Stability of the vortex in micromagnetics and related models

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Abstract

We consider line-energy models of Ginzburg-Landau type in a two-dimensional simply-connected bounded domain. Configurations of vanishing energy have been characterized by Jabin, Otto and Perthame: the domain must be a disk, and the configuration a vortex. We prove a quantitative version of this statement in the class of $C^{1,1}$ domains, improving on previous results by Lorent. In particular, the deviation of the domain from a disk is controlled by a power of the energy, and that power is optimal. The main tool is a Lagrangian representation introduced by the second author, which allows to decompose the energy along characteristic curves.

1 Introduction

1.1 Models

Several models arising in a variety of physical applications (micromagnetics, smectic liquid crystals, blistering) have in common that, as a characteristics length scale ε tends to 0, bounded-energy configurations converge to two-dimensional vector fields $m\colon \Omega \to \mathbb{R}^2$ satisfying the eikonal equation

$$|m| = 1 \text{ a.e. in } \Omega, \qquad \nabla \cdot (\mathbf{1}_\Omega m) = 0 \text{ in } \mathbb{R}^2. \tag{1}$$

Here $\Omega \subset \mathbb{R}^2$ is a smooth, simply connected bounded domain, and the divergence constraint on the trivially extended field $\mathbf{1}_\Omega m$ amounts to $\nabla \cdot m = 0$ in Ω and $m \cdot n_{\partial\Omega} = 0$ on $\partial\Omega$, where $n_{\partial\Omega}$ is the exterior unit normal (and the last condition makes sense whenever m admits a strong trace on $\partial\Omega$). Examples of such models include:

- The Aviles-Giga functional, introduced in [4] as a simplified model for smectic liquid crystals and proposed as a model for thin film blisters in [30] (see the introduction of [19] for other applications),

$$\begin{aligned} &E_\varepsilon^{AG}(m;\Omega) = \frac{\varepsilon}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_\Omega (1-|m|^2)^2, \\ &m\colon \Omega \to \mathbb{R}^2, \qquad \nabla \cdot (\mathbf{1}_\Omega m) = 0 \text{ in } \mathbb{R}^2. \end{aligned} \tag{2}$$

  Note that the Aviles-Giga functional is more often expressed in terms of u such that $\nabla^\perp u = m$ in Ω and $u = 0$ on $\partial\Omega$, however in a simply connected domain the two formulations are equivalent.

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- A micromagnetics model studied in [34, 35],

$$E_\varepsilon^{RS}(m;\Omega) = \frac{\varepsilon}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_{\mathbb{R}^2} |H|^2, \tag{3}$$
$$m\colon \Omega \to \mathbb{S}^1 \subset \mathbb{R}^2, \qquad H\colon \mathbb{R}^2 \to \mathbb{R}^2,$$
$$\nabla \times H = 0 \text{ and } \nabla\cdot(H + 1_\Omega m) = 0 \text{ in } \mathbb{R}^2.$$

- A more general micromagnetics model studied in [1],

$$E_\varepsilon^{ARS}(m;\Omega) = \frac{\varepsilon}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_{\mathbb{R}^2} |H|^2 + \frac{1}{2c_\varepsilon}\int_\Omega |m_3|^2, \tag{4}$$
$$m\colon \Omega \to \mathbb{S}^2 \subset \mathbb{R}^3, \quad 0 < c_\varepsilon \le \varepsilon^{1+\delta},$$
$$H\colon \mathbb{R}^2 \to \mathbb{R}^2, \qquad \nabla\times H = 0 \text{ and } \nabla\cdot(H + (m_1,m_2)\mathbf{1}_\Omega) = 0 \text{ in } \mathbb{R}^2.$$

For all these models, sequences of bounded energy as $\varepsilon \to 0$ are precompact in $L^2(\Omega)$ [1, 2, 11, 34], and limits of converging subsequences satisfy the eikonal equation (1). (In the case of (4) the limit satisfies $m_3 = 0$ so we can identify it with an $\mathbb{R}^2$-valued map.)

A large literature is devoted to understanding the behavior of minimizers m_ε of $E_\varepsilon^{AG}(\cdot;\Omega)$ as $\varepsilon \to 0$. In particular it is conjectured in [30] that the minimizers m_ε converge to $m_* = \nabla^\perp \operatorname{dist}(\cdot,\partial\Omega)$ when Ω is convex (counterexamples in nonconvex domains are given in [16, Theorem 7]). A positive answer is obtained in [17] when Ω is a disk, and in [27] for some special domains including ellipses (under the additional boundary constraint $m_{\lfloor\partial\Omega} = -in_{\partial\Omega}$). For E_ε^{RS} much more is known: that conjecture has been verified [35], and limits of non-minimizing sequences also have a well-understood structure [28].

In [17] the authors characterize zero-energy states, that is, limits of sequences with energy (2) converging to 0 as $\varepsilon \to 0$. In addition to the eikonal equation (1), zero-energy states satisfy the kinetic equation

$$e^{is}\cdot\nabla_x \mathbf{1}_{m(x)\cdot e^{is}>0} = 0 \qquad \text{in } \Omega, \text{ for all } s \in \mathbb{R}. \tag{5}$$

This is also valid for zero-energy states of (3) [17] and of (4) [1] (see Appendix A). It is shown in [17] that, if a smooth bounded simply connected domain Ω admits a zero energy state, that is, a solution of (1) and (5), then Ω must be a disk $\Omega = B_R(x_0)$, and m must be a vortex $m(x) = \pm i(x-x_0)/|x-x_0|$, or equivalently $m = \pm\nabla^\perp \operatorname{dist}(\cdot,\partial\Omega)$. Various generalizations can be found in [6, 20, 21, 26].

1.2 Main results

The main purpose of this work is to provide a quantitative version of the characterization of zero-energy states from [17]: estimate how much Ω differs from a disk and m from a vortex, in terms of the energy of an approximating sequence $m_\varepsilon \to m$. Previous results in this direction are proven in [23, 24]. Under the assumption that Ω is a C^2 convex domain renormalized to satisfy $\operatorname{diam}(\Omega) = 2$, it is shown in [24] that there exists $x_* \in \mathbb{R}^2$ such that

$$|\Omega\Delta B_1(x_*)| + \int_\Omega \left| m + i\frac{x-x_*}{|x-x_*|}\right|^2 dx \le C E_\varepsilon^{AG}(m;\Omega)^\delta, \tag{6}$$
$$\text{whenever } \nabla\cdot m = 0 \text{ in } \Omega \quad \text{and } m\cdot\tau = -1 \text{ on } \partial\Omega,$$

for some absolute constants $C > 0$, $\delta = 2^{-9}$, and $\tau = in_{\partial\Omega}$ a unit tangent to $\partial\Omega$. Note that the boundary condition $m\cdot\tau = -1$, commonly imposed in the study of the Aviles-Giga functional (see e.g. [2, 5, 7, 19]), is more restrictive than the condition $m\cdot n_{\partial\Omega} = 0$ enforced in (2) (which is natural in micromagnetics models).

Our goal is to obtain an estimate similar to (6), but with a sharp exponent δ, in the limit $\varepsilon \to 0$. To present our results in a unified setting, we consider the energy functional

$$\begin{aligned} F_\varepsilon(m;\Omega) &= \frac{\varepsilon}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_{\mathbb{R}^2} |H|^2 + \frac{1}{2\varepsilon}\int_\Omega (1-|m|^2)^2 + \frac{1}{2\varepsilon}\int_\Omega |m_3|^4, \\ & m\colon \Omega \to \mathbb{R}^3, \quad H\colon \mathbb{R}^2 \to \mathbb{R}^2, \\ & \nabla \times H = 0 \text{ and } \nabla\cdot(H + (m_1,m_2)\mathbf{1}_\Omega) = 0 \text{ in } \mathbb{R}^2. \end{aligned} \tag{7}$$

This functional satisfies $F_\varepsilon \le E_\varepsilon^{AG}, E_\varepsilon^{RS}, E_\varepsilon^{ARS}$ (for $F_\varepsilon \le E_\varepsilon^{ARS}$, note that any $m \in \mathbb{S}^2$ satisfies $|m_3|^4 \le |m_3|^2$) and the compactness proof of [11] applies (see Appendix A) to show that any sequence (m_ε) with bounded energy $F_\varepsilon(m_\varepsilon;\Omega) \le C$ is precompact in $L^2(\Omega)$, and its limits $m = \lim m_\varepsilon$ are $\mathbb{R}^2$-valued and satisfy the eikonal equation (1). We obtain a sharp bound for the L^2-distance between the unit normal to $\partial\Omega$ and the unit normal to a disk, in terms of the limit of $F_\varepsilon(m_\varepsilon;\Omega)$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set of class $C^{1,1}$ with $\mathcal{H}^1(\partial\Omega) = 2\pi$ and $\sup_{\partial\Omega} |\kappa| \le K$ for some $K > 0$, where κ denotes the curvature of $\partial\Omega$. There exists $c > 0$ depending only on K such that*

$$\inf_{x_* \in \mathbb{R}^2} \int_{\partial\Omega} \left| n_{\partial\Omega}(x) - \frac{x - x_*}{|x - x_*|} \right|^2 d\mathcal{H}^1(x) \le c \liminf_{\varepsilon\to 0} \inf_{H^1(\Omega;\mathbb{R}^3)} F_\varepsilon(\cdot;\Omega), \tag{8}$$

where F_ε is the functional defined in (7).

Moreover, this estimate is sharp:

Proposition 1.2. *There exist a family of convex domains $\{\Omega_N\}_{N\ge 3}$ of class $C^{1,1}$ with uniformly bounded curvature such that*

$$\begin{aligned} \frac{c_1}{N^2} &\ge \inf_{x_*\in\mathbb{R}^2} \int_{\partial\Omega_N} \left| n_{\partial\Omega_N}(x) - \frac{x - x_*}{|x - x_*|}\right|^2 d\mathcal{H}^1(x) \\ &\ge c_2 \liminf_{\varepsilon\to 0} \inf_{H^1(\Omega;\mathbb{R}^3)} F_\varepsilon(\cdot;\Omega_N) \ge \frac{c_3}{N^2}, \end{aligned}$$

for some absolute constants $c_1, c_2, c_3 > 0$.

Remark 1.3. The estimate (8) is sharp also when replacing $\inf F_\varepsilon$ with any of the (larger) $\inf E_\varepsilon^{AG}$, $\inf E_\varepsilon^{RS}$ or $\inf E_\varepsilon^{ARS}$, where the infimums are taken over all admissible maps for the corresponding functionals as described in (2), (3) and (4). This will be clear from the explicit description of the Ω_N's in § 6.3.

As corollaries of Theorem 1.1 and its proof we obtain two other estimates, which are however probably not sharp. The first corollary provides a bound on the distance of the boundary $\partial\Omega$ to the boundary of a disk, which is perhaps a more natural way of measuring how close Ω is to a disk.

Corollary 1.4. *Let Ω be as in Theorem 1.1. Then*

$$\inf_{x_*\in\mathbb{R}^2} \operatorname{dist}(\partial\Omega, \partial B_1(x_*)) \le c \liminf_{\varepsilon\to 0} \inf F_\varepsilon(\cdot;\Omega)^{\frac{1}{2}}.$$

for some constant $c = c(K) > 0$.

The second corollary provides a bound on the distance of a limiting map m from a vortex.

Corollary 1.5. *Let Ω be as in Theorem 1.1 and $m = \lim m_\varepsilon$ as $\varepsilon \to 0$, where (m_ε) is a sequence of admissible maps for the functional F_ε. Then there exists $\alpha \in \{\pm 1\}$ and $x_* \in \mathbb{R}^2$ such that*

$$\int_\Omega \left| m(x) - \alpha\, i \frac{x - x_*}{|x - x_*|} \right|^4 dx \leq c \liminf_{\varepsilon \to 0} F_\varepsilon(m_\varepsilon; \Omega)^{\frac{2}{3}},$$

for some constant $c = c(K) > 0$.

Remark 1.6. In comparison with the estimate (6) for E_ε^{AG} from [24], we don't require Ω to be convex, and impose only the boundary condition $m \cdot n_{\partial\Omega} = 0$ on limit maps. However, we only obtain bounds in the limit $\varepsilon \to 0$, while (6) is valid for any fixed $\varepsilon > 0$. Note that the constant c in (8) depends on K, while the constant C in (6) is absolute; on the other hand it is not possible to obtain an absolute constant if we drop the assumption of Ω being convex. Indeed if $\Omega_\delta = B_1((-1+\delta, 0)) \cup B_1((1-\delta, 0))$ (or rather, a mollification of this domain at scale much smaller than δ), then $\liminf_{\varepsilon \to 0} \inf F_\varepsilon(\cdot, \Omega_\delta)$ tends to 0 as $\delta \to 0$. This can be checked by using the solution of (1) in Ω_δ given by $m_\delta = i\nabla d_\delta$, where $d_\delta(x) = \mathrm{dist}(x, \partial\Omega_\delta)$, and the upper bound (see e.g. [7, 32]) $\liminf_{\varepsilon \to 0} \inf F_\varepsilon(\cdot, \Omega_\delta) \leq C \int_{J_\delta} |[m_\delta]|^3 d\mathcal{H}^1$, where J_δ is the jump set of m_δ.

Our proofs of Theorem 1.1 and its corollaries rely on a generalization of the zero-energy kinetic equation (5) to limits $m = \lim_{\varepsilon \to 0} m_\varepsilon$ of bounded energy sequences:

$$\begin{aligned} &e^{is} \cdot \nabla_x \mathbf{1}_{m(x) \cdot e^{is} > 0} = \partial_s \sigma, \qquad \sigma \in \mathcal{M}(\Omega \times \mathbb{R}/2\pi\mathbb{Z}), \\ &|\sigma|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}) \leq c_0 \liminf_{\varepsilon \to 0} F_\varepsilon(m_\varepsilon; \Omega), \end{aligned} \tag{9}$$

where $c_0 > 0$ is an absolute constant. This kinetic formulation, inspired by the field of scalar conservation laws [22], was first obtained in [18] for the Aviles-Giga functional (2) (see also [12]) and in [35] for the micromagnetics model (3). It also applies to the more general functional F_ε (see Appendix A). It is worth noting that it implies that m admits strong traces along 1-rectifiable subsets (see [36] or [10]), and in particular along $\partial\Omega$.

Among the measures σ satisfying (9), we consider the measure $\sigma_{\min}$ with minimal total variation $|\sigma|(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$ (the uniqueness of $\sigma_{\min}$ is proven in [27]), and set

$$\nu = (p_x)_\sharp |\sigma_{\min}|, \tag{10}$$

where $p_x : \Omega \times \mathbb{R}/2\pi\mathbb{Z} \to \Omega$ denotes the standard projection. In particular we have

$$\nu(\Omega) = |\sigma_{\min}|(\Omega \times \mathbb{R}/2\pi\mathbb{Z}) \leq c_0 \liminf_{\varepsilon \to 0} F_\varepsilon(m_\varepsilon; \Omega). \tag{11}$$

With these notations we may reformulate our main estimate as follows.

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^2$ be a simply connected open set of class $C^{1,1}$ with $\mathcal{H}^1(\partial\Omega) = 2\pi$ and $\sup_{\partial\Omega} |\kappa| \leq K$ for some $K > 0$, where κ denotes the curvature of $\partial\Omega$. If there exists $m \colon \Omega \to \mathbb{R}^2$ solving the eikonal equation (1) and the kinetic formulation (9), then*

$$\inf_{x_* \in \mathbb{R}^2} \int_{\partial\Omega} \left| n_{\partial\Omega}(x) - \frac{x - x_*}{|x - x_*|} \right|^2 d\mathcal{H}^1(x) \leq c\, \nu(\Omega), \tag{12}$$

for some constant $c > 0$ depending only on K.

Theorem 1.7 implies Theorem 1.1 thanks to (11). Similarly, Corollaries 1.4 and 1.5 will be consequences of the estimates

$$\mathrm{dist}(\partial\Omega, \partial B_1(x_*)) \leq c\, \nu(\Omega)^{\frac{1}{2}}, \tag{13}$$

$$\int_\Omega \left| m(x) - \alpha\, i \frac{x - x_*}{|x - x_*|} \right|^4 dx \leq c\, \nu(\Omega)^{\frac{2}{3}}, \tag{14}$$

for some $x_* \in \mathbb{R}^2$ and $\alpha \in \{\pm 1\}$. Next we briefly describe our strategy to prove Theorem 1.7.

1.3 Strategy of proof

1.3.1 A basic geometric argument

At the heart of our estimates is the following basic geometric argument. Assume m is a zero-energy state, that is, a solution of (1) and (5), and assume moreover that $m = -\tau$ on $\partial\Omega$. Suppose there are three boundary points $x_k \in \partial\Omega$, $k = 1, 2, 3$, and three directions $e^{i\alpha_k}$ with the following properties:

1. the three lines $x_k + e^{i\alpha_k}\mathbb{R}$ intersect at a point $z_0 \in \Omega$,
2. the direction $e^{i\alpha_k}$ points in the half-cirle determined by the direction $m = -\tau$ at x_k, i.e. $e^{i\alpha_k} \cdot \tau(x_k) < 0$,
3. the three directions $e^{i\alpha_k}$ are not contained in the same half-circle.

Such configuration is made impossible by the kinetic equation (5), because $\mathbf{1}_{m \cdot e^{i\alpha_k} > 0}$ must be constant along the line $x_k + \mathbb{R}e^{i\alpha_k}$. By the second property, its constant value must be one for $k = 1, 2, 3$, which implies that $m(z_0)$ has positive scalar product with the three directions $e^{i\alpha_k}$, which is impossible by the third property. (To make this rigorous actually requires a bit of care and 'almost everywhere' statements, as in [17].) So there are no triplets of points satisfying that condition, and this can be seen to imply that $\partial\Omega$ must be a circle, as it forces the normal lines at any three boundary points to be concurrent.

1.3.2 A quantitative version

Our strategy is to make that basic geometric argument quantitative. Let $a(x_1, x_2, x_3) \geq 0$ quantify the above properties: $a > 0$ if there are three lines from x_k with directions $e^{i\alpha_k}$ intersecting well inside Ω, with $e^{i\alpha_k} \cdot \tau(x_k) \leq -a$ and the three directions are not contained in the a-neighborhood of any half-circle. Note that this is a purely geometric quantity, defined without any reference to a map m. Let m satisfy the eikonal equation (1) and kinetic equation (9) with a non-zero dissipation measure $\nu(\Omega) = |\sigma_{\min}|(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$. Compared to the above basic geometric argument, the assumptions on m are relaxed in two ways: $\nu(\Omega) > 0$, and the trace $m_{\lfloor\partial\Omega}$ can take values into $\{\pm\tau\}$. Then we show that

$$\int_{\partial\Omega^3} a(x_1, x_2, x_3)^2 \, d(\mathcal{H}^1)^{\otimes 3} \leq c\,\nu(\Omega) + c\,\mathcal{H}^1(\{m_{\lfloor\partial\Omega} = \tau\}), \tag{15}$$

provided Ω is a priori close enough to a disk. This a priori condition will be satisfied if $\nu(\Omega)$ is small enough thanks to a compactness argument and the characterization of zero-energy states [17]. To deal with the trace issue, (15) needs to be complemented with the estimate

$$\mathcal{H}^1(\{m_{\lfloor\partial\Omega} = \tau\}) \leq c\,\nu(\Omega), \tag{16}$$

provided the left-hand side is a priori small enough. Again, this a priori condition can be obtained by means of a compactness argument and the characterization of zero-energy states. (The compactness argument tells us that, for small $\nu(\Omega)$, one of the complementary subsets $\{m_{\lfloor\partial\Omega} = \tau\}$ or $\{m_{\lfloor\partial\Omega} = -\tau\}$ is small, here we consider without loss of generality only the first case.) Finally Theorem 1.1 is obtained by estimating the deviation of $n_{\partial\Omega}$ from the disk's normal with the geometric quantity a, which relies on purely geometric considerations (that is, independent of the map m).

1.3.3 Lagrangian representation

The quantitative estimate (15) is our main new ingredient. It relies on the Lagrangian representation introduced by the second author in [27, 29], which allows to decompose the dissipation $\nu(\Omega)$ along Lagrangian trajectories. Roughly speaking, the dissipation created by one trajectory is the amount by which it deviates from being a straight line. In particular, absence of dissipation ($\nu = 0$) is equivalent to Lagrangian trajectories being straight lines. With this interpretation in mind, the intuition behind the proof of (15) can be explained as follows. Assume for simplicity that $m_{\lfloor\partial\Omega} = -\tau$. The basic geometric argument outlined above implies that Lagrangian trajectories meeting three boundary points x_k with directions close to $e^{i\alpha_k}$ cannot be straight lines if $a > 0$: they must therefore create dissipation. More precisely, for intervals of directions of order a around each $e^{i\alpha_k}$, at least one of the corresponding three trajectories should deviate of order a from being a straight line, and summing these contributions provides a dissipation of order a^2, as expressed by (15). Many technical details are however needed to make this intuition rigorous. In particular, trajectories cannot be considered individually, but in 'packets' inside which only a certain amount of trajectories follow that intuition. Similar arguments are used to prove the trace estimate (16).

