On Aviles-Giga limit states with L^p entropy productions

Xavier Lamy* Andrew Lorent† Guanying Peng‡

Abstract

The Aviles-Giga energy provides sequences of maps converging to weak solutions $m\colon \Omega \subset \mathbb{R}^2 \to \mathbb{R}^2$ of the eikonal equation

$$\operatorname{div} m = 0 \text{ in } \mathcal{D}'(\Omega), \quad |m| = 1 \text{ a.e. in } \Omega\,,$$

whose entropy productions $\operatorname{div}\Phi(m)$ are Radon measures in Ω, controlled by the energy. Here, the entropies are all C^2 vector fields $\Phi\colon \mathbb{S}^1 \to \mathbb{R}^2$ such that $\operatorname{div}\Phi(m_*) = 0$ for any smooth solution m_*. It is conjectured that the entropy production measures are concentrated on the one-dimensional jump set of m, as follows from the chain rule if m has bounded variation. In particular, the entropy production measures should vanish if they coincide with L^p functions: this is what we establish in this note if p is not too small and under natural boundary conditions.

1 Introduction

Given a bounded domain $\Omega \subset \mathbb{R}^2$, we consider weak solutions $m : \Omega \to \mathbb{R}^2$ of the 2D eikonal equation given by

$$|m| = 1 \text{ a.e. in } \Omega, \qquad \operatorname{div} m = 0 \text{ in } \mathcal{D}'(\Omega). \tag{1.1}$$

If m is a *smooth* solution of (1.1), the chain rule ensures that $\operatorname{div}\Phi(m) = 0$ for any smooth vector field $\Phi\colon \mathbb{S}^1 \to \mathbb{R}^2$ with derivative tangent to $\mathbb{S}^1$. By analogy with

*Institut Universitaire de France (IUF) & Univ Toulouse, CNRS, IMT, Toulouse, France. Email: Xavier.Lamy@math.univ-toulouse.fr

†Department of Mathematical Sciences, University of Cincinnati, Cincinnati, OH 45221, USA. Email: lorentaw@uc.edu

‡Department of Mathematical Sciences, Worcester Polytechnic Institute, Worcester, MA 01609, USA. Email: gpeng@wpi.edu

hyperbolic conservation laws, these vector fields are called entropies. We denote C^2 entropies by

$$\mathrm{ENT} = \Big\{ \Phi \in C^2(\mathbb{S}^1;\mathbb{R}^2) \colon \exists \lambda_\Phi \in C^1(\mathbb{S}^1;\mathbb{R}), \\ \frac{d}{d\theta}\Phi(e^{i\theta}) = \lambda_\Phi(e^{i\theta}) i e^{i\theta} \quad \forall \theta \in \mathbb{R} \Big\}.$$

If m is a general weak solution, we expect the distributions $\operatorname{div}\Phi(m)$ to carry information about singularities of m, and we are particularly interested in *finite entropy solutions*, i.e., weak solutions of (1.1) satisfying the finite entropy production property

$$\operatorname{div}\Phi(m) \in \mathcal{M}(\Omega) \qquad \forall \Phi \in \mathrm{ENT}\,, \tag{1.2}$$

where $\mathcal{M}(\Omega)$ is the set of finite Radon measures on Ω. As shown in [8], this property is satisfied by limits $m = \lim_{\epsilon\to 0} m_\epsilon$ of sequences of vector fields $m_\epsilon \colon \Omega \to \mathbb{R}^2$ with $\operatorname{div} m_\epsilon = 0$ and bounded Aviles-Giga energy

$$E_\epsilon(m_\epsilon) = \int_\Omega \left(\frac{\epsilon}{2} |\nabla m_\epsilon|^2 + \frac{1}{2\epsilon}(1-|m_\epsilon|^2)^2 \right) dx \leq C\,, \tag{1.3}$$

and is also relevant in some micromagnetics models [20]. The conjecture that these energy functionals Γ-converge to a line-energy functional concentrated on the one-dimensional jump set of m has attracted sustained attention over the past decades [2, 3, 13, 1, 8, 6, 11, 17]. If true, that conjecture would imply a similar concentration property for the entropy productions, namely that

$$\operatorname{div}\Phi(m) \ll \mathcal{H}^1 \lfloor J_m\,, \tag{1.4}$$

for any weak finite-entropy solution of (1.1)-(1.2), where J_m is the 1-rectifiable jump set of m (in the sense of [7]), see [6, Conjecture 1]. If $m \in BV(\Omega)$, the concentration property (1.4) follows from the chain rule, but solutions of (1.1)-(1.2) are in general not of bounded variation, see [1]. In fact, a weak solution of (1.1) satisfies (1.2) if and only if it has the Besov regularity $B^{1/3}_{3,\infty}$ [9].

