On regularity and rigidity of 2×2 differential inclusions into non-elliptic curves

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Abstract

We study differential inclusions $Du \in \Pi$ in an open set $\Omega \subset \mathbb{R}^2$, where $\Pi \subset \mathbb{R}^{2\times 2}$ is a compact connected C^2 curve without rank-one connections, but non-elliptic: tangent lines to Π may have rank-one connections, so that classical regularity and rigidity results do not apply. For a wide class of such curves Π, we show that Du is locally Lipschitz outside a discrete set, and is rigidly characterized around each singularity. Moreover, in the partially elliptic case where at least one tangent line to Π has no rank-one connections, or under some topological restrictions on the tangent bundle of Π, there are no singularities. This goes well beyond previously known particular cases related to Burgers' equation and to the Aviles-Giga functional. The key is the identification and appropriate use of a general underlying structure: an infinite family of conservation laws, called entropy productions in reference to the theory of scalar conservation laws.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be an open set. We demonstrate regularity and rigidity properties of weakly differentiable maps $u\colon \Omega \to \mathbb{R}^2$ satisfying the differential inclusion

$$Du \in \Pi \qquad \text{a.e. in } \Omega,$$

where $\Pi \subset \mathbb{R}^{2\times 2}$ is a compact connected C^2 curve without rank-one connections, which is non-elliptic: tangent lines to Π may be generated by rank-one matrices. (Here and in the rest of the article, by curve we mean a one-dimensional submanifold, with or without boundary, in other words it is always embedded.)

Regularity of differential inclusions is a subject with a long history. The best known result is the analyticity of solutions of the Cauchy-Riemann equations – reformulated

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as a differential inclusion, this is the statement that if a function $u : \Omega \to \mathbb{R}^2$ satisfies $Du \in \mathrm{CO}_+(2) = \mathbb{R}_+\mathrm{SO}(2)$ everywhere, then u is analytic. In 1850 Liouville [Lio50] proved that if a C^3 function u satisfies the differential inclusion $Du \in \mathrm{CO}_+(3) = \mathbb{R}_+\mathrm{SO}(3)$, then it is a Möbius mapping. The optimal generalization of this result is still an open problem, which has inspired a large literature and to some extent motivated the development of the theory of higher dimensional quasiconformal mappings [BI82, Res94].

It is well known (see e.g. [Mül99]) that a necessary condition for regularity of a differential inclusion $Du \in K$ is that K should have no rank-one connections:

$$\mathrm{rank}(A - B) \neq 1 \quad \text{for all matrices } A \neq B \in K.$$

For differential inclusions in $\mathbb{R}^{2\times 2}$, a general sufficient condition is due to Šverák: if $K \subset \mathbb{R}^{2\times 2}$ is a smooth connected closed submanifold without rank-one connections that is *elliptic* (in the sense that its tangent spaces have no rank-one connections), then solutions of the differential inclusion $Du \in K$ are smooth [Š93, § 5].

For non-elliptic sets, we are not aware of any general regularity result, but we describe next two examples.

Example 1.1. Consider a bounded weak solution of Burgers' equation, which does not dissipate energy:

$$\partial_t v + \partial_x \frac{v^2}{2} = 0, \quad \partial_t \frac{v^2}{2} + \partial_x \frac{v^3}{3} = 0.$$

According to [Pan94] (see also [DLOW04]), the function v is both an entropy and anti-entropy solution of Burgers' equation, and by Oleinik's one-sided Lipschitz estimate must therefore be locally Lipschitz. In a simply connected domain, the two conservation laws satisfied by v are equivalent to the existence of u_1, u_2 such that

$$Du = \begin{pmatrix} \partial_t u_1 & \partial_x u_1 \\ \partial_t u_2 & \partial_x u_2 \end{pmatrix} = \begin{pmatrix} -\frac{v^2}{2} & v \\ -\frac{v^3}{3} & \frac{v^2}{2} \end{pmatrix} := \gamma(v),$$

so the Lipschitz regularity of v amounts to Lipschitz regularity of Du if $Du \in \Pi = \gamma([a, b])$. It can be checked that $\det(\gamma') = 0$, so this differential inclusion is nowhere elliptic.

Example 1.2. In our previous work [LLP20], motivated by connections with the Aviles-Giga functional [AG99, ADLM99, JK00, JOP02, LP18], we studied an explicit closed curve $K_0 \subset \mathbb{R}^{2\times 2}$ which has no rank-one connections, but is nowhere elliptic. There we established that solutions of $Du \in K_0$ enjoy some regularity: Du is locally Lipschitz outside a discrete set – but also some rigidity: Du is explicitly characterized in any convex neighborhood of a singularity.

Our proof in [LLP20] relied strongly on the explicit form of K_0 and its link with the eikonal equation, but since then we have been intrigued by the possibility that there might be a general result for differential inclusions into curves that do not have rank-one connections and are not necessarily elliptic.

This is what we establish in this article: we prove regularity and rigidity for differential inclusions $Du \in \Pi$, where $\Pi \subset \mathbb{R}^{2\times 2}$ is a generic compact connected C^2 curve which has no rank-one connections but may not be elliptic. More precisely, a compact connected curve $\Pi \subset \mathbb{R}^{2\times 2}$ without rank-one connections is elliptic if and only if the quadratic estimate

$$|\det(A-B)| \geq c\,|A-B|^2 \qquad \forall A, B \in \Pi,$$

is valid for some $c > 0$ [Š93, § 5]. Here we assume only a weaker quartic estimate (1.1), which allows the tangent lines to have rank-one connections, while retaining some weak nondegeneracy. Our main theorem is

Theorem 1.3. *Let $\Pi \subset \mathbb{R}^{2\times 2}$ be a compact connected C^2 curve, with or without boundary. Assume that Π has no rank-one connections, and that it satisfies the nondegeneracy estimate*

$$|\det(A-B)| \geq c\,|A-B|^4 \qquad \forall A, B \in \Pi, \tag{1.1}$$

for some constant $c > 0$. For any open set $\Omega \subset \mathbb{R}^2$ and weakly differentiable map $u\colon \Omega \to \mathbb{R}^2$, if u solves the differential inclusion

$$Du \in \Pi \qquad \text{a.e. in } \Omega, \tag{1.2}$$

then Du is locally Lipschitz away from a locally finite set $\mathcal{S}$. Moreover, the singular set $\mathcal{S}$ is empty in the following (non-disjoint) cases:

- *if Π is partially elliptic (at least one tangent line to Π has no rank-one connections);*
- *if Π has a boundary;*
- *under some topological conditions on the tangent bundle $T\Pi$, to be made explicit in Theorem 1.7.*

Remark 1.4. Without the nondegeneracy assumption (1.1), Du might fail to be Lipschitz away from singularities. Consider for instance, for $q > 0$, the scalar conservation law

$$\partial_t v + \partial_x \frac{v^2|v|^q}{2+q} = 0,$$

whose characteristic curves are given by $x = x_0 + tv_0|v_0|^q$. For any bounded continuous and monotone nondecreasing initial condition $v_0(x)$, there is a solution $v(t,x)$ constant along characteristics for $t > 0$. That solution also solves

$$\partial_t \frac{v^2}{2} + \partial_x \frac{v|v|^{q+2}}{3+q} = 0,$$

and provides a solution of the differential inclusion

$$Du \in \Pi := \left\{ \begin{pmatrix} -\frac{w^2|w|^q}{2+q} & w \\ -\frac{w|w|^{q+2}}{3+q} & \frac{w^2}{2} \end{pmatrix} : |w| \le \|v_0\|_\infty \right\},$$

which satisfies

$$\det(A-B) \ge c|A-B|^{4+q} \qquad \forall A, B \in \Pi. \tag{1.3}$$

Arguing as in [COW08, Proposition 4.3], one can choose the initial data v_0 in a way that $v(t,x)$ is Hölder continuous with Hölder exponent not better than $1/(1+q)$. Zero-energy states of generalized Aviles-Giga energies [BP17, LLP22], would also provide closed curves Π satisfying (1.3) and solutions of $Du \in \Pi$ for which Du is not better than $1/(1+q)$-Hölder continuous, see in particular [BP17, Remark 4.3].

In view of Remark 1.4, the following open question is very natural:

Question: If $\Pi \subset \mathbb{R}^{2\times 2}$ is a compact connected C^2 curve satisfying (1.3) for some $q > 0$, can one deduce that any solution u of $Du \in \Pi$ a.e. in an open set $\Omega \subset \mathbb{R}^2$ is locally $C^{1,\frac{1}{1+q}}$ away from a locally finite set?

In fact, one could even ask: if $\Pi \subset \mathbb{R}^{2\times 2}$ is a compact connected C^2 curve without rank-one connection, can one deduce that any solution of $Du \in \Pi$ is locally C^1 away from a locally finite set?

Our method fails to answer these questions because we bootstrap from an initial low regularity $Du \in B^{1/3}_{p,\infty,\mathrm{loc}}$ for some $p > 3$, which we do not know how to obtain if (1.1) is not satisfied (see Lemma A.1).

1.1 Strategy of proof: entropy productions

Applying a homothety, we can assume without loss of generality that Π has length at most 2π. We fix $\gamma\colon I \to \Pi$ an arc-length parametrization of class C^2, where I is either a segment $[a,b] \subset \mathbb{R}$ ($a < b < a + 2\pi$) in the case with boundary, or $I = \mathbb{R}/2\pi\mathbb{Z}$ in the case without boundary.

Our strategy to prove Theorem 1.3 is to obtain a large family of nonlinear conservation laws, which can then be used to deduce regularity and rigidity.

The basic principle is as follows. The differential inclusion $Du \in \Pi = \gamma(I)$ implies $Du = \gamma(\theta)$ for some real-valued function θ. The standard identity $\nabla \cdot \operatorname{cof} Du = 0$ can be interpreted as two conservation laws for the function θ, one from each row of the matrix $\operatorname{cof} Du = \operatorname{cof} \gamma(\theta)$. If θ is a smooth function, then the chain rule provides an infinite family of conservation laws $\nabla \cdot \Xi(\theta) = 0$, called entropy productions in analogy with the theory of scalar conservation laws. (Any smooth map $\Xi\colon \mathbb{R} \to \mathbb{R}^2$ such that Ξ' is a linear combination of the two rows of $\operatorname{cof} \gamma'$ has this property.) If θ is not smooth, entropy productions are distributions which can in general not be computed via the chain rule, and may not vanish. One might however expect a partial converse statement: if all entropy productions vanish, then θ is (somewhat) regular. This is the type of regularity property on which Theorem 1.3 relies.

For uniformly convex scalar conservation laws, such regularity property is a consequence of regularity features of entropy solutions [Kru70]: if all entropy productions vanish, then the solution is locally Lipschitz, see Example 1.1.

In the theory of the Aviles-Giga functional, methods based on entropy productions were introduced in [DMKO01] to obtain compactness properties, inspired by similar arguments for scalar conservation laws [Tar79, Tar83], and have been widely used since then (see e.g. [DLO03, Lor14, DLI15, GL20]). The analog of the above regularity property is the characterization by Jabin, Otto and Perthame [JOP02] of zero-energy solutions to the two-dimensional eikonal equation: if $\nabla \cdot m = 0$ and $|m| = 1$ in $\Omega \subset \mathbb{R}^2$, and all entropy productions vanish, then m is locally Lipschitz outside a discrete set, and moreover singularities are rigid. This result was improved by the last two authors in [LP18], where it was shown that only two specific entropy productions $\nabla \cdot \Sigma_1(m) = \nabla \cdot \Sigma_2(m) = 0$ are needed to obtain the same conclusion. And we improved it further in [LLP20] by showing that the original conservation law $\nabla \cdot m = 0$ is not even needed. This amounts to a regularity and rigidity result for the differential inclusion $Du \in K_0$ mentioned in Example 1.2, where $K_0 = \Sigma^\perp(\mathbb{S}^1) \subset \mathbb{R}^{2\times 2}$, and $\Sigma^\perp$ is the matrix-valued map whose two rows are $i\Sigma_1, i\Sigma_2$. Here we identify $\mathbb{R}^2 \approx \mathbb{C}$ and multiplication by i corresponds to rotation by $\pi/2$.

What we explain next is that differential inclusions into curves in $\mathbb{R}^{2\times 2}$ are endowed with a structure similar to entropy productions of the two-dimensional eikonal equation $\nabla \cdot m = 0$, $|m| = 1$, and this structure can be used to obtain analogs of [JOP02, LP18, LLP20].

Recall that $\gamma\colon I \to \mathbb{R}^{2\times 2}$ is an arc-length parametrization of Π, and consider $J = \exp(iI) \subset \mathbb{S}^1$, that is, $J = \mathbb{S}^1$ if $I = \mathbb{R}/2\pi\mathbb{Z}$ and $J = \{e^{i\theta}\}_{\theta\in[a,b]}$ if $I = [a,b]$. We let $\Gamma\colon J \to \mathbb{R}^{2\times 2}$ denote the cofactor matrix

$$\Gamma(e^{i\theta}) = \operatorname{cof} \gamma(\theta) \qquad \forall \theta \in I. \tag{1.4}$$

Any solution of the differential inclusion (1.2) satisfies $\operatorname{cof} Du \in \Gamma(J)$ a.e., so there exists $m\colon \Omega \to J \subset \mathbb{S}^1$ such that

$$\operatorname{cof} Du = \Gamma(m). \tag{1.5}$$

Therefore the identity $\nabla \cdot \operatorname{cof} Du = 0$ implies

$$\nabla \cdot \Gamma_1(m) = \nabla \cdot \Gamma_2(m) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \tag{1.6}$$

where $\Gamma_1, \Gamma_2\colon J \to \mathbb{R}^2$ are the first and second rows of the matrix-valued map Γ. So the unit vector field m satisfies two conservation laws, similarly to the examples 1.1 and 1.2. If m were smooth, then the chain rule would provide an infinite family of conservation laws

$$\nabla \cdot \Phi(m) = 0 \qquad \text{in } \mathcal{D}'(\Omega),$$

for any smooth map $\Phi\colon J \to \mathbb{R}^2$ such that

$$\partial_\theta \Phi = \alpha_\Phi^1 \partial_\theta \Gamma_1 + \alpha_\Phi^2 \partial_\theta \Gamma_2, \qquad \alpha_\Phi^1, \alpha_\Phi^2\colon J \to \mathbb{R}.$$

In analogy with scalar conservation laws, we follow the terminology of [DMKO01] and call such maps Φ entropies, the distributions $\nabla \cdot \Phi(m)$ entropy productions, and we let

$$\mathrm{ENT}_\Gamma = \Big\{ \Phi \in C^2(J;\mathbb{R}^2)\colon \exists \alpha_\Phi^1, \alpha_\Phi^2 \in C^1(J;\mathbb{R}) \text{ s.t.} \\ \partial_\theta \Phi = \alpha_\Phi^1 \partial_\theta \Gamma_1 + \alpha_\Phi^2 \partial_\theta \Gamma_2 \Big\}. \tag{1.7}$$

Our map m is not smooth enough to apply the chain rule, so it is not obvious that entropy productions should vanish. If we manage to prove that they do, then the ideas of [JOP02, GMPS23] can be applied to obtain regularity and rigidity (see § 2 and 3). Therefore, the main ingredient in our proof of Theorem 1.3 is the following proposition, which shows that entropy productions do vanish provided m has some low fractional regularity (it is classical that this starting low regularity is guaranteed by the quartic nondegeneracy estimate (1.1), see Lemma A.1). We state it here in the nowhere elliptic case where $\det(\partial_\theta \Gamma) \equiv 0$ (i.e. all tangent lines to Π are generated by rank-one matrices), under the nondegeneracy assumption that $\det(\partial_\theta^2 \Gamma)$ does not vanish, which in that case is equivalent to the quartic estimate (1.1) (see Lemma B.1). It will a posteriori be valid in the general setting of Theorem 1.3.

Proposition 1.5. *Let $J \subset \mathbb{S}^1$ be compact and connected. Assume that $\Gamma \in C^2(J;\mathbb{R}^{2\times 2})$ satisfies $|\partial_\theta \Gamma| = 1$, $\det(\partial_\theta \Gamma) = 0$ and $|\det(\partial_\theta^2 \Gamma)| > 0$ on J. Then any solution $m\colon \Omega \to J \subset \mathbb{S}^1$ of (1.6) satisfies*

$$\nabla \cdot \Phi(m) = 0 \quad \textit{in } \mathcal{D}'(\Omega), \quad \forall \Phi \in \mathrm{ENT}_\Gamma, \tag{1.8}$$

provided $m \in B^{\frac{1}{3}}_{p,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1)$ for some $p > 3$, that is,

$$\sup_{|h|\leq 1} \frac{\|D^h m\|_{L^p(U)}}{|h|^{\frac{1}{3}}} < \infty, \qquad D^h m(x) = (m(x+h) - m(x))\mathbf{1}_{x,x+h\in\Omega}\,, \tag{1.9}$$

for all $U \subset\subset \Omega$.

A less general version of the above proposition also lies at the heart of our previous work [LLP20], in a special case where $\partial_\theta \Gamma(e^{i\theta}) = \lambda(e^{i\theta}) \otimes ie^{i\theta}$ for some $\lambda\colon \mathbb{S}^1 \to \mathbb{S}^1$, and the identity mapping is an entropy: $\mathrm{id}_{\mathbb{S}^1} \in \mathrm{ENT}_\Gamma$. Here our proof follows a similar road map, but requires new ingredients to deal with the more general setting.

The basic principle is as follows. In order to apply the chain rule, we consider a mollified map m_ε, but this destroys the nonconvex constraint $|m| = 1$: the identities showing that entropy productions vanish for a smooth m are not valid for m_ε. Our task consists in proving that the error terms thus introduced are negligible as $\varepsilon \to 0$. Standard commutator estimates play an important role, as in [DLI15], but they are not enough to conclude directly: they only serve to show that entropy productions are in $L^{p/3}_{\mathrm{loc}}$. As in [LLP20], we then need to bootstrap that information into eventually obtaining that entropy productions vanish. A crucial feature in [LLP20] was the special role played by the identity mapping: an entropy that can be extended to a linear mapping of $\mathbb{R}^2$. Here we do not have this structure in general and rely on a different argument. A well-designed decomposition and careful use of commutator estimates enable us to obtain identities relating entropy productions and weak limits of some error terms, see (4.1). Testing these identities with well-chosen entropies shows that the error terms vanish.

Remark 1.6. A key observation in [LLP20] was that different choices of extensions of entropies might provide different information in the limit, but our new argument allows us to use only classical radial extension, and provides a simpler proof of the main result in [LLP20].

The details of Theorem 1.3 are quite different in the two cases where the curve is either

- nowhere elliptic,
- or partially elliptic.

Next we describe the precise statements we obtain in these two cases.

1.2 Nowhere elliptic curves

1.2.1 The case without boundary

Here we consider a closed C^2 curve $\Pi = \gamma(\mathbb{R}/2\pi\mathbb{Z})$ that is nowhere elliptic, which in terms of the arc-length parametrization γ amounts to

$$\det(\gamma'(\theta)) = 0 \qquad \forall \theta \in \mathbb{R}.$$

This is the only case where the differential inclusion (1.2) may develop singularities. Specific details about rigidity of singularities depend on topological properties of the tangent bundle $T\Pi$, which, as a loop into the set of 2×2 rank-one matrices, induces a loop into the projective line $\mathbb{RP}^1 = \mathbb{S}^1/\{\pm 1\}$.

More precisely, let $\Psi \colon \mathbb{S}^1 \to \mathbb{RP}^1 = \mathbb{S}^1/\{\pm 1\}$ be the C^1 map such that the image of the rank-one matrix $(\operatorname{cof} \gamma'(\theta))^T$ is spanned by $\Psi(e^{i\theta})$. An explicit expression of Ψ in terms of γ is provided in Lemma 2.1 and Remark 2.2. The map Ψ carries a winding number $\deg(\Psi)$, which characterizes its homotopy class in $\pi_1(\mathbb{RP}^1)$.

