On Lebesgue points of entropy solutions to the eikonal equation

Xavier Lamy* Elio Marconi†

April 28, 2023

Abstract

We consider entropy solutions to the eikonal equation $|\nabla u| = 1$ in two space dimensions. These solutions are motivated by a class of variational problems and fail in general to have bounded variation. Nevertheless they share with BV functions, several of their fine properties: we show in particular that the set of non-Lebesgue points has co-dimension at least one.

1 Introduction

We consider an open set $\Omega \subset \mathbb{R}^2$ and $m\colon \Omega \to \mathbb{R}^2$ a solution of the eikonal equation

$$|m| = 1 \text{ a.e., and } \nabla \cdot m = 0 \text{ in } \Omega. \tag{1}$$

We are interested in particular in solutions that arise as limits as $\varepsilon \to 0$ of vector fields m_ε with equi-bounded energy $\sup_{\varepsilon>0} F_\varepsilon(m_\varepsilon, \Omega) < \infty$, where

$$\begin{aligned} &F_\varepsilon(m;\Omega) = \frac{\varepsilon}{2}\int_\Omega |\nabla m|^2 + \frac{1}{2\varepsilon}\int_\Omega (1-|m|^2)^2, \\ &m\colon \Omega \to \mathbb{R}^2, \qquad \nabla \cdot m = 0, \end{aligned} \tag{2}$$

are the functionals introduced by Aviles and Giga in [5]. We refer to the introduction of [11] for a description of several physical applications.

The notion of entropy, borrowed from the field of conservation laws, plays a fundamental role in the study of the singular limit as $\varepsilon \to 0$ of these functionals. We say that a compactly supported function $\Phi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ is an *entropy* for (1) if for every open set U and every smooth $m : U \to \mathbb{R}^2$ solving $\nabla \cdot m = 0$ and $|m| = 1$ it

*Institut de Mathématiques de Toulouse; UMR 5219, Université de Toulouse; CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: xlamy@math.univ-toulouse.fr.

†Dipartimento di Matematica 'Tullio Levi Civita', Università di Padova, via Trieste 63, 35121 Padova (PD), Italy. Email: elio.marconi@unipd.it.

holds $\nabla \cdot \Phi(m) = 0$. It is shown in [3, 8] that functions with equi-bounded energy as $\varepsilon \to 0$ are pre-compact in $L^2(\Omega)$ and any limit is an entropy solution of (1): namely for every entropy $\Phi \in C^\infty(\mathbb{R}^2, \mathbb{R}^2)$ the distribution $\nabla \cdot \Phi(m)$ is a finite Radon measure. Remarkably, the same class of entropy solutions to (1) contains the asymptotic domain of other families of functionals: see [2, 17] for two micromagnetics models.

It is shown in [9] that m is an entropy solution if and only if the following kinetic equation (introduced in [10]) is satisfied:

$$e^{is} \cdot \nabla_x \mathbf{1}_{m(x)\cdot e^{is} > 0} = \partial_s \sigma, \qquad \sigma \in \mathcal{M}_{\mathrm{loc}}(\Omega \times \mathbb{R}/2\pi\mathbb{Z}). \tag{3}$$

We denote by $\nu \in \mathcal{M}_{\mathrm{loc}}(\Omega)$ the entropy dissipation measure given by

$$\nu(A) = |\sigma|(A \times \mathbb{R}/2\pi\mathbb{Z}), \qquad A \subset \Omega. \tag{4}$$

It is known [7] that $\mathcal{H}^1$-a.e. point $x \in \Omega$ at which $\nu(B_r(x))/r \to 0$ as $r \to 0^+$ is a vanishing mean oscillation (VMO) point of m, that is,

$$⨍_{B_r(x)} \left| m - ⨍_{B_r(x)} m \right| \longrightarrow 0 \quad \text{as } r \to 0^+.$$

It is conjectured in [7, *Conjecture 1*(b')] that $\mathcal{H}^1$-a.e. such point is in fact a Lebesgue point. Our main result states that this conjecture is true under the additional assumption that $\nu(B_r(x))/r$ decays algebraically to 0.

Theorem 1. *Let $m\colon \Omega \to \mathbb{R}^2$ be an entropy solution* (3) *of the eikonal equation* (1). *Then $\mathcal{H}^1$-a.e. $x \in \Omega$ such that $\lim_{r\to 0^+} \nu(B_r(x))/r^{1+a} = 0$ for some $a > 0$ is a Lebesgue point of m. In particular, the set of non Lebesgue points of m has Hausdorff dimension at most 1.*

Remark 2. After this work was submitted, we became aware that the bound on the Hausdorff dimension can also be obtained as a consequence of classical capacity estimates [1, Theorem 6.21] and of the regularity $m \in B^{1/3}_{3,\infty}$ [9] (which implies $m \in W^{s,3}$ for any $s < 1/3$). Note however that the information provided by Theorem 1 is stronger, in that it directly relates oscillations at a point x to the local energy dissipation $\nu(B_r(x))$. Also note that, as will be clear from the proof, the assumption of algebraic energy decay $\nu(B_r)/r = \mathcal{O}(r^a)$ can be relaxed to $\nu(B_r)/r = \mathcal{O}(|\ln r|^{-14})$. Via a covering argument this implies that non Lebesgue points are finite for the Hausdorff measure defined by the function $r \mapsto r|\ln r|^{-14}$ (see e.g. [1, § 5.1]), a fact which does not follow directly from capacity estimates.

