Rectifiability of entropy productions for weak solutions of the 2D eikonal equation with supercritical regularity

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Abstract

Weak solutions $m\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ of the eikonal equation

$$|m| = 1 \text{ a.e. and } \operatorname{div} m = 0\,,$$

arise naturally as sharp interface limits of bounded energy configurations in various physically motivated models, including the Aviles-Giga energy. The distributions $\mu_\Phi = \operatorname{div} \Phi(m)$, defined for a class of smooth vector fields Φ called entropies, carry information about singularities and energy cost. If these entropy productions are Radon measures, a long-standing conjecture predicts that they must be concentrated on the 1-rectifiable jump set of m – as they do if m has bounded variation (BV) thanks to the chain rule. We establish this concentration property, for a large class of entropies, under the Besov regularity assumption

$$m \in B^{1/p}_{p,\infty} \quad \Leftrightarrow \quad \sup_{h\in\mathbb{R}^2\setminus\{0\}} \frac{\|m(\cdot+h)-m\|_{L^p}}{|h|^{1/p}} < \infty\,,$$

for any $1 \leq p < 3$, thus going well beyond the BV setting ($p = 1$) and leaving only the borderline case $p = 3$ open.

1 Introduction

For an open set $\Omega \subset \mathbb{R}^2$, we consider weak solutions $m\colon \Omega \to \mathbb{R}^2$ of the eikonal equation

$$|m| = 1 \text{ a.e. in } \Omega, \quad \operatorname{div} m = 0 \text{ in } \mathcal{D}'(\Omega). \tag{1.1}$$

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If Ω is simply connected, this is equivalent to the existence of a Lipschitz function $u\colon \Omega \to \mathbb{R}$ satisfying $m = i\nabla u$ and

$$|\nabla u| = 1 \text{ a.e. in } \Omega\,,$$

which is classically referred to as the eikonal equation.

If $m\colon \Omega \to \mathbb{R}^2$ is a C^1 solution of the eikonal equation (1.1) then the chain rule provides a whole family of conservation laws: $\operatorname{div}\Phi(m) = 0$ for all $\Phi \in C^1(\mathbb{S}^1;\mathbb{R}^2)$ such that $\partial_\theta \Phi(e^{i\theta}) \cdot e^{i\theta} = 0$ for all $\theta \in \mathbb{R}$.

For a general weak solution $m\colon \Omega \to \mathbb{R}^2$ of the eikonal equation (1.1), the distributions $\operatorname{div}\Phi(m)$ may not be zero, and they carry information on how singular m is. They were first introduced in the context of the eikonal equation in [7], and called entropy productions by analogy with the theory of scalar conservation laws. We denote by

$$\mathrm{ENT} = \left\{\Phi \in C^{1,1}(\mathbb{S}^1;\mathbb{R}^2)\colon \frac{d}{d\theta}\Phi(e^{i\theta}) \cdot e^{i\theta} = 0 \ \forall \theta \in \mathbb{R}\right\}, \tag{1.2}$$

the set of all $C^{1,1}$ entropies.

Weak solutions m of the eikonal equation (1.1) whose entropy productions $\operatorname{div}\Phi(m)$ are finite Radon measures play an important role in the theory of the Aviles-Giga energy [3]. The structure of these finite-entropy solutions is not fully understood, but it is known that they share with functions of bounded variation (BV) several fine properties. Note that, if $m \in BV(\Omega;\mathbb{R}^2)$ is a weak solution of (1.1), then by the BV chain rule its entropy productions are measures concentrated on the $\mathcal{H}^1$-rectifiable jump set J_m. For a general finite-entropy solution, denote by ν the supremum measure

$$\nu = \bigvee_{\Phi \in \mathrm{ENT},\, \|\Phi\|_{C^{1,1}} \le 1} |\operatorname{div}\Phi(m)|\,.$$

In [6] the authors prove that the jump set

$$J_m \doteq \left\{x \in \Omega : \limsup_{r \to 0} \frac{\nu(B_r(x))}{r} > 0\right\}, \tag{1.3}$$

is $\mathcal{H}^1$-rectifiable and m admits left and right L^1 traces $\mathcal{H}^1$-a.e. along J_m. According to a long-standing conjecture on the Aviles-Giga energy [3], entropy productions should be concentrated on that jump set.

Among all weak solutions of (1.1), the finite-entropy solutions can be characterized, at least locally, in terms of Besov $B^{1/3}_{3,\infty}$ regularity [8]. For $s \in (0,1)$ and $p \ge 1$, a map $m \in L^p(\Omega)$ has the Besov regularity $B^s_{p,\infty}$ if and only if the seminorm

$$|m|_{B^s_{p,\infty}} = \sup_{h \in \mathbb{R}^2 \setminus \{0\}} \frac{1}{|h|^s} \|m(\cdot + h) - m\|_{L^p(\Omega \cap (\Omega - h))}\,,$$

is finite [14, §2.5.12]. Between the spaces $BV(\Omega;\mathbb{S}^1)$ and $B^{1/3}_{3,\infty}(\Omega;\mathbb{S}^1)$ lies the intermediate scale of spaces $B^{1/p}_{p,\infty}(\Omega;\mathbb{S}^1)$, for $1 < p < 3$. We prove that the concentration conjecture is true for solutions of (1.1) with that intermediate regularity, and for entropies in the class

$$\widetilde{\mathrm{ENT}} \doteq \left\{ \Phi \in \mathrm{ENT} : \frac{d}{d\theta}\Phi(-e^{i\theta}) = -\frac{d}{d\theta}\Phi(e^{i\theta}) \ \forall \theta \in \mathbb{R} \right\}, \tag{1.4}$$

which corresponds to odd entropies plus constants. This restriction is due to the same technical reasons as in [12] (where this class of entropies is denoted by $\mathcal{E}_\pi$). The fundamental entropies introduced in [10] to establish a sharp lower bound for the Aviles-Giga energy (see also [1]) are odd, and therefore covered by our result.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ a bounded open set, $m\colon \Omega \to \mathbb{R}^2$ a weak solution of the eikonal equation (1.1), and assume that $m \in B^{1/p}_{p,\infty}(\Omega)$ for some $p \in [1,3)$. Then the entropy productions of m are 1-rectifiable,*

$$\operatorname{div} \Phi(m) = \mathbf{n} \cdot \big(\Phi(m^+) - \Phi(m^-)\big)\, \mathcal{H}^1 \llcorner J_m \,,$$

for all $\Phi \in \widetilde{\mathrm{ENT}}$.

To describe the ideas behind Theorem 1.1, let us consider first a solution $m \in \mathrm{BV}(\Omega;\mathbb{S}^1)$ of (1.1). Then, by the BV chain rule we have

$$|\operatorname{div} \Phi(m)| \leq C_\Phi |m^+ - m^-|^3 d\mathcal{H}^1 \llcorner J_m.$$

for every entropy $\Phi \in \mathrm{ENT}$. Moreover, for any $p \in [1,3)$ we have

$$\int_{J_m} |m^+ - m^-|^p d\mathcal{H}^1 \leq \|m\|^p_{B^{1/p}_{p,\infty}} .$$

As a consequence, the contribution of jumps smaller than a threshold $\delta > 0$ is controlled by

$$\begin{aligned} &|\operatorname{div} \Phi(m)|(\Omega \setminus (J_m \cap \{|m^+ - m^-| \geq \delta\}) \\ &\leq C_\Phi \int_{J_m \cap \{|m^+ - m^-| \leq \delta\}} |m^+ - m^-|^3 d\mathcal{H}^1 \leq C_\Phi \delta^{3-p} \|m\|^p_{B^{1/p}_{p,\infty}} . \end{aligned} \tag{1.5}$$

Here we assumed that $m \in BV(\Omega;\mathbb{S}^1)$ to ensure that $\operatorname{div}\Phi(m)$ is concentrated on J_m, but the estimate does not depend on the total variation of m. Moreover, a structure result proved in [12] about the continuous part of the entropy production allows to interpret it as being generated by infinitesimally small jumps. It is therefore natural to conjecture that the estimate (1.5) should be true for solutions m not necessarily of bounded variation. We prove indeed a similar estimate in Proposition 4.1, and the main result then follows by letting $\delta \to 0$.

To technically implement these ideas, we actually have to argue along trajectories of a Lagrangian representation of m, also introduced in [12], and which can only provide information on entropies in (1.4). The control on the continuous part of entropy production at the level of these Lagrangian trajectories is obtained by using a singular family of entropies, whose entropy productions are uniformly bounded thanks to the supercritical Besov regularity assumption.