1.4 Plan of the article

The article is organized as follows. In Section 2 we gather purely geometrical estimates, showing in particular that (15)-(16) imply Theorem 1.1 and Corollary 1.4. In Section 3 we prove (15), under the a priori assumption that Ω is close to a disk. In Section 4 we present the compactness argument that allows to lift that a priori assumption. In Section 5 we prove the trace estimate (16). In the short Section 6.1 we gather all previous results to prove Theorem 1.7, Theorem 1.1 and Corollary 1.4. In Section 6.2 we prove Corollary 1.5. In Section 6.3 we prove the sharpness statement of Proposition 1.2. In Appendix A we recall the arguments leading to the kinetic formulation 9, showing in particular that they apply to our generalized functional F_ε. In Appendix B we recall some of the analysis of the model (4) from [1], to emphasize that in that case the total dissipation $\nu(\Omega)$ provides a sharp lower bound. In Appendix C we present a quantitative proof which allows to bypass the compactness argument of Section 4 under the additional assumption that $m = -\tau$ on $\partial\Omega$, an assumption relevant for the Aviles-Giga model (2) but not for the other models considered here.

1.5 Notations

We use the symbol $\lesssim$ to denote inequality up to an absolute multiplicative constant and we write $a \sim b$ if both $a \lesssim b$ and $b \lesssim a$ hold true. We systematically identify $\mathbb{R}^2$ and $\mathbb{C}$, multiplication by i corresponds to rotation by an angle $\pi/2$. We denote by $g\colon \mathbb{R}/2\pi\mathbb{Z} \to \partial\Omega$ a $C^{1,1}$ counterclockwise arc-length parametrization of $\partial\Omega$, and by $\tau(g(s)) = \dot{g}(s)$, $n_{\partial\Omega} = -i\tau$ the corresponding unit tangent and normal.

2 Geometric estimates

Here and in the rest of the article, we fix $B_R(x_0)$ a maximal disk contained in Ω. As explained in the introduction, the proofs of our main results rely on a geometric quantity a defined for triples of boundary points.

Definition 2.1. *Given $\hat{x} = (x_1, x_2, x_3) \in \partial\Omega^3$, we define $a(\hat{x}) \geq 0$ as the maximal value $a \geq 0$ for which there are $\alpha_1, \alpha_2, \alpha_3 \in \mathbb{R}/2\pi\mathbb{Z}$ such that*

1. *the lines $x_k + e^{i\alpha_k}\mathbb{R}$ are concurrent in $B_{R/2}(x_0)$, namely there are $t_1, t_2, t_3 \in \mathbb{R}$ such that*

$$x_1 + t_1 e^{i\alpha_1} = x_2 + t_2 e^{i\alpha_2} = x_3 + t_3 e^{i\alpha_3} = z_0 \in B_{R/2}(x_0);$$

2. $\min(0, \tau(x_k) \cdot e^{i\alpha_k}) \le -a$ *for* $k = 1, 2, 3$*;*

3. $a \le \max\{l(\alpha_1, \alpha_2, \alpha_3) - \pi, 0\}$*, where* $l(\alpha_1, \alpha_2, \alpha_3)$ *denotes the length of the shortest interval in* $\mathbb{R}/2\pi\mathbb{Z}$ *containing* $\alpha_1, \alpha_2, \alpha_3$*.*

Note that each direction $e^{i\alpha_k}$ *may be entering, i.e.* $t_k > 0$*, or exiting, i.e.* $t_k < 0$ *(equivalently,* $(x_k - z_0) \cdot \tau(x_k) > 0$ *or* $(x_k - z_0) \cdot \tau(x_k) < 0$*).*

We observe that $a(\cdot)$ is identically 0 if Ω is a disk. A useful geometric interpretation of $a(\cdot)$ is that $a(x_1, x_2, x_3)$ is bounded below by the inner radius of the triangle formed by the three normals to $\partial\Omega$ passing through x_1, x_2 and x_3. See Figure 1

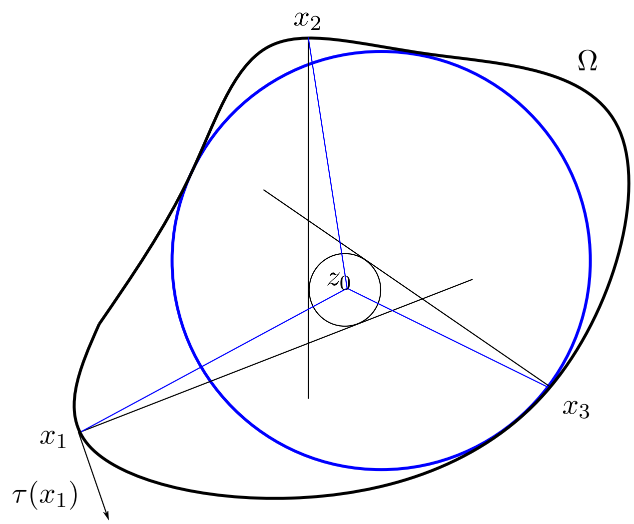

Figure 1: The black lines through the three points $x_1, x_2, x_3 \in \partial\Omega$ are the normals to $\partial\Omega$, while the blue lines have directions $e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}$ and they are concurrent in the point z_0 as in the definition of a. In this case z_0 is chosen as the center of the incircle of the triangle formed by the normals, and a is of the order of that incircle's radius.

The quantity a defined in Definition 2.1 will be useful only if the three segments $[z_0, x_k]$ are contained in $\Omega \cup \{x_k\}$. That is why we define next subsets of $\partial\Omega$ where this will be ensured. Recall that $B_R(x_0)$ is a maximal disk contained in Ω, and consider the set

$$E_* = \{x \in \partial\Omega \colon tx + (1-t)x_0 \in \Omega \;\forall t \in (0,1)\},$$

in some sense the part of the boundary that is star-shaped around x_0. And for every $\eta > 0$, we define the subset of E_* given by

$$E(\eta) = \{x \in E_* \colon |x - x_0| \le (1+\eta)R\}. \tag{17}$$

The main aim of the present section is to prove that the quantity a can be used to estimate the deviation of Ω from a disk, as follows.

Proposition 2.2. *Let* Ω *as in Theorem 1.1. There exists* $\eta_0 = \eta_0(K) > 0$ *such that, if* $E(\eta_0) = \partial\Omega$ *then*

$$\mathrm{dist}^2(\partial\Omega, \partial D_1(x_*)) + \int_{\partial\Omega} \left| n_{\partial\Omega}(x) - \frac{x - x_*}{|x - x_*|} \right|^2 \lesssim \int_{\partial\Omega^3} a^2 d(\mathcal{H}^1)^{\otimes 3}.$$

for some $x_* \in \mathbb{R}^2$*.*

2.1 A few preliminary geometric facts

First we show that boundary points close to the maximal inscribed circle must have a unit normal close to radial (with respect to the inscribed circle's center).

Lemma 2.3. *Let Ω be a $C^{1,1}$ simply connected domain with $\mathcal{H}^1(\partial\Omega) = 2\pi$, $\sup_{\partial\Omega} |\kappa| \le K$ and denote by $B_R(x_0)$ a maximal disk contained in Ω. Then $1/K \le R \le 1$ and for every $x \in \partial\Omega$ we have*

$$\left|\tau(x) \cdot \frac{x - x_0}{|x - x_0|}\right| \le 2\sqrt{K \operatorname{dist}(x, \partial B_R(x_0))}.$$

Proof of Lemma 2.3. The isoperimetric inequality ensures $R \le 1$. For a proof of the property $R \ge 1/K$ we refer to [31, 14]. Let us consider an arc-length parametrization $g : \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}^2$ of $\partial\Omega$ and let $\psi \colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ be defined by

$$\psi(s) = |g(s) - x_0| - R = \operatorname{dist}(g(s), \partial B_R(x_0)).$$

In particular we have

$$\begin{aligned} \psi'(s) &= \frac{\dot g(s) \cdot (g(s) - x_0)}{|g(s) - x_0|} = \tau(g(s)) \cdot \frac{g(s) - x_0}{|g(s) - x_0|}, \\ \psi''(s) &= \frac{\ddot g(s) \cdot (g(s) - x_0)}{|g(s) - x_0|} - \frac{|\dot g(s) \cdot (g(s) - x_0)|^2}{|g(s) - x_0|^3}. \end{aligned} \tag{18}$$

and therefore

$$\|\psi''\|_{L^\infty} \le K + \frac{1}{R} \le 2K.$$

Now consider, for any $a > 2$, the function

$$\varphi(s) = a\,\|\psi''\|_{L^\infty}\psi(s) - \psi'(s)^2,$$

which is C^1 with derivative

$$\begin{aligned} \varphi'(s) &= a\,\|\psi''\|_{L^\infty}\psi'(s) - 2\psi'(s)\psi''(s) \\ &= \left(a\,\|\psi''\|_{L^\infty} - 2\psi''(s)\right)\psi'(s). \end{aligned}$$

The first factor is positive since $a > 2$. Hence at a minimal point s_0 of φ one must have $\psi'(s_0) = 0$ and so $\varphi(s_0) = a\,\|\psi''\|_{L^\infty}\psi(s_0) \ge 0$. Therefore φ is a nonnegative function. As this is valid for any $a > 2$ we deduce that

$$\psi'(s)^2 \le 2\|\psi''\|_{L^\infty}\psi(s) \le 4K\,\psi(s).$$

Taking the square root and recalling the expression of ψ and ψ' concludes the proof. □

The next lemma ensures that $a(\hat x)$ is meaningful whenever $\hat x \in E(\eta)^3$ for sufficiently small $\eta > 0$.

Lemma 2.4. *Let Ω be a $C^{1,1}$ simply connected domain with $\mathcal{H}^1(\partial\Omega) = 2\pi$ and $\sup_{\partial\Omega} |\kappa| \le K$. There exists $\eta_0 > 0$ depending only on K such that, for all $x \in E(\eta_0)$ and any $z \in B_{2R/3}(x_0)$, the segment $[z, x]$ is included in $\Omega \cup \{x\}$.*

Proof of Lemma 2.4. Let $x \in E(\eta_0)$, and write $x_0 = x + re^{i\theta_0}$ for some $r \in [R, (1+\eta_0)R]$ and $\theta_0 \in \mathbb{R}$. Denote by $L_\theta = x + [0,\infty)e^{i\theta}$ the half line from x in direction θ. This half-line has a nontrivial intersection with $B_{2R/3}(x_0)$ if and only if $\theta \in (\theta_0 - \alpha, \theta_0 + \alpha) + 2\pi\mathbb{Z}$, where

$\alpha = \arcsin(2R/(3r)) = \arcsin(2/3) + \mathcal{O}(\eta_0) \leq \pi/4$ if η_0 is small enough. By definition (17) of $E(\eta)$ and of E_*, for all $z \in B_{2R/3}(x_0) \cap L_{\theta_0}$ the segment $[z, x]$ is included in $\Omega \cup \{x\}$. If the conclusion of Lemma 2.4 is not true, by continuity we may therefore find $\theta_1 \in (\theta_0 - \alpha, \theta_0 + \alpha)$, $\theta_1 \neq \theta_0$, and $z_1 \in B_{2R/3}(x_0) \cap L_{\theta_1}$ such that $[z_1, x]$ is not included in $\Omega \cup \{x\}$, while that property holds for $z \in B_{2R/3}(x_0) \cap L_\theta$ if $\theta \in (\theta_0, \theta_1)$. This implies the existence of $y \in [z_1, x] \cap \partial\Omega \setminus \{x\}$, with tangent vector $\tau(y) = e^{i\theta_1}$ and $|x - y| \lesssim R\eta_0$. In particular we have $(y - x_0)/|y - x_0| = e^{i\theta}$ with $|\theta - \theta_0| \lesssim R\eta_0$, and by Lemma 2.3 applied to the boundary point y we infer

$$|e^{i\theta_1} \cdot e^{i\theta_0}|^2 \lesssim KR\eta_0 \lesssim c_0 K\eta_0.$$

As $|e^{i\theta_1} \cdot e^{i\theta_0}|^2 \geq \cos^2\alpha \geq 1/2$ this implies $\eta_0 \gtrsim 1/K$, hence choosing $\eta_0 = 1/(CK)$ for a large enough absolute constant C ensures the validity of Lemma 2.4. □

We also remark that, for a connected component of $\partial\Omega \cap \overline{B}_{(1+\eta)R}(x_0)$ to be contained in $E(\eta)$, it is sufficient that one of its elements belongs to $E(\eta)$.

Lemma 2.5. *Let $\eta \in [0, 1/(4K)]$ and assume that $\bar{x} = g(\bar{s}) \in E(\eta)$ and $s_1, s_2 \in \mathbb{R}$ are such that*

$$\bar{s} \in [s_1, s_2] \qquad \text{and} \qquad |g(s) - x_0| \leq (1 + \eta)R \quad \forall s \in [s_1, s_2].$$

Then $g((s_1, s_2)) \subset E(\eta)$.

Proof of Lemma 2.5. Since we assume that $|g(s) - x_0| \leq (1 + \eta)R$ for all $s \in [s_1, s_2]$, it only remains to show that $g((s_1, s_2)) \subset E_*$, that is, for all $s \in (s_1, s_2)$ the line interval $\{(1 - t)x_0 + tg(s) \colon 0 < t < 1\}$ is contained in Ω. Consider the largest interval $I \subset (s_1, s_2)$ containing $\bar{s}$ and such that $g(s) \in E_*$ for all $s \in I$. Note that I is open and non-empty. Assume by contradiction that $(s_1, s_2) \setminus I \neq \emptyset$, and denote by $\tilde{s} \in (s_1, s_2)$ an extremity of I. Then by maximality of I the line interval $\{(1 - t)x_0 + tg(\tilde{s}) \colon 0 < t < 1\}$ intersects $\partial\Omega$: there exists $t \in (0, 1)$ such that $\tilde{x} = (1 - t)x_0 + tg(\tilde{s}) \in \partial\Omega$. By definition of I, locally near $\tilde{x}$ the C^2 curve $\partial\Omega$ stays on one side of the line $x_0 + \mathbb{R}(g(\tilde{s}) - x_0)$, hence it must be tangent to that line. Therefore we have $\tau(x) = \pm(\tilde{x} - x_0)/|\tilde{x} - x_0|$. By Lemma 2.3, and since $R \leq |\tilde{x} - x_0| < |g(\tilde{s}) - x_0| \leq (1 + \eta)R$ this implies $2\sqrt{K\eta R} > 1$, in contradiction with the assumption that $\eta \in [0, 1/(4K)]$ and the fact that $R \leq 1$ (by isoperimetric inequality). □

Finally we remark that the function a is Lipschitz.

Lemma 2.6. *The function a is Lipschitz on $\partial\Omega^3$ (with respect to the geodesic distance), with Lipschitz constant $L \lesssim K$.*

Proof. Let $\hat{x} = (x_1, x_2, x_3) \in \partial\Omega^3$, and $\hat{\alpha} = (\alpha_1, \alpha_2, \alpha_3) \in (\mathbb{R}/2\pi\mathbb{Z})^3$ as in the definition of $a(\hat{x})$. Denote by $z_0 \in B_{R/2}(x_0)$ the intersection point of the three lines $x_k + e^{i\alpha_k}\mathbb{R}$. Let $\hat{x}' = (x_1', x_2', x_3') \in \partial\Omega^3$. Since z_0 lies at a distance at least $R/2$ of each x_k, the three concurrent lines connecting x_k' to z_0 are of the form $x_k' + e^{i\alpha_k'}\mathbb{R}$ for some $\hat{\alpha}' = (\alpha_1', \alpha_2', \alpha_3') \in (\mathbb{R}/2\pi\mathbb{Z})^3$ such that

$$|\alpha_k - \alpha_k'| \lesssim \frac{1}{R}|x_k - x_k'| \lesssim K \operatorname{dist}(\hat{x}, \hat{x}') \qquad \forall k \in \{1, 2, 3\}.$$

Therefore we have

$$\begin{aligned} \max(l(\hat{\alpha}') - \pi, 0) &\geq \max(l(\hat{\alpha}) - \pi, 0) - CK \operatorname{dist}(\hat{x}, \hat{x}') \\ &\geq a(\hat{x}) - CK \operatorname{dist}(\hat{x}, \hat{x}'), \end{aligned}$$

for some absolute constant $C > 0$. Moreover by definition of K we have

$$|\tau(x_k) - \tau(x_k')| \lesssim K \operatorname{dist}(\hat{x}, \hat{x}') \qquad \forall k \in \{1, 2, 3\},$$

and therefore

$$\begin{aligned} \tau(x_k') \cdot e^{i\alpha_k'} &\leq \tau(x_k) \cdot e^{i\alpha_k} + C|\tau(x_k) - \tau(x_k')| + C|\alpha_k - \alpha_k'| \\ &\leq -a(\hat{x}) - CK \operatorname{dist}(\hat{x}, \hat{x}'). \end{aligned}$$

This shows that

$$a(\hat{x}') \geq a(\hat{x}) - CK \operatorname{dist}(\hat{x}, \hat{x}').$$

Exchanging the roles of $\hat{x}$ and $\hat{x}'$ we conclude that $|a(\hat{x}') - a(\hat{x})| \lesssim K \operatorname{dist}(\hat{x}, \hat{x}')$. □

2.2 Proof of Proposition 2.2

We start by remarking that the distance between $\partial\Omega$ and a unit circle is controlled by the L^1-difference of their normals.

Lemma 2.7. *If Ω is a simply connected C^1 domain such that $\mathcal{H}^1(\partial\Omega) = 2\pi$, then for any $x_* \in \Omega$ such that Ω is strictly star-shaped around x_*, we have*

$$\operatorname{dist}(\partial\Omega, \partial D_1(x_*)) \leq \int_{\partial\Omega} \left| n_{\partial\Omega}(x) - \frac{x - x_*}{|x - x_*|} \right| d\mathcal{H}^1(x)$$

Proof of Lemma 2.7. We choose coordinates in which $x_* = 0$ and denote

$$n_*(x) = \frac{x - x_*}{|x - x_*|} = \frac{x}{|x|}.$$

First we claim that

$$|\tau_{\partial\Omega}(x) \cdot n_*(x)| \leq |n_{\partial\Omega}(x) - n_*(x)|\,. \tag{19}$$

To prove (19), note that since Ω is strictly star-shaped around x_*, i.e. $n_{\partial\Omega} \cdot n_* > 0$ on $\partial\Omega$, we have

$$n_{\partial\Omega} \cdot n_* = \sqrt{1 - (\tau_{\partial\Omega} \cdot n_*)^2}$$

Hence we deduce

$$|n_{\partial\Omega} - n_*|^2 = 2 - 2\, n_{\partial\Omega} \cdot n_* = 2 - 2\sqrt{1 - (\tau_{\partial\Omega} \cdot n_*)^2}.$$

Estimate (19) follows from this identity and the convexity inequality

$$2 - 2\sqrt{1 - t} \geq t \qquad \forall t \in [0, 1].$$

Let $g \in C^1(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$ denote a counterclockwise arc-length parametrization of $\partial\Omega$, and let $r_{\min} = \min |g|$, $r_{\max} = \max |g|$ be the respective radii of the maximal centered disk contained in Ω and of the minimal centered disk containing Ω. As

$$\frac{d}{ds}|g(s)| = \dot{g}(s) \cdot \frac{g(s)}{|g(s)|} = \tau_{\partial\Omega}(g(s)) \cdot n_*(g(s)),$$

we infer, using also (19),

$$r_{\max} - r_{\min} \leq \frac{1}{2} \int_{\partial\Omega} |\tau_{\partial\Omega} \cdot n_*|\, d\mathcal{H}^1 \leq \frac{1}{2} \int_{\partial\Omega} |n_{\partial\Omega} - n_*|\, d\mathcal{H}^1. \tag{20}$$

Note that $r_{\min} \leq 1$ thanks to the isoperimetric inequality, so if $r_{\max} \geq 1$ then (20) directly implies the conclusion of Lemma 2.7. In what follows we may therefore assume $r_{\max} < 1$. As Ω is strictly star-shaped around $x_* = 0$, the map

$$g_*(s) = \frac{g(s)}{|g(s)|}, \qquad s \in \mathbb{R}/2\pi\mathbb{Z},$$

defines a one-to-one parametrization of the unit circle ∂D_1. In particular we must have

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} |\dot{g}_*| = 2\pi.$$

On the other hand direct calculation shows

$$|g|\, \dot{g}_* = \dot{g} - (\dot{g} \cdot g_*)\, g_*,$$

hence

$$\begin{aligned}
\int_{\mathbb{R}/2\pi\mathbb{Z}} (1 - |g|)|\dot{g}_*| &= 2\pi - \int_{\mathbb{R}/2\pi\mathbb{Z}} |\dot{g} - (\dot{g} \cdot g_*)\, g_*| \\
&\leq 2\pi - \int_{\mathbb{R}/2\pi\mathbb{Z}} |\dot{g}| + \int_{\mathbb{R}/2\pi\mathbb{Z}} |\dot{g} \cdot g_*| \\
&= \int_{\partial\Omega} |\tau_{\partial\Omega} \cdot n_*|\, d\mathcal{H}^1.
\end{aligned}$$

As $0 < 1 - r_{\max} \leq 1 - |g|$, this implies

$$2\pi(1 - r_{\max}) = (1 - r_{\max}) \int_{\mathbb{R}/2\pi\mathbb{Z}} |\dot{g}_*| \leq \int_{\partial\Omega} |\tau_{\partial\Omega} \cdot n_*|\, d\mathcal{H}^1.$$

Together with (20) this gives

$$1 - r_{\min} \leq \left(\frac{1}{2} + \frac{1}{2\pi}\right) \int_{\partial\Omega} |\tau_{\partial\Omega} \cdot n_*|\, d\mathcal{H}^1 \leq \int_{\partial\Omega} |n_{\partial\Omega} - n_*|\, d\mathcal{H}^1,$$

proving Lemma 2.7 also in the case $r_{\max} < 1$. □

Now we turn to the proof of Proposition 2.2.

Proof of Proposition 2.2. We choose coordinates in which $x_0 = 0$. We assume $\Omega = E(\eta_0)$ for some $\eta_0 = \varepsilon_0^2/K > 0$, with ε_0 to be fixed later. Hence Ω is star-shaped around x_0 and $B_R(x_0) \subset \Omega \subset B_{(1+\eta_0)R}$.

