

Lifting of \mathbb{RP}^{d-1} -valued maps in BV and applications to uniaxial Q -tensors. With an appendix on an intrinsic BV -energy for manifold-valued maps.

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Abstract

We prove that a BV map with values into the projective space \mathbb{RP}^{d-1} has a BV lifting with values into the unit sphere \mathbb{S}^{d-1} that satisfies an optimal BV -estimate. As an application to liquid crystals, this result is also stated for BV maps with values into the set of uniaxial Q -tensors. In order to quantify BV liftings, we prove an explicit formula for an intrinsic BV -energy of maps with values into any compact smooth manifold.

Keywords: Manifold-valued BV maps, lifting, Q -tensors.

1 Introduction

For a vector $n \in \mathbb{S}^{d-1}$ in the unit sphere in \mathbb{R}^d ($d \geq 2$), we denote by $[n]$ the corresponding element of the projective space $\mathbb{RP}^{d-1} = \mathbb{S}^{d-1}/\mathbb{Z}_2$, i.e.,

$$[n] = \{\pm n\}.$$

Let $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) be an open set and $u : \Omega \rightarrow \mathbb{RP}^{d-1}$ be a Lebesgue measurable map such that $u(x) \in \mathbb{RP}^{d-1}$ for a.e. $x \in \Omega$. We call *lifting* of u (or *orientation* of u), any Lebesgue measurable map $n : \Omega \rightarrow \mathbb{R}^d$ such that

$$u(x) = [n(x)] \quad \text{and} \quad n(x) \in \mathbb{S}^{d-1} \quad \text{for a.e. } x \in \Omega.$$

The following question naturally arises (motivated in particular by the theory of nematic liquid crystals, see e.g. [3, 4, 5, 18]):

Lifting question. If u has some regularity, is there a lifting n of u with the same regularity?

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For example, if Ω is simply connected and u is continuous (respectively, $u \in C^k(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ for some $k \in \mathbb{N} \cup \{\infty\}$), then it is well known that n can be chosen to be continuous (respectively, $n \in C^k(\Omega; \mathbb{S}^{d-1})$), see for example [13, p. 61, Prop. 1.33]. Moreover, in these cases, only two choices of lifting n are possible, i.e., $\{-n, n\}$. The answer is more delicate in the framework of Sobolev spaces $W^{1,p}$. If $p \geq 2$, then a map $u \in W^{1,p}(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ has exactly two liftings n and $-n$ belonging to $W^{1,p}(\Omega; \mathbb{S}^{d-1})$ provided that Ω is simply connected; however, if $1 \leq p < 2$, there exist maps $u \in W^{1,p}(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ that do not admit any lifting $n \in W^{1,p}(\Omega; \mathbb{S}^{d-1})$ (see [6] and Section 2 below).

The aim of this article is to give a positive answer to the Lifting question in the framework of BV maps together with an optimal estimate of a BV lifting. For an open set $\Omega \subset \mathbb{R}^N$ ($N \geq 2$) and any compact Riemannian manifold \mathcal{N} isometrically embedded in \mathbb{R}^D we consider the nonlinear space $BV(\Omega; \mathcal{N})$ as the set of maps $u \in L^1_{loc}(\Omega; \mathbb{R}^D)$ such that $u(x) \in \mathcal{N}$ for a.e. $x \in \Omega$ and the differential Du is a finite Radon measure. We will systematically use the decomposition of the $\mathbb{R}^{D \times N}$ -valued measure Du into its absolutely continuous part $D^a u$, its Cantor part $D^c u$ and its jump part $D^j u$:

$$Du = D^a u + D^c u + D^j u, \quad D^a u = \nabla u \mathcal{L}^N, \quad D^j u = (u^+ - u^-) \otimes \nu \mathcal{H}^{N-1} \llcorner J_u.$$

Here ∇u , the density of $D^a u$ with respect to the Lebesgue measure \mathcal{L}^N , is called the approximate gradient of u , the set $J_u \subset \Omega$ is the $(N-1)$ -rectifiable jump set of u that is oriented by the unit vector field ν , and u^\pm are the traces of u on J_u with respect to ν . The three measures $D^a u$, $D^c u$ and $D^j u$ are mutually singular. The part of Du that does not involve jumps, i.e. $\tilde{D}u = D^a u + D^c u$, is called the diffuse part. We say that $u \in SBV$ if $u \in BV$ and the Cantor part vanishes, i.e., $D^c u = 0$ in Ω .

Remark 1.1. The above definition of the nonlinear space $BV(\Omega; \mathcal{N})$ does not depend on the choice of an isometric embedding $\mathcal{N} \subset \mathbb{R}^D$. However, it is important to note that the resulting seminorm $|u|_{BV} = |Du|(\Omega)$ does depend on the embedding, through the way it measures jumps. To be more specific, let us consider two isometric embeddings $\Phi_\ell: \mathcal{N} \rightarrow \mathbb{R}^{D_\ell}$ ($\ell = 1, 2$). The diffuse part of the seminorm does not depend on the embedding: the total variations of $D^a[\Phi_\ell(u)]$ and $D^c[\Phi_\ell(u)]$ satisfy

$$|D^a[\Phi_1(u)]| = |D^a[\Phi_2(u)]| \quad \text{and} \quad |D^c[\Phi_1(u)]| = |D^c[\Phi_2(u)]| \quad \text{as measures in } \Omega \quad (1)$$

(see Lemma A.1 in the Appendix below). The jump set J_u is also independent of the embedding, *but* the total variation of the jump part is given by

$$|D^j[\Phi_\ell(u)]| = |\Phi_\ell(u^+) - \Phi_\ell(u^-)| \mathcal{H}^{N-1} \llcorner J_u \quad \text{as measures in } \Omega, \quad \ell = 1, 2.$$

In other words, the cost of a jump between u_+ and u_- is $|\Phi_\ell(u^+) - \Phi_\ell(u^-)|$ (where $|\cdot| = |\cdot|_{\mathbb{R}^{D_\ell}}$ denotes the Euclidean distance in \mathbb{R}^{D_ℓ}), which need not be the same for $\ell = 1, 2$. As an example, consider the circle $\mathcal{N} = \mathbb{S}^1$ and $u_\pm = (\pm 1, 0)$ two opposite points on the circle. For the standard embedding $\mathbb{S}^1 \subset \mathbb{R}^2$ the cost of a jump between u^+ and u^- is $|u^+ - u^-|_{\mathbb{R}^2} = 2$. However, any smooth injective curve $\gamma: \mathbb{S}^1 \simeq \mathbb{R}/2\pi\mathbb{Z} \rightarrow \mathbb{R}^D$ with $|\gamma'(t)|_{\mathbb{R}^D} \equiv 1$ provides an isometric embedding of \mathbb{S}^1 into \mathbb{R}^D and the cost of such jump is $|\gamma(0) - \gamma(\pi)|_{\mathbb{R}^D}$, which can be any arbitrary number in $(0, \pi)$. In this context, one could also wish to measure jumps in the geodesic distance which yields $\text{dist}_{\mathbb{S}^1}(u^+, u^-) = \pi$ as the cost of this jump.

The answer to the Lifting question in the framework of BV maps is positive:

Theorem 1.2. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set and $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$. Then there exists $n \in BV(\Omega; \mathbb{S}^{d-1})$ such that $u = [n]$ a.e. Moreover, in the case of a bounded Lipschitz open set Ω , if $n_0 \in L^1(\partial\Omega; \mathbb{S}^{d-1})$ is a prescribed “lifting” trace at the boundary, i.e., $u = [n_0]$ \mathcal{H}^{N-1} -a.e. on $\partial\Omega$, then there exists a lifting $n \in BV(\Omega; \mathbb{S}^{d-1})$ of u such that $n = n_0$ \mathcal{H}^{N-1} -a.e. on $\partial\Omega$.*

The main point of our article is to prove optimal BV -estimates of liftings using a method based on fine properties of BV maps. We underlined in Remark 1.1 that the total variation of the diffuse part of Du (i.e. the part $\tilde{D}u$ that does not involve jumps) does not depend on the choice of an embedding. This intrinsicality extends to the choice of a BV lifting n of $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$, i.e., the total variation of the diffuse part of Dn is independent of the lifting:

Proposition 1.3. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set and $n \in BV(\Omega; \mathbb{S}^{d-1})$. Set $u = [n]$ in Ω . Then $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ and the total variations of the diffuse parts of Dn and Du are related by*

$$|D^a n| = |D^a u|, \quad |D^c n| = |D^c u| \quad \text{as measures in } \Omega. \quad (2)$$

These equalities also hold for the partial derivative measures in any direction $\omega \in \mathbb{S}^{N-1}$, i.e., $|D_\omega^a n| = |D_\omega^a u|$ and $|D_\omega^c n| = |D_\omega^c u|$ as measures in Ω .

This has interesting consequences regarding function spaces that are useful in the modeling of liquid crystals [3, 5].

Corollary 1.4. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set. If $u \in SBV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$, then any BV lifting n of u belongs to $SBV(\Omega; \mathbb{S}^{d-1})$. If in addition, $u \in W^{1,p}(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ for some $p \geq 1$, then any BV lifting n of u belongs to $SBV^p(\Omega; \mathbb{S}^{d-1})$ and the approximate gradient of n satisfies $|\nabla n| = |\nabla u| \in L^p(\Omega)$, while the traces of n satisfy $n^+ = -n^-$ \mathcal{H}^{N-1} -a.e. on J_n .*

We highlight the fact that Ω is not necessarily simply connected in our results (in particular, in Corollary 1.4); therefore, our result covers also the case of maps $u \in W^{1,p}(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ that do not need to have a lifting $n \in W^{1,p}(\Omega; \mathbb{S}^{d-1})$ even if $p \geq 2$. This provides a generalization of Proposition 4 in [5].

We are actually interested in a more precise version of the above Theorem 1.2, with optimal BV -estimates of liftings. As $BV(\Omega; \mathcal{N})$ is a nonlinear space, it does not make sense to consider a seminorm. We will rather call BV -energy a quantity that is the nonlinear equivalent of a BV seminorm. More precisely, we consider the following two cases:

- On the one hand, a natural choice is to use an intrinsic BV -energy (see (4) below): measuring jumps in terms of the geodesic distance on both \mathbb{S}^{d-1} and $\mathbb{R}\mathbb{P}^{d-1}$ induced by the Riemannian structure. Such BV -energy is independent of the choice of an embedding.
- On the other hand, the physical motivation of our problem provides us with at least one other natural BV -energy coming from the seminorm induced by the choice of an embedding: in liquid crystals, the projective plane arises naturally as embedded into the linear space of so-called Q -tensors (which are symmetric traceless $d \times d$ matrices). That is why we will also pay special attention to the isometric embedding of $\mathbb{R}\mathbb{P}^{d-1}$ into

$d \times d$ matrices given by ¹

$$\Phi : [n] \in \mathbb{R}\mathbb{P}^{d-1} \mapsto \frac{1}{\sqrt{2}}n \otimes n \in \mathbb{R}^{d \times d}. \quad (3)$$

Here we naturally use for the target manifold \mathbb{S}^{d-1} of liftings the standard embedding $\mathbb{S}^{d-1} = \{|x| = 1\} \subset \mathbb{R}^d$ where $|\cdot|$ is the Euclidean norm in \mathbb{R}^d .

Next we present our results in the two aforementioned cases: first, when the BV -energy measures jumps in geodesic distance; second, when the jumps are measured in Euclidean distance.

1.1 Measuring jumps in geodesic distance

In the case $\mathcal{N} = \mathbb{S}^{d-1}$, we denote by $\text{dist}_{\mathbb{S}^{d-1}}(n, m)$ (or simply, $\text{dist}(n, m)$ when \mathcal{N} is implied by the context to be \mathbb{S}^{d-1}) the geodesic distance between $n, m \in \mathbb{S}^{d-1}$ with respect to the canonical Riemannian metric, which is the one induced by the usual isometric embedding $\mathbb{S}^{d-1} \subset \mathbb{R}^d$. The induced distance on $\mathcal{N} = \mathbb{R}\mathbb{P}^{d-1}$ is then given by:

$$\begin{aligned} \text{dist}_{\mathbb{R}\mathbb{P}^{d-1}}([n], [m]) &= \text{dist}_{\mathbb{S}^{d-1}}(n, m) \wedge \text{dist}_{\mathbb{S}^{d-1}}(-n, m) \\ &= \text{dist}_{\mathbb{S}^{d-1}}(n, m) \wedge (\pi - \text{dist}_{\mathbb{S}^{d-1}}(n, m)), \quad \text{for any } n, m \in \mathbb{S}^{d-1}, \end{aligned}$$

where $a \wedge b$ denotes the minimum of two real numbers a, b . Within these notations, we introduce the following BV -energy for $u \in BV(\Omega; \mathcal{N})$ defined on an open set $\Omega \subset \mathbb{R}^N$ (Ω is always endowed with the Euclidean norm $|\cdot| = |\cdot|_{\mathbb{R}^N}$): ²

$$|u|_{BV, \mathcal{N}} = \liminf_{\varepsilon \rightarrow 0} \iint_{\Omega \times \Omega} \frac{\text{dist}_{\mathcal{N}}(u(x), u(y))}{|x - y|} \rho_\varepsilon(|x - y|) dx dy < \infty, \quad (4)$$

where $\{\rho_\varepsilon\}_{\varepsilon > 0}$ is a family of radial nonnegative mollifiers satisfying,

$$\rho_\varepsilon \geq 0, \quad \int_{\mathbb{R}^N} \rho_\varepsilon(|x|) dx = 1, \quad \lim_{\varepsilon \rightarrow 0} \int_{|x| > h} \rho_\varepsilon(|x|) dx = 0, \quad \forall h > 0. \quad (5)$$

Intrinsic BV -energies of type (4) have been introduced by Korevaar and Schoen [15]. If we consider an isometric embedding $\mathcal{N} \subset \mathbb{R}^D$ and the open set Ω is bounded and Lipschitz, the BV -energy (4) of u can be expressed in the following way: $|u|_{BV, \mathcal{N}}$ represents the average over all directions $\omega \in \mathbb{S}^{N-1}$ of the total variation of the partial derivative measure $D_\omega u$ of u in direction $\omega \in \mathbb{S}^{N-1}$ where the jump cost is given by the geodesic distance in \mathcal{N} . This is valid for every compact manifold \mathcal{N} . (This averaging formula relies strongly on the radial symmetry of mollifiers in (5)).