2 Entropy productions and Besov regularity

In this section we prove that the supercritical Besov regularity assumption provides uniform control on entropy productions over families of entropies which are unbounded in $C^{1,1}$.

Proposition 2.1. *Let $\Omega \subset \mathbb{R}^2$ a bounded open set, $m\colon \Omega \to \mathbb{S}^1$ with $\operatorname{div} m = 0$ and assume that $m \in B^{1/(2+\alpha)}_{2+\alpha,\infty}(\Omega)$ for some $\alpha \in (0,1)$. Then we have*

$$\Big(\bigvee_{\|\Phi\|_{C^{1,\alpha}} \le 1} |\operatorname{div} \Phi(m)| \Big)(\Omega) \le C \|m\|^{2+\alpha}_{B^{1/(2+\alpha)}_{2+\alpha,\infty}(\Omega)} ,$$

for some $C = C(\alpha) > 0$, where the supremum of measure is taken over all $\Phi \in \mathrm{ENT}$ such that $\|\Phi\|_{C^{1,\alpha}} \le 1$.

Proposition 2.1 can be interpreted as an interpolation between the estimates

$$\Big(\bigvee_{\|\Phi\|_{C^{1,1}} \le 1} |\operatorname{div} \Phi(m)| \Big)(\Omega) \lesssim \|m\|^{3}_{B^{1/3}_{3,\infty}(\Omega)} ,$$

$$\Big(\bigvee_{\|\Phi\|_{C^{0,1}} \le 1} |\operatorname{div} \Phi(m)| \Big)(\Omega) \lesssim \|m\|^{2}_{B^{1/2}_{2,\infty}(\Omega)} .$$

The first of these estimates is proved in [8, Proposition 3.10], and the second can be established using similar calculations which rely on commutator estimates for the function

$$w_\epsilon = 1 - |m_\epsilon|^2 = (|m|^2)_\epsilon - |m_\epsilon|^2 ,$$

where the subscript ϵ denotes convolution at scale ϵ. (In the context of the eikonal equation, arguments based on commutator estimates were introduced in [5].) The interpolation argument is however a bit involved. In particular, the constant $C = C(\alpha)$ we are able to obtain in Proposition 2.1 blows up as $\alpha \to 0$ or 1, even though these borderline cases are easier to handle.

The commutator estimates we use in the proof of Proposition 2.1 are as follows.

Lemma 2.2. *Let $\Omega \subset \mathbb{R}^2$ an open set and $m\colon \Omega \to \mathbb{S}^1$. Let $\rho \in C^1_c(B_1)$, $\rho \ge 0$, $\int \rho = 1$, $|\nabla \rho| \le 2$, and define $m_\epsilon = m * \rho_\epsilon$ for $\rho_\epsilon(x) = \epsilon^{-2}\rho(x/\epsilon)$ and $\epsilon > 0$. Then the commutator*

$$w_\epsilon = 1 - |m_\epsilon|^2 = (|m|^2)_\epsilon - |m_\epsilon|^2 ,$$

satisfies

$$|w_\epsilon|(x) \lesssim \fint_{B_\epsilon} |D^h m(x)|^2 \, dh ,$$

$$|D^{\epsilon k} w_\epsilon|(x) \lesssim |k| \fint_{B_{2\epsilon}} |D^h m(x)|^2 \, dh \quad \text{for } |k| \le 1 ,$$

for all $x \in \Omega$ such that $B_{2\epsilon}(x) \subset \Omega$.

Here and in what follows we denote by D^h the finite difference operator

$$D^h m = m^h - m, \quad m^h = m(\cdot + h) \quad \text{for } h \in \mathbb{R}^2 .$$

Proof of Lemma 2.2. As in [5], these commutator estimates come from the commutator identity

$$\begin{aligned} w_\epsilon(x) = 1 - |m_\epsilon(x)|^2 &= \int_{B_\epsilon} |m(x-y) - m_\epsilon(x)|^2 \rho_\epsilon(y)\, dy \\ &= \int_\Omega \left| \int_\Omega \big(m(y) - m(z)\big) \rho_\epsilon(x-z)\, dz \right|^2 \rho_\epsilon(x-y)\, dy , \end{aligned}$$

which follows by integrating

$$\begin{aligned} 1 - |m_\epsilon(x)|^2 &= |m(x-y)|^2 - |m_\epsilon(x)|^2 \\ &= |m(x-y) - m_\epsilon(x)|^2 + 2\langle m_\epsilon(x), m(x-y) - m_\epsilon(x)\rangle , \end{aligned}$$

with respect to $\rho_\epsilon(y)\, dy$. The commutator identity directly implies the first estimate. Applying the finite difference operator, it also implies the identity

$$\begin{aligned} D^{\epsilon k} w_\epsilon(x) &= \int_\Omega \left| \int_\Omega \big(m(y) - m(z)\big) \rho_\epsilon^{\epsilon k}(x-z)\, dz \right|^2 D^{\epsilon k} \rho_\epsilon(x-y)\, dy \\ &\quad + \int_\Omega \Big\langle \int_\Omega \big(m(y) - m(z')\big)(\rho_\epsilon + \rho_\epsilon^{\epsilon k})(x-z')\, dz', \\ &\qquad\qquad \int_\Omega \big(m(y) - m(z)\big) D^{\epsilon k} \rho_\epsilon(x-z)\, dz \Big\rangle \rho_\epsilon(x-y)\, dy \\ &= \int_{B_2} \left| \int_{B_2} \big(m(x-\epsilon y) - m(x-\epsilon z)\big) \rho^k(z)\, dz \right|^2 D^k \rho(y)\, dy \\ &\quad + \int_{B_2} \Big\langle \int_\Omega \big(m(x-\epsilon y) - m(x-\epsilon z')\big)(\rho + \rho^k)(z')\, dz', \\ &\qquad\qquad \int_{B_2} \big(m(x-\epsilon y) - m(x-\epsilon z)\big) D^k \rho(z)\, dz \Big\rangle \rho(y)\, dy , \end{aligned}$$

which then provides the second estimate. □

We will also use the estimate

$$|\nabla m_\epsilon|(x) \lesssim \frac{1}{\epsilon} \fint_{B_\epsilon} |D^h m(x)|\, dh . \tag{2.1}$$

which follows from the identity

$$\nabla m_\epsilon(x) = \frac{1}{\epsilon} \int_{B_1} (m(x-\epsilon y) - m(x)) \nabla \rho(y)\, dy .$$

Proof of Proposition 2.1. Let $\Phi \in \mathrm{ENT}$ such that $\|\Phi\|_{C^{1,\alpha}} \leq 1$, and $\widehat{\Phi}$ a radial extension $\widehat{\Phi}(re^{i\theta}) = \eta(r)\Phi(e^{i\theta})$ for some $\eta \in C_c^2(0,\infty)$ with $\eta(1) = 1$. Then we have

$$\begin{aligned}\operatorname{div}\widehat{\Phi}(m_\epsilon) &= \Psi(m_\epsilon)\cdot\nabla w_\epsilon\,, \quad w_\epsilon = 1 - |m_\epsilon|^2\,,\\ \Psi(re^{i\theta}) &= \frac{1}{2r^2}\eta(r)\lambda(e^{i\theta})\,e^{i\theta} - \frac{1}{2r}\eta'(r)\Phi(e^{i\theta})\,, \quad \lambda = \partial_\theta\Phi\cdot(ie^{i\theta})\,,\end{aligned}$$

and $\|\Psi\|_{C^{0,\alpha}} \lesssim 1$.

