Global uniform estimate for the modulus of $2D$ Ginzburg-Landau vortexless solutions with asymptotically infinite boundary energy

Radu Ignat* Matthias Kurzke† Xavier Lamy ‡

October 28, 2019

Abstract

For $\varepsilon > 0$, let $u_\varepsilon : \Omega \to \mathbb{R}^2$ be a solution of the Ginzburg-Landau system

$$-\Delta u_\varepsilon = \frac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2)$$

in a Lipschitz bounded domain Ω. In an energy regime that excludes interior vortices, we prove that $1 - |u_\varepsilon|$ is uniformly estimated by a positive power of ε **globally** in Ω provided that the energy of u_ε at the boundary $\partial\Omega$ does not grow faster than $\varepsilon^{-\alpha}$ with $\alpha \in (0, 1)$.

1 Introduction

Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded open connected set (not necessarily simply connected) with the unit outer normal and tangent vector fields (ν, τ) defined a.e. on $\partial\Omega$ with

$$\tau = \nu^\perp = (-\nu_2, \nu_1)$$

so that (ν, τ) forms an oriented frame a.e. on $\partial\Omega$. For every small $\varepsilon > 0$, let $u_\varepsilon : \Omega \to \mathbb{R}^2$ be a solution of the Ginzburg-Landau system:

$$\begin{cases} -\Delta u_\varepsilon = \dfrac{1}{\varepsilon^2} u_\varepsilon (1 - |u_\varepsilon|^2) & \text{in } \Omega, \\ u_\varepsilon = g_\varepsilon & \text{on } \partial\Omega \end{cases} \tag{1}$$

with the boundary data $g_\varepsilon : \partial\Omega \to \mathbb{R}^2$. For the boundary energy

$$N_\varepsilon := \int_{\partial\Omega} \frac{1}{2} |\partial_\tau g_\varepsilon|^2 + \frac{1}{4\varepsilon^2} (1 - |g_\varepsilon|^2)^2 \, d\mathcal{H}^1 \tag{2}$$

and the interior energy

$$M_\varepsilon := \int_\Omega \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} (1 - |u_\varepsilon|^2)^2 \, dx, \tag{3}$$

*Institut de Mathématiques de Toulouse & Institut Universitaire de France, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Radu.Ignat@math.univ-toulouse.fr

†School of Mathematical Sciences, University of Nottingham, Nottingham NG7 2RD, United Kingdom. Email: matthias.kurzke@nottingham.ac.uk

‡Institut de Mathématiques de Toulouse, UMR 5219, Université de Toulouse, CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France. Email: Xavier.Lamy@math.univ-toulouse.fr

we assume that there exists a power $\alpha \in (0,1)$ such that [1]

$$M_\varepsilon \leq \kappa |\log \varepsilon| \quad \text{and} \quad N_\varepsilon \ll \frac{1}{\varepsilon^\alpha} \quad \text{as } \varepsilon \to 0, \tag{4}$$

for some small constant $\kappa > 0$ depending on the Lipschitz regularity of Ω. The first condition in (4) avoids nucleation of interior vortices of non-vanishing winding number (because the energetic cost of an interior vortex of non-zero winding number is of order $|\log \varepsilon|$, see the seminal book of Bethuel-Brezis-Hélein [4]). The second condition in (4) corresponds to an energetic regime avoiding the presence of boundary vortices: indeed, a transition of g_ε between two opposite directions at the boundary on a distance ε (the length scale of a vortex) has an energetic cost of order $\frac{1}{\varepsilon}$ (see Example 1 below). If $N_\varepsilon \lesssim \frac{1}{\varepsilon}$, then solutions u_ε of (1) may have zeros on the boundary (see Proposition 3).

1.1 Main result

Our main result is the following global uniform estimate in the regime (4) for the convergence of $|u_\varepsilon|$ to 1 in Ω, which means that $1 - |u_\varepsilon|$ behaves as a positive power of ε.

Theorem 1 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. There exists a (small) constant $\kappa > 0$ depending on the Lipschitz regularity of Ω such that for every solution u_ε of (1) satisfying (4) for some $\alpha \in (0,1)$ we have the following global estimate* [2]

$$\sup_\Omega \big||u_\varepsilon| - 1\big| \leq C\left(\varepsilon^{1-}(1 + N_\varepsilon + M_\varepsilon)(1 + M_\varepsilon)^{\frac{1}{2}-}\right)^{\frac{1}{6}-} \quad \text{as} \quad \varepsilon \to 0,$$

for some constant $C > 0$ depending only on the Lipschitz regularity [3] *of Ω. In particular, g_ε has zero winding number on $\partial\Omega$, i.e.,*[4]

$$\deg(g_\varepsilon, \partial\Omega) := \frac{1}{2\pi} \int_{\partial\Omega} \frac{g_\varepsilon^\perp}{|g_\varepsilon|} \cdot \partial_\tau \left(\frac{g_\varepsilon}{|g_\varepsilon|}\right) d\mathcal{H}^1 = 0.$$

In particular, under the assumption of Theorem 1, we deduce that

$$\sup_\Omega \big||u_\varepsilon| - 1\big| \leq C\varepsilon^{\frac{1-\alpha}{6}-} \quad \text{as} \quad \varepsilon \to 0.$$

We believe that the power $\frac{1}{6}-$ of ε in the above estimate is not optimal; moreover, the optimal power of ε is expected to be $\leq \frac{1}{2}$ (see (9) below). The proof of Theorem 1 is done in several steps. In Section 2, we obtain a preliminary estimate of the uniform convergence of $|u_\varepsilon|$ to 1, but at a much slower rate than the one claimed in Theorem 1. Thanks to this preliminary estimate, in Section 3, we will be able to use more efficiently the Ginzburg-Landau system (1) to deduce an improved rate for the convergence of $|u_\varepsilon|$ to 1, first in the L^2-norm and then in the L^∞-norm.

Remark 1 Our proof adapts with minor modifications to critical points of the energy

$$E_\varepsilon(u_\varepsilon; \Omega) := \int_\Omega \frac{1}{2} |\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2} F(|u|^2)\, dx, \tag{5}$$

where $F \in C^2([0,\infty))$ satisfies $F \geq 0$, $F(1) = 0$ and $(s-1)F'(s) \geq c(1-s)^2$ for all $s \geq 0$ and some constant $c > 0$. The typical example is $F(s) = (1-s)^2$.

[1] We write $a \ll b$ if $\frac{a}{b} \to 0$, and $a \lesssim b$ if $\frac{a}{b}$ is bounded by a universal constant.

[2] We denote by $a+$ (resp. $a-$) any number strictly bigger than a (resp. strictly smaller than a) that one can think of as close to a. The constants in inequalities involving $a+$ or $a-$ may depend on the choice of these numbers.

[3] In fact, $C > 0$ depends only on the lowest angle and lowest radius of interior and exterior cones at any point of the Lipschitz boundary $\partial\Omega$.

[4] In general, $\partial\Omega$ is not connected; the definition of the degree is coherent with the choice of the orientation $\tau = \nu^\perp$ given by the outer normal field ν.

1.2 Optimality of our assumptions

Let us discuss the optimality of the assumption (4) in Theorem 1. First, the assumption on M_ε in (4) is optimal: if the constant κ is not small enough, then solutions u_ε of (1) may vanish inside Ω. Moreover, the threshold value of κ at which this happens can be arbitrarily small depending on the Lipschitz regularity of the domain:

Proposition 2 *For any $\theta_0 \in (0, \pi)$ and any $\eta > 0$ there exists a cone shape domain Ω of opening angle θ_0, an exponent $\alpha \in (0, 1)$ and a solution u_ε of the Ginzburg-Landau system (1) such that for small $\varepsilon > 0$, $u_\varepsilon(P_\varepsilon) = 0$ for a degree-one vortex point $P_\varepsilon \in \Omega$ and (4) holds true for $\kappa = \frac{\theta_0}{2} + \eta$.*

Second, the assumption on N_ε in (4) is near-optimal in the following sense: if $N_\varepsilon \lesssim \frac{1}{\varepsilon}$, then a solution u_ε of (1) may have zeros at the boundary of any Lipschitz domain Ω.

Proposition 3 *For any Lipschitz domain Ω, there exists a solution u_ε of the Ginzburg-Landau system (1) such that for small $\varepsilon > 0$, $u_\varepsilon(x_0) = 0$ for some $x_0 \in \partial\Omega$, while $M_\varepsilon \lesssim 1$ and $N_\varepsilon \lesssim \frac{1}{\varepsilon}$.*

We also point out that solutions of (1) with $M_\varepsilon \le \kappa\,|\log\varepsilon|$ with small κ may present an interior vortex with non-zero degree if the boundary energy is of order $\frac{1}{\varepsilon}$.

Proposition 4 *For any smooth simply connected domain Ω and any small $\eta = \eta(\varepsilon) > 0$, there exists a solution u_η of the Ginzburg-Landau system (1) such that $|u_\eta| = 1$ on $\partial\Omega$ and $\deg(u_\eta, \partial\Omega) = 1$ (in particular, u_η has a vortex on non-vanishing degree in Ω), while $M_\varepsilon \lesssim \left(\frac{\eta}{\varepsilon}\right)^2 |\log\eta|$ and $N_\varepsilon \lesssim \frac{1}{\eta}$. In particular,*

- *we can choose $\eta = \eta(\varepsilon) > 0$ such that $M_\varepsilon \lesssim 1$ and $N_\varepsilon \lesssim \frac{|\log\varepsilon|^{1/2}}{\varepsilon}$;*
- *for every small $\kappa > 0$, we can choose $\eta = \eta(\varepsilon, \kappa) > 0$ such that $M_\varepsilon \le \kappa|\log\varepsilon|$ and $N_\varepsilon \lesssim \frac{1}{\varepsilon}$.*

Finally we remark that even for $\mathbb{S}^1$-valued boundary data with zero degree, if $N_\varepsilon \gg \frac{1}{\varepsilon}$ then minimizers may have bounded energy but modulus **not uniformly close to** 1. (This is related to Example 1 in Section 1.4 below.)