A first major step towards (1.4) was established in [6] by showing that J_m coincides ($\mathcal{H}^1$-a.e.) with points of positive one-dimensional density for (at least one of) the measures (1.2), hence $\operatorname{div}\Phi(m)\lfloor \Omega \setminus J_m$ vanishes on sets with finite $\mathcal{H}^1$-measure. Conjectures very similar to (1.4) were recently settled by E. Marconi in the contexts of Burgers' equation [18] and of a related micromagnetics model [19]. These methods also brought new information on the structure of $\operatorname{div}\Phi(m)\lfloor \Omega \setminus J_m$ [17, 15], but the validity of (1.4) remains open.

If true, the concentration property (1.4) would imply in particular

$$\Big(\forall\Phi\in\mathrm{ENT}\,,\ \operatorname{div}\Phi(m)\in L^p(\Omega)\Big)\implies\Big(\forall\Phi\in\mathrm{ENT}\,,\ \operatorname{div}\Phi(m)=0\Big)\,,$$

for any $p\geq 1$, and any weak solution m of the eikonal equation (1.1). But even this weaker concentration property is not known. Our main result establishes it under natural boundary conditions if $p\geq 2$. Note that $p=2$ is critical in the sense that the rescaling $m_r(x)=m(rx)$ yields

$$\|\operatorname{div}\Phi(m_r)\|_{L^p(B_1)}=r^{1-\frac{2}{p}}\|\operatorname{div}\Phi(m)\|_{L^p(B_r)}\,.$$

If the domain is a disk, our result also holds for subcritical exponents $p\geq 1.95$. More precisely, we prove the stronger implication

$$\Big(\forall\Phi\in\{\Sigma_1,\Sigma_2\}\,,\ \operatorname{div}\Phi(m)\in L^p(\Omega)\Big)\implies\Big(\forall\Phi\in\mathrm{ENT}\,,\ \operatorname{div}\Phi(m)=0\Big)\,,$$

where Σ_1,Σ_2 are the Jin-Kohn entropies introduced in [13]. This entropy pair plays a fundamental role in the Γ-convergence conjecture for the Aviles-Giga energy (1.3). The total variation of the $\mathbb{R}^2$-valued measure $\operatorname{div}\Sigma(m)=(\operatorname{div}\Sigma_1(m),\operatorname{div}\Sigma_2(m))$ provides a lower bound

$$\liminf_{\epsilon\to 0}E_\epsilon(m_\epsilon)\geq|\operatorname{div}\Sigma(m)|(\Omega)\,,$$

for any sequence $m_\epsilon\to m$ in $\mathcal{D}'(\Omega)$ with $\operatorname{div}m_\epsilon=0$ [1], and that lower bound is sharp if $m\in BV(\Omega)$, in the sense that there exists $m_\epsilon\to m$ whose energy converges to $|\operatorname{div}\Sigma(m)|(\Omega)$ [5, 21].

Remark 1.1. The Jin-Kohn entropies are polynomial fields of degree three. One way to represent them explicitly is via the matrix $\Sigma(e^{i\theta})\in\mathbb{R}^{2\times 2}$ with rows $\Sigma_1(e^{i\theta})$, $\Sigma_2(e^{i\theta})$. This matrix is characterized by its conformal-anticonformal decomposition

$$\Sigma(e^{i\theta})z=e^{i\theta}z+\frac{1}{3}e^{3i\theta}\bar{z}\,,\qquad\forall z\in\mathbb{R}^2\approx\mathbb{C}\,.$$

Here, the left-hand side is a matrix multiplication for $z=(z_1,z_2)\in\mathbb{R}^2$ and the terms in the right-hand side are complex multiplications for $z=z_1+iz_2\in\mathbb{C}$.

We impose tangential boundary conditions, which are natural in the context of the Aviles-Giga energy [13]. More precisely, we assume that Ω is a bounded simply connected open set of class C^2 and that the inner trace of m along $\partial\Omega$, which is well-defined thanks to the trace theorem of [22] and the kinetic formulation of [9], is equal to the counterclockwise unit tangent $\tau\in C^1(\partial\Omega;\mathbb{S}^1)$. The trace theorem of [22] is stated in a slightly different context, but one may also avoid referring to traces and impose an outer boundary layer condition

$$m(x) = \tau(\pi_{\partial\Omega}(x)) \quad \forall x \in \Omega_\delta \setminus \Omega\,, \quad \Omega_\delta = \{x \in \mathbb{R}^2 \colon \operatorname{dist}(x,\Omega) < \delta\}\,, \tag{1.5}$$

with $\delta > 0$ small enough that the nearest-neighbor projection $\pi_{\partial\Omega} \colon \Omega_\delta \setminus \Omega \to \partial\Omega$ is C^1. Under these boundary conditions, if $\operatorname{div}\Phi(m) = 0$ in Ω_δ for all $\Phi \in \mathrm{ENT}$, then Ω must be a disk and m a vortex, that is, $m(x) = ix/|x|$ in centered coordinates, see [20, Theorem 1.2]. Thus our main result can be stated as follows.

