On the regularity of weak solutions to Burgers' equation with finite entropy production

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Abstract

Bounded weak solutions of Burgers' equation $\partial_t u + \partial_x(u^2/2) = 0$ that are not entropy solutions need in general not be BV. Nevertheless it is known that solutions with finite entropy productions have a BV-like structure: a rectifiable jump set of dimension one can be identified, outside which u has vanishing mean oscillation at all points. But it is not known whether all points outside this jump set are Lebesgue points, as they would be for BV solutions. In the present article we show that the set of non-Lebesgue points of u has Hausdorff dimension at most one. In contrast with the aforementioned structure result, we need only one particular entropy production to be a finite Radon measure, namely $\mu = \partial_t(u^2/2) + \partial_x(u^3/3)$. We prove Hölder regularity at points where μ has finite $(1+\alpha)$-dimensional upper density for some $\alpha > 0$. The proof is inspired by a result of De Lellis, Westdickenberg and the second author : if μ_+ has vanishing 1-dimensional upper density, then u is an entropy solution. We obtain a quantitative version of this statement: if μ_+ is small then u is close in L^1 to an entropy solution.

1 Introduction

It is well-known that weak solutions of Burgers' equation

$$\partial_t u + \partial_x \frac{u^2}{2} = 0, \tag{1}$$

(and more generally scalar conservation laws) are not uniquely determined by initial data, and this is the reason why the notion of entropy solution was introduced [23]. Entropy solutions are characterized by their nonpositive entropy production : for any convex entropy $\eta\colon \mathbb{R} \to \mathbb{R}$ and associated entropy flux $q(u) = \int^u v\eta'(v)dv$, the corresponding entropy production μ_η satisfies

$$\mu_\eta = \partial_t \eta(u) + \partial_x q(u) \leq 0.$$

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This constraint ensures well-posedness of the Cauchy problem for (1) with L^∞ initial data. Entropy solutions can be equivalently characterized by Oleinik's estimate $\partial_x u \leq 1/t$ [28], and in particular they are locally in BV.

Although entropy solutions are the physically relevant solutions, general weak solutions sometimes need to be considered. For instance in [33, 27, 4] large deviation principles for stochastic approximation of entropy solutions are related to variational principles for energy functionals of the form

$$F_\varepsilon(u) = \int \left| \frac{1}{\varepsilon} |\partial_x|^{-1} \left(\partial_t u + \partial_x \frac{u^2}{2} \right) - \varepsilon \partial_x u \right|^2.$$

The Γ-limit of such functional is defined for weak solutions of (1) that need not be entropy solutions, but have finite entropy production:

$$\mu_\eta = \partial_t \eta(u) + \partial_x q(u) \quad \text{is a locally finite Radon measure,} \tag{2}$$

for any $\eta \in C^2(\mathbb{R})$ and associated flux q. An important feature of such solutions is that they enjoy a kinetic formulation (see e.g. [11]), namely there exists $m(t, x, v)$ a locally finite Radon measure such that

$$\partial_t \chi + v \partial_x \chi = \partial_v m, \quad \chi(t, x, v) = \mathbb{1}_{0 < v \leq u(t,x)} - \mathbb{1}_{u(t,x) \leq v < 0}.$$

The measure m encodes the entropy production through the formula

$$\langle \mu_\eta, \varphi \rangle = \int \eta''(v) \varphi(t, x) m(dt, dx, dv).$$

For entropy solutions it is nonpositive and the kinetic formulation was introduced in [26].

Another motivation for studying general weak solutions of (1) comes from a formal analogy with solutions of the eikonal equation

$$|\nabla \varphi| = 1, \tag{3}$$

that need not be viscosity solutions. Such solutions arise for instance in the problem of Γ-convergence of the Aviles-Giga functional

$$E_\varepsilon(\varphi) = \frac{\varepsilon}{2} \int \left| \nabla^2 \varphi \right|^2 + \frac{1}{2\varepsilon} \int \left(|\nabla \varphi|^2 - 1 \right)^2.$$

They can be endowed with a relevant concept of entropy production [22, 1, 15, 18] and a kinetic formulation [20, 19]. The Γ-limit of E_ε is conjectured to be the total entropy production, but a proof of the upper bound is still missing because not enough is known about the regularity of solutions with finite entropy production (see [6, 29] when $\nabla \varphi \in BV$). The analogy between (1) and (3) has already proven fruitful. For instance, techniques developed in [10] to understand the fine structure of solutions of (3) were adapted in [11] to the context of scalar conservation laws. See also [21, 9] for other regularity properties shared by both equations.

Unlike entropy solutions, weak solutions of (1) with finite entropy production (2) may not be in BV. They are in $B^{1/3}_{3,\infty}$ [17], but this is the best regularity one could hope for [14]. However it is shown in [24, 11] (related results can be found e.g. in [3, 30]) that they do enjoy a BV-like structure, namely: there exists an $\mathcal{H}^1$-rectifiable set $\mathcal{J} \subset \Omega$ such that u has strong one-sided traces on $\mathcal{J}$, and vanishing mean oscillation at all points outside $\mathcal{J}$. Moreover the entropy production restricted to the "jump set" $\mathcal{J}$ can be computed with the chain rule: if ν denotes a normal vector along $\mathcal{J}$ and $u^\pm$ the corresponding one-sided traces of u, then

$$\mu_\eta \lfloor \mathcal{J} = \left[(\eta(u^+) - \eta(u^-))\nu_t + (q(u^+) - q(u^-))\nu_x \right] \mathcal{H}^1 \lfloor \mathcal{J}.$$

The similarity with the structure of BV solutions is not perfect, and the two following questions are left open:

- Is μ_η supported on $\mathcal{J}$?
- Is every point outside $\mathcal{J}$ a Lebesgue point of u ?

In the present article we investigate the second question. Note that for entropy solutions of a large class of one-dimensional scalar conservation laws the corresponding questions have been answered positively [13]. Much recently the second question has also been answered for entropy solutions of multidimensional conservation laws [31].

The quadratic entropy $\eta(u) = u^2/2$ plays a special role in our analysis. In fact our methods are strongly inspired by [12] where the importance of that particular entropy is shed light upon. We consider bounded weak solutions $u(t, x)$ of (1) in a domain $\Omega \subset \mathbb{R}_t \times \mathbb{R}_x$ and denote simply by μ the corresponding entropy production

$$\mu = \partial_t \frac{u^2}{2} + \partial_x \frac{u^3}{3} \in \mathcal{M}(\Omega). \tag{4}$$

In [11] the singular set $\mathcal{J}$ is defined as the set of points with positive upper $\mathcal{H}^1$ density with respect to the measure $\nu \in \mathcal{M}(\Omega)$ given by

$$\nu(A) = |m|(A \times \mathbb{R}) = \sup_{|\eta''| \leq 1} |\mu_\eta|(A) \qquad \text{for } A \subset \Omega. \tag{5}$$

In other words, denoting by $Q_r(z)$ the square of size r centered at z, i.e.

$$Q_r(z) = (t - r, t + r) \times (x - r, x + r) \quad \text{if } z = (t, x),$$

the "regular points" of [11] are those belonging to

$$\mathcal{J}^c = \left\{ z \in \Omega \colon \lim_{r \to 0} r^{-1} \nu(Q_r(z)) = 0 \right\} \subset \left\{ z \in \Omega \colon \lim_{r \to 0} r^{-1} |\mu|(Q_r(z)) = 0 \right\}.$$

The last inclusion follows from $|\mu| \leq \nu$, and it is not clear whether it is strict or not.

