_0)\mu(\{x_0\})$ if $\mu(\{x_0\}) \neq 0$ and $\mathcal{D}(w_\infty) = 0$ if $\mu(\{x_0\}) = 0$.

Proof. Recalling (1.13), by definition $w_\infty = \lim_n w_{x_0,r_n}$, for a sequence $r_n \to 0$, where the convergence is locally uniform and in $W^{1,p}_{\mathrm{loc}}$ for all $p \in [1,\infty)$ in $\mathbb{R}^2$, see Theorem 1.1. Take any $\varphi \in C_c^\infty(\mathbb{R}^2)$ and set $\varphi_n(x) \doteq \varphi\left(\frac{x-x_0}{r_n}\right)$. Notice that φ_n converges pointwise to 0 in $\{x_0\}^c$ and to $\varphi(0)$ at x_0. On one hand:

$$\lim_n \mathcal{D}(w)(\varphi_n) = \mathcal{D}(w_\infty)(\varphi), \tag{5.5}$$

because of $\mathcal{D}(w)(\varphi_n) = \mathcal{D}(w_{x_0,r_n})(\varphi)$ and the strong $W^{1,2}$ convergence of w_{x_0,r_n} to w_∞. On the other hand:

$$\mathcal{D}(w)(\varphi_n) = \int_{B_1} \varphi_n(x)P(x)d\mu(x),$$

so that dominated convergence and (5.5) conclude the proof. ☐

6. Theorem D: Regularity of inclusions into curves

In this section we show Theorem D. The proof is quite long, so it is useful to split it into steps. Throughout, we denote by Γ the C^1 curve fulfilling (1.9) for which, as in Theorem D, w solves

$$Dw \in \Gamma \text{ a.e. in } \Omega. \tag{6.1}$$

As some results depend on the global geometry of the domain, it is convenient to consider a general open, connected, simply connected set Ω in (6.1). We let $\Gamma = \gamma(I)$ for some $I = [a, b]$ or $\mathbb{R}/L\mathbb{Z}$ and $\gamma \in C^1(I, \mathbb{R}^{2\times 2})$ a homeomorphism onto Γ with $|\gamma'| > 0$ on I. The proof will be split into steps:

- First, in §6.1, we show that, for any $a \in \mathbb{R}^2$, $\varphi \doteq (a, w)$ is a viscosity solution in the sense of [10] to two Hamilton-Jacobi equations simultaneously: $\pm f(D\varphi) = 0$, where f depends only on Γ and a;
- next, in §6.2 we will reduce the proof to the case where $\operatorname{rank}(\gamma'(t)) = 1$, for all $t \in I$;
- §6.3 considers a special case, the one where Γ is *graphical*, see the special form (6.5). The main result of that subsection is that, for such curves, solutions to (6.1) are everywhere C^1;
- In §6.4, we deduce as a consequence the partial regularity result for general curves Γ;
- The fifth and sixth step, contained in §6.5-6.6, are devoted to show that w is an *entropy solution* to $Dw \in \Gamma$ and to deduce from this the structure of the singular set, respectively.

Theorem D is then a direct consequence of Corollary 6.3, Lemma 6.4, Propositions 6.13-6.17 and Lemma 6.18.

6.1. Step 1: Viscosity properties.

We wish to establish the following viscosity-type property.

Proposition 6.1. *Assume $K \subset \mathbb{R}^{2\times 2}$ is compact and satisfies (1.9). Let $w \in \operatorname{Lip}(\Omega, \mathbb{R}^2)$ solve $Dw \in K$ a.e. and define, for any $a \in \mathbb{R}^2$, $\varphi \doteq w \cdot a$. Then, φ is a strong viscosity solution of the differential inclusion $D\varphi \in aK$, i.e. if $\zeta \in C^1(\Omega)$ is such that $\varphi - \zeta$ has an extremum at $x_0 \in \Omega$, then $D\zeta(x_0) \in aK \subset \mathbb{R}^2$.*

Let us first recall the following properties of probability measures on K.

Lemma 6.2. *If $K \subset \mathbb{R}^{2\times 2}$ satisfies (1.9), then for any probability measure μ supported in K we have*

$$\mathcal{A}(\mu) \doteq \int \det(X)\, d\mu(X) - \det\Big(\int X d\mu(X)\Big) \geq \omega\Big(\int \Big|X - \int Y d\mu(Y)\Big|^2 d\mu(X)\Big), \tag{6.2}$$

for some nondecreasing function $\omega\colon [0, \infty) \to [0, \infty)$ such that $\omega(0) = 0$ and $\omega(t) > 0$ for all $t > 0$.

Proof. This proof is essentially contained in [54, Lemma 3]. Indeed we have:

$$\begin{aligned}\iint \det(X - Y)\, d\mu(X)d\mu(Y) &= \iint \Big(\det(X) + \det(Y) - \langle X, \operatorname{cof}(Y)\rangle\Big)\, d\mu(X)d\mu(Y)\\ &= 2\int \det(X)\, d\mu(X) - \Big\langle \int X\, d\mu(X), \operatorname{cof}\Big(\int Y\, d\mu(Y)\Big)\Big\rangle = 2\mathcal{A}(\mu).\end{aligned}$$

Thus $\mathcal{A}(\mu) \geq 0$, with equality if and only if $\mu \otimes \mu$ is supported on the diagonal, which is equivalent to μ being a Dirac mass. Then one can take $\omega(t)$ to be the minimum of the weak-$*$ continuous function $\mathcal{A}$ over the weak-$*$ compact set of probability measures μ on K such that

$$\int |X - \int Y d\mu(Y)|^2\, d\mu(X) \geq t.$$

□

Proof of Proposition 6.1. If $a = 0$ there is nothing to show. If $a \neq 0$, then we can write $a = e_1 Q$, for some $Q \in \mathbb{R}^{2\times 2}$ with $\det(Q) > 0$. The map $\tilde{w} = wQ$ satisfies $D\tilde{w} \in \tilde{K} = Q^T K$, and $\tilde{K}$ still satisfies (1.9), so it suffices to prove Proposition 6.1 for $a = e_1$. Letting, as usual $w = (u, v)$, we then wish to show the property for $\varphi = u$. Let us notice that, applying the previous observation with $a = \pm e_1$, it suffices to consider the case where the extremum of $u - \zeta$ is a minimum. As a further simplification we can, without loss of generality, assume that such minimum point is at $x_0 = 0$, that $u(0) = \zeta(0)$, and that the minimum is strict. Indeed, if the latter is not satisfied, then we can simply replace ζ by $\zeta_\varepsilon(x) = \zeta(x) + \varepsilon|x|^2$ for $0 < \varepsilon \ll 1$.

Thanks to the previous reductions, we now fix $\zeta \in C^1(B_1)$ such that $u-\zeta$ has a strict minimum at 0, and we let $K_1 = \pi_1(K) \subset \mathbb{R}^2$, as in Proposition 2.2. To show $D\zeta(x_0) \in K_1$, we follow the strategy of [18, §4.2]. For small $\delta > 0$, we consider the open set $\Omega_\delta \subset\subset \Omega$ given by the connected component of $\{u-\zeta < \delta\}$ which contains 0. Denote by ν_δ and μ_δ the following probability measures:

$$\nu_\delta = \frac{1}{|\Omega_\delta|}\mathbf{1}_{\Omega_\delta}\,dx \ \text{ and } \mu_\delta = (Dw)_\sharp \nu_\delta\,.$$

Notice that ν_δ is a probability measure on Ω, while μ_δ, its pushforward through Dw, is a probability measure on K. Since $u-\zeta-\delta = 0$ on $\partial\Omega_\delta$, the divergence theorem and $\operatorname{curl}(Dv) = 0$ imply, respectively:

$$\int \partial_1(u-\zeta-\delta)\,d\nu_\delta = \int \partial_2(u-\zeta-\delta)\,d\nu_\delta = 0 \ \text{ and } \int \partial_2 v\partial_1(u-\zeta-\delta)\,d\nu_\delta = \int \partial_1 v\partial_2(u-\zeta-\delta)\,d\nu_\delta\,. \tag{6.3}$$

From these two identities we readily infer that:

$$\mathcal{A}(\mu_\delta) \overset{(6.2)}{=} \int \det(Dw)\,d\nu_\delta - \det\Big(\int Dw\,d\nu_\delta\Big) = \int \partial_2 v\Big(\partial_1\zeta - \int \partial_1\zeta\,d\nu_\delta\Big) - \partial_1 v\Big(\partial_2\zeta - \int\partial_2\zeta\,d\nu_\delta\Big)\,d\nu_\delta\,.$$

Since $\zeta \in C^1$ and $\operatorname{diam}(\Omega_\delta) \to 0$ as $\delta \to 0$, we deduce $\mathcal{A}(\mu_\delta) \to 0$ as $\delta \to 0$. Notice that the properties of ω in (6.2) ensure that if $\omega(t_k) \to 0$, then $t_k \to 0$, for any sequence $t_k \geq 0$. Thus, $\mathcal{A}(\mu_\delta) \to 0$ and (6.2) imply:

$$\int \Big|Du(x) - \int Du(y)\,d\nu_\delta(y)\Big|^2 d\nu_\delta(x) \longrightarrow 0 \quad \text{as } \delta \to 0\,.$$

Since $Du \in K_1$ a.e. and $\int D\zeta\,d\nu_\delta = \int Du\,d\nu_\delta$ by the first equation of (6.3), we infer

$$\mathrm{d}\left(\int D\zeta\,d\nu_\delta, K_1\right) \to 0\,.$$

As K_1 is closed, $D\zeta \in C^0(\Omega,\mathbb{R}^2)$, $\lim_{\delta\to 0}\operatorname{diam}(\Omega_\delta) = 0$ and $0 \in \cap_{\delta>0}\Omega_\delta$, we conclude that $D\zeta(0) \in K_1$. □

It is important to note that the viscosity property established in Proposition 6.1 is very strong: for any continuous function $f\colon \mathbb{R}^2 \to \mathbb{R}$ that vanishes on aK, the function $\varphi = w\cdot a$ is a viscosity solution of $f(D\varphi) = 0$, but also of $-f(D\varphi) = 0$. If f is strictly convex, this already implies C^1 regularity.