We adopt here, following e.g. [BCL86, § VIII.B], the convention that the winding number is a half-integer: it is given by

$$\deg(\Psi) = \frac{\varphi_\Psi(2\pi) - \varphi_\Psi(0)}{2\pi} \in \frac{1}{2}\mathbb{Z}, \tag{1.10}$$

for any continuous phase $\varphi_\Psi\colon \mathbb{R}\to\mathbb{R}$ such that $\Psi(e^{i\theta})=\{\pm e^{i\varphi_\Psi(\theta)}\}$. The map Ψ is orientable, that is, can be lifted to a C^1 map $\Psi\colon \mathbb{S}^1\to\mathbb{S}^1$, if and only if $\deg(\Psi)\in\mathbb{Z}$, and in that case $\deg(\Psi)$ corresponds to the usual winding number for loops in $\mathbb{S}^1$.

Our precise description of regularity and rigidity properties of the differential inclusion into Π depends on the value of this winding number.

Theorem 1.7. *Let $\Pi\subset\mathbb{R}^{2\times 2}$ be a closed C^2 curve of length 2π, without rank-one connections and nowhere elliptic. Denote by $\gamma\in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{2\times 2})$ an arc-length parametrization of Π and assume that $\det(\gamma''(\theta))\neq 0$ for all $\theta\in\mathbb{R}$.*

For any open set $\Omega\subset\mathbb{R}^2$ and weakly differentiable map $u\colon\Omega\to\mathbb{R}^2$, if u solves the differential inclusion

$$Du\in\Pi=\gamma(\mathbb{R}/2\pi\mathbb{Z})\qquad a.e.\ in\ \Omega,$$

then Du is locally Lipschitz outside a locally finite set $\mathcal{S}\subset\Omega$.

Moreover, letting $\Psi\colon\mathbb{S}^1\to\mathbb{RP}^1=\mathbb{S}^1/\{\pm 1\}$ be the C^1 map such that the image of the rank-one matrix $(\operatorname{cof}\gamma'(\theta))^T$ is spanned by $\Psi(e^{i\theta})$, we have that $\mathcal{S}$ is empty if $|\deg(\Psi)|\notin\{1/2,1\}$, and singularities are rigid if $|\deg(\Psi)|\in\{1/2,1\}$.

More precisely:

(a) *If $|\deg(\Psi)|\notin\{1/2,1\}$, then $\mathcal{S}=\emptyset$ and $Du=\gamma(\theta)$ is constant along characteristic lines directed by $\Psi(e^{i\theta})$.*

(b) *If $|\deg(\Psi)|=1$, then the map Ψ can be lifted to a C^1 diffeomorphism $\Psi\colon\mathbb{S}^1\to\mathbb{S}^1$. Moreover, in any convex subset $U\subset\Omega$ containing a singular point $x_0\in U\cap\mathcal{S}$, we have $Du=\gamma(\theta)$ with $e^{i\theta}=\Psi^{-1}(v)$ and $v\colon U\to\mathbb{S}^1$ is given by*

$$v(x)=\tau\frac{x-x_0}{|x-x_0|}\qquad \text{for a.e. } x\in U,$$

for some $\tau\in\{\pm 1\}$.

(c) *If $|\deg(\Psi)|=1/2$, then the map $\Psi\colon\mathbb{S}^1\to\mathbb{RP}^1=\mathbb{S}^1/\{\pm 1\}$ is a C^1 diffeomorphism. Moreover, for any disk $B_{2r}(x_0)\subset\Omega$ centered at a singular point $\{x_0\}=B_{2r}(x_0)\cap\mathcal{S}$, we have $Du=\gamma(\theta)$ with $e^{i\theta}=\Psi^{-1}(\{\pm v\})$ and $v\colon B_r(x_0)\to\mathbb{S}^1$ is given by either*

$$v(x)=\frac{x-x_0}{|x-x_0|}\qquad \text{for a.e. } x\in B_r(x_0),$$

or there exists $\zeta\in\mathbb{S}^1$ such that

$$v(x)\begin{cases} =\frac{x-x_0}{|x-x_0|} & \text{for a.e. } x\in B_r(x_0)\cap\{(x-x_0)\cdot\zeta>0\},\\ \text{is Lipschitz in } B_r(x_0)\cap\{(x-x_0)\cdot\zeta\leq 0\}. \end{cases}$$

Remark 1.8. For nowhere elliptic curves, the nondegeneracy estimate (1.1) happens to be equivalent to the condition $\det(\gamma''(\theta)) \neq 0$, see Appendix B.

Remark 1.9. The rigid singularities in parts (b) and (c) of Theorem 1.7 correspond to *zero-energy states* of Aviles-Giga functionals. In case (b), the $\mathbb{S}^1$-valued map $w = i\Psi(e^{i\theta})$ is a zero-energy state of the Aviles-Giga functional as described in [JOP02]. In case (c), the $\mathbb{RP}^1$-valued map $v = i\Psi(e^{i\theta})$ is a zero-energy state of an unoriented Aviles-Giga functional, as described in [GMPS23]. In particular, optimality of our regularity statements follows from the optimality of the regularity statements in [JOP02, GMPS23]. It is also instructive to compare Theorem 1.7 with [Iqb00], where nowhere elliptic curves are used instead to construct very irregular solutions of related differential inclusions.

1.2.2 The case with boundary

Here we consider a compact connected nowhere elliptic curve with boundary. Since the curve is not closed, the differential inclusion (1.2) cannot have singularities with nontrivial winding numbers as in parts (b) and (c) of Theorem 1.7.

Theorem 1.10. *Let $\Pi \subset \mathbb{R}^{2\times 2}$ be a compact connected C^2 curve with non-empty boundary, without rank-one connections and nowhere elliptic. Denote by $\gamma \in C^2([a,b];\mathbb{R}^{2\times 2})$ an arc-length parametrization of Π and assume that $\det(\gamma''(\theta)) \neq 0$ for all $\theta \in [a,b]$.*

For any open set $\Omega \subset \mathbb{R}^2$ and weakly differentiable map $u \colon \Omega \to \mathbb{R}^2$, if u solves the differential inclusion

$$Du \in \Pi = \gamma([a,b]) \qquad a.e.\ in\ \Omega,$$

then $Du = \gamma(\theta)$ is locally Lipschitz in Ω, and constant along characteristic lines directed by $\Psi(e^{i\theta})$, where $\Psi \colon [a,b] \to \mathbb{RP}^1$ is the C^1 map such that the image of the rank-one matrix $(\operatorname{cof}\gamma'(\theta))^T$ is spanned by $\Psi(e^{i\theta})$.

The proof of Theorem 1.10 is essentially a reproduction of the proof of Theorem 1.7 case (a). This case with boundary is crucial in the proof of Theorem 1.12 in the partially elliptic case as described in § 1.3.

1.2.3 Examples of closed nowhere elliptic curves

It is natural to wonder whether there exist many curves $\Pi \subset \mathbb{R}^{2\times 2}$ satisfying the assumptions of Theorem 1.7. One important example is the curve K_0 studied in [LLP20], which is parametrized by

$$\gamma_2(t) = \frac{1}{2}[e^{it}]_c + \frac{1}{6}[e^{3it}]_a.$$

where, for $z \in \mathbb{C}$, $[z]_c$ and $[z]_a$ are the naturally associated conformal and anticonformal matrices,

$$[z]_c = \begin{pmatrix} \mathfrak{Re}\, z & -\mathfrak{Im}\, z \\ \mathfrak{Im}\, z & \mathfrak{Re}\, z \end{pmatrix}, \quad [z]_a = \begin{pmatrix} \mathfrak{Re}\, z & \mathfrak{Im}\, z \\ \mathfrak{Im}\, z & -\mathfrak{Re}\, z \end{pmatrix}. \tag{1.11}$$

Indeed there is a family of related examples, parametrized by

$$\gamma_k(t) = \frac{1}{2}[e^{it}]_c + \frac{1}{2(k+1)}[e^{(k+1)it}]_a \qquad \forall k \in \mathbb{N},\, k \geq 1. \tag{1.12}$$

That these satisfy the assumptions will be checked in § 5.1, and similar examples will be given in § 5.2. For this curve γ_k, the map Ψ appearing in Theorem 1.7 has winding number $\deg(\Psi) = k/2$, as can be inferred from the proof of Lemma 2.1.

In general, it is easy to check whether a closed C^2 curve Π parametrized by $\gamma \in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{2\times 2})$ is nowhere elliptic, as this simply amounts to the local condition $\det(\gamma') = 0$. It is also easy to check the nondegeneracy assumption (1.1) as it is equivalent to $|\det(\gamma'')| > 0$ (see Remark 1.8 and Appendix B). Let us denote by NE_* the set of such parametrizations:

$$\mathrm{NE}_* = \left\{\gamma \in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{2\times 2})\colon\ \det(\gamma') = 0 \text{ and } |\det(\gamma'')| > 0\right\}.$$

What is usually harder is to check the condition that $\Pi = \gamma(\mathbb{R}/2\pi\mathbb{Z})$ has no rank-one connections. We show however that the subset $\mathrm{NE}_{**} \subset \mathrm{NE}_*$ which corresponds to curves without rank-one connections is somewhat large, in the sense that it is open.

Proposition 1.11. *The set*

$$\mathrm{NE}_{**} = \Big\{\gamma \in \mathrm{NE}_*\colon \gamma(\mathbb{R}/2\pi\mathbb{Z}) \textit{ has no rank-one connections}\Big\},$$

is open in NE_* *for the* C^2 *topology. In particular it contains a neighborhood of each curve* γ_k *(1.12).*

1.3 Partially elliptic curves

Here we consider a curve $\Pi = \gamma(I)$ which is partially elliptic: we divide it into elliptic and non-elliptic points,

$$\Pi = \Pi_E \cup \Pi_{NE}, \qquad \Pi_E = \{\gamma(t)\colon\ \det(\gamma'(t)) \neq 0\}, \qquad \Pi_{NE} = \Pi \setminus \Pi_E,$$

and assume, in contrast with the previous section, that $\Pi_{NE} \neq \Pi$. In this case the differential inclusion (1.2) again cannot have singularities.

Theorem 1.12. *Let* $\Pi \subset \mathbb{R}^{2\times 2}$ *be a compact connected* C^2 *curve without rank-one connections and assume that* $\Pi_E \neq \emptyset$.

(i) For any connected open set $\Omega \subset \mathbb{R}^2$ *and weakly differentiable map* $u\colon \Omega \to \mathbb{R}^2$*, if* u *solves the differential inclusion*

$$Du \in \Pi \qquad \text{a.e. in } \Omega,$$

then either $Du \in \Pi_{NE}$ *a.e. or* Du *is constant.*

(ii) Assume moreover that Π *satisfies the quartic nondegeneracy estimate* (1.1)*, then*

(a) either Du *is constant,*

(b) or Du *is locally Lipschitz in* Ω*, takes values into one single connected component of* Π_{NE}*, and we are in the situation of Theorem 1.10.*

Part (*i*) of Theorem 1.12 follows from the unique continuation principle established recently in [DPGT23]. (A weaker version of this result, namely Du is locally constant in $\Omega_E = (Du)^{-1}(\Pi_E)$, which is sufficient for our purpose, could also be deduced from regularity properties of degenerate elliptic equations in two variables established in [Lle23, Theorem 1.9].) Under a low regularity assumption which is implied by the quartic nondegeneracy estimate (1.1), this first conclusion can be strengthened to Du being either constant or with values into a single connected component of Π_{NE}.

1.4 Plan of the article

In § 2 we give the proofs of Theorems 1.7 and 1.10 about nowhere elliptic curves. In § 3 we give the proof of Theorem 1.12 about partially elliptic curves. In § 4 we show Proposition 1.5. In § 5 we provide examples of closed nowhere elliptic curves and the proof of Proposition 1.11. In the appendices we collect and prove various technical tools that are used in the course of the article, in particular in Appendix A we establish the initial low fractional regularity which follows from the nondegeneracy estimate (1.1).

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2 The nowhere elliptic case: proofs of Theorems 1.7 and 1.10

Proposition 1.5 is the main ingredient in the proofs of Theorems 1.7 and 1.10 and will be proved in § 4. Here we show how Theorems 1.7 and 1.10 follow from Proposition 1.5.

First note that thanks to Lemma A.1, under the assumptions of Theorem 1.3, in particular the quartic estimate (1.1), the map m given by $Du = \operatorname{cof}\Gamma(m)$ (1.5) has the fractional regularity $B^{1/3}_{4,\infty,\mathrm{loc}}$ and we can apply Proposition 1.5: the map m solves the family of conservation laws (1.8). In this section we use this family of conservation laws to obtain the conclusions of Theorems 1.7 and 1.10.

This is done in several steps and the map Ψ appearing in Theorems 1.7 and 1.10 plays a crucial role. First we check in § 2.1 that the map $\tilde{m} = i\Psi(m)$ is a zero-energy state of the eikonal equation, in the sense of [JOP02] if Ψ is $\mathbb{S}^1$-valued, and in the sense of [GMPS23] if Ψ is $\mathbb{RP}^1$-valued. (This idea has already appeared in [LLP22, Theorem 4], see Remark 2.6.) In particular, this gives parts (b) and (c) of Theorem 1.7, in which cases Ψ is a C^1 diffeomorphism and m can have rigid singularities. Then in § 2.2.2, we use geometric arguments to show that the singular set of m is empty in the case where the curve is closed and Ψ has a high winding number. This gives part (a) of Theorem 1.7. The same geometric arguments allow to show that the singular set of m is empty if the curve has boundary, finally giving Theorem 1.10 in § 2.3.

The different cases in the conclusion of Theorem 1.7 are due to different topological properties of the tangent bundle of the curve Π, described by the map Ψ appearing in Theorem 1.7. First we give a more explicit form of this map Ψ (see Remark 2.2), which will be convenient for the proofs.

Lemma 2.1. *Assume that* $\gamma \in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{2\times 2})$ *satisfies* $|\gamma'(\theta)| = 1$, $\det(\gamma'(\theta)) = 0$ *and* $\det(\gamma''(\theta)) \neq 0$ *for all* $\theta \in \mathbb{R}$. *Then there exist* $\hat{\lambda}, \hat{\Psi} \in C^1(\mathbb{R};\mathbb{S}^1)$, *and integers* $k, \ell \in \mathbb{Z}$, *such that, for all* $\theta \in \mathbb{R}$,

$$\operatorname{cof}\gamma'(\theta) = \partial_\theta \Gamma(e^{i\theta}) = \hat{\lambda}(\theta) \otimes \hat{\Psi}(\theta), \tag{2.1}$$

where Γ *is defined by* (1.4), *and*

$$\hat{\Psi}(\theta + 2\pi) = e^{ik\pi}\hat{\Psi}(\theta), \quad \hat{\lambda}(\theta + 2\pi) = e^{i\ell\pi}\hat{\lambda}(\theta).$$

Moreover, $\hat{\lambda}, \hat{\Psi}$ *have strictly monotone phases:*

$$\hat{\lambda} = e^{i\varphi_\lambda},\ \hat{\Psi} = e^{i\varphi_\Psi},\ \text{with } \varphi_\lambda, \varphi_\Psi \in C^1(\mathbb{R};\mathbb{R}) \text{ such that } |\varphi_\lambda'|, |\varphi_\Psi'| > 0.$$

Proof of Lemma 2.1. Let $\gamma_c, \gamma_a \in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C})$ correspond to the conformal and anticonformal parts of γ, that is,

$$\gamma = [\gamma_c]_c + [\gamma_a]_a, \tag{2.2}$$

where, for $z \in \mathbb{C}$, $[z]_c$ and $[z]_a$ are the associated conformal and anticonformal matrices as in (1.11). Since Π is nowhere elliptic and γ is an arc-length parametrization, we have $0 = \det(\gamma') = |\gamma_c'|^2 - |\gamma_a'|^2$ and $1 = |\gamma'|^2 = 2|\gamma_c'|^2 + 2|\gamma_a'|^2$, and deduce that

$$|\gamma_c'| = |\gamma_a'| = \frac{1}{2}.$$

In particular, the maps $\gamma_c', \gamma_a' \colon \mathbb{R}/2\pi\mathbb{Z} \to \frac{1}{2}\mathbb{S}^1$ have well-defined winding numbers, $\deg(\gamma_c'), \deg(\gamma_a') \in \mathbb{Z}$. We may choose $\varphi_c, \varphi_a \in C^1(\mathbb{R};\mathbb{R})$ such that

$$\gamma_c'(\theta) = \frac{1}{2}e^{i\varphi_c(\theta)}, \qquad \gamma_a'(\theta) = \frac{1}{2}e^{i\varphi_a(\theta)},$$

and we have

$$\begin{aligned}\operatorname{cof}\gamma' &= \begin{pmatrix} -\sin\left(\frac{\varphi_c+\varphi_a}{2}\right)\sin\left(\frac{\varphi_c-\varphi_a}{2}\right) & -\sin\left(\frac{\varphi_c+\varphi_a}{2}\right)\cos\left(\frac{\varphi_c-\varphi_a}{2}\right) \\ \cos\left(\frac{\varphi_c+\varphi_a}{2}\right)\sin\left(\frac{\varphi_c-\varphi_a}{2}\right) & \cos\left(\frac{\varphi_c+\varphi_a}{2}\right)\cos\left(\frac{\varphi_c-\varphi_a}{2}\right) \end{pmatrix} \\ &= ie^{i\frac{\varphi_c+\varphi_a}{2}} \otimes ie^{i\frac{\varphi_a-\varphi_c}{2}},\end{aligned}$$

so we can define

$$\hat{\lambda}(\theta) = ie^{i\frac{\varphi_c(\theta)+\varphi_a(\theta)}{2}}, \qquad \hat{\Psi}(\theta) = ie^{i\frac{\varphi_a(\theta)-\varphi_c(\theta)}{2}}.$$

By definition of the winding numbers $\deg(\gamma_a')$, $\deg(\gamma_c')$, we have

$$(\varphi_a \pm \varphi_c)(\theta + 2\pi) - (\varphi_a \pm \varphi_c)(\theta) = 2\pi(\deg(\gamma_a') \pm \deg(\gamma_c')).$$

Letting $k = \deg(\gamma_a') - \deg(\gamma_c')$ and $\ell = \deg(\gamma_a') + \deg(\gamma_c')$, this implies all claimed properties of $\hat{\lambda}, \hat{\Psi}$, except the strict monotonicity of their phases. That monotonicity follows from the nondegeneracy condition

$$0 < |\det(\gamma'')| = |\det(\hat{\lambda}' \otimes \hat{\Psi} + \hat{\lambda} \otimes \hat{\Psi}')| = |\hat{\lambda}'|\,|\hat{\Psi}'|,$$

where the penultimate equality follows from (2.1), and the last equality from the fact that $|\hat{\lambda}| = |\hat{\Psi}| = 1$. This implies $|\hat{\lambda}'|, |\hat{\Psi}'| > 0$ on $\mathbb{R}$. □

Remark 2.2. As a consequence of Lemma 2.1, we can define the map

$$\begin{aligned} \Psi \in C^1(\mathbb{S}^1;\mathbb{S}^1), \quad & \Psi(e^{i\theta}) = \hat{\Psi}(\theta) && \text{if } k \text{ is even}, \\ \Psi \in C^1(\mathbb{S}^1;\mathbb{RP}^1), \quad & \Psi(e^{i\theta}) = \{\pm\hat{\Psi}(\theta)\} && \text{if } k \text{ is odd}. \end{aligned}$$

This map has winding number $\deg(\Psi) = k/2$, and is a C^1 diffeomorphism if $|k| \in \{1, 2\}$. (Recall our convention (1.10) that the winding number of a loop in $\mathbb{RP}^1$ is a half-integer.) Similarly we can define λ, $\mathbb{S}^1$-valued or $\mathbb{RP}^1$-valued, so that $\partial_\theta\Gamma = \lambda\otimes\Psi$. It then follows that the image of $(\operatorname{cof}\gamma'(\theta))^T = (\partial_\theta\Gamma(e^{i\theta}))^T$ is spanned by $\Psi(e^{i\theta})$.