¹ If Φ is the embedding (3), then $|D\Phi([n])v|_{\mathbb{R}^{d \times d}} = \frac{1}{\sqrt{2}}|n \otimes v + v \otimes n|_{\mathbb{R}^{d \times d}} = |v|_{\mathbb{R}^d}$ for every $n \in \mathbb{S}^{d-1}$ and $v \in T_n \mathbb{S}^{d-1} \cong T_{[n]} \mathbb{R}\mathbb{P}^{d-1}$, which proves that Φ is indeed an isometry.

²It is known that the liminf in (4) is equal to the corresponding limsup as proved by Korevaar and Schoen [15] (see also Theorem 1.5) in the case of a bounded Lipschitz domain Ω .

Theorem 1.5. *Let $\Omega \subset \mathbb{R}^N$ be a bounded Lipschitz open set, \mathcal{N} be a compact smooth Riemannian manifold isometrically embedded in \mathbb{R}^D and $u \in BV(\Omega; \mathcal{N})$. For any family of radial nonnegative mollifiers $\{\rho_\varepsilon\}_{\varepsilon>0}$ satisfying (5), the $\liminf_{\varepsilon \rightarrow 0}$ in (4) is equal to the corresponding $\limsup_{\varepsilon \rightarrow 0}$ and this limit is given by*

$$|u|_{BV, \mathcal{N}} = \int_{\Omega} \int_{\mathbb{S}^{N-1}} |\nabla_{\omega} u| d\mathcal{H}^{N-1}(\omega) dx + K_N |D^c u|(\Omega) + K_N \int_{J_u} \text{dist}_{\mathcal{N}}(u^-, u^+) d\mathcal{H}^{N-1}, \quad (6)$$

where $\nabla_{\omega} u = (\nabla u)\omega$ stands for the approximate derivative of u in direction $\omega \in \mathbb{S}^{N-1}$,

$$K_N = \int_{\mathbb{S}^{N-1}} |\omega \cdot e| d\mathcal{H}^{N-1}(\omega)$$

for any $e \in \mathbb{S}^{N-1}$ and the average is denoted by $\int_{\mathbb{S}^{N-1}} := \frac{1}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})} \int_{\mathbb{S}^{N-1}}$.

This implies in particular that (4) is independent of the mollifying family $\{\rho_\varepsilon\}$ with (5). Note that our BV -energy (6) is different from the one considered by Giaquinta and Mucci [12] (see also [11, Section 6.2.2] when $\mathcal{N} = \mathbb{S}^1$).

Our main result concerning the geodesic case is the following:

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set. For any $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$, there exists a lifting $n \in BV(\Omega; \mathbb{S}^{d-1})$, i.e. $u = [n]$ a.e. in Ω , with*

$$|n|_{BV, \mathbb{S}^{d-1}} \leq 2|u|_{BV, \mathbb{R}\mathbb{P}^{d-1}}. \quad (7)$$

Moreover the constant 2 is optimal if $N \geq 2$.

Our results hold also in dimension $N = 1$, but they do not provide the optimal constant. That is why, in Section 5, we will present a different method in estimating BV liftings in the case of dimension $N = 1$ for an interval $\Omega \subset \mathbb{R}$; this method will lead to the optimal constant equal to 1 of the BV -energy of a lifting in (7). In fact, no additional jumps appear for optimal liftings n of u on intervals $\Omega \subset \mathbb{R}$, that is why the optimal constant is less than in dimension $N > 1$.

1.2 Measuring jumps in Euclidean distance

We endow $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ with the global distance corresponding to Euclidean distance in \mathbb{R}^d , and interpret $n \in BV(\Omega; \mathbb{S}^{d-1})$ as a map $n \in BV(\Omega; \mathbb{R}^d)$. We denote by $|n|_{BV, \mathbb{R}^d}$ the corresponding seminorm, i.e. the total variation norm of Dn as a $\mathbb{R}^{d \times N}$ -valued measure:

$$\begin{aligned} |n|_{BV, \mathbb{R}^d} &= \sup \left\{ \int_{\Omega} n \cdot \text{div} \varphi dx : \varphi \in C_c^1(\Omega, \mathbb{R}^{d \times N}), \|\varphi\|_{L^\infty} \leq 1 \right\} \\ &= \int_{\Omega} |\nabla n| dx + |D^c n|(\Omega) + \int_{J_n} |n^+ - n^-|_{\mathbb{R}^d} d\mathcal{H}^{N-1}, \end{aligned}$$

where the Euclidean distance is used to measure the jumps of n .

We identify $\mathbb{R}\mathbb{P}^{d-1}$ to a subset of $\mathbb{R}^{d \times d}$ through the physical embedding (3), i.e., $\Phi([n]) = \frac{1}{\sqrt{2}} n \otimes n$ for all $n \in \mathbb{S}^{d-1}$, so that $\mathbb{R}\mathbb{P}^{d-1}$ is endowed with the global distance corresponding to

Euclidean distance in $\mathbb{R}^{d \times d}$. Then we interpret $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ as a map $u \in BV(\Omega; \mathbb{R}^{d \times d})$ through the physical embedding (3), and denote by $|u|_{BV, \mathbb{R}^{d \times d}}$ the corresponding seminorm, i.e. the total variation norm of Du as a $\mathbb{R}^{d \times d \times N}$ -valued measure

$$|u|_{BV, \mathbb{R}^{d \times d}} = \int_{\Omega} |\nabla u| dx + |D^c u|(\Omega) + \int_{J_u} |u^+ - u^-|_{\mathbb{R}^{d \times d}} d\mathcal{H}^{N-1},$$

where the cost of a jump between $u^+ = [n^+]$ and $u^- = [n^-]$ is given by

$$|u^+ - u^-|_{\mathbb{R}^{d \times d}} := \frac{1}{\sqrt{2}} |n^+ \otimes n^+ - n^- \otimes n^-|_{\mathbb{R}^{d \times d}}. \quad (8)$$

Our main result concerning the Euclidean case is the following:

Theorem 1.7. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set. For any $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$, there exists a lifting $n \in BV(\Omega; \mathbb{S}^{d-1})$, i.e. $\Phi(u) \stackrel{(3)}{=} \frac{1}{\sqrt{2}} n \otimes n$ a.e. in Ω with*

$$|n|_{BV, \mathbb{R}^d} \leq \left(1 + \frac{2}{\pi}\right) \left(|D^c u|(\Omega) + |D^j u|(\Omega)\right) + C^a(N, d) \int_{\Omega} |\nabla u| dx, \quad (9)$$

where $C^a(N, d) \geq 1 + 2/\pi$, with equality if $d = 2$ or $N = 1$. In particular, for $d = 2$ and $N \geq 1$ it holds

$$|n|_{BV, \mathbb{R}^2} \leq \left(1 + \frac{2}{\pi}\right) |u|_{BV, \mathbb{R}^{2 \times 2}},$$

and the constant $1 + 2/\pi$ is optimal if $N \geq 2$.

Remark 1.8. In the proof of Theorem 1.7 we will in fact consider general embeddings $\mathbb{R}\mathbb{P}^{d-1} \subset \mathbb{R}^D$. This has the effect of modifying the constant appearing in inequality (9) in front of the jump part $|D^j u|$, and it will turn out that the physical embedding (3) provides the optimal constant $1 + \frac{2}{\pi}$. Hence, while this choice of embedding was motivated by physical reasons, our result shows that it also stands out at the pure mathematical level.

Remark 1.9. For $N \geq 2$, the constant $C^a(N, d)$ that we obtain in the proof of Theorem 1.7 is strictly greater than $1 + 2/\pi$ if $d > 2$, but we believe that this is a limitation of our method and that the optimal constant should be $1 + 2/\pi$ independently of d . However, we will prove in Proposition 4.3 that the optimal constant is $1 + 2/\pi$ (independently of d and of $N \geq 2$) if the total variation is given by an averaging formula similar to $|\cdot|_{BV, \mathcal{N}}$, i.e.,

$$\begin{aligned} |||u|||_{BV, \mathbb{R}^{d \times d}} &:= \int_{\mathbb{S}^{N-1}} |D_{\omega} u|(\Omega) d\mathcal{H}^{N-1}(\omega) \\ &= \int_{\Omega} \int_{\mathbb{S}^{N-1}} |\nabla_{\omega} u| d\mathcal{H}^{N-1}(\omega) dx + K_N |D^c u|(\Omega) + K_N \int_{J_u} |u^+ - u^-|_{\mathbb{R}^{d \times d}} d\mathcal{H}^{N-1}, \end{aligned} \quad (10)$$

where $D_{\omega} u$ is the partial derivative measure of u in direction $\omega \in \mathbb{S}^{N-1}$. The difference between $|||u|||_{BV, \mathbb{R}^{d \times d}}$ and $|u|_{BV, \mathbb{R}\mathbb{P}^{d-1}}$ lies in the jump cost: Euclidean distance vs. geodesic distance.

We restate Theorem 1.7 in the setting relevant to liquid crystals. To this end we denote by $\mathcal{S}_0 \subset \mathbb{R}^{d \times d}$ the space of traceless symmetric matrices (Q -tensors) endowed with the norm $|\cdot|_{\mathbb{R}^{d \times d}}$, and by $\mathcal{U}_\star \subset \mathcal{S}_0$ the subset of uniaxial Q -tensors with fixed orientational order $s_\star \in \mathbb{R} \setminus \{0\}$, i.e.

$$\mathcal{U}_\star = \left\{ s_\star \left(n \otimes n - \frac{1}{d} I_d \right) : n \in \mathbb{S}^{d-1} \right\},$$

that is diffeomorphic with \mathbb{RP}^{d-1} , where I_d is the identity matrix. We call a map $Q \in BV(\Omega; \mathcal{U}_\star)$ if $Q \in BV(\Omega; \mathcal{S}_0)$ and $Q(x) \in \mathcal{U}_\star$ for a.e. $x \in \Omega$. Taking into account that the (non-isometric) embedding $[n] \mapsto s_\star(n \otimes n - I_d/d)$, compared with the isometric embedding (3), multiplies distances by a factor $\sqrt{2}|s_\star|$, we have the following lifting result:

Corollary 1.10. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set. For any $Q \in BV(\Omega; \mathcal{U}_\star)$ (respectively, $Q \in SBV(\Omega; \mathcal{U}_\star)$), there exists $n \in BV(\Omega; \mathbb{S}^{d-1})$ (respectively, $n \in SBV(\Omega; \mathbb{S}^{d-1})$) such that*

$$Q = s_\star \left(n \otimes n - \frac{1}{d} I_d \right) \quad \text{a.e. in } \Omega,$$

and

$$\sqrt{2}|s_\star| |n|_{BV, \mathbb{R}^d} \leq \left(1 + \frac{2}{\pi} \right) \left(|D^c Q|(\Omega) + |D^j Q|(\Omega) \right) + C^a(N, d) \int_{\Omega} |\nabla Q| dx,$$

and $C^a(N, d) \geq 1 + \frac{2}{\pi}$ with equality if $d = 2$ or $N = 1$.

The outline of the paper is as follows. In Section 2, we discuss the optimality of the estimates we found for BV liftings. In Section 3, we prove the geodesic case, in particular, Theorem 1.6, while in Section 4 we prove the Euclidean case. In Section 5, we discuss the case of dimension $N = 1$. In Appendix A we prove the claims in Remark 1.1 and Proposition 1.3 about the diffuse part's total variation. Finally, in Appendix B we prove Theorem 1.5 giving the expression of the intrinsic BV -energy (4).

2 Optimality of our estimates

We start by considering the case $N = d = 2$ for the unit open disc $\Omega = \mathbb{D} \subset \mathbb{R}^2$ and $u = [n] \in BV(\mathbb{D}; \mathbb{RP}^1)$ with $n: \mathbb{D} \mapsto \mathbb{S}^1 \subset \mathbb{R}^2 \simeq \mathbb{C}$ given in polar coordinates by

$$n(re^{i\theta}) = e^{i\frac{\theta}{2}}, \quad 0 < r < 1, \quad 0 \leq \theta < 2\pi. \quad (11)$$

This map u describes a defect of degree $1/2$ that can be observed in liquid crystals and is depicted in Figure 1. Moreover, u belongs to $W^{1,p}(\mathbb{D}; \mathbb{RP}^1)$ for all $p < 2$. We will prove by this example that the constants obtained in Theorem 1.6 and for $d = 2$ in Theorem 1.7 are optimal.

The geodesic case. Note that n has a jump along the radius $\mathcal{R} := \{\theta = 0\} = [0, 1) \times \{0\}$ but $u = [n]$ is locally Lipschitz in $\mathbb{D} \setminus \{0\}$. Moreover, u and n are smooth away from \mathcal{R} ; since \mathbb{RP}^1 is locally isometric to \mathbb{S}^1 , any isometric embedding $\mathbb{RP}^1 \subset \mathbb{R}^D$ will be such that for any $\omega \in \mathbb{S}^1$ it holds $|\nabla_\omega u|_{\mathbb{R}^D} = |\nabla_\omega n|_{\mathbb{R}^2}$ in $\mathbb{D} \setminus \mathcal{R}$. Here $\nabla_\omega n = (\nabla n)\omega$ is the approximate gradient

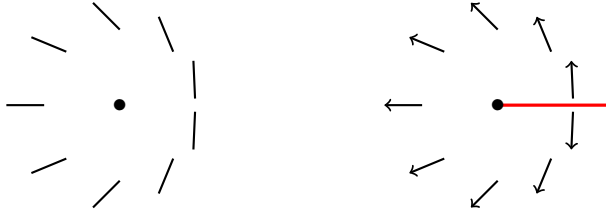


Figure 1: Defect of degree $1/2$ – representation of u (left) and n (right).

of $n \in BV(\mathbb{D}; \mathbb{R}^2)$ in direction ω , and in polar coordinates it holds $\nabla n(re^{i\theta}) = \frac{i}{2r} e^{i\frac{\theta}{2}} \otimes ie^{i\theta}$ for $0 < r < 1$, $0 \leq \theta < 2\pi$. Therefore, we have by (6) that

$$\begin{aligned} |u|_{BV, \mathbb{R}\mathbb{P}^1} &= \int_{\mathbb{D}} \int_{\mathbb{S}^1} |\nabla_{\omega} n| d\mathcal{H}^1(\omega) dx \\ &= \int_0^1 \int_0^{2\pi} \frac{1}{2r} \int_{\mathbb{S}^1} |\omega \cdot ie^{i\theta}| d\mathcal{H}^1(\omega) r d\theta dr \\ &= K_2\pi. \end{aligned}$$

On the other hand, by (6), it holds $|n|_{BV, \mathbb{S}^1} = 2K_2\pi$ as $|D^j n|(\mathbb{D}) = \pi\mathcal{H}^1(\mathcal{R}) = \pi$. To prove optimality of (7) it remains to show that other BV liftings cannot have a smaller BV -energy. Indeed, let $\tilde{n} \in BV(\mathbb{D}; \mathbb{S}^1)$ be a lifting of u . For a.e. $r \in (0, 1)$, the restriction of \tilde{n} to the circle $C(0, r)$ centered at 0 of radius r is BV . This restriction must have at least one jump between two opposite vectors since $[\tilde{n}] = u$. Such jump costs $\pi = \text{dist}(\tilde{n}, -\tilde{n})$. Moreover the absolutely continuous part of the tangential derivative of \tilde{n} has the same total variation as the one of n , i.e. $r^{-1}|\partial_{\theta} n|$. Hence using (6), the properties of one-dimensional restriction of BV maps [2, Section 3.11] and polar coordinates, we find that

$$|\tilde{n}|_{BV, \mathbb{S}^1} \geq K_2 \int_0^1 \left(\int_0^{2\pi} |\partial_{\theta} n| d\theta + \pi \right) dr = 2K_2\pi = 2|u|_{BV, \mathbb{R}\mathbb{P}^1}.$$

This shows optimality of the constant 2 in the estimate of Theorem 1.6 for $N = d = 2$.