Recall $R \leq 1$ by the isoperimetric inequality, and thanks to Lemmas 2.3 and 2.7 we have

$$\sup_{x \in \partial\Omega} \left| n_{\partial\Omega}(x) - \frac{x}{|x|} \right| \lesssim \varepsilon_0, \qquad 0 \leq 1 - R \lesssim \varepsilon_0. \tag{21}$$

For any $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ we denote by I_α the portion of $\partial\Omega$ that intersects the centered cone corresponding to angles from α to $\alpha + \pi/6$, that is

$$I_\alpha = \left\{ x \in \partial\Omega \colon \frac{x}{|x|} = e^{i\theta} \text{ for some } \theta \in [\alpha, \alpha + \pi/6] \right\}.$$

Thanks to the above we have $\mathcal{H}^1(I_\alpha) = \pi/6 + \mathcal{O}(\varepsilon_0) \geq \pi/12$ for small enough η_0, so by Fubini there exist $\bar{x}_2 \in I_{\alpha+2\pi/3}$, $\bar{x}_3 \in I_{\alpha+4\pi/3}$ such that

$$\int_{I_\alpha} a^2(x_1, \bar{x}_2, \bar{x}_3)\, d\mathcal{H}^1(x_1) \lesssim \int_{\partial\Omega^3} a^2\, d(\mathcal{H}^1)^{\otimes 3}.$$

Denote by z_α the intersection of the two normal lines at $\bar{x}_2$ and $\bar{x}_3$. It satisfies $|z_\alpha| \lesssim \varepsilon_0$. Let n_α denote the vortex centered at z_α, that is,

$$n_\alpha(x) = \frac{x - z_\alpha}{|x - z_\alpha|}.$$

We claim that

$$|n_{\partial\Omega}(x_1) - n_\alpha(x_1)| \lesssim a(x_1, \bar{x}_2, \bar{x}_3) \qquad \forall x_1 \in I_\alpha. \tag{22}$$

Let indeed $x_1 \in I_\alpha$. We denote by L_1, L_2, L_3 the normal lines to $\partial\Omega$ at $x_1, \bar{x}_2, \bar{x}_3$. As the normals are close to radial thanks to (21), the three intersection points $L_1 \cap L_2$, $L_1 \cap L_3$ and $L_2 \cap L_3$ lie in $D_{c_0\varepsilon_0}$ for some absolute constant c_0. Recall that z_α is the intersection point $z_\alpha = L_2 \cap L_3$, and denote by d its distance to the line L_1, $d = \mathrm{dist}(z_\alpha, L_1) \lesssim \varepsilon_0$. Since $x_1 \in I_\alpha$, $\bar{x}_2 \in I_{\alpha+2\pi/3}$ and $\bar{x}_3 \in I_{\alpha+4\pi/3}$, the triangle formed inside $D_{c_0\varepsilon_0}$ by the three lines L_1, L_2, L_3 has its three angles $\gtrsim 1$ for small enough ε_0. Hence the radius r of that triangle's incircle is comparable to the distance d, we have $d \lesssim r$. On the other hand, considering the three concurrent lines from $x_1, \bar{x}_2, \bar{x}_3$ to the incircle's center, we find that $r \lesssim a(x_1, \bar{x}_2, \bar{x}_3)$. Thus we have $d \lesssim r \lesssim a(x_1, \bar{x}_2, \bar{x}_3)$. Moreover, the angle between L_1 and the line from z_α to x_1 is $\lesssim d$, which shows that

$$|n_{\partial\Omega}(x_1) - n_\alpha(x_1)| \lesssim d \lesssim r \lesssim a(x_1, \bar{x}_2, \bar{x}_3),$$

and proves the claim (22). From (22) we deduce

$$\int_{I_\alpha} |n_{\partial\Omega} - n_\alpha|^2\, d\mathcal{H}^1 \lesssim \int_{I_\alpha} a^2(\cdot, \bar{x}_2, \bar{x}_3)\, d\mathcal{H}^1 \lesssim \int_{\partial\Omega^3} a^2\, d(\mathcal{H}^1)^{\otimes 3}.$$

Applying this to $\alpha = j\pi/12$ for $j = 1, \ldots, 24$, we cover $\partial\Omega$ with portions $I_1, \ldots, I_{24}$ satisfying $\mathcal{H}^1(I_j \cap I_{j+1}) = \pi/12 + \mathcal{O}(\varepsilon_0)$, and find points $z_j \in D_{c_0\varepsilon_0}$ such that

$$\int_{I_j} |n_{\partial\Omega} - n_j|^2\, d\mathcal{H}^1 \lesssim \int_{\partial\Omega^3} a^2\, d(\mathcal{H}^1)^{\otimes 3}, \qquad n_j(x) = \frac{x - z_j}{|x - z_j|}. \tag{23}$$

This implies

$$\int_{I_j \cap I_{j+1}} |n_j - n_{j+1}|^2 d\mathcal{H}^1 \lesssim \int_{\partial\Omega^3} a^2\, d(\mathcal{H}^1)^{\otimes 3}.$$

We claim that

$$|z_j - z_{j+1}|^2 \lesssim \int_{I_j \cap I_{j+1}} |n_j - n_{j+1}|^2 d\mathcal{H}^1 \lesssim \int_{\partial\Omega^3} a^2\, d(\mathcal{H}^1)^{\otimes 3}. \tag{24}$$

The second inequality was proved above, so we only need to show the first inequality in (24). First note that since $z_j, z_{j+1} \in D_{c_0\varepsilon_0}$ and $I_j \cap I_{j+1} \subset D_{1+c_0\varepsilon_0} \setminus D_{1-c_0\varepsilon_0}$, for any $x \in I_j \cap I_{j+1}$ we have

$$|(z_{j+1} - z_j) \cdot (i n_j(x))| \lesssim |n_j(x) - n_{j+1}(x)|.$$

In other words, $|n_j(x)-n_{j+1}(x)|$ controls $|z_j-z_{j+1}|$ unless x, z_j, z_{j+1} are closed to aligned. But since $I_j \cap I_{j+1}$ is a portion of curve inside $D_{1+c_0\varepsilon_0} \setminus D_{1-c_0\varepsilon_0}$ from a point of polar angle $(j+1)\pi/12$ to a point of polar angle $(j+2)\pi/12$, there is a subset $J \subset I_j \cap I_{j+1}$ satisfying $\mathcal{H}^1(J) \geq \pi/24$ and such that for $x \in J$ the three points x, z_j, z_{j+1} are far from aligned, that is,

$$|z_j - z_{j+1}| \lesssim |(z_{j+1} - z_j) \cdot (in_j(x))| \lesssim |n_j(x) - n_{j+1}(x)| \qquad \forall x \in J.$$

Taking squares and integrating over J we obtain (24). From (24) we deduce $|z_j - z_1|^2 \lesssim \int_{\partial\Omega^3} a^2\, d(\mathcal{H}^1)^{\otimes 3}$ for all $j = 1, \ldots, 24$, and therefore (23) implies

$$\int_{\partial\Omega} |n_{\partial\Omega} - n_1|^2 \lesssim \int_{\partial\Omega^3} a^2\, d(\mathcal{H}^1)^{\otimes 3}.$$

Taking $x_* = z_1$ and applying Lemma 2.7 and Cauchy-Schwarz' inequality, this concludes the proof of Proposition 2.2. $\square$

3 Lower bound on the dissipation

In this section we prove the following.

Proposition 3.1. *Let Ω and m be as in Theorem 1.7. We have the estimate*

$$\int_{E(\eta_*)^3} a^2 d(\mathcal{H}^1)^{\otimes 3} \leq C\, \nu(\Omega) + C\, \mathcal{H}^1(\{m_{\partial\Omega} = \tau\}),$$

where $\eta_* = \min(\eta_0/2, 1/(8K))$, for η_0 as in Lemma 2.4, and $C > 0$ is a constant depending only on K.

3.1 Lagrangian representation

In order to prove Proposition 3.1, we introduce the notion of Lagrangian representation for entropy solutions of the eikonal equation from [27].

Given $T > 0$ we let

$$\Gamma = \Big\{ (\gamma, t_\gamma^-, t_\gamma^+) \colon 0 \leq t_\gamma^- \leq t_\gamma^+ \leq T, \\ \gamma = (\gamma_x, \gamma_s) \in \mathrm{BV}((t_\gamma^-, t_\gamma^+); \Omega \times \mathbb{R}/2\pi\mathbb{Z}), \gamma_x \text{ is Lipschitz} \Big\}.$$

We will always consider the right-continuous representative of the component γ_s and we will write $\gamma(t_\gamma^-)$ instead of $\lim_{t \to t_\gamma^-} \gamma(t)$ and $\gamma(t_\gamma^+)$ instead of $\lim_{t \to t_\gamma^+} \gamma(t)$. For every $t \in (0, T)$ we consider the section

$$\Gamma(t) := \big\{ \big(\gamma, t_\gamma^-, t_\gamma^+\big) \in \Gamma : t \in \big(t_\gamma^-, t_\gamma^+\big) \big\}$$

and we denote by

$$\begin{aligned} e_t \colon \quad \Gamma(t) &\longrightarrow \Omega \times \mathbb{R}/2\pi\mathbb{Z} \\ (\gamma, t_\gamma^-, t_\gamma^+) &\longmapsto \gamma(t), \end{aligned}$$

the evaluation map at time t.

Definition 3.2. *Let Ω be a $C^{1,1}$ open set and m solving (1) and (9). We say that a finite non-negative Radon measure $\omega \in \mathcal{M}(\Gamma)$ is a* Lagrangian representation *of m if the following conditions are satisfied:*

1. *for every $t \in (0,T)$ we have*

$$(e_t)_\sharp\,[\omega\llcorner\Gamma(t)] = \mathbf{1}_{E_m}\,\mathcal{L}^2\times\mathcal{L}^1, \tag{25}$$

where $E_m \subset \Omega\times\mathbb{R}/2\pi\mathbb{Z}$ is the 'epigraph'

$$E_m = \left\{(x,s)\in\Omega\times\mathbb{R}/2\pi\mathbb{Z}\colon m(x)\cdot e^{is}>0\right\};$$

2. *the measure ω is concentrated on curves $(\gamma, t_\gamma^-, t_\gamma^+)\in\Gamma$ solving the characteristic equation:*

$$\dot\gamma_x(t) = e^{i\gamma_s(t)} \qquad \text{for a.e. } t\in(t_\gamma^-,t_\gamma^+); \tag{26}$$

3. *we have the integral bound*

$$\int_\Gamma \mathrm{TV}_{(0,T)}\gamma_s d\omega(\gamma) < \infty;$$

4. *for ω-a.e. $(\gamma,t_\gamma^-,t_\gamma^+)\in\Gamma$ we have*

$$t_\gamma^- > 0 \Rightarrow \gamma_x(t_\gamma^-)\in\partial\Omega, \qquad \text{and} \qquad t_\gamma^+ < T \Rightarrow \gamma_x(t_\gamma^+)\in\partial\Omega. \tag{27}$$

A useful property of the Lagrangian formulation is the possibility of decomposing the entropy dissipation measure ν along the characteristics detected by ω. More precisely, from [27] we have

Proposition 3.3. *Let Ω be a $C^{1,1}$ open set, m solving* (1) *and* (9)*, and $T>0$. Then there is a Lagrangian representation ω of m such that for every Borel set $A\subset[0,T]$ and $B\subset\Omega$ it holds*

$$\int_\Gamma \mu_\gamma(\{t\in A:\gamma_x(t)\in B\})\,d\omega(\gamma) = \mathcal{L}^1(A)\nu(B),$$

where $\mu_\gamma = |D_t\gamma_s|$.

Remark 3.4. Note that γ_s takes values into $\mathbb{R}/2\pi\mathbb{Z}$, so a few precisions about the meaning of the measure $\mu_\gamma = |D_t\gamma_s|$ are in order. It should actually be understood as the measure $|D_t\hat\gamma_s|$ where $\hat\gamma_s\in BV(I_\gamma;\mathbb{R})$ is such that $\gamma_s(t)=\hat\gamma_s(t)+2\pi\mathbb{Z}$ for all $t\in I_\gamma$, and the jumps of $\hat\gamma_s$ are such that $|\hat\gamma_s(t+)-\hat\gamma_s(t-)| = \mathrm{dist}_{\mathbb{R}/2\pi\mathbb{Z}}(\gamma_s(t-),\gamma_s(t+))$ (see e.g. [15, Theorem 1] for the existence of such a lifting, which is however not necessary to define the measure μ_γ).

We will also use that, thanks to property (27) and the trace properties of m, the pushforward of ω under evaluation at initial time t_γ^- is related to the $\mathcal{H}^1_{\lfloor\partial\Omega}$ in the following way.

Lemma 3.5. *Denote $\Gamma_{ini} = \{t_\gamma^- > 0\}\subset\Gamma$ and*

$$P_{ini}\colon\Gamma_{ini}\to(0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z}, \quad \gamma\mapsto(t_\gamma^-,\gamma(t_\gamma^-)),$$

then the pushforward measure $\mu_{ini} = P_{ini}\sharp\omega_{\lfloor\Gamma_{ini}}$ is given by

$$d\mu_{ini}(t,x,s) = \mathbf{1}_{m(x)\cdot e^{is}>0}\mathbf{1}_{i\tau(x)\cdot e^{is}>0}\,(i\tau(x)\cdot e^{is})\,dt\,d\mathcal{H}^1_{\lfloor\partial\Omega}(x)\,ds.$$

Proof of Lemma 3.5. The argument is similar to [8, Lemma 3.1]. Let $F \in C^1_c((0,T) \times \partial\Omega \times \mathbb{R}/2\pi\mathbb{Z})$, and denote also by F a C^1 extension to $(0,T) \times \overline{\Omega} \times \mathbb{R}/2\pi\mathbb{Z}$.

For small enough $h > 0$ we may find a C/h-Lipschitz function $G_h \colon \mathbb{R}^2 \to [0,1]$, with C depending only on K, such that

$$\mathbf{1}_{x\in\Omega,\,\mathrm{dist}(x,\partial\Omega)\le h} \le G_h(x) \le \mathbf{1}_{x\in\Omega},$$
$$\text{and } \nabla G_h \to i\tau\, d\mathcal{H}^1_{\lfloor \partial\Omega} \qquad \text{as } h \to 0.$$

Thanks to the trace property of m and the Lagrangian property (25) we have

$$\begin{aligned}
&\int_{(0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z}} F(t,x,s)\, \mathbf{1}_{m(x)\cdot e^{is}>0}\, (i\tau(x)\cdot e^{is})\, dt\, d\mathcal{H}^1_{\lfloor\partial\Omega}(x)\, ds \\
&= \lim_{h\to 0^+} \int_{(0,T)\times\Omega\times\mathbb{R}/2\pi\mathbb{Z}} F(t,x,s)\, \mathbf{1}_{m(x)\cdot e^{is}>0}\, (e^{is}\cdot \nabla G_h(x))\, dt\, dx\, ds \\
&= \lim_{h\to 0^+} \int_\Gamma A_h(\gamma)\, d\omega(\gamma),
\end{aligned}$$

where

$$\begin{aligned}
A_h(\gamma) &= \int_{t_\gamma^-}^{t_\gamma^+} F(t,\gamma_x(t),\gamma_s(t))\; (e^{i\gamma_s(t)}\cdot \nabla G_h(\gamma_x(t))\, dt \\
&= \int_{t_\gamma^-}^{t_\gamma^+} F(t,\gamma_x(t),\gamma_s(t))\; \frac{d}{dt}\left[G_h(\gamma_x(t))\right]\, dt.
\end{aligned}$$

For the last equality we used the characteristic equation (26). Then we integrate by parts: since $G_h(\gamma_x(t_\gamma^-)) = 0$ if $t_\gamma^- > 0$ and $G_h(t_\gamma^+) = 0$ if $t_\gamma^+ < T$ we obtain

$$A_h(\gamma) = -\int_{t_\gamma^-}^{t_\gamma^+} G_h(\gamma_x(t)) D\Phi_\gamma(dt),$$
$$\text{where } \Phi_\gamma(t) = F(t,\gamma_x(t),\gamma_s(t)).$$

In particular we have the convergence

$$A_h(\gamma) \longrightarrow A_0(\gamma) = \int_{t_\gamma^-}^{t_\gamma^+} \mathbf{1}_{\gamma_x(t)\in\Omega} D\Phi_\gamma(dt),$$

as $h \to 0^+$. By definition of the Lagrangian representation, for ω-a.e. $\gamma \in \Gamma$ we have $\gamma_x(t) \in \Omega$ for all $t \in (t_\gamma^-, t_\gamma^+)$, so

$$\begin{aligned}
A_0(\gamma) &= -\int_{t_\gamma^-}^{t_\gamma^+} D\Phi_\gamma(dt) = \Phi_\gamma(t_\gamma^-) - \Phi_\gamma(t_\gamma^+) \\
&= F(t_\gamma^-,\gamma_x(t_\gamma^-),\gamma_s(t_\gamma^-)) - F(t_\gamma^+,\gamma_x(t_\gamma^+),\gamma_s(t_\gamma^+)).
\end{aligned}$$

Thanks to the domination $|A_h(\gamma)| \le \|\nabla F\|_\infty (1 + TV(\gamma_s))$, we deduce

$$\begin{aligned}
&\int_{(0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z}} F(t,x,s)\, \mathbf{1}_{m(x)\cdot e^{is}>0}\, (i\tau(x)\cdot e^{is})\, dt\, d\mathcal{H}^1_{\lfloor\partial\Omega}(x)\, ds \\
&= \int_\Gamma \left(F(t_\gamma^-,\gamma_x(t_\gamma^-),\gamma_s(t_\gamma^-)) - F(t_\gamma^+,\gamma_x(t_\gamma^+),\gamma_s(t_\gamma^+))\right)\, d\omega(\gamma).
\end{aligned}$$

We apply this to

$$F(t,x,s) = f(t,x,s)\phi_\varepsilon(x,s), \quad \mathbf{1}_{i\tau(x)\cdot e^{is}>\varepsilon} \le \phi_\varepsilon(x,s) \le \mathbf{1}_{i\tau(x)\cdot e^{is}>0},$$

where $f \in C_c^1((0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z})$ and $\phi_\varepsilon \in C^1(\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z})$. Since for ω-a.e. $\gamma\in\Gamma$ it holds $\gamma_x(t)\in\Omega$ for all $t\in(t_\gamma^-,t_\gamma^+)$, by the characteristic speed constraint (26), we have $i\tau(\gamma_x(t_\gamma^+))\cdot e^{i\gamma_s(t_\gamma^+)} \le 0$ if $t_\gamma^+ < T$, so $\phi_\varepsilon(\gamma_x(t_\gamma^+),\gamma_s(t_\gamma^+)) = 0$, and we deduce

$$\begin{aligned}&\int_{(0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z}} f(t,x,s)\phi_\varepsilon(x,s)\,\mathbf{1}_{m(x)\cdot e^{is}>0}\,(i\tau(x)\cdot e^{is})\,dt\,d\mathcal{H}^1_{\lfloor\partial\Omega}(x)\,ds\\ &= \int_\Gamma f(t_\gamma^-,\gamma_x(t_\gamma^-),\gamma_s(t_\gamma^-))\phi_\varepsilon(\gamma_x(t_\gamma^-),\gamma_s(t_\gamma^-))\,d\omega(\gamma).\end{aligned}$$

By dominated convergence as $\varepsilon\to 0$ this implies

$$\begin{aligned}&\int_{(0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z}} f(t,x,s)\,\mathbf{1}_{i\tau(x)\cdot e^{is}>0}\,\mathbf{1}_{m(x)\cdot e^{is}>0}\,(i\tau(x)\cdot e^{is})\,dt\,d\mathcal{H}^1_{\lfloor\partial\Omega}(x)\,ds\\ &= \int_\Gamma f(t_\gamma^-,\gamma_x(t_\gamma^-),\gamma_s(t_\gamma^-))\,\mathbf{1}_{i\tau(\gamma_x(t_\gamma^-))\cdot e^{i\gamma_s(t_\gamma^-))}>0}\,d\omega(\gamma).\end{aligned}$$

As above, for ω-a.e. $\gamma\in\Gamma_{ini}$ we have $i\tau(\gamma_x(t_\gamma^-))\cdot e^{i\gamma_s(t_\gamma^-))} \ge 0$. Indeed the curves that enter tangentially into Ω, namely for which $i\tau(\gamma_x(t_\gamma^-))\cdot e^{i\gamma_s(t_\gamma^-))} = 0$, are ω-negligible (see [8, (3.5)] for details). In particular for ω-a.e. $\gamma\in\Gamma_{ini}$ we have $i\tau(\gamma_x(t_\gamma^-))\cdot e^{i\gamma_s(t_\gamma^-))} > 0$ and we infer

$$\begin{aligned}&\int_{(0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z}} f(t,x,s)\,\mathbf{1}_{i\tau(x)\cdot e^{is}>0}\,\mathbf{1}_{m(x)\cdot e^{is}>0}\,(i\tau(x)\cdot e^{is})\,dt\,d\mathcal{H}^1_{\lfloor\partial\Omega}(x)\,ds\\ &= \int_{\Gamma_{ini}} f(t_\gamma^-,\gamma_x(t_\gamma^-),\gamma_s(t_\gamma^-))\,d\omega(\gamma),\end{aligned}$$

for any $f \in C_c^1((0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z})$. By approximation this is valid for any $f \in C_c^0(0,T)\times\partial\Omega\times\mathbb{R}/2\pi\mathbb{Z})$, concluding the proof of Lemma 3.5. □

3.2 Proof of Proposition 3.1

Before proving Proposition 3.1 we set some notations and definitions.