Remark 1. For BV solutions of (1) the measures μ, m and ν can be computed explicitly using the chain rule (see e.g. [4, Remark 2.7]) and one can check that $\nu = |\mu|$ so that the inclusion is not strict.

In the present paper we need a geometric rate of decay for $r^{-1}|\mu|(Q_r(z))$ – but no bound on $r^{-1}\nu(Q_r(z))$ – to conclude that z is a Lebesgue point: our regular points are given by

$$\widetilde{\mathcal{J}}^c = \left\{ z \in \Omega \colon r^{-1}|\mu|(Q_r(z)) = \mathcal{O}(r^\alpha) \text{ for some } \alpha > 0 \right\}.$$

In order to quantify the regularity we obtain outside of $\widetilde{\mathcal{J}}$ we define, for any $\alpha, K > 0$,

$$\Omega_{\alpha,K} = \left\{ z \in \Omega \colon |\mu|(Q_r(z)) \le K r^{1+\alpha},\ \forall r \in (0, d(z) \wedge 1) \right\},$$

where $d(z) = \mathrm{dist}_\infty(z, \Omega^c)$ denotes the distance of z to the boundary of Ω with respect to the ℓ^∞ norm. In particular

$$\widetilde{\mathcal{J}} = \bigcap_{\alpha, K > 0} (\Omega_{\alpha,K})^c$$

has Hausdorff dimension at most one since $\mathcal{H}^{1+\alpha}((\Omega_{\alpha,K})^c) \lesssim K^{-1}|\mu|(\Omega)$, as follows from a covering argument (see e.g. [2, Theorem 2.56]). We show that in $\Omega_{\alpha,K}$ the function u is Hölder continuous.

Theorem 2. *Let $R, \alpha > 0$, and u be a weak solution of* (1) *in $Q_R = Q_R(0)$. If*

$$\frac{1}{r}|\mu|(Q_r) \le \left(\frac{r}{R}\right)^\alpha \qquad \forall r \in (0, R),$$

then

$$\fint_{Q_r(z)} \left(u - \fint_{Q_r(z)} u \right)^4 \le C \left(\frac{r}{R}\right)^{\frac{8\alpha}{105+2\alpha}}, \qquad \text{for all } r \in (0, R/2),$$

where $C > 0$ depends on $\|u\|_{L^\infty}$.

Remark 3. Such Campanato decay implies local Hölder continuity in $\Omega_{\alpha,K}$ in the classical sense: for any $d_0 > 0$ and $z_1, z_2 \in \Omega_{\alpha,K} \cap \{d \ge d_0\}$ it holds

$$|u(z_1) - u(z_2)| \le C|z_1 - z_2|^{\frac{2\alpha}{105+2\alpha}},$$

where $C > 0$ depends on $\|u\|_{L^\infty}$, K, α and d_0. (The proof of this implication does not require $\Omega_{\alpha,K}$ to be open and can be reproduced for instance as in [16, Theorem 5.5].)

Corollary 4. *The set of non-Lebesgue points of any bounded weak solution u of* (1) *with finite entropy production* (4) *has Hausdorff dimension at most one.*

The proof of Theorem 2 relies on the following principle : if the positive part of the entropy production (4) is small, then u should be close to an entropy solution. This principle is already present in [12] where it is shown that if μ_+ has vanishing upper $\mathcal{H}^1$-density, then u must be an entropy solution. Here we obtain, using methods inspired by [12], a quantitative version of this result: (a small power of) the total mass of μ_+ controls the L^1-distance of u to entropy solutions. This is the content of the next result, where we write Q_r for $Q_r(0,0)$.

Theorem 5. *Let u be a bounded weak solution of* (1). *Then there exists a bounded entropy solution ζ of* (1) *in Q_1 such that*

$$\int_{Q_{1/2}} (u-\zeta)^4 \leq C\mu_+(Q_1)^{\frac{4}{35}},$$

where $C > 0$ only depends on $\|u\|_{L^\infty}$.

To prove Theorem 5, the main step is to estimate the distance to entropy solutions in a rather weak sense, as explained below. This weak estimate can then be strengthened to an L^4 estimate by appropriately quantifying the compactness enforced by (4).

Remark 6. If the measure ν defined in (5), that encodes all entropy productions, is finite, then we have a $B^{1/3}_{3,\infty}$ estimate [17] so that the compactness is easily quantified. Such an assumption would simplify the proof of Theorem 5 and improve the dependence on $\mu_+(Q_1)$: we would obtain

$$\int_{Q_{3/4}} |u-\zeta|^3 \leq C\mu_+(Q_1)^{\frac{3}{28}},$$

for some constant $C > 0$ depending on $\|u\|_{L^\infty}$ and $\nu(Q_1)$. In Theorem 2, with the additional assumption $r^{-1}|\nu|(Q_r) \leq M$, we would accordingly have

$$⨍_{Q_r(z)} \left| u - ⨍_{Q_r(z)} u \right|^3 \leq C\left(\frac{r}{R}\right)^{\frac{9\alpha}{140+3\alpha}}, \qquad \text{for all } r \in (0, R/2),$$

for a constant C depending on $\|u\|_{L^\infty}$ and M.

As in [12] we make use of the correspondence between Burgers' equation and the related Hamilton-Jacobi equation

$$\partial_t h + \frac{1}{2}(\partial_x h)^2 = 0, \tag{6}$$

obtained from observing that the vector field $(-u^2/2, u)$ is curl-free and therefore can be written as a gradient field $(\partial_t h, \partial_x h)$. The aforementioned weak estimate consists in estimating the L^∞-distance of h to viscosity solutions of (6), which correspond to entropy solutions of (1) [12]. This is the very heart of our argument and we achieve this in §2 by turning the following loose statement into a rigorous one: if the positive part of the entropy production is small, then h is “not far” from being a viscosity supersolution of

$$\partial_t h + \frac{1}{2}(\partial_x h)^2 \geq -\delta, \qquad \text{for “small” } \delta.$$

If h really was a viscosity supersolution of such modified (6), then the comparison principle [8] would allow to estimate its L^∞-distance to viscosity solutions. We prove instead a weak version of the maximum principle (Lemma 8) where we need to assume some additional regularity on the subsolution to compare h with, but this turns out to be sufficient for our purposes.

The plan of the article is as follows. In §2 we derive the estimates for h, and in §3 we prove Theorems 5 and 2.

2 Estimates for the Hamilton-Jacobi equation

We denote by Q the unit square

$$Q := (0,1)_t \times (0,1)_x,$$

and consider Lipschitz functions h in $\overline{Q}$ that solve (6) almost everywhere. In particular this ensures [25, §11.1] that h restricted to the parabolic boundary

$$\partial_0 Q := \{0\}_t \times (0,1)_x \ \cup \ (0,1)_t \times \{0,1\}_x,$$

is compatible with the existence of a viscosity solution $\bar{h}$ satisfying $\bar{h} = h$ on $\partial_0 Q$.

Moreover, such viscosity solution satisfies $\bar{h} \geq h$ and

$$\left|\partial_x \bar{h}\right| \leq \|\partial_x h\|_{L^\infty(Q)}. \tag{7}$$

For a proof of (7) see Appendix A. Note that we need to consider the parabolic boundary instead of the full boundary because final values can in general not be imposed for viscosity solutions of (6) (see e.g. [5, § II.1]).

