Another regularizing property of the 2D eikonal equation

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Abstract

A weak solution of the two-dimensional eikonal equation amounts to a vector field $m\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ such that $|m| = 1$ a.e. and $\operatorname{div} m = 0$ in $\mathcal{D}'(\Omega)$. It is known that, if m has some low regularity, e.g., continuous or $W^{1/3,3}$, then m is automatically more regular: locally Lipschitz outside a locally finite set. A long-standing conjecture by Aviles and Giga, if true, would imply the same regularizing effect under the Besov regularity assumption $m \in B^{1/3}_{p,\infty}$ for $p > 3$. In this note we establish that regularizing effect in the borderline case $p = 6$, above which the Besov regularity assumption implies continuity. If the domain is a disk and m satisfies tangent boundary conditions, we also prove this for p slightly below 6.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ an open set and $m\colon \Omega \to \mathbb{R}^2$ a weak solution of the eikonal equation

$$|m| = 1 \quad \text{a.e. in } \Omega, \qquad \operatorname{div} m = 0 \quad \text{in } \mathcal{D}'(\Omega)\,. \tag{1.1}$$

In a simply connected domain, this is equivalent to the usual eikonal equation $|\nabla u| = 1$ for $u\colon \Omega \to \mathbb{R}$ such that $im = \nabla u$, where i denotes rotation by $\pi/2$.

We are interested in regularizing features of the eikonal equation (1.1), of the form: if a weak solution m has a given low regularity, then m is locally Lipschitz outside a locally finite set. The latter property corresponds to being a *zero-energy state* of the Aviles-Giga energy, as defined and characterized in [12]. We are aware of two instances of this regularizing effect:

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- If m is continuous, then m is locally Lipschitz. This follows e.g. from [4] or [2, Lemma 2.2], see also [11].
- If m is $W^{1/3,3}$, then m is a zero-energy state [5]. See also [10, 9] for the $W^{1/2,2}$ case.

The results in [15] make it natural to conjecture another instance of this regularizing effect:

$$\left.\begin{array}{r} m \text{ solves (1.1)} \\ m \in B^{1/3}_{p,\infty}(\Omega) \text{ for some } p > 3 \end{array}\right\} \quad \Rightarrow \quad m \text{ is a zero-energy state}\,, \tag{1.2}$$

where the Besov regularity $m \in B^{1/3}_{p,\infty}$ is defined by

$$m \in B^{1/3}_{p,\infty}(\Omega) \quad \Leftrightarrow \quad \sup_{|h|>0} \frac{1}{|h|^{1/3}} \|m(\cdot + h) - m\|_{L^p(\Omega\cap(\Omega-h))} < \infty\,.$$

More precisely, the validity of the conjecture (1.2) is a necessary condition for the validity of the Aviles-Giga conjecture [1, p.9], see the discussion in [15, § 1.2]. The range $p > 3$ in (1.2) is sharp, since pure jump solutions $m = m^+\mathbf{1}_{x_1>0} + m^-\mathbf{1}_{x_1<0}$ (with $m^\pm \in \mathbb{S}^1$ s.t. $m_1^+ = m_1^-$) belong to the space $B^{1/3}_{3,\infty}$, which plays a critical role in the Aviles-Giga conjecture [8].

For $p > 6$, $B^{1/3}_{p,\infty}$ regularity implies continuity [16, § 2.7.1], so (1.2) is true. For $3 < p \leq 6$, this does not follow directly from the already known regularizing effects, since $B^{1/3}_{p,\infty}$ regularity does not imply continuity, nor $W^{1/3,3}$ regularity.[1] In this note we present a short argument solving the borderline case $p = 6$.

Theorem 1.1. *Let $\Omega \subset \mathbb{R}^2$ be an open set, and $m\colon \Omega \to \mathbb{R}^2$ a weak solution of the eikonal equation* (1.1)*. If $m \in B^{1/3}_{6,\infty}(\Omega)$, then m is locally Lipschitz outside a locally finite set.*

We rely on the characterization of zero-energy solutions of (1.1) established in [12]: a weak solution m is locally Lipschitz outside a locally finite set if

$$\begin{aligned} &\operatorname{div}\Phi(m) = 0 \quad \text{in } \mathcal{D}'(\Omega)\,, \qquad \forall \Phi \in \mathrm{ENT}\,, \\ &\mathrm{ENT} = \left\{ \Phi \in C^2(\mathbb{S}^1;\mathbb{R}^2)\colon \frac{d}{d\theta}\Phi(e^{i\theta}) \cdot e^{i\theta} = 0 \quad \forall \theta \in \mathbb{R} \right\}. \end{aligned} \tag{1.3}$$

Maps in ENT are called entropies, and for a weak solution m the distributions $\operatorname{div}\Phi(m)$ are the corresponding entropy productions. Entropies are characterized by the fact that all smooth solutions of the eikonal equation have zero entropy production (1.3). They were introduced in [7] to study compactness properties of sequences with bounded Aviles-Giga energy.

[1] An example showing that $B^{1/3}_{p,\infty} \not\subset W^{1/3,3} = B^{1/3}_{3,3}$ in any bounded domain can be constructed using a wavelet basis, as recalled e.g. in [3, Corollary 2.17(ii)].

Remark 1.2. In [15, Theorem 1.4] it is shown that, for a unit divergence-free vector field m as in (1.1) and $3 < p \leq 4$, local $B^{1/3}_{p,\infty}$ regularity is equivalent to local $L^{\frac{p}{3}}$ integrability of all entropy productions $\operatorname{div}\Phi(m)$. That equivalence could not be true for $p > 6$, since in that case $B^{1/3}_{p,\infty}$ maps are continuous, but the discontinuous map $m(x) = ix/|x|$ has zero entropy productions. There is however no direct obstruction to it being true for $p = 6$. In that case, Theorem 1.1 would imply that any solution of (1.1) with L^2 entropy productions is a zero-energy state, thus partly answering [15, Conjecture 1.5].

We let m_ε denote the convolution $m_\varepsilon = m * \rho_\varepsilon$ where $\rho_\varepsilon(z) = \varepsilon^{-2}\rho(z/\varepsilon)$ and ρ is a fixed kernel supported in B_1 with $\int_{B_1} \rho = 1$, $0 \leq \rho \leq 1$ and $|\nabla \rho| \leq 1$. The main ingredient in our proof of Theorem 1.1 shows Lipschitz regularity under the assumptions that m_ε stays away from zero and that entropy productions are in L^p for some $p > 1$.

Proposition 1.3. *Let $m \in B^{1/3}_{3,\infty}(B_2; \mathbb{S}^1)$ with $\operatorname{div} m = 0$ and $p > 1$. Assume that $\operatorname{div}\Phi(m) \in L^p(B_2)$ for all $\Phi \in \mathrm{ENT}$, and*

$$\limsup_{\varepsilon \to 0} \left(\inf_{B_1} |m_\varepsilon| \right) > 0.$$

Then m is Lipschitz in $B_{1/2}$.

The proof of Theorem 1.1 follows from Proposition 1.3 combined with the following property of $B^{1/3}_{6,\infty}$.

Lemma 1.4. *For any $m \in B^{1/3}_{6,\infty}(\Omega; \mathbb{S}^1)$, the set of points $x \in \Omega$ such that*

$$\limsup_{\varepsilon \to 0} \left(\inf_{B_r(x)} |m_\varepsilon| \right) = 0 \qquad \forall\, r \in (0, \operatorname{dist}(x, \Omega^c)),$$

is Lebesgue-negligible.

Proof of Theorem 1.1 from Proposition 1.3 and Lemma 1.4. Denote by $X \subset \Omega$ the negligible set of points in Lemma 1.4. If $x \in \Omega \setminus X$, there exists $r > 0$ such that $B_{2r}(x) \subset \Omega$ and

$$\limsup_{\varepsilon \to 0} \left(\inf_{B_r(x)} |m_\varepsilon| \right) > 0,$$

thus we can apply Proposition 1.3 to m appropriately rescaled in $B_{2r}(x)$, and we deduce that m is Lipschitz in $B_{r/2}(x)$. This implies $\operatorname{div}\Phi(m) = 0$ in $B_{r/2}(x)$, for any entropy Φ. Since this is valid for a.e. $x \in \Omega$ and $\operatorname{div}\Phi(m) \in L^2$ thanks to [15, Proposition 4.1], we infer that $\operatorname{div}\Phi(m) = 0$ in Ω, and may invoke [12, Theorem 1.3] to conclude. $\square$

1.1 Improved estimate under tangent boundary conditions in a disk

If Ω is simply connected and m solves (1.1), then there exists $u\colon \Omega \to \mathbb{R}$ such that $im = \nabla u$ and $|\nabla u| = 1$ a.e. in Ω. Motivated by physical considerations, and assuming Ω has Lipschitz boundary, natural boundary conditions for this function u are

$$u = 0 \quad \text{and} \quad \frac{\partial u}{\partial n} = -1 \qquad \text{on } \partial\Omega,$$

where $\partial u/\partial n$ denotes the exterior normal derivative [14]. In terms of m this corresponds to the tangential boundary condition

$$m = \tau_{\partial\Omega} \qquad \text{on } \partial\Omega\,, \tag{1.4}$$

where $\tau_{\partial\Omega} = in_{\partial\Omega}$ is the counterclockwise unit tangent to Ω.