- We apply Proposition 3.3 for some fixed $T \ge 3\pi$, and let $h \in L^1(\partial\Omega)$ be defined by the relation

$$\int_E h\,d\mathcal{H}^1 = \int_{\{\gamma:\gamma_x(t_\gamma^-)\in E\}} |D_t\gamma_s|(I_\gamma)\,d\omega_h(\gamma), \tag{28}$$

  that is, $h(x)$ encodes the entropy dissipation generated (via Proposition 3.3) by the curves of the Lagrangian representation emanating from $x\in\partial\Omega$.
- We denote by W the 'wrong trace' set $W = \{m=\tau\}\subset\partial\Omega$, by $\mathcal{M}\mathbf{1}_W$ the maximal function

$$\mathcal{M}\mathbf{1}_W(x) = \sup_{r>0}\frac{1}{r}\int_{I_r(x)} \mathbf{1}_W d\mathcal{H}^1,$$

  where $I_r(x) = g([t-r,t+r])$ for $x = g(t)$, and for any $\varepsilon > 0$ we define the set

$$G_\varepsilon = \{\mathcal{M}\mathbf{1}_W < \varepsilon\}\subset\partial\Omega, \tag{29}$$

  of boundary points where the proportion of wrong traces is less than ε at any scale around that point. Note that the Hardy-Littlewood inequality ensures that $\mathcal{H}^1(\partial\Omega\setminus G_\varepsilon) \lesssim \varepsilon^{-1}\mathcal{H}^1(W)$.

The main, and most technical, part of Proposition 3.1's proof is encoded in the following lemma.

Lemma 3.6. *There exist $C, c, \varepsilon > 0$, depending only on K, such that, for any $\hat{x} \in (E(\eta_*) \cap G_\varepsilon)^3$ with $a(\hat{x}) > 0$, the quantity $a(\hat{x})$ provides the following lower bound on the entropy dissipation:*

$$\frac{C}{T} \int_{I(\hat{x}, a(\hat{x}))} (h(x_1) + h(x_2) + h(x_3)) \, d(\mathcal{H}^1)^{\otimes 3} \geq a(\hat{x})^5,$$

where

$$I(\hat{x}, a(\hat{x})) = I_1 \times I_2 \times I_3, \qquad I_k = I_{c\, a(\hat{x})}(x_k) = g([\bar{s}_k - c\, a(\hat{x}), \bar{s}_k + c\, a(\hat{x})]),$$

and $x_k = g(\bar{x}_k)$ for $k = 1, 2, 3$.

Proof of Lemma 3.6. Let $\hat{x} \in (E(\eta_*) \cap G_\varepsilon)^3$ with $a(\hat{x}) > 0$

First we choose the constant $c = c(K) > 0$ appearing in the definition of $I(\hat{x}, a(\hat{x}))$ in order to ensure

$$e^{i\alpha_k} \cdot \tau(x) \leq -\frac{a(\hat{x})}{2} \; \forall x \in I_k, \quad \text{and } I_k \subset E(2\eta_*),$$

where α_k are the angles in the definition of $a(\hat{x})$. The first condition can be imposed because τ is K-Lipschitz and $e^{i\alpha_k} \cdot \tau(x_k) \leq -a(\hat{x})$, and the second thanks to Lemma 2.5, since $2\eta_* \leq 1/(4K)$ (as imposed in the statement of Proposition 3.1).

We denote by z_0 the intersection point as in the definition of $a(\hat{x})$, and consider the three cylinders

$$\mathcal{C}_k := B_{c_1 a(\hat{x})}(z_0) \times [\alpha_k - c_2 a(\hat{x}), \alpha_k + c_2 a(\hat{x})] \qquad \text{for } k = 1, 2, 3,$$

where $c_1, c_2 > 0$ are small constants depending on K and chosen to ensure that:

- for any $(s_1, s_2, s_3) \in \Pi_{k=1}^3 [\alpha_k - c_2 a(\hat{x}), \alpha_k + c_2 a(\hat{x})]$, the shortest interval in $\mathbb{R}/2\pi\mathbb{Z}$ containing s_1, s_2, s_3 has length $l > \pi$;
- for all $(x, s) \in \mathcal{C}_k$, there is a boundary point $y \in I_k$ such that $x = y + te^{is}$ for some $t \in \mathbb{R}$, the segment $[x, y]$ is contained in $\overline{\Omega}$, and $\tau(y) \cdot e^{is} < 0$.

The last property is possible since $2\eta_* \leq \eta_0$ with η_0 as in Lemma 2.4.

Claim. For every $k = 1, 2, 3$ we have at least one of the following two properties:

$$\mathcal{L}^3(\mathcal{C}_k \cap E_m) \geq \frac{3}{4} \mathcal{L}^3(\mathcal{C}_k), \qquad \text{or} \qquad \frac{1}{T} \int_{I_k} h \, d\mathcal{H}^1 \gtrsim a(\hat{x})^3.$$

In other words, either most of the elements of $\mathcal{C}_k$ belong to the 'epigraph' E_m, or there must occur entropy dissipation of order $a(\hat{x})$.

We first prove the statement assuming the Claim. Since $\mathcal{H}^1(I_k) = 2c\, a(\hat{x})$ for every $k = 1, 2, 3$, then

$$\int_{I(\hat{x}, a(\hat{x}))} (h(x_1) + h(x_2) + h(x_3)) \, d(\mathcal{H}^1)^{\otimes 3} = 2c\, a(\hat{x})^2 \sum_{k=1}^{3} \int_{I_k} h \, d\mathcal{H}^1.$$

In view of the Claim, in order to conclude the proof it is sufficient to check that the first property $\mathcal{L}^3(\mathcal{C}_k \cap E_m) \geq (3/4)\mathcal{L}^3(\mathcal{C}_k)$ cannot be satisfied for all $k = 1, 2, 3$. Assume by contradiction that this is the case: for every $k = 1, 2, 3$, let

$$A_k = \{x \in B_{c_1 a(\hat{x})}(z_0) : \mathcal{L}^1(\{s : (x, s) \in \mathcal{C}_k \cap E_m\}) > 0\}.$$

In particular we have $\mathcal{L}^2(A_k) \geq \frac{3}{4}\mathcal{L}^2(B_{c_1 a(\hat{x})}(z_0))$ so that

$$\mathcal{L}^2(A_1 \cap A_2 \cap A_3) \geq \frac{1}{4}\mathcal{L}^2(B_{c_1 a(\hat{x})}(z_0)) > 0.$$

But the definition of $a(\hat{x})$ implies that $A_1 \cap A_2 \cap A_3 = \emptyset$. Indeed for every triple $(s_1, s_2, s_3) \in \prod_{k=1}^3 [\alpha_k - c_2 a(\hat{x}), \alpha_k + c_2 a(\hat{x})]$, on the one hand the choice of c_2 ensures that there is no $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$ such that $e^{i\alpha} \cdot e^{is_k} > 0$ for every $k = 1, 2, 3$, and on the other hand for $\mathcal{L}^3$-a.e. $(x, s) \in E_m$ we have $m(x) \cdot e^{is} > 0$. So this gives a contradiction.

It remains to prove the Claim. For $k = 1, 2, 3$, we consider the set of curves $G_k \subset \Gamma$ defined as follows. If the direction $e^{i\alpha_k}$ enters Ω, G_k consists of the curves which enter Ω in a way that the 'free characteristic' (i.e. straight line) entering with the same initial direction intersects the cylinder $\mathcal{C}_k$. If the direction $e^{i\alpha_k}$ exits Ω, G_k consists of the curves which exit Ω in a way that the free characteristic exiting with the same final direction intersects the cylinder $\mathcal{C}_k$. Explicitly:

$$G_k = \left\{ \gamma \in \Gamma \colon \exists \bar{t} \in \left[\frac{T}{3}, \frac{2}{3}T\right], \left(\gamma_x(t_\gamma^-) + e^{i\gamma_s(t_\gamma^-)}(\bar{t} - t_\gamma^-), \gamma_s(t_\gamma^-)\right) \in \mathcal{C}_k \right\}$$
$$\text{if } (x_k - z_0) \cdot \tau(x_k) > 0,$$
$$G_k = \left\{ \gamma \in \Gamma \colon \exists \bar{t} \in \left[\frac{T}{3}, \frac{2}{3}T\right], \left(\gamma_x(t_\gamma^+) + e^{i\gamma_s(t_\gamma^+)}(\bar{t} - t_\gamma^+), \gamma_s(t_\gamma^+)\right) \in \mathcal{C}_k \right\}$$
$$\text{if } (x_k - z_0) \cdot \tau(x_k) < 0.$$

Moreover for $\gamma \in G_k$ we denote by $t_{\mathcal{C}_k}(\gamma)$ the time spent by γ in $\mathcal{C}_k$, and by $\tilde{t}_{\mathcal{C}_k}(\gamma)$ the time spent in $\mathcal{C}_k$ by the corresponding (entering or exiting) free characteristic. Explicitly:

$$t_{\mathcal{C}_k}(\gamma) = \mathcal{L}^1 \left(\left\{ t \in \left[\frac{T}{3}, \frac{2}{3}T\right] : \gamma(t) \in \mathcal{C}_k \right\} \right)$$
$$\tilde{t}_{\mathcal{C}_k}(\gamma) = \mathcal{L}^1 \left(\left\{ t \in \left[\frac{T}{3}, \frac{2}{3}T\right], \left(\gamma_x(t_\gamma^-) + e^{i\gamma_s(t_\gamma^-)}(t - t_\gamma^-), \gamma_s(t_\gamma^-)\right) \in \mathcal{C}_k \right\} \right)$$
$$\text{if } (x_k - z_0) \cdot \tau(x_k) > 0,$$
$$\tilde{t}_{\mathcal{C}_k}(\gamma) = \mathcal{L}^1 \left(\left\{ t \in \left[\frac{T}{3}, \frac{2}{3}T\right], \left(\gamma_x(t_\gamma^+) + e^{i\gamma_s(t_\gamma^+)}(\bar{t} - t_\gamma^+), \gamma_s(t_\gamma^+)\right) \in \mathcal{C}_k \right\} \right)$$
$$\text{if } (x_k - z_0) \cdot \tau(x_k) < 0,$$

Since $T \geq 3\pi$, the choices of c_1, c_2 ensure that

$$(1 - \beta)\frac{T}{3}\mathcal{L}^3(\mathcal{C}_k) \leq \int_{G_k} \tilde{t}_{\mathcal{C}_k} d\omega \leq \frac{T}{3}\mathcal{L}^3(\mathcal{C}_k), \tag{30}$$

where $\beta \lesssim \dfrac{\mathcal{H}^1(\{x \in I_k(\hat{x}, a(\hat{x}))\colon m(x) = \tau(x)\})}{\mathcal{H}^1(I_k(\hat{x}, a(\hat{x})))} \lesssim \varepsilon.$

To prove (30), assume without loss of generality that $(x_k - z_0) \cdot \tau(x_k) > 0$. Then, by Fubini theorem, we have

$$\int_{G_k} \tilde{t}_{\mathcal{C}_k} d\omega = \int_{T/3}^{2T/3} \omega(A_t)\, dt,$$

where $A_t = \left\{ \gamma \in \Gamma \colon \left(\gamma_x(t_\gamma^-) + e^{i\gamma_s(t_\gamma^-)}(t - t_\gamma^-), \gamma_s(t_\gamma^-)\right) \in \mathcal{C}_k \right\}.$

Since $T/3 \geq \pi$ and Ω has diameter $< \pi$, for $t \in [T/3, 2T/3]$ any $\gamma \in A_t$ satisfies $t_\gamma^- > 0$, and moreover $\gamma_x(t_\gamma^-) \in I_k$ and $i\tau(\gamma_x(t_\gamma^-)) \cdot e^{i\gamma_s(t_\gamma^-)} > 0$ thanks to the choices of c_1, c_2. Invoking Lemma 3.5 this implies that

$$\omega(A_t) = \int_{\alpha_k - c_2 a(\hat{x})}^{\alpha_k + c_2 a(\hat{x})} \int_{I_k} \int_0^{2T/3} \mathbf{1}_{y+(t-t_{ini})e^{is} \in B_{c_1 a(\hat{x})}(z_0)} \cdot \mathbf{1}_{m(y)\cdot e^{is}>0} (i\tau(y) \cdot e^{is})\, dt_{ini}\, d\mathcal{H}^1(y)\, ds.$$

For any $s \in [\alpha_k - c_2 a(\hat{x}), \alpha_k + c_2 a(\hat{x})]$ the map $(t_{ini}, y) \mapsto y + (t - t_{ini})e^{is}$ is injective, its image contains $B_{c_1 a(\hat{x})}(z_0)$, and its jacobian is $i\tau(y) \cdot e^{is} > 0$, so we deduce

$$\omega(A_t) = \int_{\alpha_k - c_2 a(\hat{x})}^{\alpha_k + c_2 a(\hat{x})} \int_{B_{c_1 a(\hat{x})}(z_0)} \mathbf{1}_{m(x_s(z))\cdot e^{is}>0}\, dz\, ds,$$

where $x_s(z) \in I_k$ is the intersection point of the half-line $z - [0,\infty)\, e^{is}$ with the boundary arc I_k. For $z \in B_{c_1 a(\hat{x})}(z_0)$ and $s \in [\alpha_k - c_1 a(\hat{x}), \alpha_k + c_2 a(\hat{x})]$, recalling that $m \in \{\pm\tau\}$ on $\partial\Omega$, we have that $m(x_s(z)) \cdot e^{is} > 0$ if and only if $m(x_s(z)) = -\tau(x_s(z))$, and we deduce the validity of (30) with

$$\beta = \sup_{z \in B_{c_1 a(\hat{x})}(z_0)} \frac{\mathcal{L}^1(\{s \in [\alpha_k - c_1 a(\hat{x}), \alpha_k + c_2 a(\hat{x})] \colon m(x_s(z)) = \tau(x_s(z))\})}{\mathcal{L}^1([\alpha_k - c_1 a(\hat{x}), \alpha_k + c_2 a(\hat{x})])}.$$

The first inequality on β in the second line of (30) follows from the fact that, for all $z \in B_{c_1 a(\hat{x})}(z_0)$, the map $s \mapsto x_s(z)$ is a diffeomorphism from $[\alpha_k - c_1 a(\hat{x}), \alpha_k + c_2 a(\hat{x})]$ onto its image in I_k, with jacobian bounded from below by $R/2 \geq 1/(2K)$. The second inequality $\beta \lesssim \varepsilon$ in (30) is simply by definition (29) of the set G_ε.

Moreover, since for every $t \in [0,T]$ we have $(e_t)_\# \omega = \mathbf{1}_{E_m} \mathcal{L}^3$, then

$$\int_{G_k} t_{\mathcal{C}_k} d\omega \leq \frac{T}{3} \mathcal{L}^3(\mathcal{C}_k \cap E_m). \tag{31}$$

We now estimate $\tilde{t}_{\mathcal{C}_k} - t_{\mathcal{C}_k}$ in terms of the entropy dissipation from the curves in G_k. Assume without loss of generality that k is such that $(x_k - z_0) \cdot \tau(x_k) > 0$.

Denote by

$$\tilde{\mathcal{C}}_k = \{(x,s) \in \mathcal{C}_k : \operatorname{dist}((x,s), \partial\mathcal{C}_k) > c_4 a(\hat{x})\}$$

for some $c_4 \in (0, \min\{c_1, c_2/2\})$ and by $\tilde{G}_k \subset G_k$ the set of curves γ such that there is $t \in (t_\gamma^-, t_\gamma^+)$ such that $\gamma_x(t_\gamma^-) + e^{i\gamma_s(t_\gamma^-)}(t - t_\gamma^-) \in \tilde{\mathcal{C}}_k$. Finally denote by

$$\tilde{G}_k^* = \{\gamma \in \tilde{G}_k : \tilde{t}_{\mathcal{C}_k}(\gamma) - t_{\mathcal{C}_k}(\gamma) > c_5 a(\hat{x})\}.$$

For every $c_4, c_5 > 0$, by the characteristic constraint (26), there is $c_6 > 0$ such that for all $\gamma \in \tilde{G}_k^*$ it holds $|\mu_\gamma|((0,T)) \geq c_6 a(\hat{x})$.

We write

$$\int_{G_k} (\tilde{t}_{\mathcal{C}_k} - t_{\mathcal{C}_k}) d\omega \leq \int_{G_k \setminus \tilde{G}_k} \tilde{t}_{\mathcal{C}_k} d\omega + \int_{\tilde{G}_k \setminus \tilde{G}_k^*} (\tilde{t}_{\mathcal{C}_k} - t_{\mathcal{C}_k}) d\omega + \int_{\tilde{G}_k^*} \tilde{t}_{\mathcal{C}_k} d\omega, \tag{32}$$

and estimate each integral separately. First, the argument leading to the upper bound in (30) also implies

$$\int_{G_k \setminus \tilde{G}_k} \tilde{t}_{\mathcal{C}_k} d\omega \leq \frac{T}{3} \mathcal{L}^3(\mathcal{C}_k \setminus \tilde{\mathcal{C}}_k) \leq \frac{2T}{3} c_4 \left(\frac{1}{c_1} + \frac{1}{c_2}\right) \mathcal{L}^3(\mathcal{C}_k). \tag{33}$$

Second, by definition of $\tilde{G}_k^*$ we have

$$\int_{\tilde{G}_k \setminus \tilde{G}_k^*} (\tilde{t}_{\mathcal{C}_k} - t_{\mathcal{C}_k}) d\omega \le c_5 a(\hat{x}) \omega(\tilde{G}_k),$$

and since $\tilde{t}_{C_k}(\gamma) \ge \sqrt{c_1 c_4} a(\hat{x})$ for all $\gamma \in \tilde{G}_k$, from (30) we deduce $\sqrt{c_1 c_4} a(\hat{x}) \omega(\tilde{G}_k) \le (T/3)\mathcal{L}^3(\mathcal{C}_k)$, and plugging this into the previous equation yields

$$\int_{\tilde{G}_k \setminus \tilde{G}_k^*} (\tilde{t}_{\mathcal{C}_k} - t_{\mathcal{C}_k}) d\omega \le \frac{T}{3} \frac{c_5}{\sqrt{c_1 c_4}} \mathcal{L}^3(\mathcal{C}_k). \tag{34}$$

Third, by definition of c_6, the third integral in (32) enjoys the estimate

$$\begin{aligned}\int_{\tilde{G}_k^*} \tilde{t}_{\mathcal{C}_k} d\omega &\le \sup \tilde{t}_{\mathcal{C}_k} \, \omega(\tilde{G}_k^*) \le \frac{2c_1 a(\hat{x})}{c_6 a(\hat{x})} \int_{\tilde{G}_k^*} |\mu_\gamma|(0,T) d\omega(\gamma) \\ &\le \frac{2c_1}{c_6} \int_{I_k(\hat{x}, a(\hat{x}))} h \, d\mathcal{H}^1,\end{aligned}$$

so plugging this and (33)-(34) into (32) we infer

$$\begin{aligned}\int_{G_k} (\tilde{t}_{\mathcal{C}_k} - t_{\mathcal{C}_k}) d\omega \le \frac{2T}{3} c_4 \left(\frac{1}{c_1} + \frac{1}{c_2} \right) \mathcal{L}^3(\mathcal{C}_k) + \frac{T}{3} \frac{c_5}{\sqrt{c_1 c_4}} \mathcal{L}^3(\mathcal{C}_k) \\ + \frac{2c_1}{c_6} \int_{I_k(\hat{x}, a(\hat{x}))} h \, d\mathcal{H}^1\end{aligned}$$

We may choose c_4 and c_5 small enough so that

$$\frac{2T}{3} c_4 \left(\frac{1}{c_1} + \frac{1}{c_2} \right) \mathcal{L}^3(\mathcal{C}_k) + \frac{T}{3} \frac{c_5}{\sqrt{c_1 c_4}} \mathcal{L}^3(\mathcal{C}_k) \le \frac{T}{24} \mathcal{L}^3(\mathcal{C}_k)$$

so that by (31) and (30), we deduce

$$\begin{aligned}\frac{T}{3} \mathcal{L}^3(\mathcal{C}_k \cap E_m) &\ge \int_{G_k} \tilde{t}_{\mathcal{C}_k} d\omega - \int_{G_k} (\tilde{t}_{\mathcal{C}_k} - t_{\mathcal{C}_k}) d\omega \\ &\ge \frac{T}{3} \left(\frac{7}{8} - c\varepsilon \right) \mathcal{L}^3(\mathcal{C}_k) - \frac{2c_1}{c_6} \int_{I_k(\hat{x}, a(\hat{x}))} h \, d\mathcal{H}^1,\end{aligned}$$

for some absolute constant $c > 0$. Choosing $\varepsilon = 1/(16c)$ we deduce

$$\mathcal{L}^3(\mathcal{C}_k \cap E_m) \ge \frac{3}{4} \mathcal{L}^3(\mathcal{C}_k) + \frac{1}{16} \mathcal{L}^3(\mathcal{C}_k) - \frac{6c_1}{c_6 T} \int_{I_k(\hat{x}, a(\hat{x}))} h \, d\mathcal{H}^1.$$

This estimate implies the claim: if $\mathcal{L}^3(\mathcal{C}_k \cap E_m) \le \frac{3}{4} \mathcal{L}^3(\mathcal{C}_k)$ then $\int_{I_k(\hat{x}, a(\hat{x}))} h \, d\mathcal{H}^1 \ge (c_6 T/(6c_1)) \mathcal{L}^3(\mathcal{C}_k)/16 = (\pi T c_1 c_2 c_6/48) \, a(\hat{x})^3$. □

Now Proposition 3.1 follows from Lemma 3.6 via a covering argument.