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^2$ be a bounded, C^2, simply connected open set, and $m \colon \Omega \to \mathbb{R}^2$ a weak solution of the eikonal equation* (1.1) *subject to the tangential condition* (1.5)*.*

- *If $\operatorname{div}\Sigma_k(m) \in L^p(\Omega_\delta)$ for $k = 1, 2$ and $p = 2$ then Ω is a disk and m is a vortex.*
- *If Ω is a disk and $\operatorname{div}\Sigma_k(m) \in L^p(\Omega_\delta)$ for $k = 1, 2$ and $p = 1.95$, then m is a vortex.*

Theorem 1.2 is a consequence of results from our previous works [16, 14] and of the following new regularity result.

Theorem 1.3. *Let $\Omega \subset \mathbb{R}^2$ be a bounded open set, and m a weak solution of* (1.1)*. Assume*

$$m \in W^{1,1}(\Omega \setminus \overline{\Omega'}; \mathbb{S}^1) \quad \text{for some } \Omega' \subset\subset \Omega\,, \tag{1.6}$$

$$\text{and } \operatorname{div}\Phi(m) \in L^p(\Omega) \qquad \text{for all odd } \Phi \in \mathrm{ENT}\,, \tag{1.7}$$

for some $1 < p \le 2$. Then $m \in B^{1/3}_{3p,\infty,\mathrm{loc}}(\Omega)$, that is,

$$\sup_{|h|>0} \frac{1}{|h|^{1/3}} \|m(\cdot + h) - m\|_{L^{3p}(U \cap (U-h))} < \infty\,,$$

for all $U \subset\subset \Omega$.

The case $1 < p \le 4/3$ of Theorem 1.3 actually does not require the boundary layer assumption (1.6) and was proved in [16], as well as the converse implication, that $\operatorname{div}\Phi(m) \in L^p_{\mathrm{loc}}$ if m is a weak solution of (1.1) with $B^{1/3}_{3p,\infty,\mathrm{loc}}$ regularity. The novelty with respect to [16] resides in the way we treat boundary terms in the regularity estimate.

Proof of Theorem 1.2 from Theorem 1.3. Under the assumptions of Theorem 1.2 we can apply [16, Theorem 1.8] to deduce that $\operatorname{div}\Phi(m) \in L^p(\Omega)$ for all odd $\Phi \in \mathrm{ENT}$ (see Remark 2.1 for the representation of odd entropies corresponding to the statement in [16]). Thus we may apply Theorem 1.3 and (noting also that m is C^1 in the boundary layer) obtain that $m \in B^{1/3}_{3p,\infty}(\Omega_\delta)$. If $p = 2$, then [14, Theorem 1.1] shows that $\operatorname{div} \Phi(m) = 0$ for all $\Phi \in \mathrm{ENT}$, and [20, Theorem 1.2] implies that Ω is a disk and m a vortex. If $p = 1.95 > (47 + \sqrt{533})/36$ and Ω is a disk, then [14, Theorem 1.5] shows that m is a vortex. □

Remark 1.4. The methods of [14] can actually be improved to show a stronger version of Theorem 1.2, namely, if $\operatorname{div} \Sigma_k(m) \in L^p(\Omega_\delta)$ for $k = 1, 2$ and $p = 1.95$ then Ω must be a disk, and m a vortex. The necessary modifications would be rather lengthy, that is why we only provide the short but weaker version stated here.

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2 Proof of Theorem 1.3

The proof of Theorem 1.3 relies on a kinetic formulation of the finite-entropy condition (1.2) and on compensation identities inspired from [10] in the context of scalar conservation laws and their adaptation to the eikonal equation in [9].

2.1 A kinetic formulation

If m is a finite-entropy solution of (1.1)-(1.2), the indicator function

$$\chi(s, x) = \mathbf{1}_{m(x)\cdot e^{is}>0}\,, \quad s \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z},\ x \in \Omega\,, \tag{2.1}$$

satisfies the kinetic formulation

$$e^{is} \cdot \nabla_x \chi = \Theta \quad \text{in } \mathcal{D}'(\mathbb{T} \times \Omega)\,, \tag{2.2}$$

for some $\Theta = \partial_s \sigma(s, x)$ with $\sigma \in \mathcal{M}(\mathbb{T} \times \Omega)$ [9], see also [12].

Remark 2.1. This actually requires (1.2) only for entropies of the form

$$\Phi^\psi(z) = \int_{\mathbb{T}} \mathbf{1}_{z\cdot e^{is}>0} \psi(s) e^{is}\, ds = \frac{1}{2} \int_{\mathbb{T}} \mathbf{1}_{z\cdot e^{is}>0} (\psi(s) + \psi(\pi + s)) e^{is}\, ds\,,$$

for $\psi \in C^1(\mathbb{T})$ with $\int_{\mathbb{T}} \psi(s) e^{is}\, ds = 0$, see [9, § 3.1]. Any such Φ^ψ is odd, and reciprocally, any odd $\Phi \in \mathrm{ENT}$ can be represented as $\Phi = \Phi^\psi$. To see this, define $\psi(s) = -\frac{1}{2} e^{is} \cdot \frac{d}{ds} \Phi(ie^{is})$. The oddness of Φ implies that ψ is π-periodic, and one then checks that $\Phi - \Phi^\psi$ has zero derivative along $\mathbb{S}^1$, hence is zero, being constant and odd.