Analogs of Theorem 1 have been obtained previously in [13] for Burgers' equation, and in [16] for general scalar conservations laws. To prove Theorem 1 we follow the scheme laid out in [16], where it is shown that oscillations of averages $⨍_{B_r(x)} u$ of the solution u are controlled by the entropy dissipation. This, together with the VMO property, implies the Lebesgue point property. However, a key feature for the argument

of [16] is that the solution u takes values in the ordered set $\mathbb{R}$. Here our solution m takes values in $\mathbb{S}^1$, and adapting the argument of [16] is not enough to conclude (see Proposition 4). Our proof of Theorem 1 relies instead on the following dichotomy: either the oscillations of $⨍_{B_r(x)} m$ are controlled by the entropy dissipation ν, or m takes very different values in large subsets of $B_R(x)$ – this second alternative is ruled out by the VMO property. That dichotomy is made quantitative in the next statement.

Proposition 3. *Assume $B_1 \subset \Omega$. Let $r \in (0, 1/2)$ and*

$$h = h(r) = \max_{x_1, x_2 \in \overline{B}_{2r}} \left| ⨍_{B_r(x_1)} m - ⨍_{B_r(x_2)} m \right|. \tag{5}$$

There exist absolute constants $c, \delta > 0$ such that, if

$$R = \frac{32r}{\delta h^2} \leq 1,$$

then either

$$\nu(B_R) \geq c\, h^{11} r, \tag{6}$$

or there exist $s_0 \in \mathbb{R}$ such that

$$\left| B_R \cap \left\{ m \cdot e^{is} \geq -\frac{1}{2} \right\} \right| \geq c\, R^2 \quad \text{for } \operatorname{dist}(s, \{s_0, s_0 + \pi\}) \leq \pi/4. \tag{7}$$

Here and in the rest of the article, we denote by $|A|$ the Lebesgue measure of a measurable set $A \subset \mathbb{R}^d$. Theorem 1 is a rather direct consequence of Proposition 3, as we explain now.

Proof of Theorem 1. Let $x \in \Omega$ be a VMO point of m such that $\nu(B_r(x))/r^{1+a} \to 0$ for some $a > 0$. Translating and rescaling we assume without loss of generality that $x = 0$ and $B_1 \subset \Omega$. We claim that $h(r) = \mathcal{O}(r^b)$ for $b = a/(13 + 2a) > 0$. This, together with the fact that 0 is a VMO point of m, implies that 0 is a Lebesgue point (see [16, Lemma 4.6]). (Note, in connection with Remark 2, that $h(r) = \mathcal{O}(|\ln r|^{-1-\varepsilon})$ for some $\varepsilon > 0$ would imply the same conclusion.) To prove that $h(r) = \mathcal{O}(r^b)$ we argue by contradiction and assume that $h(r)/r^b \to \infty$ along a sequence $r \to 0^+$. Then, along the same sequence,

$$R = \frac{32r}{\delta h^2} = \frac{32}{\delta} r^{1-2b} \left(\frac{r^b}{h} \right)^2 \to 0 \quad \text{because } b < \frac{1}{2},$$
$$\text{and } \frac{R^{1+a}}{h^{11} r} = \frac{32^{1+a}}{\delta^{1+a}} \left(\frac{r^b}{h} \right)^{13+2a} \to 0.$$

Therefore, applying Proposition 3 along the sequence $R \to 0$, the condition (6) cannot be satisfied because $\nu(B_R)/R^{1+a} \to 0$, so we have (7). This contradicts the VMO property: for all small enough R, the projection $z_R \in \mathbb{S}^1$ of $⨍_{B_R} m$ onto $\mathbb{S}^1$ satisfies

$$|B_R \cap \{|m - z_R| \geq \pi/12\}| \leq \frac{c}{2} R^2. \tag{8}$$

But one can choose $s \in \mathbb{R}$ such that $\operatorname{dist}(s, \{s_0, s_0 + \pi\}) \leq \pi/4$ and

$$z \cdot e^{is} \geq -\frac{1}{2} \quad \Longrightarrow \quad |z - z_R| \geq \pi/12,$$

for any $z \in \mathbb{S}^1$ (if $z_R = e^{is_R}$, any $s \in [s_R + 3\pi/4, s_R + 5\pi/4]$ has that property). According to (7) this implies $|B_R \cap \{|m - z_R| \geq \pi/12\}| \geq cR^2$, in contradiction with (8). Hence we have proved that x is a Lebesgue point. The estimate on the Hausdorff dimension of non Lebesgue points follows via a covering argument (see e.g. [4, Theorem 2.56]). □

The proof of Proposition 3 has two main ingredients. The first ingredient consists in adapting the arguments of [16] to prove a dichotomy similar to Proposition 3, but where the second option (7) is replaced by a statement which is not strong enough to conclude.

Proposition 4. *Let $r \in (0, 1/2)$ and h be as in Proposition 3. There exist absolute constants $c, \delta > 0$ such that, if $R = 32r/(\delta h^2) \leq 1$, then we have either $\nu(B_R) \geq ch^{11} r$, or*

$$\left| B_{R/2} \cap \left\{ m \cdot m_0 \geq \frac{1}{2} \right\} \right| \geq chr^2 \quad \text{and} \quad \left| B_{R/2} \cap \left\{ m \cdot m_0 \leq -\frac{1}{2} \right\} \right| \geq chr^2, \tag{9}$$

for some $m_0 \in \mathbb{S}^1$.