Remark 2.3. If $\gamma : [a, b] = I \to \mathbb{R}^{2\times 2}$ is a curve with boundary and satisfies the assumptions of Lemma 2.1, then the same construction as in the proof of Lemma 2.1 gives $\hat{\lambda}, \hat{\Psi} \in C^1([a, b]; \mathbb{S}^1)$ such that

$$\operatorname{cof}\gamma'(\theta) = \partial_\theta\Gamma(e^{i\theta}) = \hat{\lambda}(\theta) \otimes \hat{\Psi}(\theta) \qquad \forall\theta \in [a, b].$$

Then, as in Remark 2.2, we can define $\Psi(e^{i\theta}) = \hat{\Psi}(\theta), \lambda(e^{i\theta}) = \hat{\lambda}(\theta) \in C^1(J; \mathbb{S}^1)$ for $J = \exp(iI)$.

2.1 Relation to zero-energy states of Aviles-Giga functionals

First recall the notations from § 1.1. Specifically, we assume without loss of generality that Π has length at most 2π, and $\gamma\colon I \to \Pi$ is an arc-length parametrization of class C^2, where I is either a segment $[a,b] \subset \mathbb{R}$ ($a < b < a + 2\pi$) in the case with boundary, or $I = \mathbb{R}/2\pi\mathbb{Z}$ in the case without boundary. Further, we set $J = \exp(iI) \subset \mathbb{S}^1$.

Here, for $m : \Omega \to J$ solving the family of conservation laws (1.8), we check that the map $\tilde{m} = i\Psi(m)$ is a zero-energy state of the eikonal equation, in the sense of [JOP02] if Ψ is $\mathbb{S}^1$-valued, and in the sense of [GMPS23] if Ψ is $\mathbb{RP}^1$-valued. This is due to the fact that, if $\widetilde{\Phi}\colon \mathbb{S}^1 \to \mathbb{R}^2$ is an entropy of the eikonal equation in the sense of [DMKO01], that is,

$$\partial_\theta \widetilde{\Phi}(e^{i\theta}) \cdot e^{i\theta} = 0 \qquad \forall \theta \in \mathbb{R},$$

then (restricting to even maps $\widetilde{\Phi}$ in the case where Ψ is $\mathbb{RP}^1$-valued) the map $\widetilde{\Phi} \circ (i\Psi)$ is an entropy in the sense of the present paper, and therefore $\nabla \cdot \widetilde{\Phi}(\tilde{m}) = 0$, which is enough to apply the regularity and rigidity results of [JOP02, GMPS23] to the map $\tilde{m}$.

Since Ψ, and therefore $\widetilde{\Phi} \circ (i\Psi)$, is only C^1, we must first extend (1.8) to entropies Φ which are only C^1.

Lemma 2.4. *Assume that $\Gamma \in C^2(J; \mathbb{R}^{2\times 2})$ satisfies $\det(\partial_\theta^2 \Gamma) \neq 0$ on J. If $m\colon \Omega \to J$ satisfies*

$$\nabla \cdot \Phi(m) = 0 \qquad \text{in } \mathcal{D}'(\Omega), \qquad \forall \Phi \in \mathrm{ENT}_\Gamma,$$

then this identity is valid for all maps Φ in the larger class

$$\mathrm{ENT}^1_\Gamma = \Big\{ \Phi \in C^1(J;\mathbb{R}^2) \colon \exists \alpha^1_\Phi, \alpha^2_\Phi \in C^0(J;\mathbb{R}) \text{ s.t. } \partial_\theta \Phi = \alpha^1_\Phi \partial_\theta \Gamma_1 + \alpha^2_\Phi \partial_\theta \Gamma_2 \Big\}.$$

Proof of Lemma 2.4. This follows directly from the fact that ENT_Γ is dense in ENT^1_Γ in the C^1 topology. Let indeed $\Phi \in \mathrm{ENT}^1_\Gamma$, and $\alpha^j := \alpha^j_\Phi \in C^0(J;\mathbb{R})$. There exist $\alpha^j_k \in C^1(J;\mathbb{R})$ such that $\alpha^j_k \to \alpha^j$ uniformly.

If $J = \{e^{i\theta}\}_{\theta \in [a,b]} \subsetneq \mathbb{S}^1$, the formula

$$\Phi_k(e^{i\theta}) = \Phi(e^{ia}) + \int_a^\theta \left(\sum_{j=1}^2 \alpha^j_k \partial_t \Gamma_j \right)(e^{it})\, dt \qquad \forall \theta \in [a,b],$$

defines an entropy $\Phi_k \in \mathrm{ENT}_\Gamma$, and $\Phi_k \to \Phi$ in the C^1 topology since $\Phi_k(e^{ia}) = \Phi(e^{ia})$ and $\partial_\theta \Phi_k \to \partial_\theta \Phi$ uniformly.

If $J = \mathbb{S}^1$, the average of $\sum_{j=1}^2 \alpha^j_k \partial_\theta \Gamma_j$ on $\mathbb{S}^1$ tends to the average of $\sum_{j=1}^2 \alpha^j \partial_\theta \Gamma_j = \partial_\theta \Phi$ on $\mathbb{S}^1$, which is equal to zero. Therefore, applying Lemma D.1, we obtain $\tilde{\alpha}^j_k \in$

$C^1(\mathbb{S}^1;\mathbb{R})$ such that $\tilde\alpha_k^j \to \alpha^j$ uniformly and $\sum_{j=1}^2 \tilde\alpha_k^j \partial_\theta \Gamma_j$ has zero average on $\mathbb{S}^1$. Then the formula

$$\Phi_k(e^{i\theta}) = \Phi(1) + \int_0^\theta \left(\sum_{j=1}^2 \tilde\alpha_k^j \partial_t \Gamma_j \right)(e^{it})\, dt$$

defines an entropy $\Phi_k \in \mathrm{ENT}_\Gamma$, and $\Phi_k \to \Phi$ in the C^1 topology since $\Phi_k(1) = \Phi(1)$ and $\partial_\theta \Phi_k \to \partial_\theta \Phi$ uniformly. $\square$

Now that entropies are allowed to be C^1, we can check that entropies of the eikonal equation provide entropies in our setting.

Lemma 2.5. *Assume that* $\Gamma \in C^2(J;\mathbb{R}^{2\times 2})$ *satisfies* $|\partial_\theta \Gamma| = 1$, $\det(\partial_\theta\Gamma) = 0$ *and* $|\det(\partial_\theta^2 \Gamma)| > 0$ *on* J. *Let* $\widetilde\Phi \in C^1(\mathbb{S}^1;\mathbb{R}^2)$ *be such that* $\partial_\theta \widetilde\Phi(e^{i\theta}) \cdot e^{i\theta} = 0$ *for all* $\theta \in \mathbb{R}$, *and in the case where* Ψ *is* $\mathbb{RP}^1$*-valued assume in addition that* $\widetilde\Phi$ *is even. Then* $\Phi = \widetilde\Phi \circ (i\Psi) \in \mathrm{ENT}^1_\Gamma$.

Proof of Lemma 2.5. Let $\mu(e^{i\theta}) = \partial_\theta \widetilde\Phi(e^{i\theta}) \cdot ie^{i\theta}$, so that $\mu \in C^0(\mathbb{S}^1;\mathbb{R})$ and $\partial_\theta \widetilde\Phi(e^{i\theta}) = \mu\left(e^{i\theta}\right) ie^{i\theta}$ for all $\theta \in \mathbb{R}$. Note for later use that μ is odd if $\widetilde\Phi$ is even. In all cases, we have, with the notations of Lemma 2.1 (and Remark 2.3 in the case where $J \subsetneq \mathbb{S}^1$),

$$\Phi(e^{i\theta}) = \widetilde\Phi(i\hat\Psi(\theta)) = \widetilde\Phi(e^{i(\frac{\pi}{2} + \varphi_\Psi(\theta))}) \qquad \forall \theta \in I,$$

so

$$\partial_\theta \Phi(e^{i\theta}) = -\varphi'_\Psi(\theta)\mu(i\hat\Psi(\theta))\hat\Psi(\theta) \qquad \forall \theta \in I.$$

Using that $\partial_\theta \Gamma_j(e^{i\theta}) = \hat\lambda_j(\theta)\hat\Psi(\theta)$ and $|\hat\lambda|^2 = 1$, this becomes

$$\begin{aligned} &\partial_\theta \Phi(e^{i\theta}) = \hat\alpha_1(\theta)\partial_\theta\Gamma_1(e^{i\theta}) + \hat\alpha_2(\theta)\partial_\theta\Gamma_2(e^{i\theta}), \\ &\text{where } \hat\alpha_j(\theta) = -\varphi'_\Psi(\theta)\mu(i\hat\Psi(\theta))\hat\lambda_j(\theta). \end{aligned}$$

We clearly have $\hat\alpha_j \in C^0(I;\mathbb{R})$, and show next that this function is 2π-periodic in the case $J = \mathbb{S}^1$, distinguishing the cases where $k := 2\deg(\Psi)$ is even or odd.

Note that $\varphi_\Psi(\theta + 2\pi) = \varphi_\Psi(\theta) + k\pi$, so φ'_Ψ is 2π-periodic in both cases. If k is even, then $\hat\Psi$ is 2π-periodic. Since $\hat\lambda \otimes \hat\Psi$ is 2π-periodic this implies that $\hat\lambda$ is also 2π-periodic, and therefore so is $\hat\alpha_j$. If k is odd, then $\hat\Psi(\theta + 2\pi) = -\hat\Psi(\theta)$ and again since $\hat\lambda \otimes \hat\Psi$ is 2π-periodic this implies that $\hat\lambda(\theta + 2\pi) = -\hat\lambda(\theta)$. Moreover in that case we assume that $\widetilde\Phi$ is even and therefore μ is odd, so we also find that $\hat\alpha_j$ is 2π-periodic.

Hence $\alpha_j(e^{i\theta}) = \hat\alpha_j(\theta)$ is well-defined and continuous on J, which proves that $\Phi \in \mathrm{ENT}^1_\Gamma$. $\square$

Combining Lemmas 2.4 and 2.5, if $m : \Omega \to J$ solves the family of conservation laws (1.8), then the map $\tilde m = i\Psi(m)$ is an $\mathbb{S}^1$-valued zero-energy state of the Aviles-Giga energy [JOP02, § 2], or an $\mathbb{RP}^1$-valued zero-energy state of the unoriented Aviles-Giga energy [GMPS23, § 1]. We will use this structure of $\Psi(m)$ in the next two subsections to show Theorems 1.7 and 1.10.

Remark 2.6. In [LLP22, Theorem 4] we used a similar property (link with zero-energy states of Aviles-Giga) to prove regularity of solutions to a generalized eikonal equation $N(\nabla u) = 1$, where N is a strictly convex C^1 norm on $\mathbb{R}^2$: if all entropy productions (associated to that eikonal equation as in [LLP22, § 2]) vanish then ∇u is continuous outside a locally finite set. Using an appropriate equivalent of Lemma 2.5, one could obtain the same result for solutions of any equation of the form $\mathcal{A}(\nabla u) = 1$, where $\{\mathcal{A} = 1\}$ is a closed strictly convex C^2 curve in $\mathbb{R}^2$. Vanishing of all entropy productions can, in turn, be inferred from a zero-energy assumption (as in [LLP22]), or from a regularity assumption $\nabla u \in W^{1/3,3}_{\mathrm{loc}}$ and a commutator argument as in [DLI15] (in fact $\nabla u \in B^{1/3}_{3,c_0,\mathrm{loc}}$ is enough).

2.2 The case without boundary: proof of Theorem 1.7

Recall from (1.10) and Remark 2.2, we use the convention that the winding number $\deg(\Psi)$ of a continuous loop $\Psi\colon \mathbb{S}^1 \to \mathbb{RP}^1$ is a half-integer, and Ψ can be identified with a continuous loop $\Psi\colon \mathbb{S}^1 \to \mathbb{S}^1$ if and only if $\deg(\Psi) \in \mathbb{Z}$. In this subsection, we prove the following result, from which Theorem 1.7 follows immediately.

Proposition 2.7. *Assume that* $\Gamma \in C^2(\mathbb{S}^1; \mathbb{R}^{2\times 2})$ *satisfies* $|\partial_\theta \Gamma| = 1$, $\det(\partial_\theta\Gamma) = 0$ *and* $\det(\partial^2_\theta\Gamma) \neq 0$ *on* $\mathbb{S}^1$. *Let* $m\colon \Omega \to \mathbb{S}^1$ *solve the family of conservation laws* (1.8). *Then* m *is locally Lipschitz outside a locally finite set* $\mathcal{S} \subset \Omega$.

Moreover, let $\lambda, \Psi \in C^1(\mathbb{S}^1; \mathbb{RP}^1)$ *satisfy* $\partial_\theta\Gamma = \lambda \otimes \Psi$ *and* $k := 2\deg(\Psi) \in \mathbb{Z} \setminus \{0\}$, *and identify* Ψ *with a map in* $C^1(\mathbb{S}^1; \mathbb{S}^1)$ *if* k *is even. The map* m *satisfies the following additional properties depending on the value of* k *:*

(a) If $|k| \notin \{1, 2\}$, *then* $\mathcal{S} = \emptyset$ *and* m *is constant along characteristic lines directed by* $\Psi(m)$.

(b) If $|k| = 2$, *then in any convex subset* $U \subset \Omega$ *containing a singular point* $x_0 \in U \cap \mathcal{S}$, *there exists* $\tau \in \{\pm 1\}$ *such that*

$$\Psi(m(x)) = \tau \frac{x - x_0}{|x - x_0|} \qquad \textit{for a.e. } x \in U.$$

(c) If $|k| = 1$, *then for any disk* $B_{2r}(x_0) \subset \Omega$ *centered at a singular point* $\{x_0\} = B_{2r}(x_0) \cap \mathcal{S}$, *we have* $\Psi(m) = \{\pm v\}$ *in* $B_r(x_0)$, *with* v *as in Theorem 1.7 (c).*

Note that in case (b) where $|\deg(\Psi)| = 1$, the map Ψ induces a C^1 diffeomorphism $\mathbb{S}^1 \to \mathbb{S}^1$. And in case (c) where $|\deg(\Psi)| = 1/2$, the map Ψ is a C^1 diffeomorphism $\mathbb{S}^1 \to \mathbb{RP}^1$. Therefore the proof of Theorem 1.7 follows directly from Proposition 2.7 and Proposition 1.5 via the identification $Du = \operatorname{cof}\Gamma(m) = \gamma(\theta)$ with $m = e^{i\theta}$.

In the next two subsections we provide the proof of Proposition 2.7, first obtaining some regularity for $\Psi(m)$ which readily implies the cases (b) and (c), and then dealing with the remaining case (a).

2.2.1 Regularity of $\Psi(m)$

Thanks to Lemmas 2.4 and 2.5, the map $\tilde{m} = i\Psi(m)$ solves $\nabla \cdot \widetilde{\Phi}(\tilde{m}) = 0$ for all $\widetilde{\Phi} \in C^1(\mathbb{S}^1;\mathbb{R}^2)$ such that $\partial_\theta \widetilde{\Phi}(e^{i\theta}) \cdot e^{i\theta} = 0$ (restricting to even maps $\widetilde{\Phi}$ in the case where k is odd, i.e. Ψ is $\mathbb{RP}^1$-valued). As a consequence of [JOP02, Theorem 1.3] or [GMPS23, Theorem 1.2], we conclude that $\Psi(m)$ is locally Lipschitz continuous outside a locally finite set $\mathcal{S} \subset \Omega$, and moreover:

- if k is even, i.e. $\Psi \in C^1(\mathbb{S}^1;\mathbb{S}^1)$, then $v = \Psi(m)$ is as in part (b) of Proposition 2.7;
- if k is odd, i.e. $\Psi \in C^1(\mathbb{S}^1;\mathbb{RP}^1)$, then $\Psi(m) = \{\pm v\}$ for a map $v\colon \Omega \to \mathbb{S}^1$ which is as in part (c) of Proposition 2.7.

In particular, this concludes the proof of Proposition 2.7 in cases (b) and (c), where Ψ is a C^1 diffeomorphism, and it remains to treat case (a).

2.2.2 The case $|k| \notin \{1, 2\}$

In that case, Ψ is not injective, and the family of conservation laws (1.8) contains in fact much more information than the one used when applying the results of [JOP02, GMPS23] to $\tilde{m}$. This is why we can expect more regularity.

One way of taking advantage of the extra information contained in (1.8) is to consider a specific family of nonsmooth entropies, similar to the ones used in [DMKO01, Lemma 2.5] and which are also the main tool in [JOP02]. These nonsmooth entropies are related to kinetic formulations of conservation laws [LPT94, Per02], and we will use them precisely via the kinetic equation (2.4) they provide.

Lemma 2.8. *For any $\xi \in \mathbb{S}^1$ and any open arc $A_\xi \subset \mathbb{S}^1$ with extremities $a_\xi \neq b_\xi \in \Psi^{-1}(\{\pm\xi\})$, the map $\Phi^\xi \colon \mathbb{S}^1 \to \mathbb{R}^2$ given by*

$$\Phi^\xi(z) = \xi \mathbf{1}_{z \in A_\xi},$$

is a generalized entropy in the sense that there exist $\Phi_k \in \mathrm{ENT}_\Gamma$ such that

$$\Phi_k(z) \to \Phi^\xi(z) \qquad \text{as } k \to \infty, \quad \forall z \in \mathbb{S}^1.$$

Remark 2.9. In the case where Ψ is a diffeomorphism, there are only one ($|k| = 1$) or two ($|k| = 2$) choices for a_ξ and b_ξ. But here Ψ is not injective, $|\deg(\Psi)| = d \geq 3/2$, so a_ξ and b_ξ can be chosen among $2d \geq 3$ points, and this is where we gain a lot of information.

Proof of Lemma 2.8. The proof is very close to the proof of [LLP22, Lemma 15]. Fix $\theta_a, \theta_b \in \mathbb{R}$ such that

$$a_\xi = e^{i\theta_a}, \quad b_\xi = e^{i\theta_b}, \quad \theta_a < \theta_b < \theta_a + 2\pi,$$

and denote

$$\hat{\Psi}(\theta_a) = \tau_\xi\, \xi, \quad \hat{\Psi}(\theta_b) = \sigma_\xi\, \xi, \qquad \tau_\xi, \sigma_\xi \in \{\pm 1\}.$$

We may choose $t_0 \in \mathbb{R}$ such that

$$\{\hat{\lambda} \cdot e^{it_0} = 0\} \cap \{\theta_a, \theta_b\} = \emptyset.$$

We fix a smooth nonnegative kernel $\rho \in C_c^\infty(\mathbb{R})$ with support $\operatorname{supp} \rho \subset (0,1)$ and unit integral $\int \rho = 1$, and let $\rho_\delta(\theta) = \frac{1}{\delta}\rho(\frac{\theta}{\delta})$. Then we define $\alpha_\delta^1, \alpha_\delta^2 \in C^\infty(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R})$ by setting

$$\begin{aligned}
\alpha_\delta^1(\theta) &= \frac{\cos(t_0)}{e^{it_0} \cdot \hat{\lambda}(\theta_a)} \tau_\xi\, \rho_\delta\left(\theta - \theta_a\right) - \frac{\cos(t_0)}{e^{it_0} \cdot \hat{\lambda}(\theta_b)} \sigma_\xi\, \rho_\delta\left(\theta_b - \theta\right), \\
\alpha_\delta^2(\theta) &= \frac{\sin(t_0)}{e^{it_0} \cdot \hat{\lambda}(\theta_a)} \tau_\xi\, \rho_\delta\left(\theta - \theta_a\right) - \frac{\sin(t_0)}{e^{it_0} \cdot \hat{\lambda}(\theta_b)} \sigma_\xi\, \rho_\delta\left(\theta_b - \theta\right), \quad \forall \theta \in (\theta_a, \theta_a + 2\pi],
\end{aligned}$$

and extended as 2π-periodic functions. Note that these functions are supported in $(\theta_a, \theta_a + \delta) \cup (\theta_b - \delta, \theta_b) + 2\pi\mathbb{Z}$.