Proposition 5 *For any smooth bounded domain Ω and $\eta(\varepsilon) \ll \varepsilon \ll 1$, there exists $g_\varepsilon \in H^1(\partial\Omega; \mathbb{S}^1)$ with $\deg(g_\varepsilon, \partial\Omega) = 0$ and[5] $N_\varepsilon \sim \frac{1}{\eta(\varepsilon)}$, such that any minimizer u_ε of $E_\varepsilon(\cdot; \Omega)$ under the Dirichlet boundary condition $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$ satisfies*

$$\sup_{\Omega} \left|1 - |u_\varepsilon|^2\right| \ge \frac{1}{2} \qquad \text{for } \varepsilon \ll 1,$$

while $M_\varepsilon \lesssim 1$.

The proofs of Propositions 2 to 5 can be found in Section 4. The case $N_\varepsilon \ll \frac{1}{\varepsilon}$ (i.e., $\alpha = 1$ in the regime (4)) remains open; in that case, we conjecture that our global estimate in Theorem 1 should still hold true, at least in smooth domains.

1.3 Related works

There is a huge literature on the analysis of solutions u_ε of the Ginzburg-Landau system (1). Let us only mention some of them (and apologize for omitting many other important ones).

In the seminal paper [3], Bethuel, Brezis and Hélein studied the system (1) on a smooth simply connected domain Ω for minimizers u_ε of the associated energy functional, with a fixed smooth boundary data $g_\varepsilon := g$ such that $|g| = 1$ on $\partial\Omega$ and g is of zero winding number (so $N_\varepsilon, M_\varepsilon$ are of

[5] We write $a \sim b$ if $a \lesssim b$ and $b \lesssim a$.

order 1); they proved that $\big||u_\varepsilon|-1\big|$ behaves as ε^2 **globally** in Ω and this rate is optimal. They also studied the case of non-fixed smooth boundary data $g_\varepsilon : \partial\Omega \to \mathbb{R}^2$ that is of zero winding number and has uniformly bounded energy $N_\varepsilon \lesssim 1$; then for minimizers u_ε, they deduced that $M_\varepsilon \lesssim 1$ and $\big||u_\varepsilon|-1\big|$ behaves as ε^2 **locally** in Ω. These results also hold for non-minimizing solutions if $u_\varepsilon \to u_0$ strongly in H^1 for some limit u_0, see [4, Remark A.1].

In [5], Bethuel, Orlandi and Smets considered solutions of (1) that need not be minimizing, without imposing any bounds on M_ε or N_ε. They proved local estimates on $\big||u_\varepsilon|-1\big|$, away from the boundary and from a vorticity set. In our setting, their results imply that $\big||u_\varepsilon|-1\big|$ is of order at most $\varepsilon^{2(1-\beta)}M_\varepsilon$ in the region $\{x \in \Omega\colon \operatorname{dist}(x, \partial\Omega) \geq \varepsilon^\beta\}$, for any $\beta \in (0,1)$, but do not provide a good uniform estimate up to the boundary.

In the present work we focus on obtaining, for solutions of (1) that need not be minimizing, precise uniform estimates on $\big||u_\varepsilon|-1\big|$ which hold:

- up to the boundary $\partial\Omega$ of a general Lipschitz domain,
- and in a regime that goes beyond the restrictive uniform bound $N_\varepsilon \lesssim 1$.

Estimates up to the boundary of a rectangle were obtained in [7, Appendix] in the regime $M_\varepsilon, N_\varepsilon \ll |\log\varepsilon|$. There it was proved that $\big||u_\varepsilon|-1\big|$ is of order at most $(\frac{1+N_\varepsilon+M_\varepsilon}{|\log\varepsilon|})^{\frac{1}{6}-}$ globally in Ω. In Section 2 we will follow the same approach in a general Lipschitz domain and under the less restrictive regime (4), as a first step towards the stronger estimate of Theorem 1.

1.4 Motivation

The energy functional E_ε is a simplified version of a model describing superconducting materials. We simply mention here that $\big||u_\varepsilon|-1\big|$ measures how close the system is to a superconducting state, and refer the interested reader to the monographs [4, 16].

Another motivation comes from several studies of the pattern formation in thin ferromagnetic films [11, 7, 10], where one wishes to approximate u_ε by $\mathbb{S}^1$-valued maps away from the vortices. In a vortexless region Ω (assume e.g. $E_\varepsilon(u_\varepsilon;\Omega) \ll |\log\varepsilon|$), the idea introduced in [11] consists in finding a (squared, spherical etc.) grid $\mathcal{R}_\varepsilon$, each cell of the grid having the size $\sim \varepsilon^\beta$ with $\beta \in (0,1)$ (i.e., much larger than the length-scale of a vortex) such that the energy $E_\varepsilon(u_\varepsilon;\mathcal{R}_\varepsilon)$ on the 1-dimensional grid $\mathcal{R}_\varepsilon$ is of order $E_\varepsilon(u_\varepsilon;\Omega)/\varepsilon^\beta$. Then Theorem 1 implies that $\big||u_\varepsilon|-1\big|$ behaves as a positive power of ε in Ω, and $v_\varepsilon = u_\varepsilon/|u_\varepsilon|$ is a "good" approximation of u_ε (in terms of the L^2 norm, their global Jacobian etc., see [10]). In that context, we give the following consequence of Theorem 1 for a cell of the grid leading to a key estimate needed in [10] (only a weaker version of this key estimate was needed in [11, 7]):

Corollary 6 *Let $\mathcal{C} \subset \mathbb{R}^2$ be a Lipschitz bounded domain. Let $\varepsilon > 0$, $\beta \in (0,1)$ and $\mathcal{C}_\varepsilon := \varepsilon^\beta\mathcal{C}$ be a cell of size ε^β. Assume that u_ε is a solution of (1) in $\mathcal{C}_\varepsilon$ with*

$$\int_{\partial\mathcal{C}_\varepsilon} \frac{1}{2}|\partial_\tau g_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|g_\varepsilon|^2)^2 \, d\mathcal{H}^1 \ll \frac{|\log\varepsilon|}{\varepsilon^\beta}$$

and

$$\int_{\mathcal{C}_\varepsilon} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2 \, dx \ll |\log\varepsilon|,$$

then

$$\big||u_\varepsilon|-1\big| \leq C\varepsilon^{\frac{1-\beta}{6}-} \quad in \quad \mathcal{C}_\varepsilon,$$

for some $C > 0$ depending on the Lipschitz regularity of $\mathcal{C}$. In particular, g_ε has zero winding number on $\mathcal{C}_\varepsilon$.

Proof. Denoting the rescaled map $\tilde{u}_{\tilde{\varepsilon}}(\tilde{x}) := u_\varepsilon(\varepsilon^\beta \tilde{x})$ for $\tilde{x} \in \mathcal{C}$ with $\tilde{\varepsilon} := \varepsilon^{1-\beta}$, then $\tilde{u}_{\tilde{\varepsilon}}$ satisfies the system (1) with the parameter $\tilde{\varepsilon}$ instead of ε and the boundary energy, resp. interior energy of $\tilde{u}_{\tilde{\varepsilon}}$ on $\partial\mathcal{C}$, resp. in $\mathcal{C}$ are estimated by $N_{\tilde{\varepsilon}}, M_{\tilde{\varepsilon}} \ll |\log \tilde{\varepsilon}|$. By Theorem 1, the conclusion follows. □

As already hinted at, the regime (4) is motivated by the study of boundary vortices (see e.g. [14, 10]). In the absence of interior vortices, the first nontrivial example corresponds to a dipole of two compensating boundary vortices.

Example 1 Let $\Omega \subset \mathbb{R}^2$ be a bounded smooth domain containing the upper half unit ball, more precisely,

$$\Omega \cap B(0,1) = \{x = (x_1, x_2) \in B(0,1) \, : \, x_2 > 0\},$$

where $B(0,1)$ is the unit ball centered at the origin. Let $\eta = \eta(\varepsilon) \in (0,1)$ be a parameter. Consider the boundary data $g_\varepsilon : \partial\Omega \to \mathbb{S}^1$ such that $g_\varepsilon(x) = e^{i\phi_\varepsilon}$ with

$$\phi_\varepsilon(x) = \begin{cases} 0 & \text{if } x \in \partial\Omega \setminus B(0,\eta), \\ \pi(1 - \frac{|x_1|}{\eta}) & \text{if } x = (x_1, x_2) \in \partial\Omega \cap B(0,\eta). \end{cases} \tag{6}$$

(This is the prototype of a dipole of two boundary vortices corresponding of two consecutive transitions between opposite directions τ and $-\tau$ at the boundary at a distance η). We extend ϕ_ε to the entire domain Ω by setting $\phi_\varepsilon = 0$ in $\Omega \setminus B(0,\eta)$ and $\phi_\varepsilon(x) = \pi(1 - \frac{|x|}{\eta})$ if $x \in \Omega \cap B(0,\eta)$. Then we compute that

$$N_\varepsilon = \int_{\partial\Omega} \frac{1}{2} |\partial_\tau g_\varepsilon|^2 \, d\mathcal{H}^1 \lesssim \frac{1}{\eta}, \quad E_\varepsilon(e^{i\phi_\varepsilon}; \Omega) \lesssim 1.$$

Therefore, if u_ε is a minimizer of $E_\varepsilon(\cdot;\Omega)$ under the Dirichlet boundary condition $u_\varepsilon = g_\varepsilon$ on $\partial\Omega$, we have that $E_\varepsilon(u_\varepsilon;\Omega) \leq E_\varepsilon(e^{i\phi_\varepsilon};\Omega)$ so that (4) holds provided that $\frac{1}{\eta} \ll \frac{1}{\varepsilon^\alpha}$. In this case, Theorem 1 implies that $|u_\varepsilon|$ remains close to 1 as a positive power of ε, in particular, no interior vortices appear in Ω. We highlight the fact that the regime $\varepsilon^\alpha \ll \eta$ with $\alpha \in (0,1)$ is essential in the above example for minimizers u_ε to have modulus close to 1 uniformly. This scenario changes dramatically in the opposite regime $\eta \ll \varepsilon$ (see Proposition 5).