Corollary 6.3. *Let $K \subset \mathbb{R}^{2\times 2}$ be a compact set fulfilling (1.9). Assume that there exists $a \in \mathbb{S}^1$ such that $aK \subset \mathbb{R}^2$ is contained in the boundary of a strictly convex open set. Then any $w \in \operatorname{Lip}(B_1,\mathbb{R}^2)$ solving (1.8) is C^1 in B_1.*

Proof. A strictly convex open subset of $\mathbb{R}^2$ is the sublevel set of its gauge function with respect to any interior point, see e.g. [4, Lemma 1.2], so aK is contained in the zero set of a strictly convex Lipschitz function $f\colon \mathbb{R}^2 \to \mathbb{R}$. We assume without loss of generality that $a = e_1$. According to Proposition 6.1, the function $\varphi = w_1$ is a viscosity solution of $f(D\varphi) = 0$, and so is $\tilde\varphi(x) = -\varphi(-x)$. By [5, Theorem 5.3.7] this implies that both φ and $\tilde\varphi$ are locally semiconcave in the sense of [5, Definition 2.1.1], hence φ is both locally semiconcave and locally semiconvex. We infer that φ is C^1 by [5, Theorem 3.3.7], and finally that w is C^1 since the differential inclusion $Dw \in K$ gives Dw_2 as a continuous function of Dw_1, see Proposition 2.2. □

6.2. **Step 2: Reduction to degenerate curves.**

Lemma 6.4. *If $w \in \operatorname{Lip}(\Omega,\mathbb{R}^2)$ satisfies (6.1), then either Dw is constant in B_1 or $Dw \in \Gamma_*$ a.e., where $\Gamma_* = \gamma(I_*)$ for some $I_* = [a_*,b_*]$ or $\mathbb{R}/L\mathbb{Z}$ and $\det(\gamma') = 0$ on I_*.*

Proof of Lemma 6.4. Thanks to the unique continuation result of [19], see [37, Proposition 3.1] and Remark 6.5, Dw is either constant in Ω or takes values into the degenerate part $\Gamma_d = \gamma(\{\det(\gamma') = 0\})$. We can assume that we are in the latter case. According to Theorem 1.3, Dw takes values at almost every point in one single connected component $\mathcal{C}$ of Γ_d. The required interval is then $I_* \doteq \gamma^{-1}(\mathcal{C})$. □

Remark 6.5. Note that [37, Proposition 3.1] is stated for C^2 curves. The results of [37] are restricted to C^2 curves for other reasons, but [37, Proposition 3.1] relies on [19], which does not require any smoothness, and on [36], where the proof is written for smooth curves, but works for C^1 curves modulo minor adaptations.

Thanks to Lemma 6.4 we assume from now on, in addition to (1.9), that $K = \Gamma = \gamma(I)$ for some $I = [a,b]$ or $\mathbb{R}/L\mathbb{Z}$ and $\gamma \in C^1(I, \mathbb{R}^{2\times 2})$ is a homeomorphism onto Γ with

$$|\gamma'| > 0 \text{ and } \det(\gamma') = 0 \text{ on } I. \tag{6.4}$$

6.3. **Step 3: Degenerate graphical curves.** In this subsection we assume that $I = [a,b]$ and that Γ is a graph over one of its four components. Without loss of generality, we assume that:

$$\gamma(t) = \begin{pmatrix} -f(t) & t \\ -q(t) & \eta(t) \end{pmatrix} \quad \text{for } t \in I\,, \tag{6.5}$$

for some $f, \eta, q \in C^1(I)$ satisfying, thanks to Lemma 6.4, the degeneracy condition $\det(\gamma') = q' - \eta' f' = 0$. This, together with the ellipticity condition (1.9), implies that f cannot be affine on any open interval.

Proposition 6.6. *If $w\colon \Omega \to \mathbb{R}^2$ satisfies $Dw \in \Gamma$ a.e. in Ω, then w is C^1, and Dw is constant along characteristic lines directed by $(1, f'(\partial_2 w_1))$, if $w = (w_1, w_2)$.*

Proof. Let $h = w_1$. By Proposition 6.1 h is a viscosity solution in Ω of the Hamilton-Jacobi equation $\partial_1 h + f(\partial_2 h) = 0$. This implies that $u = \partial_2 h$ is an entropy solution of the scalar conservation law

$$\partial_1 u + \partial_2 f(u) = 0\,, \tag{6.6}$$

see e.g. Lemma 6.7 below. In other words, all entropy productions of u are nonpositive distributions:

$$\partial_1 A(u) + \partial_2 B(u) \leq 0\,, \text{ for all } A, B \in C^1(\mathbb{R}) \text{ such that } B' = f'A'.$$

Since $\tilde{w}(x) = -w(-x)$ satisfies the same differential inclusion, the function $\tilde{u}(x) = u(-x)$ also has this property. As a consequence, all entropy productions of u are both nonpositive and nonnegative:

$$\partial_1 A(u) + \partial_2 B(u) = 0 \quad \text{in } \mathcal{D}'(\Omega)\,, \text{ for all } A, B \in C^1(\mathbb{R}) \text{ such that } B' = f'A'. \tag{6.7}$$

Fixing any $v \in \mathbb{R}$ and continuous ρ with support in $(0,1)$ and $\int_0^1 \rho = 1$, we use the previous equality with

$$A_k(t) = \int_{-\infty}^{t-v} k\rho(ks)\, ds, \quad B_k(t) = \int_{-\infty}^{t} f'(s) A_k'(s)\, ds.$$

It is not hard to check that $A_k(t) \to 1_{\{t>v\}}$ and $B_k(t) \to f'(v)\mathbf{1}_{\{t>v\}}$ for all $t \in \mathbb{R}$ as $k \to \infty$. By dominated convergence, we deduce that the indicator function $\chi_v(x) \doteq \mathbf{1}_{\{u(x)>v\}}$ solves the free transport equation

$$\partial_1 \chi_v + f'(v)\partial_2 \chi_v = 0 \quad \text{in } \mathcal{D}'(\Omega)\,, \tag{6.8}$$

for all $v \in \mathbb{R}$. This is the kinetic formulation [39] associated with the scalar conservation law (6.6), in the case of zero entropy production. The fact that f cannot be affine on any open interval ensures that for any $v_1 < v_2 \in [a,b]$ one can find $v_1 < \tilde{v}_1 < \tilde{v}_2 < v_2$ such that $(1, f'(\tilde{v}_1))$ and $(1, f'(\tilde{v}_2))$ are linearly independent in $\mathbb{R}^2$. This property, combined with the free transport equation, implies that u is continuous, see e.g. [17, Proposition 6]. We sketch here the argument for the readers' convenience: to prove e.g. upper semicontinuity it suffices to show, for all $v_1 < v_2$, the existence of $0 < \delta < 1$ such that

$$u(x_0) \leq v_1 \quad \Longrightarrow \quad u \leq v_2 \text{ a.e. in } B_{\delta r}(x_0)\,,$$

for any Lebesgue point $x_0 \in \Omega$ with $B_r(x_0) \subset \Omega$. (Upper semicontinuity follows thanks to the fact that δ is independent of the Lebesgue point x_0, see [17] for details, and lower semicontinuity is proved in the same way.) To prove this implication, note that x_0 is a Lebesgue point of $\chi_{\tilde{v}_1}$ with value $\chi_{\tilde{v}_1}(x_0) = 0$. Using (6.8) we deduce that all points in the line interval $\tilde{I} = [x_j + \mathbb{R}(1, f'(\tilde{v}_1))] \cap B_r(x_0)$ are Lebesgue points of $\chi_{\tilde{v}_1}$, with value 0. As $\tilde{v}_2 > \tilde{v}_1$, they are also Lebesgue points of $\chi_{\tilde{v}_2}$ with value 0, and using again (6.8) we see that all points in the set $[\tilde{I} + \mathbb{R}(1, f'(\tilde{v}_2))] \cap B_r(x_0)$ are Lebesgue points of $\chi_{\tilde{v}_2}$ with value 0. Since

$(1, f'(\tilde{v}_1))$ and $(1, f'(\tilde{v}_2))$ are linearly independent, this set contains $B_{\delta r}(x_0)$ for some $\delta \in (0,1)$ depending on $\tilde{v}_1, \tilde{v}_2$ (hence only on v_1, v_2), and we conclude that $u \leq v_2$ a.e. in $B_{\delta r}(x_0)$.

The fact that u is constant along the characteristic lines directed by $(1, f'(u))$ is also a consequence of the free transport equation. In fact, any continuous weak solution of (6.6) has this property, see [11]. We conclude that $Dw = \gamma(u)$ is continuous and constant along characteristics, as wanted. □

Lemma 6.7. *If $f \in C^1(\mathbb{R})$, $\Omega \subset \mathbb{R}^2$ is open and $h \in \mathrm{Lip}(\Omega)$ is a viscosity solution of*

$$\partial_t h + f(\partial_x h) = 0\,, \tag{6.9}$$

then $u = \partial_x h$ is an entropy solution of $\partial_t u + \partial_x[f(u)] = 0$.

Proof. Let $L = \|\partial_x h\|_{L^\infty(\Omega)}$ and $C = \max_{[-L,L]} |f'|$. To show that u is an entropy solution it suffices to do so in a neighborhood of any $(t_0, x_0) \in \Omega$. Pick $T > 0$ such that $[t_0 - T/2, t_0 + T/2] \times [x_0 - CT, x_0 + CT] \subset \Omega$. Translating the coordinates, we assume $t_0 - T/2 = 0$ and $x_0 = 0$. We choose a bounded, compactly supported, L-Lipschitz function $\tilde{h}_0 \colon \mathbb{R} \to \mathbb{R}$ such that $\tilde{h}_0(x) = h(0,x)$ for $x \in [-CT, CT]$, and we consider the unique viscosity solution $\tilde{h} \colon [0,T] \times \mathbb{R} \to \mathbb{R}$ of (6.9) in $[0,T] \times \mathbb{R}$, see for instance [10, Theorem VI.2]. Thanks to the local comparison principle [10, Theorem V.3] it coincides with h in the cone $\mathcal{C} = \{0 \leq t \leq T,\, |x| \leq C(T-t)\}$. Moreover $\tilde{h}$ can be obtained as $\tilde{h} = \lim \tilde{h}_\varepsilon$, in the sense of distributions, of the vanishing viscosity approximation $\tilde{h}_\varepsilon$ solving

$$\partial_t \tilde{h}_\varepsilon + f(\partial_x \tilde{h}_\varepsilon) = \varepsilon \partial^2_{xx} \tilde{h}_\varepsilon \quad \text{in } [0,T] \times \mathbb{R}\,, \qquad \text{and} \qquad \tilde{h}_\varepsilon(0,\cdot) = \tilde{h}_0 \quad \text{in } \mathbb{R}\,. \tag{6.10}$$

Thus the function $\tilde{u}_\varepsilon = \partial_x \tilde{h}_\varepsilon$ solves

$$\partial_t \tilde{u}_\varepsilon + \partial_x[f(\tilde{u}_\varepsilon)] = \varepsilon \partial_{xx} \tilde{u}_\varepsilon \quad \text{in } [0,T] \times \mathbb{R}\,, \qquad \text{and} \qquad \tilde{u}_\varepsilon(0,\cdot) = \tilde{u}_0 \quad \text{in } \mathbb{R}\,, \tag{6.11}$$

with $\tilde{u}_0 = \partial_x \tilde{h}_0$. Thanks to [12, § 6.3], we have $\tilde{u}_\varepsilon \to \tilde{u}$ as distributions, where $\tilde{u}$ is the unique entropy solution such that $\tilde{u}(0,\cdot) = \tilde{u}_0$. Moreover we also have $\tilde{u}_\varepsilon \to \partial_x \tilde{h}$ as distributions, and since $\tilde{h} = h$ in $\mathcal{C}$ we deduce that $\tilde{u} = u$ in $\mathcal{C}$, so u is an entropy solution in $\mathcal{C}$, and therefore in a neighborhood of (t_0, x_0). □

Remark 6.8. Lemma 6.7 is surely well-known to experts, and is in fact an "if and only if" statement. For brevity, we only recalled the argument to show the implication we used in the proof above.