Then we define $\Phi_\delta^\xi \colon (\theta_a, \theta_a + 2\pi] \to \mathbb{R}^2$ by setting

$$\Phi_\delta^\xi(\theta) = \int_{\theta_a}^{\theta} \Big(\alpha_\delta^1(t) \partial_t \Gamma_1(e^{it}) + \alpha_\delta^2(t) \partial_t \Gamma_2(e^{it}) \Big)\, dt \qquad \forall \theta \in (\theta_a, \theta_a + 2\pi]\,.$$

Using the identity (2.1) defining $\hat{\lambda}$ and $\hat{\Psi}$, we see that it satisfies

$$\begin{aligned}
\Phi_\delta^\xi(\theta) &= \tau_\xi \int \frac{e^{it_0} \cdot \hat{\lambda}(t)}{e^{it_0} \cdot \hat{\lambda}(\theta_a)} \hat{\Psi}(t) \rho_\delta(t - \theta_a)\, dt, \qquad \text{if } \theta \in [\theta_a + \delta, \theta_b - \delta], \\
\Phi_\delta^\xi(\theta) &= \tau_\xi \int \frac{e^{it_0} \cdot \hat{\lambda}(t)}{e^{it_0} \cdot \hat{\lambda}(\theta_a)} \hat{\Psi}(t) \rho_\delta(t - \theta_a)\, dt \\
&\quad - \sigma_\xi \int \frac{e^{it_0} \cdot \hat{\lambda}(t)}{e^{it_0} \cdot \hat{\lambda}(\theta_b)} \hat{\Psi}(t) \rho_\delta(\theta_b - t)\, dt, \qquad \text{if } \theta \in [\theta_b, \theta_a + 2\pi].
\end{aligned}$$

Since $\tau_\xi \hat{\Psi}(\theta_a) = \sigma_\xi \hat{\Psi}(\theta_b) = \xi$ by definition of τ_ξ, σ_ξ, we deduce the limit

$$\lim_{\delta \to 0} \Phi_\delta^\xi(\theta) = \begin{cases} \xi & \text{if } \theta \in (\theta_a, \theta_b)\,, \\ 0 & \text{if } \theta \in [\theta_b, \theta_a + 2\pi]\,. \end{cases} \tag{2.3}$$

This corresponds exactly to $\Phi^\xi(e^{i\theta})$.

The function Φ_δ^ξ may not be 2π-periodic, that is why we need to modify the functions α_δ^j. Since, by the above,

$$\int_{\mathbb{R}/2\pi\mathbb{Z}} \left(\sum_{j=1}^{2} \alpha_\delta^j \partial_\theta \Gamma_j \right) d\theta = \Phi_\delta^\xi(\theta_a + 2\pi) \to 0 \quad \text{as } \delta \to 0,$$

by Lemma D.1 there exist $\tilde{\alpha}_\delta^j \in C^1(\mathbb{S}^1;\mathbb{R})$ such that $(\tilde{\alpha}_\delta^j - \alpha_\delta^j)$ tends to 0 uniformly as $\delta \to 0$, and $\sum_{j=1}^2 \tilde{\alpha}_\delta^j \partial_\theta \Gamma_j$ has zero average on $\mathbb{S}^1$. Then the map

$$\widetilde{\Phi}_\delta^\xi(e^{i\theta}) = \int_{\theta_a}^{\theta} \Big(\tilde{\alpha}_\delta^1(e^{it})\partial_t \Gamma_1(e^{it}) + \tilde{\alpha}_\delta^2(e^{it})\partial_t \Gamma_2(e^{it}) \Big)\, dt$$

is well-defined, $\widetilde{\Phi}_\delta^\xi \in \mathrm{ENT}_\Gamma$, and $\widetilde{\Phi}_\delta^\xi(e^{i\theta}) - \Phi_\delta^\xi(\theta) \to 0$ as $\delta \to 0$, for all $\theta \in (\theta_a, \theta_a + 2\pi]$. Thanks to (2.3), we deduce that $\widetilde{\Phi}_\delta^\xi(z) \to \Phi^\xi(z)$ as $\delta \to 0$, for all $z \in \mathbb{S}^1$. □

Combining Lemma 2.8 and (1.8), we see by dominated convergence that for every $\xi \in \mathbb{S}^1$ and A_ξ an open arc with extremities $a_\xi \neq b_\xi \in \Psi^{-1}(\{\pm\xi\})$, we have

$$\xi \cdot \nabla_x \mathbf{1}_{m(x)\in A_\xi} = 0 \qquad \text{in } \mathcal{D}'(\Omega). \tag{2.4}$$

We deduce the following:

Lemma 2.10. *Let x_1 be a Lebesgue point of m and $x_2 \neq x_1$ be such that $[x_1, x_2] \subset \Omega$. Let $\xi = \frac{x_2 - x_1}{|x_2 - x_1|}$. Then, for any open arc $A_\xi \subset \mathbb{S}^1$ with extremities $a_\xi \neq b_\xi \in \Psi^{-1}(\{\pm\xi\})$, we have*

$$m(x_1) \in A_\xi \quad \Longrightarrow \quad \text{the set } \{x : m(x) \in A_\xi\} \text{ has density } 1 \text{ at } x_2.$$

Proof of Lemma 2.10. The proof is exactly the same as the proof of [JOP02, Proposition 3.1]. We include some details for the convenience of the reader. According to (2.4), the function χ^ξ given by

$$\chi^\xi(x) = \mathbf{1}_{m(x)\in A_\xi},$$

is constant in the direction of ξ in a neighborhood of the line segment $[x_1, x_2]$, that is, $\chi^\xi(x) = \tilde{\chi}(x \cdot i\xi)$ for a.e. x in a δ-neighborhood of $[x_1, x_2]$ for some $\delta > 0$ and some measurable function $\tilde{\chi}\colon (t_1 - \delta, t_1 + \delta) \to \{0, 1\}$, where $t_1 = x_1 \cdot i\xi = x_2 \cdot i\xi$. Note that since A_ξ is an open set and x_1 is a Lebesgue point of m, so we have that $t_1 = x_1 \cdot i\xi$ is a Lebesgue point of $\tilde{\chi}$ and $\tilde{\chi}(t_1) = 1$. It follows that the set $\{x : m(x) \in A_\xi\}$ has density 1 at x_2. □

What makes Lemma 2.10 more powerful in the case where Ψ has a high winding number is that there are several different choices of open arcs A_ξ. We use this flexibility in the following form.

Lemma 2.11. *There is a constant $c \in (0, 1)$ depending on the map Ψ, with the following property. For any $z_1 \neq z_2 \in \mathbb{S}^1$, there exist $\xi_1 \neq \xi_2 \in \mathbb{RP}^1$ and open arcs $A_{\xi_1}, A_{\xi_2} \subset \mathbb{S}^1$ with extremities in $\Psi^{-1}(\xi_1), \Psi^{-1}(\xi_2)$, such that:*

$$\begin{aligned} &z_1 \in A_{\xi_1},\ z_2 \in A_{\xi_2}, \quad A_{\xi_1} \cap A_{\xi_2} = \emptyset, \\ \textit{and} \quad &\mathrm{dist}_{\mathbb{RP}^1}(\xi_1, \xi_2) \geq c|z_1 - z_2|. \end{aligned}$$

Proof of Lemma 2.11. We first set some notations. Recall that $\Psi\colon \mathbb{S}^1 \to \mathbb{RP}^1$ is C^1 with uniformly monotone phase and winding number $k/2$ with $|k| \geq 3$. Let us assume that the phase is increasing and $k \geq 3$, the case of a decreasing phase and $k \leq -3$ being completely similar. For any $\xi = \{\pm e^{i\beta}\} \in \mathbb{RP}^1$ we can write the preimage $\Psi^{-1}(\xi)$ as

$$\Psi^{-1}(\{\pm e^{i\beta}\}) = \{e^{i\alpha_\ell(\beta)}\}_{\ell\in\mathbb{Z}},$$

where the angle functions $\alpha_\ell \in C^1(\mathbb{R};\mathbb{R})$, $\ell \in \mathbb{Z}$, are uniformly increasing and satisfy

$$\alpha_\ell + c_0 < \alpha_{\ell+1}, \qquad \alpha_{\ell+k} = \alpha_\ell + 2\pi, \qquad \forall \ell \in \mathbb{Z},$$

for some small enough constant $c_0 > 0$ depending on Ψ. The functions α_ℓ can simply be chosen as the inverses of the functions $\varphi_\Psi - \ell\pi$, where $\varphi_\Psi \in C^1(\mathbb{R};\mathbb{R})$ is a uniformly increasing phase of Ψ as in Lemma 2.1. They satisfy also $\alpha_\ell(\cdot + \pi) = \alpha_{\ell+1}(\cdot)$.

Without loss of generality we assume that $z_1 = e^{i\theta_1}$, $z_2 = e^{i\theta_2}$ for some $\theta_1 < \theta_2 \leq \theta_1 + \pi$, so that $|z_1 - z_2|$ is of the order of $\theta_2 - \theta_1$. By a continuity argument, we can find $\xi_0 = \{\pm e^{i\beta_0}\} \in \mathbb{RP}^1$ and $\ell_1, \ell_2 \in \mathbb{Z}$ such that $\ell_1 < \ell_2 < \ell_1 + k$ and

$$\theta_j \in [\alpha_{\ell_j}(\beta_0) + c_1(\theta_2 - \theta_1), \alpha_{\ell_j+1}(\beta_0) - c_1(\theta_2 - \theta_1)], \quad j = 1, 2,$$

for some small enough constant $c_1 > 0$ depending on Ψ.

Since $k \geq 3$, we must have either $\alpha_{\ell_1+1}(\beta_0) + c_0 < \alpha_{\ell_2}(\beta_0)$ or $\alpha_{\ell_2+1}(\beta_0) + c_0 < \alpha_{\ell_1+k}(\beta_0)$. We assume that we are in the first case,

$$\alpha_{\ell_1+1}(\beta_0) + c_0 < \alpha_{\ell_2}(\beta_0),$$

the other case being completely similar.

For $\xi \in \mathbb{RP}^1$ such that $\mathrm{dist}_{\mathbb{RP}^1}(\xi, \xi_0) < \pi/4$, we can write $\xi = \{\pm e^{i\beta}\}$ for some $\beta \in (\beta_0 - \pi/4, \beta_0 + \pi/4)$, and the angles α_ℓ are uniformly increasing functions of β. If c_1 is small enough, then we can choose $\xi_1 = \{\pm e^{i\beta_1}\}$, $\xi_2 = \{\pm e^{i\beta_2}\}$ with $\beta_1, \beta_2 \in (\beta_0 - \pi/4, \beta_0 + \pi/4)$ and

$$\alpha_{\ell_1}(\beta_1) = \alpha_{\ell_1}(\beta_0) + \frac{1}{2}c_1(\theta_2 - \theta_1), \quad \alpha_{\ell_2+1}(\beta_2) = \alpha_{\ell_2+1}(\beta_0) - \frac{1}{2}c_1(\theta_2 - \theta_1).$$

It follows that

$$\beta_1 - \beta_2 = (\beta_1 - \beta_0) + (\beta_0 - \beta_2) \geq c(\theta_2 - \theta_1),$$

for some small enough constant c depending on Ψ. Choosing c_1 smaller compared to c_0 if necessary, we may assume $\alpha_{\ell_1+1}(\beta_1) < \alpha_{\ell_2}(\beta_2)$, so that

$$\alpha_{\ell_1}(\beta_1) < \theta_1 < \alpha_{\ell_1+1}(\beta_1) < \alpha_{\ell_2}(\beta_2) < \theta_2 < \alpha_{\ell_2+1}(\beta_2).$$

Thus we have

$$\mathrm{dist}_{\mathbb{RP}^1}(\xi_1, \xi_2) = \beta_1 - \beta_2 \geq c(\theta_2 - \theta_1) \gtrsim c|z_1 - z_2|,$$

and

$$z_j \in A_{\xi_j} := \exp(i(\alpha_{\ell_j}(\beta_j), \alpha_{\ell_j+1}(\beta_j))), \quad j = 1, 2,$$

where the open arcs A_{ξ_j} have their extremities in $\Psi^{-1}(\xi_j)$.

Moreover, since $\alpha_{\ell_1+1}(\beta_1) < \alpha_{\ell_2}(\beta_2)$, and by monotonicity of the α_ℓ we also have

$$\alpha_{\ell_2+1}(\beta_2) < \alpha_{\ell_2+1}(\beta_0) \leq \alpha_{\ell_1+k}(\beta_0) = \alpha_{\ell_1}(\beta_0) + 2\pi < \alpha_{\ell_1}(\beta_1) + 2\pi,$$

we deduce $A_{\xi_1} \cap A_{\xi_2} = \emptyset$. □

We can now combine Lemmas 2.10 and 2.11 to deduce that m is locally Lipschitz.

Let $x_1, x_2 \in \Omega$ be two Lebesgue points of m, and $z_j = m(x_j)$ for $j = 1, 2$. If $m(x_1) \neq m(x_2)$, then applying Lemma 2.11, we obtain $\xi_j \in \mathbb{RP}^1$, open arcs $A_{\xi_j} \subset \mathbb{S}^1$ with extremities in $\Psi^{-1}(\xi_j)$ such that $z_j \in A_{\xi_j}$ and

$$\mathrm{dist}_{\mathbb{RP}^1}(\xi_1, \xi_2) \geq c|z_1 - z_2|. \tag{2.5}$$

As $\xi_1 \neq \xi_2$ in $\mathbb{RP}^1$, the two lines $x_1 + \mathbb{R}\xi_1$, $x_2 + \mathbb{R}\xi_2$ intersect in a point $x_0 \in \mathbb{R}^2$. If the segments $[x_0, x_1]$ and $[x_0, x_2]$ were contained in Ω, then one would deduce from Lemma 2.10 that both sets $\{m \in A_{\xi_j}\}$ have density one at x_0, but this is impossible since $A_{\xi_1} \cap A_{\xi_2} = \emptyset$. So at least one of the segments $[x_0, x_1]$, $[x_0, x_2]$ cannot be contained in Ω, which implies that

$$\mathrm{dist}(\{x_1, x_2\}, \partial\Omega) \leq \max\{|x_0 - x_1|, |x_0 - x_2|\}.$$

Let $\theta \in (0, \pi/2]$ be the angle between ξ_1 and ξ_2. From elementary trigonometry in the triangle with vertices $\{x_0, x_1, x_2\}$, we deduce that

$$\begin{aligned}\mathrm{dist}_{\mathbb{RP}^1}(\xi_1, \xi_2) = \theta \lesssim \sin(\theta) &\leq \frac{|x_1 - x_2|}{\max\{|x_0 - x_1|, |x_0 - x_2|\}} \\ &\leq \frac{|x_1 - x_2|}{\mathrm{dist}(\{x_1, x_2\}, \partial\Omega)}.\end{aligned}$$

Recalling (2.5) we deduce that

$$|m(x_1) - m(x_2)| = |z_1 - z_2| \lesssim \frac{|x_1 - x_2|}{c\,\mathrm{dist}(\{x_1, x_2\}, \partial\Omega)}.$$

The above estimate holds automatically if $m(x_1) = m(x_2)$, hence m is locally Lipschitz.

In particular, $\Psi(m)$ is locally Lipschitz, and by [JOP02, GMPS23], it is constant along characteristics directed by $\xi = \Psi(m)$. Along these characteristics, the map m is continuous with values into the finite set $\Psi^{-1}(\{\pm\xi\})$ and must therefore be constant. This concludes the proof of Proposition 2.7 part (a). □

2.3 The case with boundary: proof of Theorem 1.10

Let $J = \{e^{i\theta}\}_{\theta\in[a,b]} \subsetneq \mathbb{S}^1$, so that we have $Du = \operatorname{cof}\Gamma(m)$ (1.5) for some $m\colon \Omega \to J$. As $\Gamma(e^{i\theta}) = \operatorname{cof}\gamma(\theta)$, the assumptions of Theorem 1.10 amount to $\Gamma \in C^2(J;\mathbb{R}^{2\times 2})$ satisfying $|\partial_\theta\Gamma| = 1$, $\det(\partial_\theta\Gamma) = 0$ and $|\det(\partial_\theta^2\Gamma)| > 0$ on J. Moreover, Γ satisfies the quartic estimate (1.1), see Remark 1.8 and Appendix B.

Thanks to Lemma A.1 we have $m \in B^{1/3}_{4,\infty,\mathrm{loc}}(\Omega; J)$. Applying Proposition 1.5, we deduce that

$$\nabla\cdot\Phi(m) = 0 \qquad \forall\Phi \in \mathrm{ENT}_\Gamma.$$

This can be used as in § 2.2.2 to deduce that m is locally Lipschitz. In fact the proof is simpler in this case, and we sketch next how to adapt the main steps.

Since we have $|\partial_\theta\Gamma| = 1$, $\det(\partial_\theta\Gamma) = 0$ and $|\det(\partial_\theta^2\Gamma)| > 0$ on J, as in Lemma 2.1 (see Remark 2.3) we can find C^1 maps $\hat\lambda, \hat\Psi\colon [a,b] \to \mathbb{S}^1$ with uniformly monotone phases and such that

$$\partial_\theta\Gamma(e^{i\theta}) = \hat\lambda(\theta)\otimes\hat\Psi(\theta) \qquad \forall\theta\in[a,b].$$

This also defines $\Psi \in C^1(J;\mathbb{S}^1)$ by $\Psi(e^{i\theta}) = \hat\Psi(\theta)$.

In that setting with boundary, Lemma 2.8 and the kinetic formulation (2.4) become valid for any $\xi \in \Psi(J)$ and any arc $\exp(i\mathcal{I}) = A_\xi \subset J$, where $\mathcal{I}$ has one of the forms (θ_1,θ_2), $[a,\theta_1)$ or $(\theta_2, b]$ for $e^{i\theta_j} \in \Psi^{-1}(\{\pm\xi\})$. (The proof is actually easier since we don't need to make the approximating entropies periodic.) Writing $m(x) = e^{i\theta(x)}$ for $\theta\colon \Omega \to [a,b]$, this implies in particular

$$\hat\Psi(\alpha)\cdot\nabla_x \mathbf{1}_{\theta(x)<\alpha} = 0 \qquad \forall\alpha\in[a,b],$$

and the corresponding version of Lemma 2.10. Namely, if x_1 is a Lebesgue point of θ such that $\theta(x_1) < \alpha$, then the set $\{\theta < \alpha\}$ has density one at all points $x_2 \in x_1 + \mathbb{R}\hat\Psi(\alpha)$ such that $[x_1,x_2]\subset\Omega$. The same holds for the set $\{\theta > \alpha\}$ if $\theta(x_1) > \alpha$. Thanks to the uniform monotonicity of $\hat\Psi$'s phase, we obtain the following simpler version of Lemma 2.11. For any $\theta_1 < \theta_2 \in [a,b]$ we can find $\theta_1 < \alpha_1 < \alpha_2 < \theta_2$ such that

$$\mathrm{dist}_{\mathbb{RP}^1}(\{\pm\hat\Psi(\alpha_1)\},\{\pm\hat\Psi(\alpha_2)\}) \geq c(\theta_2-\theta_1).$$

As in § 2.2.2, we apply this to the values $\theta_j = \theta(x_j)$ at Lebesgue points x_1, x_2 of θ, deduce that the lines $x_j + \mathbb{R}\hat\Psi(\alpha_j)$ must intersect $\partial\Omega$ before crossing, and conclude that θ is locally Lipschitz, and so are m and $\Psi(m)$.