Notations

In the sequel we will use the symbol $\lesssim$ to denote an inequality up to a multiplicative constant that depends only on the Lipschitz regularity of Ω, that is, on $(\rho_0, \theta_0) \in (0,\infty) \times (0,\pi)$ such that for all $x \in \partial\Omega$ there is a cone of vertex x, radius ρ_0 and opening angle θ_0 which is included in $\overline{\Omega}$, and the opposite cone is included in $\mathbb{R}^2 \setminus \Omega$ (this is the uniform cone property, see e.g. [8, Theorem 1.2.2.2]). We also note that, thanks to the uniform cone property, the rectangle

$$R = (-\frac{\rho_0}{2} \sin \frac{\theta_0}{2}, \frac{\rho_0}{2} \sin \frac{\theta_0}{2}) \times (-\rho_0 \cos \frac{\theta_0}{2}, \rho_0 \cos \frac{\theta_0}{2}),$$

has the following property: for all $x \in \Omega$, there exists an angle $\gamma = \gamma(x) \in \mathbb{R}$ such that for all $t \in (0,1]$, the set

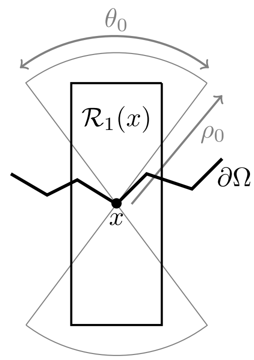
$$\mathcal{R}_t(x) = (x + te^{i\gamma} R) \cap \Omega \text{ is bi-Lipschitz homeomorphic to } tB, \tag{7}$$

where B is the unit ball, and the Lipschitz constants of the homeomorphism and its inverse are bounded by a constant depending only on (ρ_0, θ_0). See Figure 1 below. (The angle γ just serves to rotate the rectangle in order to align it with the cone; Figure 1 corresponds to $\gamma = 0$.)

We recall that for $a \in \mathbb{R}$ we write $a+$ (resp. $a-$) to denote any real number strictly greater (resp. smaller) than a but that can be chosen arbitrarily close to a. In inequalities involving such exponents, the constant will also depend on the distance of that number to a.

We write $B(x,r)$ for the ball centered at x of radius r.

Figure 1: Cone property and rectangle $\mathcal{R}_1(x)$ at a boundary point $x \in \partial\Omega$



2 A-priori global uniform estimate of $|u_\varepsilon|$ in Ω

The aim of this section is to prove the following weaker estimate of $\big||u_\varepsilon| - 1\big|$ in Ω:

Theorem 7 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. If u_ε satisfies* (1) *and* (4)*, then we have*

$$\sup_\Omega \big||u_\varepsilon| - 1\big| \lesssim \left(\frac{1 + M_\varepsilon}{|\log \varepsilon|}\right)^{\frac{1}{6}-}.$$

In particular, if κ is small enough in (4) *then $|u_\varepsilon| \geq \frac{1}{2}$ in Ω as $\varepsilon \to 0$.*

Theorem 7 is an improvement of [7, Theorem 6 in Appendix], where the boundary data satisfies the additional condition $|g_\varepsilon| \leq 1$, Ω is a square and $N_\varepsilon \ll |\log \varepsilon|$. We will follow the strategy in [7], generalizing to Lipschitz domains and general boundary data $g_\varepsilon : \partial\Omega \to \mathbb{R}^2$ with N_ε satisfying the wider regime (4). The proof of Theorem 7 is divided into three parts:

Part 1 of the proof of Theorem 7 . We prove the following upper bound of $|u_\varepsilon|$ in Ω:

$$\|u_\varepsilon\|_{L^\infty(\Omega)} - 1 \lesssim \sqrt{\varepsilon N_\varepsilon}. \tag{8}$$

For that, we start by denoting $\zeta = (1 - |g_\varepsilon|)^2$ on $\partial\Omega$. The Cauchy-Schwarz inequality yields: [6]

$$\frac{1}{2}|\partial_\tau g_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |g_\varepsilon|^2)^2 \geq \frac{1}{8\varepsilon^2}\zeta + \left(\frac{1}{8\varepsilon^2}\zeta + \frac{1}{2}\big|\partial_\tau |g_\varepsilon|\big|^2\right) \geq \frac{1}{8\varepsilon^2}\zeta + \frac{1}{4\varepsilon}|\partial_\tau \zeta| \quad \text{on } \partial\Omega.$$

Using the embedding $W^{1,1}(\partial\Omega) \subset L^\infty(\partial\Omega)$, as $\mathcal{H}^1(\partial\Omega) \geq \varepsilon$, we deduce by (2):

$$N_\varepsilon = \int_{\partial\Omega} \frac{1}{2}|\partial_\tau g_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1 - |g_\varepsilon|^2)^2 \, d\mathcal{H}^1 \gtrsim \frac{1}{\varepsilon}\|\zeta\|_{L^\infty(\partial\Omega)}, \quad \text{as} \quad \varepsilon \to 0,$$

so that

$$\delta_\varepsilon := \big\||g_\varepsilon| - 1\big\|_{L^\infty(\partial\Omega)} \lesssim \sqrt{\varepsilon N_\varepsilon}. \tag{9}$$

[6] For the more general energy (5), only the estimate $F(s) \gtrsim (1 - s)^2$ is needed, which is a consequence of $(s - 1)F'(s) \gtrsim (1 - s)^2$ and $F(1) = 0$.

Let $\tilde{\rho}_\varepsilon = 1 - |u_\varepsilon|^2$ in Ω. Then (1) implies that

$$-\Delta\tilde{\rho}_\varepsilon + \frac{2}{\varepsilon^2}|u_\varepsilon|^2\tilde{\rho}_\varepsilon = 2|\nabla u_\varepsilon|^2 \geq 0 \quad \text{in} \quad \Omega$$

and $\tilde{\rho}_\varepsilon = 1-|g_\varepsilon|^2 \geq 1-(1+\delta_\varepsilon)^2$ on $\partial\Omega$. Thus, the maximum principle[7] implies that $\tilde{\rho}_\varepsilon \geq 1-(1+\delta_\varepsilon)^2$ in Ω, i.e., $|u_\varepsilon| \leq 1+\delta_\varepsilon$ in Ω yielding (8) by (9).

Part 2 of the proof of Theorem 7 . We estimate a Hölder seminorm for u_ε.

Lemma 8 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. If u_ε satisfies* (1) *and* (4)*, then* [8]

$$|u_\varepsilon(x) - u_\varepsilon(y)| \leq C\left(\frac{|x-y|}{\varepsilon}\right)^{\frac{1}{2}-} \qquad \forall x, y \in \Omega,$$

where $C > 0$ depends only on the Lipschitz regularity of Ω.

Remark 2 In general, we don't have that $\|\nabla u_\varepsilon\|_{L^\infty(\Omega)} \leq \frac{C}{\varepsilon}$ because this estimate can be violated by the boundary condition g_ε on $\partial\Omega$. But since g_ε belongs to $H^1(\partial\Omega)$ that embeds into the Hölder space $C^{0,\frac{1}{2}}(\partial\Omega)$, we can deduce an appropriate estimate of a Hölder seminorm for u_ε in Ω.

Proof of Lemma 8. Consider the rescaled map $\hat{u}(\hat{x}) = u_\varepsilon(\varepsilon\hat{x})$ defined for $\hat{x} \in \Omega_\varepsilon = \varepsilon^{-1}\Omega$. (The map $\hat{u}$ depends on ε, we omit this dependence to simplify notation.) This map solves

$$\begin{cases} -\Delta\hat{u} = (1-|\hat{u}|^2)\hat{u} & \text{in } \Omega_\varepsilon, \\ \hat{u} = \hat{g} & \text{on } \partial\Omega_\varepsilon, \end{cases}$$

where $\hat{g}(\hat{x}) = g_\varepsilon(\varepsilon\hat{x})$ for $\hat{x} \in \partial\Omega_\varepsilon$. We fix $x_0 \in \Omega_\varepsilon$ and consider the Lipschitz domain

$$\mathcal{R} = \mathcal{R}(x_0) = \frac{1}{\varepsilon}\left((\varepsilon x_0 + \varepsilon e^{i\gamma(\varepsilon x_0)}R) \cap \Omega\right),$$

which is bi-Lipschitz homeomorphic to the unit ball B, with Lipschitz bounds uniform in ε and x_0 and depending only on the Lipschitz regularity of Ω, thanks to the definition of R, see (7). Since $|\hat{g}| \leq 1+\delta_\varepsilon \leq 2$ on $\partial\Omega_\varepsilon$ (by (9)) and $|\hat{u}| \leq 1+\delta_\varepsilon \leq 2$ in Ω_ε (by (8)) as $\varepsilon \to 0$, elliptic estimates in Lipschitz domains (see e.g. [12, 17], and [13, Section VI] for the theory of traces) yield

$$\|\hat{u}\|_{H^{\frac{3}{2}-}(\mathcal{R})} \lesssim 1 + \|\partial_\tau\hat{g}\|_{L^2(\partial\Omega_\varepsilon)}.$$

The constant depends only on the Lipschitz regularity of the domain $\mathcal{R}$ (see e.g. the proof of Theorem 2 in [17]), and is therefore bounded independently of $x_0 \in \Omega_\varepsilon$ and $\varepsilon \in (0,1]$. By Sobolev embedding we deduce that

$$\|\hat{u}\|_{C^{0,\frac{1}{2}-}(\mathcal{R})} \lesssim 1 + \|\partial_\tau\hat{g}\|_{L^2(\partial\Omega_\varepsilon)} \lesssim 1 + (\varepsilon N_\varepsilon)^{\frac{1}{2}}.$$

The constant in the Sobolev imbedding depends only on the Lipschitz regularity of Ω, since the imbedding inequalities $\|v\|_{L^{4-}(B)} \lesssim \|v\|_{H^{\frac{1}{2}-}(B)}$ and $\|v\|_{C^{0,\frac{1}{2}-}(B)} \lesssim \|v\|_{W^{1,4-}(B)}$ are valid on the unit ball $B \subset \mathbb{R}^2$ and behave well under composition by a bi-Lipschitz homeomorphism. Since any

[7]This argument adapts to critical points of the general energy (5), provided $F'(s) \geq 0$ for $s \geq 1$, see e.g. [15, Lemma 8.3].