If m further satisfies the L^p entropy production property (1.7), then the measure σ has an explicit form (see [16, 17]) in terms of $\theta\colon \Omega \to \mathbb{T}$ such that $m = e^{i\theta}$, given by

$$\sigma(s,x) = \Big(\delta_{\theta(x)+\frac{\pi}{2}}(s) + \delta_{\theta(x)-\frac{\pi}{2}}(s)\Big)\, F(x) \quad \text{for some } F \in L^p(\Omega)\,. \tag{2.3}$$

2.2 A coercive quantity

We can use the indicator function (2.1) to build a quantity that controls finite differences of m. We denote by T^h and D^h the translation and finite difference operators in the x variable:

$$T^h f(x) = f(x+h), \quad D^h f = T^h f - f\,,$$

for any function $f(x)$, and we will frequently use the identity

$$D^h(fg) = f D^h g + T^h g D^h f\,. \tag{2.4}$$

For any $\varphi \in L^1(\mathbb{T})$ odd and π-periodic, and for the indicator function (2.1), we define the function

$$\Delta^{\varphi,\chi}(x,h) = \frac{1}{2}\int_{\mathbb{T}^2} \varphi(t-s) D^h\chi(t,x) D^h\chi(s,x)\sin(t-s)\, dtds, \tag{2.5}$$

on $\Omega^* = \{(x,h) \in \Omega\times\mathbb{R}^2 \colon x+h \in \Omega\}$. This can be rewritten as

$$\Delta^{\varphi,\chi}(x,h) = \frac{1}{2}\Xi^{\varphi}(m(x), T^h m(x))\,,$$

where $\Xi^{\varphi}\colon \mathbb{S}^1\times\mathbb{S}^1 \to \mathbb{R}$ is given by

$$\Xi^{\varphi}(z_1,z_2) = \int_{\mathbb{S}^1\times\mathbb{S}^1} \varphi(t-s)\sin(t-s)\Big(\mathbf{1}_{e^{it}\cdot z_2>0} - \mathbf{1}_{e^{it}\cdot z_1>0}\Big) \cdot \Big(\mathbf{1}_{e^{is}\cdot z_2>0} - \mathbf{1}_{e^{is}\cdot z_1>0}\Big)\, dtds\,.$$

For a large class of odd π-periodic functions φ, this quantity $\Xi^{\varphi}(z_1,z_2)$ provides control on the distance $|z_1 - z_2|$.

Lemma 2.2. *If $\varphi\colon \mathbb{R}\to\mathbb{R}$ is odd, π-periodic and $\varphi \geq 0$ on $(0,\pi/2)$, then we have*

$$\Xi^{\varphi}(z_1,z_2) \gtrsim \omega_{\varphi}(|z_1-z_2|)\,, \quad \text{where } \omega_{\varphi}(t) = t\int_0^{\frac{t}{4}} s\varphi(s)\, ds\,, \quad \forall t\in[0,2]\,.$$

Proof of Lemma 2.2. This follows essentially from the calculations in [9, Lemma 3.8], which we briefly recall here. Thanks to the invariance $\Xi^\varphi(e^{i\alpha}z_1, e^{i\alpha}z_2) = \Xi^\varphi(z_1, z_2) = \Xi^\varphi(z_2, z_1)$ for all $\alpha \in \mathbb{R}$ and $z_1, z_2 \in \mathbb{S}^1$, it suffices to consider the case $z_1 = e^{-i\beta}$, $z_2 = e^{i\beta}$ for some $\beta \in [0, \pi/2]$. Then the computations in [9, Lemma 3.8] give

$$\frac{1}{8}\Xi^\varphi(e^{-i\beta}, e^{i\beta}) = \begin{cases} \displaystyle\int_0^{2\beta} \varphi(t)(2\beta - t)\sin(t)\, dt & \text{if } \beta \in [0, \pi/4]\,, \\ \displaystyle\int_0^{\frac{\pi}{2}} \varphi(t)\min(2\beta - t, \pi - 2t)\sin(t)\, dt & \text{if } \beta \in [\pi/4, \pi/2]\,. \end{cases}$$

Since all factors in the integrands are nonnegative, using that

$$\begin{aligned} & 2\beta - t \geq \beta \quad \text{for } 0 \leq t \leq \beta\,, \\ & \sin(t) \geq 2t/\pi \quad \text{for } 0 \leq t \leq \pi/2, \\ \text{and} \quad & \min(2\beta - t, \pi - 2t) \geq \pi/4 \quad \text{for } 0 \leq t \leq \pi/4 \leq \beta\,, \end{aligned}$$

we obtain

$$\Xi^\varphi(e^{-i\beta}, e^{i\beta}) \gtrsim \beta \int_0^{\frac{\beta}{2}} \varphi(t) t\, dt\,.$$

Noting that $|e^{-i\beta} - e^{i\beta}| = 2\sin\beta \leq 2\beta$, we deduce the claimed estimate. □

2.3 A compensation identity

The quantity $\Delta^{\varphi,\chi}$ defined in (2.5) satisfies a compensation identity, essentially established in [9, Lemma 3.9].