6.4. **Step 4: Partial regularity.** We infer a regularity result under small pointwise oscillation. Recall that we work, without loss of generality, with a curve Γ satisfying (6.4).

Proposition 6.9. *Let Γ satisfy (1.9) and (6.4). Then, there exists $\varepsilon = \varepsilon(\Gamma) > 0$ such that, if $w \in \mathrm{Lip}(\Omega, \mathbb{R}^2)$ solves (6.1) and $\mathrm{diam}([Dw](\Omega)) \leq \varepsilon$, then $w \in C^1(\Omega, \mathbb{R}^2)$, and is constant along characteristic lines: for any $x \notin \mathrm{Sing}(w)$, the matrix Dw is constant on the connected component of $(x + \mathbb{R}v) \cap \Omega$ containing x, where v is any vector $v \in \mathrm{ran}(\mathrm{cof}\, M)^T$ for $M \in T_{Dw(x)}\Gamma$.*

Proof. There exists $\varepsilon > 0$ such that, for any $M \in \Gamma$, the intersection $\Gamma \cap \overline{B_\varepsilon(M)}$ is a graph over one of its 4 components. Therefore the small oscillation assumption $\mathrm{diam}([Dw](B_1)) \leq \varepsilon$ allows to assume that Γ is such a graph. Possibly permuting coordinates in the domain and in the target, we may moreover assume that Γ is a graph over its $(1,2)$ component as in (6.5). We can then apply Proposition 6.6 and deduce that $w = (w_1, w_2)$ is C^1 in Ω, and constant along a segment directed by $(1, f'(\partial_2 w_1))$ and containing x. A direct calculation shows that, for γ as in (6.5) with $q' = f'\eta'$, the range of $(\mathrm{cof}\, \gamma'(t))^T$ is the line spanned by $(1, f'(t))$, so this is consistent with the characteristic lines described in the statement of Proposition 6.9. □

Note that, since (6.4) holds, the characteristic direction $v = \Psi(t) \in \mathrm{ran}(\mathrm{cof}\, \gamma'(t))^T$ is uniquely determined in the projective line $\mathbb{RP}^1$. One can also check that it depends continuously on t.

Lemma 6.10. *There exists a unique continuous map $\Psi \colon I \to \mathbb{RP}^1$ such that $\mathrm{ran}(\mathrm{cof}\, \gamma'(t))^T = \mathbb{R}\Psi(t)$ for all $t \in \mathbb{R}$. Moreover Ψ is not constant on any open interval.*

Proof. The continuity of Ψ is contained in [37, Lemma 5.1], we provide the proof for completeness. Without loss of generality we assume that γ is an arc-length parametrization, that is, $|\gamma'| = 1$ on I. This, together with the degeneracy $\det(\gamma') = 0$, implies that $\mathfrak{a} = [\gamma']_{\mathcal{H}}$ and $\mathfrak{b} = [\gamma']_{\bar{\mathcal{H}}}$ satisfy $|\mathfrak{a}| = |\mathfrak{b}| = 1/2$, compare (2.3). As continuous maps from I to $\frac{1}{2}\mathbb{S}^1$, they can be written as $2\mathfrak{a} = e^{i\alpha}$, $2\mathfrak{b} = e^{i\beta}$, for some liftings $\alpha, \beta \colon I \to \mathbb{R}$. These liftings are continuous in the case $I = [a, b]$, but may fail to be periodic in the non-simply-connected case $I = \mathbb{R}/L\mathbb{Z}$, so that they may not be continuous maps from I to $\mathbb{R}$. However, in this case they can still be identified with continuous functions on $\mathbb{R}$ which satisfy $\alpha(t + L) = \alpha(t) + 2k\pi$, $\beta(t + L) = \beta(t) + 2\ell\pi$ for some $k, \ell \in \mathbb{Z}$. Direct calculation then shows that

$$\operatorname{cof} \gamma'(t) = ie^{i\frac{\alpha+\beta}{2}} \otimes ie^{i\frac{\beta-\alpha}{2}} ,$$

so $\operatorname{ran}(\operatorname{cof} \gamma')^T = \mathbb{R}\Psi$, where $\Psi = ie^{i\frac{\beta-\alpha}{2}}$ is continuous from I to $\mathbb{S}^1$ in the case $I = [a, b]$, and identified with a continuous map from $\mathbb{R}$ to $\mathbb{S}^1$ in the case $I = \mathbb{R}/L\mathbb{Z}$. In this latter case it satisfies $\Psi(t + L) = e^{i(\ell-k)\pi}\Psi(t)$, so Ψ is continuous when seen as a map into $\mathbb{RP}^1$. Moreover, if Ψ is constantly equal to Ψ_0 on an open interval (s, t), then integrating γ' over (s, t) we find that $\operatorname{ran} \operatorname{cof}(\gamma(t) - \gamma(s))^T \subset \mathbb{R}\Psi_0$, in contradiction with the ellipticity assumption (1.9). □

Finally, Proposition 6.9 and our previous analysis imply the following partial regularity result.

Proposition 6.11. *Let w solve (6.1). Then $\operatorname{Sing}(w)$ is closed, $\mathcal{H}^1(\operatorname{Sing}(w)) = 0$ and $w \in C^1(\Omega \setminus \operatorname{Sing}(w))$.*

Proof. Thanks to Theorem A and Theorem B, it suffices to show that the set $\operatorname{Reg}(w)$ defined in Definition 1.5 is open. Let $\varepsilon > 0$ be as in Proposition 6.9. Thanks to Theorem A, if $x \in \operatorname{Reg}(w)$ then there exists $\delta = \delta(\varepsilon) > 0$ such that $\operatorname{diam}([Dw](B_\delta(x))) \le \varepsilon$, so w is C^1 in $B_\delta(x)$, hence $B_\delta(x) \subset \operatorname{Reg}(w)$. □

6.5. **Step 5: Entropy productions.** In this section and the next one, the goal is to analyze the structure of the solution w to $Dw \in \Gamma$ a.e. around singular points, thus completing the proof of Theorem D. Recall that we work under assumption (6.4) for Γ.

Heuristically, one can expect a rigid structure of w near singularities because of the local constancy of Dw along characteristic lines outside $\operatorname{Sing}(w)$ combined with the fact that generic lines do not intersect the $\mathcal{H}^1$-negligible singular set $\operatorname{Sing}(w)$, i.e. Propositions 6.9-6.11. One difficulty is that characteristic lines are not generic since their direction depends on the value of Dw, so it is not at all obvious that their typical behavior is not to intersect $\operatorname{Sing}(w)$. This difficulty can be overcome by taking inspiration from the kinetic formulation (6.8) of the scalar conservation law (6.6) used in the proof of Proposition 6.6. There, the kinetic variable v is decoupled from the value of the solution $u(x)$, and for generic v the line $x + \mathbb{R}(1, f'(v))$ will not intersect a $\mathcal{H}^1$-negligible set. Since the kinetic formulation (6.8) is related to the vanishing of entropy productions it becomes natural to introduce similar tools here. We refer the reader to [50] for a systematic treatment of the link between entropies and kinetic formulations of conservation laws.

The notion of entropy production for the differential inclusion into $\Gamma = \gamma(I)$ has been used already in [37]. It stems from the system of conservation laws $\operatorname{div} \operatorname{cof}(Dw) = 0$ satisfied by the map $Dw \colon \Omega \to \Gamma$. If Dw is C^1, an application of the chain rule provides a whole family of conservation laws $\operatorname{div} \Sigma(Dw) = 0$, for any vector field $\Sigma \in C^1(\Gamma, \mathbb{R}^2)$ whose tangential derivative $\partial_\tau \Sigma(\gamma(t)) \in \mathbb{R}^2$ is orthogonal to the kernel of $\operatorname{cof} \gamma'(t)$ for all $t \in I$. These vector fields Σ are called *entropies* and we denote their class by

$$\begin{aligned}\mathcal{E}_\Gamma &= \left\{\Sigma \in C^1(\Gamma, \mathbb{R}^2) \colon \partial_\tau \Sigma(A) \in (\ker(\operatorname{cof} M))^\perp = \operatorname{ran}(\operatorname{cof} M)^T ,\ \forall A \in \Gamma,\ M \in T_A\Gamma \setminus \{0\}\right\} \\ &= \left\{\Sigma \in C^1(\Gamma, \mathbb{R}^2) \colon (\Sigma \circ \gamma)'(t) \in (\ker(\operatorname{cof} \gamma'(t)))^\perp = \operatorname{ran}(\operatorname{cof} \gamma'(t))^T ,\ \forall t \in I\right\} .\end{aligned}$$

If w is just a Lipschitz solution to $Dw \in \Gamma$ a.e. then there is no direct reason for the entropy productions $\operatorname{div} \Sigma(Dw)$ to vanish. The first step is to show that this actually happens, as for smooth solutions.