Let $U \subset \Omega$ an open subset and $\zeta \in C_c^1(U)$. We fix an intermediate open set Ω' and $\delta \in (0,1)$ such that

$$\operatorname{supp}(\zeta) + B_{4\delta} \subset \Omega' \subset \Omega' + B_{2\delta} \subset U\,,$$

and a cut-off function $\chi \in C_c^1(\Omega)$ such that

$$\mathbf{1}_{\operatorname{supp}(\zeta)} \leq \chi \leq 1 \quad \text{and} \quad \operatorname{supp}(\chi) + B_{2\delta} \subset \Omega'\,.$$

Then, for $0 < \epsilon < \delta$, we write

$$\langle \operatorname{div}\widehat{\Phi}(m_\epsilon), \zeta\rangle + \int_{\mathbb{R}^2} w_\epsilon\chi\Psi(m_\epsilon)\cdot\nabla\zeta\,dx = \int_{\mathbb{R}^2}\chi\Psi(m_\epsilon)\cdot\nabla(\zeta w_\epsilon)\,dx\,. \tag{2.2}$$

Using the Littlewood-Paley characterization of Besov spaces and Hölder's inequality (see Lemma 2.3) we find

$$\left|\int_{\mathbb{R}^2}\chi\Psi(m_\epsilon)\cdot\nabla(\zeta w_\epsilon)\,dx\right| \lesssim \|\chi\Psi(m_\epsilon)\|_{B^\alpha_{p,\infty}}\|\zeta w_\epsilon\|_{B^{1-\alpha}_{q,1}}\,, \tag{2.3}$$

where we choose $p = (2+\alpha)/\alpha$, hence $q = (2+\alpha)/2$. To estimate the two Besov norms in the right-hand side, we come back to their finite difference characterization. We have

$$\begin{aligned}\|D^h[\chi\Psi(m_\epsilon)]\|_{L^p} = \|(D^h\chi)\Psi(m_\epsilon^h) + \chi D^h[\Psi(m_\epsilon)]\|_{L^p(\Omega'\cap(\Omega'-h))}\\ + \|\chi\Psi(m_\epsilon)\|_{L^p(\operatorname{supp}(\chi)\setminus(\Omega'-h))}\\ + \|\chi\Psi(m_\epsilon)\|_{L^p(\operatorname{supp}(\chi)\setminus(\Omega'+h))}\,.\end{aligned}$$

For $0 < |h| < \delta$, the last two terms are zero and we deduce

$$\begin{aligned}\frac{1}{|h|^\alpha}\|D^h[\chi\Psi(m_\epsilon)]\|_{L^p} &\lesssim |\Omega|^{\frac{1}{p}}\|\chi\|_{C^1}\|\Psi\|_\infty + \frac{1}{|h|^\alpha}\|\chi D^h[\Psi(m_\epsilon)]\|_{L^p(\Omega'\cap(\Omega'-h))}\\ &\lesssim |\Omega|^{\frac{1}{p}}\|\chi\|_{C^1}\|\Psi\|_\infty + \frac{1}{|h|^\alpha}\|\Psi\|_{C^{0,\alpha}}\|D^h m_\epsilon\|^\alpha_{L^{\alpha p}(\Omega'\cap(\Omega'-h))}\\ &\lesssim |\Omega|^{\frac{1}{p}}\|\chi\|_{C^1}\|\Psi\|_\infty + \|\nabla m_\epsilon\|^\alpha_{L^{\alpha p}(\Omega'+B_\delta)}\,.\end{aligned}$$

To estimate the last factor we use (2.1) which implies

$$\|\nabla m_\epsilon\|_{L^{\alpha p}(\Omega'+B_\delta)} \lesssim \frac{1}{\epsilon}\fint_{B_\epsilon}\|D^h m\|_{L^{\alpha p}(\Omega'+B_\delta)}\,dh\,.$$

Plugging this into the previous estimate for $0 < |h| < \delta$ and recalling that $p = (2+\alpha)/\alpha$ we obtain

$$
\begin{aligned}
\|\chi\Psi(m_\epsilon)\|_{B^\alpha_{p,\infty}} &\lesssim \|\chi\Psi(m_\epsilon)\|_{L^p} + \sup_{|h|>0} \frac{\|D^h[\chi\Psi(m_\epsilon)]\|_{L^p}}{|h|^\alpha} \\
&\lesssim |\Omega|^{\frac{\alpha}{2+\alpha}} \big(\|\chi\|_{C^1} + \delta^{-\alpha}\big)\|\Psi\|_\infty + \frac{1}{\epsilon^\alpha} \fint_{B_\epsilon} \|D^h m\|^\alpha_{L^{2+\alpha}(\Omega'+B_\delta)}\, dh\,. \qquad (2.4)
\end{aligned}
$$

Now we turn to estimating the second factor in the right-hand side of (2.3). As above, we have

$$
\begin{aligned}
\|D^h(\zeta w_\epsilon)\|_{L^q} &= \|(D^h\zeta) w^h_\epsilon + \zeta D^h w_\epsilon\|_{L^q(\Omega'\cap(\Omega'-h))} \\
&\quad + \|\zeta w_\epsilon\|_{L^q(\mathrm{supp}(\zeta)\setminus(\Omega'-h))} + \|\zeta w_\epsilon\|_{L^q(\mathrm{supp}(\zeta)\setminus(\Omega'+h))}\,.
\end{aligned}
$$

Since $0 < \epsilon < \delta$, the last two terms are zero if $|h| \le \epsilon$. We deduce

$$
\begin{aligned}
\|\zeta w_\epsilon\|_{B^{1-\alpha}_{q,1}} &\lesssim \|\zeta w_\epsilon\|_{L^q} + \int_{\mathbb{R}^2} \frac{\|D^h(\zeta w_\epsilon)\|_{L^q}}{|h|^{1-\alpha}} \frac{dh}{|h|^2} \\
&\lesssim \|\zeta\|_{C^1}\|w_\epsilon\|_{L^q(\Omega')} + \|\zeta\|_\infty \int_{|h|\ge\epsilon} \frac{\|w_\epsilon\|_{L^q(\mathrm{supp}(\zeta))}}{|h|^{1-\alpha}} \frac{dh}{|h|^2} \\
&\quad + \|\zeta\|_\infty \int_{|h|\le\epsilon} \frac{\|D^h w_\epsilon\|_{L^q(\mathrm{supp}(\zeta))}}{|h|^{1-\alpha}} \frac{dh}{|h|^2} \\
&\lesssim \|\zeta\|_{C^1}\|w_\epsilon\|_{L^q(\Omega')} + \frac{\|\zeta\|_\infty}{1-\alpha} \frac{\|w_\epsilon\|_{L^q(\mathrm{supp}(\zeta))}}{\epsilon^{1-\alpha}} \\
&\quad + \frac{\|\zeta\|_\infty}{\epsilon^{1-\alpha}} \int_{|k|\le 1} \frac{\|D^{\epsilon k} w_\epsilon\|_{L^q(\mathrm{supp}(\zeta))}}{|k|^{1-\alpha}} \frac{dk}{|k|^2}\,.
\end{aligned}
$$

Recalling that $q = (2+\alpha)/2$ and using the commutator estimates of Lemma 2.2 we infer

$$
\begin{aligned}
\|\zeta w_\epsilon\|_{B^{1-\alpha}_{q,1}} &\lesssim \|\zeta\|_{C^1} \sup_{|h|\le\epsilon} \|D^h m\|^2_{L^{2+\alpha}(\Omega')} \\
&\quad + \frac{\|\zeta\|_\infty}{1-\alpha} \frac{1}{\epsilon^{1-\alpha}} \fint_{B_\epsilon} \|D^h m\|^2_{L^{2+\alpha}(\Omega')}\, dh \\
&\quad + \frac{\|\zeta\|_\infty}{\epsilon^{1-\alpha}} \frac{1}{\alpha} \fint_{B_{2\epsilon}} \|D^h m\|^2_{L^{2+\alpha}(\Omega')} dh \\
&\lesssim \epsilon^{\frac{2}{2+\alpha}} \|\zeta\|_{C^1} |m|^2_{B^{1/(2+\alpha)}_{2+\alpha,\infty}} \\
&\quad + \frac{\|\zeta\|_\infty}{\alpha(1-\alpha)} \frac{1}{\epsilon^{1-\alpha}} \fint_{B_{2\epsilon}} \|D^h m\|^2_{L^{2+\alpha}(\Omega')} dh\,.
\end{aligned}
$$

Using this and (2.4) to estimate the right-hand side of (2.3), noting that the second term in the left-hand side of (2.2) converges to 0 as $\epsilon \to 0$ (since $w_\epsilon \to 0$ in L^1), and also that

$$
\frac{1}{\epsilon^{1-\alpha}} \sup_{|h|<2\epsilon} \|D^h m\|^2_{L^{2+\alpha}} \lesssim \epsilon^{\frac{2}{2+\alpha}+\alpha-1} |m|^2_{B^{1/(2+\alpha)}_{2+\alpha,\infty}} \to 0\,,
$$

we deduce

$$
\begin{aligned}
&\left|\langle \operatorname{div}\widehat{\Phi}(m_\epsilon),\zeta\rangle\right| - o(1)\\
&\lesssim \frac{\|\zeta\|_\infty}{\alpha(1-\alpha)}\frac{1}{\epsilon}\fint_{B_{2\epsilon}} \|D^h m\|^2_{L^{2+\alpha}(U)}dh \fint_{B_\epsilon} \|D^k m\|^{\alpha}_{L^{2+\alpha}(U)}dk\,,
\end{aligned}
$$

where $o(1)\to 0$ as $\epsilon\to 0$. Using Young's inequality $ab\leq a^p/p+b^q/q$ for $a,b\geq 0$, as well as Jensen's inequality, this implies

$$
\left|\langle \operatorname{div}\widehat{\Phi}(m_\epsilon),\zeta\rangle\right| \lesssim \frac{\|\zeta\|_\infty}{\alpha(1-\alpha)}\frac{1}{\epsilon}\fint_{B_{2\epsilon}}\int_U |D^h m(x)|^{2+\alpha}dx\,dh + o(1)\,.
$$