Lemma 2.3. *Let $\Omega \subset \mathbb{R}^2$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\chi \in L^\infty(\mathbb{T} \times \Omega)$ such that the distribution $\Theta \in \mathcal{D}'(\mathbb{T} \times \Omega)$ defined by*

$$e^{is} \cdot \nabla_x \chi = \Theta \quad \text{in } \mathcal{D}'(\mathbb{T} \times \Omega)\,,$$

is of the form $\Theta = \partial_s \sigma$ for some $\sigma \in L^1(\Omega; \mathcal{M}(\mathbb{T}))$. Then, for any $\varphi \in C^1(\mathbb{T})$ odd and π-periodic, the function $\Delta^{\varphi,\chi}$ defined in (2.5) satisfies

$$\frac{d}{d\tau}\Delta^{\varphi,\chi}(x, \tau\mathbf{e}_1) = I^\tau(x) + \mathrm{div}_x A^\tau(x)\,, \tag{2.6}$$

in $\mathcal{D}'(\{(x,\tau)\in\Omega\times\mathbb{R}\colon x+\tau\mathbf{e}_1\in\Omega\})$*, where* I^τ *and* $A^\tau=(A_1^\tau,A_2^\tau)$ *are given by*

$$\begin{aligned}
I^\tau(x) &= \int_{\mathbb{T}^2}(\Theta+T^{\tau\mathbf{e}_1}\Theta)(s,x)\varphi(t-s)D^{\tau\mathbf{e}_1}\chi(t,x)\sin(t)\,dsdt \\
&\qquad - D^{\tau\mathbf{e}_1}\left[\int_{\mathbb{T}^2}\Theta(s,x)\varphi(t-s)\chi(t,x)\sin(t)\,dsdt\right], \\
A_1^\tau(x) &= \int_{\mathbb{T}^2}\varphi(t-s)\sin(t)\cos(s)T^{\tau\mathbf{e}_1}\chi(t,x)D^{\tau\mathbf{e}_1}\chi(s,x)\,dsdt\,, \\
A_2^\tau(x) &= \int_{\mathbb{T}^2}\varphi(t-s)\sin(t)\,\sin(s)\,\chi(t,x)D^{\tau\mathbf{e}_1}\chi(s,x)\,dsdt\,.
\end{aligned}$$

In the expression of I^τ*, integrals* $\int\Theta(s,\cdot)\varphi(t-s)\,ds$ *should be understood in the sense of distributions, that is,* $\int\Theta(s,\cdot)\varphi(t-s)\,ds=\int\varphi'(t-s)\sigma(ds,\cdot)\in L^1(\Omega)$*.*

Proof of Lemma 2.3. Assume first that, in addition to the above, χ is C^1 with respect to x and $\nabla_x\chi\in L^\infty(\mathbb{T}\times\Omega)$. Note that this implies that Θ is C^0 with respect to x, in the sense that $\int\Theta(s,\cdot)\psi(s)\,ds=\int\psi(s)e^{is}\cdot\nabla_x\chi(s,\cdot)\,ds$ is continuous for any $\psi\in C^1(\mathbb{T})$. Then the calculations of [9, Lemma 3.9] give the validity of (2.6) pointwise, see also [4, Lemma 2.6] which uses the same notation as here.

For a general $\chi\in L^\infty(\mathbb{T}\times\Omega)$, one may therefore apply (2.6) to a regularization $\chi_\epsilon=\chi*_x\rho_\epsilon$ by convolution with respect to the x variable, and the corresponding $\Theta_\epsilon=\partial_s\sigma_\epsilon$. Then, for a fixed $\Omega'\subset\subset\Omega$, we have that χ_ϵ is uniformly bounded, $\chi_\epsilon\to\chi$ a.e., and for any test function $\psi\in C^1(\mathbb{T})$, the functions $g_\epsilon=\int_{\mathbb{T}}\psi(s)\sigma_\epsilon(ds,\cdot)$ converge to $g=\int_{\mathbb{T}}\psi(s)\sigma(ds,\cdot)$ strongly in $L^1(\Omega')$. One can therefore, by dominated convergence, let $\epsilon\to 0$ in the distributional formulation of (2.6) for χ_ϵ, and conclude that (2.6) holds for χ in the sense of distributions. □

2.4 Estimates for the right-hand side of the compensation identity

We give estimates on A^τ and I^τ appearing in the right-hand side of (2.6). For A^τ we simply invoke [4, Lemma 2.7], which we restate here.