Proof of Proposition 3.1. Denote by L the Lipschitz constant of a from Lemma 2.6. Let $\varepsilon > 0$ be fixed as in Lemma 3.6, and consider the covering

$$\{I(\hat{x}, a(\hat{x})) : \hat{x} \in (E(\eta_0) \cap G_\varepsilon)^3\}$$

of the set $X^* := \{\hat{x} \in (E(\eta_0) \cap G_\varepsilon)^3 \colon a(x) > 0\}$. Since for every $\hat{x} \in X^*$, the diameter of $I(\hat{x}, a(\hat{x}))$ is $\lesssim a(\hat{x})$ and the function a is Lipschitz by Lemma 2.6, we have

$$\int_{I(\hat{x},a(\hat{x}))} a^2 \, d(\mathcal{H}^1)^{\otimes 3} \lesssim a(\hat{x})^2 (\mathcal{H}^1)^{\otimes 3}(I(\hat{x}, a(\hat{x}))) \lesssim a(\hat{x})^5. \tag{35}$$

By Besicovitch covering theorem, there is a subcovering

$$\{I(\hat{x}_i, a(\hat{x}_i)) : i \in I\}$$

of X^* with finite overlap. By (35), Lemma 3.6 and the finite overlap property we obtain

$$\begin{aligned}
\int_{X^*} a^2 \, d(\mathcal{H}^1)^{\otimes 3} &\leq \sum_{i\in I} \int_{I(\hat{x}_i, a(\hat{x}_i))} a^2 d(\mathcal{H}^1)^{\otimes 3} \lesssim \sum_{i \in I} a(\hat{x}_i)^5 \\
&\lesssim \sum_{i\in I} \int_{I(\hat{x}_i, r_+(\hat{x}_i))} (h(x_1) + h(x_2) + h(x_3)) \, d(\mathcal{H}^1)^{\otimes 3} \\
&\lesssim \int_{\partial\Omega^3} (h(x_1) + h(x_2) + h(x_3)) \, d(\mathcal{H}^1)^{\otimes 3} \lesssim \int_{\partial\Omega} h \, d\mathcal{H}^1.
\end{aligned}$$

The definition (28) of h and Proposition 3.3 ensure that $\int_{\partial\Omega} h \, d\mathcal{H}^1 \lesssim \nu(\Omega)$, so we deduce

$$\int_{(E(\eta_0)\setminus G_\varepsilon)^3} a^2 \, d(\mathcal{H}^1)^{\otimes 3} \lesssim \nu(\Omega),$$

which implies, since $0 \leq a \leq \pi$,

$$\int_{E(\eta_0)^3} a^2 \, d(\mathcal{H}^1)^{\otimes 3} \lesssim \nu(\Omega) + \mathcal{H}^1(\partial\Omega \setminus G_\varepsilon).$$

Recalling the definition (29) of G_ε, thanks to the the Hardy-Littlewood inequality the last term is $\lesssim \varepsilon^{-1}\mathcal{H}^1(\{m = \tau\})$, and this concludes the proof of Proposition 3.1. □

4 Compactness argument

In this section we use the characterization of zero-energy states [17] and a compactness argument to 'initialize' our analysis of the previous sections: if $\nu(\Omega)$ is small enough, then Ω must be close to a disk and m close to a vortex.

Lemma 4.1. *For any $K, \varepsilon > 0$, there exists $\delta = \delta(\varepsilon, K) > 0$ with the following property. If Ω is a $C^{1,1}$ simply connected domain with $\mathcal{H}^1(\partial\Omega) = 2\pi$, $\sup_{\partial\Omega} |\kappa| \leq K$ which admits a map m solving* (1) *and* (9) *and its dissipation measure ν defined in* (10) *satisfies $\nu(\Omega) \leq \delta$, then*

$$\mathrm{dist}(\partial\Omega, \partial B_1(x_0)) + \sup_{x\in\partial\Omega} \left| n_{\partial\Omega}(x) - \frac{x - x_0}{|x - x_0|} \right| \leq \varepsilon, \tag{36}$$

for some $x_0 \in \mathbb{R}^2$, and there is $\alpha \in \{\pm 1\}$ such that

$$\int_\Omega \mathbf{1}_{\mathrm{dist}(\cdot,\partial\Omega)\leq \frac{1}{2K}} |m - \alpha\tau_{\partial\Omega} \circ \pi_{\partial\Omega}| \, dx \leq \varepsilon, \tag{37}$$

where $\pi_{\partial\Omega}(x) \in \partial\Omega$ is the nearest-point projection of x onto $\partial\Omega$, well-defined for $\mathrm{dist}(x, \partial\Omega) \leq 1/(2K)$.

Proof of Lemma 4.1. Assume by contradiction that there exists a sequence of C^2 simply connected domains Ω_k such that $\mathcal{H}^1(\partial\Omega_k) = 2\pi$, $\sup_{\partial\Omega_k} |\kappa| \le K$ with maps $m_k : \Omega_k \to \mathbb{S}^1$ satisfying $\nabla \cdot m_k = 0$ in Ω, $m_k \cdot n_{\partial\Omega_k} = 0$ on $\partial\Omega_k$ and $\nu_{m_k}(\Omega_k) = \delta_k \to 0$, such that

$$\inf_{x_0\in\mathbb{R}^2} \left\{ \mathrm{dist}(\partial\Omega_k, \partial B_1(x_0)) + \sup_{x\in\partial\Omega_k} \left| n_{\partial\Omega_k}(x) - \frac{x-x_0}{|x-x_0|} \right| \right\} \ge \varepsilon,$$

or $\pi_{\partial\Omega_k}(x)$ is not well-defined (that is, not unique) for $\mathrm{dist}(x, \partial\Omega_k) \le 1/(2K)$, or

$$\min_{\alpha\in\{\pm1\}} \int_{\Omega_k} \mathbf{1}_{\mathrm{dist}(\cdot,\partial\Omega_k)\le\frac{1}{2K}} |m - \alpha\tau_{\partial\Omega_k} \circ \pi_{\partial\Omega_k}| \, dx \ge \varepsilon.$$

Let $g_k \in C^2(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$ be a counterclockwise arc-length parametrization of $\partial\Omega_k$. Up to a translation we may assume that $\int_{\mathbb{R}/2\pi\mathbb{Z}} g_k(t)\, dt = 0$. Since $|\ddot{g}_k| \le K$ there exists a subsequence (which we don't relabel) such that g_k convergence in $C^1(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$ to a curve $g \in C^{1,1}(\mathbb{R}/2\pi\mathbb{Z}; \mathbb{R}^2)$ with $|\dot{g}| = 1$. The curve g can self-intersect, but not self-cross, so at a multiple point all tangents must be parallel.

Each domain Ω_k contains a disk of radius $\ge 1/K$ [14, Proposition 2.1], so $\mathbb{R}^2 \backslash g(\mathbb{R}/2\pi\mathbb{Z})$ has an open bounded simply connected component containing a disk of radius $\ge 1/K$, which we denote by Ω. Since $\partial\Omega \subset g(\mathbb{R}/2\pi\mathbb{Z})$, the boundary $\partial\Omega$ is C^1 except at multiple points of the C^1 curve g. We distinguish two types of singular points: a singular point $z \in \partial\Omega$ is of type I if there exists $\delta > 0$ such that all connected components ω_δ of $\Omega \cap B_\delta$ are such that $\partial\omega_\delta \cap \partial\Omega$ is C^1, and of type II otherwise. See Figure 2. The rest of the proof is divided in 4 steps.

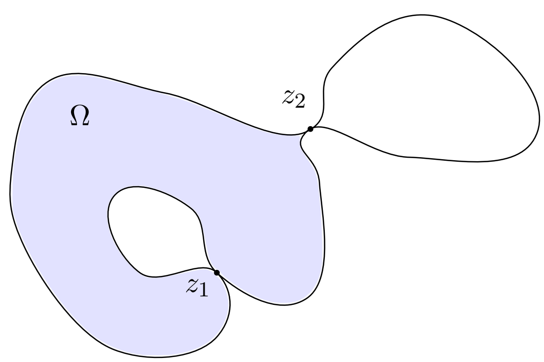

Figure 2: The point z_1 is a type I singularity, while z_2 is a type II singularity.

Step 1. Singular points of type II are isolated in $\partial\Omega$.

Let $z_0 \in \partial\Omega$ a singular point. In particular it is a multiple point: $g^{-1}(\{z_0\})$ contains strictly more than one element. If $t_1 \in g^{-1}(\{z_0\})$, then $g(t) \ne z_0$ for $|t - t_1| < 2\pi/K$ because $|\dot{g}| = 1$ and $|\ddot{g}| \le K$. This implies that $g^{-1}(\{z_0\})$ is finite, $g^{-1}(\{z_0\}) = \{t_1, \ldots, t_N\}$ for some $N \ge 2$. Since g cannot self-cross, we have $\dot{g}(t_j) \in \{\pm\tau\}$ for all $j = 1, \ldots, N$ and one fixed $\tau = \dot{g}(t_1) \in \mathbb{S}^1$.

We consider, for $\delta > 0$ small enough, the open set $g^{-1}(B_\delta(z_0))$. From a Taylor expansion around each t_j, we deduce the existence of $\eta = \eta(K) > 0$ such that for $j = 1, \ldots, N$, the subset $g^{-1}(B_\delta(z_0)) \cap (t_j - \eta, t_j + \eta)$ is an open interval I_δ^j. For small enough $\delta > 0$, the open set $g^{-1}(B_\delta(z_0))$ is exactly the union of these N open intervals. Maybe I would give for granted this proof, but since it is already there, we can also keep it. We prove this by contradiction: otherwise, there would exist a sequence $\delta \to 0$ and t_δ such that

$|g(t_\delta)-z_0|<\delta$ but $t_\delta\notin I_j^\delta$ for any $j=1,\ldots,N$. Extracting a subsequence $t_\delta\to t_*$, we must have $g(t_*)=z_0$, so $t_*\in\{t_1,\ldots,t_N\}$, and therefore $t_\delta\in(t_j-\eta,t_j+\eta)$ for some $j\in\{1,\ldots,N\}$ and all small enough δ. By the above property of η this implies $t_\delta\in I_\delta^j$ and gives a contradiction, so $g^{-1}(B_\delta(z_0))$ is indeed the union of the N intervals I_δ^j.

We choose coordinates (x,y) in which $z_0=0$ and $\tau=e_1$. Then for small enough $\delta>0$ we have

$$g(\mathbb{R}/2\pi\mathbb{Z})\cap B_\delta=\bigcup_{j=1}^N\{y=f_j(x)\}\cap B_\delta,$$

for some C^1 functions $f_1\leq\cdots\leq f_N$ such that $f_j(0)=f_j'(0)=0$ for $j=1,\ldots,N$. We have $z_0=0\in\partial\Omega$, and any $(x_0,y_0)\in B_\delta\cap\Omega$ must satisfy $y_0\notin\{f_1(x_0),\ldots,f_N(x_0)\}$. Moreover, if $f_j(x_0)<y_0<f_{j+1}(x_0)$ for some $j\in\{1,\ldots,N-1\}$, we deduce that $f_j(x)<f_{j+1}(x)$ for all $x\in(0,x_0)$ since Ω is connected. Therefore, possibly taking a smaller value of δ, the connected components of $\Omega\cap B_\delta$ are among the sets

$$\begin{aligned}&\{y<f_1(x)\}\cap B_\delta,\\&\{y>f_N(x)\}\cap B_\delta,\\&\{f_j(x)<y<f_{j+1}(x)\}\cap\{x>0\}\cap B_\delta,\\&\{f_j(x)<y<f_{j+1}(x)\}\cap\{x<0\}\cap B_\delta.\end{aligned}$$

Note that the singular point $z_0=0$ is of type I if and only if the two last types of connected components do not arise. Moreover, if z_0 is of type II, this description of $\Omega\cap B_\delta$ shows that $\partial\Omega\cap B_\delta$ contains no other singular points of type II. This proves Step 1.

Step 2. There exists $m\colon\Omega\to\mathbb{S}^1$ such that $\nabla\cdot m=0$ in Ω, $\nu(\Omega)=0$, with a strong L^1 trace $m_{\lfloor\partial\Omega}$ satisfying $m_{\lfloor\partial\Omega}\cdot n_{\partial\Omega}=0$ a.e. on $\partial\Omega$. More precisely, this makes sense in any C^1 portion of $\partial\Omega$, singular points of type II are negligible by Step 1, and around a singular point of type I, Ω is, in adapted coordinates, locally of the form $\{y<f(x)\}\cup\{y>\tilde f(x)\}$ for some C^1 functions $f\leq\tilde f$ with $f(0)=\tilde f(0)=f'(0)=\tilde f'(0)=0$, and the trace $m_{\partial\Omega}$ may differ from one side to another, but both traces satisfy $m\cdot n_{\partial\Omega}=0$ a.e. for any choice of unit normal $n_{\partial\Omega}$.

For any $z\in\Omega$ and $B_r(z)\subset\Omega$, the sequence $m_{k\lfloor D_r(z)}$ has bounded entropy production and is therefore compact in $L^1(D_r(z))$ [11], so $m_{k\lfloor\Omega}$ is compact in $L^1_{\mathrm{loc}}(\Omega)$. After extracting a subsequence converging in L^1_{loc} and a.e., its limit $m\colon\Omega\to\mathbb{S}^1$ satisfies $\nabla\cdot m=0$ in Ω and $\nabla\cdot\Phi(m)=0$ for any entropy Φ (see Appendix A), so $\nu(\Omega)=0$. This last property implies that m has an L^1 trace along any C^1 portion of $\partial\Omega$, and along both sides of any portion of $\partial\Omega$ around a singular point of type I (see e.g. [17, § 3.2]). It remains to prove that this trace satisfies $m_{\lfloor\partial\Omega}\cdot n_{\partial\Omega}=0$ a.e. on $\partial\Omega$.

To that end consider first a C^1 point $z_0\in\partial\Omega$, and a disk $B_{2r}(z_0)$ such that $\partial\Omega\cap B_{2r}(z_0)=g(I)$ for some open interval I. Possibly choosing a smaller r and adapted coordinates (x,y) in which $z_0=0$ and $g(I)$ is close to horizontal, we write $\Omega\cap B_{2r}$ as a subgraph

$$\Omega\cap B_{2r}=\{y<f(x)\}\cap B_{2r},$$

where $f(0)=f'(0)=0$ and f is C^1. Since $g_k\to g$ in C^1, we can write $g_k(I)$ as a graph $\{y=f_k(x)\}$ for some functions f_k converging to f in C^1, and define

$$\widetilde\Omega_k=\{y<f_k(x)\}\cap B_r.$$

Then we have $\widetilde\Omega_k\subset\Omega_k$ and $B_r\cap\partial\widetilde\Omega_k\subset\partial\Omega_k$, so $m_k\cdot n_{\partial\widetilde\Omega_k}=0$ on $B_r\cap\partial\widetilde\Omega_k\subset\partial\Omega_k$, implying that $\nabla\cdot(m_k\mathbf{1}_{\widetilde\Omega_k})=0$ in B_r. By dominated convergence we have $m_k\mathbf{1}_{\widetilde\Omega_k}\to m\mathbf{1}_{\Omega\cap B_r}$ in

$L^1(B_r)$. We deduce that $\nabla \cdot (m\mathbf{1}_\Omega) = 0$ in B_r, which implies that $m_{\lfloor \partial\Omega} \cdot n_{\partial\Omega} = 0$ a.e. on $\partial\Omega \cap B_r$. This is valid around any C^1 point of $\partial\Omega$. Around a singular point z_0 of type I, the same argument can be applied in both connected components of $\Omega \cap B_{2r}(z_0)$. And singular points of type II are isolated by Step 1, so $m_{\lfloor \partial\Omega} \cdot n_{\partial\Omega} = 0$ a.e. on $\partial\Omega$.

Step 3. There are no singular points of type II.

If $z_0 \in \partial\Omega$ is a singular point of type II, then by the analysis in Step 1 we may choose coordinates (x, y) in which $z_0 = 0$ and there exist $\delta > 0$ and a connected component ω_δ of $\Omega \cap B_\delta$ such that

$$\omega_\delta = \{f_1(x) < y < f_2(x)\} \cap \{x > 0\} \cap B_\delta,$$

where f_1, f_2 are C^1 functions such that $f_1(0) = f_2(0) = f_1'(0) = f_2'(0) = 0$ and $f_1(x) < f_2(x)$ for $x > 0$. For all small $x > 0$, the normal $N_{f_1,x}$ to the graph of f_1 at x intersects the graph of f_2, at a point $(x', f_2(x'))$. There must be at least a value of x at which the normal $N_{f_2,x'}$ to the graph of f_2 at x' is not parallel to $N_{1,x}$: otherwise the distance from $(x, f_1(x))$ to the graph of f_2 would be a constant function of x, contradicting the fact that the two graphs intersect at 0. Therefore, considering x'' slightly larger or smaller than x' (depending on the sign of the angle between the two normals $N_{f_1,x}$ and $N_{f_2,x'}$) we have that $N_{f_2,x''}$ intersects $N_{f_1,x}$ at a point $z_1 \in \omega_\delta$. The proof of [17, Theorem 1.2] implies that, for every point $z \in \partial\Omega$ such that $n_{\partial\Omega}(x)$ is defined and $[z_1, z] \setminus \{z\}$ is contained in Ω, $n_{\partial\Omega}(z)$ must be equal to $(z - z_1)/|z - z_1|$. By a continuation argument, we deduce that the graphs of f_1 and f_2 are, until they meet, arcs of circles centered at z_1, but this contradicts the fact that they have the same tangent at their intersection point. This contradiction proves that $\partial\Omega$ contains no singular points of type II.

Step 4. Conclusion.

If there are no singular points of type I, Ω is C^1 with bounded curvature and [14, Proposition 2.1] ensures the existence of a tangent inscribed disk centered at a focal point: explicitly we have a disk $B_r(z_*) \subset \Omega$ and a tangency point $z_b = g(t_0) \in \partial B_r(z_*) \cap \partial\Omega$, such that for any $\varepsilon > 0$ and $z_\varepsilon = z_* - \varepsilon(z_b - z_*)$, the function $t \mapsto |g(t) - z_\varepsilon|$ doesn't have a local minimum at t_0. In fact it can be checked that the proof of [14, Proposition 2.1] works even if Ω has singular points of type I. (One may also approximate Ω with C^1 domains, apply [14, Proposition 2.1] and pass to the limit.) So we do have a disk $B_r(z_*) \subset \Omega$ and a tangency point $z_b = g(t_0) \in \partial B_r(z_*) \cap \partial\Omega$, such that for any $\varepsilon > 0$ and $z_\varepsilon = z_* - \varepsilon(z_b - z_*)$, the function $t \mapsto |g(t) - z_\varepsilon|$ doesn't have a local minimum at t_0. The derivative of that function cannot be nondecreasing near t_0, and this implies the existence of $t \neq t_0$ arbitrarily close to t_0 such that the normal line to $\partial\Omega$ at $g(t)$ intersects $[z_\varepsilon, z_b]$. Denote by $z_1 \in \Omega$ one such intersection point. Applying again the argument in the proof of [17, Theorem 1.2], we have the following geometric property: for every point $z \in \partial\Omega$ such that $[z_1, z] \setminus \{z\}$ is contained in Ω, $n_{\partial\Omega}(z)$ must be equal to $(z - z_1)/|z - z_1|$. Let $I \subset \mathbb{R}$ be the connected component of t_0 in the open set of all $t \in \mathbb{R}$ such that $[z_1, g(t)] \setminus \{g(t)\}$ is contained in Ω. Thanks to the above, we deduce that $g(I)$ is an arc of circle centered at z_1. Assume I doesn't coincide with $\mathbb{R}$, it means that there is a half-line from z_1 which intersects $\partial\Omega$ for the first time tangentially, but this is impossible by the above geometric property. We conclude that $I = \mathbb{R}$ and $\partial\Omega = g(\mathbb{R}/2\pi\mathbb{Z})$ is a circle $\partial B_R(z_1)$. Since g has length 2π we must have $R = 1$, hence $\Omega = B_1(z_1)$. Further, from [17, Theorem 1.2] there exists $\alpha \in \{\pm 1\}$ such that

$$m(x) = \alpha i \frac{x - z_1}{|x - z_1|} = \alpha \tau_{\partial\Omega} \circ \pi_{\partial\Omega}(x) \qquad \text{for a.e. } x \in \Omega.$$

Therefore, the convergence of g_k to g in $C^1(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^2)$ implies

$$\operatorname{dist}(\partial\Omega_k, \partial B_1(z_1)) + \sup_{x\in\partial\Omega_k} \left| n_{\partial\Omega_k}(x) - \frac{x-z_1}{|x-z_1|} \right| \longrightarrow 0,$$

and also that $\pi_{\partial\Omega_k}(x)$ is well defined for $\operatorname{dist}(x,\partial\Omega_k) \leq 1/(2K)$ and large enough k. Moreover by dominated convergence we have

$$\int_{\Omega_k} \mathbf{1}_{\operatorname{dist}(\cdot,\partial\Omega_k)\leq\frac{1}{2K}} |m_k - \alpha\tau_{\partial\Omega_k}\circ\pi_{\partial\Omega_k}|\, dx \longrightarrow 0.$$

This contradicts the assumptions on (Ω_k, m_k) and concludes the proof of Lemma 4.1. □

5 Trace estimate

The estimate (37) provided by the compactness argument is not enough to handle the trace term in Proposition 3.1. In this section we explain how to strengthen it to a quantitative trace estimate, using the Lagrangian representation introduced in § 3.1.