The main result of this section is the following estimate for $\left\|h - \bar{h}\right\|_\infty$.

Proposition 7. *For all $L \geq 0$ there exists a constant $C > 0$ such that, for any function h with* $\mathrm{Lip}(h) \leq L$ *solving*

$$\partial_t h + \frac{1}{2}(\partial_x h)^2 = 0 \qquad a.e. \text{ in } Q,$$

if $u = \partial_x h$ is such that $\mu = \partial_t(u^2/2) + \partial_x(u^3/3)$ is a Radon measure in Q, then it holds

$$\sup_{Q \cap \{t \leq 7/8\}} \left|h - \bar{h}\right| \leq C \mu_+(Q)^{1/7},$$

where $\bar{h}$ is the viscosity solution of

$$\partial_t \bar{h} + \frac{1}{2}(\partial_x \bar{h})^2 = 0, \qquad \bar{h} = h \text{ on } \partial_0 Q.$$

As explained in the introduction, the proof of Proposition 7 is about showing that if μ_+ is small, then h is “not far” from being a viscosity supersolution of (6) with small negative right-hand side. Such property is interesting because super- and subsolutions in the viscosity sense enjoy a comparison principle. In fact instead of proving a supersolution property, we directly prove a comparison principle. The main difference with the comparison principle for viscosity solutions is that we have to assume some additional regularity on the subsolution we are comparing h with, namely semiconvexity. We say that a function ζ is $(1/r)$-semiconvex if for all points z, z' it holds

$$\zeta(\theta z + (1-\theta)z') - \theta\zeta(z) - (1-\theta)\zeta(z') \leq \frac{1}{r}, \qquad 0 \leq \theta \leq 1.$$

This is equivalent to $\zeta(z) + |z|^2/(2r)$ being convex, and allows to prove the following maximum principle.

Lemma 8. *For all $L \geq 0$ there exists a constant $C > 0$ such that for any $R > 0$ and $\delta, r \in (0,1]$ the following holds true.*

- *Let a function h with $\mathrm{Lip}(h) \leq L$ solve*

  $$\partial_t h + \frac{1}{2}(\partial_x h)^2 = 0 \qquad \textit{a.e. in } B_R,$$

  and, denoting $u = \partial_x h$ and $\mu = \partial_t(u^2/2) + \partial_x(u^3/3)$, assume that the non-negative part of the entropy production is small enough in the sense that

  $$C\mu_+(B_R) \leq \delta^6 r. \tag{8}$$

- *Then, for any viscosity subsolution ζ of*

  $$\partial_t \zeta + \frac{1}{2}(\partial_x \zeta)^2 \leq -\delta \qquad \textit{in } B_R,$$

  with $\mathrm{Lip}(\zeta) \leq L$ and the additional regularity assumption that ζ be $(1/r)$-semiconvex, the function $(h - \zeta)$ can not *attain its minimum at 0.*

With Lemma 8 at hand, Proposition 7 will follow by regularizing $\bar{h}$ (using sup-convolution) and appropriately balancing the scale of regularization with the smallness of μ_+ and the smallness of the negative right-hand side modification of (6).

Proof of Lemma 8. By scaling, we assume without loss of generality that $R = 1$. We assume moreover that $L = 1$, the general case entailing no additional difficulty.

Suppose that (8) holds in B_1 for some constant $C > 0$, and that $(h - \zeta)$ attains its minimum at 0. We are going to obtain a contradiction if the constant C is large enough. Without loss of generality we assume that $h(0) = \zeta(0) = 0$.

Step 1. There exists an affine function ζ_a with $\mathrm{Lip}(\zeta_a) \leq 1$ and such that

$$\partial_t \zeta_a + \frac{1}{2}(\partial_x \zeta_a)^2 \leq -\delta, \tag{9}$$

$$\zeta_a(0,0) = \zeta(0,0), \qquad \zeta(t,x) \geq \zeta_a(t,x) - \frac{1}{2r}(t^2 + x^2). \tag{10}$$

Since $z \mapsto \zeta(z) + |z|^2/(2r)$ is convex there exists an affine function ζ_a satisfying (10). Then the smooth function $\varphi(z) = \zeta_a(z) - |z|^2/(2r)$ is such that $\zeta - \varphi$ has a maximum at $(0,0)$. By the viscosity subsolution property of ζ we deduce

$$\partial_t \varphi + \frac{1}{2}(\partial_x \varphi)^2 \leq -\delta \qquad \text{at } (0,0),$$

which yields (9). It remains to show that $\mathrm{Lip}(\zeta_a) \leq 1$. Since $\mathrm{Lip}(\zeta) \leq 1$, (10) implies

$$z \cdot \nabla \zeta_a = \zeta_a(z) - \zeta_a(0) \leq \zeta(z) - \zeta(0) + \frac{|z|^2}{2r} \leq |z| + \frac{|z|^2}{2r} \qquad \forall z \in B_\rho.$$

Applying this to $z = t\nabla\zeta_a/|\nabla\zeta_a|$ for $t \to 0$ yields $|\nabla\zeta_a| \leq 1$.

Step 2. For any height $\eta \in (0, r/4)$, letting

$$\widetilde{\zeta}(z) := \zeta_a(z) - \frac{1}{r}|z|^2,$$

and defining as in [12] the set

$$\Omega_\eta := B_1 \cap \left\{ \widetilde{\zeta} + \eta \geq h \right\},$$

it holds

$$B_{\eta/3} \subset \Omega_\eta \subset B_{2(r\eta)^{1/2}} \subset\subset B_1. \tag{11}$$

Since h and ζ_a are 1-Lipschitz and $\eta \leq 3r$, in $B_{\eta/3}$ we obtain

$$\begin{aligned} \widetilde{\zeta} + \eta - h &\geq \eta - 2|z| - \frac{1}{r}|z|^2 = \eta - \left(2 + \frac{|z|}{r}\right)|z| \\ &\geq \eta - \left(2 + \frac{\eta}{3r}\right)|(t,x)| \geq \eta - 3|z| > 0, \end{aligned}$$

which implies $B_{\eta/3} \subset \Omega_\eta$.

The strict inclusion $B_{2(r\eta)^{1/2}} \subset\subset B_1$ follows from $\eta < r/4$ and $r \leq 1$. Moreover since $h \geq \zeta$ in B_1 and (10) holds, in $B_1 \setminus B_{2(r\eta)^{1/2}}$ we have

$$h - \widetilde{\zeta} \geq \frac{1}{2r}|z|^2 \geq 2\eta,$$

which shows $\Omega_\eta \subset B_{2(r\eta)^{1/2}}$.