Lemma 2.4 ([4])**.** *Let* $m\colon\Omega\to\mathbb{S}^1$ *measurable and* $\chi(s,x)=\mathbf{1}_{e^{is}\cdot m(x)>0}$*. Then the vector field* A^τ *in* (2.6) *satisfies*

$$|A^\tau|\lesssim\|\varphi\|_{L^1(\mathbb{T})}\,|D^{\tau\mathbf{e}_1}m|\qquad\text{a.e. in }\Omega\cap(\Omega-\tau\mathbf{e}_1)\,.$$

For I^τ, we establish the following estimate.

Lemma 2.5. *Let* $m\colon\Omega\to\mathbb{S}^1$ *such that* $\chi(s,x)=\mathbf{1}_{e^{is}\cdot m(x)>0}$ *satisfies* (2.2) *with* $\Theta=\partial_s\sigma$ *and* σ *given by* (2.3)*. Assume that the odd* π*-periodic function* φ *in* (2.5) *is* C^1 *and satisfies* $|\varphi(t)| \lesssim |t|^\gamma$ *for some* $\gamma > 0$ *and all* $t \in (0, \pi/2]$. *Then, for any* $\eta \in C^1_c(\Omega; [0,1])$, *the function* I^τ *in* (2.6) *satisfies*

$$\begin{aligned}\int_\Omega I^\tau\, \eta^2\, dx \lesssim \big\| \eta^2 |D^{\tau \mathbf{e}_1} m|^\gamma \big\|_{L^{p'}(\Omega)} \|F\|_{L^p(\Omega)} \\ + |\tau| \|\varphi'\|_{L^1(\mathbb{T})} \|\nabla \eta\|_\infty \|F\|_{L^1(\Omega)}\,,\end{aligned}$$

for $|\tau| \leq \operatorname{dist}(\operatorname{supp}(\eta), \partial\Omega)$.

Proof of Lemma 2.5. Recalling the expression of I^τ in (2.6), plugging in $\Theta = \partial_s \sigma$ as in (2.3) and using that φ' is π-periodic, we have

$$\begin{aligned}I^\tau &= I_1^\tau + I_2^\tau - D^{\tau \mathbf{e}_1} I_3\,,\\ I_1^\tau(x) &= 2F(x) \int_\mathbb{T} \varphi'(t - \theta(x) - \frac{\pi}{2}) D^{\tau \mathbf{e}_1} \chi(t,x) \sin(t)\, dt\,,\\ I_2^\tau(x) &= 2F(x + \tau \mathbf{e}_1) \int_\mathbb{T} \varphi'(t - \theta(x + \tau\mathbf{e}_1) - \frac{\pi}{2}) D^{\tau \mathbf{e}_1} \chi(t,x) \sin(t)\, dt\,,\\ I_3(x) &= 2F(x) \int_\mathbb{T} \varphi'(t - \theta(x) - \frac{\pi}{2}) \chi(t,x) \sin(t)\, ds dt\,.\end{aligned}$$

Recalling that $\chi(t,x) = \mathbf{1}_{e^{it} \cdot m(x) > 0}$, we rewrite these as

$$\begin{aligned}I_1^\tau(x) &= 2F(x) \Big(G(im(x), m(x + \tau\mathbf{e}_1)) - G(im(x), m(x)) \Big)\,,\\ I_2^\tau(x) &= 2F(x + \tau\mathbf{e}_1) \Big(G(im(x + \tau\mathbf{e}_1), m(x + \tau\mathbf{e}_1)) - G(im(x + \tau\mathbf{e}_1), m(x)) \Big)\,,\\ I_3(x) &= 2F(x) G(im(x), m(x))\,,\end{aligned}$$

where

$$G(e^{i\alpha}, e^{i\theta}) = \int_{\theta - \frac{\pi}{2}}^{\theta + \frac{\pi}{2}} \varphi'(t - \alpha) \sin(t)\, dt\,. \tag{2.7}$$

Note that $G\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ is well-defined: its expression does not depend on the choice of α modulo 2π since φ is 2π-periodic, nor on the choice of θ modulo 2π since $\int_0^{2\pi} \varphi'(t - s) \sin(t)\, dt = 0$ as a consequence of the π-periodicity of φ. Using that

$$\frac{d}{dt} \Big[G(e^{i\alpha}, e^{it}) \Big] = 2\varphi'(t + \frac{\pi}{2} - \alpha) \sin(t + \frac{\pi}{2})\,,$$

we find, for any $z = e^{i\theta} \in \mathbb{S}^1$,

$$\begin{aligned}\frac{G(iz, e^{i\beta}z) - G(iz, z)}{2} &= \frac{1}{2}\int_0^\beta \frac{d}{dt}\Big[G(e^{i(\theta+\frac{\pi}{2})}, e^{i(\theta+t)})\Big]\,dt \\ &= \int_0^\beta \varphi'(t)\sin(\theta + t + \frac{\pi}{2})\,dt \\ &= \varphi(\beta)\cos(\theta+\beta) + \int_0^\beta \varphi(t)\sin(\theta+t)\,dt\,,\end{aligned}$$