[8]For the general energy (5) this argument only uses the fact that F is C^1 and the validity of a uniform upper bound $\|u_\varepsilon\|_\infty \lesssim 1$, implied e.g. by (8) which is valid as soon as $F'(s) \geq 0$ for $s \geq 1$.

two points $x, y \in \Omega_\varepsilon$ which are close enough are contained in a domain $\mathcal{R}(x_0)$ for some $x_0 \in \Omega_\varepsilon$, recalling once more that $|\hat{u}| \le 2$ in Ω_ε (by (8)) we infer

$$\|\hat{u}\|_{C^{0,\frac{1}{2}-}(\Omega_\varepsilon)} \lesssim 1 + (\varepsilon N_\varepsilon)^{\frac{1}{2}} \lesssim 1 \quad \text{as} \quad \varepsilon \to 0.$$

The last inequality is due to our assumption (4). The conclusion follows by scaling back to $u_\varepsilon(x) = \hat{u}(\varepsilon^{-1}x)$. □

Part 3 of the proof of Theorem 7. We start by estimating the normal derivative of u_ε at the boundary $\partial\Omega$:

Lemma 9 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. If u_ε satisfies* (1)*, then we have* [9]

$$\int_{\partial\Omega} |\partial_\nu u_\varepsilon|^2 \, d\mathcal{H}^1 \lesssim M_\varepsilon + N_\varepsilon.$$

Proof of Lemma 9. We use the Pohozaev identity for u_ε in the spirit of [3, Proposition 3], the only difference is to adapt that result to the setting of Lipschitz domains Ω. More precisely, we consider a map $V : \Omega \to \mathbb{R}^2$ that is C^1 in the closed domain $\bar{\Omega}$ and such that $V \cdot \nu \ge a > 0$ on $\partial\Omega$ for some $a > 0$ depending only on the Lipschitz regularity of Ω (see e.g. [8, Lemma 1.5.1.9]). Multiplying the equation (1) by $(V(x) \cdot \nabla)u_\varepsilon$ and integrating by parts, as $V \in C^1(\bar{\Omega})$, we deduce by (2) and (3):

$$\Big|\frac{1}{\varepsilon^2}\int_\Omega u_\varepsilon(1-|u_\varepsilon|^2)\cdot(V(x)\cdot\nabla)u_\varepsilon\,dx\Big| = \Bigg|\frac{1}{4\varepsilon^2}\int_\Omega \nabla\cdot V(1-|u_\varepsilon|^2)^2\,dx - \frac{1}{4\varepsilon^2}\int_{\partial\Omega} V(x)\cdot\nu(1-|g_\varepsilon|^2)^2\,d\mathcal{H}^1\Bigg| \lesssim M_\varepsilon + N_\varepsilon, \quad (10)$$

$$\int_\Omega \Delta u_\varepsilon\cdot(V(x)\cdot\nabla)u_\varepsilon\,dx = \int_{\partial\Omega}\left((\nu\cdot\nabla)u_\varepsilon\cdot(V\cdot\nabla)u_\varepsilon - \frac{1}{2}V\cdot\nu|\nabla u_\varepsilon|^2\right)d\mathcal{H}^1 + \int_\Omega\left(\frac{1}{2}\nabla\cdot V|\nabla u_\varepsilon|^2 - \sum_{j=1,2}\partial_j u_\varepsilon\cdot(\partial_j V\cdot\nabla)u_\varepsilon\right)dx. \quad (11)$$

For $x \in \partial\Omega$, we decompose $V = s(x)\nu + t(x)\tau$ where $s, t \in L^\infty(\partial\Omega)$, $s(x) = V \cdot \nu \ge a > 0$ for a.e. $x \in \partial\Omega$, and $\nabla u_\varepsilon = \nu \otimes \partial_\nu u_\varepsilon + \tau \otimes \partial_\tau g_\varepsilon$ on $\partial\Omega$. By (1), (2), (3), (10) and (11), as $V \in C^1(\bar{\Omega})$, we conclude by Young's inequality:

$$M_\varepsilon + N_\varepsilon \gtrsim \int_{\partial\Omega}\left(\frac{s(x)}{2}|\partial_\nu u_\varepsilon|^2 - \frac{s(x)}{2}\big|\partial_\tau g_\varepsilon\big|^2 + t(x)\partial_\nu u_\varepsilon\cdot\partial_\tau g_\varepsilon\right)d\mathcal{H}^1 \gtrsim \int_{\partial\Omega}|\partial_\nu u_\varepsilon|^2\,d\mathcal{H}^1 - N_\varepsilon.$$

□

We use Lemma 9 to prove the following estimate of the potential energy in small balls (of radius $\ll \varepsilon^\alpha$). To simplify notation, we denote the energy density by

$$e_\varepsilon(u_\varepsilon) := \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2, \quad u_\varepsilon : \Omega \to \mathbb{R}^2.$$

(In the context of the energy (5), only the assumption $F \in C^1$ is needed for the following estimate).

[9] In the context of the general energy (5), we need only the assumption that $F \in C^1$.

Lemma 10 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain and u_ε be a solution of (1) in the regime (4). Fix $1 > \alpha_1 > \alpha_2 > \alpha > 0$. There exists $C \geq 1$ such that for every $x_0 \in \Omega$, we can find $r_0 = r_0(x_0) \in (\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$ such that*

$$\int_{\partial\left(B(x_0,r_0)\cap\Omega\right)} e_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \leq \frac{C(1+M_\varepsilon)}{r_0|\log\varepsilon|} \tag{12}$$

for every $\varepsilon \leq \varepsilon_0$ with $\varepsilon_0 = \varepsilon_0(\alpha_2, \alpha) > 0$. Moreover, we have that

$$\frac{1}{\varepsilon^2}\int_{B(x_0,r_0)\cap\Omega} (1-|u_\varepsilon|^2)^2\, dx \leq \frac{\tilde{C}(1+M_\varepsilon)}{|\log\varepsilon|} \tag{13}$$

for some $\tilde{C} \geq 1$.

Proof of Lemma 10. We distinguish two steps:

Step 1. *Proof of* (12). Assume by contradiction that for every $C \geq 1$ there exists $x \in \Omega$ such that for every $r \in (\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$ we have

$$\int_{\partial\left(B(x,r)\cap\Omega\right)} e_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \geq \frac{C(1+M_\varepsilon)}{r|\log\varepsilon|}.$$

Since $N_\varepsilon\varepsilon^\alpha \ll 1$, by (2) and Lemma 9, there exists $c_1 > 0$ such that

$$\int_{\partial\Omega} e_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \leq c_1(M_\varepsilon + N_\varepsilon) \leq \frac{1+M_\varepsilon}{2\varepsilon^{\alpha_2}|\log\varepsilon|} \leq \frac{C(1+M_\varepsilon)}{2r|\log\varepsilon|}, \quad \forall r \in (\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$$

for every $\varepsilon \leq \varepsilon_0$ (with $\varepsilon_0 > 0$ depending on α_2 and α). Therefore, we deduce that

$$\int_{\partial B(x,r)\cap\Omega} e_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \geq \frac{C(1+M_\varepsilon)}{2r|\log\varepsilon|}.$$

Integrating in $r \in (\varepsilon^{\alpha_1}, \varepsilon^{\alpha_2})$, we obtain by (3):

$$M_\varepsilon = \int_\Omega e_\varepsilon(u_\varepsilon)\, dx \geq \int_{B(x,\varepsilon^{\alpha_2})\cap\Omega} e_\varepsilon(u_\varepsilon)\, dx \geq \int_{\varepsilon^{\alpha_1}}^{\varepsilon^{\alpha_2}} dr \int_{\partial B(x,r)\cap\Omega} e_\varepsilon(u_\varepsilon)\, d\mathcal{H}^1 \geq \frac{C(\alpha_1-\alpha_2)(1+M_\varepsilon)}{2}$$

which is a contradiction with the fact that C can be arbitrary large.