Finally, Lemmas 2.4 and 2.5 ensure that $\tilde m = i\Psi(m)$ is an $\mathbb{S}^1$-valued zero-energy state of the Aviles-Giga energy in the sense of [JOP02, § 2]. Arguing exactly as the end of § 2.2.2, we deduce that m is constant along characteristics directed by $\Psi(m)$, and this concludes the proof of Theorem 1.10 for $Du = \operatorname{cof}\Gamma(m)$. □

3 The partially elliptic case: proof of Theorem 1.12

3.1 Regularity at elliptic values

In this section we prove part (i) of Theorem 1.12, that $Du \in \Pi_{NE}$ or Du is constant. For this part we do not need to assume the quartic estimate (1.1), but only the fact that Π has no rank-one connections. Recall from the beginning of § 1.1 that we assume without loss of generality that Π has length at most 2π, and $\gamma\colon I \to \Pi$ is an arc-length parametrization of class C^2, where I is either a segment $[a, b] \subset \mathbb{R}$ ($a < b < a + 2\pi$) in the case with boundary, or $I = \mathbb{R}/2\pi\mathbb{Z}$ in the case without boundary.

Proposition 3.1. *Let $\Pi \subset \mathbb{R}^{2\times 2}$ be a compact connected C^2 curve of length at most 2π, without rank-one connections. Denote by $\gamma \in C^2(I; \mathbb{R}^{2\times 2})$ an arc-length parametrization of Π, and by $\Pi_E = \gamma(\{\det(\gamma') \neq 0\}) \subset \Pi$ the subset of elliptic values. For any connected open set $\Omega \subset \mathbb{R}^2$ and weakly differentiable map $u\colon \Omega \to \mathbb{R}^2$, if u solves the differential inclusion*

$$Du \in \Pi = \gamma(I) \qquad a.e. \ in\ \Omega,$$

and $\Omega_E = (Du)^{-1}(\Pi_E)$ has positive Lebesgue measure, then Du is constant.

Proof of Proposition 3.1. The proof we present here, based on the notion of quasiconformal envelope [FS08, KS08] and a recent unique continuation result of [DPGT23], has been pointed out to us by Riccardo Tione. (Another possible proof, based on [Lle23, Theorem 1.9], would require a refined, and rather technical to demonstrate, version of Kirszbraun's extension theorem.)

As in the beginning of Appendix A we assume without loss of generality that

$$\det(A - B) > 0 \qquad \forall A \neq B \in \Pi. \tag{3.1}$$

We assume that Ω_E has positive Lebesgue measure: then we can pick a Lebesgue point x_0 of Du such that $Du(x_0) \in \Pi_E$. Since Π_E is open in Π, for every neighborhood of $Du(x_0)$ in Π_E, its preimage by Du has positive Lebesgue measure. Therefore we may fix $a' < b' < a' + 2\pi$ such that $\det(\gamma') \neq 0$ on $[a', b'] \subset I$ and, letting

$$\mathcal{J} = \gamma([a', b']) \subset \Pi_E,$$

we have that

$$\operatorname{dist}(Du, \mathcal{J})^{-1}(\{0\}) \text{ has positive Lebesgue measure.} \tag{3.2}$$

For every $t_0 \in [a', b']$ and $h \to 0$, we have

$$\det(\gamma(t_0 + h) - \gamma(t_0)) = h^2 \det(\gamma'(t_0)) + o(h^2).$$

According to (3.1) this implies in particular $\det(\gamma') > 0$ on $[a', b']$, and by compactness of $[a', b']$ we deduce the existence of $\delta_0 > 0$ and $K_0 > 0$ such that

$$|A - B|^2 \leq K_0 \det(A - B) \qquad \forall A \in \mathcal{J} \text{ and } B \in \Pi \text{ s.t. } |A - B| \leq \delta_0.$$

Thanks to (3.1) we have

$$c_0 := \min\left\{\frac{\det(A-B)}{|A-B|^2} : A\in\mathcal{J},\, B\in\Pi,\, |A-B|\geq\delta_0\right\} > 0,$$

and setting $K=\max(K_0,1/c_0)$ we deduce that

$$|A-B|^2 \leq K\det(A-B) \qquad \forall A\in\mathcal{J} \text{ and } B\in\Pi. \tag{3.3}$$

In the terminology of [FS08, KS08], this means that the curve Π is included in the K-quasiconformal envelope $\mathcal{E}_{\mathcal{J}}$ of the arc $\mathcal{J}\subset\mathbb{R}^{2\times 2}$. Here, as in [DPGT23], we omit the dependence on the fixed $K>0$ from the notation. Therefore the differential inclusion $Du\in\Pi$ implies

$$Du\in\mathcal{E}_{\mathcal{J}} \qquad \text{a.e. in } \Omega. \tag{3.4}$$

Moreover, the inequality (3.3) for $A,B\in\mathcal{J}$ means that $\mathcal{J}$ is *elliptic* and satisfies therefore a *rigidity estimate*: there exists $c>0$ such that

$$\inf_{A\in\mathcal{J}}\int_{B_{1/2}}|Du-A|^2\,dx \leq c\int_{B_1}\operatorname{dist}^2(Du,\mathcal{J})\,dx \qquad \forall u\in W^{1,2}(B_1;\mathbb{R}^2).$$

This follows from a minor adaptation of [LLP23] presented in Appendix E. As a consequence, we can apply the unique continuation principle of [DPGT23, Theorem 1.3] to the differential inclusion (3.4), and deduce that either $\operatorname{dist}(Du,\mathcal{J})>0$ a.e. in Ω, or $\operatorname{dist}(Du,\mathcal{J})=0$ a.e. in Ω and Du is constant. The first possibility is ruled out by (3.2), so we conclude that Du is constant equal to $Du(x_0)$. □

3.2 Improvement under low fractional regularity

In this section we explain that the regularity at elliptic values obtained in the previous section automatically improves to Du being either constant or with values in a single connected component of $\Pi_{NE}=\gamma(\{\det(\gamma')=0\})$, under an extra fractional regularity assumption.

Proposition 3.2. *Let $\Pi\subset\mathbb{R}^{2\times 2}$ be a compact connected C^2 curve of length at most 2π, without rank-one connections. Denote by $\gamma\in C^2(I;\mathbb{R}^{2\times 2})$ an arc-length parametrization of Π. For any connected open set $\Omega\subset\mathbb{R}^2$ and weakly differentiable map $u\colon\Omega\to\mathbb{R}^2$, if u solves the differential inclusion*

$$Du\in\Pi=\gamma(I) \qquad \textit{a.e. in } \Omega,$$

and satisfies in addition $Du\in B^s_{p,\infty,\mathrm{loc}}(\Omega)$ for some $s\in(0,1)$ and $p>1/s$, then Du is either constant, or takes values into a single connected component of Π_{NE}.

Proof of Proposition 3.2. More precisely, we show that $Du(\Omega\cap\mathcal{G})$ is either a point or a connected subset of Π_{NE}, where $\mathcal{G}$ is the set of Lebesgue points of Du. If $Du(\Omega\cap\mathcal{G})\cap\Pi_E\neq\emptyset$, then $Du^{-1}(\Pi_E)$ must have positive Lebesgue measure since Π_E is open in Π, and Proposition 3.1 then implies that Du is constant. So we assume without loss of generality that

$$Du(\Omega\cap\mathcal{G})\subset\Pi_{NE},$$

and show that $Du(\Omega\cap\mathcal{G})$ is actually contained in a single connected component of Π_{NE}.

By connectedness of Ω, it suffices to show that for any fixed open ball $B\subset\Omega$, $Du(B\cap\mathcal{G})$ is contained in a connected component of Π_{NE}. Let us therefore fix an open ball $B=B_r(x)\subset\Omega$, and assume without loss of generality that $x=0$.

The proof relies on properties of restrictions of Du to one-dimensional lines. For $a,b\in(-r,r)$ we define the vertical and horizontal intervals

$$B_a^1=B\cap(\{a\}\times\mathbb{R}),\quad B_b^2=B\cap(\mathbb{R}\times\{b\}).$$

For any $\sigma\in(1/p,s)$, we have $B^s_{p,\infty}(B)\subset W^{\sigma,p}(B)$ [Tri83, Proposition 2.3.2/2], and therefore $Du|_{B_a^1}\in W^{\sigma,p}(B_a^1)$ for a.e. $a\in(-r,r)$ [Tri83, Theorem 2.5.13/(i)]. Since $\sigma p>1$ this implies that $Du|_{B_a^1}$ agrees a.e. with a continuous map. By Fubini's theorem we also have that $\mathcal{H}^1(B_a^1\setminus\mathcal{G})=0$ for a.e. $a\in(-r,r)$.

Fix $a\in(-r,r)$ such that both $\mathcal{H}^1(B_a^1\setminus\mathcal{G})=0$ and $Du|_{B_a^1}$ agrees a.e. with a continuous map. Since Π_E is open, if the image of B_a^1 under the continuous representative of $Du|_{B_a^1}$ intersects Π_E, then there exists a point $x\in B_a^1\cap\mathcal{G}$ such that $Du(x)\in\Pi_E$, but we have ruled out this possibility. Therefore, the image of B_a^1 under the continuous representative of $Du|_{B_a^1}$ lies in a connected component $\mathcal{J}_a^1\subset\Pi_{NE}$, hence there exists a full measure set $X_a\subset B_a^1$, $\mathcal{H}^1(B_a^1\setminus X_a)=0$, such that $Du(B_a^1\cap X_a)\subset\mathcal{J}_a^1$.

The same argument shows that, for a.e. $b\in(-r,r)$, there exist a connected component $\mathcal{J}_b^2\subset\Pi_{NE}$ and a full measure set $Y_b\subset B_b^2$, $\mathcal{H}^1(B_b^2\setminus Y_b)=0$, such that $Du(B_b^2\cap Y_b)\subset\mathcal{J}_b^2$.

By Fubini's theorem, the sets $X=\bigcup_a X_a, Y=\bigcup_b Y_b\subset B$ have full Lebesgue measure in B. For any $\varepsilon\in(0,1)$, this implies the existence of $a\in(-\varepsilon,\varepsilon)$ such that the vertical set X_a intersects the horizontal set Y_b for a.e. $b\in(-r+\varepsilon,r-\varepsilon)$. As a consequence, the connected components $\mathcal{J}_b^2,\mathcal{J}_a^1\subset\Pi_{NE}$ have a non-empty intersection and therefore $\mathcal{J}_b^2=\mathcal{J}_a^1$ does not depend on $b\in(-r+\varepsilon,r-\varepsilon)$. Letting $\varepsilon\to 0$, we deduce that $Du(Y)$ is contained in a single connected component $\mathcal{J}\subset\Pi_{NE}$. Since Y has full measure this implies $Du(B\cap\mathcal{G})\subset\mathcal{J}$. □

3.3 Conclusion of proof of Theorem 1.12

In this section we assume that Π satisfies the quartic estimate (1.1), and complete the proof of part (*ii*) of Theorem 1.12. Since the quartic estimate implies via Lemma A.1 that $Du=\operatorname{cof}\Gamma(m)$ with $m\in B^{1/3}_{4,\infty,\mathrm{loc}}(\Omega)$, we can apply Proposition 3.2 and deduce that either Du is constant (case $(ii)(a)$ of Theorem 1.12), or Du takes values into one single connected component of Π_{NE} (case $(ii)(b)$ of Theorem 1.12). $\square$

4 Proof of Proposition 1.5

The starting point of Proposition 1.5 is the identity which follows from the chain rule,

$$\nabla \cdot \Phi(m) = \sum_{j=1,2} \alpha_\Phi^j(m) \nabla \cdot \Gamma_j(m) \qquad \text{if } m\colon \Omega \to J \subset \mathbb{S}^1 \text{ is smooth.}$$

In our case, m is not smooth: in order to apply the chain rule, we use smooth approximating maps $m_\varepsilon \to m$. The maps m_ε will in general not be $\mathbb{S}^1$-valued, which introduces error terms in the above identity, and the proof of Proposition 1.5 consists in estimating these error terms efficiently. If the quantity $|h|^{-1/3}\|D^h m\|_{L^p}$ vanishes as $|h| \to 0$, this follows directly from commutator estimates as in [CET94, DLI15]. Here we only know that this quantity is bounded (1.9), and basic commutator estimates are not enough to conclude directly: they only provide the information that entropy productions are in $L^{p/3}$, see § 4.1.1. However they also provide identities involving weak limits of error terms, and we can take advantage of these identities and the structure of ENT_Γ to conclude.

4.1 The case without boundary

We first prove Proposition 1.5 in the case of a closed curve, $J = \mathbb{S}^1$. The case with boundary will be treated in § 4.2.

4.1.1 First step: entropy productions are in $L^{p/3}_{\mathrm{loc}}$

In this first step we fix $\Phi \in \mathrm{ENT}_\Gamma$ and prove, using the regularity $m \in B^{1/3}_{p,\infty}$ and commutator estimates, that $\nabla \cdot \Phi(m) \in L^{p/3}_{\mathrm{loc}}(\Omega)$. In fact we do not use all the assumptions on Γ in Proposition 1.5 and prove a more precise statement.

Proposition 4.1. *Let $\Gamma \in C^2(\mathbb{S}^1; \mathbb{R}^{2\times 2})$ and $m \in B^{\frac{1}{3}}_{p,\infty,\mathrm{loc}}(\Omega; \mathbb{S}^1)$ for some $p > 3$, such that*

$$\nabla \cdot \Gamma_1(m) = \nabla \cdot \Gamma_2(m) = 0 \qquad \text{in } \mathcal{D}'(\Omega).$$

Then there exist $f_1, f_2 \in L^{p/3}_{\mathrm{loc}}(\Omega; \mathbb{R}^2)$ depending on Γ and m such that, for any entropy $\Phi \in \mathrm{ENT}_\Gamma$ (1.7), we have

$$\nabla \cdot \Phi(m) = \sum_{j=1}^{2} \partial_\theta \alpha_\Phi^j(m)\, im \cdot f_j \in L^{p/3}_{\mathrm{loc}}(\Omega), \tag{4.1}$$

where $\alpha_\Phi^1, \alpha_\Phi^2 \in C^1(\mathbb{S}^1; \mathbb{R})$ are as in (1.7).

Proof of Proposition 4.1. Direct calculations using polar coordinates $x = re^{i\theta}$ in $\mathbb{R}^2$ show that, for any C^1 vector fields $w\colon \Omega \to \mathbb{R}^2$ and $F\colon \mathbb{R}^2 \to \mathbb{R}^2$ such that $\nabla F(0) = 0$, we have

$$\nabla \cdot F(w) = ((\partial_r F(w) \cdot \nabla)\, w) \cdot \frac{w}{|w|} + ((\partial_\theta F(w) \cdot \nabla)\, w) \cdot \frac{iw}{|w|^2}. \tag{4.2}$$

We apply (4.2) to vector fields F which are 'flat radial' extensions of Φ and Γ_j $(j = 1, 2)$ given in polar coordinates by

$$\widehat{\Phi}(re^{i\theta}) = \eta(r)\Phi(e^{i\theta}), \qquad \widehat{\Gamma}_j(re^{i\theta}) = \eta(r)\Gamma_j(e^{i\theta}),$$

for some smooth $\eta \in C_c^\infty((0,\infty); [0,1])$ such that $\eta(r) \equiv 1$ for $1/2 \le r \le 2$. Recall from the definition (1.7) of ENT_Γ that

$$\partial_\theta \Phi = \sum_{j=1}^{2} \alpha_\Phi^j \partial_\theta \Gamma_j, \qquad \alpha_\Phi^j \in C^1(\mathbb{S}^1; \mathbb{R}).$$

From (4.2) applied to $\widehat{\Phi}$ and $\widehat{\Gamma}_j$ we therefore obtain, for any smooth $w\colon \Omega \to \mathbb{R}^2$,

$$\begin{aligned}
\nabla \cdot \widehat{\Phi}(w) &= \eta'(|w|) \left(\left(\Phi\left(\frac{w}{|w|}\right) \cdot \nabla \right) w \right) \cdot \frac{w}{|w|} \\
&\qquad + \eta(|w|) \sum_{j=1}^{2} \alpha_\Phi^j \left(\frac{w}{|w|}\right) \left(\left(\partial_\theta \Gamma_j \left(\frac{w}{|w|}\right) \cdot \nabla \right) w \right) \cdot \frac{iw}{|w|^2}, \\
\nabla \cdot \widehat{\Gamma}_j(w) &= \eta'(|w|) \left(\left(\Gamma_j \left(\frac{w}{|w|}\right) \cdot \nabla \right) w \right) \cdot \frac{w}{|w|} \\
&\qquad + \eta(|w|) \left(\left(\partial_\theta \Gamma_j \left(\frac{w}{|w|}\right) \cdot \nabla \right) w \right) \cdot \frac{iw}{|w|^2},
\end{aligned}$$

which implies

$$\nabla \cdot \widehat{\Phi}(w) - \sum_{j=1,2} \alpha_\Phi^j \left(\frac{w}{|w|}\right) \nabla \cdot \widehat{\Gamma}_j(w) = \eta'(|w|) \left(\left(H\left(\frac{w}{|w|}\right) \cdot \nabla \right) w \right) \cdot \frac{w}{|w|},$$

where $H = \Phi - \sum \alpha_\Phi^j \Gamma_j \in C^1(\mathbb{S}^1; \mathbb{R}^2)$. Now we apply this to $w = m_\varepsilon$, where

$$m_\varepsilon = m * \rho_\varepsilon, \quad \rho_\varepsilon(x) = \varepsilon^{-2}\rho(x/\varepsilon),$$

for some smooth kernel ρ with support in B_1, and deduce

$$\begin{aligned}
\nabla \cdot \widehat{\Phi}(m_\varepsilon) &= R_\varepsilon^1 + R_\varepsilon^2, \\
R_\varepsilon^1 &= \sum_{j=1,2} \alpha_\Phi^j \left(\frac{m_\varepsilon}{|m_\varepsilon|}\right) \nabla \cdot \left(\widehat{\Gamma}_j(m_\varepsilon) - \widehat{\Gamma}_j(m)_\varepsilon \right), \\
R_\varepsilon^2 &= \eta'(|m_\varepsilon|) \left(\left(H\left(\frac{m_\varepsilon}{|m_\varepsilon|}\right) \cdot \nabla \right) m_\varepsilon \right) \cdot \frac{m_\varepsilon}{|m_\varepsilon|}.
\end{aligned} \tag{4.3}$$

To obtain the expression of R^1_ε we used the fact that $\nabla\cdot\widehat{\Gamma}_j(m) = \nabla\cdot\Gamma_j(m) = 0$. Because $\eta'(r)\equiv 0$ in a neighborhood of $r=1$ we have

$$|R^2_\varepsilon| \leq C(1-|m_\varepsilon|^2)^2|Dm_\varepsilon|,$$

for some $C>0$ depending on η and H, and noting that $1=|m|^2=(|m|^2)_\varepsilon$ we deduce from commutator estimates (Lemma C.1 applied to $G=|\cdot|^2$, $w=m$, $\alpha=2$, $\beta=1$, $s=1/3$, $p>3$) that

$$R^2_\varepsilon \longrightarrow 0 \qquad \text{in } L^1_{\mathrm{loc}}(\Omega). \tag{4.4}$$