and can therefore further rewrite I_1^τ and I_2^τ as

$$\begin{aligned}I_1^\tau(x) &= 4F(x)\,H(m(x), m(x+\tau\mathbf{e}_1)/m(x))\,,\\ I_2^\tau(x) &= 4F(x+\tau\mathbf{e}_1)\,H(m(x+\tau\mathbf{e}_1), m(x+\tau\mathbf{e}_1)/m(x))\,,\\ \text{where } H(e^{i\theta}, e^{i\alpha}) &= \varphi(\alpha)\cos(\theta+\alpha) + \int_0^\alpha \varphi(t)\sin(\theta+t)\,dt\,. \end{aligned} \tag{2.8}$$

Here $z_1/z_2 = z_1\bar{z}_2$ for $z_1, z_2 \in \mathbb{S}^1$ is the division of complex numbers. Note that H is well-defined on $\mathbb{S}^1 \times \mathbb{S}^1$ since its expression is 2π-periodic with respect to both θ and α, thanks to the π-periodicity of φ. Summarizing, we have

$$\begin{aligned}I_1^\tau(x) &= 4F(x)\,H(m(x), m(x+\tau\mathbf{e}_1)/m(x))\,,\\ I_2^\tau(x) &= 4F(x+\tau\mathbf{e}_1)\,H(m(x+\tau\mathbf{e}_1), m(x+\tau\mathbf{e}_1)/m(x))\,,\\ I_3(x) &= 2F(x)\,G(im(x), m(x))\,,\end{aligned}$$

with G as in (2.7) and H as in (2.8). The function G satisfies

$$|G| \lesssim \|\varphi'\|_{L^1}\,.$$

And, recalling $|\varphi(t)| \lesssim |t|^\gamma$, the function H satisfies

$$|H(w, e^{i\alpha})| \lesssim |\alpha|^\gamma\,, \qquad \text{hence } |H(w, z'/z)| \lesssim |z'/z - 1|^\gamma = |z' - z|^\gamma\,,$$

for all $w, z, z' \in \mathbb{S}^1$. Thus we have

$$\begin{aligned}|I_1^\tau| &\lesssim |F|\,|D^{\tau\mathbf{e}_1}m|^\gamma\,,\\ |I_2^\tau| &\lesssim T^{\tau\mathbf{e}_1}|F|\,|D^{\tau\mathbf{e}_1}m|^\gamma\,,\\ |I_3| &\lesssim \|\varphi'\|_{L^1(\mathbb{T})}|F|\,.\end{aligned}$$

Integrating I^τ against $\eta^2\,dx$, performing a discrete integration by parts using (2.4) in the term involving $D^{\tau\mathbf{e}_1}I_3$, and using Hölder's inequality, we obtain the claimed estimate. □

2.5 Regularity estimate

Proof of Theorem 1.3. Fix some $\Omega' \subset\subset U \subset\subset \Omega$, and a smooth cut-off function $\eta \in C_c^\infty(\Omega)$ such that $\mathbf{1}_U \le \eta \le \mathbf{1}_\Omega$.

Thanks to the assumption (1.7), the indicator function χ satisfies the kinetic formulation (2.2) with $\Theta = \partial_s \sigma$ as in (2.3). For any odd π-periodic $\varphi \in C^1(\mathbb{R})$ we may therefore apply the compensation identity (2.6). Let $h = \tau \mathbf{e}_1$ with $0 < \tau < r_0 = \min\{\mathrm{dist}(\mathrm{supp}(\eta), \partial\Omega), \mathrm{dist}(\Omega', \partial U)\}$, then we have

$$\int_\Omega \Delta^{\varphi,\chi}(x,h)\, \eta^2\, dx = \int_0^\tau \left(\int_\Omega I^{\tau'} \eta^2\, dx - 2\int_\Omega (A^{\tau'} \cdot \nabla \eta)\, \eta\, dx \right) d\tau'. \tag{2.9}$$

We assume also that $|\varphi(t)| \lesssim |t|^\gamma$ for all $t \in (0, \pi/2]$. Then, from Lemma 2.5 and Lemma 2.4, we have

$$\begin{aligned}
\int_0^\tau \int_\Omega I^{\tau'} \eta^2\, dx\, d\tau' &\lesssim \tau \sup_{\tau' \le \tau} \left\| \eta^2 |D^{\tau' \mathbf{e}_1} m|^\gamma \right\|_{L^{p'}(\Omega)} \|F\|_{L^p(\Omega)} \\
&\quad + \tau^2 \|\varphi'\|_{L^1(\mathbb{T})} \|\nabla \eta\|_\infty \|F\|_{L^1(\Omega)}\,, \\
\int_0^\tau \int_\Omega |A^{\tau'} \cdot \nabla \eta|\, \eta\, dx\, d\tau' &\lesssim \tau \|\varphi\|_{L^1(\mathbb{T})} \sup_{\tau' \le \tau} \int_\Omega |D^{\tau' \mathbf{e}_1} m|\, |\nabla \eta|\, \eta\, dx\,. \\
&\lesssim \tau^2 \|\varphi\|_{L^1(\mathbb{T})} \|\nabla \eta\|_\infty \|\nabla m\|_{L^1(\Omega \setminus \overline{\Omega'})}\,,
\end{aligned}$$