Step 3. Denoting by $\langle f \rangle$ the average of a function f in Ω_η and assuming

$$\eta \ll \delta^2 r, \tag{12}$$

(where the symbol $\ll$ denotes inequality up to a small constant) it holds

$$\delta \le \left\langle (u - \langle u \rangle)^2 \right\rangle . \tag{13}$$

Since $\nabla\widetilde{\zeta}(z) - \nabla\zeta_a(z) = -z/r$ and $|\nabla\zeta_a| \le 1$, for $z \in B_{2(r\eta)^{1/2}}$ it holds

$$\begin{aligned}\left| \left(\partial_t\widetilde{\zeta} + \frac{1}{2}(\partial_x\widetilde{\zeta})^2 \right) - \left(\partial_t\zeta_a + \frac{1}{2}(\partial_x\zeta_a)^2 \right) \right| &\lesssim \left(1 + \frac{|z|}{r}\right)\frac{|z|}{r} \\ &\lesssim \left(1 + \left(\frac{\eta}{r}\right)^{1/2}\right)\left(\frac{\eta}{r}\right)^{1/2} \\ &\le \frac{\delta}{2},\end{aligned}$$

where the last inequality follows from (12). Recalling (9) we deduce

$$\partial_t\widetilde{\zeta} + \frac{1}{2}(\partial_x\widetilde{\zeta})^2 \le -\frac{\delta}{2} \qquad \text{in } B_{2(r\eta)^{1/2}}.$$

By (11) this holds in particular in Ω_η and therefore using Jensen's inequality we have

$$-\frac{\delta}{2} \ge \langle \partial_t\widetilde{\zeta} \rangle + \frac{1}{2}\langle (\partial_x\widetilde{\zeta})^2 \rangle \ge \langle \partial_t\widetilde{\zeta} \rangle + \frac{1}{2}\langle \partial_x\widetilde{\zeta} \rangle^2.$$

Moreover since $u = \partial_x h$ and $(\widetilde{\zeta} + \eta - h)_+$ has compact support in B_1 it holds

$$\langle \partial_x\widetilde{\zeta} \rangle - \langle u \rangle = \frac{1}{|\Omega_\eta|}\int_{B_1} \partial_x \left[(\widetilde{\zeta} + \eta - h)_+ \right] = 0, \tag{14}$$

and similarly $\langle \partial_t\widetilde{\zeta} \rangle = \langle -u^2/2 \rangle$. This implies

$$-\frac{\delta}{2} \ge -\frac{1}{2}\left\langle (u - \langle u \rangle)^2 \right\rangle,$$

and proves (13).

Step 4. It holds

$$\left\langle (u - \langle u \rangle)^2 \right\rangle^2 \lesssim \left(\frac{\eta}{r}\right)^{1/2} + \frac{\|\mu_+\|}{\eta}. \tag{15}$$

The argument relies as in [12] on a quantification of Tartar's application of the div-curl lemma to equations of Burgers type [32]. By Hölder's inequality and [12, Proposition 3.2] we have

$$\begin{aligned}\left\langle (u - \langle u \rangle)^2 \right\rangle^2 &\le \left\langle (u - \langle u \rangle)^4 \right\rangle \\ &\lesssim \left\langle \begin{pmatrix} -\frac{u^2}{2} \\ u \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} -\frac{u^2}{2} \\ u \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle \\ &= \left\langle \begin{pmatrix} \partial_t h \\ \partial_x h \end{pmatrix} \cdot \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle - \left\langle \begin{pmatrix} \partial_t h \\ \partial_x h \end{pmatrix} \right\rangle \cdot \left\langle \begin{pmatrix} u^2/2 \\ u^3/3 \end{pmatrix} \right\rangle\end{aligned}$$

Recalling (14) and its counterpart for the t-derivative, we deduce

$$\begin{aligned}\left\langle (u-\langle u\rangle)^2\right\rangle^2 &\lesssim \left\langle \begin{pmatrix} \partial_t(h-\widetilde{\zeta}-\eta) \\ \partial_x(h-\widetilde{\zeta}-\eta)\end{pmatrix}\cdot\begin{pmatrix} u^2/2 \\ u^3/3\end{pmatrix}\right\rangle \\ &\quad + \left\langle \begin{pmatrix} \partial_t\widetilde{\zeta}-\langle\partial_t\widetilde{\zeta}\rangle \\ \partial_x\widetilde{\zeta}-\langle\partial_x\widetilde{\zeta}\rangle\end{pmatrix}\cdot\begin{pmatrix} u^2/2 \\ u^3/3\end{pmatrix}\right\rangle \\ &\le \frac{1}{|\Omega_\eta|}\int_{\Omega_\eta}(\widetilde{\zeta}+\eta-h)d\mu + \frac{2}{r}\,\mathrm{diam}(\Omega_\eta)\end{aligned}$$

For the last inequality we used the fact that, ζ_a being affine, we have

$$\nabla\widetilde{\zeta}-\langle\nabla\widetilde{\zeta}\rangle = -\frac{1+\delta}{r}\begin{pmatrix} t \\ x\end{pmatrix}.$$

Since in Ω_η it holds $0\le\widetilde{\zeta}+\eta-h\le\eta+\zeta-h\le\eta$, we find

$$\left\langle (u-\langle u\rangle)^2\right\rangle^2 \lesssim \frac{\eta}{|\Omega_\eta|}\|\mu_+\| + \frac{2}{r}\,\mathrm{diam}(\Omega_\eta).$$

Using the inclusions (11) satisfied by Ω_η we obtain (15).

Step 5. Conclusion.

We choose $\eta = r^{1/3}\|\mu_+\|^{2/3}$ in order to balance the two terms on the right-hand side of (15). Since

$$r^{1/3}\|\mu_+\|^{2/3} \le \frac{1}{C^{2/3}}\delta^4 r,$$

the restrictions (12) and $\eta < r/4$ are indeed satisfied provided C is large enough. Moreover combining (13) and (15) with the smallness assumption (8) on μ_+, we obtain

$$\delta^2 \lesssim \left(\frac{\|\mu_+\|}{r}\right)^{1/3} \lesssim \frac{1}{C^{1/3}}\delta^2,$$

and therefore the desired contradiction for large enough C. □

Proof of Proposition 7. We assume that $L=1$, hence $\mathrm{Lip}(h)\le 1$ (the proof in the general case is the same but this simplifies notations). Note that since $h\le\bar{h}$ we only need to estimate $(h-\bar{h})$ from below. The viscosity solution $\bar{h}$ is in general not semiconvex, that is why, in order to apply Lemma 8, we regularize $\bar{h}$ as follows. Given $\rho\in(0,1)$ we consider the sup-convolution

$$\bar{h}_\rho(t,x) = \sup_{(s,y)\in Q}\left\{\bar{h}(s,y) - \frac{1}{2\rho}\left((t-s)^2+(x-y)^2\right)\right\}.$$

As a supremum of functions of (t,x) which are $(1/\rho)$-semiconvex, this function $\bar{h}_\rho$ is $(1/\rho)$-semiconvex. We also introduce a parameter $\delta\in(0,1)$ and define

$$\zeta(t,x) = \bar{h}_\rho(t,x) - \delta t - 2^9\cdot((t-7/8)_+)^2 - 9\rho,$$

so that ζ is $(1/r)$-semiconvex with $1/r = 1/\rho + 2^{10}$. We want to use Lemma 8 to deduce that $h \geq \zeta$ in Q and from there obtain the desired lower bound on $(h - \bar{h})$. We split the proof in the following way : in Step 1 we prove that, far enough from the boundary, ζ is a viscosity subsolution as in Lemma 8; in Step 2 we control the Lipschitz constant of ζ in the relevant region; then we show that $h \geq \zeta$ near the boundary, dealing with the parabolic boundary $\partial_0 Q$ in Step 3 and the remaining boundary in Step 4; eventually in Step 5 we obtain $h \geq \zeta$ in Q and optimize the choices of ρ and δ in order to conclude.