The main idea behind the argument in [16] is that a large value of h implies the existence of a configuration which would be impossible in the absence of entropy dissipation. In the presence of dissipation, such configuration provides a lower bound on the dissipation, and there is no dichotomy. Here instead, not all configurations created by large values of h can be ruled out in the absence of dissipation: in particular the vortex solution $m(x) = x^\perp/|x|$ has zero dissipation but the values of $h(r)$ around the origin are not vanishing. This is reflected in the second alternative (9) of the dichotomy.

The second ingredient in our proof of Proposition 3 consists in using the methods developed in [15, 14, 6, 12] in order to pass from (9) to (7).

Proposition 5. *Let $m_0 = e^{is_0} \in \mathbb{S}^1$, and $R > 0$ such that $B_R \subset \Omega$. Then we have either*

$$\nu(B_R) \geq \frac{c}{R} \min(|X_+|, |X_-|), \qquad X_\pm = B_{R/2} \cap \{\pm m \cdot m_0 \geq 1/2\}, \tag{10}$$

or $\nu(B_R) \geq cR$, or (7), for some absolute constant $c > 0$.

Proposition 3 follows readily from Proposition 4 and Proposition 5. Thanks to Proposition 4, we know indeed that either $\nu(B_R) \geq ch^{11}r$, in which case we are done, or estimate (9) is valid. But according to Proposition 5, if (9) is satisfied, then we have either $\nu(B_R) \geq chr^2/R \geq ch^{11}r$, or $\nu(B_R) \geq cR \geq ch^{11}r$, or (7). In all cases, Proposition 3 is verified.

The proofs of Proposition 4 and Proposition 5 are presented in Section 2 and Section 3.

Notations.

We denote by $|A|$ the Lebesgue measure of a set $A \subset \mathbb{R}^d$. We use the symbol $\gtrsim$ to signify inequality up to an absolute multiplicative constant.

Acknowledgements.

We would like to thank Bogdan Raiţă for pointing out to us the capacity argument in Remark 2. Part of this work was completed during X.L.'s stay at the EPFL's Institute of Mathematics, which he thanks for their hospitality. X.L. received support from ANR project ANR-22-CE40-0006. E.M. acknowledges the support received from the SNF Grant 182565 and the European Union's Horizon 2020 research and innovation program under the Marie Skłodowska-Curie grant No. 101025032.

2 Proof of Proposition 4

Let x_1, x_2 attain the maximum in the definition (5) of h, and define, for $j = 1, 2$, $\rho_j(s)$ as the proportion of points $x \in B_r(x_j)$ at which $m(x)$ lies in the semi-circle of direction e^{is}, that is, for every $s \in \mathbb{R}/2\pi\mathbb{Z}$, we set

$$\rho_j(s) = \frac{1}{|B_r|}\left|B_r(x_j) \cap \left\{m \cdot e^{is} > 0\right\}\right| = \frac{1}{|B_r|}\int_{B_r(x_j)} \mathbf{1}_{E_m}(x,s)dx,$$

where

$$E_m = \left\{(x,s) \in \Omega \times \mathbb{R}/2\pi\mathbb{Z} \colon m(x) \cdot e^{is} > 0\right\}. \tag{11}$$

Note that $|\rho_j| \leq 1$ and, since for every $x \in \Omega$ it holds $|D_s \mathbf{1}_{E_m}(x,\cdot)|(\mathbb{R}/2\pi\mathbb{Z}) = 2$, then $\rho_j \in BV(\mathbb{R}/2\pi\mathbb{Z})$ with $|D\rho_j|(\mathbb{R}/2\pi\mathbb{Z}) \leq 2$. Moreover, by Fubini theorem, these functions satisfy the identities

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} e^{is}\rho_j(s)\,ds = \int_{\mathbb{R}/2\pi\mathbb{Z}} \fint_{B_r(x_j)} \mathbf{1}_{E_m}(x,s)e^{is}dxds = 2\fint_{B_r(x_j)} m(x)\,dx.$$

For $s \in \mathbb{R}$ and $\rho > 0$ we denote by $I_\rho(s)$ the segment

$$I_\rho(s) = [s-\rho, s+\rho].$$

For a small enough absolute constant $\delta \in (0,1)$, the subset $S \subset \mathbb{R}/2\pi\mathbb{Z}$ given by

$$S = \left\{ s \in \mathbb{R}/2\pi\mathbb{Z} \colon (|D\rho_1| + |D\rho_2|)(I_{\delta h^2}(s)) \geq \frac{h}{4\pi} \right\},$$

satisfies $|S| \leq h/2$ (as follows e.g. from a Besicovitch covering argument). Thus we have

$$\begin{aligned} h &= \frac{1}{2} \left| \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{is} \rho_1(s)\, ds - \int_{\mathbb{R}/2\pi\mathbb{Z}} e^{is} \rho_2(s)\, ds \right| \leq \frac{1}{2} \int_{\mathbb{R}/2\pi\mathbb{Z}} |\rho_1(s) - \rho_2(s)|\, ds \\ &\leq \frac{1}{2} \int_{(\mathbb{R}/2\pi\mathbb{Z})\setminus S} |\rho_1(s) - \rho_2(s)|\, ds + \frac{h}{2}. \end{aligned}$$

We may therefore find $s \in \mathbb{R}/2\pi\mathbb{Z}$ such that $s \notin S$ and $|\rho_1(s) - \rho_2(s)| \geq h/2\pi$. We assume without loss of generality that $\rho_1(s) - \rho_2(s) \geq h/2\pi$, and by definition of S we deduce

$$\inf_{I_{\delta h^2}(s)} \rho_1 - \sup_{I_{\delta h^2}(s)} \rho_2 \geq \frac{h}{4\pi}.$$