Step 2. *Proof of* (13). Let ν be the outer unit normal vector at the boundary of the domain

$$\mathcal{D} := B(x_0, r_0) \cap \Omega.$$

As in the proof of Lemma 9, we use the Pohozaev identity for the solution u_ε of (1). Indeed, multiplying the equation by $(x - x_0)\cdot\nabla u_\varepsilon$ and integrating by parts, we deduce:

$$\int_{\mathcal{D}} -\Delta u_\varepsilon \cdot \Big((x-x_0)\cdot\nabla u_\varepsilon\Big)\, dx = \int_{\partial\mathcal{D}} \left(\frac{1}{2}(x-x_0)\cdot\nu|\nabla u_\varepsilon|^2 - \partial_\nu u_\varepsilon \cdot \big((x-x_0)\cdot\nabla\big)u_\varepsilon\right) d\mathcal{H}^1,$$

$$\frac{1}{\varepsilon^2}\int_{\mathcal{D}} u_\varepsilon(1-|u_\varepsilon|^2)\cdot\Big((x-x_0)\cdot\nabla u_\varepsilon\Big)\, dx = \frac{1}{2\varepsilon^2}\int_{\mathcal{D}}(1-|u_\varepsilon|^2)^2\, dx - \frac{1}{4\varepsilon^2}\int_{\partial\mathcal{D}}(x-x_0)\cdot\nu(1-|u_\varepsilon|^2)^2\, d\mathcal{H}^1.$$

Since $|x - x_0| \leq r_0$ on $\partial\mathcal{D}$, by (12), we deduce that (13) holds true. □

The conclusion of Theorem 7 comes from the following result:

Lemma 11 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. If u_ε satisfies (1) and (4), then we have* [10]

$$\||u_\varepsilon|^2 - 1\|_{L^\infty(\Omega)} \lesssim \left(\frac{1+M_\varepsilon}{|\log\varepsilon|}\right)^{\frac{1}{6}-}.$$

Proof. Let $x_0 \in \Omega$ and set $1 > A \geq 0$ such that

$$4C\, A^{\frac{1}{2}-} = \frac{\left|1-|u_\varepsilon(x_0)|^2\right|}{2},$$

where $C \geq 1$ is given by Lemma 8. By Lemma 8, we obtain for any $y \in B(x_0, A\varepsilon) \cap \Omega$

$$\left|1-|u_\varepsilon(y)|^2\right| \geq \left|1-|u_\varepsilon(x_0)|^2\right| - 4C\, A^{\frac{1}{2}-} = \frac{\left|1-|u_\varepsilon(x_0)|^2\right|}{2}$$

as $|u_\varepsilon(y)| + |u_\varepsilon(x_0)| \leq 4$. Hence, for small ε,

$$\begin{aligned}\int_{B(x_0,A\varepsilon)\cap\Omega} (1-|u_\varepsilon(y)|^2)^2\,dy &\geq C(\Omega)A^2\varepsilon^2(1-|u_\varepsilon(x_0)|^2)^2 \\ &\geq \tilde{C}(\Omega)\varepsilon^2(1-|u_\varepsilon(x_0)|^2)^{6+},\end{aligned} \tag{14}$$

where $C(\Omega), \tilde{C}(\Omega) > 0$. We have that $B(x_0, A\varepsilon) \subset B(x_0, r_0)$ for $\varepsilon \leq \varepsilon_0$ with ε_0 depending only on α_1 in Lemma 10. Thus, by (13), we obtain

$$(1-|u_\varepsilon(x_0)|^2)^{6+} \leq \hat{C}\frac{1+M_\varepsilon}{|\log\varepsilon|}$$

and the conclusion follows. □

3 Proof of Theorem 1

The main idea is to improve the convergence of $|u_\varepsilon|$ to 1 locally in L^2-norm; this involves improving the local estimate of the potential energy (13) to a positive power of ε and then the argument in Lemma 11 yields the conclusion (i.e., the desired estimate in L^∞-norm of $|u_\varepsilon| - 1$ in our main result).

Let $x_0 \in \Omega$ and $\varepsilon > 0$. By Fubini's theorem we may choose $t \in [1/2, 1]$ such that the domain

$$\mathcal{R} = \mathcal{R}_t(x_0) \tag{15}$$

defined in (7) satisfies

$$\int_{\partial\mathcal{R}\cap\Omega} \frac{1}{2}|\nabla u_\varepsilon|^2 + \frac{1}{4\varepsilon^2}(1-|u_\varepsilon|^2)^2\,d\mathcal{H}^1 \lesssim M_\varepsilon. \tag{16}$$

Recall that $\mathcal{R}$ is bi-Lipschitz homeomorphic to the unit ball B, in particular it is simply connected. Moreover by Theorem 7 if κ is small enough then u_ε does not vanish. So we may write

$$u_\varepsilon = \rho_\varepsilon e^{i\varphi_\varepsilon} \quad \text{in } \mathcal{R},$$

with $\rho_\varepsilon, \varphi_\varepsilon \in H^1(\mathcal{R})$ (moreover, ρ_ε^2 and φ_ε are smooth in $\mathcal{R}$ as u_ε is smooth by standard elliptic regularity). The system (1) writes in terms of ρ_ε and φ_ε:

$$\begin{cases} -\Delta\rho_\varepsilon + \rho_\varepsilon|\nabla\varphi_\varepsilon|^2 = \frac{1}{\varepsilon^2}\rho_\varepsilon(1-\rho_\varepsilon^2) \\ \nabla\cdot(\rho_\varepsilon^2\nabla\varphi_\varepsilon) = 0 \end{cases} \quad \text{in } \mathcal{R}. \tag{17}$$

[10] For the general energy (5) we only need here $(s-1)^2 \lesssim F(s)$.

Step 1. *We prove the following estimate* [11] *of $\nabla\varphi_\varepsilon$ in $L^q(\mathcal{R})$, where $q = 4-$:*

$$\|\nabla\varphi_\varepsilon\|_{L^q(\mathcal{R})} \lesssim 1 + N_\varepsilon^{\frac{1}{2}} + M_\varepsilon^{\frac{1}{2}}. \tag{18}$$

Indeed, by (2), (9), Lemma 9 and (16), we note that

$$\begin{aligned} &\int_{\partial\Omega\cap\mathcal{R}} |\nabla\varphi_\varepsilon|^2 \, d\mathcal{H}^1 \lesssim \int_{\partial\Omega} |\nabla u_\varepsilon|^2 \, d\mathcal{H}^1 \lesssim N_\varepsilon + M_\varepsilon \\ \text{and } &\int_{\Omega\cap\partial\mathcal{R}} |\nabla\varphi_\varepsilon|^2 \, d\mathcal{H}^1 \lesssim \int_{\partial\mathcal{R}\cap\Omega} |\nabla u_\varepsilon|^2 \, d\mathcal{H}^1 \lesssim M_\varepsilon. \end{aligned} \tag{19}$$

Therefore, by the Poincaré-Wirtinger inequality, up to adding a constant to φ_ε, we can assume that

$$\|\varphi_\varepsilon\|_{H^1(\partial\mathcal{R})} \lesssim 1 + N_\varepsilon^{\frac{1}{2}} + M_\varepsilon^{\frac{1}{2}}. \tag{20}$$

By the theory of traces in Lipschitz domains (see e.g. [13, Section VI.2]), for $s = 1-$ there is a continuous extension operator from $H^s(\partial\mathcal{R})$ to $H^{s+1/2}(\mathcal{R})$, and its operator norm is bounded by a constant depending only on the Lipschitz regularity of $\mathcal{R}$, hence only on the Lipschitz regularity of Ω. Thus there exists an extension $\Phi \in H^{\frac{3}{2}-}(\mathcal{R})$ of $\varphi_\varepsilon\big|_{\partial\mathcal{R}}$ such that

$$\|\Phi\|_{H^{\frac{3}{2}-}(\mathcal{R})} \lesssim 1 + N_\varepsilon^{\frac{1}{2}} + M_\varepsilon^{\frac{1}{2}}.$$

By Sobolev embedding $H^{\frac{1}{2}-}(\mathcal{R}) \subset L^{4-}(\mathcal{R})$ we deduce the bound

$$\|\nabla\Phi\|_{L^q(\mathcal{R})} \lesssim 1 + N_\varepsilon^{\frac{1}{2}} + M_\varepsilon^{\frac{1}{2}}. \tag{21}$$

The constant in the Sobolev embedding depends only on the Lipschitz regularity of Ω since $\mathcal{R}$ is bi-Lipschitz homeomorphic to the unit ball (with Lipschitz constants depending only on the Lipschitz regularity of Ω). Denoting

$$\psi := \varphi_\varepsilon - \Phi \in H^1_0(\mathcal{R}),$$

by (17), ψ solves

$$\Delta\psi = \nabla \cdot \big((1-\rho_\varepsilon^2)\nabla\varphi_\varepsilon - \nabla\Phi\big) \quad \text{in} \quad \mathcal{R},$$

so that elliptic estimates in Lipschitz domains (see e.g. [12, Theorem 0.5] or [17, Theorem 2]) yield

$$\|\nabla\varphi_\varepsilon\|_{L^q(\mathcal{R})} \leq C(1 + \big\|(1-\rho_\varepsilon^2)\nabla\varphi_\varepsilon\big\|_{L^q(\mathcal{R})} + \|\nabla\Phi\|_{L^q(\mathcal{R})}).$$

By Theorem 7, $C\left|1-\rho_\varepsilon^2\right| \leq \frac{1}{2}$ in $\mathcal{R}$ for $\kappa > 0$ small enough. This implies

$$\|\nabla\varphi_\varepsilon\|_{L^q(\mathcal{R})} \lesssim 1 + \|\nabla\Phi\|_{L^q(\mathcal{R})}.$$

The last term can be estimated by (21) and this proves (18).

Step 2. *An improved local estimate of the potential energy.* We will prove the following:

Lemma 12 *Let $\Omega \subset \mathbb{R}^2$ be a Lipschitz bounded domain. If u_ε satisfies* (1) *and* (4)*, then*

$$\frac{1}{\varepsilon^2}\int_{\mathcal{R}} (1-|u_\varepsilon|^2)^2 \, dx \lesssim \varepsilon^{1-}(1 + N_\varepsilon + M_\varepsilon)(1+M_\varepsilon)^{\frac{1}{2}-},$$

for every point $x_0 \in \Omega$ with the associated domain $\mathcal{R}$ in (15).