Step 1. The function ζ is a viscosity subsolution of

$$\partial_t \zeta + \frac{1}{2}(\partial_x \zeta)^2 \leq -\delta \quad \text{ in } \widetilde{Q} := Q \cap \{\operatorname{dist}(\cdot, \partial Q) > 4\rho\}.$$

It suffices to show that $\bar{h}_\rho$ is a viscosity subsolution of

$$\partial_t \bar{h}_\rho + \frac{1}{2}(\partial_x \bar{h}_\rho)^2 \leq 0 \quad \text{ in } \widetilde{Q}. \tag{16}$$

The fact that sup convolution preserves the viscosity subsolution property is well-known, see e.g. [7, Lemma A.5]. For the convenience of the reader we provide a proof of (16) in our setting. Let $\varphi(t,x)$ be a smooth function such that $(\bar{h}_\rho - \varphi)$ attains its maximum at $(t_0, x_0) \in \widetilde{Q}$, and assume w.l.o.g. that $\varphi(t_0,x_0) = \bar{h}_\rho(t_0,x_0)$. For any $(s,y) \in U$ with

$$d := |(s,y) - (t_0,x_0)| = \sqrt{(t_0-s)^2 + (x_0-y)^2} \geq 2\rho,$$

since by (7) and the equation (6) satisfied by $\bar{h}$ we have $\operatorname{Lip}(\bar{h}) \leq 1$, it holds

$$\begin{aligned}\bar{h}(s,y) - \frac{1}{2\rho}d^2 &\leq \bar{h}(t_0,x_0) + \left(1 - \frac{d}{2\rho}\right) d \\ &\leq \bar{h}(t_0,x_0) \leq \bar{h}_\rho(t_0,x_0).\end{aligned}$$

Hence the supremum in the definition of $\bar{h}_\rho(t_0,x_0)$ is attained at some $(s_0,y_0) \in B_{2\rho}(t_0,x_0) \subset Q$, and

$$\varphi(t_0,x_0) = \bar{h}_\rho(t_0,x_0) = \bar{h}(s_0,y_0) - \frac{1}{2\rho}\left((t_0-s_0)^2 + (x_0-y_0)^2\right). \tag{17}$$

Moreover since $(\bar{h}_\rho - \varphi)$ is maximal at (t_0,x_0) with value zero, it holds

$$\bar{h}(s,y) - \frac{1}{2\rho}\left((t-s)^2 + (x-y)^2\right) \leq \varphi(t,x) \qquad \forall (t,x),(s,y) \in Q. \tag{18}$$

In particular for all $(s,y) \in B_{2\rho}(s_0,y_0) \subset Q$ we may choose

$$(t,x) = (s - s_0 + t_0, y - y_0 + x_0) \in B_{2\rho}(t_0,x_0) \subset Q,$$

in (18) and obtain

$$\bar{h}(s,y) \leq \varphi(s - s_0 + t_0, y - y_0 + x_0) + \frac{1}{2\rho}\left((t_0-s_0)^2 + (x_0-y_0)^2\right) =: \psi(s,y).$$

Moreover (17) ensures $\psi(s_0, y_0) = \bar{h}(s_0, y_0)$, hence $\bar{h} - \psi$ has a local maximum at (s_0, y_0). Since $\bar{h}$ is a viscosity solution we deduce that

$$\partial_t \varphi(t_0, x_0) + \frac{1}{2}(\partial_x \varphi(t_0, x_0))^2 = \partial_t \psi(s_0, y_0) + \frac{1}{2}(\partial_x \psi(s_0, y_0))^2 \leq 0,$$

which proves (16).

Step 2. We have

$$\mathrm{Lip}(\zeta) \leq 3 + 2^{10} \qquad \text{in } \widetilde{Q},$$

where $\widetilde{Q} = Q \cap \{\mathrm{dist}(\cdot, \partial Q) > 4\rho\}$ as in Step 1.

It was shown in Step 1 that for (t_0, x_0) in $\widetilde{Q}$, the supremum in the definition of $\bar{h}_\rho(t_0, x_0)$ is attained at some $(s_0, y_0) \in B_{2\rho}(t_0, x_0)$. It follows that for any small (t, x) we have

$$\begin{aligned}
&\bar{h}_\rho(t_0, x_0) - \bar{h}_\rho(t_0 + t, x_0 + x) \\
&\qquad = \bar{h}(s_0, y_0) - \frac{1}{2\rho}|(t_0 - s_0, x_0 - y_0)|^2 \\
&\qquad\quad - \sup_{(s,y)\in Q} \left\{ \bar{h}(s, y) - \frac{1}{2\rho}|(t_0 + t - s, x_0 + x - y)|^2 \right\} \\
&\qquad \leq \frac{1}{2\rho}\left(2(t_0 - s_0)t + 2(x_0 - y_0)x + |(t, x)|^2\right) \\
&\qquad \leq 2|(t, x)| + \frac{1}{2\rho}|(t, x)|^2.
\end{aligned}$$

This implies $\left|\nabla \bar{h}_\rho\right| \leq 2$ in $\widetilde{Q}$. Therefore in $\widetilde{Q}$ it holds

$$\mathrm{Lip}(\zeta) \leq 2 + \delta + 2 \cdot 2^9,$$

which concludes the proof of Step 2 since $\delta \leq 1$.

Note that thanks to Step 1 and Step 2, Lemma 8 ensures the existence of a universal constant $C > 0$ such that if $C\mu_+(Q) \leq \delta^6 r$ then the minimum of $(h - \zeta)$ in $\overline{Q}$ cannot be attained in $\widetilde{Q}$.

Step 3. It holds

$$\zeta \leq h \quad \text{in } Q \cap \{\mathrm{dist}(\cdot, \partial_0 Q) \leq 4\rho\}.$$

Since $\mathrm{Lip}(\bar{h}) \leq 1$, the definition of $\bar{h}_\rho$ implies

$$\begin{aligned}
\bar{h}_\rho(t, x) - \bar{h}(t, x) &= \sup_{(s,y)\in Q} \left\{ \bar{h}(s, y) - \bar{h}(t, x) - \frac{1}{2\rho}\left((t - s)^2 + (x - y)^2\right) \right\} \\
&\leq \sup_{d \geq 0} \left\{ d - \frac{1}{2\rho} d^2 \right\} = \frac{\rho}{2}.
\end{aligned}$$

By definition of ζ this yields

$$\zeta(t, x) - h(t, x) \leq -2^9 \cdot ((t - 7/8)_+)^2 - 9\rho + \frac{\rho}{2} + \bar{h}(t, x) - h(t, x),$$

so that by $\bar{h} = h$ on $\partial_0 Q$ and the Lipschitz continuities of h and $\bar{h}$ (7) we obtain

$$\zeta(t,x) - h(t,x) \leq -2^9 \cdot ((t-7/8)_+)^2 - 8\rho + 2\,\mathrm{dist}((t,x),\partial_0 Q). \tag{19}$$

Therefore if $\mathrm{dist}((t,x),\partial_0 Q) \leq 4\rho$ we have $\zeta(t,x) - h(t,x) \leq 0$.

Step 4. It holds

$$\zeta \leq h \quad \text{in } Q \cap \{t \geq 15/16\}.$$

For $t \geq 15/16$ we have by (19)

$$\zeta(t,x) - h(t,x) \leq 2 - 2^9 \cdot (1/16)^2 \leq 0.$$

Step 5. Conclusion.