In particular, setting $s_0 = s - \pi/2 - 3\delta h^2/4$, we have

$$\inf_{I_{\delta h^2/4}(s_0+\pi/2)} \rho_1 - \sup_{I_{\delta h^2/4}(s_0+\pi/2+\delta h^2)} \rho_2 \geq \frac{h}{4\pi},$$

$$\inf_{I_{\delta h^2/4}(s_0+\pi/2+\delta h^2)} \rho_1 - \sup_{I_{\delta h^2/4}(s_0+\pi/2)} \rho_2 \geq \frac{h}{4\pi}.$$

As $\rho_j(s+\pi) = 1 - \rho_j(s)$ for a.e. $s \in \mathbb{R}/2\pi\mathbb{Z}$, this implies

$$\operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0+\pi/2)} \rho_1 + \operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0-\pi/2+\delta h^2)} \rho_2 \geq 1 + \frac{h}{4\pi}, \tag{12}$$

$$\operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0+\pi/2+\delta h^2)} \rho_1 + \operatorname*{ess\,inf}_{I_{\delta h^2/4}(s_0-\pi/2)} \rho_2 \geq 1 + \frac{h}{4\pi}. \tag{13}$$

The relevance of (12)-(13) comes from the following geometric observation. Given two directions $s_1 \in I_{\delta h^2/4}(s_0 + \pi/2)$ and $s_2 \in I_{\delta h^2/4}(s_0 - \pi/2 + \delta h^2)$ and two points $y_1 \in B_r(x_1) \cap \{m \cdot e^{is_1} > 0\}$, $y_2 \in B_r(x_2) \cap \{m \cdot e^{is_2} > 0\}$, we have $|s_1 - s_2| \geq \delta h^2$, and the two lines $y_j + \mathbb{R}e^{is_j}$ intersect at a point $z \in B_{8r/(\delta h^2)}$. In the absence of dissipation, one would have $m(z) \cdot e^{is_j} > 0$ for $j = 1, 2$, and therefore $m(z) \cdot e^{is_0} \geq \cos(2\delta h^2) \geq 1/2$. The last lower bound is valid provided $\delta \leq \pi/24$, since $|h| \leq 2$. The same argument with $s_1 \in I_{\delta h^2/4}(s_0 + \pi/2 + \delta h^2)$ and $s_2 \in I_{\delta h^2/4}(s_0 - \pi/2)$ implies instead $m(z) \cdot e^{is_0} \leq -1/2$.

Thanks to the techniques in [16], in the presence of dissipation this can be made quantitative. The main idea is that (3) provides an estimate on the difference between the 'epigraph' E_m defined in (11) and its free transport $\mathrm{FT}(E_m, t)$, where the free transport operator $\mathrm{FT}(\cdot, t)$ is defined for $t \in \mathbb{R}$ by

$$\mathrm{FT}(E,t) = \left\{ (x,s) \in \Omega \times \mathbb{R}/2\pi\mathbb{Z} \colon (x - te^{is}, s) \in E \right\}.$$

Lemma 6. *Let $t \in \mathbb{R}$ and $\rho > 0$ such that $B_{\rho+|t|} \subset \Omega$. For all $\phi \in C^1_c(B_\rho \times \mathbb{R}/2\pi\mathbb{Z})$ we have*

$$\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \phi(x,s)\left(\mathbf{1}_{\mathrm{FT}(E_m,t)} - \mathbf{1}_{E_m}\right) dxds \le \left(|t|\,\|\partial_s\phi\|_\infty + t^2\|\nabla_x\phi\|_\infty\right)\nu(B_{\rho+|t|}).$$

Proof of Lemma 6. Define $\chi, \chi^{\mathrm{FT}} \colon [-|t|,|t|]\times\Omega\times\mathbb{R}/2\pi\mathbb{Z} \to \mathbb{R}$ by

$$\chi(\tau,x,s) = \mathbf{1}_{(x,s)\in E_m}, \qquad \chi^{\mathrm{FT}}(\tau,x,s) = \mathbf{1}_{(x,s)\in \mathrm{FT}(E_m,\tau)} = \chi(x-\tau e^{is},s).$$

so we have, in the sense of distributions,

$$\partial_\tau\chi + e^{is}\cdot\nabla_x\chi = \partial_s\sigma(x,s), \qquad \partial_\tau\chi^{\mathrm{FT}} + e^{is}\cdot\nabla_x\chi^{\mathrm{FT}} = 0.$$

Setting $\hat\chi = \chi^{\mathrm{FT}} - \chi$, and $\psi(\tau,x,s) = \phi(x+e^{is}(t-\tau),s)$ which satisfies $\partial_\tau\psi + e^{is}\cdot\nabla_x\psi = 0$, we deduce

$$\partial_\tau\left[\psi\hat\chi\right] + e^{is}\cdot\nabla_x\left[\psi\hat\chi\right] = -\psi\partial_s\sigma.$$

Integrating with respect to (x,s) (this is formal but makes sense distributionnally) we deduce

$$\frac{d}{d\tau}\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \psi\hat\chi\,dxds = \int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \partial_s\psi\,d\sigma(x,s).$$

Integrating this from 0 to t and recalling $\nu(A) = |\sigma|(A\times\mathbb{R}/2\pi\mathbb{Z})$ for $A\subset\Omega$, we obtain

$$\begin{aligned}\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \phi(x,s)\left(\mathbf{1}_{\mathrm{FT}(E_m,t)} - \mathbf{1}_{E_m}\right) dxds &= \int_0^t\int_{\Omega\times\mathbb{R}/2\pi\mathbb{Z}} \partial_s\psi d\sigma\,d\tau \\ &\le |t|\,\|\partial_s\psi\|_\infty |\nu|(B_{\rho+|t|}\times\mathbb{R}/2\pi\mathbb{Z}).\end{aligned}$$