This is valid for any $\Phi\in$ ENT with $\|\Phi\|_{C^{1,\alpha}}\leq 1$, any open $U\subset\Omega$ and any $\zeta\in C^1_c(U)$. Letting $\epsilon\to 0$ and taking the supremum over functions ζ with $\|\zeta\|_\infty\leq 1$, we deduce

$$
|\operatorname{div}\Phi(m)|(U)\lesssim \frac{1}{\alpha(1-\alpha)}\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\fint_{B_{2\epsilon}}\int_U |D^h m(x)|^{2+\alpha}dx\,dh\,.
$$

Applying this to any finite collection $\Phi_1,\ldots,\Phi_N\in$ ENT with $\|\Phi_j\|_{C^{1,\alpha}}\leq 1$, any disjoint collection of open subsets $U_1,\ldots,U_N\subset V\subset\subset\Omega$, we deduce that

$$
\begin{aligned}
\sum_{j=1}^N |\operatorname{div}\Phi(m)|(U_j) &\lesssim \frac{1}{\alpha(1-\alpha)}\liminf_{\epsilon\to 0}\frac{1}{\epsilon}\fint_{B_{2\epsilon}}\int_V |D^h m(x)|^{2+\alpha}dx\,dh\\
&\lesssim \frac{1}{\alpha(1-\alpha)}\|m\|^{2+\alpha}_{B^{1/(2+\alpha)}_{2+\alpha,\infty}(\Omega)}\,.
\end{aligned}
$$

The desired estimate on the supremum measure follows thanks to the inner regularity of Radon measures. ☐

Lemma 2.3. *For any $p,p'\in[1,\infty]$ such that $1/p+1/p'=1$ and any $\alpha\in(0,1)$ we have*

$$
\left|\int_{\mathbb{R}^2} f\nabla g\,dx\right| \lesssim \|f\|_{B^\alpha_{p,\infty}}\|g\|_{B^{1-\alpha}_{p',1}}\,,
$$

for all $(f,g)\in B^\alpha_{p,\infty}(\mathbb{R}^2)\times B^{1-\alpha}_{p',1}(\mathbb{R}^2)$.

Proof of Lemma 2.3. This inequality essentially amounts to the inclusion of $B^{-\alpha}_{p',1}$ into the dual of $B^\alpha_{p,\infty}$. The elementary proof is basically contained in [14, § 2.11], where the dual of $B^\alpha_{p,q}$ is shown to be equal to $B^{-\alpha}_{p',q'}$ for all $p,q\in(1,\infty)$. Here we have $q=\infty$ and only one inclusion is true, that is why this statement is not stated explicitly there. We provide the proof for the readers' convenience.

It relies on the Littlewood-Paley characterization of Besov spaces, which we start by recalling. We fix a smooth partition of unity $\{\chi_j\}_{j\geq 0}\subset C^\infty_c(\mathbb{R}^2)$ with the properties that

$$\sum_{j=0}^{\infty} \chi_j(x) = 1,$$
$$|\chi_0(\xi)| \le \mathbf{1}_{|\xi|\le 2}\,,$$
$$|\chi_j(\xi)| \le \mathbf{1}_{2^{-j-1}\le|\xi|\le 2^{j+1}} \text{ for } j\ge 1\,,$$
$$\sup_{j\ge 0} 2^{jk} \sup_{\mathbb{R}^2} |\nabla^k \chi_j| < \infty \quad \forall k \ge 0\,.$$

Then, for any $\gamma\in\mathbb{R}$ and $p,q\in[1,\infty]$, the Besov space $B^{\gamma}_{p,q}(\mathbb{R}^2)$ consists of all tempered distributions $\varphi\in\mathcal{S}'(\mathbb{R}^2)$ such that the norm

$$\|\varphi\|_{B^{\gamma}_{p,q}} = \big\|\big(2^{j\gamma}\|\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\|_{L^p}\big)_{j\ge 0}\big\|_{\ell^q}\,,$$

is finite [14, § 2.3.1], where $\mathcal{F}$ denotes the Fourier transform on $\mathcal{S}'(\mathbb{R}^2)$. Moreover, for $\gamma\in(0,1)$, these norms (which depend on the system $\{\chi_j\}$) are equivalent to

$$\|\varphi\|_{B^{\gamma}_{p,q}} = \|\varphi\|_{L^p} + \big\||h|^{-\gamma}\|D^h\varphi\|_{L^p}\big\|_{L^q(dh/h^2)}\,,$$

see e.g. [14, §2.5.12].

To prove the claimed inequality, we use the decomposition

$$\varphi = \sum_{j\ge 0}\mathcal{F}^{-1}\chi_j\mathcal{F}\varphi\,,$$

and the fact that $\chi_j\chi_k\equiv 0$ for $|j-k|\ge 2$, to rewrite the integral as

$$\begin{aligned}
\int_{\mathbb{R}^2} f\nabla g\,dx = \langle f,\nabla g\rangle &= \sum_{j,k\ge 0}\langle \mathcal{F}^{-1}\chi_j\mathcal{F}f, \mathcal{F}^{-1}\chi_k i\xi\mathcal{F}g\rangle\\
&= \sum_{j,k\ge 0}\langle \chi_j\mathcal{F}f, \chi_k i\xi\mathcal{F}g\rangle\\
&= \sum_{r=-1}^{1}\sum_{j\ge 0}\langle \chi_{j+r}\mathcal{F}f, \chi_j i\xi\mathcal{F}g\rangle\\
&= \sum_{r=-1}^{1}\sum_{j\ge 0}\langle \mathcal{F}^{-1}\chi_{j+r}\mathcal{F}f, \mathcal{F}^{-1}\chi_j i\xi\mathcal{F}g\rangle\,.
\end{aligned}$$

Recalling that χ_j is supported in $2^{j-1}\le|\xi|\le 2^{j+1}$ for $j\ge 2$, and applying Hölder's inequality, we infer

$$\begin{aligned}
\left|\int_{R^2} f\nabla g\,dx\right| &\le \sum_{r=-1}^{1}\sum_{j\ge 0}\|\mathcal{F}^{-1}\chi_{j+r}\mathcal{F}f\|_{L^p}\|\mathcal{F}^{-1}\chi_j i\xi\mathcal{F}g\|_{L^{p'}}\\
&\lesssim \sum_{r=-1}^{1}\sum_{j\ge 0}2^j\|\mathcal{F}^{-1}\chi_{j+r}\mathcal{F}f\|_{L^p}\|\mathcal{F}^{-1}\chi_j\mathcal{F}g\|_{L^{p'}}\,.
\end{aligned}$$

The last inequality follows from the properties of $\{\chi_j\}$ and a Fourier multiplier theorem (see e.g. [14, §1.5]). Writing $2^j = 2^{\alpha j} 2^{(1-\alpha)j}$, we deduce

$$\left| \int_{R^2} f \nabla g \, dx \right| \lesssim \sup_{k \geq 0} 2^{\alpha k} \| \mathcal{F}^{-1} \chi_k \mathcal{F} f \|_{L^p} \sum_{j \geq 0} 2^{(1-\alpha)j} \| \mathcal{F}^{-1} \chi_j \mathcal{F} g \|_{L^{p'}} ,$$

which corresponds to the claimed inequality. □

3 Kinetic formulation and Lagrangian representation

In this section we recall the notions of kinetic formulation and of Lagrangian representation introduced in [9, 8] and [12] respectively. We prove moreover some properties which relate the traces on the jump set J_m to the traces of Lagrangian trajectories.