Recall that thanks to Step 1 and Step 2 the minimum of $(h-\zeta)$ in $\overline{Q}$ cannot be attained in $\widetilde{Q}$ provided that $C\mu_+(Q) \leq \delta^6 r$. Moreover, if $4\rho \leq 1/16$, then by Steps 3 and 4 it must hold $h - \zeta \geq 0$ outside of $\widetilde{Q}$. We deduce that if $\rho \leq 2^{-6}$ and $C\mu_+(Q) \leq \delta^6 r$ then $h - \zeta \geq 0$ in $\overline{Q}$. Hence for $t \leq 7/8$ it holds

$$\begin{aligned} h - \bar{h} &\geq \zeta - \bar{h} \\ &= \bar{h}_\rho - \bar{h} - \delta t - 9\rho \\ &\geq -9\rho - \delta, \end{aligned}$$

and therefore

$$\sup_{Q \cap \{t \leq t_1\}} \left| h - \bar{h} \right| \leq 9\rho + \delta.$$

Choosing $\delta = \rho$ and recalling that $r = \rho/(1 + 2^{10} \cdot \rho) \geq \rho/(1+2^4)$ we conclude that for $\rho \leq 2^{-6}$,

$$2^5 C\mu_+(Q) \leq \rho^7 \quad \Longrightarrow \quad \sup_{Q \cap \{t \leq t_1\}} \left| h - \bar{h} \right| \leq 10\rho.$$

If $\mu_+(Q) \leq 2^{-47}/C$ we can apply this to $\rho = (2^5 C\mu_+(Q))^{1/7}$ to finish the proof. If $\mu_+(Q) > 2^{-47}/C$ we can simply invoke the fact that $\left|h - \bar{h}\right|$ is bounded by a universal constant. □

3 Proofs of Theorems 5 and 2

In this section we use the symbol $\lesssim$ to denote inequality up to a constant depending only on $\|u\|_\infty$. The following lemma quantifies the compactness induced by our assumption of finite entropy production (4).

Lemma 9. *For any bounded weak solution u of (1) in Q_1 it holds*

$$\int_{Q_{1/2}} (u - u_r)^4 \lesssim r(1 + \mu_+(Q_1)) \qquad \forall r \in (0, 1/4),$$

where $(\cdot)_r$ denotes convolution with $\varphi_r = r^{-2}\varphi(\cdot/r)$ for some even nonnegative kernel $\varphi \in C_c^\infty(Q_1)$, $\int \varphi = 1$.

Proof. The proof relies on a "div-curl" argument inspired by Tartar's compactness result [32]. Fix $\eta \in C_c^1(Q_{3/4})$ and $r \in (0, 1/4)$. We will prove below the two following identities:

$$\begin{aligned}
&\iint \varphi_r(x-y)\eta(x)(u(x)-u(y))^4\,dxdy \\
&= \iint \varphi_r(x-y)(\eta(x)-\eta(y))(u^4(y)-4u^3(y)u(x)+3u^2(y)u^2(x))\,dxdy \\
&\quad + 2\int \eta(x)(u^4(x)-4u^3(x)u_r(x)+3u^2(x)(u^2)_r(x)),
\end{aligned} \tag{20}$$

and, with h such that $u = \partial_x h$ and $-u^2/2 = \partial_t h$,

$$\begin{aligned}
&\frac{1}{12}(u^4 - 4u^3 u_r + 3u^2(u^2)_r) \\
&= \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix} \cdot \left[(h-h_r)\begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 \end{pmatrix}\right] - (h-h_r)\mu,
\end{aligned} \tag{21}$$

in $\mathcal{D}'(Q_{3/4})$. Combining (20) and (21) we obtain

$$\begin{aligned}
&\iint \varphi_r(x-y)\eta(x)(u(x)-u(y))^4\,dxdy \\
&= \iint \varphi_r(x-y)(\eta(x)-\eta(y))(u^4(y)-4u^3(y)u(x)+3u^2(y)u^2(x))\,dxdy \\
&\quad - \int (h-h_r)\begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 \end{pmatrix} \cdot \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}\eta + \int \eta(h-h_r)\,d\mu \\
&\lesssim r\left(\|\nabla\eta\|_\infty + \|\eta\|_\infty|\mu|(Q_{3/4})\right),
\end{aligned}$$

since $|u|, |\nabla h| \leq C$. Choosing $\eta \geq 0$ with $\eta \equiv 1$ in $Q_{1/2}$ and applying Jensen's inequality we deduce

$$\int_{Q_{1/2}} (u-u_r)^4 \lesssim r\left(1+|\mu|(Q_{3/4})\right). \tag{22}$$

Moreover, fixing $\chi \in C_c^1(Q_1)$ with $0 \leq \chi \leq 1$ and $\chi \equiv 1$ on $Q_{3/4}$, we have for any $\zeta \in C_c^1(Q_{3/4})$ with $|\zeta| \leq 1$,

$$\begin{aligned}
\int \zeta d\mu &= \int (\zeta+\chi)d\mu + \int \left(\frac{1}{2}u^2\partial_t\chi + \frac{1}{3}u^3\partial_x\chi\right) \\
&\lesssim \mu_+(Q_1) + 1,
\end{aligned}$$

since $\zeta + \chi \geq 0$ and $|u| \leq C$. This implies that

$$|\mu|(Q_{3/4}) \lesssim 1 + \mu_+(Q_1),$$

and therefore (22) proves Lemma 9. It remains to justify the identities (20) and

(21). Validity of (20) follows from

$$
\begin{aligned}
&\iint \varphi_r(x-y)\eta(x)(u(x)-u(y))^4\,dxdy \\
&= \iint \varphi_r(x-y)\eta(x)\Big(u^4(x)-4u^3(x)u(y) \\
&\qquad\qquad\qquad + 6u^2(x)u^2(y)-4u(x)u^3(y)+u^4(y)\Big)\,dxdy \\
&= \iint \varphi_r(x-y)\eta(x)(u^4(x)-4u^3(x)u(y)+3u^2(x)u^2(y))\,dxdy \\
&\quad + \iint \varphi_r(x-y)\eta(x)(u^4(y)-4u^3(y)u(x)+3u^2(y)u^2(x))\,dxdy \\
&= \iint \varphi_r(x-y)\eta(x)(u^4(x)-4u^3(x)u(y)+3u^2(x)u^2(y))\,dxdy \\
&\quad + \iint \varphi_r(x-y)(\eta(x)-\eta(y))(u^4(y)-4u^3(y)u(x)+3u^2(y)u^2(x))\,dxdy \\
&\quad + \iint \varphi_r(x-y)\eta(y)(u^4(y)-4u^3(y)u(x)+3u^2(y)u^2(x))\,dxdy, \\
&= \iint \varphi_r(x-y)(\eta(x)-\eta(y))(u^4(y)-4u^3(y)u(x)+3u^2(y)u^2(x))\,dxdy \\
&\quad + 2\iint \varphi_r(x-y)\eta(x)(u^4(x)-4u^3(x)u(y)+3u^2(x)u^2(y))\,dxdy.
\end{aligned}
$$

The last equality was obtained interverting x and y in the last integral, and using the fact that φ_r is even. To prove (21) we write

$$
\begin{aligned}
&\frac{1}{12}(u^4-4u^3u_r+3u^2(u^2)_r) \\
&= \left(\begin{pmatrix} -\frac{1}{2}u^2 \\ u \end{pmatrix} - \begin{pmatrix} -\frac{1}{2}u^2 \\ u \end{pmatrix}_r\right)\cdot\begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 \end{pmatrix} \\
&= \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}(h-h_r)\cdot\begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 \end{pmatrix} \\
&= \begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}\cdot\left[(h-h_r)\begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 \end{pmatrix}\right] - (h-h_r)\begin{pmatrix} \partial_t \\ \partial_x \end{pmatrix}\cdot\begin{pmatrix} \frac{1}{2}u^2 \\ \frac{1}{3}u^3 \end{pmatrix}.
\end{aligned}
$$

□

Proof of Theorem 5. Let h be the Lipschitz solution of (6) such that $u=\partial_x h$ and $-u^2/2=\partial_t h$, let $\bar{h}$ be as in §2 the viscosity solution of (6) in Q_1 with $\bar{h}=h$ on the parabolic boundary $\partial_0 Q_1$. By Proposition 7 it holds

$$
\sup_{Q_{3/4}}\left|\bar{h}-h\right| \lesssim \mu_+(Q_1)^{\frac{1}{7}}. \tag{23}
$$

This estimate tells us that u is close to the entropy solution $\zeta=\partial_x\bar{h}$ in a weak sense. The fact that ζ is an entropy solution follows from the correspondence between entropy solutions of (1) and viscosity solutions of (6) (see e.g. [12]).