Noting that $\|\partial_s\psi\|_\infty \le \|\partial_s\phi\|_\infty + |t|\,\|\nabla_x\phi\|_\infty$ completes the proof. □

Equipped with Lemma 6 we continue the proof of Proposition 4. First we make use of (12). We define $\hat z \in \mathbb{R}^2$ as the intersection of the lines $x_1 + \mathbb{R}e^{i(s_0+\pi/2)}$ and $x_2 + \mathbb{R}e^{i(s_0-\pi/2+\delta h^2)}$, that is,

$$x_1 + t_1 e^{i(s_0+\pi/2)} = x_2 + t_2 e^{i(s_0-\pi/2+\delta h^2)} = \hat z,$$

for some $t_1, t_2 \in \mathbb{R}$. Since $|x_1 - x_2| \le 4r$, we have

$$|t_1|, |t_2| \le \frac{4r}{\sin(\delta h^2)} \le \frac{8r}{\delta h^2}, \tag{14}$$

and therefore $B_r(\hat z) \subset B_{R/2}$. We will use Lemma 6 to compare E_m with $\mathrm{FT}(E_m,t_1)$ and $\mathrm{FT}(E_m,t_2)$ on $B_r(\hat z)$. We define

$$\begin{aligned}&C_1 = B_r(\hat z)\times I_c(s_0+\pi/2), \qquad C_2 = B_r(\hat z)\times I_c(s_0-\pi/2+\delta h^2),\\ &A_1 = E_m\cap C_1, \qquad A_2 = E_m\cap C_2\end{aligned}$$

with $c = \delta h^3/128\pi \le \delta h^2/4$. and their free transport counterparts

$$A_1^{\mathrm{FT}} = \mathrm{FT}(E_m, t_1) \cap C_1, \qquad A_2^{\mathrm{FT}} = \mathrm{FT}(E_m, t_2) \cap C_2.$$

We estimate

$$\begin{aligned}
|A_1^{\mathrm{FT}}| &= |E_m \cap \mathrm{FT}(\cdot, t_1)^{-1}(C_1)| \\
&\ge |E_m \cap (B_r(x_1) \times I_c(s_0 + \pi/2))| - |\mathrm{FT}(\cdot, t_1)^{-1}(C_1) \setminus (B_r(x_1) \times I_c(s_0 + \pi/2))| \\
&= \int_{s_0+\frac{\pi}{2}-c}^{s_0+\frac{\pi}{2}+c} \rho_1(s)ds - |\mathrm{FT}(\cdot, t_1)^{-1}(C_1) \setminus (B_r(x_1) \times I_c(s_0 + \pi/2))|.
\end{aligned}$$

Moreover

$$\begin{aligned}
|\mathrm{FT}(\cdot, t_1)^{-1}(C_1) \setminus (B_r(x_1) \times I_c(s_0 + \pi/2))| &= \int_{s_0+\frac{\pi}{2}-c}^{s_0+\frac{\pi}{2}+c} |B_r(\hat{z} - t_1 e^{is}) \setminus B_r(x_1)|ds \\
&\le 2r \int_{s_0+\frac{\pi}{2}-c}^{s_0+\frac{\pi}{2}+c} |\hat{z} - t_1 e^{is} - x_1|ds \\
&\le 2r \int_{s_0+\frac{\pi}{2}-c}^{s_0+\frac{\pi}{2}+c} |t_1||e^{i(s_0+\pi/2)} - e^{is}|ds \\
&\le 32 \frac{c^2 r^2}{\delta h^2},
\end{aligned}$$

where in the last inequality we used (14). Therefore we have

$$|A_1^{\mathrm{FT}}| \ge \int_{s_0+\frac{\pi}{2}-c}^{s_0+\frac{\pi}{2}+c} \rho_1(s)ds - 32\frac{c^2 r^2}{\delta h^2}, \tag{15}$$

and similarly

$$|A_2^{\mathrm{FT}}| \ge \int_{s_0-\frac{\pi}{2}+\delta h^2-c}^{s_0-\frac{\pi}{2}+\delta h^2+c} \rho_2(s)ds - 32\frac{c^2 r^2}{\delta h^2}. \tag{16}$$

From (12) we know that

$$\rho_1\left(s_0 + \frac{\pi}{2} + s\right) + \rho_2\left(s_0 - \frac{\pi}{2} + \delta h^2 + s\right) \ge 1 + \frac{h}{4\pi} \qquad \text{for all } |s| \le \frac{\delta}{4}h^2.$$

Integrating this inequality in $s \in [-c, c]$, it follows from (15),(16) that

$$|A_1^{\mathrm{FT}}| + |A_2^{\mathrm{FT}}| \ge 2c|B_r|\left(1 + \frac{h}{4\pi}\right) - 64\frac{c^2 r^2}{\delta h^2} \ge 2c|B_r|\left(1 + \frac{h}{8\pi}\right), \tag{17}$$

by the choice $c = \delta h^3/128\pi$. Next we consider two cases, depending on whether A_1 and A_2 satisfy a similar inequality.