Remark 2.1. For arbitrary $N \geq 2$ and $d \geq 2$ it suffices to extend the above example constantly in the additional variables, i.e., consider the cylindrical domain $\Omega = \mathbb{D} \times (0, 1)^{N-2} \subset \mathbb{R}^N$ and

$$u = [n] \text{ with } n(re^{i\theta}, y) = e^{i\theta/2}, r \in (0, 1), \theta \in [0, 2\pi), y \in (0, 1)^{N-2},$$

and identify its target \mathbb{S}^1 with $\mathbb{S}^1 \times \{0_{\mathbb{R}^{d-2}}\} \subset \mathbb{S}^{d-1}$. Note that $u = [n] \in W^{1,p}(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ for all $p < 2$ and u admits no $W^{1,p}$ lifting (see [6]), but only BV liftings.

The Euclidean case. Let $u = [n]$ within the isometric embedding (3), i.e., $\Phi(u) = \frac{1}{\sqrt{2}}n \otimes n$ where n is given in (11). By the above computation, it holds

$$|u|_{BV, \mathbb{R}^{2 \times 2}} = \int_{\mathbb{D}} |\nabla n| dx = \pi,$$

where as above ∇n is the approximate gradient of n . For any BV lifting \tilde{n} of u , the restriction of \tilde{n} to a.e. circle $C(0, r)$ with $r \in (0, 1)$ must have at least one jump between two opposite

vectors, and such jump costs $2 = |\tilde{n} - (-\tilde{n})|$. Moreover the absolutely continuous part of the tangential derivative of \tilde{n} has the total variation $r^{-1}|\partial_\theta n|$, therefore $|D^a \tilde{n}| \geq r^{-1}|\partial_\theta n| dx$ as measures in \mathbb{D} . Hence, we have

$$|\tilde{n}|_{BV, \mathbb{R}^2} \geq |n|_{BV, \mathbb{R}^2} = \int_0^1 \left(\int_0^{2\pi} |\partial_\theta n| d\theta + 2 \right) dr = \pi + 2 = \left(1 + \frac{2}{\pi} \right) |u|_{BV, \mathbb{R}^2 \times 2}.$$

This shows optimality of the constant in Theorem 1.7 for $N = d = 2$. For arbitrary $N \geq 2$ and $d \geq 2$, it suffices to extend the above example constantly in the additional variables in a cylindrical domain (as in Remark 2.1).

3 “Geodesic” lifting. Proof of Theorem 1.6

The proof of Theorem 1.6 relies on ideas introduced in [9] where the case of BV liftings of \mathbb{S}^1 -valued maps was analyzed. Our main contribution in this paper consists in adapting those ideas to the case of $\mathbb{R}\mathbb{P}^{d-1}$ -valued maps, using new tools based on the group of special rotations $G := SO(d)$ endowed with the Haar measure. More precisely, we start by considering a measurable map $F: \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$ such that

$$F(n) = \begin{cases} n & \text{if } n \cdot e_d > 0, \\ -n & \text{if } n \cdot e_d < 0 \end{cases} \quad \text{and} \quad F(n) = F(-n), \quad \forall n \in \mathbb{S}^{d-1}, \quad (12)$$

where $e_d = (0, \dots, 0, 1) \in \mathbb{R}^d$. Since F is symmetric, there exists a measurable map $L: \mathbb{R}\mathbb{P}^{d-1} \rightarrow \mathbb{S}^{d-1}$ such that

$$L([n]) = F(n), \quad \forall n \in \mathbb{S}^{d-1}.$$

Given $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$, the map $n = L(u)$ satisfies $[n] = u$ a.e. in Ω , but since L is not Lipschitz one cannot in general expect n to belong to $BV(\Omega; \mathbb{S}^{d-1})$. To remedy this problem we consider the following symmetric map for any special rotation $R \in G := SO(d)$:

$$F_R: \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}, \quad F_R(n) = R^{-1}F(Rn),$$

and the corresponding lifting map $L_R: \mathbb{R}\mathbb{P}^{d-1} \rightarrow \mathbb{S}^{d-1}$ given by

$$L_R([n]) = F_R(n), \quad \forall n \in \mathbb{S}^{d-1}.$$

We claim that for any $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ one may choose $R \in G$ such that $n := L_R(u)$ belongs to $BV(\Omega; \mathbb{S}^{d-1})$ and satisfies the estimate (7). The main ingredient is the following averaging inequality over the group G endowed with the normalized Haar measure μ . We recall that μ is the unique regular Borel measure μ on G satisfying

$$\mu(R \cdot A) = \mu(A \cdot R) = \mu(A), \quad \forall A \in \text{Bor}(G), \forall R \in G,$$

and $\mu(G) = 1$. In particular, the pushforward measure of μ under the map $R \in G \mapsto Rn \in \mathbb{S}^{d-1}$ (for an arbitrary fixed $n \in \mathbb{S}^{d-1}$) is a rotation-invariant measure on \mathbb{S}^{d-1} and therefore, proportional to $\mathcal{H}^{d-1} \llcorner \mathbb{S}^{d-1}$; in other words, for every $n \in \mathbb{S}^{d-1}$ and any Borel set $S \in \text{Bor}(\mathbb{S}^{d-1})$,

$$\mu(\{R: Rn \in S\}) = \frac{1}{\lambda_d} \mathcal{H}^{d-1}(S), \quad (13)$$

where $\lambda_d = \mathcal{H}^{d-1}(\mathbb{S}^{d-1})$.

Lemma 3.1. For any $u \in BV(\Omega, \mathbb{R}\mathbb{P}^{d-1})$ it holds

$$\begin{aligned} \int_G \iint_{\Omega \times \Omega} \frac{\text{dist}_{\mathbb{S}^{d-1}}(L_R(u(x)), L_R(u(y)))}{|x-y|} \rho_\varepsilon(|x-y|) dx dy d\mu(R) \\ \leq 2 \iint_{\Omega \times \Omega} \frac{\text{dist}_{\mathbb{R}\mathbb{P}^{d-1}}(u(x), u(y))}{|x-y|} \rho_\varepsilon(|x-y|) dx dy, \end{aligned}$$

where ρ_ε is any family of nonnegative radial functions.

Proof of Theorem 1.6. This is a direct consequence of Lemma 3.1 and Fatou's lemma when passing to the liminf as $\varepsilon \rightarrow 0$: indeed, by averaging over G , there exists $R_0 \in G$ such that

$$\begin{aligned} |L_{R_0}(u)|_{BV, \mathbb{S}^{d-1}} &\stackrel{(4)}{\leq} \int_G \liminf_{\varepsilon \rightarrow 0} \iint_{\Omega \times \Omega} \frac{\text{dist}_{\mathbb{S}^{d-1}}(L_R(u(x)), L_R(u(y)))}{|x-y|} \rho_\varepsilon(|x-y|) dx dy d\mu(R) \\ &\leq \liminf_{\varepsilon \rightarrow 0} \int_G \iint_{\Omega \times \Omega} \frac{\text{dist}_{\mathbb{S}^{d-1}}(L_R(u(x)), L_R(u(y)))}{|x-y|} \rho_\varepsilon(|x-y|) dx dy d\mu(R) \\ &\leq 2 \liminf_{\varepsilon \rightarrow 0} \iint_{\Omega \times \Omega} \frac{\text{dist}_{\mathbb{R}\mathbb{P}^{d-1}}(u(x), u(y))}{|x-y|} \rho_\varepsilon(|x-y|) dx dy = 2|u|_{BV, \mathbb{R}\mathbb{P}^{d-1}}, \end{aligned}$$

where the last inequality is due to Lemma 3.1. □

In order to prove Lemma 3.1 we start by proving the following:

Lemma 3.2. For any $n, m \in \mathbb{S}^{d-1}$ it holds

$$\int_G \text{dist}_{\mathbb{S}^{d-1}}(F(Rn), F(Rm)) d\mu(R) = \frac{2}{\pi} \text{dist}_{\mathbb{S}^{d-1}}(n, m) \text{dist}_{\mathbb{S}^{d-1}}(-n, m).$$

Proof of Lemma 3.2. Given $n \in \mathbb{S}^{d-1}$ we split G into the partition:

$$G = G_n^+ \sqcup G_n^- \sqcup Z_n, \tag{14}$$

where

$$G_n^+ = \{R \in G : (Rn) \cdot e_d > 0\}, \quad G_n^- = \{R \in G : (Rn) \cdot e_d < 0\},$$

and Z_n is μ -negligible since

$$\mu(Z_n) = \mu(\{R \in G : Rn \cdot e_d = 0\}) \stackrel{(13)}{=} \frac{1}{\lambda_d} \mathcal{H}^{d-1} \left(\left\{ \omega \in \mathbb{S}^{d-1} : \omega \cdot e_d = 0 \right\} \right) = 0.$$

Splitting the integral according to (14), we obtain

$$\begin{aligned} \int_G d\mu(R) \text{dist}(F(Rn), F(Rm)) &= \mu((G_n^+ \cap G_m^+) \sqcup (G_n^- \cap G_m^-)) \text{dist}(n, m) \\ &\quad + \mu((G_n^+ \cap G_m^-) \sqcup (G_n^- \cap G_m^+)) \text{dist}(-n, m). \end{aligned}$$

We claim that it holds

$$\mu((G_n^+ \cap G_m^-) \sqcup (G_n^- \cap G_m^+)) = \frac{1}{\pi} \text{dist}(n, m). \tag{15}$$

Since $\mu(G) = 1$ and $\text{dist}(-n, m) = \pi - \text{dist}(n, m)$ this will imply

$$\mu((G_n^+ \cap G_m^+) \sqcup (G_n^- \cap G_m^-)) = \frac{1}{\pi} \text{dist}(-n, m),$$

which completes the proof of Lemma 3.2, up to proving the claim (15). For that, we will make repeated use of the (double-sided) G -invariance of μ and the fact that

$$G_{Rn}^\pm = G_n^\pm R^{-1} \quad \text{for all } R \in G \text{ and } n \in \mathbb{S}^{d-1}. \quad (16)$$

If n and m are not collinear³, choosing R_π to be the rotation of angle π in the 2-plane $\langle n, m \rangle$ spanned by n and m and the identity in its orthogonal, we find that

$$(G_n^+ \cap G_m^-)R_\pi^{-1} = G_n^- \cap G_m^+,$$

and therefore

$$\mu((G_n^+ \cap G_m^-) \sqcup (G_n^- \cap G_m^+)) = 2\mu(G_n^+ \cap G_m^-),$$

so that (15) reduces to

$$\mu(G_n^+ \cap G_m^-) = \frac{1}{2\pi} \text{dist}(n, m). \quad (17)$$

To show (17) we define the continuous function $\varphi: \mathbb{S}^{d-1} \times \mathbb{S}^{d-1} \rightarrow [0, 1]$ given by

$$\varphi(n, m) := \mu(G_n^+ \cap G_m^-), \quad \forall n, m \in \mathbb{S}^{d-1}.$$

Using again (16) and the G -invariance of μ we obtain

$$\varphi(Rn, Rm) = \varphi(n, m), \quad \forall n, m \in \mathbb{S}^{d-1}, R \in G.$$

Therefore $\varphi(n, m)$ is a function of the scalar product $(n \cdot m)$, or equivalently a function of $\text{dist}(n, m) = \arccos(n \cdot m) \in [0, \pi]$. In other words, there exists a continuous function $\psi: [0, \pi] \rightarrow [0, 1]$ such that

$$\varphi(n, m) = \psi(\text{dist}(n, m)) \quad \forall n, m \in \mathbb{S}^{d-1}.$$

The function ψ can be expressed as

$$\psi(\theta) = \varphi(e_d, \cos \theta e_d + \sin \theta e_{d-1}) = \varphi(e_d, R_\theta e_d) \quad \forall \theta \in [0, \pi],$$

where $R_\theta \in G$ is the rotation that maps e_d to $(\cos \theta e_d + \sin \theta e_{d-1})$ and acts as the identity on the subspace of \mathbb{R}^d spanned by $\langle e_1, \dots, e_{d-2} \rangle$. Let $\theta \in [0, \pi]$ and $\xi \in [0, \pi - \theta]$. For any $n \in \mathbb{S}^{d-1}$ one can check the following implications

$$\begin{aligned} (n \cdot e_d > 0 \text{ and } n \cdot R_\theta e_d < 0) &\implies n \cdot R_{\theta+\xi} e_d < 0, \\ (n \cdot R_{\theta+\xi} e_d < 0 \text{ and } n \cdot R_\theta e_d > 0) &\implies n \cdot e_d > 0, \end{aligned}$$

³ If $n = m$ (respectively, $n = -m$), then $G_n^+ \cap G_m^- = G_n^- \cap G_m^+ = \emptyset$ (respectively, $(G_n^+ \cap G_m^-) \cup (G_n^- \cap G_m^+) = G_n^+ \cup G_n^- \stackrel{(14)}{=} G \setminus Z_n$) so that (15) is obvious.