Theorem 3.1. *Let $m \in B^{\frac{1}{3}}_{3,\infty}(\Omega)$ be a solution of* (1.1). *Then there is $\sigma \in \mathcal{M}(\Omega \times \mathbb{T})$ such that*

$$e^{is} \cdot \nabla_x \chi = \partial_s \sigma, \tag{3.1}$$

where

$$\chi(x,s) = \begin{cases} 1 & \textit{if } e^{is} \cdot m(x) > 0, \\ 0 & \textit{otherwise.} \end{cases}$$

Among the measures σ satisfying (3.1)*, there is a unique $\sigma_{\min}$ minimizing $\|\sigma\|$ and it has the following structure:*

$$\sigma_{\min} = \nu_{\min} \otimes (\sigma_{\min})_x, \qquad \textit{where} \quad \nu_{\min} = (p_x)_\sharp |\sigma_{\min}|$$

and for $\nu_{\min}$-a.e. $x \in \Omega \setminus J_m$ it holds

$$(\sigma_{\min})_x = \pm \frac{1}{2} \left(\delta_{\mathfrak{s}(x)} + \delta_{\mathfrak{s}(x)+\pi} \right) \tag{3.2}$$

for some $\mathfrak{s} : \Omega \to \mathbb{T}$ uniquely defined $\nu_{\min} \llcorner J_m^c$-a.e., and for $\nu_{\min}$-a.e. $x \in J_m$ it holds

$$(\sigma_{\min})_x = \mathbf{n} \cdot e^{i\bar{s}} \bar{g}_\beta(s - \bar{s}) \mathcal{L}^1,$$

where $\mathbf{n}(x)$ is the normal to J_m at x, the values $\bar{s} \in \mathbb{T}$, $\beta \in (0, \frac{\pi}{2})$ are uniquely determined by

$$m^+(x) = e^{i(\bar{s}+\beta)}, \quad m^-(x) = e^{i(\bar{s}-\beta)}$$

and $\bar{g}_\beta$ is a Lipschitz function. If moreover $\beta \in (0, \frac{\pi}{4})$, then $\bar{g}_\beta$ is supported on $[-\frac{\pi}{2} - \beta, -\frac{\pi}{2} + \beta] \cup [\frac{\pi}{2} - \beta, \frac{\pi}{2} + \beta]$ and is non-negative.

We refer to [8] for the kinetic formulation (3.1) and to [12] for the structure of the kinetic measure $\sigma_{\min}$, where the explicit expression of $\bar{g}_\beta$ is computed (see also [11]). We observe also that $\bar{g}_\beta$ approaches $\frac{1}{2}\delta_{\pi/2} + \frac{1}{2}\delta_{-\pi/2}$ as $\beta \to 0$ matching (3.2).

For future use, we recall that the kinetic measure $\sigma_{\min}$ encodes the dissipation of the entropies for a large class of entropies: to any $\psi \in C^1(\mathbb{T};\mathbb{R})$ we can associate the entropy

$$\Phi_\psi(z) = \int_{\mathbb{T}} \mathbf{1}_{e^{is}\cdot z>0}\psi(s)e^{is}\, ds\,. \tag{3.3}$$

An entropy Φ belongs to the class $\widetilde{\mathrm{ENT}}$ defined in (1.4) if and only if $\Phi = \Phi_\psi$ for some π-periodic $\psi \in C^1(\mathbb{T};\mathbb{R})$, see [12]. Integrating (3.1) in the s-variable tested with $\psi(s)$, we obtain

$$\operatorname{div}\Phi_\psi(m) = \left(-\int_{\mathbb{T}} \psi'(s)d(\sigma_{\min})_x(s)\right)\nu_{\min} \tag{3.4}$$

We now recall the notion of Lagrangian representation. Given $T > 0$ we let

$$\Gamma = \Big\{(\gamma, t^-_\gamma, t^+_\gamma) \colon 0 \le t^-_\gamma \le t^+_\gamma \le T,\\ \gamma = (\gamma_x, \gamma_s) \in \mathrm{BV}((t^-_\gamma, t^+_\gamma); \Omega \times \mathbb{R}/2\pi\mathbb{Z}), \gamma_x \text{ is Lipschitz}\Big\}.$$

We will always consider the right-continuous representative of the component γ_s and we will write $\gamma(t^-_\gamma)$ instead of $\lim_{t\to t^-_\gamma}\gamma(t)$ and $\gamma(t^+_\gamma)$ instead of $\lim_{t\to t^+_\gamma}\gamma(t)$. For every $t \in (0, T)$ we consider the section

$$\Gamma(t) := \left\{\left(\gamma, t^-_\gamma, t^+_\gamma\right) \in \Gamma : t \in \left(t^-_\gamma, t^+_\gamma\right)\right\}$$

and we denote by

$$\begin{aligned} e_t \colon \quad \Gamma(t) &\longrightarrow \Omega \times \mathbb{R}/2\pi\mathbb{Z} \\ (\gamma, t^-_\gamma, t^+_\gamma) &\longmapsto \gamma(t), \end{aligned}$$

the evaluation map at time t.

Definition 1. *Let Ω be a $C^{1,1}$ open set and m solving* (1.1) *and* (3.1)*. We say that a finite non-negative Radon measure $\omega \in \mathcal{M}(\Gamma)$ is a* Lagrangian representation *of m if the following conditions are satisfied:*

1. *for every $t \in (0, T)$ we have*

$$(e_t)_\sharp\left[\omega \llcorner \Gamma(t)\right] = \mathbf{1}_{E_m}\mathcal{L}^2 \times \mathcal{L}^1, \tag{3.5}$$

where $E_m \subset \Omega \times \mathbb{R}/2\pi\mathbb{Z}$ is the 'epigraph'

$$E_m = \left\{(x, s) \in \Omega \times \mathbb{R}/2\pi\mathbb{Z} \colon m(x)\cdot e^{is} > 0\right\};$$

2. *the measure ω is concentrated on curves $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ solving the characteristic equation:*

$$\dot{\gamma}_x(t) = e^{i\gamma_s(t)} \qquad \text{for a.e. } t \in (t_\gamma^-, t_\gamma^+); \tag{3.6}$$

3. *we have the integral bound*

$$\int_\Gamma \mathrm{TV}_{(0,T)} \gamma_s d\omega(\gamma) < \infty;$$

4. *for ω-a.e. $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ we have*

$$t_\gamma^- > 0 \Rightarrow \gamma_x(t_\gamma^-) \in \partial\Omega, \qquad \text{and} \qquad t_\gamma^+ < T \Rightarrow \gamma_x(t_\gamma^+) \in \partial\Omega. \tag{3.7}$$

Moreover, we say that a Lagrangian representation ω of m is minimal if

$$\int_\Gamma \mathrm{TV}_{(0,T)} \gamma_s d\omega(\gamma) = T|\sigma_{\min}|(\Omega),$$

where $\sigma_{\min}$ is defined in Theorem 3.1.

Given $v \in BV(I; \mathbb{R}^n)$ for some interval $I \subset \mathbb{R}$ we consider the decomposition of the derivative

$$Dv = \tilde{D}v + D^j v,$$

where $\tilde{D}v$ is the sum of the absolutely continuous and Cantor part of the measure Dv and $D^j v$ is the jump part of Dv, see for example [2].

Theorem 3.2 ([13, 12])**.** *Let Ω be a $C^{1,1}$ open set and $m \in B^{1/3}_{3,\infty}(\Omega)$ solving (1.1) and (3.1). Then there is a minimal Lagrangian representation ω of m. Moreover the following equalities between measures hold:*

$$\begin{aligned} \mathcal{L}^1 \llcorner [0,T] \times \sigma_{\min} &= -\int_\Gamma \sigma_\gamma d\omega(\gamma), \\ \mathcal{L}^1 \llcorner [0,T] \times |\sigma_{\min}| &= \int_\Gamma |\sigma_\gamma| d\omega(\gamma), \end{aligned} \tag{3.8}$$

where

$$\sigma_\gamma = (\mathrm{Id}, \gamma)_\sharp \tilde{D}_t \gamma_s + \mathcal{H}^1 \llcorner E_\gamma^+ - \mathcal{H}^1 \llcorner E_\gamma^- \tag{3.9}$$

and

$$\begin{aligned} E_\gamma^+ &:= \{(t,x,s) \in (0,T) \times \Omega \times \mathbb{T} : \\ &\qquad \gamma_x(t) = x \text{ and } \gamma_s(t-) \le s \le \gamma_s(t+) \le \gamma_s(t-) + \pi\}, \\ E_\gamma^- &:= \{(t,x,s) \in (0,T) \times \Omega \times \mathbb{T} : \\ &\qquad \gamma_x(t) = x \text{ and } \gamma_s(t+) \le s \le \gamma_s(t-) < \gamma_s(t+) + \pi\}. \end{aligned} \tag{3.10}$$

Accordingly, for any $\psi \in C^1(\mathbb{T})$ we can disintegrate the entropy production of Φ_ψ defined in (3.3) *along the Lagrangian curves:*

$$\langle \operatorname{div} \Phi_\psi(m), \zeta \rangle = -\frac{1}{T} \int_\Gamma \int_{I_\gamma} \zeta(\gamma_x(t)) D(\psi \circ \gamma_s)(dt)\, d\omega(\gamma)\,, \tag{3.11}$$

for any $\zeta \in C^1(\Omega)$.