Case 1. Assume first that

$$|A_1| + |A_2| \geq 2c|B_r| \left(1 + \frac{h}{16\pi}\right),$$

then

$$|\pi_x(A_1)| + |\pi_x(A_2)| \geq |B_r| \left(1 + \frac{h}{16\pi}\right).$$

Moreover, since $\pi_x(A_1) \cup \pi_x(A_2) \subset B_r(\hat{z})$, it follows that $A := \pi_x(A_1) \cap \pi_x(A_2)$ satisfies $|A| \geq h|B_r|/16$. By construction, we have

$$\begin{aligned} A = \Big\{ x \in B_r(\hat{z}) \colon \exists s_1 \in I_c(s_0 + \pi/2),\, s_2 \in I_c(s_0 - \frac{\pi}{2} + \delta h^2), \\ m(x) \cdot e^{is_1} > 0 \text{ and } m(x) \cdot e^{is_2} > 0 \Big\} \\ \subset B_r(\hat{z}) \cap \{ m \cdot e^{is_0} \geq \cos(2\delta h^2) \}, \end{aligned}$$

so this implies

$$\left| B_{R/2} \cap \left\{ m \cdot m_0 \geq \frac{1}{2} \right\} \right| \gtrsim hr^2. \tag{18}$$

Case 2. Assume now that

$$|A_1| + |A_2| < 2c|B_r| \left(1 + \frac{h}{16\pi}\right).$$

Then using (17) we obtain

$$|A_1^{\mathrm{FT}}| - |A_1| + |A_2^{\mathrm{FT}}| - |A_2| > 2c|B_r| \frac{h}{16\pi},$$

so either $|A_1^{\mathrm{FT}}| - |A_1|$ or $|A_2^{\mathrm{FT}}| - |A_2|$ is larger than half the right-hand side. We consider without loss of generality only the first case:

$$|A_1^{\mathrm{FT}}| - |A_1| > |B_r| \frac{ch}{16\pi}.$$

This implies a lower bound on the entropy dissipation $\nu(B_R)$ thanks to Lemma 6. Specifically, we apply Lemma 6 to $t = t_1$ and $\phi \in C_c^\infty(B_{2r(\hat{z})} \times I_{2c}(s_0 + \pi/2))$ such that

$$\mathbf{1}_{x \in B_r(\hat{z})} \mathbf{1}_{s \in I_c(s_0 + \pi/2)} \leq \phi(x, s) \leq \mathbf{1}_{x \in B_{(1+\varepsilon)r}(\hat{z})} \mathbf{1}_{s \in I_{(1+\varepsilon)c}(s_0 + \pi/2)},$$

and $|\partial_s \phi| \leq 2/(\varepsilon c)$, $|\nabla_x \phi| \leq 2/(\varepsilon r)$. We choose $\varepsilon = h/192\pi$ to ensure

$$\left| \big(B_{(1+\varepsilon)r}(\hat{z}) \times I_{(1+\varepsilon)c}\big) \setminus \big(B_r(\hat{z}) \times I_c\big) \right| \leq \frac{ch}{32\pi} |B_r|.$$

Since $|t_1| \leq 8r/(\delta h^2)$ and $B_{2r+|t_1|}(\hat{z}) \subset B_R$, we deduce $\nu(B_R) \gtrsim \delta^3 h^{11} r \gtrsim h^{11} r$.

Similarly, using (13) we have two cases: either

$$\left| B_{R/2} \cap \left\{ m \cdot m_0 \leq -\frac{1}{2} \right\} \right| \gtrsim hr^2, \tag{19}$$

or $\nu(B_R) \gtrsim h^{11} r$. So gathering all cases, we see that either both (18) and (19) are satisfied, or $\nu(B_R) \gtrsim h^{11} r$, which is exactly the dichotomy of Proposition 4. □

3 Proof of Proposition 5

In order to prove Proposition 5, we briefly recall from [14] the notion of *Lagrangian representation* of an entropy solution m of the eikonal equation. In [14, 15] the second author shows the existence of a finite non-negative Radon measure ω on the set of curves

$$\Gamma = \Big\{(\gamma, t_\gamma^-, t_\gamma^+)\colon 0 \le t_\gamma^- \le t_\gamma^+ \le 1, \\ \gamma = (\gamma_x, \gamma_s) \in \mathrm{BV}((t_\gamma^-, t_\gamma^+); \Omega \times \mathbb{R}/2\pi\mathbb{Z}), \\ \gamma_x \text{ is Lipschitz}\Big\},$$

with the following three properties:

- for every $t \in (0,1)$, the pushforward of ω, restricted to the section $\Gamma(t) = \{(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma\colon t_\gamma^- < t < t_\gamma^+\}$, by the evaluation map $e_t\colon \gamma \mapsto \gamma(t)$ (a right-continuous representative of γ_s is always considered), is uniform on the 'epigraph' $E_m = \{m(x) \cdot e^{is} > 0\}$, that is,

$$(e_t)_\sharp\, [\omega \lfloor \Gamma(t)] = \mathbf{1}_{m(x)\cdot e^{is}>0}\, dx\, ds; \tag{20}$$

- the measure ω is concentrated on curves $(\gamma, t_\gamma^-, t_\gamma^+) \in \Gamma$ solving the characteristic equation,

$$\dot\gamma_x(t) = e^{i\gamma_s(t)} \qquad \text{for a.e. } t \in (t_\gamma^-, t_\gamma^+); \tag{21}$$

- the entropy dissipation measure (4) disintegrates along the Lagrangian curves as

$$\nu(A) = \int_\Gamma \mu_\gamma(\gamma_x^{-1}(A))\, d\omega(\gamma) \qquad \text{for all measurable } A \subset \Omega, \tag{22}$$

  where $\mu_\gamma = |D_t\gamma_s|$, with the convention that a jump of γ_s from s^- to s^+ at time $t_0 \in (t_\gamma^-, t_\gamma^+)$ contributes $\mathrm{dist}_{\mathbb{R}/2\pi\mathbb{Z}}(s^-, s^+)\delta_{t=t_0}$ to the jump part of μ_γ (see [14, Proposition 2.5]).