Proposition 6.12. *If $w \in \operatorname{Lip}(\Omega, \mathbb{R}^2)$ satisfies (6.1), then $\operatorname{div} \Sigma(Dw) = 0$ in $\mathcal{D}'(\Omega)$ for all $\Sigma \in \mathcal{E}_\Sigma$.*

Proof of Proposition 6.12. Recall from Proposition 6.11 that w is C^1 outside a closed $\mathcal{H}^1$-negligible set $\mathrm{Sing}(w)$. We fix $\Sigma \in \mathcal{E}_\Gamma$ and start by showing that, for any $x \in \Omega \setminus \mathrm{Sing}(w)$ there exists $r > 0$ such that $\operatorname{div} \Sigma(Dw) = 0$ in $\mathcal{D}'(B_r(x))$. Since Dw is continuous in $\Omega \setminus \mathrm{Sing}(w)$, we may choose $r > 0$ such that $Dw(B_r(x))$ is contained in a portion $\Gamma_* = \gamma_*(I_*) \subset \Gamma$ where γ^* can be written as in (6.5), without loss of generality. With that notation, the kernel of $\operatorname{cof} \gamma_*'(t)$ is spanned by the vector $(-f'(t), 1)$, and the condition $\partial_\tau \Sigma(\gamma) \perp \ker \operatorname{cof}(\gamma')$ therefore implies $\partial_\tau \Sigma_2(\gamma) = f' \partial_\tau \Sigma_1(\gamma)$ on I_*, that is, $(\Sigma_2 \circ \gamma_*)' = f'(\Sigma_1 \circ \gamma_*)'$ on I_*. Now recall from the proof of Proposition 6.6 that $u = \partial_2 w_1$ solves (6.7), namely

$$\partial_1 A(u) + \partial_2 B(u) = 0 \quad \text{in } B_r(x)\,, \text{ for all } A, B \in C^1(\mathbb{R}) \text{ such that } B' = f'A'.$$

Applying this to $A = \Sigma_1 \circ \gamma$ and $B = \Sigma_2 \circ \gamma$, and noting that $\Sigma(Dw) = \Sigma(\gamma(u))$, we get $\operatorname{div} \Sigma(Dw) = 0$ in $B_r(x)$. This is valid for any $x \in \Omega \setminus \mathrm{Sing}(w)$ and some $r = r(x) > 0$, hence $\operatorname{div} \Sigma(Dw) = 0$ in $\mathcal{D}'(\Omega \setminus \mathrm{Sing}(w))$. Since $\Sigma(Dw) \in L^\infty(\Omega)$ and $\mathcal{H}^1(\mathrm{Sing}(w)) = 0$, [27, Theorem 4.1(b)] yields $\operatorname{div} \Sigma(Dw) = 0$ in $\mathcal{D}'(\Omega)$. □

Through this Proposition, we could in principle apply the strategy of [37, §7] to conclude the proof. However, in [37] the curve Γ is assumed to have C^2 regularity, and some nontrivial adaptations are required to deal with our lower C^1 regularity.

6.5.1. *The case of a nonclosed curve.* In the case of a nonclosed curve, $I = [a, b]$, Proposition 6.12 implies quite directly the continuity of Dw, the argument being essentially the same as the one of Proposition 6.6.

Proposition 6.13. *Assume $I = [a, b]$ and w solves (6.1). Then w is C^1 in Ω.*

Proof. Since $I = [a, b]$, we can lift the map Ψ from Lemma 6.10 to a continuous map from I to $\mathbb{S}^1$, which we still denote by Ψ. For any $\alpha \in [a, b]$, the map $\Sigma^\alpha \colon \Gamma \to \mathbb{R}^2$ given by $\Sigma^\alpha \circ \gamma(t) \doteq \Psi(\alpha) \mathbf{1}_{\{t > \alpha\}}$ is a pointwise limit $\Sigma^\alpha = \lim \Sigma_j$ of entropies $\Sigma_j \in \mathcal{E}_\Gamma$. One can see this for instance by setting:

$$\Sigma_j = \tilde{\Sigma}_j \circ \gamma^{-1}, \qquad \text{with } \tilde{\Sigma}_j(t) = \int_a^t \rho_j(s - \alpha) \Psi(s)\, ds\,,$$

where $\rho_j(s) = j\rho(sj)$ for some continuous nonnegative function ρ with support in $(0, 1)$ and unit integral. Let $\theta \colon \Omega \to I$ be such that $Dw = \gamma(\theta)$. By Proposition 6.12, we have $\operatorname{div} \Sigma_j(\gamma(\theta)) = 0$ in $\mathcal{D}'(\Omega)$ for all $j \geq 1$ and therefore, by dominated convergence, $\operatorname{div} \Sigma^\alpha(\gamma(\theta)) = 0$. This amounts to the kinetic formulation

$$\Psi(\alpha) \cdot D(\mathbf{1}_{\{\theta > \alpha\}}) = 0 \quad \text{in } \mathcal{D}'(\Omega)\,, \text{ for all } \alpha \in [a, b]. \tag{6.12}$$

Recall from Lemma 6.10 that Ψ is not constant on any open interval. Thus, for any $\alpha_1 < \alpha_2 \in [a, b]$, there exist $\alpha_1 < \tilde{\alpha}_1 < \tilde{\alpha}_2 < \alpha_2$ such that $\Psi(\tilde{\alpha}_1)$ and $\Psi(\tilde{\alpha}_2)$ are linearly independent in $\mathbb{R}^2$. Combining this with (6.12), we conclude that θ is continuous by [17, Proposition 6] (whose argument is recalled in the proof of Proposition 6.6), and hence so is $Dw = \gamma(\theta)$. □

6.5.2. *The case of a closed curve.* We focus now on the case $I = \mathbb{R}/L\mathbb{Z}$. The first step is to observe, as in [37], that the map $v = \Psi(\theta) \colon \Omega \to \mathbb{RP}^1$, which indicates the direction of characteristics, is a zero-state of the unoriented Aviles-Giga functional, as defined in [25].

Lemma 6.14. *The map v satisfies*

$$\operatorname{div} \Phi(v) = 0 \quad \forall \Phi \in C^1(\mathbb{S}^1, \mathbb{R}^2) \textit{ even such that } \frac{d}{dt} \Phi(e^{it}) \cdot ie^{it} = 0 \quad \forall t \in \mathbb{R}\,. \tag{6.13}$$

Proof. Denote $\Psi(t) = e^{i\psi(t)}$ with $\psi \in C^0(\mathbb{R})$ such that $\psi(t + L) = \psi(t) + k\pi$ for some $k \in \mathbb{Z}$. Define also

$$\lambda(e^{it}) = e^{it} \cdot \frac{d}{dt} \Phi(e^{it})\,,$$

so that $\lambda \in C^0(\mathbb{S}^1)$ is odd since Φ is even. If Γ is C^2, then Ψ and ψ are C^1, and we may simply apply Proposition 6.12 to $\Sigma = \Phi \circ \Psi \circ \gamma^{-1}$, which satisfies

$$(\Sigma \circ \gamma)'(t) = (\Phi \circ \Psi)'(t) = \lambda(e^{i\psi(t)}) \psi'(t) e^{i\psi(t)}\,,$$

and belongs therefore to $\mathcal{E}_\Sigma$ by definition of $\Psi = e^{i\psi}$. Given that Γ is merely C^1, we cannot use this direct calculation. It would be natural to argue by approximation, but it is not clear how to construct a sequence of entropies $\Sigma_j \in \mathcal{E}_\Gamma$ converging pointwise to Σ. We rely instead on the removability of $\mathrm{Sing}(w)$ as in Proposition 6.12 and on the kinetic formulation obtained in Proposition 6.13 for nonclosed curves. Thus, as in Proposition 6.12, it suffices to show that, for fixed $x \in \Omega \setminus \mathrm{Sing}(w)$:

$$\operatorname{div}\Phi(v) = 0, \quad \text{in } \mathcal{D}'(B_r(x)) \text{ for some } r = r(x) > 0\,. \tag{6.14}$$

We choose $r > 0$ small enough that $Dw(B_r(x)) \subset \gamma([a,b])$ for some $a < b < a + L$ such that Ψ admits a continuous lifting from $[a,b]$ into the arc $A = \{e^{is} \colon s_1 < s < s_2\}$ for some $s_1 < s_2 < s_1 + \pi$. We write $\Psi = e^{i\psi}$ on $[a,b]$, with $\psi \in C^0([a,b])$ such that $s_1 < \psi < s_2$. Then, by the proof of Proposition 6.13, the function $\theta \colon \Omega \to [a,b]$ such that $Dw = \gamma(\theta)$ satisfies (6.12), namely

$$\operatorname{div}\left(\Psi(\alpha)\mathbf{1}_{\{\theta>\alpha\}}\right) = 0 \quad \text{in } \mathcal{D}'(B_r(x))\,, \qquad \text{for all } \alpha \in [a,b].$$

Similarly, we get that $\Psi(\alpha)\mathbf{1}_{\{\theta\le\alpha\}}$ and $\Psi(\alpha)\mathbf{1}_{\{\theta<\alpha\}}$ are divergence-free in $B_r(x)$. Moreover, we can assume, without loss of generality, that $|\{\theta = a\} \cup \{\theta = b\}| = 0$. These observations lead to

$$\operatorname{div}\left(\xi\mathbf{1}_{\{\alpha<\theta<\beta\}}\right) = 0 \quad \text{in } \mathcal{D}'(B_r(x))\,, \quad \text{for all } \xi \in A \text{ and } \alpha,\beta \in \{a,b\} \cup \Psi^{-1}(\{\xi\}),\ \alpha < \beta. \tag{6.15}$$

Given $s \in (s_1, s_2)$ and $\xi = e^{is}$, the open set $\{t : \psi(t) > s\} \subset (a,b)$ is a countable union of intervals (α_j, β_j) with $\alpha_j < \beta_j \in \{a,b\} \cup \Psi^{-1}(\{\xi\})$. By dominated convergence and (6.15) we infer

$$\operatorname{div}\left(e^{is}\mathbf{1}_{\{\psi(\theta)>s\}}\right) = 0 \quad \text{in } \mathcal{D}'(B_r(x))\,, \quad \forall s \in (s_1, s_2)\,. \tag{6.16}$$

Moreover for $s \in (s_1, s_2)$ we have

$$\Phi(e^{is}) = \Phi(e^{is_1}) + \int_{s_1}^{s_2} \lambda(e^{i\tau})\mathbf{1}_{\{\tau<s\}} e^{i\tau}\, d\tau\,,$$

so we deduce, for any $\zeta \in C^1_c(B_r(x_0))$ using Fubini's theorem,

$$\langle \operatorname{div}\Phi(v), \zeta\rangle = \int_{s_1}^{s_2} \lambda(e^{i\tau})\langle \operatorname{div}\left(e^{i\tau}\mathbf{1}_{\{\psi(\theta)>\tau\}}\right), \zeta\rangle\, d\tau \overset{(6.16)}{=} 0\,,$$

thus concluding the proof. □

Lemma 6.14 allows us to use [25, Theorem 6.5], which we recall for the reader's convenience.