For a solution $m \in B^{1/3}_{3,\infty}(\Omega)$, the kinetic formulation [13, 8] allows to define a one-sided trace of m on $\partial\Omega$ by the arguments in [17] or [6], and therefore make sense of this tangential boundary condition. If $m \in B^{1/3}_{p,\infty}(\Omega)$ for some $p > 3$, then m automatically has a trace on $\partial\Omega$ (see e.g. [16, § 3.3.3]). Here we will replace these trace considerations by requiring that m is extended equal to $i\nabla \operatorname{dist}(\cdot, \partial\Omega)$ outside Ω.

Specializing to the case of the disk $\Omega = B_1$, we therefore consider $m\colon B_4 \to \mathbb{S}^1$ such that $\operatorname{div} m = 0$ and

$$m(x) = i\frac{x}{|x|} \qquad \forall x \in B_4 \setminus B_1\,. \tag{1.5}$$

Under these boundary conditions, we have

Theorem 1.5. *Let $m \in B^{1/3}_{q,\infty}(B_4; \mathbb{S}^1)$ for some $(47 + \sqrt{553})/12 < q \le 6$ such that $\operatorname{div} m = 0$ and (1.5) holds. Then*

$$m(x) = i\frac{x}{|x|} \qquad \forall x \in B_1 \setminus \{0\}\,. \tag{1.6}$$

The vortex configuration given in (1.6) is the only zero-energy state under the boundary condition (1.5) over a disk, as characterized in [12, Theorem 1.2].

Remark 1.6. The ideas we use to prove Theorem 1.5 can be elaborated on to obtain that, under general assumptions on a smooth domain Ω, if m solves (1.1) in Ω with tangential boundary conditions (1.4) and has the regularity $m \in B^{1/3}_{q,\infty}$ for some $q > (47 + \sqrt{553})/12$, then Ω must be a disk and m given by (1.6). This works for instance if Ω is uniformly convex or analytic. We present only Theorem 1.5 in order to keep the presentation short and not-too-technical.

Plan of the article. In § 2 we present the proof of Proposition 1.3, relying on a lemma proved in § 3. In § 4 we give the proof of Lemma 1.4, and in § 5 we prove a refined version of a regularity estimate from [8]. Finally, the proof of Theorem 1.5 is given in § 6.

Notation. The notation $A \lesssim B$ stands for the existence of an absolute constant $C > 0$ such that $A \le C\,B$.

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2 Proof of Proposition 1.3

The proof of Proposition 1.3 uses the link between entropy productions and a kinetic formulation discovered in [13], and further explored in [8, 15]. Relevant to us are the following properties.

Proposition 2.1 ([8],[15, Proposition 4.2])**.** *Let $m \in B^{1/3}_{3,\infty}(\Omega;\mathbb{R}^2)$ a weak solution of the eikonal equation (1.1). There exists $\sigma \in \mathcal{M}(\Omega \times \mathbb{S}^1)$ such that*

$$e^{is} \cdot \nabla_x \mathbf{1}_{m(x)\cdot e^{is}>0} = \partial_s \sigma \quad \text{in } \mathcal{D}'(\Omega \times \mathbb{S}^1).$$

If $\operatorname{div} \Phi(m) \in L^p(\Omega)$ for all $\Phi \in \mathrm{ENT}$ and some $p > 1$, then $\sigma \in L^p(\Omega; \mathcal{M}(\mathbb{S}^1))$, that is, the measure

$$\nu = (\mathrm{proj}_\Omega)_\sharp |\sigma| = \int_{\mathbb{S}^1} |\sigma|(\cdot, ds) \in \mathcal{M}(\Omega)\,, \tag{2.1}$$

has an L^p density with respect to the Lebesgue measure.

With these notations, the main ingredient in the proof of Proposition 1.3 is the following lemma, which shows that any integral curve of the curl-free vector field im_ε must be almost straight, provided $\nu \in L^p$ and $|m_\varepsilon|$ stays away from zero along the curve.

Lemma 2.2. *Let $m \in B^{1/3}_{3,\infty}(B_2;\mathbb{S}^1)$ with $\operatorname{div} m = 0$, hence $im = \nabla u$ for some 1-Lipschitz function $u\colon B_2 \to \mathbb{R}$. Assume that $\nu \in L^p(B_2)$ for some $p > 1$, where ν is defined in (2.1). Let $\varepsilon \in (0,1)$. If, for some $T > 0$ and $c_0 \in (0,1)$, there is an integral curve $\gamma\colon [0,T] \to B_1$ such that*

$$\begin{aligned} &\dot\gamma = \nabla u_\varepsilon(\gamma) \quad \text{in } [0,T]\,, \\ \text{and} \quad &|m_\varepsilon| \ge c_0 > 0 \quad \text{on } \gamma([0,T])\,, \end{aligned}$$

then we have

$$u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0)) \ge |\gamma(T) - \gamma(0)| - \delta\,, \tag{2.2}$$

$$\text{and} \quad \gamma([0,T]) \subset [\gamma(0), \gamma(T)] + B_{\delta + \sqrt{\delta T}}\,, \tag{2.3}$$

$$\text{where} \quad \delta = C(\|\nu\|_{L^p}/c_0^2)^{\frac{p}{9p-6}}\,\varepsilon^{\frac{p-1}{9p-6}}\,T^{\frac{9p-7}{9p-6}}\,,$$

for some absolute constant $C > 0$.

Proof of Proposition 1.3 from Lemma 2.2. We let $u\colon B_2 \to \mathbb{R}$ be such that $\nabla u = im$. By assumption, there exist $c_0 > 0$ and a sequence $\varepsilon \to 0$ such that $|m_\varepsilon| \geq c_0$ in B_1. For any ε in that sequence, consider the maximal integral curve $\gamma_\varepsilon\colon (S_\varepsilon, T_\varepsilon) \to B_1$ solving

$$\gamma_\varepsilon(0) = 0, \quad \dot{\gamma}_\varepsilon = \nabla u_\varepsilon(\gamma_\varepsilon).$$

Since u_ε is 1-Lipschitz and $(d/dt)[u_\varepsilon(\gamma_\varepsilon)] = |\nabla u_\varepsilon|^2(\gamma_\varepsilon) \geq c_0^2$, we have $T_\varepsilon - S_\varepsilon \leq 2/c_0^2$. We fix $S_\varepsilon^* < 0 < T_\varepsilon^*$ such that $\gamma_\varepsilon((S_\varepsilon^*, T_\varepsilon^*)) \subset B_{1/2}$ and $X_\varepsilon = \gamma_\varepsilon(S_\varepsilon^*) \in \partial B_{1/2}$, $Y_\varepsilon = \gamma_\varepsilon(T_\varepsilon^*) \in \partial B_{1/2}$. Thanks to Lemma 2.2 applied on the time intervals $[S_\varepsilon^*, 0]$ and $[0, T_\varepsilon^*]$, these points $X_\varepsilon, Y_\varepsilon \in \partial B_{1/2}$ satisfy

$$u_\varepsilon(Y_\varepsilon) - u_\varepsilon(X_\varepsilon) \geq 1 - c\varepsilon^{\frac{p-1}{9p-6}},$$

for some constant c depending on $\|\nu\|_{L^p}$ and c_0. Extracting a subsequence $\varepsilon \to 0$, we deduce the existence of $X, Y \in \partial B_{1/2}$ such that $u(Y) - u(X) \geq 1$. Since u is 1-Lipschitz and $|X - Y| \leq 1$, this implies that $|X - Y| = 1$ and u is affine with slope 1 along the segment $[X, Y]$. Rescaling, we can apply this argument to deduce that, for any $x \in B_1$ such that $B_{2r}(x) \subset B_1$, there exists a direction $w_x \in \mathbb{S}^1$ such that u restricted to $x + [-rw_x, rw_x]$ is affine with slope 1. This implies that ∇u is constant (equal to w_x) along that segment. Two such segments starting from points in $B_{1/2}$ cannot cross inside $B_{2/3}$, and this implies that m is locally Lipschitz, see e.g. the proof of [12, Lemma 5.1]. $\square$