Using that $\eta(r)\equiv 1$ in a neighborhood of $r=1$, the same argument shows that, for the term R^1_ε in (4.3), we have

$$\begin{aligned}&(1-\eta(|m_\varepsilon|))R^1_\varepsilon\\ &\quad = (1-\eta(|m_\varepsilon|))\sum_{j=1,2}\alpha^j_\Phi\left(\frac{m_\varepsilon}{|m_\varepsilon|}\right)\nabla\cdot\widehat{\Gamma}_j(m_\varepsilon)\to 0 \qquad \text{in } L^1_{\mathrm{loc}}(\Omega).\end{aligned}\tag{4.5}$$

It remains to estimate $\eta(|m_\varepsilon|)R^1_\varepsilon$, which, defining $\hat{\alpha}^j_\Phi(re^{i\theta})=\eta(r)\alpha^j_\Phi(e^{i\theta})\in C^1_c(\mathbb{R}^2;\mathbb{R})$, we rewrite as

$$\begin{aligned}\eta(|m_\varepsilon|)R^1_\varepsilon &= \nabla\cdot\left(\sum_{j=1,2}\hat{\alpha}^j_\Phi(m_\varepsilon)\left(\widehat{\Gamma}_j(m_\varepsilon)-\widehat{\Gamma}_j(m)_\varepsilon\right)\right)\\ &\qquad -\sum_{j=1,2}\left(\nabla\hat{\alpha}^j_\Phi(m_\varepsilon)Dm_\varepsilon\right)\cdot\left(\widehat{\Gamma}_j(m_\varepsilon)-\widehat{\Gamma}_j(m)_\varepsilon\right).\end{aligned}$$

The first line in the right-hand side is the divergence of a sequence which tends to 0 in $L^1_{\mathrm{loc}}(\Omega)$. Each term in the second line is bounded in $L^{p/3}_{\mathrm{loc}}$, independently of ε, by commutator estimates (Lemma C.1 applied to $G=\widehat{\Gamma}_j$, $w=m$, $s=1/3$, $\alpha=\beta=p/3$). More precisely, there exist $f_1,f_2\in L^{p/3}_{\mathrm{loc}}(\Omega;\mathbb{R}^2)$ and a sequence $\varepsilon_k\to 0$ such that

$$Dm_\varepsilon\left(\widehat{\Gamma}_j(m)_\varepsilon-\widehat{\Gamma}_j(m_\varepsilon)\right)\rightharpoonup f_j \qquad \text{in } L^{\frac{p}{3}}_{\mathrm{loc}}(\Omega),$$

and therefore

$$\eta(|m_{\varepsilon_k}|)R^1_{\varepsilon_k}\to\sum_{j=1}^2\nabla\hat{\alpha}^j_\Phi(m)\cdot f_j=\sum_{j=1}^2\partial_\theta\alpha^j_\Phi(m)\,im\cdot f_j \qquad \text{in } \mathcal{D}'(\Omega).$$

Plugging this and (4.4)-(4.5) into (4.3), we deduce

$$\nabla\cdot\widehat{\Phi}(m_{\varepsilon_k})\to\sum_{j=1}^2\partial_\theta\alpha^j_\Phi(m)\,im\cdot f_j \qquad \text{in } \mathcal{D}'(\Omega).$$

Since $\widehat{\Phi}(m_\varepsilon)\to\Phi(m)$ in $L^1_{\mathrm{loc}}(\Omega)$, this implies (4.1). □

4.1.2 Conclusion: entropy productions vanish

Here we show that all entropy productions vanish. If $\Phi \in \mathrm{ENT}_\Gamma$ is such that there exist $\beta^1_\Phi, \beta^2_\Phi \in C^1(\mathbb{S}^1;\mathbb{R})$ for which

$$\partial_\theta \Phi = \beta^1_\Phi \partial_\theta \Gamma_1 = \beta^2_\Phi \partial_\theta \Gamma_2, \tag{4.6}$$

then applying (4.1) we obtain

$$\begin{pmatrix} \partial_\theta \beta^1_\Phi(m) \\ -\partial_\theta \beta^2_\Phi(m) \end{pmatrix} \cdot \begin{pmatrix} im \cdot f_1 \\ im \cdot f_2 \end{pmatrix} = 0.$$

So if we can find two such maps $\Phi, \overline{\Phi}$ such that

$$\det \begin{pmatrix} \partial_\theta \beta^1_\Phi(m) & \partial_\theta \beta^1_{\overline{\Phi}}(m) \\ -\partial_\theta \beta^2_\Phi(m) & -\partial_\theta \beta^2_{\overline{\Phi}}(m) \end{pmatrix} \neq 0 \quad \text{a.e.},$$

then we deduce that $im \cdot f_1 = im \cdot f_2 = 0$ a.e. and the conclusion (1.8) follows from (4.1).

Note that here all assumptions of Lemma 2.1 are satisfied, so we have (recalling Remark 2.2)

$$\partial_\theta \Gamma = \lambda \otimes \Psi,$$

for $\lambda, \Psi \in C^1(\mathbb{S}^1;\mathbb{RP}^1)$. Assume for example that there are entropies $\Phi, \overline{\Phi}$ such that

$$\partial_\theta \Phi = \lambda_1^2 \lambda_2 \Psi, \qquad \partial_\theta \overline{\Phi} = \lambda_2^2 \lambda_1 \Psi. \tag{4.7}$$

The right-hand sides belong to $C^1(\mathbb{S}^1;\mathbb{R}^2)$ because $\lambda \in C^1(\mathbb{S}^1;\mathbb{RP}^1)$ and $\lambda \otimes \Psi \in C^1(\mathbb{S}^1;\mathbb{R}^{2\times 2})$. These maps $\Phi, \overline{\Phi}$ would then satisfy (4.6), with

$$\beta^1_\Phi = \lambda_1 \lambda_2, \quad \beta^2_\Phi = \lambda_1^2, \quad \beta^1_{\overline{\Phi}} = \lambda_2^2, \quad \beta^2_{\overline{\Phi}} = \lambda_1 \lambda_2,$$

and the above determinant is equal to

$$\begin{aligned} \det \begin{pmatrix} \partial_\theta \beta^1_\Phi & \partial_\theta \beta^1_{\overline{\Phi}} \\ -\partial_\theta \beta^2_\Phi & -\partial_\theta \beta^2_{\overline{\Phi}} \end{pmatrix} &= -(\lambda_2 \partial_\theta \lambda_1 + \lambda_1 \partial_\theta \lambda_2)^2 + 4\lambda_1 \lambda_2 \partial_\theta \lambda_1 \partial_\theta \lambda_2 \\ &= -(\lambda_2 \partial_\theta \lambda_1 - \lambda_1 \partial_\theta \lambda_2)^2 = -(i\lambda \cdot \partial_\theta \lambda)^2 \\ &= -|\partial_\theta \lambda|^2 < 0. \end{aligned}$$

The last equality follows from the fact that $|\lambda| = 1$, and the last inequality follows from the monotonicity of its phase as shown in Lemma 2.1.

In general there might however not exist entropies whose derivatives are exactly as in (4.7), because the right-hand sides in (4.7) may not have zero average. But for any fixed $x \in \Omega$, we can modify the maps $\lambda_1^2 \lambda_2 \Psi, \lambda_2^2 \lambda_1 \Psi \in C^1(\mathbb{S}^1;\mathbb{R}^2)$ on a subset of $\mathbb{S}^1$ which does not contain a small interval around $m(x)$, so that the modified maps have zero averages and are derivatives of entropies $\Phi, \overline{\Phi}$ as above. At the point x, the determinant has the same value as before, so we deduce that $im(x) \cdot f_1(x) = im(x) \cdot f_2(x) = 0$. And we can do this for any fixed x, so the right-hand side of (4.1) is zero a.e., and this concludes the proof that all entropy productions vanish.

Here is how we modify the maps. Recall from Lemma 2.1 that λ, Ψ have uniformly monotone phases. This implies that $\lambda_1^{-1}(\{0\})$, $\lambda_2^{-1}(\{0\})$ and $\Psi^{-1}(\{\pm\Psi(z)\})$ are finite subsets of $\mathbb{S}^1$ for any $z \in \mathbb{S}^1$. As a consequence, we can find $z_1 = e^{i\theta_1}, z_2 = e^{i\theta_2} \in \mathbb{S}^1 \setminus \{m(x)\}$ such that $\det(\partial_\theta \Gamma_1(z_1), \partial_\theta \Gamma_2(z_2)) \neq 0$. Denote $\alpha_1 = \lambda_1\lambda_2$, $\alpha_2 = 0$, so that $\sum_{j=1}^2 \alpha_j \partial_\theta \Gamma_j = \lambda_1^2 \lambda_2 \Psi$. Then Lemma D.1 gives $\tilde{\alpha}_1, \tilde{\alpha}_2 \in C^1(\mathbb{S}^1; \mathbb{R})$ such that $\int_{\mathbb{S}^1} \left(\sum_{j=1}^2 \tilde{\alpha}_j \partial_\theta \Gamma_j \right) d\theta = 0$ and $\sum_{j=1}^2 \tilde{\alpha}_j \partial_\theta \Gamma_j = \sum_{j=1}^2 \alpha_j \partial_\theta \Gamma_j$ away from a sufficiently small neighborhood of $\{z_1, z_2\}$ (see Remark D.2). In particular, we have $\sum_{j=1}^2 \tilde{\alpha}_j \partial_\theta \Gamma_j = \lambda_1^2 \lambda_2 \Psi$ in a neighborhood of $m(x)$. Therefore we can define $\Phi(e^{i\theta}) = \int_0^\theta \left(\sum_{j=1}^2 \tilde{\alpha}_j \partial_t \Gamma_j \right) dt$, which satisfies the first equation of (4.7) on an open interval around $m(x)$. An entropy $\overline{\Phi}$ satisfying the second equation of (4.7) on an open interval around $m(x)$ is obtained by the same argument. □

4.2 The case with boundary

We now prove Proposition 1.5 in the case with boundary, that is, $J \subsetneq \mathbb{S}^1$ is compact and connected.

We start by showing that there exist $f_1, f_2 \in L^{p/3}_{\mathrm{loc}}(\Omega)$ such that

$$\nabla \cdot \Phi(m) = \sum_{j=1}^{2} \partial_\theta \alpha_\Phi^j(m)\, im \cdot f_j, \qquad \forall \Phi \in \mathrm{ENT}_\Gamma. \tag{4.8}$$

To prove (4.8) we extend Γ to a map $\widetilde{\Gamma} \in C^2(\mathbb{S}^1; \mathbb{R}^{2\times 2})$. Then the map m satisfies $\widetilde{\Gamma}(m) = \Gamma(m)$, hence $\nabla \cdot \widetilde{\Gamma}(m) = 0$. Applying Proposition 4.1 to $\widetilde{\Gamma}$ and m, we deduce that

$$\nabla \cdot \widetilde{\Phi}(m) = \sum_{j=1}^{2} \partial_\theta \alpha_{\widetilde{\Phi}}^j(m)\, im \cdot f_j, \qquad \forall \widetilde{\Phi} \in \mathrm{ENT}_{\widetilde{\Gamma}}, \tag{4.9}$$

for some $f_1, f_2 \in L^{p/3}_{\mathrm{loc}}(\Omega)$. Now let $\Phi \in \mathrm{ENT}_\Gamma$, that is,

$$\partial_\theta \Phi = \sum_{j=1}^{2} \alpha_\Phi^j \partial_\theta \Gamma_j \qquad \text{on } J,$$

for some $\alpha_\Phi^j \in C^1(J; \mathbb{R})$. There exist extensions $\tilde{\alpha}_\Phi^j \in C^1(\mathbb{S}^1; \mathbb{R})$ of α_Φ^j such that

$$\int_{\mathbb{S}^1} \left(\sum_{j=1}^{2} \tilde{\alpha}_\Phi^j \partial_\theta \widetilde{\Gamma}_j \right) d\theta = 0. \tag{4.10}$$

Indeed, one can choose any extensions $\tilde{\alpha}_\Phi^j$ and modify them in a neighborhood of a point $z_0 \in \mathbb{S}^1 \setminus J$ at which $\det(\partial_\theta \widetilde{\Gamma}) \neq 0$ in order to ensure (4.10) (see Remark D.3). If such a point z_0 does not exist in the first place, one can simply modify $\widetilde{\Gamma}$ away from J by adding a term of the form ηI for $I \in \mathbb{R}^{2\times 2}$ the identity matrix and for some non-zero $\eta \in C_c^2(\mathbb{S}^1 \setminus J; \mathbb{R})$ with $\|\eta'\|_\infty$ sufficiently large.

Thanks to (4.10) one can then define $\widetilde{\Phi} \in \mathrm{ENT}_{\widetilde{\Gamma}}$ such that $\widetilde{\Phi} = \Phi$ on J, by setting

$$\widetilde{\Phi}(e^{i\theta}) = \Phi(e^{i\theta_0}) + \int_{\theta_0}^{\theta} \left(\sum_{j=1}^{2} \tilde{\alpha}_\Phi^j \partial_t \widetilde{\Gamma}_j \right) dt,$$

for some $e^{i\theta_0} \in J$. Applying (4.9) to this $\widetilde{\Phi}$ and using that m takes values into J we deduce (4.8).

Now we can use (4.8) exactly as in § 4.1.2 and conclude that $f_1 = f_2 = 0$. In fact it is even easier because entropies do not have to be periodic, so we can directly find entropies $\Phi, \overline{\Phi} \in \mathrm{ENT}_\Gamma$ such that

$$\partial_\theta \Phi = \lambda_1^2 \lambda_2 \Psi, \quad \partial_\theta \overline{\Phi} = \lambda_2^2 \lambda_1 \Psi,$$

where $\lambda, \Psi \in C^1(J; \mathbb{S}^1)$ are such that $\partial_\theta \Gamma = \lambda \otimes \Psi$ on J (see Remark 2.3). □

5 Examples of nowhere elliptic curves without rank-one connections

In this section we give several examples of curves satisfying the assumptions of Theorem 1.7, and prove Proposition 1.11 which states that, among nondegenerate nowhere elliptic curves, the subset of curves without rank-one connections is open.

5.1 The curves γ_k

Lemma 5.1. *The curves parametrized by*

$$\gamma_k(t) = \frac{1}{2}[e^{it}]_c + \frac{1}{2(k+1)}[e^{(k+1)it}]_a,$$

for any integer $k \geq 1$, have no rank-one connections, are nowhere elliptic, and satisfy $\det(\gamma_k'') < 0$ on $\mathbb{R}/2\pi\mathbb{Z}$. That is, they satisfy the assumptions of Theorem 1.7.

Proof of Lemma 5.1. Since

$$\gamma_k'(t) = \frac{1}{2}\left(\left[ie^{it}\right]_c + \left[ie^{(k+1)it}\right]_a \right),$$
$$\gamma_k''(t) = \frac{1}{2}\left(\left[-e^{it}\right]_c + \left[-(k+1)e^{(k+1)it}\right]_a \right),$$

it is straightforward to check that $\det(\gamma_k') = 0$ and $\det(\gamma_k'') = \frac{1}{4}(1-(k+1)^2) < 0$ for all $k \geq 1$.

It remains to check that the curve $\gamma_k(\mathbb{R}/2\pi\mathbb{Z})$ does not have rank-one connections. Note that, for all $t, h \in \mathbb{R}$, we have

$$\begin{aligned}\det(\gamma_k(t+h)-\gamma_k(t)) &= \frac{1}{4}\left|e^{i(t+h)}-e^{it}\right|^2 - \frac{1}{4(k+1)^2}\left|e^{(k+1)i(t+h)}-e^{(k+1)it}\right|^2 \\ &= \frac{1}{4}\left(\left|e^{ih}-1\right|^2-(k+1)^{-2}\left|e^{(k+1)ih}-1\right|^2\right) \\ &= \sin^2\left(\frac{h}{2}\right)-(k+1)^{-2}\sin^2\left(\frac{(k+1)h}{2}\right) =: f(h).\end{aligned}$$

We show next that $f(h) > 0$ for all $h \in (0, 2\pi)$, which implies that the curve $\gamma_k(\mathbb{R}/2\pi\mathbb{Z})$ has no rank-one connections.

First note that $f(2\pi - h) = f(h)$, so it suffices to show that $f > 0$ on $(0, \pi]$. Then we argue separately on $(\pi/2, \pi]$ and $(0, \pi/2]$.

For $h \in (\pi/2, \pi]$ we have, using that $\sin(x) \geq \frac{2x}{\pi}$ for all $x \in (0, \pi/2]$,

$$f(h) \geq \frac{h^2}{\pi^2} - (k+1)^{-2}\sin^2\left(\frac{(k+1)h}{2}\right) > \frac{1}{4} - \frac{1}{(k+1)^2} \geq 0.$$

On $(0, \pi/2]$ we show that $f' > 0$, which completes the proof since $f(0) = 0$. The derivative is given by

$$f'(h) = \frac{1}{2}\left(\sin(h) - \frac{\sin((k+1)h)}{k+1}\right).$$

Using that $\sin(x) > \frac{2x}{\pi}$ for all $x \in \left(0, \frac{\pi}{2}\right)$, we know $\sin\left(\frac{\pi}{2(k+1)}\right) > \frac{1}{k+1}$ and it follows that

$$\sin(h) > \sin\left(\frac{\pi}{2(k+1)}\right) > \frac{\sin((k+1)h)}{k+1} \qquad \forall h \in \left(\frac{\pi}{2(k+1)}, \frac{\pi}{2}\right].$$

Note that $\cos(x) > \cos((k+1)x)$ for $x \in \left(0, \frac{\pi}{2(k+1)}\right]$ since the cosine function is decreasing on $\left(0, \frac{\pi}{2}\right]$. By integrating this inequality we have that

$$\sin(h) > \frac{\sin((k+1)h)}{k+1} \qquad \forall h \in \left(0, \frac{\pi}{2(k+1)}\right].$$

Putting the above two estimates together gives $f'(h) > 0$ for all $h \in (0, \pi/2]$. □

5.2 Further examples

Here we show how to construct many other examples of curves satisfying the assumptions of Theorem 1.7.

Lemma 5.2. *Let $\gamma \in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C})$ be an arc-length parametrization of a simple closed curve, and let $\tilde{\gamma} \in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{C})$ be such that $|\tilde{\gamma}'| \equiv 1$ and $|\tilde{\gamma}''| > 0$ on $\mathbb{R}/2\pi\mathbb{Z}$. Then there exists $k_0 = k_0(\gamma,\tilde{\gamma}) \in \mathbb{N}$ such that for every integer $k \geq k_0$, the curve in $\mathbb{R}^{2\times 2}$ parametrized by*

$$\alpha_k(t) := [\gamma(t)]_c + k^{-1}[\tilde{\gamma}(kt)]_a$$

is nowhere elliptic, satisfies $\det(\alpha_k'') < 0$ and has no rank-one connections.

Proof of Lemma 5.2. To simplify notation let $\hat{\gamma}_k(t) = k^{-1}\tilde{\gamma}(kt)$. First note that $|\hat{\gamma}_k'| = |\gamma'| = 1$, so $\det(\alpha_k') = |\gamma'|^2 - |\hat{\gamma}_k'|^2 = 0$, that is, the curve is nowhere elliptic.