Moreover, the Lagrangian property (20) implies that ω is concentrated on curves γ such that $\gamma_x(t)$ is a Lebesgue point of m with $m(\gamma_x(t)) \cdot e^{i\gamma_s(t^+)} > 0$, for a.e. $t \in (0,1)$ [14, Lemma 2.7]. We denote by $\Gamma_g \subset \Gamma$ the full-measure subset of Lagrangian curves which satisfy that property together with the characteristic equation (21).

The proof of Proposition 5 is based on two main tools. The first, Lemma 7, is a dichotomy stating that either Lagrangian curves passing through a given set create a lot of dissipation, or one can find an almost-straight Lagrangian curve passing through that set. The second ([12, Lemma 5.2], a slightly more precise version of [14, Lemma 3.1], itself adapted from [15, Lemma 22]) is another dichotomy: given an almost-straight Lagrangian curve, either the density of points at which m lies in the semi-circle indicated

by the s-component of that curve is high, or a lot of dissipation must be created. The succession of these two dichotomies is reflected in the three alternatives in the conclusion of Proposition 5. We first state and prove the first tool, and then proceed to the proof of Proposition 5.

Lemma 7. *For any $R > 0$ such that $B_R \subset \Omega$, any measurable set $A \subset B_R \times \mathbb{R}/2\pi\mathbb{Z}$, and any $\eta \in (0,1)$, we have either*

$$\nu(B_R) \gtrsim \frac{\eta}{R} \left| \{(x,s) \in A \colon m(x) \cdot e^{is} > 0\} \right|, \tag{23}$$

or there exists a curve $\gamma \in \Gamma_g$ and a connected component J of $\gamma_x^{-1}(B_R)$ such that

$$J \cap \gamma^{-1}(A) \neq \emptyset \quad \text{and} \quad \mu_\gamma(J) \leq \eta.$$

Proof of Lemma 7. Assume that the second alternative of Lemma 7 is not verified: for every curve $\gamma \in \Gamma_g$ and every connected component J of $\gamma_x^{-1}(B_R)$ intersecting $\gamma^{-1}(A)$, we have $\mu_\gamma(J) > \eta$. Then we claim that

$$\mu_\gamma(\gamma_x^{-1}(B_R)) \gtrsim \frac{\eta}{R} T(\gamma), \qquad T(\gamma) = \left| \{t \in (t_\gamma^-, t_\gamma^+) \colon \gamma(t) \in A\} \right|, \tag{24}$$

for all $\gamma \in \Gamma_g$. To prove (24), denote by $J_k = (t_k^-, t_k^+)$ the connected components of $\gamma_x^{-1}(B_R)$ which intersect $\gamma^{-1}(A)$. We show next that $\mu_\gamma(J_k) \gtrsim \eta |J_k|/R$ for all k. On the one hand, if $|J_k| \leq 4R$ then $\mu_\gamma(J_k) \gtrsim \eta |J_k|/R$ because $\mu_\gamma(J_k) > \eta$ by assumption. On the other hand, from the characteristic equation (21) and the definition of $\mu_\gamma = |D_t \gamma_s|$, we have the inequality

$$|\gamma_x(t_2) - \gamma_x(t_1) - e^{i\gamma_s(t_1)}(t_2 - t_1)| \leq \mu_\gamma([t_1, t_2]) |t_2 - t_1|,$$

and we deduce that in any any interval $J \subset (0,1)$ such that $\gamma_x(J) \subset B_R$ and $|J| \geq 4R$, we must have $\mu_\gamma(J) \geq 1/2$. Therefore, if $|J_k| \geq 4R$, cutting J_k in disjoint subintervals of length between $4R$ and $8R$, we obtain that $\mu_\gamma(J_k) \gtrsim |J_k|/R \geq \eta |J_k|/R$. So we have

$$\mu_\gamma(\gamma_x^{-1}(B_R)) \geq \sum_k \mu_\gamma(J_k) \gtrsim \frac{\eta}{R} \sum_k |J_k|,$$

which implies (24) since $\gamma^{-1}(A) \subset \bigcup_k J_k$. From (24) and the fact that $\omega(\Gamma \setminus \Gamma_g) = 0$ we infer

$$\nu(B_R) = \int_\Gamma \mu_\gamma(\gamma_x^{-1}(B_R))\, d\omega(\gamma) \gtrsim \frac{\eta}{R} \int_\Gamma T(\gamma)\, d\omega(\gamma),$$

where the first equality comes from the disintegration (22). Making use of the Lagrangian property (20) to rewrite the last expression, we see that it is precisely equal to the right-hand side of (23), which concludes the proof of Lemma 7. □

Proof of Proposition 5. We recall that $m_0 = e^{is_0}$ and the sets $X_\pm$ are defined by

$$X_\pm = B_{R/2} \cap \{\pm m \cdot e^{is_0} \geq 1/2\}.$$

For any $\hat{s} \in [s_0 - \pi/4, s_0 + \pi/4]$, we apply Lemma 7 to $A(\hat{s}) = B_{R/2} \times I_\eta(\hat{s})$, where $I_\eta(\hat{s}) = [\hat{s}-\eta, \hat{s}+\eta]$. If $\eta \in (0, \pi/12)$ then we have $m(x)\cdot e^{is} > 0$ for all $(x,s) \in X_+ \times I_\eta(\hat{s})$, and therefore

$$\big|\{(x,s) \in A(\hat{s})\colon m(x)\cdot e^{is} > 0\}\big| \geq \eta |X_+|.$$