Proof of Lemma 2.2. Since u_ε is 1-Lipschitz and $(d/dt)[u_\varepsilon(\gamma)] = |\nabla u_\varepsilon|^2(\gamma) \geq c_0^2$, we know that

$$|\gamma(t) - \gamma(s)| \geq c_0^2|t - s| \qquad \text{for all } s, t \in [0, T]\,. \tag{2.4}$$

For $r \in [\varepsilon, \min(1/4, T)]$, to be fixed later, we decompose the time interval $[0, T]$ into $N - 1$ subintervals $[t_j, t_{j+1}]$, with

$$0 = t_1 < t_2 < \cdots < t_N = T, \quad \frac{r}{2} \leq t_{j+1} - t_j \leq r, \quad N \leq \frac{2T}{r}\,.$$

Setting $X_j = \gamma(t_j)$, we have $\gamma([t_j, t_{j+1}]) \subset B_r(X_j)$ since $|\dot{\gamma}| \leq 1$. Moreover, the inequality (2.4) implies $|X_i - X_j| \geq c_0^2|i - j|r/2$, and ensures therefore the bounded intersection property

$$\sum_{j=1}^{N} \mathbf{1}_{B_{4r}(X_j)} \lesssim \frac{1}{c_0^2}\mathbf{1}_{\bigcup_{j=1}^{N} B_{4r}(X_j)} \lesssim \frac{1}{c_0^2}\mathbf{1}_{\gamma([0,T]) + B_{4r}}\,. \tag{2.5}$$

Applying the estimate (3.1) in Lemma 3.1 in the next section, on each time interval $[t_j, t_{j+1}]$, we find, for any $\alpha > 0$,

$$u_\varepsilon(X_{j+1}) - u_\varepsilon(X_j) \geq (1 - \sqrt{\alpha})|X_{j+1} - X_j| - \frac{C}{\alpha}(\nu(B_{4r}(X_j)) + \varepsilon^{1/2}r^{1/2})\,,$$

where $C > 0$ is a generic absolute constant which may change from line to line in what follows. Summing over j, using that $N \leq 2T/r$, that $|\gamma(T) - \gamma(0)| \leq T$ and the property (2.5), we deduce

$$\begin{aligned}
&u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0)) \\
&\geq (1 - \sqrt{\alpha})|\gamma(T) - \gamma(0)| - \frac{C}{\alpha}\sum_{j=1}^{N} \nu(B_{4r}(X_j)) - C\frac{T}{\alpha}\sqrt{\frac{\varepsilon}{r}} \\
&\geq |\gamma(T) - \gamma(0)| - \sqrt{\alpha}T - \frac{C}{\alpha}\left(\frac{\nu(\gamma([0,T]) + B_{4r})}{c_0^2} + T\sqrt{\frac{\varepsilon}{r}}\right).
\end{aligned}$$

Choosing

$$\alpha = \left(\frac{\nu(\gamma([0,T]) + B_{4r})}{c_0^2 T} + \sqrt{\frac{\varepsilon}{r}}\right)^{\frac{2}{3}},$$

we obtain

$$u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0)) \geq |\gamma(T) - \gamma(0)| - CT\left(\frac{\nu(\gamma([0,T]) + B_{4r})}{c_0^2 T} + \sqrt{\frac{\varepsilon}{r}}\right)^{\frac{1}{3}}.$$

Using that ν has an L^p density and the Lebesgue measure of $\gamma([0,T]) + B_{4r}$ is at most $16rT$, this implies

$$u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0)) \geq |\gamma(T) - \gamma(0)| - CT\left(\frac{\|\nu\|_{L^p(B_2)}}{c_0^2}\frac{(rT)^{1-\frac{1}{p}}}{T} + \sqrt{\frac{\varepsilon}{r}}\right)^{\frac{1}{3}}.$$

Finally we choose

$$r = \left(\frac{c_0^2}{\|\nu\|_{L^p(B_2)}}\right)^{\frac{2p}{3p-2}} \varepsilon^{\frac{p}{3p-2}} T^{\frac{2}{3p-2}},$$

which gives

$$u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0)) \geq |\gamma(T) - \gamma(0)| - C\left(\frac{\|\nu\|_{L^p(B_2)}}{c_0^2}\right)^{\frac{p}{9p-6}} \varepsilon^{\frac{p-1}{9p-6}} T^{\frac{9p-7}{9p-6}}.$$

This proves (2.2). To show (2.3), for any $t \in (0,T)$ we apply (2.2) on the intervals $[0,t]$ and $[t,T]$ and, since u_ε is 1-Lipschitz, deduce the chain of inequalities

$$\begin{aligned}
|\gamma(T) - \gamma(0)| &\leq |\gamma(T) - \gamma(t)| + |\gamma(t) - \gamma(0)| \\
&\leq u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(t)) + u_\varepsilon(\gamma(t)) - u_\varepsilon(\gamma(0)) + 2\delta \\
&= u_\varepsilon(\gamma(T)) - u_\varepsilon(\gamma(0)) + 2\delta \\
&\leq |\gamma(T) - \gamma(0)| + 2\delta.
\end{aligned}$$

So we have the approximate reverse triangle inequality

$$|\gamma(T)-\gamma(t)|+|\gamma(t)-\gamma(0)|\leq|\gamma(T)-\gamma(0)|+2\delta\,,$$

which implies that $\gamma(t)$ must be close to the segment $[\gamma(0),\gamma(T)]$. More precisely, let $d=\operatorname{dist}(\gamma(t),[\gamma(0),\gamma(T)])=|\gamma(t)-X|$ for some $X\in[\gamma(0),\gamma(T)]$. Assume first that $X\notin\{\gamma(0),\gamma(T)\}$. Then $\ell_1=|\gamma(0)-X|$ and $\ell_2=|\gamma(T)-X|$ satisfy

$$\begin{aligned}&|\gamma(t)-\gamma(0)|=\sqrt{\ell_1^2+d^2}\,,\quad |\gamma(T)-\gamma(t)|=\sqrt{\ell_2^2+d^2}\,,\\ &\text{and } \ell_1+\ell_2=|\gamma(T)-\gamma(0)|\,.\end{aligned}$$

If $d\leq|\gamma(T)-\gamma(0)|$, then, using that $\sqrt{1+x}\geq 1+x/3$ for all $x\in[0,1]$, we deduce

$$\begin{aligned}|\gamma(T)-\gamma(t)|+|\gamma(t)-\gamma(0)|&\geq\ell_1\sqrt{1+d^2/\ell_1^2}+\ell_2\sqrt{1+d^2/\ell_2^2}\\ &\geq(\ell_1+\ell_2)\sqrt{1+\frac{d^2}{(\ell_1+\ell_2)^2}}\\ &\geq|\gamma(T)-\gamma(0)|+\frac{d^2}{3|\gamma(T)-\gamma(0)|}\,,\end{aligned}$$

hence $d\leq\sqrt{6\delta T}$. If $d\geq|\gamma(T)-\gamma(0)|$, then we have

$$|\gamma(T)-\gamma(t)|+|\gamma(t)-\gamma(0)|\geq 2d\geq|\gamma(T)-\gamma(0)|+d,$$

hence $d\leq 2\delta$. And if $X\in\{\gamma(0),\gamma(T)\}$, then we also have

$$|\gamma(T)-\gamma(t)|+|\gamma(t)-\gamma(0)|\geq|\gamma(T)-\gamma(0)|+d\,,$$

and $d\leq 2\delta$. In all cases we have $d\leq 2\delta+\sqrt{6\delta T}$, and, after adjusting the absolute constant C, this gives (2.3). □

3 Proof of Lemma 3.1

In this section we prove the following.

Lemma 3.1. *Let $r\in(0,1)$ and $m\in B^{1/3}_{3,\infty}(B_{4r};\mathbb{S}^1)$ such that $\operatorname{div} m=0$, hence $im=\nabla u$ for some 1-Lipschitz function $u\colon B_{4r}\to\mathbb{R}$.*

For $0<\varepsilon\leq r$, let $\gamma\colon[t_1,t_2]\to B_r$ solve $\dot\gamma=\nabla u_\varepsilon(\gamma)$, and denote $X_j=\gamma(t_j)$, $j=1,2$. Then for all $\alpha>0$ we have

$$\begin{aligned}u_\varepsilon(X_2)-u_\varepsilon(X_1)\geq(1-\sqrt{\alpha})|X_2-X_1|-\frac{C}{\alpha}(\nu(B_{4r})+\varepsilon^{1/2}r^{1/2})\\ +\int_{\gamma([t_1,t_2])\cap\{|m_\varepsilon|\leq 1-\sqrt{\alpha}\}}|m_\varepsilon|\,d\mathcal{H}^1\,,\end{aligned}\tag{3.1}$$

where $C>0$ is an absolute constant.