[11] For the general energy (5), no modification is required for this step since the equation satisfied by φ_ε stays the same.

Proof. Multiplying (17) by $1-\rho_\varepsilon^2$, as $\rho_\varepsilon \geq 1/2$ in $\mathcal{R}$ (by Theorem 7), integration by parts yields[12]

$$\begin{aligned}
\frac{1}{2\varepsilon^2}\int_{\mathcal{R}}(1-\rho_\varepsilon^2)^2\,dx &\leq \frac{1}{\varepsilon^2}\int_{\mathcal{R}}\rho_\varepsilon(1-\rho_\varepsilon^2)^2\,dx \\
&= -\int_{\mathcal{R}}(1-\rho_\varepsilon^2)\Delta\rho_\varepsilon\,dx + \int_{\mathcal{R}}\rho_\varepsilon(1-\rho_\varepsilon^2)|\nabla\varphi_\varepsilon|^2\,dx \\
&= -\int_{\partial\mathcal{R}}(1-\rho_\varepsilon^2)\partial_\nu\rho_\varepsilon\,d\mathcal{H}^1 - 2\int_{\mathcal{R}}\rho_\varepsilon\left|\nabla\rho_\varepsilon\right|^2\,dx + \int_{\mathcal{R}}\rho_\varepsilon(1-\rho_\varepsilon^2)\left|\nabla\varphi_\varepsilon\right|^2\,dx \\
&\leq \left\|1-\rho_\varepsilon^2\right\|_{L^2(\partial\mathcal{R})}\left\|\partial_\nu\rho_\varepsilon\right\|_{L^2(\partial\mathcal{R})} + 2\left\|\nabla\varphi_\varepsilon\right\|_{L^q(\mathcal{R})}^2\left\|1-\rho_\varepsilon^2\right\|_{L^{\frac{q}{q-2}}(\mathcal{R})} \\
&\lesssim \varepsilon(M_\varepsilon+N_\varepsilon) + \varepsilon^{1-}(1+N_\varepsilon+M_\varepsilon)M_\varepsilon^{\frac{1}{2}-}
\end{aligned}$$

for $q = 4-$, where we used

- (2) and (16) yielding $\left\|1-\rho_\varepsilon^2\right\|_{L^2(\partial\mathcal{R})} \lesssim \varepsilon(M_\varepsilon+N_\varepsilon)^{\frac{1}{2}}$;
- (19) yielding $\left\|\partial_\nu\rho_\varepsilon\right\|_{L^2(\partial\mathcal{R})} \lesssim (M_\varepsilon+N_\varepsilon)^{\frac{1}{2}}$;
- (18) and the interpolation inequality for $\lambda = \frac{2(q-2)}{q} = 1-$

$$\left\|1-\rho_\varepsilon^2\right\|_{L^{\frac{q}{q-2}}(\mathcal{R})} \leq \left\|1-\rho_\varepsilon^2\right\|_{L^\infty(\mathcal{R})}^{1-\lambda}\left\|1-\rho_\varepsilon^2\right\|_{L^2(\mathcal{R})}^{\lambda} \overset{(3),(8)}{\lesssim} \varepsilon^\lambda M_\varepsilon^{\frac{\lambda}{2}}$$

yielding the last estimate. □

Step 3. *Conclusion of Theorem 1.* Applying the arguments in the proof of Lemma 11 in the domain $\mathcal{R} = \mathcal{R}_t(x_0)$ defined at (15), we find

$$(|u_\varepsilon(x_0)|^2-1)^{6+} \lesssim \frac{1}{\varepsilon^2}\int_{\mathcal{R}}(1-\rho_\varepsilon^2)^2\,dx \lesssim \varepsilon^{1-}(1+N_\varepsilon+M_\varepsilon)(1+M_\varepsilon)^{\frac{1}{2}-}.$$

The last inequality follows from the previous step. Since $x_0 \in \Omega$ is arbitrary and the constant depends only on the Lipschitz regularity of Ω, this proves Theorem 1. □

4 Optimality of the regime (4)

In this section, we prove Propositions 2 to 5:

Proof of Proposition 2. Let Ω be a cone of opening angle θ_0 and height 1, see Figure 4. Consider the point P_ε on the medial axis at distance s_ε from the corner, where

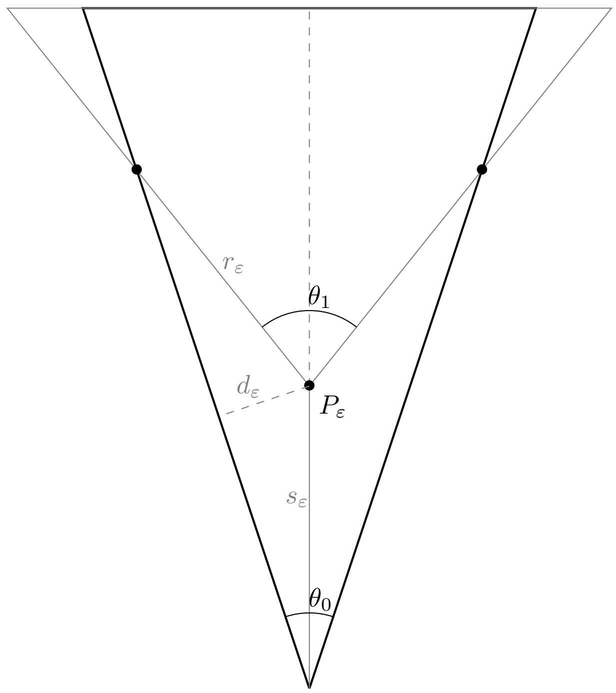
$$s_\varepsilon = \varepsilon^\mu \quad \text{with } 0<\mu<1.$$

Set $\alpha = \frac{1+\mu}{2} \in (0,1)$. We also denote by d_ε the distance of P_ε to the boundary $\partial\Omega$. For $\theta_1 = \theta_0+\eta$ (where, possibly lowering η, we may assume $\eta < \theta_0$) consider the cone K_1 of opening θ_1 and height 1 centered at P_ε and with the same medial axis. The boundaries of the two cones intersect in two points at a distance r_ε from P_ε. It follows that $\Omega \subset B(P_\varepsilon, r_\varepsilon) \cup K_1$ (as $s_\varepsilon < r_\varepsilon$),

$$d_\varepsilon = s_\varepsilon \sin\frac{\theta_0}{2} \sim \varepsilon^\mu \quad \text{and} \quad r_\varepsilon = s_\varepsilon\frac{\sin\frac{\theta_0}{2}}{\sin\frac{\eta}{2}} \sim \varepsilon^\mu.$$

[12]For the general energy (5), this estimate holds thanks to the assumption $(s-1)F'(s) \gtrsim (s-1)^2$ for $s \geq 0$.

Figure 2: The cones Ω and K_1 of opening angles θ_0 and θ_1 respectively.



We consider the following degree-one vortex solution u_ε of (1):

$$u_\varepsilon(x) = f\left(\frac{|x-P_\varepsilon|}{\varepsilon}\right)\frac{x-P_\varepsilon}{|x-P_\varepsilon|} \quad \text{for every} \quad x \in \mathbb{R}^2,$$

where P_ε is the vortex point (i.e., $u_\varepsilon(P_\varepsilon)=0$), $f:[0,\infty)\to[0,1)$ is the smooth radial profile given by the unique solution of

$$-f'' - \frac{1}{r}f' + \frac{1}{r^2}f = f(1-f^2) \quad \text{for every} \quad r \in (0,\infty),$$

with $f(0)=0$ and $\lim_{r\to\infty} f(r)=1$; f and f' have the following asymptotics for $r\to\infty$ (see [6, 9])

$$f(r) = 1 - \frac{1}{2r^2} - \frac{9}{8r^4} + O(r^{-6}), \quad f'(r) = \frac{1}{r^3} + \frac{9}{2r^5} + O(r^{-7}).$$

In a point $x\in\mathbb{R}^2$ with $|x-P_\varepsilon|=t$, the Ginzburg-Landau energy density is given by

$$e_\varepsilon(u_\varepsilon(x)) = \frac{1}{2}\left(\frac{|f'(\frac{t}{\varepsilon})|^2}{\varepsilon^2} + \frac{|f(\frac{t}{\varepsilon})|^2}{t^2}\right) + \frac{1}{4\varepsilon^2}\left(1-|f(\tfrac{t}{\varepsilon})|^2\right)^2,$$

so that for $t\geq\varepsilon$, we find

$$e_\varepsilon(u_\varepsilon(x)) = \frac{1}{2t^2} + \frac{1}{\varepsilon^2}O(\frac{\varepsilon^4}{t^4}) \tag{22}$$

Recalling that $r_\varepsilon \gg \varepsilon$, we obtain by integrating over $K_1\setminus B(P_\varepsilon, r_\varepsilon)$:

$$\int_{K_1\setminus B(P_\varepsilon,r_\varepsilon)} e_\varepsilon(u_\varepsilon)\,dx \leq \theta_1\int_{r_\varepsilon}^2 t\left(\frac{1}{2t^2} + \frac{1}{\varepsilon^2}O(\frac{\varepsilon^4}{t^4})\right)\,dt \leq \frac{\theta_1}{2}\log\frac{2}{r_\varepsilon} + O(\frac{\varepsilon^2}{r_\varepsilon^2}).$$