which yield

$$\begin{aligned} G_{e_d}^+ \cap G_{R_{\theta+\xi}e_d}^- \cap G_{R_{\theta}e_d}^- &= G_{e_d}^+ \cap G_{R_{\theta}e_d}^-, \\ G_{e_d}^+ \cap G_{R_{\theta+\xi}e_d}^- \cap G_{R_{\theta}e_d}^+ &= G_{R_{\theta+\xi}e_d}^- \cap G_{R_{\theta}e_d}^+ \stackrel{(16)}{=} \left(G_{R_{\xi}e_d}^- \cap G_{e_d}^+ \right) R_{\theta}^{-1}. \end{aligned}$$

As a consequence, the definition of φ implies $\psi(\theta + \xi) = \psi(\theta) + \psi(\xi)$. As ψ is continuous, we deduce that $\psi(\theta) = \lambda\theta$ for some $\lambda \in \mathbb{R}$. Now, we claim that $\psi(\pi/2) = 1/4$, so that $\lambda = 1/(2\pi)$ and this proves (17). To prove that $\psi(\pi/2) = 1/4$ it suffices to remark that for $\theta = \pi/2$ we have $R_{\pi/2}e_d = e_{d-1}$ and $R_{\pi/2}e_{d-1} = -e_d$, so that the sets $(G_{e_d}^+ \cap G_{e_{d-1}}^-)$ and $(G_{e_d}^+ \cap G_{e_{d-1}}^+)$ have the same measure under μ (so, equal to $\frac{1}{2}\mu(G_{e_d}^+) = \frac{1}{4}\mu(G) = \frac{1}{4}$) because it holds

$$\left(G_{e_d}^+ \cap G_{e_{d-1}}^- \right) R_{\pi/2}^{-1} \stackrel{(16)}{=} G_{e_{d-1}}^+ \cup G_{e_d}^+.$$

□

Proof of Lemma 3.1. Pick one measurable map n such that $[n] = u$ a.e. Then, using the same argument as in [17], by Fubini's theorem and Lemma 3.2 we have

$$\begin{aligned} \int_G d\mu(R) \iint_{\Omega \times \Omega} dx dy \frac{\text{dist}(L_R(u(x)), L_R(u(y)))}{|x-y|} \rho_\varepsilon(|x-y|) \\ &= \iint_{\Omega \times \Omega} dx dy \frac{\rho_\varepsilon(|x-y|)}{|x-y|} \int_G d\mu(R) \text{dist}(F(Rn(x)), F(Rn(y))) \\ &= \frac{2}{\pi} \iint_{\Omega \times \Omega} \frac{\rho_\varepsilon(|x-y|)}{|x-y|} \text{dist}(n(x), n(y)) \text{dist}(-n(x), n(y)) dx dy \\ &= \frac{2}{\pi} \iint_{\Omega \times \Omega} \frac{\rho_\varepsilon(|x-y|)}{|x-y|} \text{dist}(u(x), u(y)) (\pi - \text{dist}(u(x), u(y))) dx dy \\ &\leq 2 \iint_{\Omega \times \Omega} \frac{\text{dist}(u(x), u(y))}{|x-y|} \rho_\varepsilon(|x-y|) dx dy. \end{aligned}$$

□

Proof of Theorem 1.2. The existence of a BV lifting of $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ for an arbitrary open set $\Omega \subset \mathbb{R}^N$ is proved in Theorem 1.6. Assume now that Ω is bounded and Lipschitz and $n_0 \in L^1(\partial\Omega; \mathbb{S}^{d-1})$ is a prescribed lifting of u at the boundary. Let $\tilde{n} \in BV(\Omega; \mathbb{S}^{d-1})$ be a lifting of u in Ω (not necessarily equal to n_0 at the boundary). By the trace theorem for BV functions (see e.g. [2, Theorem 3.88]), we know that \tilde{n} has an $L^1(\partial\Omega; \mathbb{S}^{d-1})$ trace at the boundary and $[\tilde{n}] = [n_0] = u$ \mathcal{H}^{N-1} -a.e. on $\partial\Omega$. Set $f = \tilde{n} \cdot n_0$ on $\partial\Omega$. Then f takes only the values $\{\pm 1\}$ as \tilde{n} and n_0 are two possible orientations of the same line field u at the boundary $\partial\Omega$. In particular, $f \in L^1(\partial\Omega; \{\pm 1\})$. Then one chooses an extension $\tilde{f} \in W^{1,1}(\Omega; [-1, 1])$ of f in Ω (for example, the harmonic extension of f in Ω satisfies that property). By the co-area formula, \tilde{f} has almost every level set of finite perimeter, in particular there exists $\alpha \in (-1, 1)$ such that the characteristic function $\mathbf{1}_{\{\tilde{f} > \alpha\}}$ is of bounded variation in Ω . Set $\bar{f} = 2\mathbf{1}_{\{\tilde{f} > \alpha\}} - 1$ in Ω . Then $\bar{f} \in BV(\Omega; \{\pm 1\})$ and $\bar{f} = \tilde{f} = f$ \mathcal{H}^{N-1} -a.e. on $\partial\Omega$. Now, one considers $n = \bar{f}\tilde{n}$. As \bar{f} and \tilde{n} are $BV \cap L^\infty$ maps in Ω , then their product n is BV in Ω with values into \mathbb{S}^{d-1} ; moreover, $[n] = [\tilde{n}] = u$ a.e. in Ω and $n = (\bar{f})^2 n_0 = n_0$ \mathcal{H}^{N-1} -a.e. on $\partial\Omega$. □

4 “Euclidean” lifting. Proof of Theorem 1.7.

As explained in the introduction, when we measure the jumps in the BV -energy, we may want to use Euclidean distances instead of geodesic distances. In that case, the choice of an isometric embedding is crucial. For \mathbb{S}^{d-1} we stick to the canonical embedding $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, and for $n \in BV(\Omega; \mathbb{S}^{d-1})$ we denote by $|n|_{BV, \mathbb{R}^d}$ the usual BV -seminorm of $n \in BV(\Omega; \mathbb{R}^d)$, i.e. the total variation norm of Dn as a $\mathbb{R}^{d \times N}$ -valued measure.

For \mathbb{RP}^{d-1} it is not obvious what a canonical embedding should be. Physics provides us with the natural embedding (3), but to understand better the effect of this choice we will also consider general isometric embeddings

$$\Phi: \mathbb{RP}^{d-1} \rightarrow \mathbb{R}^D.$$

We denote by $\bar{\Phi}: \mathbb{S}^{d-1} \rightarrow \mathbb{R}^D$ the canonically associated map on the sphere \mathbb{S}^{d-1} , i.e., $\bar{\Phi}(n) = \Phi([n])$ for all $n \in \mathbb{S}^{d-1}$. For $u \in BV(\Omega; \mathbb{RP}^{d-1})$ we will identify u with $\Phi(u) \in BV(\Omega; \mathbb{R}^D)$ and denote by $|u|_{BV, \Phi}$ the usual BV seminorm of $u \in BV(\Omega; \mathbb{R}^D)$, i.e. the total variation norm of Du as a $\mathbb{R}^{D \times N}$ -valued measure. We also denote by $D^c u$ the Cantor part, by $D^j u$ the jump part of the differential Du of $u \in BV(\Omega; \mathbb{R}^D)$ and by ∇u its approximate gradient.

Theorem 4.1. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set. For any $u \in BV(\Omega; \mathbb{RP}^{d-1})$, there exists a lifting $n \in BV(\Omega; \mathbb{S}^{d-1})$ with $u = [n]$ a.e. in Ω and*

$$|n|_{BV, \mathbb{R}^d} \leq \left(1 + \frac{2}{\pi}\right) |D^c u|(\Omega) + C^j(\Phi) |D^j u|(\Omega) + C^a(N, d) \int_{\Omega} |\nabla u| dx, \quad (18)$$

where

$$C^j(\Phi) = \frac{2}{\pi} \sup \left\{ \frac{\theta \cos \frac{\theta}{2} + (\pi - \theta) \sin \frac{\theta}{2}}{|\bar{\Phi}(n) - \bar{\Phi}(m)|} : n, m \in \mathbb{S}^{d-1}, \theta = \arccos(n \cdot m) \in [0, \pi] \right\},$$

$$C^a(N, d) = 1 + \frac{2}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} \sup_{\substack{v_k \in \mathbb{R}^{d-1} \\ \sum_{k=1}^N |v_k|^2 = 1}} \left\{ \int_{\mathbb{S}^{d-2}} \left(\sum_{k=1}^N (\omega \cdot v_k)^2 \right)^{1/2} d\mathcal{H}^{d-2}(\omega) \right\}.$$

The constants C^j and C^a satisfy $C^j, C^a \geq 1 + 2/\pi$. For the tensorial embedding Φ in (3) it holds $C^j = 1 + 2/\pi$. For $d = 2$ it holds $C^a(N, d = 2) = 1 + 2/\pi$ (independently of Φ) and this constant is optimal if $N \geq 2$.

Remark 4.2. For $d > 2$ and $N \geq 2$, the formula for C^a found in Theorem 4.1 leads ⁴ to $C^a > 1 + 2/\pi$, but we conjecture that the optimal constant should be $1 + 2/\pi$ for any $d, N \geq 2$. Note that $C^a(1, d) = 1 + 2/\pi$ for every $d \geq 2$ (see the proof of (21)). However, $1 + 2/\pi$ is not the optimal constant when estimating the BV seminorm of liftings in dimension $N = 1$; for example, the optimal constant is $\sqrt{2}$ in the case of the tensorial embedding (3) (see Section 5).

⁴ For $d = 3$, choosing $v_1 = e_1/\sqrt{2}$ and $v_2 = e_2/\sqrt{2}$ in the supremum of the formula for C^a yields

$$C^a(N, d = 3) \geq C^a(2, d = 3) \geq 1 + \frac{2}{\mathcal{H}^2(\mathbb{S}^2)} \int_{\mathbb{S}^1} \frac{|\omega|}{\sqrt{2}} d\mathcal{H}^1(\omega) = 1 + 1/\sqrt{2} > 1 + 2/\pi$$

for all $N \geq 2$.

As mentioned in Remark 1.9, we prove that we always obtain the optimal constant $1 + 2/\pi$ in dimension $N \geq 2$ provided that ⁵ the total variation is measured as the average over all directions ω of the sphere \mathbb{S}^{N-1} of the total variation of partial derivative measure in direction ω (the jumps being measured by the Euclidean distance).

Proposition 4.3. *Let $\Omega \subset \mathbb{R}^N$ ($N \geq 1$) be an open set. For any $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$, there exists a lifting $n \in BV(\Omega; \mathbb{S}^{d-1})$ with $u = [n]$ a.e. in Ω and*

$$\| \|n\| \|_{BV, \mathbb{R}^d} \leq \left(1 + \frac{2}{\pi}\right) \| \|u\| \|_{BV, \mathbb{R}^{d \times d}}$$

where the seminorm $\| \| \cdot \| \|_{BV, \mathbb{R}^D}$ was introduced in (10).

Proof of Theorem 4.1. The main change with respect to the geodesic case in Theorem 1.6 consists in computing the total variation of BV liftings in Euclidean case using some truncation maps as in [9]. More precisely, for $\varepsilon > 0$ we introduce a Lipschitz approximation $F_\varepsilon: \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$ of the symmetric map $F: \mathbb{S}^{d-1} \rightarrow \mathbb{S}^{d-1}$ introduced in (12), that is given by

$$F_\varepsilon(n) = \begin{cases} n & \text{if } n \cdot e_d \geq \varepsilon, \\ \frac{1}{\varepsilon}(n \cdot e_d)n & \text{if } |n \cdot e_d| < \varepsilon, \\ -n & \text{if } n \cdot e_d \leq -\varepsilon, \end{cases}$$

so that F_ε is symmetric on \mathbb{S}^{d-1} . We also introduce for any $R \in SO(d)$ the map $F_{\varepsilon, R}: \mathbb{S}^{d-1} \rightarrow \mathbb{R}^d$ given by

$$F_{\varepsilon, R}(n) = R^{-1}F_\varepsilon(Rn),$$

and the corresponding map $L_{\varepsilon, R}: \mathbb{R}\mathbb{P}^{d-1} \rightarrow \mathbb{R}^d$, $L_{\varepsilon, R}([n]) = F_{\varepsilon, R}(n)$ for every $n \in \mathbb{S}^{d-1}$. Note that F_ε , $F_{\varepsilon, R}$ and $L_{\varepsilon, R}$ are not \mathbb{S}^{d-1} -valued maps; however, this property will be satisfied almost everywhere in the limit $\varepsilon \rightarrow 0$. We will prove

$$\int_G |L_{\varepsilon, R}(u)|_{BV, \mathbb{R}^d} d\mu(R) \leq \left(1 + \frac{2}{\pi}\right) |D^c u|(\Omega) + C^j |D^j u|(\Omega) + C^a \int_\Omega |\nabla u| dx + o(1), \quad \text{as } \varepsilon \rightarrow 0, \quad (19)$$

which implies (18) by arguing as in [9]. For convenience of the reader, we sketch the argument here: any rotation $R \in G$ defines an ‘‘equator’’ $E_R = \{[n]: Rn \in \mathbb{S}^{d-2} \times \{0\} \subset \mathbb{S}^{d-1}\} \subset \mathbb{R}\mathbb{P}^{d-1}$, outside of which $L_{\varepsilon, R}$ converges towards L_R . For μ -a.e. $R \in G$, the set $\{x \in \Omega: u(x) \in E_R\}$ has zero Lebesgue measure, which allows to deduce by the lower semicontinuity of the seminorm $|\cdot|_{BV, \mathbb{R}^d}$ under L^1_{loc} topology that $|L_R(u)|_{BV, \mathbb{R}^d} \leq \liminf_{\varepsilon \rightarrow 0} |L_{\varepsilon, R}(u)|_{BV, \mathbb{R}^d}$. Thus from (19) we may conclude by Fatou’s lemma that

$$\int_G |L_R(u)|_{BV, \mathbb{R}^d} d\mu(R) \leq \left(1 + \frac{2}{\pi}\right) |D^c u|(\Omega) + C^j |D^j u|(\Omega) + C^a \int_\Omega |\nabla u| dx,$$

and by the averaging theorem, one can choose a rotation R for which (18) holds for $n = L_R(u)$.

⁵Repeating the arguments at Section 2, then one concludes that indeed the constant $1 + 2/\pi$ is achieved when using the seminorm $\| \| \cdot \| \|_{BV, \mathbb{R}^D}$ for u and its liftings n in any dimension $N, d \geq 2$.