The proof of Lemma 3.1 relies on the two next lemmas, where we denote by D^h the finite difference operator

$$D^h f(x) = f(x+h) - f(x)\,, \tag{3.2}$$

for $h \in \mathbb{R}^2$.

Lemma 3.2. *For any $m\colon B_{4r} \to \mathbb{S}^1$ and $0 < \varepsilon \le r$ we have*

$$\int_{-r}^{r} \left(\sup_{\{x_1\}\times(-r,r)} (1-|m_\varepsilon|)^2 \right) dx_1 \lesssim \sup_{|h|\le\varepsilon} \frac{1}{|h|} \int_{B_{2r}} |D^h m|^3 \, dx.$$

Lemma 3.3. *For any $r \in (0,1)$ and $m \in B^{1/3}_{3,\infty}(B_{4r};\mathbb{S}^1)$ such that* $\operatorname{div} m = 0$, *we have*

$$\frac{1}{|h|} \int_{B_{2r}} |D^h m|^3 \, dx \lesssim \nu(B_{4r}) + r^{1/2}|h|^{1/2} \quad \forall h \in B_r,$$

where ν is defined in (2.1).

As a consequence of Lemmas 3.2 and 3.3, under the assumptions of Lemma 3.1 we have

$$\int_{-r}^{r} \left(\sup_{\{x_1\}\times(-r,r)} (1-|m_\varepsilon|)^2 \right) dx_1 \lesssim \nu(B_{4r}) + \varepsilon^{1/2} r^{1/2}\,, \tag{3.3}$$

and we can proceed to prove Lemma 3.1.

Proof of Lemma 3.1. We write

$$\begin{aligned} u_\varepsilon(X_2) - u_\varepsilon(X_1) &= \int_{t_1}^{t_2} \dot\gamma(t)\cdot\nabla u_\varepsilon(\gamma(t))\,dt \\ &= \int_{\gamma([t_1,t_2])} |m_\varepsilon|\, d\mathcal{H}^1 \\ &\ge (1-\sqrt{\alpha})\mathcal{H}^1\left(\gamma([t_1,t_2]) \cap \{(1-|m_\varepsilon|)^2 \le \alpha\}\right) \\ &\quad + \int_{\gamma([t_1,t_2])\cap\{|m_\varepsilon|\le 1-\sqrt{\alpha}\}} |m_\varepsilon|\, d\mathcal{H}^1\,. \end{aligned}$$

Then we assume without loss of generality that $X_2 - X_1$ is along the x_1-axis, denote by π_1 the projection onto it, and use (3.3) to estimate

$$\begin{aligned} &\mathcal{H}^1\left(\gamma([t_1,t_2]) \cap \{(1-|m_\varepsilon|)^2 \le \alpha\}\right) \\ &\ge \mathcal{H}^1\left(\pi_1\left[\gamma([t_1,t_2]) \cap \{(1-|m_\varepsilon|)^2 \le \alpha\}\right]\right) \\ &\ge |X_2 - X_1| - \frac{C}{\alpha}\int_{-r}^{r}\left(\sup_{\{x_1\}\times(-r,r)} (1-|m_\varepsilon|)^2 \right) dx_1 \\ &\ge |X_2 - X_1| - \frac{C}{\alpha}\left(\nu(B_{4r}) + \varepsilon^{1/2} r^{1/2}\right)\,. \end{aligned}$$

Plugging this into the above estimate concludes the proof. ∎

Finally we give the proofs of Lemmas 3.2 and 3.3.

Proof of Lemma 3.2. For any fixed $x_1 \in (-r, r)$ we have

$$\begin{aligned}\sup_{\{x_1\}\times(-r,r)} (1-|m_\varepsilon|)^2 &\leq \frac{1}{r}\int_{-r}^{r} (1-|m_\varepsilon|)^2\, dx_2 + \int_{-r}^{r} \left| \frac{d}{dx_2} \left[(1-|m_\varepsilon|)^2\right] \right| dx_2 \\ &\leq \frac{1}{r}\int_{-r}^{r} (1-|m_\varepsilon|)^{3/2}\, dx_2 + 2\int_{-r}^{r} (1-|m_\varepsilon|)|\nabla m_\varepsilon|\, dx_2.\end{aligned}$$

Integrating with respect to x_1 we deduce

$$\begin{aligned}&\int_{-r}^{r} \left(\sup_{\{x_1\}\times(-r,r)} (1-|m_\varepsilon|)^2 \right) dx_1 \\ &\leq \frac{1}{r}\int_{B_{2r}} (1-|m_\varepsilon|)^{3/2}\, dx + 2\int_{B_{2r}} (1-|m_\varepsilon|)|\nabla m_\varepsilon|\, dx \\ &\leq \frac{1}{r}\int_{B_{2r}} (1-|m_\varepsilon|)^{3/2}\, dx + 2\left(\int_{B_{2r}} (1-|m_\varepsilon|)^{3/2}\, dx\right)^{\frac{2}{3}} \left(\int_{B_{2r}} |\nabla m_\varepsilon|^3\, dx\right)^{\frac{1}{3}}.\end{aligned}$$

The conclusion follows from the estimates

$$\begin{aligned}\int_{B_{2r}} (1-|m_\varepsilon|)^{3/2}\, dx &\lesssim \varepsilon \sup_{|h|\leq\varepsilon} \frac{1}{|h|} \int_{B_{2r}} |D^h m|^3\, dx\,, \\ \int_{B_{2r}} |\nabla m_\varepsilon|^3\, dx &\lesssim \frac{1}{\varepsilon^2} \sup_{|h|\leq\varepsilon} \frac{1}{|h|} \int_{B_{2r}} |D^h m|^3\, dx\,,\end{aligned}$$

see e.g. [5, Step 6 in Proposition 3], and the fact that $\varepsilon \leq r$. □

Proof of Lemma 3.3. This follows from keeping track more precisely of each step in the proof of [8, Proposition 3.7], see Lemma 5.1. Choosing a test function ϕ in Lemma 5.1 such that $\mathbf{1}_{B_{2r}} \leq \phi \leq \mathbf{1}_{B_{3r}}$ and $|\nabla\phi| \lesssim 1/r$ gives Lemma 3.3. □

4 Proof of Lemma 1.4

Lemma 1.4 follows from a classical covering argument which provides the following.

Lemma 4.1. *Let $m \in B^s_{q,\infty}(\Omega; \mathbb{S}^1)$ for some $s \in (0,1)$ and $q \geq 1$ and $U \subset\subset \Omega$. For any $0 < \varepsilon < \mathrm{dist}(U, \Omega^c)/3$, there is a finite set $X^U_\varepsilon \subset U$ such that*

$$|m_\varepsilon| \geq \frac{1}{2} \quad \text{in } U \setminus \bigcup_{x\in X^U_\varepsilon} B_{5\varepsilon}(x)\,, \quad \text{and} \quad \mathrm{card}(X^U_\varepsilon) \lesssim \|m\|^q_{B^s_{q,\infty}(\Omega)} \varepsilon^{sq-2}\,.$$

Proof of Lemma 1.4 from Lemma 4.1. Thanks to Lemma 4.1 applied to $m \in B^{\frac{1}{3}}_{6,\infty}$ we can select a sequence $\varepsilon_k \to 0$ and a finite set X^U_* such that

$$\mathrm{dist}(X^U_{\varepsilon_k}, X^U_*) \to 0 \qquad \text{as } k \to \infty.$$

For $x \in U \setminus X_*^U$ and $r > 0$ such that $B_{2r}(x) \subset U \setminus X_*^U$, we have

$$B_r(x) \subset U \setminus \bigcup_{x \in X_\varepsilon^U} B_{5\varepsilon_k}(x)\,,$$

for large enough k, and therefore

$$\limsup_{\varepsilon \to 0} \left(\inf_{B_r(x)} |m_\varepsilon| \right) \geq \frac{1}{2}.$$

We conclude that the set of points considered in Lemma 1.4 is locally finite. $\square$

Proof of Lemma 4.1. Given $\alpha > 0$, to be fixed later, define

$$\mathcal{B}_\varepsilon^U := \left\{ x \in U : \fint_{B_\varepsilon} \fint_{B_\varepsilon} |m(x+y) - m(x+z)|^q \, dydz > \alpha \right\}. \tag{4.1}$$

Since $|m| = 1$ a.e. in Ω and $|\rho_\varepsilon| \leq 1/\varepsilon^2$, for any $x \in U \setminus \mathcal{B}_\varepsilon^U$ we have

$$\begin{aligned} \big|1 - |m_\varepsilon(x)|\big| &= \Big| \int_{B_\varepsilon} \Big(|m(x-z)| - \Big| \int_{B_\varepsilon} m(x-y)\, \rho_\varepsilon(y) dy \Big| \Big) \rho_\varepsilon(z)\, dz \Big| \\ &\leq \int_{B_\varepsilon} \int_{B_\varepsilon} |m(x-z) - m(x-y)|\, \rho_\varepsilon(y) \rho_\varepsilon(z) dydz \\ &\lesssim \left(\fint_{B_\varepsilon} \fint_{B_\varepsilon} |m(x-z) - m(x-y)|^q \, dydz \right)^{\frac{1}{q}} \lesssim \alpha^{1/q}\,. \end{aligned}$$