In $B(P_\varepsilon, r_\varepsilon)$, using (22) and the fact that $f(0) = 0$ and $|f'| \lesssim 1$ (in particular, $|f(t)| \lesssim t$ for $t > 0$), we estimate

$$\int_{B(P_\varepsilon,r_\varepsilon)} e_\varepsilon(u_\varepsilon)\,dx \le \pi\Big(\int_0^\varepsilon + \int_\varepsilon^{r_\varepsilon}\Big)\left[\Big(\frac{|f'(\frac{t}{\varepsilon})|^2}{\varepsilon^2} + \frac{|f(\frac{t}{\varepsilon})|^2}{t^2}\Big) + \frac{1}{2\varepsilon^2}\left(1-|f(\tfrac{t}{\varepsilon})|^2\right)^2\right] t\,dt$$
$$\le C\int_0^\varepsilon \frac{t}{\varepsilon^2}\,dt + \pi\log\frac{r_\varepsilon}{\varepsilon} + O(1) = \pi\log\frac{r_\varepsilon}{\varepsilon} + O(1).$$

As $\Omega \subset B(P_\varepsilon, r_\varepsilon) \cup K_1$, it follows that the interior energy M_ε is estimated as:

$$M_\varepsilon = \int_\Omega e_\varepsilon(u_\varepsilon)\,dx \le \pi\log\frac{r_\varepsilon}{\varepsilon} + \frac{\theta_1}{2}\log\frac{2}{r_\varepsilon} + O(1) \le \left(\pi(1-\mu) + \frac{\theta_1}{2}\mu\right)|\log\varepsilon| + C$$

where $C > 0$ is a constant depending only on η and θ_0. Note that for μ sufficiently close to 1 and ε small enough, this implies

$$M_\varepsilon \le (\frac{\theta_0}{2} + \eta)|\log\varepsilon|.$$

To estimate the boundary energy N_ε, we write $\partial\Omega = \Gamma_1 \cup \Gamma_2^+ \cup \Gamma_2^-$, where Γ_1 is the basis of the cone, and $\Gamma_2^\pm$ are the two sides of the cone adjacent to its vertex. Since P_ε is at distance ~ 1 of Γ_1, it holds

$$\int_{\Gamma_1} e_\varepsilon(u_\varepsilon)\,d\mathcal{H}^1 = O(1).$$

On the rest of the boundary, note that for every point $x \in \Gamma_2^\pm$ that has a distance s from the orthogonal projections of P_ε onto $\Gamma_2^\pm$, we have

$$e_\varepsilon(u_\varepsilon(x)) = \frac{1}{2t^2} + \frac{1}{\varepsilon^2}O(\frac{\varepsilon^4}{t^4}), \qquad \text{where } t = |x - P_\varepsilon| = \sqrt{s^2 + d_\varepsilon^2},$$

since $t \ge d_\varepsilon \sim \varepsilon^\mu \gg \varepsilon$. We can thus estimate

$$N_\varepsilon \le 2\int_{-\infty}^{\infty}\left(\frac{1}{2(s^2+d_\varepsilon^2)} + C\frac{\varepsilon^2}{(s^2+d_\varepsilon^2)^2}\right)ds + O(1) \lesssim \frac{1}{d_\varepsilon} + \frac{\varepsilon^2}{d_\varepsilon^3} \lesssim \frac{1}{s_\varepsilon} \sim \frac{1}{\varepsilon^\mu} \ll \frac{1}{\varepsilon^\alpha}$$

as α was chosen such that $\alpha = \frac{1+\mu}{2} < 1$. So (4) holds with $\kappa = \frac{\theta_0}{2} + \eta$, while $u_\varepsilon(P_\varepsilon) = 0$. □

Remark 3 Applying the construction in the proof of Proposition 2 to a half-space domain, we deduce that a necessary condition in order that Theorem 1 holds true is given by $\kappa \le \frac{\pi}{2}$ in (4) (even for smooth domains Ω).

Proof of Proposition 3. Let $f : [0,\infty) \to [0,1]$ be a smooth function with $f(0) = 0$, $f(r) = 1$ for $r \ge 1$ and $|f'(r)| \le C$. Let $x_0 \in \partial\Omega$ and consider $v_\varepsilon(x) = f(\frac{x-x_0}{\varepsilon})$ for every $x \in \mathbb{R}^2$. Let $g_\varepsilon = (v_\varepsilon, 0)$ on $\partial\Omega$ and let u_ε be a minimizer of the Ginzburg-Landau energy with Dirichlet boundary conditions g_ε, in particular, $u_\varepsilon(x_0) = g_\varepsilon(x_0) = 0$. Then u_ε satisfies (1) and (by minimality) $M_\varepsilon \le E_\varepsilon(v_\varepsilon;\Omega) \lesssim 1$ while $N_\varepsilon \lesssim \frac{1}{\varepsilon}$. □

Proof of Proposition 4. Since Ω is smooth, bounded and simply connected, upon applying a conformal diffeomorphism we assume $\Omega = B(0,1)$ is the unit disk.[13] Our example is strongly inspired by [1, Example 1] and [2, Lemma 4.1]. For $\eta > 0$ small we consider

$$v_\eta(z) = \frac{z-(1-\eta)}{1-(1-\eta)z} \qquad \text{for } z \in B(0,1) \subset \mathbb{C},$$

[13] Only the bulk energy changes and is bounded (up to a multiplicative constant depending on Ω) by the new bulk energy in the disk $B(0,1)$.

and denote by g_η its boundary datum $g_\eta(\theta) = v_\eta(e^{i\theta})$, which satisfies $|g_\eta| = 1$ on $\partial B(0,1)$. As explained in [2, Lemma 4.1], the map v_η satisfies

$$\frac{1}{2}\int_{B(0,1)} |\nabla v_\eta|^2 \, dx = \pi, \qquad \text{and } \deg(v_\eta, \partial B(0,1)) = 1.$$

Lengthy but direct computations show that for $\theta \in (-\pi,\pi)$ we have

$$\left|g_\eta'(\theta)\right|^2 = \frac{(2-\eta)^2}{\eta^2}\frac{1}{\left(1+2\frac{1-\eta}{\eta^2}(1-\cos\theta)\right)^2}.$$

Moreover $1-\cos\theta \geq \frac{2}{\pi^2}\theta^2$ for $\theta \in (-\pi,\pi)$, and therefore, for small η,

$$\begin{aligned}\int_{-\pi}^{\pi} \left|g_\eta'(\theta)\right|^2 d\theta &\lesssim \frac{1}{\eta^2}\int_{-\pi}^{\pi}\frac{d\theta}{(1+\frac{\theta^2}{\eta^2})^2} = \frac{1}{\eta}\int_{-\frac{\pi}{\eta}}^{\frac{\pi}{\eta}}\frac{dt}{(1+t^2)^2} \\ &\lesssim \frac{1}{\eta}.\end{aligned}$$

For $0<r<1$ and $\theta \in (-\pi,\pi)$, setting $a = 1-\eta$ we have

$$1-\left|v_\eta(re^{i\theta})\right|^2 = (1-a^2)\frac{1-r^2}{(1+a^2r^2-2ar\cos\theta)}.$$

In order to estimate $\int(1-|v_\eta|^2)^2$ we first compute

$$I := \int_{-\pi}^{\pi}\frac{d\theta}{(1+a^2r^2-2ar\cos\theta)^2}.$$

The change of variable $x = \tan\frac{\theta}{2}$ gives $d\theta = 2dx/(1+x^2)$ and $\cos\theta = (1-x^2)/(1+x^2)$, so

$$\begin{aligned}I &= 2\int_{-\infty}^{\infty}\frac{1+x^2}{\left((1-ar)^2+(1+ar)^2x^2\right)^2}dx \\ &= \frac{2}{(1+ar)^4}\int_{-\infty}^{\infty}\frac{1+x^2}{(A^2+x^2)^2}dx, \qquad A := \frac{1-ar}{1+ar}.\end{aligned}$$

We have

$$\begin{aligned}\int_{-\infty}^{\infty}\frac{1+x^2}{(A^2+x^2)^2}dx &= \int_{-\infty}^{\infty}\frac{dx}{A^2+x^2} + (1-A^2)\int_{-\infty}^{\infty}\frac{dx}{(A^2+x^2)^2} \\ &= \frac{1}{A}\int_{-\infty}^{\infty}\frac{dt}{1+t^2} + (1-A^2)\frac{1}{A^3}\int_{-\infty}^{\infty}\frac{dt}{(1+t^2)^2} \\ &= \frac{1}{A}\left(\pi + \frac{1-A^2}{A^2}\frac{\pi}{2}\right) = \frac{\pi}{2}\frac{1+A^2}{A^3}.\end{aligned}$$

Since $1+ar \geq 1$, plugging this back into I yields

$$I = 2\pi\frac{1+a^2r^2}{(1+ar)^3}\frac{1}{(1-ar)^3} \leq \frac{2\pi}{(1-ar)^3}.$$

We deduce

$$\begin{aligned}\int_{B(0,1)}(1-|v_\eta|^2)^2\,dx &= (1-a^2)^2\int_0^1 (1-r^2)^2 I\,r\,dr\\ &\lesssim \eta^2\int_0^1\frac{(1-r)^2}{(1-ar)^3}dr = \eta^2\int_0^1\frac{t^2}{(at+\eta)^3}dt\\ &= \frac{1}{\eta}\int_0^1\frac{t^2}{(\frac{a}{\eta}t+1)^3}dt = \frac{\eta^2}{a^3}\int_0^{\frac{a}{\eta}}\frac{s^2}{(s+1)^3}ds\\ &\lesssim \eta^2\log\frac{1}{\eta}.\end{aligned}$$

In particular,

• choosing $\eta=\varepsilon/|\log\varepsilon|^{\frac12}$ and u_ε minimizing the Ginzburg-Landau energy with Dirichlet condition $u_\varepsilon=g_\eta$ on $\partial B(0,1)$, we have that u_ε is of modulus one and degree one on $\partial B(0,1)$ while $M_\varepsilon\lesssim 1$ and $N_\varepsilon\lesssim\frac{|\log\varepsilon|^{\frac12}}{\varepsilon}$;

• choosing $\eta:=\frac{\varepsilon\sqrt{\kappa}}{L}$ with $L>0$ big enough and u_ε minimizing the Ginzburg-Landau energy with Dirichlet condition $u_\varepsilon=g_\eta$ on $\partial B(0,1)$, we have that u_ε is of modulus one and degree one on $\partial B(0,1)$ while $M_\varepsilon\le\kappa|\log\varepsilon|$ and $N_\varepsilon\lesssim\frac{1}{\varepsilon}$.