Theorem 6.15 ([25])**.** *If $v \colon \Omega \to \mathbb{RP}^1$ satisfies (6.13), then it has the following properties.*

(1) The map v is locally Lipschitz in $\Omega \setminus \mathcal{S}_v$ for a locally finite set $\mathcal{S}_v \subset \Omega$.
(2) For $x \in \Omega \setminus \mathcal{S}_v$, $v \equiv v(x)$ on the connected component of $[x + \mathbb{R}v(x)] \cap (\Omega \setminus \mathcal{S}_v)$ containing x;
(3) if $B \doteq B_r(x_0)$, $B_{2r}(x_0) \subset \Omega$, and $2B \cap \mathcal{S}_v = \{x_0\}$, then either
  - *(a) $v(x) = V^{x_0}(x) \doteq \frac{x-x_0}{|x-x_0|}$ in $B \setminus \{x_0\}$;*
  - *(b) or there exists $\xi \in \mathbb{S}^1$ such that*
    - *$v(x) = V^{x_0}(x)$ in $\{x \in B \colon (x - x_0, \xi) > 0\}$;*
    - *v is Lipschitz in $\{x \in B \colon (x - x_0, \xi) < 0\}$.*

Remark 6.16. Here and in what follows we implicitly identify a map $v \colon \Omega \to \mathbb{RP}^1 = \mathbb{S}^1/\{\pm 1\}$ with any lifting $\tilde{v} \colon \Omega \to \mathbb{S}^1$ such that $v = \{\pm\tilde{v}\}$. In [25, Theorem 6.5], this result is stated for the map $\tilde{v}^2$ which uniquely determines v and still satisfies (6.13). Theorem 6.15 is a direct translation of that statement, taking into account the aforementioned implicit identification.

Proposition 6.17. *Assume $I = \mathbb{R}/L\mathbb{Z}$ and w solves (6.1), for Γ fulfilling (1.9). Then, the field $v = \Psi(\theta)$ of characteristic lines is continuous in $\Omega \setminus \mathcal{S}_v$, where $\mathcal{S}_v$ is locally finite. Moreover, $\mathcal{S}_v = \mathrm{Sing}(w)$.*

Proof. Lemma 6.14 and Theorem 6.15 imply that the field v of characteristic lines is continuous in $\Omega \setminus \mathcal{S}_v$, where $\mathcal{S}_v$ is locally finite. Let us show that $\mathcal{S}_v = \mathrm{Sing}(w)$. Clearly $\mathcal{S}_v \subset \mathrm{Sing}(w)$, since $v = \Psi \circ \gamma^{-1}(Dw)$ and $\Psi \circ \gamma^{-1}$ is continuous by Lemma 6.10. To show the other inclusion we fix $x_0 \in \Omega \setminus \mathcal{S}_v$ and argue that x_0 must be a regular point of w. Since $Dw = \gamma(\theta)$ and $v = \Psi(\theta)$ we can write, for all $\delta > 0$,

$$[Dw](B_\delta(x_0)) \subset \gamma(\Psi^{-1}(A_\delta)), \text{ where } A_\delta = v(\overline{B_\delta(x_0)}).$$

By continuity of v at x_0, $\bigcap_{\delta>0} A_\delta = \{v(x_0)\}$, and hence $[Dw](x_0) \subset \gamma(\Psi^{-1}(\{v(x_0)\}))$. By Theorem 1.3 and (1.9) we infer that $[Dw](x_0)$ is connected, so it is contained in a single connected component of $\gamma(\Psi^{-1}(\{v(x_0)\}))$, which must be of the form $\gamma(\mathcal{C})$ for a single connected component $\mathcal{C}$ of $\Psi^{-1}(\{v(x_0)\}) \subset I$. Thus $\mathcal{C}$ is an interval, and since Ψ is not constant on any open interval by Lemma 6.10, we get that $\mathcal{C}$ is a singleton, thus $\mathrm{diam}([Dw](x_0)) = 0$. This yields $x_0 \in \mathrm{Reg}(w)$ and concludes the proof. □

6.6. **Step 6: Structure of the singular set.** Elementary geometric considerations give a quite strong constraint on the structure of the characteristic lines. To see this, let $\theta \colon \Omega \to I$ be such that $Dw = \gamma(\theta)$, and let $v = \Psi \circ \theta \colon \Omega \to \mathbb{RP}^1$, where Ψ is the characteristic direction defined in Lemma 6.10. The map v is continuous in $\Omega \setminus \mathcal{S}_v$ and locally constant in its own direction.

Lemma 6.18. *If $\Omega \subset \mathbb{R}^2$ is convex and $v \colon \Omega \to \mathbb{RP}^1$ is continuous and locally constant in its own direction in $\Omega \setminus \mathcal{S}_v$, for a locally finite set $\mathcal{S}_v$ of discontinuity points, then $\mathcal{S}_v$ contains at most two points, and for any $x_0 \in \mathcal{S}_v$ there exists $\xi_0 \in \mathbb{S}^1$ such that $v(x_0 + x) = x/|x|$ for all $x \in \Omega - x_0$ with $x \cdot \xi_0 > 0$.*

Proof. From Theorem 6.15, we know that there exist $\delta > 0$ and $\xi_0 \in \mathbb{S}^1$ such that

$$v(x) = V^{x_0}(x) = \frac{x - x_0}{|x - x_0|} \text{ for all } x \in B_\delta(x_0) \text{ with } (x - x_0) \cdot \xi_0 > 0\,. \tag{6.17}$$

The following crucial property is used to show (6.17) and we will make use of it in our proof as well: due to the convexity of Ω, for any $x \neq y \in \Omega \setminus \mathcal{S}_v$, if the characteristic lines $x + \mathbb{R}v(x)$ and $y + \mathbb{R}v(y)$ intersect at a single point $z \in \Omega$, then at least one of the segments $[x, z]$ and $[y, z]$ contains a singular point.

As a preliminary step in our proof, we claim that formula (6.17) is actually valid in the whole half-domain

$$\Omega_{x_0,\xi_0} \doteq \{x \in \Omega \colon (x - x_0) \cdot \xi_0 > 0\}\,.$$

By convexity of Ω and local constancy of v in its own direction, this is equivalent to $\Omega_{x_0,\xi_0} \cap \mathcal{S}_v = \emptyset$. Assume, by contradiction, that there exists $x_1 \in \Omega_{x_0,\xi_0} \cap \mathcal{S}_v$. We can choose x_1 to be the closest element of $\mathcal{S}_v$ to x_0, and hence a neighborhood of $[x_0, x_1]$ contains only x_0, x_1 as singular points. Employ again Theorem 6.15 at x_1: there exist $\delta_1 > 0$ and $\xi_1 \in \mathbb{S}^1$ such that $B_{\delta_1}(x_1) \cap \mathcal{S}_v = \{x_1\}$ and $v = V^{x_1}$ in $B_{\delta_1}(x_1) \cap \Omega_{x_1,\xi_1}$. For small $\varepsilon > 0$ and any $y \in B_\varepsilon(x_1) \cap \Omega_{x_1,\xi_1}$ such that $y - x_1$ is not parallel to $x_1 - x_0$, the segment $[y, x_0]$ contains no singular point, so v must be constant along it, that is, equal to $(y - x_0)/|y - x_0|$, in contradiction with the form of v near x_1. Hence our claim holds: (6.17) is valid in Ω_{x_0,ξ_0}.

Assume now that $\mathcal{S}_v$ contains at least two singular points $x_1 \neq x_2 \in \mathcal{S}_v$. Then there exist $\xi_1, \xi_2 \in \mathbb{S}^1$ such that $v = V^{x_j}$ in Ω_{x_j,ξ_j} for $j = 1, 2$. This implies in particular that the two sets Ω_{x_1,ξ_1} and Ω_{x_2,ξ_2} must be disjoint. Consider, for $j = 1, 2$, the two lines $L_j = x_j + \mathbb{R}\xi_j^\perp$ which bound Ω_{x_j,ξ_j}. Assume by contradiction that there exists a third, distinct singular point x_3, and let $\xi_3 \in \mathbb{S}^1$ be such that $v = V^{x_3}$ in Ω_{x_3,ξ_3}. Clearly, this set must be disjoint from Ω_{x_j,ξ_j}, $j = 1, 2$. From this, we deduce immediately that $x_3 \notin L_j$. We notice that if $L_1 = L_2$, then $\Omega = \overline{\Omega_{x_1,\xi_1}} \cup \overline{\Omega_{x_2,\xi_2}}$. In that case, since $(L_1 \cup L_2) \cap \mathcal{S}_v = \{x_1, x_2\}$, $\mathcal{S}_v = \{x_1, x_2\}$. We can therefore assume that $L_1 \neq L_2$, so that the open set lying between them is nonempty:

$$U = \Omega \setminus \left(\overline{\Omega_{x_1,\xi_1}} \cup \overline{\Omega_{x_2,\xi_2}}\right) \neq \emptyset.\Big)\,,$$

Necessarily $x_3 \in U \cap \mathcal{S}_v$, and we can assume without loss of generality that x_3 is the point in $U \cap \mathcal{S}_v$ closest to $[x_1, x_2]$. As Ω_{x_3,ξ_3} must be disjoint from $\Omega_{x_1,\xi_1} \cup \Omega_{x_2,\xi_2}$, x_3 does not belong to $[x_1, x_2]$, hence the open interval $X \doteq [x_1, x_2] \setminus \{x_1, x_2\}$ does not contain any singular point (since x_3 is a singular point in U closest to that segment), and x_1, x_2 are the only singular points in a neighborhood of the segment

$[x_1, x_2]$. Assume without loss of generality that $[x_1, x_2]$ is horizontal, that x_1 lies left of x_2, and that x_3 lies above $[x_1, x_2]$. Hence the lines L_1, L_2 enclosing U are not horizontal. For $x \in X$, the line $\ell_x = x + \mathbb{R}v(x)$ cannot be horizontal: otherwise, the characteristic line from any point above and close enough to x is nearly horizontal, its intersection with U is contained in the regular neighborhood of $[x_1, x_2]$, and intersects the characteristic lines L_1 and L_2 at regular points, which is impossible. Hence, each line ℓ_x, for $x \in X$, intersects the horizontal line H passing through x_3 at a point $y(x)$. Since v is continuous along X, the map $x \mapsto y(x)$ is continuous on X. As $\lim_{x\to x_i, x\in X} v(x) = \xi_i^\perp$, $y(x)$ lies left/right of x_3 on H for $x \in X$ sufficiently close to x_1/x_2. As a consequence, there exists $x \in X$ such that $y(x) = x_3$. This yields a characteristic segment $\hat{L}$ connecting x to x_3. Set $L_3 = x_3 + \mathbb{R}\xi_3^\perp$. Observe that $x \in \Omega_{x_3,-\xi_3}$ and that $\hat{L} \cap L_3 = \{x_3\}$, otherwise $x \in \overline{\Omega_{x_3,\xi_3}}$ and thus either x_1 or x_2 belongs to Ω_{x_3,ξ_3}, a contradiction. Therefore, $\hat{L} \subset \overline{\Omega_{x_3,-\xi_3}}$, and $\hat{L} \cap L_3 = \{x_3\}$. This implies that all characteristic lines starting from $z \in \Omega_{x_3,-\xi_3}$ close enough to x_3 must stay in a sector delimited by $\hat{L}$ and $L_3 = x_3 + \mathbb{R}\xi_3^\perp$, and must therefore intersect x_3. Hence, $v = V^{x_3}$ in $B_r(x_3) \cap \Omega_{x_3,-\xi_3}$ for some small $r > 0$ and the same holds for Ω_{x_3,ξ_3} by the first claim of the proof. For the same reason, $v = V^{x_3}$ in Ω, in contradiction with the fact that $\{x_1, x_2\} \subset \mathcal{S}_v$. Thus such x_3 could not exist, and we conclude the proof. ☐