where the last inequality follows from the facts that $\mathrm{supp}(\eta |\nabla \eta|) + B_{r_0} \subset \Omega \setminus \overline{\Omega'}$ and $m \in W^{1,1}(\Omega \setminus \overline{\Omega'})$. Plugging these inequalities into (2.9) gives

$$\begin{aligned}
\int_\Omega \Delta^{\varphi,\chi}(x,h)\, \eta^2\, dx &\lesssim \tau \sup_{\tau' \le \tau} \left\| \eta^2 |D^{\tau' \mathbf{e}_1} m|^\gamma \right\|_{L^{p'}(\Omega)} \|F\|_{L^p(\Omega)} \\
&\quad + \tau^2 \|\varphi'\|_{L^1(\mathbb{T})} \|\nabla \eta\|_\infty \|F\|_{L^1(\Omega)} \\
&\quad + \tau^2 \|\varphi\|_{L^1(\mathbb{T})}\, \|\nabla \eta\|_\infty\, \|\nabla m\|_{L^1(\Omega \setminus \overline{\Omega'})}\,.
\end{aligned}$$

This estimate is valid for any φ odd, π-periodic and C^1 such that $|\varphi(t)| \lesssim |t|^\gamma$ for $t \in (0, \pi/2]$. By approximation, this is also true for $\varphi \colon \mathbb{R} \to \mathbb{R}$ odd, π-periodic, and C^1 on $(0, \pi)$ such that $\varphi \ge 0$ on $(0, \pi/2)$ and $\varphi(t) = t^\gamma$ for $t \in (0, \pi/4)$, with $\gamma = 3p - 3$. (If $\gamma > 1$ this φ is C^1 on $\mathbb{R}$ and no approximation is needed.) For such a choice, Lemma 2.2 ensures that

$$\Delta^{\varphi,\chi}(x,h) \ge c |D^h m(x)|^{3+\gamma} = c |D^h m(x)|^{3p}\,,$$

so we deduce

$$\begin{aligned}
\int_\Omega |D^{\tau \mathbf{e}_1} m|^{3p}\, \eta^2\, dx &\lesssim \tau \sup_{\tau' \le \tau} \left\| \eta^2 |D^{\tau' \mathbf{e}_1} m|^\gamma \right\|_{L^{p'}(\Omega)} \|F\|_{L^p(\Omega)} \\
&\quad + \tau^2 \|\varphi'\|_{L^1(\mathbb{T})} \|\nabla \eta\|_\infty \|F\|_{L^1(\Omega)} \\
&\quad + \tau^2 \|\varphi\|_{L^1(\mathbb{T})}\, \|\nabla \eta\|_\infty\, \|\nabla m\|_{L^1(\Omega \setminus \overline{\Omega'})}\,.
\end{aligned}$$

Note that $\gamma\, p' = 3p$. Taking a supremum over τ in the left hand side, and using Young's inequality and the fact that $\eta \leq 1$ in the first term on the right hand side, we obtain

$$\begin{aligned}\sup_{\tau'\leq\tau}\int_\Omega |D^{\tau'\mathbf{e}_1} m|^{3p}\,\eta^2\,dx \leq \frac{1}{2}\sup_{\tau'\leq\tau}\int_\Omega |D^{\tau'\mathbf{e}_1} m|^{3p}\,\eta^2\,dx + C\,\tau^p\|F\|^p_{L^p(\Omega)} \\ + C\tau^2\|\varphi'\|_{L^1(\mathbb{T})}\|\nabla\eta\|_\infty\|F\|_{L^1(\Omega)} \\ + C\tau^2\|\varphi\|_{L^1(\mathbb{T})}\,\|\nabla\eta\|_\infty\,\|\nabla m\|_{L^1(\Omega\setminus\overline{\Omega'})}\,,\end{aligned}$$

hence, recalling $1 < p \leq 2$,

$$\int_\Omega |D^h m|^{3p}\,\eta^2\,dx \leq C|h|^p \quad \text{for } h = \tau\mathbf{e}_1,\ 0 < \tau < r_0\,,$$

where the constant $C > 0$ depends on p, $\|F\|_{L^p(\Omega)}$, Ω, Ω', U and $m_{\lfloor\Omega\setminus\overline{\Omega'}}$. Rotating the variable, this holds for every $0 < |h| < r_0$, and since $|D^h m| \leq 2$, this holds for every $|h| > 0$, with a possibly larger constant $C > 0$. Recalling that $\eta \equiv 1$ on U, we deduce that $m \in B^{1/3}_{3p,\infty}(U)$. □

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