Proposition 5.1. *Let $\Omega \subset \mathbb{R}^2$ be a $C^{1,1}$ simply connected domain with $\mathcal{H}^1(\partial\Omega) = 2\pi$, $\sup_{\partial\Omega}|\kappa| \leq K$. For any map m solving* (1) *and* (9)*, there is $\alpha \in \{\pm 1\}$ for which*

$$\mathcal{H}^1(\{x \in \partial\Omega : m(x) = -\alpha\tau(x)\}) \leq C\nu(\Omega), \tag{38}$$

where $C > 0$ is a constant depending only on K.

We start by showing a preliminary lemma, which is a more precise version of [27, Lemma 3.1] (see also [28, Lemma 22]). As in [27, Lemma 2.7] we denote by $\Gamma_g \subset \Gamma$ the full measure set of curves γ such that for a.e. $t \in (t_\gamma^-, t_\gamma^+)$ we have that $\gamma_x(t)$ is a Lebesgue point of m with $m(\gamma_x(t)) \cdot e^{i\gamma_s(t)} > 0$.

Lemma 5.2. *Let $r > 0$, $\bar\gamma \in \Gamma_g$ and $\bar t \in (t_{\bar\gamma}^-, t_{\bar\gamma}^+)$ be such that $B_r(\bar\gamma_x(\bar t)) \subset \Omega$ and denote by (t_r^-, t_r^+) the connected component of $\gamma_x^{-1}(B_r(\bar\gamma_x(\bar t)))$ in $(t_{\bar\gamma}^-, t_{\bar\gamma}^+)$ containing $\bar t$. Then there exists an absolute constant $c > 0$ such that for every $\beta \in \left(\operatorname{Osc}_{(t_r^-,t_r^+)}\gamma_s, \pi/4\right)$ at least one of the following holds:*

1. *$\nu(B_r(\bar\gamma_x(\bar t))) \geq c\beta^3 r$;*
2. *$\mathcal{L}^2(\{x \in B_r(\bar\gamma_x(\bar t)) : e^{i\bar\gamma_s(\bar t)} \cdot m(x) \geq -2\beta\}) \geq c\beta r^2$.*

Proof of Lemma 5.2. We let $\bar x = \bar\gamma_x(\bar t)$, $\bar s = \bar\gamma_s(\bar t)$, and $\mathcal{C}_{\bar\gamma}$ be the image curve $\mathcal{C}_{\bar\gamma} = \bar\gamma_x((t_r^-, t_r^+)) \subset \Omega$. For $x = \bar\gamma(t) \in \mathcal{C}_{\bar\gamma}$ we denote by $\tau_{\bar\gamma}(x) = \dot{\bar\gamma}_x(t)$ the unit tangent vector determined by the parametrization $\bar\gamma$. In particular, $\tau_{\bar\gamma}(\bar x) = e^{i\bar s}$.

For $\mathcal{H}^1$-a.e. $x \in \mathcal{C}_{\bar\gamma} \cap B_{r/2}(\bar x)$ we have $m(x) \cdot \tau_{\bar\gamma}(x) \geq 0$, therefore recalling that $\beta \in \left(\operatorname{Osc}_{(t_r^-,t_r^+)}\gamma_s, \pi/4\right)$ one of the following holds:

$$\begin{aligned} & m(x) \cdot e^{is} e^{i\bar s} > 0 \quad \forall s \in \left(\frac{5\beta}{4}, \frac{7\beta}{4}\right), \\ \text{or} \quad & m(x) \cdot e^{is} e^{i\bar s} > 0 \quad \forall s \in \left(-\frac{7\beta}{4}, -\frac{5\beta}{4}\right). \end{aligned}$$

One of these two conditions must be satisfied for at least half the points in $\mathcal{C}_{\bar\gamma} \cap B_{r/2}(\bar x)$, and we assume without loss of generality that

$$\mathcal{H}^1\left(\left\{x \in \mathcal{C}_{\bar\gamma} \cap B_{r/2}(\bar x) : m(x) \cdot e^{is} e^{i\bar s} > 0 \ \forall s \in \left(\frac{5\beta}{4}, \frac{7\beta}{4}\right)\right\}\right) \geq \frac{r}{2}. \tag{39}$$

We define

$$
\begin{aligned}
I(\gamma) &= \{t \in (0,T) : \gamma_x(t) \in B_r(\bar{x})\},\\
I'(\gamma) &= \{t \in I(\gamma) : \gamma_s(t) \in (\bar{s}+\beta, \bar{s}+2\beta)\},
\end{aligned}
$$

and $T(\gamma) = \mathcal{L}^1(I'(\gamma))$. We moreover consider

$$
N(\gamma) = \# \left\{ t \in (0,T) : \gamma_x(t) \in \mathcal{C}_{\bar{\gamma}} \cap B_{r/2}(\bar{x}), \gamma_s(t) \in \left(\bar{s} + \frac{5\beta}{4}, \bar{s} + \frac{7\beta}{4} \right) \right\}. \tag{40}
$$

This cardinal is finite for ω-a.e. $\gamma \in \Gamma$ thanks to [8, Proposition 3.3], and we denote by $t_1(\gamma) < \ldots < t_{N(\gamma)}(\gamma)$ the elements of the above set. We show that for every $\gamma \in \Gamma$ we have

$$
\frac{\mu_\gamma(I(\gamma))}{\beta} + \frac{T(\gamma)}{r} \geq \frac{1}{4} N(\gamma), \tag{41}
$$

where $\mu_\gamma = |D_t \gamma_s| \in \mathcal{M}(t_\gamma^-, t_\gamma^+)$ can be interpreted as the entropy dissipation along γ thanks to Proposition 3.3. It follows from the characteristic equation (26) that for every $i = 1, \ldots N(\gamma)$ there is a neighbourhood I_i of $t_i(\gamma)$ of size at least $r/2$ such that $I_i \subset I(\gamma)$ and at least one of the following holds:

$$
I_i \subset I'(\gamma) \qquad \text{or} \qquad \mu_\gamma(I_i) \geq \frac{\beta}{4}.
$$

The neighborhoods I_i are not necessarily disjoint, but if $i = 1, \ldots N(\gamma) - 1$ is such that $t_{i+1}(\gamma) - t_i(\gamma) < r$, then (26) implies that $[t_i(\gamma), t_{i+1}(\gamma)] \subset I(\gamma)$ and $\mu_\gamma([t_i(\gamma), t_{i+1}(\gamma)]) \geq \beta/4$. This establishes (41).

Next we integrate (41) with respect to ω. From Proposition 3.3 we deduce

$$
\int_\Gamma \mu_\gamma(I(\gamma))\, d\omega \leq \nu(B_r(\bar{x}))T,
$$

and from the Lagrangian property (25) we infer

$$
\begin{aligned}
\int_\Gamma T(\gamma)\, d\omega &\leq T\mathcal{L}^3(\{(x,s) \in B_r(\bar{x}) \times (\bar{s}+\beta, \bar{s}+2\beta) \colon m(x) \cdot e^{is} > 0\})\\
&\leq T\beta \mathcal{L}^2(\{x \in B_r(\bar{x}) \colon m(x) \cdot e^{i\bar{s}} > -2\beta\}).
\end{aligned}
$$

Therefore integrating (41) we obtain

$$
\frac{1}{T} \int_\Gamma N(\gamma)\, d\omega \leq \frac{4}{\beta} \nu(B_r(\bar{x})) + \frac{4\beta}{r} \mathcal{L}^2(\{x \in B_r(\bar{x}) : m(x) \cdot e^{i\bar{s}} > -2\beta\}). \tag{42}
$$

To estimate from below the left-hand side of (42) we use its link with the Lagrangian flux across the curve $\mathcal{C}_{\bar{\gamma}}$. Specifically, for any $f \in C_c^1((0,T) \times \Omega \times \mathbb{R}/2\pi\mathbb{Z})$, we have

$$
\int f(t,x,s) \mathbf{1}_{m(x)\cdot e^{is} > 0} (i\tau_{\bar{\gamma}}(x) \cdot e^{is})\, ds\, d\mathcal{H}^1_{\lfloor \mathcal{C}_{\bar{\gamma}}}(x)\, dt = \int_\Gamma \langle F_\gamma, f \rangle\, d\omega(\gamma),
$$

where F_γ is given by

$$
\begin{aligned}
\langle F_\gamma, f \rangle &= \sum_{t \in X^+} f(t, \gamma_x(t), \gamma_s(t^+)) - \sum_{t \in X^-} f(t, \gamma_x(t), \gamma_s(t^-))\\
X^+ &= \left\{ t \in (t_\gamma^-, t_\gamma^+) \colon \gamma_x(t) \in \mathcal{C}_{\bar{\gamma}},\ i\tau_{\bar{\gamma}}(\gamma_x(t)) \cdot e^{i\gamma_s(t)} > 0 \right\}\\
X^- &= \left\{ t \in (t_\gamma^-, t_\gamma^+) \colon \gamma_x(t) \in \mathcal{C}_{\bar{\gamma}},\ i\tau_{\bar{\gamma}}(\gamma_x(t)) \cdot e^{i\gamma_s(t^-)} < 0 \right\}.
\end{aligned}
$$

The set X^+ corresponds to intersection times of γ with $\mathcal{C}_{\bar\gamma}$ where γ exits $\mathcal{C}_{\bar\gamma}$ in direction of the normal $i\tau_{\bar\gamma}$, and the set X^- to intersection times where γ enters $\mathcal{C}_{\bar\gamma}$ in the opposite direction. Note that these two sets may not be disjoint since γ could 'bounce' on $\mathcal{C}_{\bar\gamma}$. The proof of this flux formula is similar to the proof of Lemma 3.5 for the boundary flux, and details are provided in [8, Theorem 1.4] in a very similar setting. Applying this flux formula to

$$f(t,x,s) \approx \mathbf{1}_{t\in(0,T)}\mathbf{1}_{x\in B_{r/2}(\bar x)}\mathbf{1}_{s\in(\bar s+5\beta/4,\bar s+7\beta/4)},$$

we see that there are no contributions from X^- and obtain

$$\int_\Gamma N(\gamma)d\omega = T\int \mathbf{1}_{m(x)\cdot e^{is}>0}(i\tau(x)\cdot e^{is})\mathbf{1}_{s\in[\bar s+\frac{5\beta}{4},\bar s+\frac{7\beta}{4}]}\, ds\, d\mathcal{H}^1_{\lfloor \mathcal{C}_{\bar\gamma}\cap B_{r/2}(\bar x)}(x).$$

Using also (39) we deduce

$$\frac{1}{T}\int_\Gamma N(\gamma)d\omega \geq \sin\left(\frac{\beta}{4}\right)\frac{\beta}{2}\frac{r}{2} \geq \frac{1}{8\pi}\beta^2 r.$$

Combining this with (42) we get

$$\frac{1}{32\pi}\beta^2 r \leq \frac{1}{\beta}\nu(B_r(\bar x)) + \frac{\beta}{r}\mathcal{L}^2(\{x\in B_r(\bar x) : m(x)\cdot e^{i\bar s} > -2\beta\}).$$

This implies the statement of Lemma 5.2. □

With Lemma 5.2 at hand, we turn to the proof of Proposition 5.1.

Proof of Proposition 5.1. It is sufficient to prove the statement for $\nu(\Omega) < \delta$ for some small δ. We choose α satisfying (37) and we prove that (37) implies (38), provided δ is sufficiently small. Assume without loss of generality that $\alpha = -1$ and let us consider the set of curves

$$G = \Big\{\gamma\in\Gamma\colon 0 < t_\gamma^- < T-1,\ m(\gamma_x(t_\gamma^-)) = \tau(\gamma_x(\tau_\gamma^-)),$$
$$\text{and } e^{i\gamma_s(t_\gamma^-)}\cdot e^{i\frac{\pi}{4}}\tau(\gamma_x(\tau_\gamma^-)) \geq \cos\left(\frac{\pi}{16}\right)\Big\}.$$

By Lemma 3.5 we have

$$\omega(G) \geq \cos\left(\frac{\pi}{4}+\frac{\pi}{16}\right)\frac{\pi(T-1)}{16}\mathcal{H}^1(\{x\in\partial\Omega : m(x)=\tau(x)\}). \tag{43}$$

Claim. If $\nu(\Omega)$ is sufficiently small, then ω-a.e. $\gamma\in G$ satisfies $\mu_\gamma((t_\gamma^-,t_\gamma^+)) \geq \frac{1}{32}$.

The Claim implies the statement since

$$T\nu(\Omega) \geq \int_G \mu_\gamma((t_\gamma^-,t_\gamma^+))d\omega \geq \frac{\omega(G)}{32}$$

and eventually (38) follows by (43).

It remains to prove the Claim. Let $\varepsilon > 0$ small to be chosen later and assume $\nu(\Omega) < \delta$ where $\delta = \min\{\delta', \delta(\varepsilon, K)\}$ where $\delta(\varepsilon, K)$ is provided by Lemma 4.1 and $\delta' > 0$ is chosen later. Assume by contradiction that there is $\bar\gamma \in G\cap\Gamma_g$ such that $\mu_{\bar\gamma}((t_{\bar\gamma}^-,t_{\bar\gamma}^+)) < \frac{1}{32}$. The constraints (36) and (27) imply that $t_{\bar\gamma}^+ - t_{\bar\gamma}^- \geq \frac{1}{K}$. Moreover setting $\bar t = t_{\bar\gamma}^- + 4r$, and $\bar x = \bar\gamma_x(\bar t)$, we have that if we assume $r \leq 1/(100K)$, then $B_r(\bar x)\subset\{x\in\Omega : \mathrm{dist}(x,\partial\Omega) < \frac{1}{2K}\}$. We can therefore apply Lemma 5.2 with $\beta = \frac{1}{32}$ and get that one of the following holds true:

1. $\nu(\Omega) \geq \nu(B_r(\bar{x})) \geq c\beta^3 r$;
2. the set $A = \{x \in B_r(\bar{x}) \colon e^{i\bar{\gamma}_s(\bar{t})} \cdot m(x) \geq -2\beta\}$ satisfies

$$\mathcal{L}^2(A) \geq c\beta r^2. \tag{44}$$

The first case is incompatible with $\nu(\Omega) < \delta'$, provided $\delta' < c\beta^3 r$. Therefore we take $\delta' = \frac{c}{2}\beta^3 r$, with $r \leq 1/(100K)$ to be fixed later.

Let us then consider the second case: we are going to show that (44) is contradicts (37) for ε sufficiently small. First we observe that every $x \in B_r(\bar{x})$ satisfies

$$\operatorname{dist}(\pi_{\partial\Omega}(x), \bar{\gamma}_x(t^-_{\bar{\gamma}})) \leq \tilde{c}(1+K)r$$

for some absolute constant $\tilde{c} > 0$, and therefore

$$\begin{aligned} e^{i\bar{\gamma}_s(\bar{t})} \cdot \tau(\pi_{\partial\Omega}(x)) &\geq e^{i\bar{\gamma}_s(\bar{t})} \cdot \tau(\bar{\gamma}_x(t^-_{\bar{\gamma}})) - |\tau(\bar{\gamma}_x(t^-_{\bar{\gamma}})) - \tau(\pi_{\partial\Omega}(x))| \\ &\geq e^{i\bar{\gamma}_s(\bar{t})} \cdot \tau(\bar{\gamma}_x(t^-_{\bar{\gamma}})) - \tilde{c}K(1+K)r. \end{aligned}$$

Moreover, using that $\gamma \in G$ and $\mu_{\bar{\gamma}}((t^-_{\bar{\gamma}}, t^+_{\bar{\gamma}})) < \frac{1}{32}$ we infer

$$\begin{aligned} e^{i\bar{\gamma}_s(\bar{t})} \cdot \tau(\pi_{\partial\Omega}(x)) &\geq e^{i\bar{\gamma}_s(t^-_{\bar{\gamma}})} \cdot \tau(\bar{\gamma}_x(t^-_{\bar{\gamma}})) - |e^{i\bar{\gamma}_s(t^-_{\bar{\gamma}})} - e^{i\bar{\gamma}_s(\bar{t})}| - \tilde{c}K(1+K)r \\ &\geq \cos\left(\frac{\pi}{16} + \frac{\pi}{4}\right) - \frac{1}{32} - \tilde{c}K(1+K)r \\ &\geq \frac{1}{4} - \tilde{c}K(1+K)r, \end{aligned}$$

for all $x \in B_r(\bar{x})$. Let us choose

$$r = \min\left\{\frac{1}{100K}, \frac{1}{8\tilde{c}K(1+K)}\right\},$$

so that by the above

$$e^{i\bar{\gamma}_s(\bar{t})} \cdot \tau(\pi_{\partial\Omega}(x)) \geq \frac{1}{8} \qquad \forall x \in B_r(\bar{x}).$$

We deduce in particular

$$\begin{aligned} \int_\Omega \mathbf{1}_{\operatorname{dist}(\cdot,\partial\Omega)\leq\frac{1}{2K}} |m + \tau \circ \pi_{\partial\Omega}|\, dx &\geq \int_A (m + \tau \circ \pi_{\partial\Omega}) \cdot e^{i\bar{\gamma}_s(\bar{t})}\, dx \\ &\geq \left(\frac{1}{8} - 2\beta\right) \mathcal{L}^2(A) = \frac{1}{16}\mathcal{L}^2(A) \\ &\geq \frac{c}{16}\beta r^2. \end{aligned}$$

The last inequality follows from (44). Choosing $\varepsilon = c\beta r^2/32$ contradicts (37) and concludes the proof of the Claim and of Proposition 5.1. □

6 Proof of the main results

We collect the results from the previous sections to prove Theorem 1.1 and Corollary 1.4. We moreover prove Corollary 1.5 and Proposition 1.2.

6.1 Proof of Theorem 1.1 and Corollary 1.4

Let m solve (1) and (9). Without loss of generality, we may assume that the constant α provided by the trace estimate Proposition 5.1 is equal to -1, hence

$$\mathcal{H}^1(\{x \in \partial\Omega : m(x) = \tau(x)\}) \leq C\nu(\Omega).$$

Lemma 4.1 ensures that, if $\nu(\Omega)$ is small enough, then $\Omega = E(\eta)$, with $\eta = \min(\eta_0, \eta_*)$, η_0 as in Proposition 2.2 and η_* as in Proposition 3.1. Gathering the results of both said Propositions together with the above trace estimate, we obtain Theorem 1.7 and (13), which imply Theorem 1.1 and Corollary 1.4 as explained in the introduction.

6.2 Proof of Corollary 1.5

In this section we prove (14), which implies Corollary 1.5. We rely on a div-curl argument involving the entropies $\Sigma_1, \Sigma_2 \colon \mathbb{S}^1 \to \mathbb{R}^2$ introduced in [19], given by

$$\Sigma_1(m) = \frac{4}{3}(m_2^3, m_1^3),$$
$$\Sigma_2(m) = \frac{2}{3}\left(-m_1^3 - 3m_1m_2^2, m_2^3 + 3m_2m_1^2\right).$$

Lemma 6.1. *For any $m_1, m_2 \colon \Omega \to \mathbb{S}^1$ with strong L^1 traces on $\partial\Omega$ we have*

$$\|m_1 - m_2\|_{L^4(\Omega)}^3 \leq c_0\|\nabla\cdot\Sigma(m_1) - \nabla\cdot\Sigma(m_2)\|_{\mathcal{M}(\Omega)} + c_0K\|m_1 - m_2\|_{L^1(\partial\Omega)},$$

where $\Sigma = (\Sigma_1, \Sigma_2)$ and $\nabla\cdot\Sigma(m) = (\nabla\cdot\Sigma_1(m), \nabla\cdot\Sigma_2(m))$. The constant $c_0 = c_0(\Omega)$ depends on the norm of the Sobolev embedding $W_0^{1,4}(\Omega) \subset L^\infty(\Omega)$, and on $K = \max_{\partial\Omega}|\kappa|$.