Corollary 6.19. *If v is as in Lemma 6.18 and $\Omega = \mathbb{R}^2$, then there exist $s = (s_1, s_2), t = (t_1, t_2)$, $s_1 \in [-\infty, +\infty)$, $t_1 \in (-\infty, +\infty]$ with $s_1 \le t_1$, $t_2, s_2 \in \mathbb{R}$, and $Q \in SO(2)$ such that for $x \ne s, t$*

$$Q^T v(Qx) = \begin{cases} \frac{x-(s_1,s_2)}{|x-(s_1,s_2)|} & \text{if } x_1 < s_1\,, \\ e_2 & \text{if } s_1 \le x_1 \le t_1\,, \\ \frac{x-(t_1,t_2)}{|x-(t_1,t_2)|} & \text{if } x > t_1\,. \end{cases} \tag{6.18}$$

Remark 6.20. Corollary 6.19 characterizes all possible entire configurations of characteristic lines with locally finite singular set, as members of a 5-dimensional family. Special cases are the constants, corresponding to $t_1 = -s_1 = +\infty$, the single half-vortices, corresponding to $(s_1, t_1) \in (\{-\infty\} \times \mathbb{R}) \cup (\mathbb{R} \times \{+\infty\})$, and the single vortices, corresponding to $(s_1 - t_1, s_2 - t_2) = (0, 0)$.

Proof of Corollary 6.19. We distinguish three cases depending on the cardinality of $\mathcal{S}_v$.

If $\mathcal{S}_v = \emptyset$ then the characteristic lines cannot intersect and must then all be parallel to a single direction Qe_2 for some $Q \in SO(2)$, which corresponds to the case $t_1 = -s_1 = +\infty$.

If $\mathcal{S}_v = \{x_0\}$, then there exists $\xi_0 \in \mathbb{S}^1$ such that $v = V^{x_0}$ in Ω_{x_0,ξ_0}. Writing $\xi_0 = -Qe_1$ for some $Q \in SO(2)$ and replacing v by $Q^T v \circ Q$ we assume without loss of generality that $\xi_0 = -e_1$. Hence $v(x) = V^{x_0}(x)$ for $x_1 < s_1 \doteq x_0 \cdot e_1$. Let $s_2 = x_0 \cdot e_2$, so that $x_0 = (s_1, s_2)$. The characteristic lines starting from any $x \in \mathbb{R}^2$ with $x_1 > s_1$ can only intersect the characteristic vertical line $\{x_1 = s_1\}$ at $x = x_0$. So they must either all be vertical, namely $t_1 = +\infty$, or all pass through x_0, i.e. $(t_1, t_2) = (s_1, s_2)$.

Assume finally that $\mathcal{S}_v = \{x_1, x_2\}$. Let $\xi_j \in \mathbb{S}^1$ be such that $v = V^{x_j}$ in Ω_{x_j,ξ_j} for $j = 1, 2$. Since these two half-planes must be disjoint, their boundaries are parallel, hence $\xi_2 = -\xi_1$. Applying a rotation, we assume without loss of generality that $\xi_2 = -\xi_1 = e_1$. Then $v(x)$ is as in (6.18) for $x_1 < s_1$ and $x_1 > t_1$, where $s_k = x_1 \cdot e_k$ and $t_k = x_2 \cdot e_k$ for $k = 1, 2$. If $s_1 = t_1$ we are done. If $s_1 < t_1$, the characteristic lines starting from any point $x \in \{s_1 < x_1 < t_1\}$ can intersect the vertical characteristic lines which form the stripe's boundary only at x_1 or x_2. But if such an intersection happens, then v must be a vortex in $\mathbb{R}^2$, and $\mathcal{S}_v$ contains only one element. So they must all be vertical, and we conclude the proof. ☐

7. Solutions of degenerate equations

Let $G\colon \mathbb{R}^2 \to \mathbb{R}^2$ a continuous and strictly monotone vector field and let $u \in \mathrm{Lip}(B_1)$ be a solution of (1.1). In [34] it is shown that, for $x_0 \in B_1$, any blow-up limit $u_\infty \in \mathcal{B}(u)(x_0)$ is either affine or satisfies

$$Du_\infty \in \mathcal{D} = \mathcal{D}_- \cap \mathcal{D}_+ \quad \text{a.e.},$$

where $\mathcal{D}_\pm$ are defined in (1.16). Recalling from Proposition 2.2 the correspondence between (1.1) and (1.8), there is a direct reformulation in terms of differential inclusions.

Theorem 7.1 ([34]). *Let $K \subset \mathbb{R}^{2\times 2}$ satisfy (1.9), $w \in \mathrm{Lip}(B_1, \mathbb{R}^2)$ solve $Dw \in K$ a.e. in B_1, and $x_0 \in B_1$. Any blow-up limit $w_\infty \in \mathcal{B}(w)(x_0)$ is either affine or satisfies $Dw_\infty \in \mathcal{K}_*$ a.e., where*

$$\mathcal{K}_* = \mathcal{K}_*^1 \cap \mathcal{K}_*^2\,, \qquad \mathcal{K}_*^j = \bigcap_{\lambda>0} \overline{\left\{ A \in K\colon \liminf_{K\ni B\to A} \frac{\det(A-B)}{|L_j(A-B)|^2} \leq \lambda \right\}}\,, \tag{7.1}$$

and $L_j(M)$ denotes the j-th row of a matrix M, for $j = 1, 2$.

In this section, we combine this property of blow-up limits with Theorems A-B-C-D and structural assumptions on the degenerate set $\mathcal{D}$, or equivalently $\mathcal{K}_*$, in order to deduce partial regularity properties of (1.8), or equivalently (1.1), obtaining in particular Theorem E.

7.1. **Regularity threshold.** For any $K \subset \mathbb{R}^{2\times 2}$, let us introduce the nonnegative number

$$\varepsilon_*(K) \doteq \sup\left\{ \varepsilon \geq 0\colon \left(Dw \in K \text{ a.e. in } B_1 \text{ and } \operatorname{diam}([Dw]) \leq \varepsilon \right) \Rightarrow w \in C^1(B_1) \right\}, \tag{7.2}$$

which is the threshold for regularity of solutions of (1.8) with small gradient oscillations. With this notation, the first statement in Proposition 6.9 can be reformulated as $\varepsilon_*(\Gamma) > 0$ for any compact connected C^1 curve Γ satisfying (1.9). Moreover, Corollary 6.3 and Proposition 6.13 imply $\varepsilon_*(\Gamma) = +\infty$ if there exists $a \in \mathbb{S}^1$ such that the projection $a\Gamma \subset \mathbb{R}^2$ is the boundary of a strictly convex open set or if Γ is simply connected. We can also reformulate partial regularity in terms of positivity of $\varepsilon_*(K)$:

Lemma 7.2. *Let $K \subset \mathbb{R}^{2\times 2}$ satisfy (1.9) and $\varepsilon_*(K) > 0$. For any Lipschitz solution w of (1.8), we have that $\mathrm{Sing}(w)$ is closed, $\mathcal{H}^1(\mathrm{Sing}(w)) = 0$ and $w \in C^1(\Omega \setminus \mathrm{Sing}(w))$.*

Proof. Thanks to Theorem A and Theorem B, it suffices to show that the set $\mathrm{Reg}(w)$ defined in Definition 1.5 is open. Let $\varepsilon = \varepsilon_*(K)/2 > 0$. By Theorem A, if $x \in \mathrm{Reg}(w)$ then there exists $\delta > 0$ such that $\operatorname{diam}([Dw](B_\delta(x))) \leq \varepsilon$, so w is C^1 in $B_\delta(x)$ by definition (7.2) of $\varepsilon_*(K)$, hence $B_\delta(x) \subset \mathrm{Reg}(w)$. □

Thanks to Theorem 7.1 and Theorem A, we can relate the regularity threshold ε_* of K with the regularity thresholds of the connected components of $\mathcal{K}_*$.

Proposition 7.3. *For any $K \subset \mathbb{R}^{2\times 2}$ satisfying (1.9), let $\mathcal{K}_*$ as in (7.1). Then,*

$$\varepsilon_*(K) \geq \inf\left\{ \varepsilon_*(\mathcal{C})\colon \mathcal{C} \text{ connected component of } \mathcal{K}_* \right\}.$$

Proof. If the infimum is zero, there is nothing to show. Assume therefore that it is positive, and fix $0 < \varepsilon < \inf\{\varepsilon_*(\mathcal{C})\}$. Let $w\colon B_1 \to \mathbb{R}^2$ satisfy $Dw \in K$ a.e. and $\operatorname{diam}([Dw](B_1)) \leq \varepsilon$. This pointwise constraint is preserved under blow-up: for any $x_0 \in B_1$ and $w_\infty \in \mathcal{B}(w)(x_0)$ we have $\operatorname{diam}([Dw_\infty])(B_1) \leq \varepsilon$. By Theorem 7.1, the blow-up limit w_∞ is either affine or satisfies $Dw_\infty \in \mathcal{K}_*$ a.e., hence $Dw_\infty \in \mathcal{C}$ a.e. for some connected component $\mathcal{C}$ of $\mathcal{K}_*$ by Theorem 1.3. By definition of $\varepsilon_*(\mathcal{C})$ this implies $w_\infty \in C^1(B_1, \mathbb{R}^2)$, hence $x_0 \in \mathrm{Reg}(w)$ by Definition 1.5. Thus $\mathrm{Reg}(w) = B_1$ and $\varepsilon_*(K) \geq \varepsilon$. □

Of course, Proposition 7.3 is only interesting if we know that the connected components of $\mathcal{K}_*$ have a positive regularity threshold (7.2). Thanks to Proposition 6.9, this is the case if we assume that

$$\text{every connected component of } \mathcal{K}_* \text{ is included in a } C^1 \text{ curve.} \tag{7.3}$$

Under this assumption, every connected component $\mathcal{C}$ of $\mathcal{K}_*$ is either a point, or satisfies $\mathcal{C} \subset \Gamma = \gamma(I)$ for some $I = [a, b]$ or $\mathbb{R}/L\mathbb{Z}$ and $\gamma\colon I \to \Gamma$ a C^1 homeomorphism with $|\gamma'| > 0$. In the latter case, since $\mathcal{C}$ is compact and connected and γ is a homeomorphism, we must in fact have $\mathcal{C} = \gamma(J)$ for some compact interval $J \subset I$. So assumption (7.3) is actually equivalent to every connected component $\mathcal{C}$ of $\mathcal{K}_*$ being either a point or a compact C^1 curve, which implies $\varepsilon_*(\mathcal{C}) > 0$ by Proposition 6.9. In fact, we will see in the next §7.2 that under assumption (7.3), the property $\varepsilon_*(K) > 0$ implies a much stronger conclusion than that of Lemma 7.2.