The last inequality follows from the fact that $x \in U \setminus \mathcal{B}_\varepsilon^U$ and the definition (4.1) of $\mathcal{B}_\varepsilon^U$. Hence, we may fix a small enough absolute constant $\alpha > 0$ so that

$$|m_\varepsilon| \geq \frac{1}{2} \qquad \text{in } U \setminus \mathcal{B}_\varepsilon^U\,. \tag{4.2}$$

For any $x \in \mathcal{B}_\varepsilon^U$ and $\tilde{x} \in B_\varepsilon(x)$ we have

$$\begin{aligned} &\fint_{B_{2\varepsilon}} \fint_{B_{2\varepsilon}} |m(\tilde{x}+y) - m(\tilde{x}+z)|^q \, dydz \\ &\geq \frac{1}{16\pi^2\varepsilon^4} \int_{B_\varepsilon} \int_{B_\varepsilon} |m(x+y) - m(x+z)|^q \, dydz \geq \frac{\alpha}{16}\,, \end{aligned}$$

so that

$$\int_{B_\varepsilon(x)} \fint_{B_{2\varepsilon}} \fint_{B_{2\varepsilon}} |m(\tilde{x}+y) - m(\tilde{x}+z)|^q \, dydz\, d\tilde{x} \geq \frac{\pi\varepsilon^2\alpha}{16}\,, \qquad \forall x \in \mathcal{B}_\varepsilon^U\,. \tag{4.3}$$

By the Vitali covering lemma there exists a finite set $X_\varepsilon^U \subset \mathcal{B}_\varepsilon^U$ such that

$$\mathcal{B}_\varepsilon^U \subset \bigcup_{x \in X_\varepsilon^U} B_{5\varepsilon}(x) \tag{4.4}$$

and the disks $\{B_\varepsilon(x)\colon x\in X_\varepsilon^U\}$ are pairwise disjoint. Recalling (4.3) we infer

$$\varepsilon^2\,\mathrm{card}(X_\varepsilon^U)\lesssim \int_{U+B_\varepsilon} \fint_{B_{2\varepsilon}}\fint_{B_{2\varepsilon}} |m(x+y)-m(x+z)|^q\,dydz\,dx \lesssim \|m\|^q_{B^s_{q,\infty}}\varepsilon^{sq}.$$

The last inequality follows from the definition of $B^s_{q,\infty}$ regularity. This implies the bound $\mathrm{card}(X_\varepsilon^U)\lesssim \|m\|^q_{B^s_{q,\infty}(\Omega)}\varepsilon^{sq-2}$. Combining this with (4.2) and the inclusion (4.4) concludes the proof. □

5 Refined Besov estimate

The proof of $B^{1/3}_{3,\infty}$ regularity in [8, Proposition 3.7] provides an estimate which can be expressed more precisely than the one stated there.

Lemma 5.1. *Let $m\in B^{1/3}_{3,\infty}(\Omega;\mathbb{R}^2)$ a weak solution of the eikonal equation* (1.1) *and $\nu\in\mathcal{M}(\Omega)$ as in* (2.1). *For any $\phi\in C^1_c(\Omega)$ and $0<\eta<\mathrm{dist}(\mathrm{supp}(\phi),\Omega^c)$, we have*

$$\sup_{|h|\le\eta}\int_\Omega |m(x+h)-m(x)|^3\phi^2(x)\,dx \lesssim \eta\sup_{|h|\le\eta}\int_\Omega \phi^2(x+h)\,\nu(dx) + \eta^{3/2}\int_\Omega |\phi|^{1/2}(x)|\nabla\phi|^{3/2}(x)\,dx\,. \tag{5.1}$$

Proof of Lemma 5.1. Recall the finite difference operator D^h defined in (3.2). The calculations in [8, Lemma 3.9] provide an identity for h-derivatives of the quantity

$$\Delta^{\varepsilon,\delta}(h,x)=\iint_{\mathbb{S}^1\times\mathbb{S}^1}\varphi_\delta(s-t)D^h\chi_\varepsilon(x,t)D^h\chi_\varepsilon(x,s)\,e^{it}\wedge e^{is}\,dt\,ds,$$

where

$$\varphi_\delta=\varphi*\gamma_\delta,\quad \varphi(t)=\mathbf{1}_{\cos(t)\sin(t)>0}-\mathbf{1}_{\cos(t)\sin(t)<0},$$
$$\chi_\varepsilon(x,t)=\left(\mathbf{1}_{e^{it}\cdot m(x)>0}\right)*\rho_\varepsilon(x)\,.$$

Here $\gamma_\delta(t)=\delta^{-1}\gamma_\delta(t/\delta)$ for a fixed smooth even kernel $\gamma_\delta\in C^1_c(-1,1)$, and ρ_ε is a two-dimensional convolution kernel as above. Thanks to Proposition 2.1, the function χ_ε solves the kinetic equation

$$e^{it}\cdot\nabla_x\chi_\varepsilon=\partial_s\sigma_\varepsilon,\qquad \sigma_\varepsilon=\sigma*_x\rho_\varepsilon\,.$$

The calculations in [8, Lemma 3.9] imply, for $0<\varepsilon<\mathrm{dist}(\mathrm{supp}(\phi),\Omega^c)-\eta$, and scalar $h\in(-\eta,\eta)$,

$$\frac{d}{dh}\int_\Omega \Delta^{\varepsilon,\delta}(he_1,x)\phi^2(x)\,dx = \int_\Omega I^{\varepsilon,\delta}(h,x)\phi^2(x)\,dx - 2\int_\Omega \phi(x)\nabla\phi(x)\cdot A^{\varepsilon,\delta}(h,x)\,dx\,, \tag{5.2}$$

where $I^{\varepsilon,\delta}$ and $A^{\varepsilon,\delta}$ are given by

$$
\begin{aligned}
I^{\varepsilon,\delta} &= -2\int_{\mathbb{S}^1}\left(\int_{\mathbb{S}^1}\varphi_\delta'(s-t)\chi_\varepsilon(x,s)\sin s\,ds\right)\sigma_\varepsilon(x+he_1,dt)\\
&\quad+2\int_{\mathbb{S}^1}\left(\int_{\mathbb{S}^1}\varphi_\delta'(s-t)\chi_\varepsilon(x+he_1,s)\sin s\,ds\right)\sigma_\varepsilon(x,dt)\,,\\
A_1^{\varepsilon,\delta} &= 2\iint_{\mathbb{S}^1\times\mathbb{S}^1}\varphi_\delta(s-t)\sin s\,\cos t\,\chi_\varepsilon(x+he_1,s)D^{he_1}\chi_\varepsilon(x,t)\,dsdt\,,\\
A_2^{\varepsilon,\delta} &= 2\iint_{\mathbb{S}^1\times\mathbb{S}^1}\varphi_\delta(s-t)\sin s\,\sin t\,\chi_\varepsilon(x,s)D^{he_1}\chi_\varepsilon(x,t)\,dsdt\,.
\end{aligned}
$$

In [8], the second component of the vector field $A^{\varepsilon,\delta}$ has a slightly different expression but can be put in this form using the fact that φ_δ is an odd function. Using that φ_δ' is bounded in $L^1(\mathbb{S}^1)$, that $|\chi_\varepsilon|\le 1$, and the definition of $\nu=\int_{\mathbb{S}^1}|\sigma|(\cdot,ds)$, we see that

$$
\int_\Omega I^{\varepsilon,\delta}(h,x)\phi^2(x)\,dx \lesssim \sup_{|z|<\eta}\int_\Omega \phi^2(x+z)\,\nu(dx)\,. \tag{5.3}
$$

Further, the vector field $A^{\varepsilon,\delta}$ can be rewritten as

$$
A^{\varepsilon,\delta}(h,x)=\int_{B_\varepsilon}\left(F^{\varepsilon,\delta,x,h}(m(x+z+he_1))-F^{\varepsilon,\delta,x,h}(m(x+z))\right)\rho_\varepsilon(z)\,dz,
$$

for a Lipschitz vector field $F^{\varepsilon,\delta,x,h}$ (details can be found in [15, Lemma 4.10]), and this implies

$$
|A^{\varepsilon,\delta}(h,x)|\lesssim\int_{B_\varepsilon}|D^{he_1}m(x+z)|\,\rho_\varepsilon(z)\,dz\,.
$$