Next we use the quantitative compactness proved in Lemma 9 to turn (23) into an L^4 estimate on

$$u-\zeta=\partial_x(h-\bar h).$$

We introduce a smooth, even, nonnegative kernel $\varphi(z)$ with compact support in Q_1 and unit integral and define

$$u_r=u*\varphi_r,\qquad \zeta_r=\zeta*\varphi_r,\qquad \text{where } \varphi_r(z)=r^{-2}\varphi(z/r).$$

From Lemma 9 we deduce that for $r\in(0,1/4)$ it holds

$$\int_{Q_{1/2}}(u-u_r)^4\lesssim r(1+\mu_+(Q_1)). \tag{24}$$

Moreover, in $Q_{3/4}$ the second derivative of $\bar h$ in any direction is bounded from above by a universal constant [25, Theorem 13.1] and therefore, in conjunction with $\mathrm{Lip}(\bar h)\lesssim 1$,

$$\int_{Q_{3/4}}\left|\nabla^2\bar h\right|\lesssim 1.$$

This implies that

$$\int_{Q_{1/2}}|\zeta-\zeta_r|^4\lesssim r. \tag{25}$$

Combining (24) and (25) yields

$$\begin{aligned}\int_{Q_{1/2}}(u-\zeta)^4&\lesssim r(1+\mu_+(Q_1))+\int_{Q_{1/2}}\left((h-\bar h)*\partial_x\varphi_r\right)^4\\&\lesssim r(1+\mu_+(Q_1))+\frac{1}{r^4}\mu_+(Q_1)^{\frac{4}{7}},\end{aligned}$$

where we used (23) in the second step. We choose $r=[\mu_+(Q_1)]^{4/35}(1+\mu_+(Q_1))^{-1/5}$, which is admissible since without loss of generality $\mu+(Q_1)\ll 1$, to find our conclusion

$$\int_{Q_{1/2}}(u-\zeta)^4\lesssim[\mu_+(Q_1)]^{4/35}.$$

□

Proof of Theorem 2. **Step 1.** If u is as in Theorem 5, then for any $\theta\in(0,1/2)$ it holds

$$⨍_{|t|\le\theta}⨍_{|x|\le\theta}⨍_{|y|\le\theta}(u(t,x)-u(t,y))^4\,dxdydt\lesssim\theta^4+\frac{1}{\theta^2}|\mu|(Q_1)^{\frac{4}{35}}.$$

To prove this estimate we apply Theorem 5 to u and to $-u(-t,x)$ and deduce the existence of an entropy solution $\overline{\zeta}$ and an anti-entropy solution $\underline{\zeta}$ of (1) in Q_1 with

$$\int_{Q_{1/2}} (u-\overline{\zeta})^4 + \int_{Q_{1/2}} (u-\underline{\zeta})^4 \lesssim |\mu|(Q_1)^{\frac{4}{35}}. \tag{26}$$

Here we call anti-entropy solution a function $\underline{\zeta}(t,x) = -\widetilde{\zeta}(-t,x)$, where $\widetilde{\zeta}$ is an entropy solution of (1). Since $\overline{\zeta}$ is an entropy solution and $\underline{\zeta}$ an anti-entropy solution it holds by Oleinik's principle [28]

$$\partial_x\overline{\zeta} \lesssim 1 \quad \text{and} \quad -\partial_x\underline{\zeta} \lesssim 1 \qquad \text{in } Q_{1/2}. \tag{27}$$

For $-1/2 < x < y < 1/2$ and $|t| < 1/2$, it holds

$$\begin{aligned}
|u(t,x)-u(t,y)| &= (u(t,y)-u(t,x))_+ + (u(t,x)-u(t,y))_+ \\
&\le (u(t,y)-\overline{\zeta}(t,y))_+ + (\overline{\zeta}(t,y)-\overline{\zeta}(t,x))_+ + (\overline{\zeta}(t,x)-u(t,x))_+ \\
&\quad + (u(t,x)-\underline{\zeta}(t,x))_+ + (\underline{\zeta}(t,x)-\underline{\zeta}(t,y))_+ + (\underline{\zeta}(t,y)-u(t,y))_+ \\
&\le \left|(u-\overline{\zeta})(t,x)\right| + \left|(u-\overline{\zeta})(t,y)\right| + \left|(u-\underline{\zeta})(t,x)\right| + \left|(u-\underline{\zeta})(t,y)\right| \\
&\quad + \left(\sup_{Q_{1/2}}(\partial_x\overline{\zeta})_+ + \sup_{Q_{1/2}}(-\partial_x\underline{\zeta})_+\right)(y-x).
\end{aligned}$$

Arguing similarly for $x > y$ and using the above estimates (26) and (27), we deduce

$$\begin{aligned}
⨍_{|t|\le\theta}⨍_{|x|\le\theta}⨍_{|y|\le\theta} (u(t,x)-u(t,y))^4 &\lesssim \frac{1}{\theta^2}\int_{Q_{3/4}} (u-\overline{\zeta})^4 + \frac{1}{\theta^2}\int_{Q_{3/4}} (u-\underline{\zeta})^4 \\
&\quad + ⨍_{|t|\le\theta}⨍_{|x|\le\theta}⨍_{|y|\le\theta} |x-y|^4 \\
&\lesssim \frac{1}{\theta^2}|\mu|(Q_1)^{\frac{4}{35}} + \theta^4.
\end{aligned}$$

Step 2. If u is as in Theorem 5, then for any $\theta \in (0,1/2)$ it holds

$$⨍_{Q_\theta}\left(u - ⨍_{Q_\theta} u\right)^4 \lesssim \theta^4 + \frac{1}{\theta^2}|\mu|(Q_1)^{\frac{4}{35}}.$$

We may use the equation (1) to transfer the estimate on oscillations in the space variable obtained in Step 1, to the time variable. We sketch here the standard argument.