Remark 7.4. Condition (1.9) is not sufficient to ensure $\varepsilon_*(K) > 0$. Consider indeed, for any $n \in \mathbb{N}$,

$$f_n(z) \doteq \frac{z^n}{|z|^{n-1}},\ h(z) \doteq -\frac{1}{3}f_3(z),\ w(z) \doteq f_2(z). \tag{7.4}$$

Direct computations yield

$$\partial_z f_n(z) = \frac{n+1}{2}\frac{z^{n-1}}{|z|^{n-1}} \text{ and } \partial_{\bar z} f_n(z) = \frac{1-n}{2}\frac{z^{n+1}}{|z|^{n+1}} \tag{7.5}$$

so that w solves (2.8) for h defined in (7.4). Through (7.5)-(2.3), we deduce that w is Lipschitz. Moreover, for any $v \in \mathbb{S}^1$, (7.5) and (2.2) imply that

$$|Dh(z)v|^2 = \left|\frac{2}{3}v - \frac{1}{3}\frac{z^2}{|z|^2}\bar v\right|^2 = \frac{5}{9} - \frac{4}{9}\operatorname{Re}\left(\frac{z^2}{|z|^2}(\bar v)^2\right) \le 1,\text{ with equality if and only if } (v,z) = 0.$$

In particular, if $a, b \in \mathbb{C}$ are such that $|h(b) - h(a)| = |b-a|$, we see that

$$(b-a, a+t(b-a)) = 0,\text{ for all } t \in [0,1]\,.$$

This can only happen if $a = b$, and hence we conclude that h fulfills (2.10). Since h is one-homogeneous, we have that $w_\alpha(z) \doteq \alpha w(z)$ solves (2.8) for h as in (7.4) for all $\alpha \in \mathbb{R}$. Moreover $\operatorname{Sing}(w_\alpha) = \{0\}$, regardless of how small α is, showing that no ε-regularity theorem can hold for the PDE (2.8) for such h. In other words, the regularity threshold (7.2) of the set $K = \{(a, h(a)) \colon |a| \le 1\}$ is $\varepsilon_*(K) = 0$. The problem in this case is, in fact, fully degenerate: for h as in (7.4), passing to the equivalent formulation in terms of a monotone field G as in (1.1) (see Propositions 2.2-2.3), we find that $\mathcal{D}_+ \cap \mathcal{D}_- = \mathbb{R}^2$, or equivalently $K = \mathcal{K}_*$.

7.2. **Characterization of blow-up limits.** Since blow-up limits w_∞ are entire solutions of $Dw_\infty \in \mathcal{C}$ a.e. for some connected component $\mathcal{C}$ of $\mathcal{K}_*$, under assumption (7.3) we know by Theorem D that they are C^1 away from at most two singular points. This statement is valid for any entire solution, but blow-up limits have the additional property that their Hessian determinant vanishes away from the origin by Theorem C. Using this, we establish that they have in fact at most one singularity.

Proposition 7.5. *Assume* (1.9) *and* (7.3)*. Let* $w \in \operatorname{Lip}(B_1, \mathbb{R}^2)$ *solve* $Dw \in K$ *a.e., and* $x_0 \in B_1$*. Any blow-up limit* $w_\infty \in \mathcal{B}(w)(x_0)$ *can only be singular at* 0 *and* Dw_∞ *is 0-homogenenous.*

The proof of Proposition 7.5 uses the structure of characteristic lines established in Corollary 6.19, combined with Theorem C and the following lemma.

Lemma 7.6. *Let* $u \in C^1(B_1 \setminus \{0\})$ *be Lipschitz,* 1*-homogeneous, and satisfy* $\mathcal{D}(u,u) = 0$ *in* B_1*. If* u *coincides with a linear function on* $B_1^- = B_1 \cap \{x_1 < 0\}$*, then* u *is linear.*

Proof. Write $u(re^{i\theta}) = rf(\theta)$ for some $f \in C^1(\mathbb{T})$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$. Recalling (5.1), $\mathcal{D}(u,u) = 0$ is equivalent to

$$\int_{B_1} (D^2\varphi JDu, JDu)\,dx = 0, \quad \forall \varphi \in C_c^2(B_1)\,. \tag{7.6}$$

Since JDu is divergence-free, this identity is preserved after substracting a linear function from u, so we may without loss of generality assume that $u = 0$ on B_1^-, that is, $f = 0$ on $[\pi/2, 3\pi/2]$. Choosing radial test functions $\varphi(re^{i\theta}) = \psi(r)$ with $\psi \in C_c^2([0,1))$ such that $\psi'(0) = \psi''(0) = 0$ and $\psi(0) = 1$, (7.6) becomes

$$\begin{aligned} 0 &= \int_0^1 \int_{\mathbb{T}} \left(f'(\theta)^2\psi''(r) + f(\theta)^2\frac{\psi'(r)}{r}\right) d\theta\, r dr \\ &= \int_{\mathbb{T}} f'(\theta)^2\,d\theta \int_0^1 \psi''(r)\, r dr + \int_{\mathbb{T}} f(\theta)^2\,d\theta \int_0^1 \psi'(r)\,dr = \int_{\mathbb{T}} f'(\theta)^2\,d\theta - \int_{\mathbb{T}} f(\theta)^2\,d\theta\,. \end{aligned}$$

As $f \equiv 0$ on $[\pi/2, 3\pi/2]$, we see that f fulfills the equality case in Wirtinger's inequality on $[-\pi/2, \pi/2]$ with Dirichlet conditions. We infer $f(\theta) = a\mathbf{1}_{|\theta|<\pi/2}\cos\theta$ for some $a \in \mathbb{R}$, thus $f = 0$ since f is C^1 on $\mathbb{T}$. $\square$

Proof of Proposition 7.5. As argued above, by Theorem 7.1 and Theorem 1.3, under assumptions (1.9)-(7.3), the blow-up limit w_∞ is either linear or it satisfies $Dw_\infty \in \Gamma$ a.e. in $\mathbb{R}^2$, where $\Gamma = \gamma(I)$ is a compact connected C^1 curve satisfying (1.9), as in § 6. We can assume we are in the latter case, as otherwise there is nothing to show, and for the same reason we can assume that $0 \in \mathrm{Sing}(w_\infty)$, since otherwise $x_0 \in \mathrm{Reg}(w)$ and w_∞ is affine by Theorem A. Letting $\theta = \gamma^{-1}(Dw_\infty)$ and Ψ as in Lemma 6.10, we have therefore that the field of characteristic lines $v = \Psi \circ \theta$ must be of the form (6.18). On any subset $A \subset \mathbb{R}^2 \setminus \mathrm{Sing}(w_\infty)$ on which v is constant, say $v = v_0$, Dw_∞ must belong to $\Psi^{-1}(\{v_0\})$, a totally disconnected set by Lemma 6.10, and therefore be constant. Combining this with the structure (6.18) of the map v, we deduce that Dw_∞ is either constant, or 0-homogeneous, or that it has two discontinuity points, 0 and $y_0 \neq 0$.

To prove Proposition 7.5 we just need to rule out the latter case. To do so, by Theorem A it suffices to show that there exists a linear blow-up of w_∞ at y_0. Let W be any such blow-up. Due to the structure of the characteristic lines of w_∞ (6.18), we infer that DW is 0-homogeneous and constant in $P_\alpha \doteq \{(x, e^{i\alpha}) < 0\}$ for some $e^{i\alpha} \in \mathbb{S}^1$. Up to a rotation, we can assume without loss of generality that $\alpha = 0$. Hence its first component, $u \in C^1(\mathbb{R}^2 \setminus \{0\})$, is 1-homogeneous and coincides with a linear function on P_0. In addition, since $w_\infty \in \mathcal{B}(w)(x_0)$, then $y_0 \notin \mathrm{spt}(\mathcal{D}(w_\infty))$ by Corollary 5.2, and hence for the same reason $\mathcal{D}(u, u) = 0$ in $\mathbb{R}^2$. Combining this information with the structure of u we can infer from Lemma 7.6 that u is linear, and hence $y_0 \notin \mathrm{Sing}(w_\infty)$, a contradiction. □

If we know in addition that the regularity threshold (7.2) of K is positive, Proposition 7.5 implies discreteness of the singular set.

Corollary 7.7. *Assume (1.9), (7.3), and $\varepsilon_*(K) > 0$. Then any Lipschitz solution w of (1.8) has a locally finite singular set $\mathrm{Sing}(w)$.*

Proof. Let $x_0 \in \mathrm{Sing}(w)$ such that $B_\delta(x_0) \subset \Omega$, and assume that x_0 is not isolated. Then there exists a sequence $x_j \in \mathrm{Sing}(w) \setminus \{x_0\}$ such that $x_j \to x_0$. Let $r_j = |x_j - x_0| \to 0$ and consider the maps $w_j \doteq w_{r_j,x_0}$, see (1.13). They satisfy $0, y_j \in \mathrm{Sing}(w_j)$ where $y_j = (x_j - x_0)/r_j \in \mathbb{S}^1$. Consider a (non relabeled) subsequence such that $w_j \to w_\infty \in \mathcal{B}(w)(x_0)$ and $y_j \to y_\infty \in \mathbb{S}^1$. By Theorem A and the assumption $\varepsilon_*(K) > 0$ we have $0, y_\infty \in \mathrm{Sing}(w_\infty)$, in contradiction with Proposition 7.5. □

Combining Corollary 7.7 with Propositions 7.3 and 6.9, we deduce the following partial regularity result.