Using this and (5.3), integrating (5.2) with respect to h and passing to the limit as $\varepsilon\to 0$ and $\delta\to 0$, we infer

$$
\begin{aligned}
\frac{1}{\eta}\int_\Omega \Delta(he_1,x)\phi^2(x)\,dx &\lesssim \sup_{|z|\le\eta}\int_\Omega\phi^2(x+z)\,\nu(dx)\\
&\quad+\sup_{|z|\le\eta}\int_\Omega\phi(x)|D^z m(x)|\,|\nabla\phi|(x)\,dx,
\end{aligned}
$$

for all $h\in(-\eta,\eta)$, where $\Delta\gtrsim|D^{he_1}m|^3$ thanks to [8, Lemma 3.8]. This estimate does not depend on the specific choice of the direction e_1, so we deduce

$$
\begin{aligned}
\frac{1}{\eta}\sup_{|h|\le\eta}\int_\Omega|D^h m(x)|^3\phi^2(x)\,dx &\lesssim \sup_{|h|\le\eta}\int_\Omega\phi^2(x+h)\,\nu(dx)\\
&\quad+\sup_{|h|\le\eta}\int_\Omega\phi(x)|D^h m(x)|\,|\nabla\phi|(x)\,dx.
\end{aligned}
$$

Thanks to Young's inequality $3ab \le a^3 + 2b^{3/2}$, for any $\lambda > 0$ we have

$$\phi|D^h m||\nabla\phi| \le \frac{1}{3}\frac{\lambda}{\eta}\phi^2|D^h m|^3 + \frac{2}{3}\frac{\eta^{1/2}}{\lambda^{1/2}}\phi^{1/2}|\nabla\phi|^{3/2}\,.$$

Choosing λ small enough allows to absorb the term containing $\phi^2|D^h m|^3$ into the left-hand side, and infer (5.1). $\square$

6 Proof of Theorem 1.5

In this section, we give the proof of Theorem 1.5. Recall that m satisfies the boundary condition (1.5), which we copy here:

$$m(x) = i\frac{x}{|x|} \qquad \forall x \in B_4 \setminus B_1\,, \tag{6.1}$$

Then there is a 1-Lipschitz function $u\colon B_4 \to \mathbb{R}$ which satisfies $im = \nabla u$ and

$$u(x) = 1 - |x| \qquad \forall x \in B_4 \setminus B_1\,. \tag{6.2}$$

In the following lemma, we combine Lemma 2.2 with the fact that, in $B_3 \setminus B_1$, the vector field ∇u obtained from (6.2) points towards the origin, to show that, if $\nu \in L^p$ for some $p > 1$, and $|m_\varepsilon| \ge 1/2$ in a not-too-thin horizontal strip which lies above the origin, then we can find many integral curves of $im_\varepsilon = \nabla u_\varepsilon$ crossing the strip from top to bottom.

Lemma 6.1. *Let $m \in B^{1/3}_{3,\infty}(B_4;\mathbb{S}^1)$ such that $\operatorname{div} m = 0$ and (6.1) holds, hence $im = \nabla u$ for some 1-Lipschitz function $u\colon B_4 \to \mathbb{R}$ satisfying (6.2). Assume that $\nu \in L^p(B_2)$ for some $p > 1$, where ν is defined in (2.1). Then there exists a constant $K \ge 1$ depending on $\|\nu\|_{L^p}$ and p such that the following holds true. Let $\varepsilon \in (0,1)$ and assume that*

$$|m_\varepsilon| \ge \frac{1}{2} \quad \text{in the strip } S_{a,b} = \{a < x_2 < b\} \cap B_3\,,$$
$$\text{for some } 0 < a < b \le 1 \text{ such that } b - a \ge 2K\varepsilon^\alpha\,,$$
$$\text{where } \alpha = \alpha_p = \frac{p-1}{18p-12}\,.$$

Then, for every $\xi \in (-\sqrt{1-b^2}, \sqrt{1-b^2})$, there exists

$$x \in B_{K\varepsilon^\alpha}((\xi, b))\,,$$

and an integral curve $\gamma\colon [0,T] \to B_1$ such that

$$\dot\gamma = \nabla u_\varepsilon(\gamma)\,, \quad \gamma(0) = x\,, \quad \gamma(T) = (\xi', a)\,,$$

for some $\xi' \in (-\sqrt{1-a^2}, \sqrt{1-a^2})$.

Proof of Lemma 6.1. We assume

$$b - a \geq 2K\varepsilon^\alpha ,$$

for some constant $K \geq 1$ that will be adjusted during the proof. We let

$$y = \frac{a+b}{2} ,$$

and consider, for any $\zeta \in (-1,1)$, the maximal integral curve $\gamma_\zeta \colon (T_1^\zeta, T_2^\zeta) \to S_{a,b}$ solving

$$\dot\gamma_\zeta = \nabla u_\varepsilon(\gamma_\zeta) \quad \text{in } (T_1^\zeta, T_2^\zeta) , \qquad \gamma_\zeta(0) = (\zeta, y) .$$

Since $|m_\varepsilon| \geq 1/2$ in $S_{a,b}$, we have $(d/dt)u_\varepsilon(\gamma_\zeta) \geq 1/4$ and therefore

$$T_2^\zeta - T_1^\zeta \leq 4|\gamma_\zeta(T_2^\zeta) - \gamma_\zeta(T_1^\zeta)| \leq 16 ,$$

because u_ε is 1-Lipschitz. This implies in particular that γ_ζ can be extended continuously to $[T_1^\zeta, T_2^\zeta]$ and $\gamma_\zeta(T_i^\zeta) \in \partial S_{a,b}$ for $i = 1, 2$. Since (ζ, y) lies at distance at least $(b-a)/2$ from $\partial S_{a,b}$ and γ_ζ is 1-Lipschitz, we deduce also

$$\min(T_2^\zeta, -T_1^\zeta) \geq \frac{b-a}{2} \geq K\varepsilon^\alpha .$$

Thanks to Lemma 2.2, we know that u_ε increases with almost unit speed along the curve γ_ζ, which must therefore be almost straight:

$$u_\varepsilon(\gamma_\zeta(t)) - u_\varepsilon(\gamma_\zeta(s)) \geq |\gamma_\zeta(t) - \gamma_\zeta(s)| - \kappa\varepsilon^{2\alpha} , \tag{6.3}$$

$$\operatorname{dist}(\gamma_\zeta([s,t]), [\gamma_\zeta(s), \gamma_\zeta(t)]) \leq \frac{\kappa}{2}\varepsilon^\alpha \qquad \text{for all } s < t \in [T_1^\zeta, T_2^\zeta] , \tag{6.4}$$

for some constant $\kappa > 0$ depending on $\|\nu\|_{L^p}$ and p. This implies in particular that the image of γ_ζ is contained in a thin band around the line

$$L_\zeta = (\zeta, y) + \mathbb{R}w_\zeta , \quad w_\zeta = \frac{\gamma_\zeta(T_2^\zeta) - \gamma_\zeta(T_1^\zeta)}{|\gamma_\zeta(T_2^\zeta) - \gamma_\zeta(T_1^\zeta)|} ,$$

namely,

$$\gamma_\zeta([T_1^\zeta, T_2^\zeta]) \subset L_\zeta + B_{\kappa\varepsilon^\alpha} . \tag{6.5}$$

Next we gather some information about these integral curves. First, due to the explicit expression (6.2) of u outside B_1, there the vector field ∇u_ε always points towards the inside of B_1, and so any integral curve which intersects B_1 must stay in B_1 at later times. This implies that

$$\gamma_\zeta(T_2^\zeta) \in B_1 \cap \partial S_{a,b} = B_1 \cap \{x_2 = a \text{ or } b\} , \qquad \text{for } |\zeta| \leq \sqrt{1 - y^2} .$$

Second, if $K > 2\kappa$, then for any $\zeta \in [-\sqrt{1-y^2}, \sqrt{1-y^2}]$, the entering point $\gamma_\zeta(T_1^\zeta)$ and the exit point $\gamma_\zeta(T_2^\zeta)$ cannot lie both on the top horizontal line $\mathbb{R} \times \{b\}$ or both on the bottom horizontal line $\mathbb{R} \times \{a\}$. Indeed, in that case we would have $L_\zeta = \mathbb{R} \times \{b\}$ or $\mathbb{R} \times \{a\}$, hence the thin band $L_\zeta + B_{\kappa\varepsilon^\alpha}$ would not intersect the horizontal line $\mathbb{R} \times \{y\}$ which contains $\gamma_\zeta(0)$, contradicting the fact that by (6.5) the image of γ_ζ must be contained in that thin band.