We fix a smooth cut-off function $\eta(x)$, set $\eta_\theta(x) = \theta^{-1}\eta(\theta^{-1}x)$ and notice that

$$\frac{d}{dt}\left[\int u(t,x)\eta_\theta(x)dx\right] = \frac{1}{2}\int u^2(t,x)(\eta_\theta)'(x)dx.$$

In particular $t \mapsto \int u(t,x)\eta_\theta(x)dx$ is Lipschitz, and it holds

$$\begin{aligned}
\int (u(s,x)-u(t,x))\eta_\theta(x)dx &= \frac{s-t}{2}\int_0^1\int u^2(\tau s+(1-\tau)t,x)(\eta_\theta)'(x)dx\,d\tau \\
&= \frac{s-t}{2\theta}\int_0^1\int \left(u^2(\tau s+(1-\tau)t,x) - u^2(\tau s+(1-\tau)t,y)\right)(\eta')_\theta(x)dx\,d\tau,
\end{aligned}$$

where we used the fact that η_θ' has zero average and where y can be choosen arbitrarily. Hence together with $|u| \lesssim 1$ we obtain from averaging over $|t| \le \theta$, $|s| \le \theta$ and $|y| \le \theta$,

$$\fint_{|s|\le\theta}\fint_{|t|\le\theta}\left(\int (u(s,x)-u(t,x))\eta_\theta(x)\,dx\right)^4 \lesssim \fint_{|t|\le\theta}\fint_{|x|\le\theta}\fint_{|y|\le\theta}(u(t,x)-u(t,y))^4\,dxdydt.$$

Combining this with the estimate

$$\fint_{|y|\le\theta}\left(u(t,y)-\int u(t,x)\eta_\theta(x)\right)^4 \lesssim \fint_{|x|\le\theta}\fint_{|y|\le\theta}(u(t,x)-u(t,y))^4\,,$$

we deduce

$$\fint_{|s|\le\theta}\fint_{|t|\le\theta}\fint_{|y|\le\theta}(u(t,y)-u(s,y))^4 \lesssim \fint_{|s|\le\theta}\fint_{|x|\le\theta}\fint_{|y|\le\theta}(u(t,x)-u(t,y))^4\,,$$

which thanks to Step 1 implies the claim in Step 2.

Step 3. Conclusion. By scaling we assume without loss of generality that $R=1$.

For any $\rho\in(0,1)$ we apply Step 1 to $z\mapsto u(\rho z)$ and use $\rho^{-1}|\mu|(Q_\rho)\le\rho^\alpha$ to obtain

$$\fint_{Q_{\frac12\theta\rho}}\left(u-\fint_{Q_{\frac12\theta\rho}}u\right)^4 \lesssim \theta^4+\frac{1}{\theta^2}\rho^{\frac{4\alpha}{35}} \qquad \text{for all } \theta\in(0,1).$$

We choose $\theta=\rho^{\frac{2\alpha}{105}}$ to balance the two terms. For $r=\frac12\theta\rho$ this yields

$$\fint_{Q_r}\left(u-\fint_{Q_r}u\right)^4 \lesssim r^{\frac{8\alpha}{2\alpha+105}},$$

which is valid for all $r\in(0,1/2)$. □

A Lipschitz estimate for the viscosity solution

Let $h\in W^{1,\infty}(Q)$ solve (6) almost everywhere. The viscosity solution $\bar h\in W^{1,\infty}(Q)$ of (6) with $\bar h=h$ on $\partial_0 Q$ is given [25, §11] by the Hopf-Lax formula

$$\bar h(t,x)=\inf\left\{h(s,y)+\frac{(x-y)^2}{2(t-s)}:(s,y)\in\partial_0Q,\ s<t\right\}.$$

Note that for $(t,x)\in Q$ the infimum is attained. Let $L:=\|\partial_x h\|_{L^\infty(Q)}$, so that the initial data $h(0,\cdot)$ has Lipschitz constant $\le L$ and the boundary data $h(\cdot,0)$ and $h(\cdot,1)$ have Lipschitz constants $\le L^2/2$.

Lemma 10. *It holds* $\left|\partial_x \bar{h}\right| \le L$ *a.e.*

Proof. Let $(t_0, x_0) \in Q$ and denote by (s_0, y_0) a point at which the infimum defining $\bar{h}(t_0, x_0)$ is attained. Then for any small x it holds

$$\begin{aligned}\bar{h}(t_0, x_0 + x) - \bar{h}(t_0, x_0) &= \bar{h}(t_0, x_0 + x) - h(s_0, y_0) - \frac{(x_0 - y_0)^2}{2(t_0 - s_0)} \\ &\le \frac{(x_0 + x - y_0)^2}{2(t_0 - s_0)} - \frac{(x_0 - y_0)^2}{2(t_0 - s_0)} \\ &= \frac{x_0 - y_0}{t_0 - s_0} x + \frac{1}{2(t_0 - s_0)} x^2,\end{aligned}$$

so that $\left|\partial_x \bar{h}(t_0, x_0)\right| \le |x_0 - y_0|/(t_0 - s_0)$ and to prove (7) it suffices to show that the infimum defining $\bar{h}(t_0, x_0)$ is attained at some (s_0, y_0) with

$$\frac{|x_0 - y_0|}{t_0 - s_0} \le L. \tag{28}$$

We show that for any $(s, y) \in \partial_0 Q \cap \{s < t_0\}$ with

$$\frac{|x_0 - y|}{t_0 - s} > L, \tag{29}$$

there exists $(\tilde{s}, \tilde{y}) \in \partial_0 Q \cap \{s < t_0\}$ satisfying

$$\frac{|x_0 - \tilde{y}|}{t_0 - \tilde{s}} < \frac{|x_0 - y|}{t_0 - s} \tag{30}$$

$$\text{and} \quad h(s_0, \tilde{y}) + \frac{(x_0 - \tilde{y})^2}{2(t_0 - \tilde{s})} \le h(s_0, y) + \frac{(x_0 - y)^2}{2(t_0 - s)}, \tag{31}$$

which proves (28).

There are two cases to consider, depending on which part of the parabolic boundary (s, y) belongs to.

Case 1 : $(s, y) \in \{0\} \times [0, 1]$. We look for $(\tilde{s}, \tilde{y})$ defined through $\tilde{s} = 0$ and

$$\frac{x_0 - \tilde{y}}{t_0} = (1 - \varepsilon) D, \quad D := \frac{x_0 - y}{t_0},$$

for some small $\epsilon > 0$, so that (30) is satisfied. On the other hand since $h(0, \cdot)$ has Lipschitz constant $\le L$, to show (31) it suffices to establish

$$L|\tilde{y} - y| \le \frac{(x_0 - y)^2 - (x_0 - \tilde{y})^2}{2t_0} \quad \Longleftrightarrow \quad \frac{L}{D} \le (1 - \frac{1}{2}\varepsilon),$$

which is satisfied for small enough ε since (29) amounts to $|D| > L$.

Case 2 : $(s, y) \in (0, 1) \times \{0, 1\}$. We assume $y = 0$, the case $y = 1$ being similar. We look for $(\tilde{s}, \tilde{y})$ defined through $\tilde{y} = 0$ and

$$\frac{x_0}{t_0 - \tilde{s}} = (1 - \varepsilon) D, \quad D := \frac{x_0}{t_0 - s},$$

for some small $\epsilon > 0$, so that (30) is satisfied. On the other hand since $h(1, \cdot)$ has Lipschitz constant $\leq L^2/2$, to show (31) it suffices to establish

$$\frac{L^2}{2}|s - \tilde{s}| \leq \frac{x_0^2}{2}\left(\frac{1}{t_0 - s} - \frac{1}{t_0 - \tilde{s}}\right) \quad \Longleftrightarrow \quad \frac{L^2}{D^2} \leq 1 - \varepsilon,$$

which is satisfied for small enough ε since (29) amounts to $|D| > L$. □

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