In general it is not true that for any $\zeta \in C^1(\Omega)$ and any $\psi \in C^1(\mathbb{T})$ it holds

$$\langle |\operatorname{div} \Phi_\psi(m)|, \zeta \rangle = \frac{1}{T} \int_\Gamma \int_{I_\gamma} \zeta(\gamma_x(t)) |D(\psi \circ \gamma_s)|(dt)\, d\omega(\gamma). \tag{3.12}$$

Before discussing the validity of a weaker version of the above formula, we prove a result stating that the decomposition in jump part and continuous part of $\nu_{\min}$ is compatible with the corresponding decomposition of Lagrangian trajectories.

Lemma 3.3. *Let ω be a minimal Lagrangian representation of $m \in B^{1/3}_{3,\infty}(\Omega)$ solving* (1.1)*. Then*

$$\nu_{\min} \llcorner J_m = \frac{1}{T} \int_\Gamma (\gamma_x)_\sharp |D^j \gamma_s| d\omega(\gamma),$$
$$\nu_{\min} \llcorner (\Omega \setminus J_m) = \frac{1}{T} \int_\Gamma (\gamma_x)_\sharp |\tilde{D}\gamma_s| d\omega(\gamma).$$

Proof. It is sufficient to prove that for ω-a.e. γ the following holds:

$$|D^j \gamma_s|(\{\gamma_x \in \Omega \setminus J_m\}) = 0, \qquad |\tilde{D}\gamma_s|(\{\gamma_x \in J_m\}) = 0.$$

The first equality follows from [12, Lemma 3.5] which states that for ω-a.e. $\gamma \in \Gamma$ the following holds: for every $t \in I_\gamma$ such that $\gamma_s(t+) \neq \gamma_s(t-)$, we have $\gamma_x(t) \in J_m$. The second equality follows from the fact that for ω-a.e. $\gamma \in \Gamma$ the set $\{\gamma_x \in J_m\}$ is at most countable, since J_m is countably 1-rectifiable and $\{\gamma_x \in \Sigma\}$ is finite for any Lipschitz curve Σ, see the proof of [4, Lemma 3.4] which adapts directly to our setting. □

We now prove that (3.12) holds outside the jump set.

Lemma 3.4. *Let $m \in B^{1/3}_{3,\infty}(\Omega)$ solving* (1.1) *and* (3.1) *and ω be a minimal Lagrangian representation of m. Then for every π-periodic $\psi \in C^1(\mathbb{T})$ and any $A \subset \Omega \setminus J_m$ it holds*

$$[\operatorname{div} \Phi_\psi(m)]_+(A) = \frac{1}{T} \int_\Gamma [D(\psi \circ \gamma_s)]_-(\gamma_x \in A)\, d\omega(\gamma),$$
$$[\operatorname{div} \Phi_\psi(m)]_-(A) = \frac{1}{T} \int_\Gamma [D(\psi \circ \gamma_s)]_+(\gamma_x \in A)\, d\omega(\gamma).$$

In particular

$$|\operatorname{div} \Phi_\psi(m)|(A) = \frac{1}{T} \int_\Gamma |D(\psi \circ \gamma_s)|(\gamma_x \in A) d\omega(\gamma) \qquad \text{for any } A \subset \Omega \setminus J_m.$$

Proof. Thanks to Theorem 3.1, the measure $(\sigma_{\min})_x$ has a definite sign on $\mathbb{T}$ for $\nu_{\min}$-a.e. $x \in J_m^c$. More precisely, we may fix two disjoint Borel sets $\tilde\Omega^\pm \subset \Omega\backslash J_m$ such that

$$
\begin{aligned}
&\nu_{\min}\big(\Omega \setminus (J_m \cup \tilde\Omega^+ \cup \tilde\Omega^-)\big) = 0\,,\\
&(\sigma_{\min})_x = \frac{1}{2}\left(\delta_{\mathfrak{s}(x)} + \delta_{\mathfrak{s}(x)+\pi}\right) \geq 0 \text{ on } \mathbb{T} \text{ for all } x \in \tilde\Omega^+\,,\\
&(\sigma_{\min})_x = -\frac{1}{2}\left(\delta_{\mathfrak{s}(x)} + \delta_{\mathfrak{s}(x)+\pi}\right) \leq 0 \text{ on } \mathbb{T} \text{ for all } x \in \tilde\Omega^-\,.
\end{aligned}
$$

Combining this with (3.4) and the fact that ψ is π-periodic, we see that

$$
\operatorname{div}\Phi_\psi(m)\llcorner(\Omega \setminus J_m) = \big(\mathbf{1}_{\tilde\Omega^-} - \mathbf{1}_{\tilde\Omega^+}\big)\big(\psi' \circ \mathfrak{s}\big)\,\nu_{\min}\llcorner(\Omega \setminus J_m)\,,
$$

hence the positive, resp. negative, part of the measure $\operatorname{div}\Phi_\psi(m)$ outside J_m is concentrated on the set $\tilde A^+$, resp. $\tilde A^-$, where

$$
\begin{aligned}
\tilde A^+ &= \big\{(\mathbf{1}_{\tilde\Omega^-} - \mathbf{1}_{\tilde\Omega^+})(\psi' \circ \mathfrak{s}) > 0\big\}\,,\\
\tilde A^- &= \big\{(\mathbf{1}_{\tilde\Omega^-} - \mathbf{1}_{\tilde\Omega^+})(\psi' \circ \mathfrak{s}) < 0\big\}\,.
\end{aligned}
$$

Moreover, thanks to (3.11) and Lemma 3.3, for any Borel set $A \subset \Omega \setminus J_m$, we have

$$
\big(\operatorname{div}\Phi_\psi(m)\big)(A) = -\frac{1}{T}\int_\Gamma \tilde D(\psi \circ \gamma_s)(\{\gamma_x \in A\})\,d\omega(\gamma)\,.
$$

Further note that, thanks to (3.8)-(3.9) and the expression of $(\sigma_{\min})_x$ for $x \notin J_m$, for ω-a.e. $\gamma \in \Gamma$ we have $\gamma_s(t) \in \{\mathfrak{s}(\gamma_x(t)), \mathfrak{s}(\gamma_x(t)) + \pi\}$ for $\tilde D\gamma_s$-a.e. $t \in I_\gamma$, so the above integrand can be rewritten as

$$
\tilde D(\psi \circ \gamma_s)(\{\gamma_x \in A\}) = \int_{I_\gamma} (\psi' \circ \mathfrak{s})(\gamma_x(t))\,\mathbf{1}_{\gamma_x(t)\in A}\tilde D\gamma_s(dt)\,.
$$

Moreover, thanks again to (3.8)-(3.9) and Lemma 3.3 we have

$$
(\tilde D\gamma_s)_+(\{\gamma_x \in \tilde\Omega^+\}) = (\tilde D\gamma_s)_-(\{\gamma_x \in \tilde\Omega^-\}) = 0 \quad \text{for } \omega\text{-a.e. } \gamma \in \Gamma\,.
$$

From the two last equations and the definitions of $\tilde A^\pm$ we deduce, for ω-a.e. $\gamma \in \Gamma$, that

$$
\tilde D(\psi \circ \gamma_s)(\{\gamma_x \in A\}) \begin{cases} \geq 0 & \text{if } A \subset \tilde A^-\,,\\ \leq 0 & \text{if } A \subset \tilde A^+\,, \end{cases}
$$

and this implies the claimed decomposition. □

The proof of the above lemma relies on the following properties: for ν-a.e. $x \in \Omega \setminus J_m$ the measure $(\sigma_{\min})_x$ has a definite sign and ψ' does not change sign on $\operatorname{supp}(\sigma_{\min})_x$. Hence the analysis extends to the case of small shocks for particular entropies: given $\delta > 0$ denote by

$$
J_m^\delta = \{x \in J_m : m^\pm(x) = e^{is^\pm(x)},\ |s^+(x) - s^-(x)| > \delta\}. \tag{3.13}
$$