The explicit expression (6.2) of u in $B_3 \setminus B_1$, also implies that for (ζ, y) outside B_1, the entering point $\gamma_\zeta(T_1^\zeta)$ of γ_ζ lies on the top horizontal line $\mathbb{R} \times \{b\}$. By the previous remark, for these curves the exit point $\gamma_\zeta(T_2^\zeta)$ must lie on the bottom horizontal line $\mathbb{R} \times \{a\}$. Since integral curves cannot intersect, for each $\zeta \in (-1, 1)$ we deduce the alternative

$$\begin{aligned} &\gamma_\zeta(T_1^\zeta) \in \mathbb{R} \times \{b\} \text{ and } \gamma_\zeta(T_2^\zeta) \in \mathbb{R} \times \{a\}, \\ \text{or} \quad &\gamma_\zeta(T_2^\zeta) \in \mathbb{R} \times \{b\} \text{ and } \gamma_\zeta(T_1^\zeta) \in \mathbb{R} \times \{a\}. \end{aligned} \tag{6.6}$$

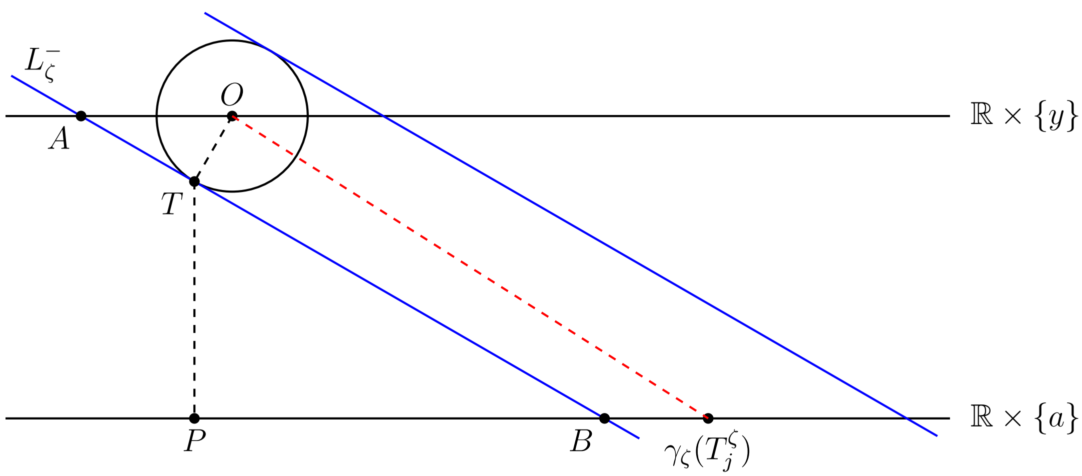

Figure 1: Estimating the horizontal width of the thin band $L_\zeta + B_{\kappa\varepsilon^\alpha}$.

Next we show that, in the alternative (6.6), the second case actually never happens. To this end, given $\zeta \in (-1, 1)$, we first estimate the horizontal width h_ζ of the thin band $L_\zeta + B_{\kappa\varepsilon^\alpha}$, in terms of the lengths of the segments $[\gamma_\zeta(0), \gamma_\zeta(T_j^\zeta)]$, for $j = 1, 2$.

Assume without loss of generality that the slope of L_ζ is negative. Let A and B denote the intersection points of the left-most line L_ζ^- of the band $L_\zeta + B_{\kappa\varepsilon^\alpha}$ with the horizontal lines $\mathbb{R} \times \{y\}$ and $\mathbb{R} \times \{a\}$, respectively (see Figure 1). Further, let $O = \gamma_\zeta(0)$, let T be the orthogonal projection of O onto the line L_ζ^-, and let P be the orthogonal projection of T onto the horizontal line $\mathbb{R} \times \{a\}$. Since the triangles OAT and TBP are similar, we have

$$\frac{|O - A|}{|O - T|} = \frac{|T - B|}{|T - P|}.$$

Denoting by $h_\zeta = 2|O-A|$, and using $|O-T| = \kappa\varepsilon^\alpha$ and $|T-P| \geq \frac{b-a}{2} - \kappa\varepsilon^\alpha \geq (K-\kappa)\varepsilon^\alpha$, we obtain

$$h_\zeta \leq \frac{\kappa}{K-\kappa}\, 2|T-B| \leq \frac{|T-B|}{2}\,,$$

provided $K > 5\kappa$. Now assuming $\gamma_\zeta(T_j^\zeta) \in \mathbb{R}\times\{a\}$, we have $|\gamma_\zeta(0) - \gamma_\zeta(T_j^\zeta)| \geq |T-B|$ and deduce from the above that

$$h_\zeta \leq \frac{|\gamma_\zeta(0) - \gamma_\zeta(T_j^\zeta)|}{2}\,. \tag{6.7}$$

Repeating the above argument between the horizontal lines $\mathbb{R}\times\{b\}$ and $\mathbb{R}\times\{y\}$, we obtain the estimate (6.7) for $j = 1, 2$.

For $|\zeta| \geq \sqrt{1-y^2}$ we have seen already that we are in the first case in (6.6). Assume by contradiction that the second case does happen. Then we can find $\zeta < \zeta' \in (-1,1)$ such that $|\zeta' - \zeta| \leq \kappa\varepsilon^\alpha$ and

$$\gamma_\zeta(T_1^\zeta), \gamma_{\zeta'}(T_2^{\zeta'}) \in \mathbb{R}\times\{b\} \quad \text{ and } \quad \gamma_\zeta(T_2^\zeta), \gamma_{\zeta'}(T_1^{\zeta'}) \in \mathbb{R}\times\{a\}\,.$$

According to (6.5) the thin bands

$$L_\zeta + B_{\kappa\varepsilon^\alpha} \quad \text{and} \quad L_{\zeta'} + B_{\kappa\varepsilon^\alpha}\,,$$

cannot fully cross inside the horizontal strip $\mathbb{R}\times[a,b]$. The size of the horizontal segment formed by the two intersection points of the left-most line of the left thin band and the right-most line of the right thin band with the horizontal line $\mathbb{R}\times\{z\}$ is an affine function $h(z)$ of z. So its minimum on $[a,b]$ is attained at a or b, and this implies

$$\min\big(h(a), h(b)\big) \leq h(y) \leq h_\zeta + h_{\zeta'}\,,$$

where the last inequality follows from the assumption $|\zeta' - \zeta| \leq \kappa\varepsilon^\alpha$. Assume for instance that the minimum in the above left-hand side is attained by $h(a)$, then using the property (6.7) we obtain

$$|\gamma_\zeta(T_2^\zeta) - \gamma_{\zeta'}(T_1^{\zeta'})| \leq h(a) \leq \frac{|\gamma_\zeta(0) - \gamma_\zeta(T_2^\zeta)|}{2} + \frac{|\gamma_{\zeta'}(0) - \gamma_{\zeta'}(T_1^{\zeta'})|}{2}\,.$$

Using the increasing property (6.3) of u_ε along these integral curves, we find

$$\begin{aligned}
&u_\varepsilon(\gamma_\zeta(T_2^\zeta)) - u_\varepsilon(\gamma_{\zeta'}(T_1^{\zeta'}))\\
&\geq u_\varepsilon(\gamma_\zeta(T_2^\zeta)) - u_\varepsilon(\gamma_\zeta(0)) + u_\varepsilon(\gamma_{\zeta'}(0)) - u_\varepsilon(\gamma_{\zeta'}(T_1^{\zeta'})) - |\gamma_\zeta(0) - \gamma_{\zeta'}(0)|\\
&\geq |\gamma_\zeta(0) - \gamma_\zeta(T_2^\zeta)| + |\gamma_{\zeta'}(0) - \gamma_{\zeta'}(T_1^{\zeta'})| - 3\kappa\varepsilon^\alpha\,.
\end{aligned}$$

But since u_ε is 1-Lipschitz and $|\gamma_\zeta(0) - \gamma_\zeta(T_2^\zeta)| \geq K\varepsilon^\alpha$, $|\gamma_{\zeta'}(0) - \gamma_{\zeta'}(T_1^{\zeta'})| \geq K\varepsilon^\alpha$, the above two estimates lead to the contradiction

$$K\varepsilon^\alpha \leq \frac{|\gamma_\zeta(0) - \gamma_\zeta(T_2^\zeta)|}{2} + \frac{|\gamma_{\zeta'}(0) - \gamma_{\zeta'}(T_1^{\zeta'})|}{2} \leq 3\kappa\varepsilon^\alpha\,.$$

This demonstrates our claim that all curves γ_ζ are in the first case of the above alternative (6.6), namely,

$$\gamma_\zeta(T_1^\zeta) \in \mathbb{R} \times \{b\} \text{ and } \gamma_\zeta(T_2^\zeta) \in \mathbb{R} \times \{a\}, \qquad \forall \zeta \in (-1,1).$$