Proof of Lemma 6.1. The proof is inspired by [13]. Let $\chi \in C_c^\infty(\Omega)$ such that $|\chi| \leq 1$, and apply the div-curl estimate of [12, Lemma 4.2] to the vector fields

$$E = \chi(\Sigma_1(m_1) - \Sigma_1(m_2)), \qquad B = \chi(\Sigma_2(m_1) - \Sigma_2(m_2)).$$

for $p = 4$. This yields

$$\begin{aligned}\int E\wedge B &\lesssim \|\chi(\Sigma(m_1) - \Sigma(m_2))\|_{L^4(\Omega)}\|\nabla\cdot(\chi(\Sigma(m_1) - \Sigma(m_2))\|_{W^{-1,4/3}(\Omega)}\\ &\lesssim \|\chi(m_1 - m_2)\|_{L^4(\Omega)}\|\nabla\cdot(\chi(\Sigma(m_1) - \Sigma(m_2))\|_{W^{-1,4/3}(\Omega)}.\end{aligned}$$

we used $|\nabla\Sigma| \lesssim 1$ for the last inequality. Moreover, thanks to [26, Lemma 7 and (92)] we have

$$E\wedge B = \chi^2\det(\Sigma(m_1) - \Sigma(m_2)) \gtrsim \chi^2|m_1 - m_2|^4 \geq \chi^4|m_1 - m_2|^4.$$

Therefore we deduce from the previous inequality that

$$\|\chi(m_1 - m_2)\|_{L^4}^3 \lesssim \|\nabla\cdot(\chi(\Sigma(m_1) - \Sigma(m_2))\|_{W^{-1,4/3}(\Omega)}. \tag{45}$$

Next we compute, for any $\zeta \in W_0^{1,4}(\Omega) \subset C_0(\Omega)$,

$$\begin{aligned}&\langle\nabla\cdot(\chi(\Sigma(m_1) - \Sigma(m_2)), \zeta\rangle\\ &= \int \zeta\nabla\chi\cdot(\Sigma(m_1) - \Sigma(m_2)) + \langle\nabla\cdot\Sigma(m_1) - \nabla\cdot\Sigma(m_2), \chi\zeta\rangle\\ &\lesssim \|\zeta\|_\infty\|\nabla\chi(m_1 - m_2)\|_{L^1} + \|\zeta\|_\infty\|\nabla\cdot\Sigma(m_1) - \nabla\cdot\Sigma(m_2)\|_{\mathcal{M}(\Omega)}\\ &\lesssim c_0(\Omega)\|\nabla\zeta\|_{L^4}\left(\|\nabla\chi(m_1 - m_2)\|_{L^1} + \|\nabla\cdot\Sigma(m_1) - \nabla\cdot\Sigma(m_2)\|_{\mathcal{M}(\Omega)}\right),\end{aligned}$$

where $c_0(\Omega)$ is the norm of the Sobolev embedding $W_0^{1,4}(\Omega) \subset L^\infty(\Omega)$. By definition of $W^{-1,4/3} = (W_0^{1,4})'$ this implies

$$\begin{aligned}
&\|\nabla \cdot (\chi(\Sigma(m_1) - \Sigma(m_2))\|_{W^{-1,4/3}(\Omega)} \\
&\lesssim c_0(\Omega) \left(\|\nabla \chi (m_1 - m_2)\|_{L^1} + \|\nabla \cdot \Sigma(m_1) - \nabla \cdot \Sigma(m_2)\|_{\mathcal{M}(\Omega)} \right).
\end{aligned}$$

Plugging this into (45) we obtain

$$\|\chi(m_1 - m_2)\|_{L^4}^3 \lesssim c_0(\Omega) \left(\|\nabla \chi (m_1 - m_2)\|_{L^1} + \|\nabla \cdot \Sigma(m_1) - \nabla \cdot \Sigma(m_2)\|_{\mathcal{M}(\Omega)} \right).$$

Finally, choosing $\chi = \chi_\varepsilon$ such that $\chi_\varepsilon(x) = 1$ for $\operatorname{dist}(x, \partial\Omega) > \varepsilon$ and $|\nabla \chi| \lesssim K/\varepsilon$ for $\varepsilon \to 0$ and using the trace property of m_1, m_2 we obtain the result. □

Applying Lemma 6.1 to $m_1 = m$ and $m_2 = m_* = i(x - x_*)/|x - x_*|$ we deduce, using Theorem 1.7 to estimate $\|m - m_*\|_{L^1(\partial\Omega)}$,

$$\|m - m_*\|_{L^4(\Omega)}^3 \leq C \, \|\nabla \cdot \Sigma(m)\|_{\mathcal{M}(\Omega)} + C \, \nu(\Omega)^{\frac{1}{2}}.$$

Moreover since Σ_1, Σ_2 are entropies (see Appendix A), the first term in the right-hand side is controlled by $\nu(\Omega)$, and we directly deduce (14). □

6.3 Proof of Proposition 1.2

Given $N \geq 3$, we define Ω_N as the convex hull of the union of the disks $D_{1/2}(e^{2ik\pi/N}/2)$, $k = 0, \ldots, N-1$, rescaled by a factor $1 + \mathcal{O}(1/N^2)$ in order to have perimeter 2π. In other words, Ω_N is obtained from the regular N-gon replacing sharp corners by arcs of circles, see Figure 3. The set Ω_N is $C^{1,1}$ with $\sup_{\partial\Omega_N} |\kappa| \leq 2$.

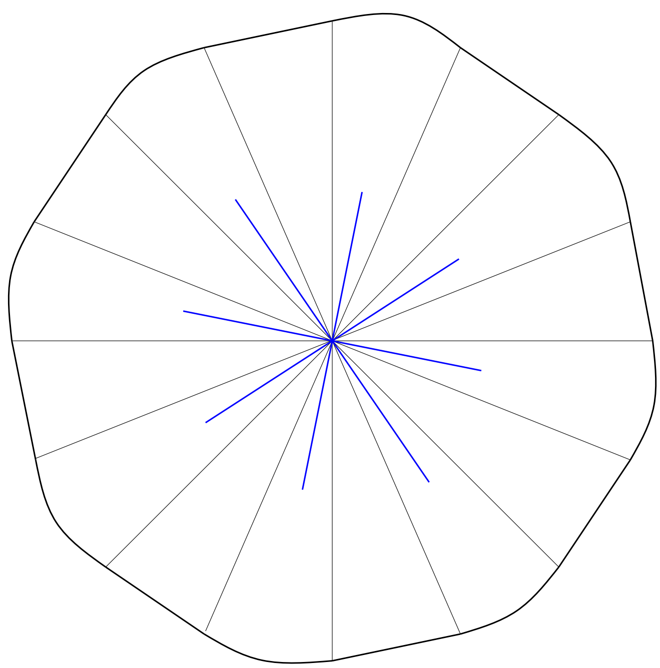

Figure 3: The figure represents Ω_N with $N = 8$. Its boundary is composed by 8 segments and 8 circular arcs. The blue segments form the set where $\nabla \operatorname{dist}(\cdot, \partial\Omega)$ is discontinuous.

The unit normal $n_{\partial\Omega_N}$ is equal to the disk's unit normal $x/|x|$ at the $2N$ points of polar angle $e^{i\pi\ell/N}$ ($\ell = 0, \ldots, 2N-1$), and it differs from it by $\sim 1/N$ in $2N$ boundary arcs of length $\sim 1/N$ away from those points. Therefore we have

$$\int_{\partial\Omega_N} \left| n_{\partial\Omega_N}(x) - \frac{x}{|x|} \right|^2 d\mathcal{H}^1(x) \sim \frac{1}{N^2}. \tag{46}$$

One particular solution m_N of (1) and (9) in Ω_N is given by

$$m_N(x) = i\nabla \operatorname{dist}(\cdot, \partial\Omega_N).$$

This map m_N is BV, its jump set J_N is the union of N segments,

$$J_N = \bigcup_{k=0}^{N-1} [0, x_k], \qquad x_k = \left(\frac{1}{2} + \mathcal{O}\left(\frac{1}{N^2}\right)\right) e^{2ik\pi/N}/2,$$

with jump amplitude

$$|m_N^+ - m_N^-| \lesssim \frac{1}{N} \qquad \text{on } J_N.$$

Replacing the sharp jump along J_N with a well-chosen smooth transition at scale ε, one obtains maps $\tilde{m}_{\varepsilon,N} \to m_N$ as $\varepsilon \to 0$, with

$$\limsup_{\varepsilon\to 0} F_\varepsilon(\tilde{m}_{\varepsilon,N}; \Omega_N) \lesssim \int_{J_N} |m_N^+ - m_N^-|^3 \, d\mathcal{H}^1 \lesssim N \cdot \frac{1}{N^3} = \frac{1}{N^2}. \tag{47}$$

Details of such construction can be found e.g. in [32, 7] for the Aviles-Giga functional E_ε^{AG}, which is enough to obtain an upper bound on F_ε. For the functional E_ε^{RS} a similar construction is performed in [34], and the methods in [32] apply for E_ε^{ARS}. (Note that for our explicit map m_N the technical details of such construction can be significantly simplified because the jump set J_N is particularly simple and stays away from the boundary, and m_N is smooth outside of it.) Combining (46), (47) and (8) we obtain Proposition 1.2. □

Remark 6.2. We cannot prove that the minimizers $m_{\varepsilon,N}$ of $E_\varepsilon^{AG}(\cdot;\Omega_N)$ and $E_\varepsilon^{ARS}(\cdot;\Omega_N)$ converge to m_N as $\varepsilon \to 0$, but from the proof above we have that m_N and the (possibly different and not unique) limit of $m_{\varepsilon,N}$ go to 0 with the same order as $N \to \infty$.

Appendix A Entropy productions, compactness and kinetic formulation

The kinetic formulation (9) is intimately linked to the notion of entropy, also borrowed from conservation laws, and introduced in [11] for the eikonal equation. A smooth map $\Phi\colon \mathbb{S}^1 \to \mathbb{R}^2$ is an entropy for the eikonal equation (1) if and only if it preserves the divergence-free quality of smooth solutions :

$$(|m| = 1 \text{ and } \nabla \cdot m = 0) \quad \Rightarrow \quad \nabla \cdot \Phi(m) = 0,$$

for any open $\Omega \subset \mathbb{R}^2$ and smooth $m\colon \Omega \to \mathbb{R}^2$. Direct calculation shows that this is equivalent to the existence of a smooth function $\lambda_\Phi\colon \mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ such that

$$\frac{d}{d\theta}\Phi(e^{i\theta}) = \lambda_\Phi(\theta) i e^{i\theta} \qquad \forall \theta \in \mathbb{R}.$$

To any $f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z})$ one may associate an entropy Φ_f given by

$$\Phi_f(z) = \int_{\mathbb{R}/2\pi\mathbb{Z}} f(s)\mathbf{1}_{z\cdot e^{is}>0}\, ds \qquad \forall z \in \mathbb{S}^1, \tag{48}$$

and the kinetic formulation (9) is equivalent to

$$\langle \nabla \cdot \Phi_f(m), \zeta \rangle = -\langle \sigma, f'(s)\zeta(x) \rangle \qquad \forall \zeta \in C_c^\infty(\Omega), f \in C^\infty(\mathbb{R}/2\pi\mathbb{Z}). \tag{49}$$

An entropy Φ, whenever extended to $\mathbb{R}^2$ by setting $\widehat{\Phi}(re^{i\theta}) = \eta(r)\Phi(e^{i\theta})$ for some fixed real-valued cut-off function $\eta \in C_c^\infty(0,\infty)$ with $\eta(1) = 1$, satisfies (see e.g. [17, 9])

$$\nabla \cdot \widehat{\Phi}(m) = \Psi(m) \cdot \nabla(1 - |m|^2) + \alpha(m)\nabla \cdot m \qquad \forall m \in H^1(\Omega; \mathbb{R}^2),$$

where $\Psi \colon \mathbb{R}^2 \to \mathbb{R}^2$ and $\alpha \colon \mathbb{R}^2 \to \mathbb{R}$ are such that

$$\|\nabla\Psi\|_{C^1(\mathbb{R}^2)} + \|\nabla\alpha\|_{C^1(\mathbb{R}^2)} \le c\|\lambda_\Phi\|_{C^1(\mathbb{R}/2\pi\mathbb{Z})},$$

for some constant $c > 0$ depending only on the cut-off function η. Applying this to $m'_\varepsilon = (m^1_\varepsilon, m^2_\varepsilon)$ for some sequence $m_\varepsilon \in H^1(\Omega;\mathbb{R}^3)$ with $F_\varepsilon(m_\varepsilon) \le C$, we find

$$\begin{aligned}\nabla \cdot \widehat{\Phi}(m'_\varepsilon) &= \nabla \cdot \left[\Psi(m'_\varepsilon)(1 - |m'_\varepsilon|^2) - \alpha(m'_\varepsilon)H_\varepsilon\right] \\ &\quad - (1 - |m'_\varepsilon|^2)\nabla \cdot \Psi(m'_\varepsilon) - H_\varepsilon \cdot \nabla[\alpha(m'_\varepsilon)] \qquad \text{in } \mathcal{D}'(\Omega),\end{aligned} \tag{50}$$

where $H_\varepsilon \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the curl-free vector field such that $\nabla \cdot H_\varepsilon = -\nabla \cdot (\mathbf{1}_\Omega m'_\varepsilon)$. Boundedness of the energy $F_\varepsilon(m_\varepsilon;\Omega)$ implies that the first line in the right-hand side of (50) tends to 0 in $H^{-1}(\Omega)$, while the second line is bounded in $L^1(\Omega)$. One can then argue exactly as in [11], to deduce that $\nabla \cdot \widehat{\Phi}(m'_\varepsilon)$ is precompact in $H^{-1}(\Omega)$ and that m'_ε is precompact in $L^2(\Omega)$. This gives the precompactness of m_ε since $m^3_\varepsilon \to 0$ in $L^2(\Omega)$. Moreover taking the limit $\varepsilon \to 0$ in (50) along a converging subsequence $m_\varepsilon \to m$, one infers

$$\langle \operatorname{div} \Phi(m), \zeta \rangle \lesssim \|\zeta\|_\infty \|\lambda_\Phi\|_{C^1} \liminf_{\varepsilon\to 0} F_\varepsilon(m_\varepsilon;\Omega) \qquad \forall \zeta \in C_c^\infty(\Omega).$$

Using the arguments of [12, § 3.1] (see [25, Appendix B] for more details), this estimate provides the existence of $\sigma \in \mathcal{M}(\Omega \times \mathbb{R}/2\pi\mathbb{Z})$ satisfying (9).

Appendix B On the sharp lower bound for E_ε^{ARS}

The analysis recalled in Appendix A provides an energy lower bound in terms of the kinetic dissipation measure of the limit map. In the case of E_ε^{ARS} (4) from [1], these arguments can be refined to obtain a sharp lower bound: for any $m = \lim m_\varepsilon$ we have

$$\frac{1}{2}\nu(\Omega) \le \liminf_{\varepsilon\to 0} E_\varepsilon^{ARS}(m_\varepsilon;\Omega), \tag{51}$$

where ν is the minimal kinetic dissipation measure associated to m as defined in (10). Moreover this lower bound is sharp if $m \in BV(\Omega;\mathbb{R}^2)$, in the sense of Γ-convergence: there exists $m_\varepsilon \to m$ such that

$$\limsup_{\varepsilon\to 0} E_\varepsilon^{ARS}(m_\varepsilon;\Omega) \le \frac{1}{2}\nu(\Omega). \tag{52}$$

The sharp lower bound (51) is contained in [1], but not explicitly stated, so we briefly recall here why it is valid. The key step is [1, Lemma 2.2], which ensures the existence of $\tilde{m}_\varepsilon \in W^{1,p}(\Omega;\mathbb{S}^1)$ for $1 \le p < 2$, such that $\tilde{m}_\varepsilon \to m$ and

$$\int_\Omega |\nabla \tilde{m}_\varepsilon| \cdot |\tilde{H}_\varepsilon|\, dx \le E_\varepsilon^{ARS}(m_\varepsilon;\Omega) + o(1). \tag{53}$$

Here $\tilde{H}_\varepsilon \colon \mathbb{R}^2 \to \mathbb{R}^2$ is the curl-free vector field such that $\nabla \cdot \tilde{H}_\varepsilon = -\nabla \cdot (\mathbf{1}_\Omega \tilde{m}_\varepsilon)$. The gain provided by this lemma is that $\tilde{m}_\varepsilon$ takes values into $\mathbb{S}^1$, so one can directly compute entropy productions (without using an extension $\widehat{\Phi}$ as in the previous section). Specifically, for an entropy Φ we have

$$\begin{aligned}\nabla \cdot \Phi(\tilde{m}_\varepsilon) &= \lambda_\Phi(\tilde{m}_\varepsilon) \nabla \cdot \tilde{m}_\varepsilon \\ &= -\nabla \cdot \left[\lambda_\Phi(\tilde{m}_\varepsilon)\tilde{H}_\varepsilon\right] + \lambda'_\Phi(\tilde{m}_\varepsilon) H_\varepsilon \cdot \nabla \tilde{m}_\varepsilon \qquad \text{in } \mathcal{D}'(\Omega).\end{aligned}$$

Here $\lambda_\Phi(e^{i\theta}) = ie^{i\theta} \cdot (d/d\theta)\Phi(e^{i\theta})$ as in Appendix A. As in Appendix A this implies

$$|\nabla \cdot \Phi(m)|(\Omega) \le \|\lambda'_\Phi\|_\infty \liminf_{\varepsilon \to 0} E_\varepsilon^{ARS}(m_\varepsilon; \Omega).$$

This is also the argument in Step 1 of the proof of [1, Theorem 1]. A natural refinement of that argument (see e.g. the proof of [20, Proposition 2]) leads to

$$\left(\bigvee_{\Phi \in S} |\nabla \cdot \Phi(m)| \right)(\Omega) \le \liminf_{n \to \infty} E_\varepsilon^{ARS}(m_\varepsilon; \Omega),$$

where S is any class of entropies Φ with $\|\lambda'_\Phi\|_\infty \le 1$, and $\bigvee$ denotes the lowest upper bound measure of a family of measures [3, Definition 1.68]. Applying this to entropies Φ_f as in (48), which satisfy $\lambda_{\Phi_f}(\theta) = f(\theta + \pi/2) + f(\theta - \pi/2)$, we deduce

$$\left(\bigvee_{|f'| \le 1/2} |\nabla \cdot \Phi_f(m)| \right)(\Omega) \le \liminf_{n \to \infty} E_\varepsilon^{ARS}(m_\varepsilon; \Omega).$$

Recalling (49) and (10), we see that the left-hand side is equal to $(1/2)\nu(\Omega)$, which proves (51).

For a BV map m, we let J_m denote its jump set and $m^\pm$ the traces of m along J_m. Then, the calculations in [27, Corollary 3.4] imply that we have

$$\frac{1}{2}\nu(\Omega) = \int_{J_m} c(|m_+ - m_-|)\, d\mathcal{H}^1,$$
$$\text{where } c(2\sin X) = \begin{cases} 2\,|\sin X - X\cos X| & \text{if } 0 \le X \le \pi/4, \\ 2\left|(X - \pi/2)\cos X - \sin X + \sqrt{2}\right| & \text{if } \pi/4 \le X \le \pi/2. \end{cases}$$

This is exactly the expression of the lower bound in [1, Theorem 1]. Moreover, that lower bound is shown to be optimal in [1, Theorem 2], in the sense that the energy cost $A(X) = c(2\sin X)$ corresponds to the asymptotic energy per unit-length of an ideal wall transition between to limit values $m^\pm$ with $|m^+ - m^-| = 2\sin X$. This implies the Γ-upper bound (52) using e.g. the techniques in [33].

Remark B.1. A closer look at Step 1 in the proof of [1, Theorem 1] reveals that only entropies of the form Φ_{f^σ} are used to obtain the lower bound (51), where $2f^\sigma(s) = g(s-\sigma)$ for any $\sigma \in \mathbb{R}$ and g is π-periodic with $g(s) = \pi/4 - |s - \pi/4|$ for $s \in [-\pi/4, 3\pi/4]$. This should come as no surprise, since, as a consequence of the disintegration of $\sigma_{\min}$ in [27, Corollary 3.4], it can be checked that the identity

$$\frac{1}{2}\nu(\Omega) = \bigvee_{|f'| \le 1/2} |\nabla \cdot \Phi_f(m)| = \bigvee_{\sigma \in \mathbb{R}} |\nabla \cdot \Phi_{f^\sigma}(m)|,$$

is valid for any m satisfying the kinetic formulation (9).

Appendix C Quantitative alternative to the compactness argument under a restrictive trace assumption

In this appendix we prove that, if the integral of a is small enough, then Ω is close enough to a disk. This provides a quantitative proof of the estimate (36) obtained via the compactness argument of Lemma 4.1. We are however not able to prove (37) without a compactness argument, so that this only leads to a quantitative proof of Theorem 1.7 under the additional trace assumption that $m \cdot \tau$ is constant on $\partial\Omega$.

Proposition C.1. *Let Ω as in Theorem 1.1. For any $\eta > 0$ there is $c = c(\eta, K) > 0$ such that if $\int_{E(\eta)^3} a\, d(\mathcal{H}^1)^{\otimes 3} \leq c$, then $E(\eta) = \partial\Omega$.*

The main ingredient to prove Proposition C.1 is the following lower bound on a at one boundary triple, if Ω fails to be close enough to ∂D.

Lemma C.2. *For any $\eta > 0$ there is a constant $a_0 = a_0(\eta, K) > 0$ such that, if $E(\eta) \neq \partial\Omega$ then there exists $\hat{x} \in E(\eta)^3$ with*

$$a(\hat{x}) \geq a_0.$$

Proof. We choose coordinates in which $x_0 = 0$ and consider $\eta \leq \varepsilon_0/K$ for some small absolute constant $\varepsilon_0 > 0$ to be adjusted during the proof: for larger values of η we can then simply take $a_0 = a_0(\varepsilon_0/K, K)$.

We assume that $E(\eta) \neq \partial\Omega$ and prove the existence of $\hat{x}$ satisfying $a(\hat{x}) \geq a_0$ in several steps. During the proof we denote by c_0 a generic small constant that depends only on η and K. We are going to construct a triple $\hat{x} = (x_1, x_2, x_3) \in E_*^3$ and three directions $e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}$ that can be used in the definition of $a(\hat{x})$ to show that $a(\hat{x}) \geq a_0$. We divide this construction in 5 steps.

Step 1. There exists $y \in E(\eta)$ such that

$$\tau(y) \cdot \frac{y}{|y|} > \frac{\eta R}{2\pi}.$$

Pick a tangency point $\bar{x} = g(\bar{s}) \in \partial\Omega \cap \partial B_R$, and let $(s_1, s_2) \subset \mathbb{R}$ denote the largest interval containing $\bar{s}$ and such that $g((s_1, s_2)) \subset E(\eta)$. Since $E(\eta) \neq \partial\Omega$ we know that $s_2 - s_1 < 2\pi$. Moreover, by Lemma 2.5 if $\varepsilon_0 \leq 1/4$ we must have

$$|g(s_1)| = |g(s_2)| = (1 + \eta)R.$$

Consider the function $\psi(s) = |g(s)| - R = \mathrm{dist}(g(s), \partial B_R)$ as in Lemma 2.3. We have $\psi(s_1) = \psi(s_2) = \eta R$ and $\psi(\bar{s}) = 0$, so there must exist $s_* \in (\bar{s}, s_2)$ such that $\psi'(s_*) \geq \eta R/(s_2 - s_1) > \eta R/(2\pi)$. Setting $y = g(s_*)$ and recalling the expression (18) of ψ', we have $\psi'(s_*) = \tau(y) \cdot y/|y| > \eta R/(2\pi)$, proving Step 1.