Proof of (19). By the rank-one property of BV -maps [1] (see also the recent simple proof in [10]), the Cantor part $D^c u$ of Du can be decomposed as $D^c u = a \otimes \eta |D^c u|$ for some \mathbb{S}^{D-1} -valued map a and \mathbb{S}^{N-1} -valued map η , and the chain rule gives

$$\begin{aligned} |L_{\varepsilon,R}(u)|_{BV} &= \int_{\Omega} |DL_{\varepsilon,R}(u)a| d|D^c u| + \int_{\Omega} |DL_{\varepsilon,R}(u)\nabla u| dx \\ &\quad + \int_{J_u} |L_{\varepsilon,R}(u^+) - L_{\varepsilon,R}(u^-)| d\mathcal{H}^{N-1}. \end{aligned}$$

Here the differential $DL_{\varepsilon,R}(u): T_u \mathbb{R}\mathbb{P}^{d-1} \rightarrow \mathbb{R}^d$ is identified with $DL_{\varepsilon,R}(u)\Pi: \mathbb{R}^D \rightarrow \mathbb{R}^d$, where Π is the orthogonal projection $\mathbb{R}^D \rightarrow T_u \mathbb{R}\mathbb{P}^{d-1}$. In particular the product $DL_{\varepsilon,R}(u)\nabla u$ is a $d \times N$ matrix. Moreover, we write $\nabla u = g|\nabla u|$ for a $\mathbb{R}^{D \times N}$ -valued map g with $|g|_{\mathbb{R}^{D \times N}} = 1$ a.e. Next we show that as $\varepsilon \rightarrow 0$, for any fixed $u, u^+, u^- \in \mathbb{R}\mathbb{P}^{d-1}$, $a \in \mathbb{S}^{D-1}$ and $g \in \mathbb{R}^{D \times N}$ with $|g|_{\mathbb{R}^{D \times N}} = 1$, it holds

$$\int_G |DL_{\varepsilon,R}(u)g| d\mu(R) \leq C^a + o(1), \quad (20)$$

$$\int_G |DL_{\varepsilon,R}(u)a| d\mu(R) \leq \left(1 + \frac{2}{\pi}\right) + o(1), \quad (21)$$

$$\int_G |L_{\varepsilon,R}(u^+) - L_{\varepsilon,R}(u^-)| d\mu(R) \leq C^j |\Phi(u^+) - \Phi(u^-)| + o(1), \quad (22)$$

from which (19) follows (where $o(1)$ are quantities independent of u, u^+, u^-, a and g that converge to 0 as $\varepsilon \rightarrow 0$).

Proof of (20). Let $n \in \mathbb{S}^{d-1}$ be such that $u = \bar{\Phi}(n)$. Then $DL_{\varepsilon,R}(u) = DF_{\varepsilon,R}(n)D\bar{\Phi}(n)^{-1}$, where $D\bar{\Phi}(n)$ is viewed as a map from $T_n \mathbb{S}^{d-1}$ to $T_u \mathbb{R}\mathbb{P}^{d-1}$, and it is an isometry. Therefore it holds

$$|DL_{\varepsilon,R}(u)g| = |DF_{\varepsilon,R}(n)\bar{g}|, \quad \text{with } \bar{g} = D\bar{\Phi}(n)^{-1}\Pi g \in \mathbb{R}^{d \times N}, \text{ and } |\bar{g}| \leq |g| = 1.$$

As $DF_{\varepsilon,R}(n) = R^{-1}DF_{\varepsilon}(Rn)R$, we obtain

$$\begin{aligned} \int_G |DL_{\varepsilon,R}(u)g| d\mu(R) &= \int_G |DF_{\varepsilon,R}(n)\bar{g}| d\mu(R) = \int_G |R^{-1}DF_{\varepsilon}(Rn)R\bar{g}| d\mu(R) \\ &= \int_{\{|Rn \cdot e_d| > \varepsilon\}} |\bar{g}| d\mu(R) + \frac{1}{\varepsilon} \int_{\{|Rn \cdot e_d| \leq \varepsilon\}} |n \otimes {}^t(R\bar{g})e_d + (Rn \cdot e_d)\bar{g}| d\mu(R) \\ &= \int_{\{|Rn \cdot e_d| > \varepsilon\}} |\bar{g}| d\mu(R) + \frac{1}{\varepsilon} \int_{\{|Rn \cdot e_d| \leq \varepsilon\}} |(Rn \cdot e_d)\bar{g}| d\mu(R) \\ &\quad + \frac{1}{\varepsilon} \int_{\{|Rn \cdot e_d| \leq \varepsilon\}} |n \otimes {}^t(R\bar{g})e_d| d\mu(R) \\ &\leq 1 + \frac{1}{\varepsilon} \int_{\{|Rn \cdot e_d| \leq \varepsilon\}} |{}^t\bar{g}{}^t R e_d| d\mu(R), \end{aligned} \quad (23)$$

where we denoted by ${}^t(\cdot)$ the transpose of a matrix (\cdot) and we used the triangle inequality

and $|\bar{g}| \leq |g| = 1$. Next we compute

$$\begin{aligned} \frac{1}{\varepsilon} \int_{\{|R_n \cdot e_d| \leq \varepsilon\}} |{}^t \bar{g}^t R e_d| d\mu(R) &= \frac{1}{\varepsilon \mathcal{H}^{d-1}(\mathbb{S}^{d-1})} \int_{\{\omega \in \mathbb{S}^{d-1}: |\omega \cdot n| \leq \varepsilon\}} |{}^t \bar{g} \omega| d\mathcal{H}^{d-1}(\omega) \\ &= \frac{2}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} \int_{\mathbb{S}^{d-2} \times \{0\}} |{}^t \tilde{g} \omega'| d\mathcal{H}^{d-2}(\omega') + o(1), \end{aligned} \quad (24)$$

where $\tilde{g} = R_n^{-1} \bar{g}$ with $R_n \in SO(d)$ such that $n = R_n e_d$. Note also that for every $\omega' \in \mathbb{S}^{d-2} \times \{0\}$, $|{}^t \tilde{g} \omega'| = |{}^t h \omega'|$ with $h = p \tilde{g} \in \mathbb{R}^{(d-1) \times N}$, where p is the matrix of the orthogonal projection $\mathbb{R}^d \rightarrow \mathbb{R}^{d-1}$. Hence, gathering (23) and (24), we find that

$$\begin{aligned} \int_G |DL_{\varepsilon, R}(u)g| d\mu(R) &\leq \left(1 + \frac{2}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} L\right) + o(1), \quad \text{as } \varepsilon \rightarrow 0, \\ \text{where } L &:= \sup_{\substack{h \in \mathbb{R}^{(d-1) \times N} \\ |h|^2 = 1}} \left\{ \int_{\mathbb{S}^{d-2}} |{}^t h \omega| d\mathcal{H}^{d-2}(\omega) \right\}, \end{aligned}$$

with the convention that $\mathbb{S}^{d-2} = \{\pm 1\}$ for $d = 2$. Denoting by v_1, \dots, v_N the columns of h , this proves (20). Note that if $d = 2$ then $L = 2$ and thus $C^a(N, d = 2) = 1 + 2/\pi$ for every $N \geq 1$. The estimate of the general case $C^a(N, d)$ for $d \geq 2$ is done below (see (25)).

Proof of (21). We consider the special case of rank-one matrices $g := a \otimes \eta$, $|a| = |\eta| = 1$ in the above computation, which leads to the same estimate, with the supremum defining the constant L restricted to rank-one matrices $h = b \otimes \eta$, $|b| = |\eta| = 1$, hence

$$\begin{aligned} \int_G |DL_{\varepsilon, R}(u)a| d\mu(R) &\leq \left(1 + \frac{2}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} M\right) + o(1), \\ \text{where } M &:= \sup_{b \in \mathbb{S}^{d-2}} \int_{\mathbb{S}^{d-2}} |b \cdot \omega| d\mathcal{H}^{d-2}(\omega). \end{aligned}$$

If $d = 2$ then $M = 2$ and we obtain (21). If $d \geq 3$, by rotational invariance we have by integrating over $\omega = (\omega_1, \dots, \omega_{d-1}) \in \mathbb{S}^{d-2}$:

$$M = \int_{\mathbb{S}^{d-2}} |\omega_{d-1}| d\mathcal{H}^{d-2}(\omega) = 2 \int_{B^{d-2}} \sqrt{1 - |\xi|^2} \frac{d\xi}{\sqrt{1 - |\xi|^2}} = 2\mathcal{H}^{d-2}(B^{d-2}),$$

and since

$$\frac{\mathcal{H}^{d-2}(B^{d-2})}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} = \frac{\pi^{\frac{d-2}{2}}}{\Gamma\left(\frac{d-2}{2} + 1\right)} \frac{\Gamma\left(\frac{d}{2}\right)}{2\pi^{\frac{d}{2}}} = \frac{1}{2\pi},$$

we obtain (21). Note that this shows also that

$$C^a(N, d) \geq C^a(1, d) = 1 + \frac{2}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} M = 1 + 2/\pi. \quad (25)$$

Proof of (22). Let $n, m \in \mathbb{S}^{d-1}$ be such that $u^+ = \bar{\Phi}(n)$ and $u^- = \bar{\Phi}(m)$. Then we find

$$\int_G |L_{\varepsilon, R}(u^+) - L_{\varepsilon, R}(u^-)| d\mu(R) = \int_G |F(Rn) - F(Rm)| d\mu(R) + o(1), \quad \text{as } \varepsilon \rightarrow 0,$$

where we used

$$\mu(\{|Rn \cdot e_d| \leq \varepsilon\}) = \frac{1}{\mathcal{H}^{d-1}(\mathbb{S}^{d-1})} \mathcal{H}^{d-1}(\{\omega \in \mathbb{S}^{d-1} : |\omega \cdot n| \leq \varepsilon\}) = o(1), \quad \text{as } \varepsilon \rightarrow 0.$$

Arguing as in the proof of Lemma 3.2 and denoting by θ the angle $\theta = \arccos(n \cdot m) \in [0, \pi]$ we obtain

$$\begin{aligned} \int_G |F(Rn) - F(Rm)| d\mu(R) &= \frac{\pi - \theta}{\pi} |n - m| + \frac{\theta}{\pi} |n + m| \\ &= \frac{\pi - \theta}{\pi} \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} + \frac{\theta}{\pi} \sqrt{(1 + \cos \theta)^2 + \sin^2 \theta} \\ &= \frac{2}{\pi} \left((\pi - \theta) \sin \frac{\theta}{2} + \theta \cos \frac{\theta}{2} \right) \\ &\leq C^j |\bar{\Phi}(n) - \bar{\Phi}(m)| = C^j |\Phi(u^+) - \Phi(u^-)|. \end{aligned}$$

Finally, we check that $C^j \geq 1 + 2/\pi$ for every isometric embedding $\bar{\Phi} : \mathbb{S}^{d-1} \rightarrow \mathbb{R}^D$. Indeed, it suffices to consider $n = e_d$, $m = \cos \theta e_d + \sin \theta e_{d-1}$, to compute

$$\begin{cases} \theta \cos \frac{\theta}{2} + (\pi - \theta) \sin \frac{\theta}{2} = \left(1 + \frac{\pi}{2}\right) \theta + o(\theta), \\ |\bar{\Phi}(n) - \bar{\Phi}(m)| = \theta |D\bar{\Phi}(e_d)e_{d-1}| + o(\theta), \end{cases} \quad \text{as } \theta \rightarrow 0^+,$$

and to remark that $|D\bar{\Phi}(e_d)e_{d-1}| = 1$ since $\bar{\Phi}$ is an isometric embedding. Moreover, in the case of the tensorial embedding (3) one has

$$|\bar{\Phi}(n) - \bar{\Phi}(m)| = \frac{1}{\sqrt{2}} |n \otimes n - m \otimes m| = \sin \theta, \quad (26)$$

and it can be checked that

$$\theta \cos \frac{\theta}{2} + (\pi - \theta) \sin \frac{\theta}{2} \leq \left(1 + \frac{\pi}{2}\right) \sin \theta \quad \forall \theta \in [0, \pi],$$

so that $C^j = 1 + 2/\pi$ for the embedding Φ in (3). \square

Proof of Proposition 4.3. We give two proofs, the first one works under the additional assumption on Ω being bounded and Lipschitz (because this method is based on Theorem 1.5), while the second method works for general open set Ω .

First method for a bounded Lipschitz open set Ω : Considering $\mathbb{S}^{d-1} \subset \mathbb{R}^d$ endowed with the Euclidean distance $|\cdot|_{\mathbb{R}^d}$ and $\mathbb{R}\mathbb{P}^{d-1} \subset \mathbb{R}^{d \times d}$ endowed with the distance (8), we will use the technique presented in the proof of Theorem 1.6 combined with Theorem 1.5. More precisely, by the proof of (22), we have that for every $n, m \in \mathbb{S}^{d-1}$:

$$\int_G |F(Rn) - F(Rm)|_{\mathbb{R}^d} d\mu(R) = \frac{\pi - \theta}{\pi} |n - m|_{\mathbb{R}^d} + \frac{\theta}{\pi} |n + m|_{\mathbb{R}^d} \leq (1 + 2/\pi) |[n] - [m]|_{\mathbb{R}^{d \times d}}.$$

This inequality combined with Lemma 3.1 lead to

$$\begin{aligned} &\int_G \iint_{\Omega \times \Omega} \frac{|L_R(u(x)) - L_R(u(y))|_{\mathbb{R}^d}}{|x - y|} \rho_\varepsilon(|x - y|) dx dy d\mu(R) \\ &\leq (1 + 2/\pi) \iint_{\Omega \times \Omega} \frac{|u(x) - u(y)|_{\mathbb{R}^{d \times d}}}{|x - y|} \rho_\varepsilon(|x - y|) dx dy, \end{aligned}$$

where ρ_ε is any family of nonnegative radial functions. By Theorem 1.5 and the definition (10), one has the representation formula for $\|u\|_{BV, \mathbb{R}^{d \times d}}$ respectively of $\|L_R(u)\|_{BV, \mathbb{R}^d}$ in terms of (4) for the distance (8), respectively $|\cdot|_{\mathbb{R}^d}$. The conclusion follows as in the proof of Theorem 1.6.