We obtain therefore two functions $\xi_1, \xi_2 \colon (-1,1) \to \mathbb{R}$, characterized by

$$\gamma_\zeta(T_1^\zeta) = (\xi_1(\zeta), b), \qquad \gamma_\zeta(T_2^\zeta) = (\xi_2(\zeta), a).$$

As integral curves cannot cross, both functions ξ_1, ξ_2 are monotone increasing, and thanks to the explicit expression (6.2) of u outside B_1 we have

$$\begin{aligned} &\xi_1(-\sqrt{1-y^2}) < -\sqrt{1-b^2}, \quad \xi_1(\sqrt{1-y^2}) > \sqrt{1-b^2}, \\ &\xi_2(-\sqrt{1-y^2}) > -\sqrt{1-a^2}, \quad \xi_2(\sqrt{1-y^2}) < \sqrt{1-a^2}. \end{aligned}$$

Since ξ_2 is increasing, this implies $|\xi_2(\zeta)| < \sqrt{1-a^2}$ if $|\zeta| < \sqrt{1-y^2}$. And since ξ_1 is increasing, this implies that for any $\xi \in (-\sqrt{1-b^2}, \sqrt{1-b^2})$, we can find $\zeta_* \in (-\sqrt{1-y^2}, \sqrt{1+y^2})$ such that

$$\lim_{\substack{\zeta\to\zeta_*\\ \zeta<\zeta_*}} \xi_1(\zeta) = \xi_1(\zeta_*^-) \le \xi \le \xi_1(\zeta_*^+) = \lim_{\substack{\zeta\to\zeta_*\\ \zeta>\zeta_*}} \xi_1(\zeta).$$

By continuity of the flow generated by ∇u_ε, we can fix $\delta > 0$ and $T_1^* \in (T_1^{\zeta_*}, 0)$ such that, for all $\zeta \in (\zeta_* - \delta, \zeta_* + \delta)$, we have

$$T_1^\zeta < T_1^* \quad \text{and} \quad \gamma_\zeta(T_1^*) \in B_{\kappa\varepsilon^\alpha}(\xi_1(\zeta_*)).$$

We fix $\zeta' \in (\zeta_* - \delta, \zeta_*)$ and $\zeta'' \in (\zeta_*, \zeta_* + \delta)$, so that $\xi_1(\zeta') < \xi < \xi_1(\zeta'')$, and $\gamma_{\zeta'}(T_1^*)$ and $\gamma_{\zeta''}(T_1^*)$ belong to the thin horizontal band $\mathbb{R} \times [b - \kappa\varepsilon^\alpha, b]$. Thanks to the property (6.4) we deduce that the set

$$\Gamma = \gamma_{\zeta'}([T_1^{\zeta'}, T_1^*]) \cup \gamma_{\zeta''}([T_1^{\zeta''}, T_1^*]),$$

is contained in the thin horizontal band $\mathbb{R} \times [b - 2\kappa\varepsilon^\alpha, b]$. Moreover, the orthogonal projection of Γ onto the line $\mathbb{R} \times \{b\}$ contains $[\xi_1(\zeta'), \xi_1(\zeta'')]$ minus an interval of size at most $2\kappa\varepsilon^\alpha$, so that projection must intersect the interval $[\xi - \kappa\varepsilon^\alpha, \xi + \kappa\varepsilon^\alpha] \times \{b\}$. Thus we can find $\tilde\zeta \in \{\zeta', \zeta''\}$ and $\tilde T \in [T_1^{\tilde\zeta}, T_1^*]$ such that

$$x = \gamma_{\tilde\zeta}(\tilde T) \in B_{3\kappa\varepsilon^\alpha}((\xi, b)) \subset B_{K\varepsilon^\alpha}((\xi, b)),$$

provided $K \ge 3\kappa$. The curve $\gamma(t) = \gamma_{\tilde\zeta}(\tilde T + t)$ satisfies the conclusion of Lemma 6.1, with $T = T_2^{\tilde\zeta} - \tilde T$ and $\xi' = \xi_2(\tilde\zeta)$. $\square$

Proof of Theorem 1.5. If $m \in B^{1/3}_{q,\infty}$ for some $q > \frac{(47+\sqrt{553})}{12}$, then, applying Lemma 4.1, there exists a finite set $X_\varepsilon \subset B_1$ such that

$$|m_\varepsilon| \geq \frac{1}{2} \text{ in } B_2 \setminus \bigcup_{x \in X_\varepsilon} B_{5\varepsilon}(x), \qquad \operatorname{card}(X_\varepsilon) \lesssim \|m\|^q_{B^{1/3}_{q,\infty}(B_2)} \varepsilon^{\frac{q}{3}-2}.$$

Further, we have $\nu \in L^p$ for $p = q/3$ by [15, Proposition 4.2]. One can check directly that, for $5.876 \approx (47 + \sqrt{553})/12 < q \leq 6$, we have

$$2 - \frac{q}{3} < \alpha = \alpha_p = \frac{p-1}{18p-12}.$$

Then, for small enough $\varepsilon > 0$, we can find

$$0 < \varepsilon < a_N < b_N < a_{N-1} < b_{N-1} < \cdots < a_1 < b_1 \leq 1,$$

such that

$$\begin{aligned} &|m_\varepsilon| \geq \frac{1}{2} \quad \text{in } \left(\mathbb{R} \times \bigcup_{j=1}^N (a_j, b_j)\right) \cap B_2, \\ &N \leq \operatorname{card}(X_\varepsilon), \quad b_j - a_j > 2K\varepsilon^\alpha, \quad \mathcal{L}^1\left([0,1] \setminus \bigcup_{j=1}^N (a_j, b_j)\right) \leq \varepsilon^\delta, \end{aligned} \tag{6.8}$$

where $\delta = (\alpha - 2 + q/3)/2 > 0$.

Then, inductively applying Lemma 6.1 on each strip $\{a_j < x_2 < b_j\}$ starting from the point (ξ_1, b_1) with $\xi_1 = 0$, we build integral curves $\gamma_j : [0, T_j] \to B_1$ such that

$$\begin{aligned} &\dot{\gamma}_j = \nabla u_\varepsilon(\gamma_j), \quad \gamma_j(0) = X_j, \quad \gamma_j(T_j) = (\xi'_j, a_j) = Y_j, \\ &X_j \in B_{K\varepsilon^\alpha}((\xi_j, b_j)), \quad |\xi'_j| < \sqrt{1 - a_j^2}, \quad \xi_{j+1} = \xi'_j, \end{aligned}$$

for $j = 1, \ldots, N$. Finally, we set $\zeta = \xi'_N$ and write

$$\begin{aligned} u_\varepsilon(\zeta, 0) - u_\varepsilon(0, 1) = u_\varepsilon(\zeta, 0) - u_\varepsilon(Y_N) &+ \sum_{j=1}^N (u_\varepsilon(Y_j) - u_\varepsilon(X_j)) \\ &+ \sum_{j=2}^N (u_\varepsilon(X_j) - u_\varepsilon(Y_{j-1})) + (u_\varepsilon(X_1) - u_\varepsilon(0,1)). \end{aligned} \tag{6.9}$$

Using the increasing property (6.3) of the integral curves γ_j and (6.8), we have

$$\sum_{j=1}^N (u_\varepsilon(Y_j) - u_\varepsilon(X_j)) \geq \sum_{j=1}^N (b_j - a_j) - N(K + \kappa)\varepsilon^\alpha \geq \sum_{j=1}^N (b_j - a_j) - \varepsilon^\delta,$$

for small enough $\varepsilon > 0$. Next, since u_ε is 1-Lipschitz and using the properties (6.8) and $X_j \in B_{K\varepsilon^\alpha}((\xi_j, b_j))$, we deduce that

$$
\begin{aligned}
&|u_\varepsilon(\zeta,0) - u_\varepsilon(Y_N)| + \sum_{j=2}^{N} |u_\varepsilon(X_j) - u_\varepsilon(Y_{j-1})| + |u_\varepsilon(X_1) - u_\varepsilon(0,1)| \\
&\qquad \le \mathcal{L}^1\left([0,1] \setminus \bigcup_{j=1}^{N} (a_j, b_j)\right) + N\,K\varepsilon^\alpha \le \varepsilon^\delta\,.
\end{aligned}
$$

Putting the above two estimates into (6.9) and using $|u_\varepsilon(0,1)| \le \varepsilon$, we obtain

$$
u_\varepsilon(\zeta,0) \ge 1 - c\varepsilon^\delta.
$$

Letting $\varepsilon \to 0$ we deduce that the supremum of u on B_1 is at least 1, which forces $u(x) = 1 - |x|$ in B_1. This translates to (1.6) through $\nabla u = im$ as desired. □

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