Moreover, we have

$$\det(\alpha_k'') = |\gamma''|^2 - k^2|\tilde{\gamma}''|^2 \leq \sup|\gamma''|^2 - k^2 \inf|\tilde{\gamma}''|^2 < 0,$$

provided $k \geq k_0$ for k_0 large enough. Thanks to the identity (B.3) in the proof of Lemma B.1 and by compactness of $\mathbb{R}/2\pi\mathbb{Z}$, this implies the existence of $\delta_1 = \delta_1(\gamma,\tilde{\gamma}) > 0$ such that

$$\det(\alpha_k(s) - \alpha_k(t)) > 0 \qquad \text{for any } s,t \in \mathbb{R}/2\pi\mathbb{Z} \text{ s.t. } |e^{is} - e^{it}| \leq \delta_1. \tag{5.1}$$

Also note that since γ parametrizes a simple closed curve, there exists another constant $\beta_0 = \beta_0(\gamma) > 0$ such that

$$|\gamma(s) - \gamma(t)| \geq \beta_0 \qquad \text{for any } s,t \in \mathbb{R}/2\pi\mathbb{Z} \text{ s.t. } |e^{is} - e^{it}| \geq \delta_1. \tag{5.2}$$

For s,t as in (5.2) we infer

$$\begin{aligned}\det(\alpha_k(s) - \alpha_k(t)) &= |\gamma(s) - \gamma(t)|^2 - \frac{1}{k^2}|\tilde{\gamma}(ks) - \tilde{\gamma}(kt)|^2 \\ &\geq \beta_0^2 - \frac{4}{k^2}\sup|\tilde{\gamma}|^2,\end{aligned}$$

and deduce that

$$\det(\alpha_k(s) - \alpha_k(t)) > 0 \qquad \text{for any } s,t \in \mathbb{R}/2\pi\mathbb{Z} \text{ s.t. } |e^{is} - e^{it}| \geq \delta_1, \tag{5.3}$$

provided $k \geq k_0$ for k_0 large enough. Putting (5.3) and (5.1) together shows that the curve parametrized by α_k has no rank-one connections. □

5.3 Proof of Proposition 1.11

Recall the definitions of the set

$$\mathrm{NE}_* = \left\{\gamma \in C^2(\mathbb{R}/2\pi\mathbb{Z};\mathbb{R}^{2\times 2})\colon\ \det(\gamma') = 0 \text{ and } |\det(\gamma'')| > 0\right\},$$

and its subset

$$\mathrm{NE}_{**} = \Big\{\gamma \in \mathrm{NE}_* \colon \gamma(\mathbb{R}/2\pi\mathbb{Z}) \text{ has no rank-one connections}\Big\}.$$

In this section we prove Proposition 1.11, which asserts that NE_{**} is open in NE_* for the C^2 topology.

Let $\gamma, \bar{\gamma} \in \mathrm{NE}_*$. It is clear from the expansion (B.5), the explicit form of $\tilde{\varepsilon}(t,h)$ and the fact that $\det(\gamma') = \det(\bar{\gamma}') = 0$, that for all t, h in $\mathbb{R}$ we have

$$\begin{aligned}&|\det(\gamma(t+h) - \gamma(t)) - \det(\bar{\gamma}(t+h) - \bar{\gamma}(t))| \\ &\leq C(\|\gamma''\|_\infty + \|\bar{\gamma}''\|_\infty)\|\gamma'' - \bar{\gamma}''\|_\infty h^4,\end{aligned}$$

for some absolute constant $C > 0$.

Now fix $\bar{\gamma} \in \mathrm{NE}_{**}$ and assume without loss of generality that $\det(\bar{\gamma}(t) - \bar{\gamma}(s)) > 0$ for all $s \neq t \in \mathbb{R}/2\pi\mathbb{Z}$, then the last inequality implies that, if $\|\gamma'' - \bar{\gamma}''\|_\infty \leq \delta$ for $\delta > 0$ small enough, then for all $s, t \in \mathbb{R}$ we have

$$\det(\gamma(t) - \gamma(s)) \geq \det(\bar{\gamma}(t) - \bar{\gamma}(s)) - C(\bar{\gamma})\delta|e^{it} - e^{is}|^4.$$

By definition of NE_{**} and Lemma B.1 there exists $\bar{\kappa} > 0$ such that

$$\det(\bar{\gamma}(t) - \bar{\gamma}(s)) \geq \bar{\kappa}|e^{it} - e^{is}|^4,$$

so if $\|\gamma'' - \bar{\gamma}''\|_\infty \leq \delta$ for $\delta > 0$ small enough we deduce

$$\det(\gamma(t) - \gamma(s)) \geq \frac{\bar{\kappa}}{4}|e^{it} - e^{is}|^4.$$

That is, $\gamma \in \mathrm{NE}_{**}$, and this shows that NE_{**} is open in NE_* with respect to the C^2 topology. □

Appendix A Fractional regularity of m

In this appendix we establish the initial low fractional regularity which is required to apply Proposition 1.5. Since Π has no rank-one connections, [Š93, Lemma 1] implies that $\det(A - B)$ has a constant sign over all matrices $A \neq B \in \Pi$. Composing u with a reflection does not affect Theorem 1.3, so we may assume without loss of generality

$$\det(A - B) > 0 \qquad \forall A \neq B \in \Pi,$$

and the coercivity of the determinant (1.1) can be rephrased as

$$\det(A - B) \geq c\,|A - B|^4 \qquad \forall A, B \in \Pi.$$

Quite classically, this implies some fractional regularity for the map m defined in (1.5), which satisfies $Du = \operatorname{cof}\Gamma(m)$.

Lemma A.1. *Let $J \subset \mathbb{S}^1$ be compact and connected. If $m\colon \Omega \to J$ solves (1.6) where $\Gamma \in C^1(J;\mathbb{R}^{2\times 2})$ satisfies*

$$\det(\Gamma(z) - \Gamma(z')) \geq c|z-z'|^p \qquad \forall z, z' \in J, \tag{A.1}$$

for some $p > 1$ and $c > 0$, then $m \in B^{\frac{1}{p-1}}_{p,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1)$, that is,

$$\sup_{|h|\leq 1} \frac{\|D^h m\|_{L^p(U)}}{|h|^{\frac{1}{p-1}}} < \infty \qquad \textit{for all } U \subset\subset \Omega,$$

where $D^h m(x) = (m(x+h) - m(x))\mathbf{1}_{x,x+h\in\Omega}$. In particular, if $p = 4$ then $m \in B^{\frac{1}{3}}_{4,\infty,\mathrm{loc}}$.

Proof of Lemma A.1. The proof is essentially the same as the first step in [Š93, Theorem 5]. Here, for the reader's convenience, we reproduce the proof with minor adaptions.

Since this is a local result, we may assume without loss of generality that Ω is simply connected. Since $\nabla \cdot \Gamma_j(m) = 0$, we infer that $\mathrm{curl}\,(i\Gamma_j(m)) = 0$, and thus there exists $F_j : \Omega \to \mathbb{R}$ with

$$\nabla F_j = i\Gamma_j(m) \qquad \text{a.e. in } \Omega. \tag{A.2}$$

For any given $U \subset\subset \Omega$ and $h \in \mathbb{R}^2$ with $|h|$ sufficiently small, e.g. $|h| < \frac{1}{3}\mathrm{dist}(U, \partial\Omega)$, by (A.1) we have

$$\det\left(\Gamma(m(x+h)) - \Gamma(m(x))\right) \geq c|D^h m(x)|^p$$

for a.e. $x \in \Omega$ with $\mathrm{dist}(x, \partial\Omega) > |h|$. By (A.2) we have

$$\det\left(\Gamma(m(\cdot + h)) - \Gamma(m(\cdot))\right) = (iD^h\Gamma_1(m)) \cdot (D^h\Gamma_2(m)) = D^h\nabla F_1 \cdot D^h\Gamma_2(m).$$

Hence gathering the two above equations, we obtain

$$\left|D^h m\right|^p \lesssim D^h\nabla F_1 \cdot D^h\Gamma_2(m) \qquad \text{for a.e. } x \in \Omega \text{ with } \mathrm{dist}(x, \partial\Omega) > |h|. \tag{A.3}$$

Let $\chi \in C_c^\infty(\Omega)$ be a test function with $\mathrm{dist}(\mathrm{supp}\,\chi, \partial\Omega) > 2h$ and $\mathbf{1}_U \leq \chi \leq \mathbf{1}_\Omega$. Integrating by parts and using that $\nabla \cdot \Gamma_2(m) = 0$ (and thus $\int_\Omega \nabla \left(D^h F_1 \chi^{\frac{p}{p-1}}\right) \cdot D^h\Gamma_2(m) dx = 0$), we have

$$\begin{aligned}
\int_\Omega \chi^{\frac{p}{p-1}} D^h\nabla F_1 \cdot D^h\Gamma_2(m)\, dx &= -\int_\Omega D^h F_1\, D^h\Gamma_2(m) \cdot \nabla(\chi^{\frac{p}{p-1}})\, dx \\
&\lesssim |h|\, \|\nabla F_1\|_{L^\infty(\Omega)} \|\partial_\theta \Gamma_2\|_{L^\infty(J)} \|\nabla\chi\|_{L^\infty(\Omega)} \int_\Omega \chi^{\frac{1}{p-1}} \left|D^h m\right| dx \\
&\lesssim |h|\, \|\nabla\chi\|_{L^\infty(\Omega)} \left(\int_\Omega \chi^{\frac{p}{p-1}} \left|D^h m\right|^p dx\right)^{\frac{1}{p}}.
\end{aligned}$$

Recalling (A.3) we deduce

$$\int_\Omega \chi^{\frac{p}{p-1}} |D^h m|^p \, dx \lesssim |h| \, \|\nabla\chi\|_{L^\infty(\Omega)} \left(\int_\Omega \chi^{\frac{p}{p-1}} |D^h m|^p \, dx \right)^{\frac{1}{p}},$$

and thus

$$\left(\int_\Omega \chi^{\frac{p}{p-1}} |D^h m|^p \, dx \right)^{\frac{1}{p}} \lesssim |h|^{\frac{1}{p-1}} \, \|\nabla\chi\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}.$$

As $\mathbf{1}_U \leq \chi$, it follows that

$$\sup_{|h|\leq t} \frac{\|D^h m\|_{L^p(U)}}{|h|^{\frac{1}{p-1}}} \lesssim \|\nabla\chi\|_{L^\infty(\Omega)}^{\frac{1}{p-1}}$$

for $t > 0$ sufficiently small. For larger $|h|$ values, the boundedness of m implies that $\sup_{t<|h|\leq 1} |h|^{-\frac{1}{p-1}} \|D^h m\|_{L^p(U)}$ is bounded. Thus $m \in B^{\frac{1}{p-1}}_{p,\infty}(U)$ for all $U \subset\subset \Omega$. □

Appendix B The nondegeneracy condition (1.1) for nowhere elliptic curves

As in the beginning of Appendix A, we may assume without loss of generality

$$\det(A - B) > 0 \qquad \forall A \neq B \in \Pi.$$

Here we show that the nondegeneracy estimate (1.1) is equivalent, in the nowhere elliptic setting, to the assumption that $\det(\gamma'')$ does not vanish.

Lemma B.1. *Let $I = [a, b]$ for some $a < b \leq a+2\pi$ and $\gamma \in C^2(I; \mathbb{R}^{2\times 2})$. If $\det(\gamma') \equiv 0$ on I and*

$$\det(\gamma(t) - \gamma(s)) > 0 \qquad \forall t \neq s \in I, \tag{B.1}$$

then the estimate

$$\det(\gamma(t) - \gamma(s)) \geq c|e^{it} - e^{is}|^4 \qquad \forall t, s \in I \tag{B.2}$$

is satisfied if and only if $\det(\gamma'')$ does not vanish on I.

Remark B.2. The assumptions of Lemma B.1, in particular, the condition (B.1), imply that $\det(\gamma'') < 0$, see (B.3). The case $\det(\gamma'') > 0$ would correspond to the opposite estimate

$$-\det(\gamma(t) - \gamma(s)) \geq c|e^{it} - e^{is}|^4 \qquad \text{for some } c > 0.$$

Proof of Lemma B.1. First we prove that, for any $t \in I$,

$$\lim_{s\to t, s\in I} \frac{\det(\gamma(s)-\gamma(t))}{(s-t)^4} = -\frac{1}{12}\det(\gamma''(t)). \tag{B.3}$$

Under the additional regularity assumption $\gamma \in C^3(I;\mathbb{R}^{2\times 2})$ this follows directly from a Taylor expansion. Indeed, derivating twice the identity $\det(\gamma') \equiv 0$, we obtain

$$\begin{aligned}
&\operatorname{cof}\gamma' : \gamma'' = 0,\\
&\operatorname{cof}\gamma' : \gamma^{(3)} = -\operatorname{cof}\gamma'' : \gamma'' = -2\det(\gamma''),
\end{aligned}$$

and then the Taylor expansion

$$\begin{aligned}
&\det(\gamma(t+h)-\gamma(t)) = \det\left(h\gamma'(t) + \frac{h^2}{2}\gamma''(t) + \frac{h^3}{6}\gamma^{(3)}(t) + o(h^3)\right)\\
&= h^2\det(\gamma'(t)) + \frac{h^3}{2}\operatorname{cof}\gamma'(t) : \gamma''(t) + \frac{h^4}{4}\det(\gamma''(t)) + \frac{h^4}{6}\operatorname{cof}\gamma'(t) : \gamma^{(3)}(t) + o(h^4)\\
&= -\frac{h^4}{12}\det(\gamma''(t)) + o(h^4),
\end{aligned}$$

implies (B.3).

In the general case where $\gamma \in C^2(I;\mathbb{R}^{2\times 2})$, one may actually use the bilinearity of the determinant to obtain the same outcome with only two derivatives. Specifically, for $t, t+h \in I$ we have

$$\begin{aligned}
\det(\gamma(t+h)-\gamma(t)) &= \det\left(h\gamma'(t) + h^2\int_0^1 (1-s)\gamma''(t+sh)\,ds\right)\\
&= h^2\det(\gamma'(t)) + h^3\int_0^1 (1-s)\operatorname{cof}\gamma'(t) : \gamma''(t+sh)\,ds\\
&\quad + h^4\det\left(\int_0^1 (1-s)\gamma''(t+sh)\,ds\right).
\end{aligned} \tag{B.4}$$

Using that $\operatorname{cof}\gamma' : \gamma'' = (\det(\gamma'))'$, we can rewrite the penultimate term as

$$\begin{aligned}
&h^3\int_0^1 (1-s)\operatorname{cof}\gamma'(t) : \gamma''(t+sh)\,ds\\
&= h^2\int_0^1 (1-s)\frac{d}{ds}[\det(\gamma'(t+sh)]\,ds\\
&\quad - h^4\int_0^1 s(1-s)\operatorname{cof}\left(\frac{\gamma'(t+sh)-\gamma'(t)}{sh}\right) : \gamma''(t+sh)\,ds\\
&= h^2\int_0^1 \det(\gamma'(t+sh))\,ds - h^2\det(\gamma'(t)) - \frac{h^4}{3}\det\gamma''(t) + h^4\varepsilon(t,h),
\end{aligned}$$

where

$$\begin{aligned}\varepsilon(t,h) &= \int_0^1 s(1-s)\left[\operatorname{cof}\gamma''(t)\colon \gamma''(t) - \operatorname{cof}\left(\frac{\gamma'(t+sh)-\gamma'(t)}{sh}\right) : \gamma''(t+sh)\right] ds \\ &= \int_0^1\int_0^1 s(1-s)\left[\operatorname{cof}\gamma''(t)\colon \gamma''(t) - \operatorname{cof}\gamma''(t+\tau sh)\colon \gamma''(t+sh)\right] d\tau\, ds,\end{aligned}$$

hence $\sup_t |\varepsilon(t,h)| \to 0$ as $h \to 0$ thanks to the uniform continuity of γ''. Plugging this into (B.4) gives

$$\begin{aligned}\det(\gamma(t+h)-\gamma(t)) = h^2 \int_0^1 \det(\gamma'(t+sh))\, ds - \frac{h^4}{12}\det(\gamma''(t)) \\ + h^4\tilde{\varepsilon}(t,h),\end{aligned} \tag{B.5}$$

where

$$\begin{aligned}\tilde{\varepsilon}(t,h) = \varepsilon(t,h) + \frac{1}{2}\int_0^1 (1-s)\operatorname{cof}\gamma''(t)\colon (\gamma''(t+sh)-\gamma''(t))\, ds \\ + \det\left(\int_0^1 (1-s)(\gamma''(t+sh)-\gamma''(t))\, ds\right),\end{aligned}$$

so $\sup_t |\tilde{\varepsilon}(t,h)| \to 0$ as $h \to 0$ thanks again to the uniform continuity of γ''. Since $\det(\gamma') \equiv 0$ on I, this implies (B.3).

A first consequence of (B.3) is that the estimate (B.2) implies that $\det(\gamma'')$ does not vanish.

For the converse, assume that $\det(\gamma'')$ does not vanish. Since the left-hand side of (B.3) must be nonnegative according to (B.1), we deduce that $\det(\gamma'') < 0$, and for all $t \in I$ there exists $\delta_t > 0$ such that

$$\det(\gamma(t+h)-\gamma(t)) \geq \frac{h^4}{24}\inf_I |\det(\gamma'')| \qquad \forall h \in (-\delta_t, \delta_t),\ t+h \in I.$$

By compactness of I this δ_t can be chosen independent of t, so we have

$$\det(\gamma(t)-\gamma(s)) \geq c|e^{it}-e^{is}|^4 \qquad \text{if } |e^{it}-e^{is}| \leq \eta, \text{ for } s,t \in I,$$

for some constants $c, \eta > 0$ depending on γ. Moreover by compactness of the set of couples $(t,s) \in I \times I$ such that $|e^{is}-e^{it}| \geq \eta$, there exists $c > 0$ such that

$$\det(\gamma(t)-\gamma(s)) \geq c \geq \frac{c}{2^4}|e^{it}-e^{is}|^4 \qquad \text{if } |e^{it}-e^{is}| \geq \eta,$$

and this proves Lemma B.1. □

Appendix C A commutator estimate

Lemma C.1. *Let $\Omega \subset \mathbb{R}^n$ be open and $w \in B^s_{p,\infty,\mathrm{loc}} \cap L^\infty(\Omega;\mathbb{R}^k)$ for some $s \in (0,1)$ and $p > 1$. Let $\rho \in C^1_c(\mathbb{R}^n)$ with $\operatorname{supp}\rho \subset B_1$ and $\int \rho\, dx = 1$ and denote convolution with $\rho_\varepsilon(x) = \varepsilon^{-n}\rho(x/\varepsilon)$ by a subscript ε. Then for any $\alpha, \beta \geq 1$, any C^2 map $G\colon \mathbb{R}^k \to \mathbb{R}$, we have*

$$\int_U |G(w_\varepsilon) - G(w)_\varepsilon|^\alpha |\nabla w_\varepsilon|^\beta \, dx \leq c\varepsilon^{s\min(p,2\alpha+\beta)-\beta} \qquad \forall U \subset\subset \Omega,$$

for some constant $c > 0$ depending on $\|G\|_{C^2}$, $\|w\|_\infty$, $|w|_{B^s_{p,\infty}(U)}$, $\|\rho\|_{C^1}$, α, β, p and U.