Lemma 3.5. *Let $m \in B^{1/3}_{3,\infty}(\Omega)$ solving (1.1) and (3.1) and ω be a minimal Lagrangian representation of m. Let $\psi \in C^{0,1}(\mathbb{T})$ be π-periodic and let $\Phi_\psi \in \widetilde{\mathrm{ENT}}$ be as in (3.3). Let $\delta \in \left(0, \frac{\pi}{2}\right)$ and assume that for every $\tilde{s} \in \mathbb{T}$ one of the two conditions hold:*

$$\psi'(s) \geq 0 \quad \text{for a.e. } s \in [\tilde{s}, \tilde{s}+\delta], \qquad \text{or} \qquad \psi'(s) \geq 0 \quad \text{for a.e. } s \in [\tilde{s}, \tilde{s}+\delta].$$

Then for any $A \subset J_m \setminus J_m^\delta$ it holds

$$[\operatorname{div} \Phi_\psi(m)]_+(A) = \frac{1}{T}\int_\Gamma [D(\psi \circ \gamma_s)]_-(\gamma_x \in A)\, d\omega(\gamma),$$
$$[\operatorname{div} \Phi_\psi(m)]_-(A) = \frac{1}{T}\int_\Gamma [D(\psi \circ \gamma_s)]_+(\gamma_x \in A)\, d\omega(\gamma).$$

In particular, we have

$$|\operatorname{div} \Phi_\psi(m)|(A) = \frac{1}{T}\int_\Gamma |D(\psi \circ \gamma_s)|(\gamma_x \in A) d\omega(\gamma) \quad \text{for any } A \subset J_m \setminus J_m^\delta.$$

Proof. With the same notation as in Theorem 3.1, let

$$\tilde{J}^+ = \{x \in J_m \setminus J_m^\delta : \mathbf{n} \cdot e^{i\bar{s}} > 0\}, \quad \tilde{J}^- = \{x \in J_m \setminus J_m^\delta : \mathbf{n} \cdot e^{i\bar{s}} < 0\}.$$

For each $x \in J_m \setminus J_m^\delta$ the corresponding half-amplitude $\beta \in (0, \frac{\delta}{2}) \subset (0, \frac{\pi}{4})$. Since in this range of β we have $g_\beta \geq 0$, then

$$\nu_{\min}\big(J_m \setminus (J_m^\delta \cup \tilde{J}^+ \cup \tilde{J}^-)\big) = 0\,,$$
$$(\sigma_{\min})_x = \mathbf{n} \cdot e^{i\bar{s}} g_\beta(\cdot - \bar{s})\mathcal{L}^1 \geq 0 \text{ on } \mathbb{T} \text{ for all } x \in \tilde{J}^+\,,$$
$$(\sigma_{\min})_x = \mathbf{n} \cdot e^{i\bar{s}} g_\beta(\cdot - \bar{s})\mathcal{L}^1 \leq 0 \text{ on } \mathbb{T} \text{ for all } x \in \tilde{J}^-\,.$$

By (3.4) it follows that

$$\operatorname{div} \Phi_\psi(m) \llcorner (J_m \setminus J_m^\delta) = \left(\int_\mathbb{T} \big(\mathbf{1}_{\tilde{\Omega}^-} - \mathbf{1}_{\tilde{\Omega}^+}\big)\psi'(s) d|(\sigma_{\min})_x|(s)\right) \nu_{\min} \llcorner (J_m \setminus J_m^\delta).$$

Moreover, by the assumption on ψ, we have that ψ' has constant sign on $\operatorname{supp}(\sigma_{\min})_x \subset [\bar{s} - \frac{\delta}{2}, \bar{s} + \frac{\delta}{2}] \cup [\bar{s} - \frac{\delta}{2} + \pi, \bar{s} + \frac{\delta}{2} + \pi]$. Therefore the positive and the negative parts of $\operatorname{div} \Phi_\psi(m)$ restricted to $J_m \setminus J_m^\delta$ are concentrated respectively on the sets

$$A^{j,+} = \big\{x \in J_m \setminus J_m^\delta : \exists s \in \operatorname{supp}(\sigma_{\min})_x\,,\ \big(\mathbf{1}_{\tilde{J}^-}(x) - \mathbf{1}_{\tilde{J}^+}(x)\big)\psi'(s) > 0\big\}\,,$$
$$A^{j,-} = \big\{x \in J_m \setminus J_m^\delta : \exists s \in \operatorname{supp}(\sigma_{\min})_x\,,\ \big(\mathbf{1}_{\tilde{J}^-}(x) - \mathbf{1}_{\tilde{J}^+}(x)\big)\psi'(s) > 0\big\}\,.$$

Thanks to (3.11) and Lemma 3.3, for any Borel set $A \subset J_m \setminus J_m^\delta$, we have

$$\begin{aligned}\big(\operatorname{div} \Phi_\psi(m)\big)(A) &= -\frac{1}{T}\int_\Gamma D^j(\psi \circ \gamma_s)(\{\gamma_x \in A\})\, d\omega(\gamma) \qquad (3.14)\\ &= -\frac{1}{T}\int_\Gamma \sum_{t \in J_\gamma} \left(\psi(\gamma_s(t+)) - \psi(\gamma_s(t-))\right) \mathbf{1}_{\{\gamma_x(t) \in A\}}\, d\omega(\gamma).\end{aligned}$$

Further note that, thanks to (3.8)-(3.9) and the properties of $(\sigma_{\min})_x$ for $x \in J_m \setminus J_m^\delta$, for ω-a.e. $\gamma \in \Gamma$ we have $\gamma_s(t+), \gamma_s(t-) \in [\bar{s}(\gamma_x(t)) - \frac{\delta}{2}, \bar{s}(\gamma_x(t)) + \frac{\delta}{2}]$ or $\gamma_s(t+), \gamma_s(t-) \in [\bar{s}(\gamma_x(t)) - \frac{\delta}{2} + \pi, \bar{s}(\gamma_x(t)) + \frac{\delta}{2} + \pi]$ for $D^j\gamma_s$-a.e. $t \in I_\gamma$. Moreover, thanks again to (3.8)-(3.9) and Lemma 3.3 we have

$$(D^j\gamma_s)_+(\{\gamma_x \in \tilde{J}^+\}) = (D^j\gamma_s)_-(\{\gamma_x \in \tilde{J}^-\}) = 0 \quad \text{for } \omega\text{-a.e. } \gamma \in \Gamma\,.$$

From the above property, the definitions of $A^{j,\pm}$ and (3.14)we deduce, for ω-a.e. $\gamma \in \Gamma$, that

$$D^j(\psi \circ \gamma_s)(\{\gamma_x \in A\}) \begin{cases} \geq 0 & \text{if } A \subset A^{j,-}\,, \\ \leq 0 & \text{if } A \subset A^{j,+}\,, \end{cases}$$

and this implies the claimed decomposition. □

4 Rectifiability

The proof of Theorem 1.1 relies on the following estimate.

Proposition 4.1. *Let $m \in B^{1/3}_{3,\infty}(\Omega;\mathbb{R}^2)$ a weak solution of the eikonal equation (1.1) and ω a minimal Lagrangian representation of m.*

For every $\delta \in (0, \frac{\pi}{8})$ and $\alpha \in (0,1)$, we have

$$\begin{aligned} &\int_\Gamma |D\gamma_s|\big(\{t \in I_\gamma \colon \gamma_x(t) \in \Omega \setminus J_m^\delta,\ |D\gamma_s|(\{t\}) \leq \delta\}\big)\, d\omega(\gamma) \\ &\lesssim \delta^{1-\alpha}\Bigg(\bigvee_{\|\Phi\|_{C^{1,\alpha}} \leq 1} |\operatorname{div}\Phi(m)|\Bigg)(\Omega \setminus J_m^\delta)\,, \end{aligned}$$

where the supremum of measures is taken over all entropies $\Phi \in \widetilde{\mathrm{ENT}}$ such that $\|\Phi\|_{C^{1,\alpha}} \leq 1$.