Second method for an arbitrary open set Ω : We repeat the argument of the proof of Theorem 4.1. Within those notations, the chain rule implies for small $\varepsilon > 0$:

$$\begin{aligned} \int_G \int_{\mathbb{S}^{N-1}} |D_\omega[L_{\varepsilon,R}(u)]|(\Omega) d\mathcal{H}^{N-1}(\omega) d\mu(R) &= \int_G \int_{\mathbb{S}^{N-1}} \int_\Omega |DL_{\varepsilon,R}(u)\xi| |\nabla_\omega u| dx d\mathcal{H}^{N-1}(\omega) d\mu(R) \\ &+ \int_G \int_{\mathbb{S}^{N-1}} \int_\Omega |\eta \cdot \omega| |DL_{\varepsilon,R}(u)a| |D^c u| d\mathcal{H}^{N-1}(\omega) d\mu(R) \\ &+ \int_G \int_{\mathbb{S}^{N-1}} \int_{J_u} |\omega \cdot \nu| |L_{\varepsilon,R}(u^+(x)) - L_{\varepsilon,R}(u^-(x))|_{\mathbb{R}^d} d\mathcal{H}^{N-1}(x) d\mathcal{H}^{N-1}(\omega) d\mu(R), \end{aligned}$$

where $\nabla_\omega u = \xi |\nabla_\omega u|$, $D^c u = a \otimes \eta |D^c u|$ with $\xi = \xi(\omega)$, a, η are unit length maps and ν is a unit normal vector at J_u . By (21) and (22) (with $C^j = 1 + 2/\pi$), it entails that

$$\int_G \int_{\mathbb{S}^{N-1}} |D_\omega[L_{\varepsilon,R}(u)]|(\Omega) d\mathcal{H}^{N-1}(\omega) d\mu(R) \leq (1 + 2/\pi) \|u\|_{BV, \mathbb{R}^{d \times d}} + o(1)$$

as $\varepsilon \rightarrow 0$. As in the proof of Theorem 4.1, one concludes that there exists a rotation $R \in G$ such that the lifting $n = L_R(u) = \lim_{\varepsilon \rightarrow 0} L_{\varepsilon,R}(u)$ of u satisfies

$$\|L_R(u)\|_{BV, \mathbb{R}^d} \stackrel{(10)}{=} \int_{\mathbb{S}^{N-1}} |D_\omega L_R(u)|(\Omega) d\mathcal{H}^{N-1}(\omega) \leq (1 + 2/\pi) \|u\|_{BV, \mathbb{R}^{d \times d}}.$$

□

5 The one-dimensional case

When the definition domain is an interval $\Omega = I \subset \mathbb{R}$, the situation is simpler, since it is possible to lift any map $u \in BV(I; \mathbb{R}\mathbb{P}^{d-1})$ without creating additional jumps for optimal BV liftings n (in contrast e.g. with the example in Section 2). Moreover, we will prove that the optimal constant in the estimate of a BV lifting in dimension $N = 1$ is strictly less than the ones found in Theorems 1.6 and 1.7. To show this, we start by fixing an open cap around the north pole $(0, \dots, 0, 1)$ of the sphere \mathbb{S}^{d-1} :

$$U = \left\{ \omega \in \mathbb{S}^{d-1} : \text{dist}_{\mathbb{S}^{d-1}}(\omega, e_d) < \pi/4 \right\} \subset \mathbb{S}^{d-1}.$$

This cap has the property that for any $n^\pm \in \bar{U}$ of the closure of U , the distance between n^+ and n^- (either geodesic or Euclidean) is the smallest of the distances between any other representants of the classes $[n^\pm] \in \mathbb{R}\mathbb{P}^{d-1}$, namely

$$\text{dist}(n^+, n^-) = \min \left\{ \text{dist}(n, m) : n = \pm n^+, m = \pm n^- \right\}, \quad \forall n^\pm \in \bar{U},$$

where $\text{dist} = \text{dist}_{\mathbb{S}^{d-1}}$ or $\text{dist}_{\mathbb{R}^d}$. Moreover, for any $n^\pm \in \mathbb{S}^{d-1}$, one can always choose $R \in SO(d)$ and $\tau \in \{\pm 1\}$ such that n^+ and τn^- both belong to the set $R^{-1} \cdot \bar{U}$.

Next we fix an isometric embedding of $\mathbb{R}\mathbb{P}^{d-1}$ into \mathbb{R}^D (whose choice will not play any role in the outcome) so that we may consider the \mathbb{R}^D -valued vector measure Du and its diffuse part $D^a u + D^c u$. We prove the following:

Proposition 5.1. *Let $I \subset \mathbb{R}$ be an open interval and $u \in BV(I; \mathbb{R}\mathbb{P}^{d-1})$. Then there exists a lifting $n \in BV(I; \mathbb{S}^{d-1})$ such that*

$$|D^a n|(I) = |D^a u|(I), \quad |D^c n|(I) = |D^c u|(I), \quad J_n = J_u,$$

and at every jump point $x \in J_u (= J_n)$, the traces $n^\pm(x)$ belong to $R^{-1} \cdot \bar{U}$ for some rotation $R \in SO(d)$ depending on x .

Proof. As usual, $u \in BV$ is identified with its precise representative away from J_u , i.e., u is continuous away from J_u (see [2]). We denote by Π the canonical projection $\Pi: \mathbb{S}^{d-1} \rightarrow \mathbb{R}\mathbb{P}^{d-1}$. The family $\{\Pi(R^{-1} \cdot U): R \in SO(d)\}$ is an open covering of $\mathbb{R}\mathbb{P}^{d-1}$, and since $u \in BV$, there exists $\delta > 0$ such that for any open interval $(a, b) \subset I$,

$$|Du|((a, b)) \leq \delta \quad \Rightarrow \quad \exists R \in SO(d) \text{ such that } u((a, b)) \subset \Pi(R^{-1} \cdot U).$$

Moreover, we may find numbers $a_0 < a_1 < \dots < a_k$ such that

$$\begin{aligned} I &= (a_0, a_k) = I_0 \cup I_1 \cup \dots \cup I_{k-1}, & I_\ell &= (a_\ell, a_{\ell+1}), \\ &\text{and } |Du|((a_\ell, a_{\ell+1})) \leq \delta, & \forall \ell &\in \{0, \dots, k-1\}. \end{aligned}$$

At the points a_1, \dots, a_{k-1} the map u is either continuous or has a jump.

For each $\ell \in \{0, \dots, k-1\}$ we denote by u_ℓ the restriction of u to I_ℓ . By the above there exists $R_\ell \in SO(d)$ such that the image of u_ℓ lies in $V_\ell := \Pi(R_\ell^{-1} \cdot U)$. The map $L_\ell := L_{R_\ell}$ (defined in Section 3) is smooth on that set V_ℓ (as F is smooth on U), so that by the chain rule, we may define the BV lifting $n_\ell = L_\ell(u_\ell) \in BV(I_\ell; \mathbb{S}^{d-1})$, which takes values into $R_\ell^{-1} \cdot U$. At every $\xi \in V_\ell$ the differential $DL_\ell(\xi)$ is simply the identity on $T_\xi \mathbb{R}\mathbb{P}^{d-1} \cong T_{L_\ell(\xi)} \mathbb{S}^{d-1}$, so that by the chain rule it holds

$$|D^a n_\ell|(I_\ell) = |D^a u_\ell|(I_\ell), \quad |D^c n_\ell|(I_\ell) = |D^c u_\ell|(I_\ell), \quad \text{and } J_{n_\ell} = J_{u_\ell} \quad \text{in } I_\ell.$$

Note that the map $\tilde{n}_\ell = -n_\ell$ is also a lifting of u_ℓ with the same properties (with R_ℓ modified accordingly). Next we glue all these liftings together by choosing a sequence of signs $\tau_0, \dots, \tau_{k-1}$ inductively, ensuring that the local liftings $\bar{n}_\ell = \tau_\ell n_\ell$ are such that

$$\begin{cases} \bar{n}_{\ell-1}(a_\ell^-) = \bar{n}_\ell(a_\ell^+) & \text{if } u \text{ is continuous at } a_\ell, \\ \text{or } \bar{n}_{\ell-1}(a_\ell^-), \bar{n}_\ell(a_\ell^+) \in \bar{R}_\ell^{-1} \cdot \bar{U} \text{ for some } \bar{R}_\ell \in SO(d) & \text{if } u \text{ has a jump at } a_\ell, \end{cases}$$

where $\bar{n}_\ell(a_\ell^+)$ and $\bar{n}_\ell(a_{\ell+1}^-)$ are the traces of \bar{n}_ℓ at a_ℓ , respectively at $a_{\ell+1}$. Finally, we define the lifting $n \in BV(I; \mathbb{S}^{d-1})$ by $n = \bar{n}_\ell$ on each interval I_ℓ ; then n satisfies the desired conclusion. \square

5.1 Optimal constants on an interval Ω

We distinguish two cases:

1. **“Geodesic” lifting:** When measuring jumps in geodesic distances, the lifting obtained in Proposition 5.1 gives the estimate

$$|n|_{BV, \mathbb{S}^{d-1}} \leq |u|_{BV, \mathbb{R}\mathbb{P}^{d-1}}.$$

Therefore, the optimal constant in dimension $N = 1$ is 1, so less than the constant found at Theorem 1.6.

2. **“Euclidean” lifting:** When measuring jumps in Euclidean distances, since for any $n, m \in R^{-1} \cdot \bar{U}$ it holds $\theta := \arccos(n \cdot m) \in [0, \pi/2]$ and

$$|n - m|_{\mathbb{R}^d} = \sqrt{(1 - \cos \theta)^2 + \sin^2 \theta} = 2 \sin \frac{\theta}{2},$$

we obtain the estimate

$$|n|_{BV, \mathbb{R}^d} \leq C(\Phi) |u|_{BV, \Phi},$$

$$C(\Phi) = \sup \left\{ \frac{2 \sin \frac{\theta}{2}}{|\Phi(n) - \Phi(m)|} : n, m \in \mathbb{S}^{d-1}, \theta = \arccos n \cdot m \in (0, \pi/2) \right\} \geq 1.$$

The fact that $C(\Phi) \geq 1$ can be checked by considering $n = e_d$, $m = \cos \theta e_d + \sin \theta e_{d-1}$ so that $|\Phi(n) - \Phi(m)| = \theta + o(1)$ as $\theta \rightarrow 0^+$ (see the proof of (22)). For the physical embedding (3), by (26), the constant $C(\Phi)$ is

$$C = \sup_{0 \leq \theta \leq \pi/2} \frac{2 \sin \frac{\theta}{2}}{\sin \theta} = \sqrt{2},$$

yielding

$$|n|_{BV, \mathbb{R}^d} \leq \sqrt{2} |u|_{BV, \mathbb{R}^{d \times d}}.$$

Note that $\sqrt{2} < 1 + \frac{2}{\pi}$ which was the optimal constant C^j in Theorem 4.1 achieved for the tensorial embedding (3). In particular, for Q -tensors (as in Corollary 1.10) we obtain

$$|s_\star| |n|_{BV, \mathbb{R}^d} \leq |Q|_{BV, \mathcal{S}_0}.$$

Remark 5.2. If the definition domain is $\Omega = \mathbb{S}^1$ (so, still of dimension 1 but not a simply connected domain), then the situation is different from the one explained above for an interval. In fact, it is similar to the case of dimension $N = 2$ in Theorems 1.6 and 1.7 because a BV map $u : \mathbb{S}^1 \rightarrow \mathbb{R}\mathbb{P}^{d-1}$ can create additional jumps for any optimal BV lifting as in the example in Section 2. (The corresponding situation for BV maps with values into \mathbb{S}^1 was studied in [14].)

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A The diffuse part of the BV seminorm

In this first part of the Appendix, we prove the claim (1) in Remark 1.1 that the total variation of the diffuse part of Du for $u \in BV(\Omega; \mathcal{N})$ is independent of the choice of an embedding $\mathcal{N} \subset \mathbb{R}^D$. Furthermore, we prove Proposition 1.3 stating that the total variation of the diffuse part of Dn for any lifting $n \in BV(\Omega; \mathbb{S}^{d-1})$ of a map $u \in BV(\Omega; \mathbb{R}\mathbb{P}^{d-1})$ is independent of the lifting n .

Lemma A.1. *Let $\mathcal{N}_1 \subset \mathbb{R}^{D_1}$ and $\mathcal{N}_2 \subset \mathbb{R}^{D_2}$ be two smooth compact submanifolds and $\Psi: \mathcal{N}_1 \rightarrow \mathcal{N}_2$ be a smooth local isometry, that is, $\nabla\Psi(y): T_y\mathcal{N}_1 \rightarrow T_{\Psi(y)}\mathcal{N}_2$ is a linear isometry for all $y \in \mathcal{N}_1$. If $u_1 \in BV(\Omega; \mathcal{N}_1)$ for an open set $\Omega \subset \mathbb{R}^N$, then the map $u_2 = \Psi(u_1)$ belongs to $BV(\Omega; \mathcal{N}_2)$ and*

$$|D^a u_1| = |D^a u_2| \quad \text{and} \quad |D^c u_1| = |D^c u_2| \quad \text{as measures in } \Omega.$$

In particular, the above equality also holds in terms of partial derivatives in direction $\omega \in \mathbb{S}^{N-1}$, i.e., $|D_\omega^a u_1| = |D_\omega^a u_2|$ and $|D_\omega^c u_1| = |D_\omega^c u_2|$ as measures in Ω .

As a consequence of Lemma A.1, the claim (1) follows by setting $u_\ell = \Phi_\ell(u)$, $\mathcal{N}_\ell = \Phi_\ell(\mathcal{N})$, $\ell = 1, 2$ and $\Psi = \Phi_2 \circ \Phi_1^{-1}$.