So we have either $\nu(B_R) \gtrsim \eta^2 |X_+|/R$, or there exists a curve $\gamma \in \Gamma_g$ and a connected component J of $\gamma_x^{-1}(B_R)$ intersecting $A(\hat{s})$ such that $\mu_\gamma(J) < \eta$. In that second case, applying [12, Lemma 5.2] we deduce that either $\nu(B_R) \gtrsim \eta^3 R$ or $|B_R \cap \{m \cdot e^{i\hat{s}} \geq -2\eta\}| \gtrsim \eta R^2$. We fix $\eta = 1/4$ and summarize the preceding discussion: for all $\hat{s} \in [s_0 - \pi/4, s_0 + \pi/4]$, we have

$$\nu(B_R) \gtrsim \frac{|X_+|}{R}, \quad \text{or } \nu(B_R) \gtrsim R, \quad \text{or } \big|B_R \cap \{m \cdot e^{i\hat{s}} \geq -1/2\}\big| \gtrsim R^2.$$

Similarly, for all $\hat{s} \in [s_0 + 3\pi/4, s_0 + 5\pi/4]$, we have

$$\nu(B_R) \gtrsim \frac{|X_-|}{R}, \quad \text{or } \nu(B_R) \gtrsim R, \quad \text{or } \big|B_R \cap \{m \cdot e^{i\hat{s}} \geq -1/2\}\big| \gtrsim R^2.$$

We conclude that we have either (10), or $\nu(B_R) \gtrsim R$, or

$$\big|B_R \cap \{m \cdot e^{is} \geq -1/2\}\big| \gtrsim R^2,$$

for all $s \in [s_0 - \pi/4, s_0 + \pi/4] \cup [s_0 + 3\pi/4, s_0 + 5\pi/4]$. This corresponds exactly to the three alternatives in the statement of Proposition 5. □

References

[1] Adams, D. R., and Hedberg, L. I. *Function spaces and potential theory*, vol. 314 of *Grundlehren Math. Wiss.* Berlin: Springer-Verlag, 1995.

[2] Alouges, F., Rivière, T., and Serfaty, S. Néel and cross-tie wall energies for planar micromagnetic configurations. *ESAIM Control Optim. Calc. Var. 8* (2002), 31–68. A tribute to J. L. Lions.

[3] Ambrosio, L., De Lellis, C., and Mantegazza, C. Line energies for gradient vector fields in the plane. *Calc. Var. Partial Differential Equations 9*, 4 (1999), 327–255.

[4] Ambrosio, L., Fusco, N., and Pallara, D. *Functions of bounded variation and free discontinuity problems*. Oxford Mathematical Monographs. The Clarendon Press, Oxford University Press, New York, 2000.

[5] Aviles, P., and Giga, Y. A mathematical problem related to the physical theory of liquid crystal configurations. In *Miniconference on geometry and partial differential equations, 2 (Canberra, 1986)*, vol. 12 of *Proc. Centre Math. Anal. Austral. Nat. Univ.* Austral. Nat. Univ., Canberra, 1987, pp. 1–16.

[6] Contreras Hip, A. A., Lamy, X., and Marconi, E. Generalized characteristics for finite entropy solutions of Burgers' equation. *Nonlinear Anal. 219* (2022), Paper No. 112804.

[7] De Lellis, C., and Otto, F. Structure of entropy solutions to the eikonal equation. *J. Eur. Math. Soc. (JEMS) 5*, 2 (2003), 107–145.

[8] DeSimone, A., Müller, S., Kohn, R. V., and Otto, F. A compactness result in the gradient theory of phase transitions. *Proc. Roy. Soc. Edinburgh Sect. A 131*, 4 (2001), 833–844.

[9] Ghiraldin, F., and Lamy, X. Optimal Besov differentiability for entropy solutions of the eikonal equation. *Commun. Pure Appl. Math. 73*, 2 (2020), 317–349.

[10] Jabin, P.-E., and Perthame, B. Compactness in Ginzburg-Landau energy by kinetic averaging. *Comm. Pure Appl. Math. 54*, 9 (2001), 1096–1109.

[11] Jin, W., and Kohn, R. V. Singular perturbation and the energy of folds. *J. Nonlinear Sci. 10*, 3 (2000), 355–390.

[12] Lamy, X., and Marconi, E. Stability of the vortex in micromagnetics and related models. *arXiv:2209.09662* (2022).

[13] Lamy, X., and Otto, F. On the regularity of weak solutions to Burgers' equation with finite entropy production . *Calc. Var. Partial Differential Equations 57*, 4 (2018).

[14] Marconi, E. Characterization of minimizers of Aviles-Giga functionals in special domains. *Arch. Ration. Mech. Anal. 242*, 2 (2021), 1289–1316.

[15] Marconi, E. Rectifiability of entropy defect measures in a micromagnetics model. *Advances in Calculus of Variations* (2021), 000010151520210012.

[16] Marconi, E. On the structure of weak solutions to scalar conservation laws with finite entropy production. *Calc. Var. Partial Differ. Equ. 61*, 1 (2022), 30. Id/No 32.

[17] Rivière, T., and Serfaty, S. Limiting domain wall energy for a problem related to micromagnetics. *Comm. Pure Appl. Math. 54*, 3 (2001), 294–338.