Proof of Theorem 1.1. Let $m \colon \Omega \to \mathbb{R}^2$ a weak solution of the eikonal equation (1.1) such that $m \in B^{1/p}_{p,\infty}(\Omega)$ for some $p \in [1,3)$. Since $B^{1/p}_{p,\infty} \cap L^\infty \subset B^{1/q}_{q,\infty}$ for all $q \geq p$, we assume without loss of generality that $2 < p < 3$ and may write $p = 2 + \alpha$ for some $\alpha \in (0,1)$. Thanks to Proposition 2.1, the supremum measure appearing in Proposition 4.1 is finite. Letting $\delta \to 0$, we deduce

$$\int_\Gamma |D\gamma_s|\big(\{t \in I_\gamma \colon \gamma_x(t) \in \Omega \setminus J_m,\ |D\gamma_s|(\{t\}) = 0\}\big)\, d\omega(\gamma) = 0\,.$$

In other words, for ω-a.e. $\gamma \in \Gamma$, the measure $|D\gamma_s|$ only has a jump part. Thanks to the representation formula (3.11) and Lemma 3.3, this implies that the entropy dissipation is concentrated on the jump set J_m for every entropy $\Phi \in \mathrm{ENT}$. □

Proof of Proposition 4.1. We fix $\delta \in (0,\pi/16)$ and start by defining a family of functions ψ bounded in $C^{0,\alpha}$ and to which Lemma 3.5 can be applied. We define $\psi_0 \in C^{0,1}(\mathbb{T};\mathbb{R})$, odd, π-periodic, even with respect to $\pi/4$, and given on $[0,\pi/4]$ by the nondecreasing piecewise affine function

$$\psi_0(t) = \begin{cases} t/\delta^{1-\alpha} & \text{for } 0 \le t \le 2\delta, \\ 2\delta^\alpha & \text{for } 2\delta < t \le \pi/4\,. \end{cases}$$

This function ψ_0 generates the family $\psi_{\bar s} = \psi_0(\cdot - \bar s)$, for $\bar s \in \mathbb{T}$. The corresponding entropies $\Phi_{\bar s} = \Phi_{\psi_{\bar s}}$, defined as in (3.3), satisfy the bound $\|\psi_{\bar s}\|_{C^{1,\alpha}} \lesssim 1$.

For any $\bar s \in \mathbb{T}$ and any $\gamma \in \Gamma$ we have

$$\begin{aligned} &|D\gamma_s|\big(\{t \in I_\gamma \colon \gamma(t^-) \text{ and } \gamma(t^+) \in [\bar s - \delta, \bar s + \delta]\}\big) \\ &\le \delta^{1-\alpha} |D(\psi_{\bar s} \circ \gamma_s)|\big(\{t \in I_\gamma \colon \gamma(t^-) \text{ and } \gamma(t^+) \in [\bar s - \delta, \bar s + \delta]\}\big)\,. \end{aligned}$$

We fix a uniform partition $0 = \bar s_1 < \bar s_2 < \cdots < \bar s_N < \bar s_{N+1} = 2\pi$ of $[0,2\pi]$ such that $\delta/2 < \bar s_{j+1} - \bar s_j \le \delta$. Then we have the covering

$$\mathbb{T} \subset \bigcup_{j=1}^N \bar I_j, \quad \bar I_j = [\bar s_j - \delta, \bar s_j + \delta]\,,$$

each interval $\bar I_j$ intersects at most 4 of the other intervals from this covering, and any interval of length δ is contained in one of these intervals. Thus we have

$$\begin{aligned} &|D\gamma_s|\big(\{t \in I_\gamma \colon |D\gamma_s|(\{t\}) \le \delta\}\big) \\ &\le \delta^{1-\alpha} \sum_{j=1}^N |D(\psi_{\bar s_j} \circ \gamma_s)|\big(\{t \in I_\gamma \colon \gamma(t^-) \text{ and } \gamma(t^+) \in \bar I_j\}\big)\,. \end{aligned}$$

Invoking Lemma 4.2 below, we see that

$$\{t \in I_\gamma \colon \gamma(t^-) \text{ and } \gamma(t^+) \in \bar I_j\} \subset \{t \in I_\gamma \colon \gamma_x(t) \in A_j\} \cup Z_j\,,$$

where $|D\gamma_s|(Z_j) = 0$ and

$$\begin{aligned} A_j = \Big\{ x \in \Omega \colon \exists \tilde s \in \bar I_j\,,\ & x \text{ is either} \\ & \text{a jump point of } m \text{ with normal } \mathbf{n} \in \{ie^{i\tilde s}, e^{i\tilde s}\}\,, \\ & \text{or a Lebesgue point of } \mathfrak{s} \text{ with value } \mathfrak{s}(x) \in \{\tilde s, \tilde s + \pi\} \Big\}\,. \end{aligned}$$

In the definition above, we refer to the function $\mathfrak{s}$ defined in Theorem 3.1 and we consider Lebesgue points with respect to the measure $\nu \llcorner (\Omega \setminus J_m)$. The finite intersection property of the intervals $\bar I_j$ implies that each set A_j intersects at most 16 of the other sets A_k, $k \neq j$. Moreover, thanks to Lemma 3.4 and Lemma 3.5, we have

$$|\operatorname{div} \Phi_{\bar s_j}(m)|(A_j \setminus J_m^\delta) = \int_\Gamma |D(\psi_{\bar s_j} \circ \gamma_s)|(\{\gamma_x \in A_j \setminus J_m^\delta\})\, d\omega(\gamma)\,.$$

Gathering the above properties, we deduce

$$\int_\Gamma |D\gamma_s|\big(\{t \in I_\gamma : \gamma_x(t) \in \Omega \setminus J_m^\delta,\ |D\gamma_s|(\{t\}) \le \delta\}\big)\, d\omega(\gamma)$$
$$\lesssim \delta^{1-\alpha} \sum_{j=1}^N |\operatorname{div} \Phi_{\bar{s}_j}(m)|(A_j \setminus J_m^\delta)\,.$$

And the finite intersection property of the A_j's, together with the boundedness of the family $\Phi_{\bar{s}}$ in $C^{1,\alpha}$ implies that the last sum is controlled by the supremum measure appearing in Proposition 4.1. □

Lemma 4.2. *Let $m \in B^{1/3}_{3,\infty}(\Omega;\mathbb{R}^2)$ be a weak solution of the eikonal equation (1.1) and ω be a minimal Lagrangian representation of m. Then for ω-a.e. $\gamma \in \Gamma$ we have the following.*

For any $\delta > 0$, any $\bar{s} \in \mathbb{T}$ and $|D\gamma_s|$-a.e. $t \in I_\gamma$, if

$$\gamma(t^-) \text{ and } \gamma(t^+) \in [\bar{s} - \delta, \bar{s} + \delta]\,, \qquad \text{and} \qquad \gamma_x(t) \in \Omega \setminus J_m^\delta,$$

then there exists $\tilde{s} \in [\bar{s} - \delta, \bar{s} + \delta]$ such that $x = \gamma_x(t)$ is either a jump point of m with normal $n \in \{ie^{i\tilde{s}}, e^{i\tilde{s}}\}$ or a Lebesgue point of $\mathfrak{s}$ with value $\mathfrak{s}(x) \in \{\tilde{s}, \tilde{s} + \pi\}$.

Proof of Lemma 4.2. It follows from Theorem 3.2 that for ω-a.e. $\gamma \in \Gamma$ the following holds: for $|D\gamma_s|$-a.e. $t \in I_\gamma$

$$\gamma_s(t+), \gamma_s(t-) \in \operatorname{supp}(\sigma_{\min})_{\gamma_x(t)}.$$

Recall that $\nu_{\min}(A) = 0$ implies $\int_\Gamma |D\gamma_s|(\{\gamma_x \in A\}) d\omega(\gamma) = 0$. Since for $\nu_{\min}$-a.e. $x \in J_m \setminus J_m^\delta$ the support of $(\sigma_{\min})_x$ is contained in $[\tilde{s} - \frac{\pi}{2} - \delta, \tilde{s} - \frac{\pi}{2} + \delta] \cup [\tilde{s} + \frac{\pi}{2} - \delta, \tilde{s} + \frac{\pi}{2} + \delta]$ where $\tilde{s} \in \mathbb{T}$ is such that $e^{i\tilde{s}} = \mathbf{n}(x)$ and for $\nu_{\min}$-a.e. $x \in \Omega \setminus J_m$ the support of $(\sigma_{\min})_x$ is contained in $\{\mathfrak{s}(x) - \frac{\pi}{2}, \mathfrak{s}(x) + \frac{\pi}{2}\}$, then the claim follows. □

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