Proof of Lemma A.1. One may extend Ψ to a 1-Lipschitz map $\tilde{\Psi}: \mathbb{R}^{D_1} \rightarrow \mathbb{R}^{D_2}$ in such a way that

$$\nabla\tilde{\Psi}(y) = \nabla\Psi(y)\Pi_{T_y\mathcal{N}_1} \quad \forall y \in \mathcal{N}_1, \quad (27)$$

where $\Pi_{T_y\mathcal{N}_1}$ denotes the orthogonal projection matrix on the tangent space $T_y\mathcal{N}_1$ in \mathbb{R}^{D_1} . By the chain rule [2, Theorem 3.96], as $\tilde{\Psi}$ is Lipschitz on \mathbb{R}^{D_1} , we have that $u_2 = \tilde{\Psi}(u_1)$ belongs to $BV(\Omega; \mathbb{R}^{D_2})$ with $u_2 \in \mathcal{N}_2$ a.e. in Ω and

$$\begin{aligned} D^a u_2 &= \nabla\tilde{\Psi}(u_1)\nabla u_1 \mathcal{L}^N, \\ D^c u_2 &= \nabla\tilde{\Psi}(u_1)g |D^c u_1|, \quad g := \frac{d(D^c u_1)}{d|D^c u_1|}. \end{aligned} \quad (28)$$

In particular, $D_\omega^a u_2 = \nabla\tilde{\Psi}(u_1)\nabla_\omega u_1 \mathcal{L}^N$ and $D_\omega^c u_2 = \nabla\tilde{\Psi}(u_1)(g \cdot \omega) |D^c u_1|$ as measures in Ω , for every direction $\omega \in \mathbb{S}^{N-1}$. The chain rule also implies that for any Lipschitz function $F: \mathbb{R}^{D_1} \rightarrow \mathbb{R}$ that vanishes on \mathcal{N}_1 (in particular, $F(u_1) = 0$ in Ω), it holds

$$\nabla F(u_1)\nabla u_1 = 0 \quad \mathcal{L}^N\text{-a.e.} \quad \text{and} \quad \nabla F(u_1)g = 0 \quad |D^c u_1|\text{-a.e.}$$

For any $z \in \mathcal{N}_1$ we may choose functions $\{F_k\}_{k=1, \dots, D_1 - \dim \mathcal{N}_1}$ vanishing on \mathcal{N}_1 and such that $\{\nabla F_k(z)\}$ spans the normal space of \mathcal{N}_1 at z . In particular, applying this to $z = u_1(x)$, we deduce that

$$\Pi_{T_{u_1}\mathcal{N}_1}\nabla u_1 = \nabla u_1 \quad \mathcal{L}^N\text{-a.e.} \quad \text{and} \quad \Pi_{T_{u_1}\mathcal{N}_1}g = g \quad |D^c u_1|\text{-a.e.} \quad (29)$$

Combining this with (27) and the fact that $\nabla\Psi(u_1)$ is an isometry on $T_{u_1}\mathcal{N}_1$, we deduce that

$$\left| D\tilde{\Psi}(u_1)\nabla u_1 \right| = |\nabla u_1| \quad \mathcal{L}^N\text{-a.e.} \quad \text{and} \quad \left| D\tilde{\Psi}(u_1)g \right| = |g| \quad |D^c u_1|\text{-a.e.},$$

(as well as $\left| D\tilde{\Psi}(u_1)\nabla_\omega u_1 \right| = |\nabla_\omega u_1| \mathcal{L}^N\text{-a.e.}$ and $\left| D\tilde{\Psi}(u_1)(g \cdot \omega) \right| = |g \cdot \omega| |D^c u_1|\text{-a.e.}$) which, recalling (28), implies the conclusion. \square

Proof of Proposition 1.3. By (1) we may fix the canonical embedding $\mathbb{S}^{d-1} \subset \mathbb{R}^d$, so that $n \in BV(\Omega; \mathbb{R}^d)$ satisfies $|n|^2 = 1$ a.e. We also fix an isometric smooth embedding $\Phi: \mathbb{R}\mathbb{P}^{d-1} \hookrightarrow \mathbb{R}^D$ and denote by $\bar{\Phi}: \mathbb{S}^{d-1} \rightarrow \mathbb{R}^D$ the induced symmetric map (i.e., $\bar{\Phi}(n) = \Phi([n])$ for every $n \in \mathbb{S}^{d-1}$) and we identify $\bar{\Phi}(\mathbb{S}^{d-1}) \simeq \mathbb{R}\mathbb{P}^{d-1}$. Then $\nabla\bar{\Phi}(n): T_n\mathbb{S}^{d-1} \rightarrow T_{\bar{\Phi}(n)}\bar{\Phi}(\mathbb{R}\mathbb{P}^{d-1})$ is a linear isometry for any $n \in \mathbb{S}^{d-1}$, and it holds $u = \bar{\Phi}(n)$ so we may apply Lemma A.1 to conclude that $|D^a u| = |D^a n|$ and $|D^c u| = |D^c n|$ as well as $|D_\omega^a u| = |D_\omega^a n|$ and $|D_\omega^c u| = |D_\omega^c n|$ as measures in Ω , for every direction $\omega \in \mathbb{S}^{N-1}$. \square

B Representation formula for the intrinsic BV -energy

In this part of the Appendix, we prove Theorem 1.5 which gives a representation formula for the intrinsic BV -energy $|u|_{BV, \mathcal{N}}$ for any compact submanifold $\mathcal{N} \subset \mathbb{R}^D$. In the case of scalar functions $u : \Omega \rightarrow \mathbb{R}$ this is proved in [8] (see also [19], [7]). A corresponding formula for $W^{1,p}$ for $p \geq 1$ maps with values into a metric space is proved in [16] and our proof is inspired by their methods.

Some notations: For $u \in BV(\Omega; \mathcal{N})$, we consider the following measures

$$m^\varepsilon = \left(\int_{\Omega} \frac{\text{dist}(u(x), u(y))}{|x-y|} \rho_\varepsilon(|x-y|) dy \right) dx \in \mathcal{M}(\Omega), \quad \text{for } \varepsilon > 0,$$

$$\mu_\omega = |\nabla_\omega u| \mathcal{L}^N + |D_\omega^c u| + |\omega \cdot \nu| \text{dist}(u^+, u^-) \mathcal{H}^{N-1} \llcorner J_u \in \mathcal{M}(\Omega), \quad \text{for } \omega \in \mathbb{S}^{N-1}.$$

Here ν denotes a unit normal vector to the rectifiable jump set J_u of u , while u^\pm are the traces of u along J_u relative to this normal vector ν . Moreover $\nabla_\omega u = (\nabla u)\omega$ is the approximate derivative of u in direction ω , and similarly $D_\omega^c u = (D^c u)\omega$ is the Cantor part of the distributional derivative of u in direction ω . By Alberti's rank one theorem, there exists an $\mathbb{S}^{D-1} \times \mathbb{S}^{N-1}$ -valued map (a, b) such that $D^c u = a \otimes b |D^c u|$. Hence, $D_\omega^c u = (\omega \cdot b) a |D^c u|$ and

$$\int_{\mathbb{S}^{N-1}} |D_\omega^c u|(\Omega) d\mathcal{H}^{N-1}(\omega) = |D^c u|(\Omega) \int_{\mathbb{S}^{N-1}} |\omega \cdot b| d\mathcal{H}^{N-1}(\omega) = K_N |D^c u|(\Omega).$$

Therefore, Theorem 1.5 amounts to prove that

$$\lim_{\varepsilon \rightarrow 0} m^\varepsilon(\Omega) = \int_{\mathbb{S}^{N-1}} \mu_\omega(\Omega) d\mathcal{H}^{N-1}(\omega). \quad (30)$$

As Ω is a Lipschitz bounded open set, by even reflection across the boundary $\partial\Omega$, we may extend u in a neighborhood of $\partial\Omega$ so that we may assume

$$u \in BV(\Omega_H; \mathcal{N}) \text{ for some } H > 0 \text{ and } |Du|(\partial\Omega) = 0$$

(see [2, Proposition 3.21]) where we denote by

$$\Omega_h = \{x \in \mathbb{R}^N : \text{dist}(x, \Omega) < h\} \quad \text{for any } h \in (0, H].$$

In the proof of (30) we use the following two lemmas:

Lemma B.1. *Let $u \in BV(\Omega; \mathcal{N})$. For any $\omega \in \mathbb{S}^{N-1}$, the measure $\mu_\omega \in \mathcal{M}(\Omega)$ is the least upper bound of the family of measures*

$$\{|D_\omega f_\xi|\}_{\xi \in \mathcal{N}}, \quad \text{where } f_\xi(x) = \text{dist}(u(x), \xi), \quad x \in \Omega, \quad \xi \in \mathcal{N},$$

i.e., on the one hand $|D_\omega f_\xi| \leq \mu_\omega$ as measures in Ω for every $\xi \in \mathcal{N}$, and on the other hand every measure $\sigma \in \mathcal{M}(\Omega)$ with $|D_\omega f_\xi| \leq \sigma$ in Ω for every $\xi \in \mathcal{N}$ satisfies $\mu_\omega \leq \sigma$. As a consequence,

$$\mu_\omega(\Omega) = \sup \left\{ \sum_i |D_\omega f_{\xi_i}|(U_i) \right\},$$

where the supremum is taken over all finite families $\{\xi_i\} \subset \mathcal{N}$ and $\{U_i\}$ of open subsets with pairwise disjoint compact closures $\bar{U}_i \subset \Omega$.

Lemma B.2. For $r \in (0, H)$, it holds

$$\int_{\Omega} \text{dist}(u(x+r\omega), u(x)) dx \leq r\mu_{\omega}(\Omega_r), \quad \forall \omega \in \mathbb{S}^{N-1}.$$

Proof of Lemma B.1. We will denote

$$\gamma_{\xi}(z) = \text{dist}(z, \xi) \quad \text{for all } \xi, z \in \mathcal{N}.$$

By the triangle inequality, γ_{ξ} is 1-Lipschitz on \mathcal{N} , so it can be extended to a Lipschitz function on \mathbb{R}^D such that $|\nabla \gamma_{\xi}| \leq 1$ on \mathcal{N} ; we still denote this extension by γ_{ξ} . By the chain rule applied to $u : \Omega \rightarrow \mathbb{R}^D$, we have

$$\begin{aligned} |D_{\omega} f_{\xi}| &= |\nabla \gamma_{\xi}(u) \cdot \nabla_{\omega} u| \mathcal{L}^N + |\nabla \gamma_{\xi}(u) \cdot D_{\omega}^c u| + |\omega \cdot \nu| |\gamma_{\xi}(u^+) - \gamma_{\xi}(u^-)| \mathcal{H}^{N-1} \llcorner J_u \\ &\leq |\nabla_{\omega} u| \mathcal{L}^N + |D_{\omega}^c u| + |\omega \cdot \nu| |\gamma_{\xi}(u^+) - \gamma_{\xi}(u^-)| \mathcal{H}^{N-1} \llcorner J_u \quad \text{as measures in } \Omega. \end{aligned}$$

It yields $|D_{\omega} f_{\xi}| \leq \mu_{\omega}$, $\forall \xi \in \mathcal{N}$ since $|\gamma_{\xi}(u^+) - \gamma_{\xi}(u^-)| \leq \text{dist}(u^+, u^-)$ by triangle inequality.

We now show that any measure σ such that $|D_{\omega} f_{\xi}| \leq \sigma$ for all $\xi \in \mathcal{N}$ must satisfy $\mu_{\omega} \leq \sigma$. Let σ be such a measure. Then, letting

$$g = \frac{d(D_{\omega}^c u)}{d|D_{\omega}^c u|}, \quad \sigma^a = \frac{d\sigma}{d\mathcal{L}^N}, \quad \sigma^c = \frac{d\sigma}{d|D_{\omega}^c u|}, \quad \sigma^j = \frac{d\sigma}{d\mathcal{H}^{N-1} \llcorner J_u},$$

we have for all $\xi \in \mathcal{N}$:

$$\begin{aligned} \sigma^a(x) &\geq |\nabla \gamma_{\xi}(u(x)) \cdot \nabla_{\omega} u(x)| && \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \\ \sigma^c(x) &\geq |\nabla \gamma_{\xi}(u(x)) \cdot g(x)| && \text{for } |D_{\omega}^c u|\text{-a.e. } x \in \Omega, \\ \sigma^j(x) &\geq |\omega \cdot \nu| |\gamma_{\xi}(u^+(x)) - \gamma_{\xi}(u^-(x))| && \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in J_u. \end{aligned}$$

Choosing $\xi = u^-(x)$ in the last inequality gives

$$\sigma^j(x) \geq |\omega \cdot \nu| \text{dist}(u^+(x), u^-(x)) \quad \text{for } \mathcal{H}^{N-1}\text{-a.e. } x \in J_u. \quad (31)$$

To use the first two inequalities we remark that given any unit vector $v \in T_{u(x)}\mathcal{N}$, choosing $\xi = \exp_{u(x)}(tv)$ for a small enough $t > 0$ we have $\nabla \gamma_{\xi}(u(x)) = -v$. Therefore, taking the supremum over all $\xi \in \mathcal{N}$, we deduce that

$$\begin{aligned} \sigma^a(x) &\geq \left| \Pi_{T_{u(x)}\mathcal{N}} \nabla_{\omega} u(x) \right| && \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \\ \sigma^c(x) &\geq \left| \Pi_{T_{u(x)}\mathcal{N}} g(x) \right| && \text{for } |D_{\omega}^c u|\text{-a.e. } x \in \Omega, \end{aligned}$$

where $\Pi_{T_{u(x)}\mathcal{N}}$ is the projection matrix on the tangent space $T_{u(x)}\mathcal{N}$. Recall by (29) (in the proof of Lemma A.1) that $\nabla_{\omega} u(x) \in T_{u(x)}\mathcal{N}$ for \mathcal{L}^N -a.e. $x \in \Omega$ and $g(x) \in T_{u(x)}\mathcal{N}$ for $|D_{\omega}^c u|\text{-a.e. } x \in \Omega$. Hence the above becomes

$$\begin{aligned} \sigma^a(x) &\geq |\nabla_{\omega} u(x)| && \text{for } \mathcal{L}^N\text{-a.e. } x \in \Omega, \\ \sigma^c(x) &\geq |g(x)| && \text{for } |D_{\omega}^c u|\text{-a.e. } x \in \Omega. \end{aligned}$$

Combining this with σ^j and the fact that \mathcal{L}^N , $|D_{\omega}^c u|$ and $\mathcal{H}^{N-1} \llcorner J_u$ are mutually singular, we deduce that $\sigma \geq \mu_{\omega}$.