Step 2. For all angles $|\theta| \leq \eta R/(8\pi^2 K)$, the ray $\{te^{i\theta} y\}_{t>0}$ does not contain any tangency point $x \in \partial\Omega \cap \partial B_R$.

Recall from (18) that $|\psi''| \leq 2K$. As $\psi'(s_*) > \eta R/(2\pi)$ this implies $\psi'(s) > 0$ for all s such that $|s - s_*| \leq \eta R/(4\pi K)$, hence $g(s)$ is not a tangency point. Further, as g is 1-Lipschitz and $|g(s_*)| \leq (1 + \eta)R$ we have $|g(s)| \leq (1 + 2\eta)R$ for all s such that $|s - s_*| \leq \eta R$, which by Lemma 2.5 implies $g((s_* - \eta R, s_* + \eta R)) \subset E(2\eta)$, provided $\varepsilon_0 \leq 1/8$. Since $K \geq 1/R \geq 1$ we deduce that whenever $|s - s_*| \leq \eta R/(4\pi K)$ we have $g(s) \in E(2\eta)$ and the ray $\{tg(s)\}_{t>0}$ does not contain any tangency point.

Let $\theta\colon \mathbb{R} \to \mathbb{R}$ be the C^1 function such that $\theta(s_*) = 0$ and

$$\frac{g(s)}{|g(s)|} = e^{i\theta(s)} \frac{y}{|y|}.$$

It satisfies

$$\theta'(s) = \frac{1}{|g(s)|} \frac{ig(s)}{|g(s)|} \cdot \dot{g}(s).$$

If $|s - s_*| \leq \eta R/(4\pi K)$ we have $ig(s) \cdot \dot{g}(s) \geq 0$ and

$$\frac{ig(s)}{|g(s)|} \cdot \dot{g}(s) = \sqrt{1 - \left(\frac{g(s)}{|g(s)|} \cdot \dot{g}(s)\right)^2} \geq \sqrt{1 - 8K\eta R}.$$

The last inequality follows from Lemma 2.3 and the fact that $g(s) \in E(2\eta)$. Since $R \leq 1$, provided $\varepsilon_0 \leq 1/16$ we deduce $\dot{g}(s) \cdot ig(s)/|g(s)| \geq 1/2$, and therefore

$$\theta'(s) \geq \frac{1}{2|g(s)|} \geq \frac{1}{2\pi},$$

using that $|g(s)| \leq \pi$ as a consequence of $\mathcal{H}^1(\partial\Omega) = 2\pi$. We deduce that

$$\left[-\frac{\eta R}{8\pi^2 K}, \frac{\eta R}{8\pi^2 K}\right] \subset \theta\left(\left[s_* - \frac{\eta R}{4\pi K}, s_* + \frac{\eta R}{4\pi K}\right]\right),$$

Therefore, if $|\theta| \leq \eta R/(8\pi^2 K)$ then there exists s such that $|s - s_*| \leq \eta R/(4\pi K)$ and the ray $\{te^{i\theta} y\}_{t>0}$ coincides with the ray $\{tg(s)\}_{t>0}$, which does not contain any tangency point. This proves Step 2.

Step 3. There exists a tangency point $\tilde{x} \in \partial\Omega \cap \partial B_R$ such that

$$\frac{\tilde{x}}{|\tilde{x}|} = e^{i\tilde{\theta}_0} \frac{y}{|y|} \qquad \text{with } \frac{\eta R}{8\pi^2 K} \leq \tilde{\theta}_0 \leq \pi - \frac{\eta R}{8\pi^2 K}.$$

By maximality of the inscribed disk $B_R \subset \Omega$, the tangency points cannot be all contained in an arc of angle less than π, so there must be at least one tangency point $\tilde{x} \in \partial\Omega \cap \partial B_R$ such that $\tilde{x}/|\tilde{x}| = e^{i\tilde{\theta}_0} y/|y|$ for some $\tilde{\theta}_0 \in [-\eta R/(8\pi^2 K), \pi - \eta R/8\pi^2 K]$. Thanks to Step 2, it must satisfy also $\tilde{\theta}_0 \geq \eta R/(8\pi^2 K)$, proving Step 3.

Step 4. There are constants $c_1, c_2, c_3 > 0$ depending only on K, with the following property. For any tangency point $\tilde{x} \in \partial\Omega \cap \partial B_R$, any $t \in (0, 1/2)$ and $z = t\tilde{x}$, and any $\delta \in (0, 1/(8K))$, there exist $x_1, x_2 \in E(2\delta^2)$ such that

$$\frac{x_1 - z}{|x_1 - z|} = e^{i\theta_1} \frac{\tilde{x}}{|\tilde{x}|} \quad \text{for some } \theta_1 \in (-c_2\delta, -c_1\delta t),$$

$$\frac{x_2 - z}{|x_2 - z|} = e^{i\theta_2} \frac{\tilde{x}}{|\tilde{x}|} \quad \text{for some } \theta_2 \in (c_1\delta t, c_2\delta),$$

$$\tau(x_1) \cdot \frac{x_1 - z}{|x_1 - z|} \leq -c_3\delta t, \qquad \tau(x_2) \cdot \frac{x_2 - z}{|x_2 - z|} \geq c_3\delta t.$$

These will be used in Step 5 as illustrated by Figure 4.

Write $\tilde{x} = g(\tilde{s})$ for some $\tilde{s} \in \mathbb{R}$. The map g is 1-Lipschitz and $|g(\tilde{s})| = R$, so by Lemma 2.5 we have $g(s) \in E_*$ for $|s - \tilde{s}| \leq \delta \leq 1/(4K)$.

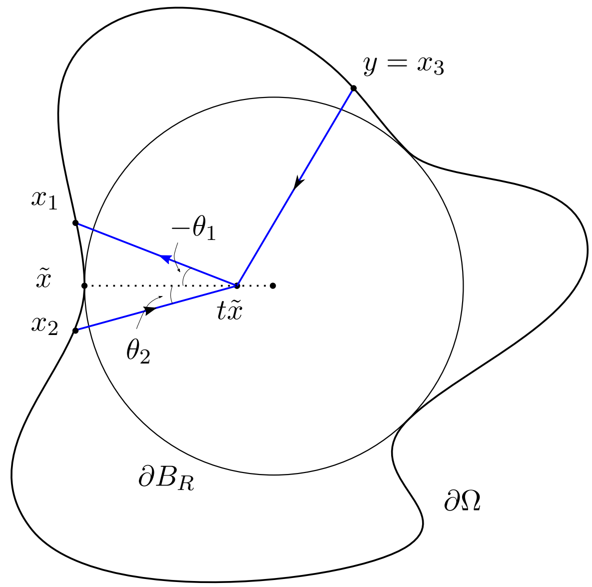

Figure 4: The blue arrows denote the directions $e^{i\alpha_1}, e^{i\alpha_2}, e^{i\alpha_3}$ in Step 5. The idea is that the two directions $e^{i\alpha_1}$ and $e^{i\alpha_2}$ are almost opposite and $e^{i\alpha_3}$ is not close to $e^{i\alpha_1}$ and $e^{i\alpha_2}$ and belongs to the longest of the two intervals with endpoints at $e^{i\alpha_1}$ and $e^{i\alpha_2}$.

Consider the C^1 function $\hat{\theta}\colon \mathbb{R} \to \mathbb{R}$ such that $\hat{\theta}(\tilde{s}) = 0$ and

$$\frac{g(s)}{|g(s)|} = e^{i\hat{\theta}(s)} \frac{\tilde{x}}{|\tilde{x}|}.$$

As in Step 2 we have

$$\hat{\theta}'(s) = \frac{1}{|g(s)|} \frac{ig(s)}{|g(s)|} \cdot \dot{g}(s).$$

Since $|g(\tilde{s})| = R$ and g is 1-Lipschitz, Lemma 2.3 implies

$$\left(\frac{g(s)}{|g(s)|} \cdot \dot{g}(s) \right)^2 \leq 4K|s - \tilde{s}| \leq \frac{1}{2} \qquad \text{for } |s - \tilde{s}| \leq \delta \leq \frac{1}{8K}.$$

We deduce as in Step 2 that $\hat{\theta}'(s) \geq 1/(2\pi)$ for $|s - \tilde{s}| \leq \delta$. Using also $|g| \geq R$ we therefore have

$$\frac{1}{2\pi} \leq \hat{\theta}'(s) \leq \frac{1}{R} \qquad \text{for } |s - \tilde{s}| \leq \delta.$$

On the other hand, setting

$$\varphi(s) = |g(s) - z|^2 = \left| \frac{|g(s)|}{R} e^{i\hat{\theta}(s)} \tilde{x} - t\tilde{x} \right|^2,$$

we have

$$\begin{aligned}
\varphi(s)-\varphi(\tilde{s}) &= \left|\frac{|g(s)|}{R}\tilde{x}-t\tilde{x}-\frac{|g(s)|}{R}(1-e^{i\hat{\theta}(s)})\tilde{x}\right|^2-(R-Rt)^2 \\
&= (|g(s)|-Rt)^2-(R-Rt)^2+|g(s)|^2|1-e^{i\hat{\theta}(s)}|^2 \\
&\quad -2|g(s)|(|g(s)|-Rt)(1-\cos\hat{\theta}(s)) \\
&= (|g(s)|-Rt)^2-(R-Rt)^2+2Rt|g(s)|(1-\cos\hat{\theta}(s)) \\
&\geq 2tR^2(1-\cos\hat{\theta}(s)),
\end{aligned}$$

where the last inequality follows from $|g|\geq R$ and $t\leq 1$. As $\hat{\theta}(\tilde{s})=0$ and $|\hat{\theta}'|\leq 1/R$, for $|s-\tilde{s}|\leq\delta$ we have $|\hat{\theta}(s)|\leq\delta/R\leq\delta K\leq 1/8$, and this implies $1-\cos\hat{\theta}(s)\geq\hat{\theta}(s)^2/4$, so

$$\varphi(s)-\varphi(\tilde{s})\geq\frac{t}{2}R^2\hat{\theta}(s)^2\qquad\text{for } |s-\tilde{s}|\leq\delta.$$

Using moreover that $\hat{\theta}'\geq 1/(2\pi)$ we deduce

$$\varphi(s)-\varphi(\tilde{s})\geq\frac{tR^2}{8\pi^2}(s-\tilde{s})^2\qquad\text{for } |s-\tilde{s}|\leq\delta.$$

Hence there exist $s_1\in(\tilde{s}-\delta,\tilde{s})$, $s_2\in(\tilde{s},\tilde{s}+\delta)$ such that

$$\begin{aligned}
\varphi'(s_1) &= -\frac{1}{\delta}(\varphi(\tilde{s}-\delta)-\varphi(\tilde{s}))\leq-\frac{R^2}{8\pi^2}\delta t, \\
\varphi'(s_2) &= \frac{1}{\delta}(\varphi(\tilde{s}+\delta)-\varphi(\tilde{s}))\geq\frac{R^2}{8\pi^2}\delta t.
\end{aligned}$$

Since $\varphi'(s)=2\tau(g(s))\cdot(g(s)-z)$ and $|g(s)-z|\leq\pi$, setting $x_1=g(s_1)$, $x_2=g(s_2)$ we obtain

$$\begin{aligned}
\tau(x_1)\cdot\frac{x_1-z}{|x_1-z|} &= \frac{\varphi'(s_1)}{2|g(s_1)-z|}\leq -c_3\delta t \\
\tau(x_2)\cdot\frac{x_2-z}{|x_2-z|} &= \frac{\varphi'(s_2)}{2|g(s_2)-z|}\geq c_3\delta t,\qquad c_3=\frac{R^2}{16\pi^3}.
\end{aligned}$$

This proves the last assertion of Step 4. Moreover, since $g'(\tilde{s})=0$, $\delta\in(0,1/(8K)$ and $|\ddot{g}|\leq K$, the points x_1,x_2 lie outside of the disk of radius $1/K$ tangent to $D_R(x_0)$ at $\tilde{x}$, and since $R\geq 1/K$ we infer that they are at distance at most $2K\delta^2$ from $D_R(x_0)$, and thanks to Lemma 2.5 they belong to $E(2K\delta^2/R)\subset E(2\delta^2)$.

It remains to show that

$$\frac{x_j-z}{|x_j-z|}=e^{i\theta_j}\frac{\tilde{x}}{|\tilde{x}|}\quad\text{for } j=1,2,\quad -c_2\delta<\theta_1<-c_1\delta t,\ c_1\delta t<\theta_2<c_2\delta.$$

By definition of $\hat{\theta}$ we know that

$$\frac{x_j}{|x_j|}=e^{i\hat{\theta}(s_j)}\frac{\tilde{x}}{|\tilde{x}|},$$

so we relate $\hat{\theta}(s_j)$ to θ_j and estimate $\hat{\theta}(s_j)$. To do the first, consider, for any fixed $s\in\mathbb{R}$, the C^1 function $\alpha_s\colon[0,1/2]\to\mathbb{R}$ such that $\alpha_s(0)=\hat{\theta}(s)$ and

$$\frac{g(s)-t\tilde{x}}{|g(s)-t\tilde{x}|}=e^{i\alpha_s(t)}\frac{\tilde{x}}{|\tilde{x}|}.$$

That way, we have $\hat\theta(s_j)=\alpha_{s_j}(0)$ and can choose $\theta_j=\alpha_{s_j}(t)$. Moreover we have

$$\alpha_s'(t) = -\frac{i(g(s)-t\tilde x)}{|g(s)-t\tilde x|^2}\cdot\tilde x = \frac{g(s)\cdot(i\tilde x)}{|g(s)-t\tilde x|^2} = \frac{R|g(s)|}{|g(s)-t\tilde x|^2}\sin\hat\theta(s).$$

Note that since $\hat\theta(\tilde s)=0$ and $1/(2\pi)\le\hat\theta'\le 1/R$ we have

$$0<\operatorname{sign}(s-\tilde s)\hat\theta(s)\le\frac{\delta}{R}\le\delta K\le\frac18 \quad \text{for } |s-\tilde s|\le\delta.$$

In particular, using $|g|\le\pi$, $|g-t\tilde x|\ge R/2$ and $|\sin\hat\theta|\le|\hat\theta|$, we deduce

$$0<\operatorname{sign}(s-\tilde s)\alpha_s'(t)\le\frac{4\pi}{R^2}\delta\le\frac{\pi}{2}K \qquad \text{for } |s-\tilde s|\le\delta,$$

hence, recalling $\theta_j-\hat\theta(s_j)=\int_0^t\alpha_{s_j}'$, we infer

$$-(1+\pi/2)K\delta\le\theta_2\le\hat\theta(s_1)<0<\hat\theta(s_2)\le\theta_2\le(1+\pi/2)K\delta.$$

The proof of Step 4 will be complete once we show that $|\hat\theta(s_j)|\ge c_1\delta t$ for $j=1,2$. Because $\varphi'(\tilde s)=0$ and $|\varphi''|\le 2K+2\le 4K$, we must have

$$|s_j-\tilde s|\ge\frac{|\varphi'(s_j)|}{4K}\ge 2\pi c_1 t,\quad c_1=\frac{R^2}{32\pi^2K}.$$

Combining this with $\hat\theta'\ge 1/(2\pi)$ on $[s_1,s_2]$ we deduce that $|\hat\theta(s_j)|\ge c_1\delta t$ and conclude the proof of Step 4.

Step 5. We choose $t\in(0,1/2)$ and $\delta\in(0,1/(8K))$ such that $2\delta^2\le\eta$ and, for the tangency point $\tilde x\in\partial\Omega\cap\partial B_R$ obtained in Step 3 and $z=t\tilde x$, letting $x_1,x_2\in\partial\Omega$ provided by Step 4, and $x_3=y$ provided by Step 1, the three concurring lines from x_1,x_2,x_3 through z can be used to show $a(x_1,x_2,x_3)\ge a_0$. See Figure 4.

Let $\alpha_1,\alpha_2,\alpha_3\in\mathbb{R}/2\pi\mathbb{Z}$ such that

$$\frac{x_1-z}{|x_1-z|}=e^{i\alpha_1},\quad \frac{x_2-z}{|x_2-z|}=-e^{i\alpha_2},\quad \frac{y-z}{|y-z|}=-e^{i\alpha_3}.$$

By definition, the three lines $x_j+e^{i\alpha_j}\mathbb{R}$ are concurrent in $z\in B_{R/2}$. Moreover by Step 4 we have

$$\tau(x_j)\cdot e^{i\alpha_j}\le -c_3\delta t \qquad \text{for } j=1,2.$$

The function $t\mapsto\tau(y)\cdot(y-t\tilde x)/|y-t\tilde x|$ is 2-Lipschitz on $[0,1/2]$ since $|\tilde x|=R$ and $|y-t\tilde x|\ge R/2$ for $t\in[0,1/2]$. Since $\tau(y)\cdot y/|y|>\eta R/(2\pi)$ by Step 1, choosing $t\in(0,\eta R/(8\pi))$ ensures

$$\tau(y)\cdot e^{i\alpha_3}=-\tau(y)\cdot\frac{y-t\tilde x}{|y-t\tilde x|}<-\frac{\eta R}{4\pi}.$$

Recall from Step 3 that we have

$$\frac{\tilde x}{|\tilde x|}=e^{i\tilde\theta_0}\frac{y}{|y|}\qquad\text{with } \frac{\eta R}{8\pi^2K}\le\tilde\theta_0\le\pi-\frac{\eta R}{8\pi^2K}.$$

The C^1 function $\tilde\theta\colon[0,1/2]\to\mathbb{R}$ such that $\tilde\theta(0)=\tilde\theta_0$ and

$$\frac{\tilde x}{|\tilde x|}=e^{i\tilde\theta(t)}\frac{y-t\tilde x}{|y-t\tilde x|},$$

satisfies, arguing as in previous steps, $|\tilde{\theta}'| \le 2$, so choosing

$$t = \frac{\eta R}{32\pi^2 K} \in (0, \eta R/(8\pi)),$$

ensures

$$\frac{\tilde{x}}{|\tilde{x}|} = e^{i\tilde{\theta}_t} \frac{y-z}{|y-z|}, \qquad \text{with } \frac{\eta R}{16\pi^2 K} \le \tilde{\theta}_t \le \pi - \frac{\eta R}{16\pi^2 K}$$

From this identity, the definitions of θ_1, θ_2 in Step 4, and the definitions of $\alpha_1, \alpha_2, \alpha_3$, we obtain

$$\frac{\tilde{x}}{|\tilde{x}|} = e^{i(\alpha_3 + \tilde{\theta}_t - \pi)} = e^{i(\alpha_2 - \theta_2 - \pi)} = e^{i(\alpha_1 - \theta_1)}.$$

So we have, recalling from Step 4 the inequalities satisfied by θ_1, θ_2,

$$\begin{aligned} e^{i\alpha_2} &= e^{i(\pi + \theta_2 - \theta_1)} e^{i\alpha_1}, & \pi + \theta_2 - \theta_1 &\in [\pi + 2c_1\delta t, \pi + 2c_2\delta], \\ e^{i\alpha_3} &= e^{i(\pi - \tilde{\theta}_t - \theta_1)} e^{i\alpha_1}, & \pi - \tilde{\theta}_t - \theta_1 &\in [\eta R/(16\pi^2 K), \pi - \eta R/(16\pi^2 K) + c_2\delta]. \end{aligned}$$

Choosing

$$\delta = \min\left(\frac{\eta R}{32 c_2 \pi^2 K}, \frac{1}{2c_2}, \frac{1}{8K}, \sqrt{\frac{\eta}{2}} \right),$$

this implies that the shortest interval in $\mathbb{R}/2\pi\mathbb{Z}$ containing $\alpha_1, \alpha_2, \alpha_3$ is of length

$$l(\alpha_1, \alpha_2, \alpha_3) \ge \pi + \min\left(2c_1\delta t, \frac{\eta R}{32\pi^2 K} \right).$$

Letting

$$a_0 = \min\left(2c_1\delta t, \frac{\eta R}{32\pi^2 K}, c_3\delta t \right),$$

and gathering the above, we conclude that $a(x_1, x_2, x_3) \ge a_0$. □

The proof of Proposition C.1 will be a combination of Lemma C.2 and of the fact, proven in Lemma 2.6, that a is Lipschitz.

Proof of Proposition C.1. We assume that $E(\eta) \ne \partial\Omega$, and prove that $\int_{E(\eta)^3} a d(\mathcal{H}^1)^{\otimes 3} \ge c$ for some $c = c(\eta, K) > 0$. As $E(\eta/2) \subset E(\eta) \subsetneq \partial\Omega$, applying Lemma C.2 we find $\hat{x} = (x_1, x_2, x_3) \in E(\eta/2)^3$ such that $a(\hat{x}) \ge a_0$, where $a_0 = a_0(\eta, K) > 0$. Let $x_k = g(\bar{s}_k)$ for $k = 1, 2, 3$. Thanks to Lemma 2.5 and the Lipschitz quality of a (Lemma 2.6) we may choose $\delta = \delta(\eta, K) > 0$ such that

$$a \ge \frac{a_0}{2} \qquad \text{on } \prod_{k=1}^{3} C([\bar{s}_k - \delta, \bar{s}_k + \delta]) \subset E(\eta)^3.$$

This implies

$$\int_{E(\eta)^3} a\, d(\mathcal{H}^1)^{\otimes 3} \ge \delta^3 \frac{a_0}{2},$$

concluding the proof of Proposition C.1. □

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