Proof of Lemma C.1. For any $U \subset\subset \Omega$ the regularity $B^s_{p,\infty,\mathrm{loc}}$ of w amounts to

$$\int_U |D^h w|^p \leq |w|^p_{B^s_{p,\infty}(U)} |h|^{sp},$$

for all $|h| \leq 1$. Moreover, letting $R = \|w\|_\infty$, for any $\varepsilon > 0$ we have (see e.g. [LLP20, Lemma 17])

$$|G(w_\varepsilon) - G(w)_\varepsilon| \leq c_0 \|G\|_{C^2(B_R)} \int_{B_1} |D^{\varepsilon y} w|^2 \rho(y)\, dy,$$

$$|\nabla w_\varepsilon| \leq \frac{c_0}{\varepsilon} \int_{B_1} |D^{\varepsilon z} w|\, |\nabla \rho|(z)\, dz,$$

for some absolute constant $c_0 > 0$, so applying Jensen's inequality and Fubini's theorem we deduce

$$\begin{aligned}
&\int_U |G(w_\varepsilon) - G(w)_\varepsilon|^\alpha |\nabla w_\varepsilon|^\beta \, dx \\
&\leq \frac{c}{\varepsilon^\beta} \int_{B_1}\int_{B_1} \rho(y)|\nabla\rho|(z) \int_U |D^{\varepsilon y} w(x)|^{2\alpha} |D^{\varepsilon z} w(x)|^\beta \, dx\, dy dz,
\end{aligned}$$

for some constant c depending on $\|G\|_{C^2(B_R)}$ and $\|\rho\|_{C^1}$. For any $y, z \in B_1$ we have, by Hölder's inequality with exponents $q = (2\alpha+\beta)/(2\alpha)$ and $q' = (2\alpha+\beta)/\beta$,

$$\begin{aligned}
\int_U |D^{\varepsilon y} w|^{2\alpha} |D^{\varepsilon z} w|^\beta \, dx &\leq \left(\int_U |D^{\varepsilon y} w|^{2\alpha+\beta}\, dx\right)^{\frac{1}{q}} \left(\int_U |D^{\varepsilon z} w|^{2\alpha+\beta}\, dx\right)^{1-\frac{1}{q}} \\
&\leq c|w|^{\min(p,2\alpha+\beta)}_{B^s_{p,\infty}(U)} \varepsilon^{s\min(p,2\alpha+\beta)}.
\end{aligned}$$

In the last line we used the Besov regularity assumption, and the constant c depends on $\|w\|_\infty$, $|U|, \alpha, \beta$ and p. Plugging this back into the previous estimate concludes the proof. □

Appendix D A modification lemma

Lemma D.1. *Let* $\Gamma \in C^2(\mathbb{S}^1;\mathbb{R}^{2\times 2})$ *be such that* $\det(\partial_\theta^2\Gamma)$ *is not identically zero. Let* Γ_1, Γ_2 *denote the two rows of the matrix* Γ. *For any* $\alpha_1, \alpha_2 \in C^1(\mathbb{S}^1;\mathbb{R})$, *there exist* $\tilde{\alpha}_1, \tilde{\alpha}_2 \in C^1(\mathbb{S}^1;\mathbb{R})$ *such that*

$$\int_{\mathbb{S}^1} \left(\sum_{j=1}^{2} \tilde{\alpha}_j \partial_\theta \Gamma_j \right) d\theta = 0$$

$$\text{and } \sum_{j=1}^{2} \sup_{\mathbb{S}^1} |\alpha_j - \tilde{\alpha}_j| \le C \left| \int_{\mathbb{S}^1} \left(\sum_{j=1}^{2} \alpha_j \partial_\theta \Gamma_j \right) d\theta \right|,$$

for some constant C *depending only on* Γ.

Proof of Lemma D.1. We denote by $v \in \mathbb{R}^2$ the integral

$$v = \int_{\mathbb{S}^1} \left(\sum_{j=1}^{2} \alpha_j \partial_\theta \Gamma_j \right) d\theta,$$

and prove the existence of $\beta_1, \beta_2 \in C^1(\mathbb{S}^1;\mathbb{R})$ such that

$$\int_{\mathbb{S}^1} \left(\sum_{j=1}^{2} \beta_j \partial_\theta \Gamma_j \right) d\theta = v \qquad \text{and} \qquad \sum_{j=1}^{2} \sup_{\mathbb{S}^1} |\beta_j| \le C|v|. \tag{D.1}$$

Then it suffices to define $\tilde{\alpha}_j = \alpha_j - \beta_j$.

We look for β_j in the form $\beta_j(e^{i\theta}) = b_j f_j(e^{i\theta})$, for some $b \in \mathbb{R}^2$ and $f_1, f_2 \in C^1(\mathbb{S}^1;\mathbb{R})$, then the first identity in (D.1) amounts to

$$Ab = v, \quad \text{where } A = \left(\int_{\mathbb{S}^1} f_1 \partial_\theta \Gamma_1 \, d\theta \;\middle|\; \int_{\mathbb{S}^1} f_2 \partial_\theta \Gamma_2 \, d\theta \right) \in \mathbb{R}^{2\times 2}.$$

So it suffices to choose f_1, f_2 such that $\det(A) \neq 0$, then we simply set $b = A^{-1}v$, and the second estimate in (D.1) follows from $|b| \le \|A^{-1}\||v|$.

To find f_1, f_2 such that $\det(A) \neq 0$, we can pick for instance $z_1, z_2 \in \mathbb{S}^1$ such that $\det(\partial_\theta \Gamma_1(z_1), \partial_\theta \Gamma_2(z_2)) \neq 0$ and choose, for $j = 1, 2$, f_j approximating the Dirac mass at z_j. The existence of such z_1, z_2 follows from the fact that $\det(\partial_\theta^2\Gamma)$ is not identically zero: if $\det(\partial_\theta \Gamma_1(z_1), \partial_\theta \Gamma_2(z_2)) = 0$ for all $z_1, z_2 \in \mathbb{S}^1$ then $\partial_\theta \Gamma_1(\mathbb{S}^1)$ and $\partial_\theta \Gamma_2(\mathbb{S}^1)$ are contained in a single line, which implies that $\partial_\theta \Gamma v_0 = 0$ is identically zero for some fixed $v_0 \in \mathbb{S}^1$, and therefore $\det(\partial_\theta^2\Gamma)$ is identically zero. □

Remark D.2. In the above proof, we only modify α_j around z_j, where $z_1, z_2 \in \mathbb{S}^1$ satisfy $\det(\partial_\theta \Gamma_1(z_1), \partial_\theta \Gamma_2(z_2)) \neq 0$. As a consequence, we can choose $\tilde{\alpha}_j$ such that $\sum_{j=1}^2 \tilde{\alpha}_j \partial_\theta \Gamma_j = \sum_{j=1}^2 \alpha_j \partial_\theta \Gamma_j$ away from any neighborhood of $\{z_1, z_2\}$.

Remark D.3. If we assume that $\det(\partial_\theta \Gamma(z_0)) \neq 0$ for some $z_0 \in \mathbb{S}^1$, then we can choose f_1, f_2 approximating a Dirac mass at z_0, and deduce the existence of $\tilde{\alpha}_j$ as in Lemma D.1, with the additional condition that $\tilde{\alpha}_j = \alpha_j$ away from any neighborhood of z_0.

Appendix E Rigidity estimate for elliptic arcs

We provide here the extension of [LLP23, Theorem 1.1] needed in the proof of Proposition 3.1, namely, a rigidity estimate for C^2 elliptic arcs $\mathcal{J} \subset \mathbb{R}^{2\times 2}$. In [LLP23] this was established for smooth closed elliptic curves.

Proposition E.1. *Let $a < b$ and $\gamma \in C^2([a,b];\mathbb{R}^{2\times 2})$ be injective such that $\mathcal{J} = \gamma([a,b])$ is elliptic:*

$$\det(A-B) \geq c_0|A-B|^2 \qquad \forall A, B \in \mathcal{J}, \tag{E.1}$$

for some $c_0 > 0$. Then $\mathcal{J}$ satisfies a rigidity estimate: there exists $C = C(\mathcal{J}) > 0$ such that

$$\inf_{A\in\mathcal{J}} \int_{B_{1/2}} |Du - A|^2\, dx \leq C \int_{B_1} \operatorname{dist}^2(Du, \mathcal{J})\, dx,$$

for all $u \in W^{1,2}(B_1;\mathbb{R}^2)$.

Proof of Proposition E.1. We will show in Lemma E.2 that we can find a closed C^2 curve $\Gamma \subset \mathbb{R}^{2\times 2}$ such that $\mathcal{J} \subset \Gamma$ and Γ is elliptic:

$$\det(A-B) \geq c_1|A-B|^2 \qquad \forall A, B \in \Gamma. \tag{E.2}$$

Granted this, we can directly apply [LLP23, Theorem 1.1] which provides $C = C(\Gamma) > 0$ such that

$$\inf_{A\in\Gamma} \int_{B_{1/2}} |Du - A|^2\, dx \leq C \int_{B_1} \operatorname{dist}^2(Du, \Gamma)\, dx \leq C \int_{B_1} \operatorname{dist}^2(Du, \mathcal{J})\, dx, \tag{E.3}$$

for all $u \in W^{1,2}(B_1;\mathbb{R}^2)$. In [LLP23, Theorem 1.1] this is stated for a smooth curve Γ, but the proof only requires C^2 regularity (and could probably be modified to require only C^1 regularity). Moreover, given $u \in W^{1,2}(B_1;\mathbb{R}^2)$ and $A \in \Gamma$ attaining the infimum in the left-hand side of (E.3), we can integrate on $B_{1/2}$ the elementary inequality

$$\operatorname{dist}^2(A, \mathcal{J}) \leq 2|Du - A|^2 + 2\operatorname{dist}^2(Du, \mathcal{J}),$$

and use (E.3) to deduce

$$|B_{1/2}| \operatorname{dist}^2(A, \mathcal{J}) \leq (2C+2) \int_{B_1} \operatorname{dist}^2(Du, \mathcal{J})\, dx.$$

Taking $\tilde{A} \in \mathcal{J}$ such that $|A - \tilde{A}| = \operatorname{dist}(A, \mathcal{J})$ and combining the above estimate with (E.3), we obtain

$$\int_{B_{1/2}} |Du - \tilde{A}|^2\, dx \leq (6C+4) \int_{B_1} \operatorname{dist}^2(Du, \mathcal{J})\, dx,$$

thus proving Proposition E.1. □

Lemma E.2. *Let $a < b$ and $\gamma \in C^2([a,b];\mathbb{R}^{2\times 2})$ be injective such that $\mathcal{J} = \gamma([a,b])$ satisfies* (E.1)*. Then there exists a closed C^2 curve $\Gamma \subset \mathbb{R}^{2\times 2}$ such that $\mathcal{J} \subset \Gamma$ and Γ satisfies* (E.2) *for some positive constant $c_1 < c_0$.*

Proof. First we extend the C^2 curve γ to $[a-\varepsilon, b+\varepsilon]$ for some $\varepsilon > 0$, while conserving the ellipticity (E.1) for $\mathcal{J} = \gamma([a-\varepsilon, b+\varepsilon])$. To that end we set $\gamma(a-t) = \gamma(a) - t\gamma'(a) + t^2\gamma''(a)/2$ for $0 < t \le \varepsilon$. That way (E.1) is satisfied with a possibly smaller constant c_0 (not renamed) for $A, B \in \gamma([a-\varepsilon, a+\varepsilon])$ provided ε is small enough, because $\det(\gamma'(a+h)) = \det(\gamma'(a)) + o(1) \neq 0$ for $h \to 0$. And (E.1) is satisfied also for $A \in \gamma([a-\varepsilon, a])$ and $B \in \gamma([a+\varepsilon, b])$ because $A \approx \gamma(a)$ and $\det(\gamma(a) - B)$ is bounded from below by a positive constant. The same arguments apply for the extension to $[b, b+\varepsilon]$. Note that γ is smooth in $[a-\varepsilon, a) \cup (b, b+\varepsilon]$.

Then we use the classical fact [Zha97, FS08] that the ellipticity (E.1) of $\mathcal{J}$ implies that the conformal-anticonformal decomposition (2.2) of γ is given by

$$\gamma = [\gamma_c]_c + [H \circ \gamma_c]_a,$$

where $\gamma_c \in C^2([a-\varepsilon, b+\varepsilon];\mathbb{C})$ is injective with $|\gamma_c'| > 0$ on $[a-\varepsilon, b+\varepsilon]$, and

$$H \colon \gamma_c([a-\varepsilon, b+\varepsilon]) \to \mathbb{C},$$

is k-Lipschitz for $k = \sqrt{(1-2c_0)/(1+2c_0)} \in (0,1)$. Next we extend H to a k-Lipschitz map over $\mathbb{C}$ with a possibly larger $k < 1$ such that the extension is smooth outside a sufficiently small neighborhood of $\gamma_c([a,b])$. The proof is very similar to the proof of [LLP23, Lemma 3.1], and we sketch below the key steps.

In the first step, possibly after a reparametrization, we may assume without loss of generality that $\gamma_c(t)$ is an arc-length parametrization of $\gamma_c([a-\varepsilon, b+\varepsilon])$. For fixed ε and small enough $\delta > 0$, denote $R_\delta := [a,b] \times (-\delta, \delta)$ and $R_\delta^\varepsilon := ([a-\varepsilon, a) \cup (b, b+\varepsilon]) \times (-\delta, \delta)$. The map

$$\varphi \colon (t, r) \mapsto \gamma_c(t) + ri\gamma_c'(t),$$

is a C^1 diffeomorphism between $R_\delta \cup R_\delta^\varepsilon$ and $\varphi(R_\delta \cup R_\delta^\varepsilon)$. Since γ is smooth in $[a-\varepsilon, a) \cup (b, b+\varepsilon]$, this map φ is further a smooth diffeomorphism between R_δ^ε and $\varphi(R_\delta^\varepsilon)$. For $z \in \varphi(R_\delta \cup R_\delta^\varepsilon)$, define $\widetilde{H}(z) = H(\gamma_c(t))$ where $z = \varphi(t, r)$. This $\widetilde{H}$ agrees with H on $\gamma_c([a-\varepsilon, b+\varepsilon])$, and is C^1 in $\varphi(R_\delta \cup R_\delta^\varepsilon)$ and smooth in $\varphi(R_\delta^\varepsilon)$ by the regularity of $\gamma_a = H \circ \gamma_c$ and φ^{-1}. Further, we have $\|D(\varphi^{-1})\| \le 1 + C\delta$ for some constant C depending on γ_c, and it follows that $\widetilde{H}$ is $\tilde{k}$-Lipschitz with $\tilde{k} = (1 + C\delta)k < 1$ for small enough δ.

In the second step, we extend $\widetilde{H}$ to $\mathbb{C}$ such that the extension is smooth outside a sufficiently small neighborhood of $\gamma_c([a,b])$. By Kirszbraun's theorem, we can first extend $\widetilde{H}$ to a $\tilde{k}$-Lipschitz map (not renamed) over $\mathbb{C}$. For $\alpha > 0$, we define $H_\alpha(z) = \int_{\mathbb{C}} \widetilde{H}(z + \alpha\chi(z)y)\rho(y)\,dy$ for a smooth kernel $\rho \ge 0$ with support in B_1 and $\int \rho(y)\,dy = 1$, and some smooth cut-off function χ with $\mathbf{1}_{\mathcal{U}_{\delta^2/4}} \le 1 - \chi \le \mathbf{1}_{\mathcal{U}_{\delta^2/2}}$ and $\|\nabla\chi\|_\infty \le 8/\delta^2$, where $\mathcal{U}_\delta := \{z \in \mathbb{C} : \mathrm{dist}(z, \gamma_c([a,b])) < \delta\}$. Note that H_α agrees with $\widetilde{H}$ in $\mathcal{U}_{\delta^2/4}$ and is smooth in $\left(\mathbb{C} \setminus \overline{\mathcal{U}_{\delta^2/2}}\right) \bigcup \varphi(R_{\delta/2}^{\varepsilon/2} \setminus R_{\delta/2}^{\delta^2/8})$ for α small enough. In particular, H_α agrees with H on $\gamma_c([a - \frac{3\delta^2}{16}, b + \frac{3\delta^2}{16}])$. Further we have

$$\begin{aligned}|H_\alpha(z) - H_\alpha(z')| &\le \int_{B_1} |\widetilde{H}(z + \alpha\chi(z)y) - \widetilde{H}(z' + \alpha\chi(z')y)|\, \rho(y)\, dy \\ &\le \tilde{k}\left(1 + \frac{8\alpha}{\delta^2}\right) \int_{B_1} |z - z'|\, \rho(y)\, dy = \tilde{k}\left(1 + \frac{8\alpha}{\delta^2}\right) |z - z'|,\end{aligned}$$

so H_α is k_α-Lipschitz for $k_\alpha = \tilde{k}\left(1 + \frac{8\alpha}{\delta^2}\right) < 1$ for α sufficiently small.

Finally we use the above H_α to construct the closed curve Γ containing $\mathcal{J}$. To that end, we claim that $\gamma_c([a - \varepsilon/3, b + \varepsilon/3])$ can be extended to a closed C^2 curve in $\mathbb{C}$ parametrized by $\tilde{\gamma}_c \in C^2(\mathbb{R}/L\mathbb{Z}; \mathbb{C})$ for some $L > b - a + 2\varepsilon/3$. This can be obtained as a consequence of Jordan's theorem stating that the complement of a Jordan arc in the plane is connected, but we sketch here a simpler proof in our C^2 context. Recall that φ is a C^1 diffeomorphism between $R_\delta \cup R_\delta^\varepsilon$ and $\varphi(R_\delta \cup R_\delta^\varepsilon)$. We can find an open neighborhood $\mathcal{V}$ of $R_{\delta/2} \cup R_{\delta/2}^{2\varepsilon/3}$ with $\mathcal{V} \subset\subset R_\delta \cup R_\delta^\varepsilon$, and extend the straight segment $[a - \varepsilon/3, b + \varepsilon/3]$ into $\mathcal{V} \setminus \left(R_{\delta^2} \cup R_{\delta^2}^{\varepsilon/3}\right)$ to form a smooth closed curve and apply φ to obtain a C^1 closed curve inside $\varphi(\mathcal{V})$ extending $\gamma_c([a - \varepsilon/3, b + \varepsilon/3])$. Finally smoothen that curve outside $[a - \varepsilon/3, b + \varepsilon/3]$ (using the same method we used to smoothen $\widetilde{H}$ via a cut-off function of the form of χ) to turn it into a C^2 closed curve $\tilde{\gamma}_c$ extending $\gamma_c([a - \varepsilon/3, b + \varepsilon/3])$.

We are now equipped with $\tilde{\gamma}_c \in C^2(\mathbb{R}/L\mathbb{Z}; \mathbb{C})$ injective such that $\tilde{\gamma}_c = \gamma_c$ on $[a - \varepsilon/3, b + \varepsilon/3]$. Then we define $\tilde{\gamma}_a = H_\alpha \circ \tilde{\gamma}_c$. Since H_α is smooth in $\left(\mathbb{C} \setminus \overline{\mathcal{U}_{\delta^2/2}}\right) \bigcup \varphi(R_{\delta/2}^{\varepsilon/2} \setminus R_{\delta/2}^{\delta^2/8})$, we know $\tilde{\gamma}_a$ is C^2 outside the interval $[a - \frac{\delta^2}{8}, b + \frac{\delta^2}{8}]$. Further, since H_α agrees with H on $\gamma_c([a - \frac{3\delta^2}{16}, b + \frac{3\delta^2}{16}])$, it follows that $\tilde{\gamma}_a = H_\alpha \circ \tilde{\gamma}_c = H \circ \gamma_c = \gamma_a \in C^2([a - \frac{3\delta^2}{16}, b + \frac{3\delta^2}{16}])$. Thus, we deduce that $\tilde{\gamma}_a \in C^2(\mathbb{R}/L\mathbb{Z}; \mathbb{C})$. Setting

$$\tilde{\gamma} = [\tilde{\gamma}_c]_c + [\tilde{\gamma}_a]_a,$$

it follows that $\Gamma = \tilde{\gamma}(\mathbb{R}/L\mathbb{Z})$ is a closed C^2 elliptic curve containing $\mathcal{J}$. The ellipticity follows from the fact that H_α is k_α-Lipschitz with $k_\alpha \in (0, 1)$. □

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