The last statement of the lemma is a consequence of the properties of the least upper bound of a family of measures (see e.g. [2, Definition 1.68]) and the inner regularity of the measures $|D_{\omega} f_{\xi}|$. \square

Proof of Lemma B.2. This is the equivalent of Lemma 2.2 in [16]; for completeness, we present the proof. We use the properties of one-dimensional restrictions of BV functions as presented in [2, Section 3.11]. We denote by Ω_ω the orthogonal projection of Ω onto the hyperplane ω^\perp , and define for any $y \in \Omega_\omega$ the one dimensional section $\Omega_y^\omega = \{t \in \mathbb{R} : y + t\omega \in \Omega\}$. Given $f \in BV(\Omega; \mathbb{R})$, then for almost every $y \in \Omega_\omega$ the one-dimensional restriction f_y^ω given by $f_y^\omega(t) = f(y + t\omega)$ is BV in Ω_y^ω , and the measure $D_\omega f$ can be disintegrated as

$$D_\omega f = \mathcal{L}^{N-1} \llcorner \Omega_\omega \otimes Df_y^\omega.$$

Applying this to the functions f_ξ for ξ in a dense countable subset $\mathcal{D} \subset \mathcal{N}$, we write

$$D_\omega f_\xi = \mathcal{L}^{N-1} \llcorner \Omega_\omega \otimes D(f_\xi)_y^\omega.$$

Since the set \mathcal{D} is countable, the negligible set of $y \in \Omega_\omega$ outside which $D(f_\xi)_y^\omega$ is not well defined can be taken independent of $\xi \in \mathcal{D}$. Moreover it is easily checked that Lemma B.1 characterizing μ_ω as lowest upper bound of the family $\{|D_\omega f_\xi|\}_{\xi \in \mathcal{N}}$ still holds when restricting this family to the dense subset $\xi \in \mathcal{D}$. Thus we infer a corresponding disintegration of μ_ω as

$$\mu_\omega = \mathcal{L}^{N-1} \llcorner \Omega_\omega \otimes (\mu_\omega)_y^\omega,$$

and the measure $(\mu_\omega)_y^\omega$ defined on Ω_y^ω satisfies $(\mu_\omega)_y^\omega \geq |D(f_\xi)_y^\omega|$. Note also that similar disintegration formulas hold with Ω replaced by Ω_r , for $r \in (0, H)$.

For almost all $y \in \Omega_\omega$ and $t \in \Omega_y^\omega$, and for all $\xi \in \mathcal{D}$ we have

$$|f_\xi(y + t\omega + r\omega) - f_\xi(y + t\omega)| \leq |D(f_\xi)_y^\omega|([t, t+r]) \leq (\mu_\omega)_y^\omega([t, t+r]),$$

since $[y + t\omega, y + t\omega + r\omega] \subset \Omega_r$. Applying this for ξ arbitrarily close to $u(y + t\omega)$ yields

$$\text{dist}(u(y + t\omega + r\omega), u(y + t\omega)) \leq (\mu_\omega)_y^\omega([t, t+r]).$$

Hence we obtain

$$\begin{aligned} \int_\Omega \text{dist}(u(x + r\omega), u(x)) dx &= \int_{\Omega_\omega} \int_{\Omega_y^\omega} \text{dist}(u(y + t\omega + r\omega), u(y + t\omega)) dt dy \\ &\leq \int_{\Omega_\omega} \int_{\Omega_y^\omega} (\mu_\omega)_y^\omega([t, t+r]) dt dy. \end{aligned}$$

Remarking that

$$\begin{aligned} \int_{\Omega_y^\omega} (\mu_\omega)_y^\omega([t, t+r]) dt &= \int_{\mathbb{R}} \int_{\Omega_y^\omega} \mathbf{1}_{t \leq z < t+r} dt d(\mu_\omega)_y^\omega(z) = \int_{\mathbb{R}} \int_{-r}^0 \mathbf{1}_{s \in \Omega_y^\omega - z} ds d(\mu_\omega)_y^\omega(z) \\ &= \int_{-r}^0 (\mu_\omega)_y^\omega(\Omega_y^\omega - s) ds = \int_0^r (\mu_\omega)_y^\omega(\Omega_y^\omega + s) ds, \end{aligned}$$

we conclude

$$\begin{aligned} \int_\Omega \text{dist}(u(x + r\omega), u(x)) dx &\leq \int_{\Omega_\omega} \int_0^r (\mu_\omega)_y^\omega(\Omega_y^\omega + s) ds dy \\ &\leq r \int_{\Omega_\omega} (\mu_\omega)_y^\omega((\Omega_r)_y^\omega) dy \\ &\leq r \int_{(\Omega_r)_\omega} (\mu_\omega)_y^\omega((\Omega_r)_y^\omega) dy = r \mu_\omega(\Omega_r). \end{aligned}$$

□

Proof of Theorem 1.5. As outlined above it suffices to prove (30).

*Step 1. Proof of the inequality “ \leq ” in (30).*⁶ We follow the ideas in [16]. Denoting the diameter of the compact manifold \mathcal{N} by $\text{diam } \mathcal{N} = \sup\{\text{dist}(u, w) : u, w \in \mathcal{N}\}$, it holds for any $h \in (0, H)$ and any $\varepsilon > 0$:

$$\begin{aligned}
m^\varepsilon(\Omega) &\leq \int_{x \in \Omega} \left(\int_{|z| \leq h} + \int_{|z| \geq h, z+x \in \Omega} \right) \frac{\text{dist}(u(x), u(x+z))}{|z|} \rho_\varepsilon(|z|) dz dx \\
&\leq \int_{\mathbb{S}^{N-1}} \int_0^h \frac{1}{r} \int_{\Omega} \text{dist}(u(x), u(x+r\omega)) dx \rho_\varepsilon(r) r^{N-1} dr d\mathcal{H}^{N-1}(\omega) \\
&\quad + \frac{\text{diam}(\mathcal{N}) \mathcal{H}^N(\Omega)}{h} \int_{|z| \geq h} \rho_\varepsilon(|z|) dz \\
&\leq \int_{\mathbb{S}^{N-1}} \mu_\omega(\Omega_h) d\mathcal{H}^{N-1}(\omega) + \frac{\text{diam}(\mathcal{N}) \mathcal{H}^N(\Omega)}{h} \int_{|z| \geq h} \rho_\varepsilon(|z|) dz, \tag{32}
\end{aligned}$$

where we used Lemma B.2 and the fact that $\int_0^h \rho_\varepsilon(r) r^{N-1} dr \leq \frac{1}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})}$. By the definition of mollifiers, we deduce by passing at the limsup $\varepsilon \downarrow 0$:

$$\limsup_{\varepsilon \rightarrow 0} m^\varepsilon(\Omega) \leq \int_{\mathbb{S}^{N-1}} \mu_\omega(\Omega_h) d\mathcal{H}^{N-1}(\omega).$$

Finally, passing at the limit $h \downarrow 0$, as $\mu_\omega(\Omega_h) \in L^1(\omega \in \mathbb{S}^{N-1})$ (because $u \in BV(\Omega_H; \mathcal{N})$), we conclude by the monotone convergence theorem

$$\limsup_{\varepsilon \rightarrow 0} m^\varepsilon(\Omega) \leq \lim_{h \downarrow 0} \int_{\mathbb{S}^{N-1}} \mu_\omega(\Omega_h) d\mathcal{H}^{N-1}(\omega) = \int_{\mathbb{S}^{N-1}} \mu_\omega(\overline{\Omega}) d\mathcal{H}^{N-1}(\omega).$$

Recalling that $|Du|(\partial\Omega) = 0$ hence $\mu_\omega(\partial\Omega) = 0$ for every $\omega \in \mathbb{S}^{N-1}$, we obtain the upper bound in (30).

Step 2. Proof of the inequality “ \geq ” in (30). Let $\omega \in \mathbb{S}^{N-1}$. In the following we will use Lemma B.1. For that, we fix a finite family of directions $\{\xi_i\} \subset \mathcal{N}$ and a finite family $\{U_i\}$ of open subsets with pairwise disjoint compact closures $\overline{U}_i \subset \Omega$ (in particular, $\text{dist}(U_i, \partial\Omega) > 0$). For every i , let $\varphi_i \in C_c^\infty(U_i)$ with $|\varphi_i| \leq 1$. Recalling that $\gamma_\xi(z) = \text{dist}(z, \xi)$ for $\xi, z \in \mathcal{N}$, it holds for any $\varepsilon > 0$ and $h \in (0, H \wedge \min_i \text{dist}(U_i, \partial\Omega) \wedge \min_i \text{dist}(\text{supp } \varphi_i, \partial U_i))$:

$$\begin{aligned}
\langle D_\omega f_{\xi_i}, \varphi_i \rangle &= \int_{U_i} (\omega \cdot \nabla \varphi_i) \gamma_{\xi_i}(u) dx = \int_{U_i} \int_{z \in \mathbb{R}^N} (\omega \cdot \nabla \varphi_i) \rho_\varepsilon(|z|) dz \gamma_{\xi_i}(u) dx \\
&= \int_{U_i} \int_{|z| \leq h} \frac{\varphi_i(x+|z|\omega) - \varphi_i(x)}{|z|} \rho_\varepsilon(|z|) dz \gamma_{\xi_i}(u(x)) dx \\
&\quad - \int_{U_i} \int_{|z| \leq h} \left(\frac{\varphi_i(x+|z|\omega) - \varphi_i(x)}{|z|} - \omega \cdot \nabla \varphi_i(x) \right) \rho_\varepsilon(|z|) dz \gamma_{\xi_i}(u(x)) dx \\
&\quad + \int_{U_i} \int_{|z| \geq h} \omega \cdot \nabla \varphi_i(x) \rho_\varepsilon(|z|) dz \gamma_{\xi_i}(u(x)) dx \\
&=: I + II + III. \tag{33}
\end{aligned}$$

⁶We use only at this Step 1 the assumption of Ω being Lipschitz.

Treating the term I . As $h < \text{dist}(\text{supp } \varphi_i, \partial U_i)$, then $\varphi_i \equiv 0$ on $U_i \setminus (U_i + r\omega)$ for every $r \in (0, h)$ so that we have

$$\begin{aligned}
\frac{I}{\mathcal{H}^{N-1}(\mathbb{S}^{N-1})} &= \int_{U_i} \int_0^h \frac{\varphi_i(x + r\omega) - \varphi_i(x)}{r} \rho_\varepsilon(r) r^{N-1} dr \gamma_{\xi_i}(u(x)) dx \\
&= \int_0^h \rho_\varepsilon(r) r^{N-1} \frac{1}{r} \left[\int_{U_i} \varphi_i(x + r\omega) \gamma_{\xi_i}(u(x)) dx - \int_{U_i} \varphi_i(x) \gamma_{\xi_i}(u(x)) dx \right] dr \\
&= \int_0^h \rho_\varepsilon(r) r^{N-1} \frac{1}{r} \left[\int_{U_i + r\omega} \varphi_i(x) \gamma_{\xi_i}(u(x - r\omega)) dx - \int_{U_i} \varphi_i(x) \gamma_{\xi_i}(u(x)) dx \right] dr \\
&= \int_0^h \rho_\varepsilon(r) r^{N-1} \frac{1}{r} \left[\int_{U_i \cap (U_i + r\omega)} \varphi_i(x) (\gamma_{\xi_i}(u(x - r\omega)) - \gamma_{\xi_i}(u(x))) dx \right] dr \\
&= \int_{U_i} \varphi_i(x) \int_0^h \frac{\gamma_{\xi_i}(u(x - r\omega)) - \gamma_{\xi_i}(u(x))}{r} \rho_\varepsilon(r) r^{N-1} dr dx.
\end{aligned}$$

The triangle inequality implies $|\gamma_{\xi_i}(u(x - r\omega)) - \gamma_{\xi_i}(u(x))| \leq \text{dist}(u(x - r\omega), u(x))$, which combined with $|\varphi_i| \leq 1$ and the fact that $U_i - h\omega \subset \Omega$, yield:

$$|I| \leq \mathcal{H}^{N-1}(\mathbb{S}^{N-1}) \int_{U_i} \int_0^h \frac{\text{dist}(u(x - r\omega), u(x))}{r} \rho_\varepsilon(r) r^{N-1} dr dx \leq m_\omega^\varepsilon(U_i), \quad (34)$$

where m_ω^ε is the following positive measure on Ω of density

$$\frac{dm_\omega^\varepsilon}{dx} : x \in \Omega \mapsto \mathcal{H}^{N-1}(\mathbb{S}^{N-1}) \int_{r>0, x-r\omega \in \Omega} \frac{\text{dist}(u(x), u(x - r\omega))}{r} \rho_\varepsilon(r) r^{N-1} dr.$$

Treating the term II . As $|\varphi_i(x + r\omega) - \varphi_i(x) - r\omega \cdot \nabla \varphi_i(x)| \leq \frac{r^2}{2} \|\nabla^2 \varphi_i\|_{L^\infty}$, we deduce that

$$|II| \leq \frac{h}{2} \text{diam}(\mathcal{N}) \mathcal{H}^N(\Omega) \|\nabla^2 \varphi_i\|_{L^\infty}. \quad (35)$$

Treating the term III . We have

$$|III| \leq \text{diam}(\mathcal{N}) \mathcal{H}^N(\Omega) \|\nabla \varphi_i\|_{L^\infty} \int_{|z| \geq h} \rho_\varepsilon(|z|) dz. \quad (36)$$

Conclusion. By (33)-(36), passing first to \liminf as $\varepsilon \rightarrow 0$ and then second to the limit $h \rightarrow 0$, we deduce that $|\langle D_\omega f_{\xi_i}, \varphi_i \rangle| \leq \liminf_{\varepsilon \rightarrow 0} m_\omega^\varepsilon(U_i)$; moreover, taking the supremum over all φ_i , this entails $|D_\omega f_{\xi_i}|(U_i) \leq \liminf_{\varepsilon \rightarrow 0} m_\omega^\varepsilon(U_i)$. Since the open sets U_i have pairwise disjoint closures in Ω this implies

$$\sum_i |D_\omega f_{\xi_i}|(U_i) \leq \sum_i \liminf_{\varepsilon \rightarrow 0} m_\omega^\varepsilon(U_i) \leq \liminf_{\varepsilon \rightarrow 0} \sum_i m_\omega^\varepsilon(U_i) \leq \liminf_{\varepsilon \rightarrow 0} m_\omega^\varepsilon(\Omega).$$

Now taking the supremum over all finite families $\{\xi_i\}$ and $\{U_i\}$ we deduce by Lemma B.1:

$$\mu_\omega(\Omega) \leq \liminf_{\varepsilon \rightarrow 0} m_\omega^\varepsilon(\Omega).$$

Then Fatou's lemma implies

$$\int_{\mathbb{S}^{N-1}} \mu_\omega(\Omega) d\mathcal{H}^{N-1}(\omega) \leq \liminf_{\varepsilon \rightarrow 0} \int_{\mathbb{S}^{N-1}} m_\omega^\varepsilon(\Omega) d\mathcal{H}^{N-1}(\omega) \leq \liminf_{\varepsilon \rightarrow 0} m^\varepsilon(\Omega),$$

where we used

$$\int_{\mathbb{S}^{N-1}} m_\omega^\varepsilon(\Omega) d\mathcal{H}^{N-1}(\omega) = m^\varepsilon(\Omega).$$

Steps 1 and 2 prove in particular that $\lim_{\varepsilon \rightarrow 0} m^\varepsilon(\Omega)$ exists and is given by (30). \square

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