Corollary 7.8. *Assume (1.9), (7.3), and that all but a finite number of connected components $\mathcal{C}$ of the degenerate set $\mathcal{K}_*$ satisfy $\varepsilon_*(\mathcal{C}) = +\infty$. Then any $w \in \mathrm{Lip}(B_1, \mathbb{R}^2)$ solving (1.8) has a locally finite singular set $\mathrm{Sing}(w)$.*

Finally, recalling that $\Gamma \subset \mathbb{R}^{2\times 2}$ has $\varepsilon_*(\Gamma) = +\infty$ if its first-row projection is the boundary of a strictly convex open set, or if it is a compact and simply connected C^1 curve $\Gamma \subset \mathbb{R}^{2\times 2}$ satisfying (1.9), we see that Theorem E follows from Corollary 7.8 and Proposition 2.2.

References

[1] Astala, K. Area distortion of quasiconformal mappings. *Acta Mathematica 173*, 1 (1994), 37–60.

[2] Astala, K., Iwaniec, T., and Martin, G. *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane (PMS-48).* Princeton Mathematical Series. Princeton University Press, 2008.

[3] Bojarski, B., and Iwaniec, T. Analytical foundations of the theory of quasiconformal mappings in $\mathbb{R}^n$. *Annales Academiae Scientiarum Fennicae Series A I Mathematica 8* (1983), 257–324.

[4] Brezis, H. *Functional analysis, Sobolev spaces and partial differential equations.* Universitext. New York, NY: Springer, 2011.

[5] Cannarsa, P., and Sinestrari, C. *Semiconcave functions, Hamilton-Jacobi equations, and optimal control*, vol. 58 of *Prog. Nonlinear Differ. Equ. Appl.* Boston, MA: Birkhäuser, 2004.

[6] Cao, W., Hirsch, J., and Inauen, D. $C^{1,1/3-}$ very weak solutions to the two dimensional Monge-Ampère equation. *Calc. Var. Partial Differ. Equ. 64*, 5 (May 2025).

[7] Cao, W., and Székelyhidi Jr., L. Very weak solutions to the two-dimensional Monge-Ampère equation. *Science China Mathematics 62*, 6 (Apr. 2019), 1041–1056.
[8] Chipot, M., and Evans, L. C. Linearisation at infinity and Lipschitz estimates for certain problems in the calculus of variations. *Proc. R. Soc. Edinb. A: Math. 102*, 3-4 (1986), 291–303.
[9] Colombo, M., and Tione, R. Non-classical solutions of the p-Laplace equation. *J. Eur. Math. Soc.* (Apr. 2024).
[10] Crandall, M. G., and Lions, P.-L. Viscosity solutions of Hamilton-Jacobi equations. *Trans. Am. Math. Soc. 277*, 1 (1983), 1–42.
[11] Dafermos, C. M. Continuous solutions for balance laws. *Ric. Mat. 55*, 1 (July 2006), 79–92.
[12] Dafermos, C. M. *Hyperbolic conservation laws in continuum physics*, 4th ed., vol. 325 of *Grundlehren Math. Wiss.* Berlin: Springer, 2016.
[13] De Blasi, F. S., and Pianigiani, G. The Baire method for the prescribed singular values problem. *J. Lond. Math. Soc., II. Ser. 70*, 3 (2004), 719–734.
[14] De Filippis, C., and Mingione, G. Nonuniformly elliptic Schauder theory. *Inventiones mathematicae 234*, 3 (Sept. 2023), 1109–1196.
[15] De Filippis, C., and Mingione, G. The sharp growth rate in nonuniformly elliptic Schauder theory. *Duke Mathematical Journal 174*, 9 (June 2025).
[16] De Giorgi, E. Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari. *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat.* (1957).
[17] De Lellis, C., Otto, F., and Westdickenberg, M. Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Ration. Mech. Anal. 170*, 2 (2003), 137–184.
[18] De Lellis, C., Otto, F., and Westdickenberg, M. Minimal entropy conditions for Burgers equation. *Q. Appl. Math. 62*, 4 (2004), 687–700.
[19] De Philippis, G., Guerra, A., and Tione, R. Unique continuation for differential inclusions. *Ann. Inst. Henri Poincaré(C) Anal. Non Linéaire* (2023).
[20] de Silva, D., and Savin, O. Minimizers of convex functionals arising in random surfaces. *Duke Math. J. 151*, 3 (2010), 487–532.
[21] Evans, L. C., and Gariepy, R. F. *Measure theory and fine properties of functions.* Chapman & Hall/CRC, 2015.
[22] Faraco, D., and Székelyhidi, L. Tartar's conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2\times 2}$. *Acta Math. 200*, 2 (2008), 279–305.
[23] Fonseca, I., and Gangbo, W. *Degree Theory in Analysis and Applications.* Oxford University Press, Nov. 1995.
[24] Gilbarg, D., and Trudinger, N. S. *Elliptic Partial Differential Equations of Second Order.* Springer, 1977.
[25] Goldman, M., Merlet, B., Pegon, M., and Serfaty, S. Compactness and structure of zero-states for unoriented Aviles-Giga functionals. *J. Inst. Math. Jussieu 23*, 2 (2024), 941–982.
[26] Hartman, P., and Nirenberg, L. On spherical image maps whose jacobians do not change sign. *American Journal of Mathematics 81*, 4 (Oct. 1959), 901.
[27] Harvey, R., and Polking, J. Removable singularities of solutions of linear partial differential equations. *Acta Mathematica 125*, 0 (1970), 39–56.
[28] Iwaniec, T., and Martin, G. *Geometric function theory and non-linear analysis.* Clarendon press, 2001.
[29] Jerrard, R. L. Some Rigidity Results Related to Monge-Ampère Functions. *Canadian Journal of Mathematics 62*, 2 (Apr. 2010), 320–354.
[30] Johansson, C. Wild solutions to scalar Euler-Lagrange equations. *Trans. Am. Math. Soc.* (2024).
[31] Kirchheim, B. Rigidity and Geometry of Microstructures. *Habilitation Thesis* (2003).
[32] Kirchheim, B., and Székelyhidi, L. j. On the gradient set of Lipschitz maps. *J. Reine Angew. Math. 625* (2008), 215–229.

[33] Koch, H., Zhang, Y. R.-Y., and Zhou, Y. An asymptotic sharp Sobolev regularity for planar infinity harmonic functions. *J. Math. Pures Appl. 132* (Dec. 2019), 457–482.
[34] Lacombe, T. Average gradient localisation for degenerate elliptic equations in the plane. *arXiv:2601.03078* (2026).
[35] Lacombe, T., and Lamy, X. On C^1 regularity for degenerate elliptic equations in the plane. *arXiv:2407.00775* (2024).
[36] Lamy, X., Lorent, A., and Peng, G. Quantitative rigidity of differential inclusions in two dimensions. *Int. Math. Res. Not. 2024*, 8 (2024), 6325–6349.
[37] Lamy, X., Lorent, A., and Peng, G. On regularity and rigidity of 2×2 differential inclusions into non-elliptic curves. *Ann. Inst. H. Poincaré C Anal. Non Linéaire* (2025).
[38] Lewicka, M., and Pakzad, M. R. Convex integration for the Monge-Ampère equation in two dimensions. *Analysis & PDE 10*, 3 (Apr. 2017), 695–727.
[39] Lions, P. L., Perthame, B., and Tadmor, E. A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Am. Math. Soc. 7*, 1 (1994), 169–191.
[40] Lledos, B. Regularity of the stress field for degenerate and/or singular elliptic problems. *Ann. Math. Blaise Pascal 31*, 1 (2024), 83–135.
[41] Marino, G., and Mosconi, S. Lipschitz regularity for solutions of a general class of elliptic equations. *Calc. Var. Partial Differ. Equ. 63*, 1 (Dec. 2023).
[42] Mingione, G. Regularity of minima: an invitation to the dark side of the calculus of variations. *Appl. Math., Praha 51*, 4 (2006), 355–425.
[43] Mooney, C. Minimizers of convex functionals with small degeneracy set. *Calc. Var. Partial Differ. Equ. 59*, 2 (Mar. 2020).
[44] Morrey, C. B. j. On the solutions of quasilinear elliptic partial differential equations. *Trans. Am. Math. Soc. 43* (1938), 126–166.
[45] Moser, J. A new proof of de Giorgi's theorem concerning the regularity problem for elliptic differential equations. *Communications on Pure and Applied Mathematics 13*, 3 (Aug. 1960), 457–468.
[46] Müller, S. Variational Models for Microstructure and Phase Transitions. Lectures at the CIME Summer School Calculus of Variations and Geometric Evolution Problems, 1999.
[47] Müller, S., and Šverák, V. Convex integration for Lipschitz mappings and counterexamples to regularity. *Annals of Mathematics 157*, 3 (2003), 715–742.
[48] Nash, J. Continuity of Solutions of Parabolic and Elliptic Equations. *Am. J. Math. 80*, 4 (Oct. 1958).
[49] Pakzad, M. R. On the Sobolev Space of Isometric Immersions. *J. Differ. Geom. 66*, 1 (Jan. 2004).
[50] Perthame, B. *Kinetic formulation of conservation laws*, vol. 21 of *Oxf. Lect. Ser. Math. Appl.* Oxford: Oxford University Press, 2002.
[51] Rickman, S. *Quasiregular Mappings.* Springer Berlin Heidelberg, 1993.
[52] Székelyhidi, L. The Regularity of Critical Points of Polyconvex Functionals. *Arch. Ration. Mech. Anal. 172*, 1 (Jan. 2004), 133–152.
[53] Székelyhidi, L. Rank-one convex hulls in $\mathbb{R}^{2\times 2}$. *Calc. Var. Partial Differ. Equ. 28*, 4 (Jan. 2007).
[54] Šverák, V. On Tartar's conjecture. *Ann. Inst. H. Poincaré C Anal. Non Linéaire 10*, 4 (1993), 405–412.
[55] Zhang, K. On connected subsets of $M^{2\times 2}$ without rank-one connections. *Proc. R. Soc. Edinb., Sect. A, Math. 127*, 1 (1997), 207–216.

Xavier Lamy
Institut Universitaire de France (IUF) & Univ Toulouse, CNRS, IMT, Toulouse, France.
Email address: xavier.lamy@math.univ-toulouse.fr

Riccardo Tione
Università degli Studi di Torino, dipartimento di Matematica "Giuseppe Peano", Via Carlo Alberto 10, 10123 Torino, Italy
Email address: riccardo.tione@unito.it