□

Proof of Proposition 5. Upon locally flattening the boundary and rescaling the domain (thus only introducing multiplicative constants in all estimates), we may assume that Ω is as in Example 1, i.e. it contains the upper half unit ball:

$$\Omega\cap B(0,1)=\{x=(x_1,x_2)\in B(0,1)\,:\,x_2>0\}.$$

As in Example 1, for $\eta=\eta(\varepsilon)\ll\varepsilon$ we consider the boundary data $g_\varepsilon:\partial\Omega\to\mathbb{S}^1$ such that $g_\varepsilon(x)=e^{i\phi_\varepsilon}$ with

$$\phi_\varepsilon(x)=\begin{cases}0 & \text{if } x\in\partial\Omega\setminus B(0,\eta),\\ \pi(1-\frac{|x_1|}{\eta}) & \text{if } x=(x_1,x_2)\in\partial\Omega\cap B(0,\eta).\end{cases}$$

As in Example 1 we then have that $N_\varepsilon\sim\frac{1}{\eta}$ and any map u_ε minimizing $E_\varepsilon(\cdot;\Omega)$ with Dirichlet boundary data $u_\varepsilon=g_\varepsilon$ on $\partial\Omega$ satisfies $M_\varepsilon\lesssim 1$.

It remains to show that

$$\sup_\Omega\left|1-|u_\varepsilon|^2\right|\ge\frac12\qquad\text{for }\varepsilon\ll 1. \tag{23}$$

The idea is to use the decomposition $u_\varepsilon=v_\varepsilon+w_\varepsilon$ with

$$\begin{cases}-\Delta v_\varepsilon=\dfrac{1}{\varepsilon^2}(1-|u_\varepsilon|^2)u_\varepsilon & \text{in }\Omega,\\ \quad v_\varepsilon=0 & \text{on }\partial\Omega,\end{cases}\qquad\text{and}\qquad\begin{cases}\Delta w_\varepsilon=0 & \text{in }\Omega,\\ w_\varepsilon=g_\varepsilon & \text{on }\partial\Omega,\end{cases}$$

(see also the proof of Lemma 3 in [7]).

The maximum principle can be used to see that $|w_\varepsilon|\le 1$ and $|u_\varepsilon|\le 1$ in Ω. Indeed, since $-\Delta|w_\varepsilon|^2=-2|\nabla w_\varepsilon|^2\le 0$ in Ω and $|w_\varepsilon|=1$ on $\partial\Omega$, the maximal principle implies that $|w_\varepsilon|\le 1$ in Ω. Moreover, recalling that $\tilde\rho_\varepsilon=1-|u_\varepsilon|^2$ satisfies

$$-\Delta\tilde\rho_\varepsilon+\frac{2}{\varepsilon^2}|u_\varepsilon|^2\tilde\rho_\varepsilon=2|\nabla u_\varepsilon|^2\ge 0\quad\text{in}\quad\Omega,$$

and $\tilde{\rho}_\varepsilon = 0$ on $\partial\Omega$, the maximum principle ensures that $\tilde{\rho}_\varepsilon \geq 0$ in Ω, i.e. $|u_\varepsilon| \leq 1$ in Ω.

We deduce that $|v_\varepsilon| \leq 2$ in Ω. Using this together with the equation satisfied by v_ε, and the interpolation inequality [3]

$$\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \lesssim \|v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{2}} \|\Delta v_\varepsilon\|_{L^\infty(\Omega)}^{\frac{1}{2}},$$

we find that $\|\nabla v_\varepsilon\|_{L^\infty(\Omega)} \lesssim \frac{1}{\varepsilon}$. In particular, since $v_\varepsilon = 0$ on $\partial\Omega$, we deduce that $|v_\varepsilon| \lesssim \frac{\eta}{\varepsilon} \ll 1$ in $\Omega \cap B(0, 2\eta)$. As a consequence we have

$$\sup_{\Omega \cap B(0,2\eta)} \left| |u_\varepsilon|^2 - |w_\varepsilon|^2 \right| \ll 1,$$

and to prove (23) it suffices to show that

$$|w_\varepsilon(0,\eta)|^2 < \frac{1}{2} \qquad \text{for } \varepsilon \ll 1. \tag{24}$$

To this end we rescale w_ε, setting $\tilde{w}_\eta(x) = w_\varepsilon(\eta x)$ so that

$$\begin{cases} \Delta \tilde{w}_\eta = 0 & \text{in } B(0, \frac{1}{\eta}) \cap \{x_2 > 0\}, \\ \tilde{w}_\eta = e^{i\tilde{\phi}} & \text{on } B(0, \frac{1}{\eta}) \cap \{x_2 = 0\}, \end{cases}$$

$$\text{where } \tilde{\phi}(x_1, 0) = \begin{cases} 0 & \text{if } |x_1| > 1, \\ \pi(1 - |x_1|) & \text{if } |x_1| \leq 1. \end{cases}$$

Since $|\tilde{w}_\eta| \leq 1$ and $\tilde{\phi}$ is Lipschitz, elliptic estimates ensure that $\tilde{w}_\eta$ is bounded in $C^{0,\alpha}_{loc}(\{x_2 \geq 0\})$ for any $\alpha \in (0,1)$ and therefore admits a subsequence converging locally uniformly to a map $\tilde{w}_0 \colon \{x_2 \geq 0\} \to \mathbb{C}$ which solves

$$\begin{cases} \Delta \tilde{w}_0 = 0 & \text{in } \{x_2 > 0\}, \\ \tilde{w}_0 = e^{i\tilde{\phi}} & \text{on } \{x_2 = 0\}, \end{cases}$$

and satisfies $|\tilde{w}_0| \leq 1$. This system has a unique bounded solution, given by the Poisson formula

$$\tilde{w}_0(x_1, x_2) = \frac{x_2}{\pi} \int_{-\infty}^{\infty} \frac{e^{i\tilde{\phi}(t,0)}}{(x_1 - t)^2 + x_2^2}\, dt.$$

In particular no subsequence is needed for the locally uniform convergence, and we have

$$w_\varepsilon(0,\eta) = \tilde{w}_\eta(0,1) \longrightarrow \tilde{w}_0(0,1) \qquad \text{as } \varepsilon \to 0.$$

Using the explicit expression of $\tilde{\phi}$ we compute

$$\begin{aligned} \tilde{w}_0(0,1) &= \frac{2}{\pi} \left(\int_0^1 \frac{e^{i\pi(1-t)}}{1+t^2}\, dt + \int_1^\infty \frac{1}{1+t^2}\, dt \right) \\ &= \frac{2}{\pi} \int_0^1 \frac{1 - e^{-i\pi t}}{1+t^2}\, dt. \end{aligned}$$

It can be checked that

$$\left| \int_0^1 \frac{1 - e^{-i\pi t}}{1+t^2}\, dt \right| \leq 1,$$

and we infer that

$$\lim_{\varepsilon \to 0} |w_\varepsilon(0,\eta)|^2 \leq \frac{4}{\pi^2} < \frac{1}{2},$$

which implies (24). □

Acknowledgements

We thank Petru Mironescu, Roger Moser and Luc Nguyen for interesting comments. R.I. acknowledges partial support by the ANR project ANR-14-CE25-0009-01. X.L. acknowledges partial support by the ANR project ANR-18-CE40-0023.

References

[1] Berlyand, L., and Mironescu, P. Ginzburg-Landau minimizers with prescribed degrees. Capacity of the domain and emergence of vortices. *J. Funct. Anal. 239*, 1 (2006), 76–99.

[2] Berlyand, L., and Mironescu, P. Ginzburg-Landau minimizers in perforated domains with prescribed degrees. hal-00747687, June 2008.

[3] Bethuel, F., Brezis, H., and Hélein, F. Asymptotics for the minimization of a Ginzburg-Landau functional. *Calc. Var. Partial Differential Equations 1*, 2 (1993), 123–148.

[4] Bethuel, F., Brezis, H., and Hélein, F. *Ginzburg-Landau vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser Boston Inc., Boston, MA, 1994.

[5] Bethuel, F., Orlandi, G., and Smets, D. Improved estimates for the Ginzburg-Landau equation: the elliptic case. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5) 4*, 2 (2005), 319–355.

[6] Chen, X., Elliott, C. M., and Qi, T. Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation. *Proc. Roy. Soc. Edinburgh Sect. A 124*, 6 (1994), 1075–1088.

[7] Côte, R., Ignat, R., and Miot, E. A thin-film limit in the Landau-Lifshitz-Gilbert equation relevant for the formation of Néel walls. *J. Fixed Point Theory Appl. 15*, 1 (2014), 241–272.

[8] Grisvard, P. *Elliptic problems in nonsmooth domains*, vol. 69 of *Classics in Applied Mathematics*. Society for Industrial and Applied Mathematics (SIAM), Philadelphia, PA, 2011.

[9] Hervé, R.-M., and Hervé, M. Étude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg-Landau. *Ann. Inst. H. Poincaré Anal. Non Linéaire 11*, 4 (1994), 427–440.

[10] Ignat, R., and Kurzke, M. Global Jacobian and Γ-convergence in a Ginzburg-Landau model for boundary vortices. arXiv:1910.06039, 2019.

[11] Ignat, R., and Otto, F. A compactness result for Landau state in thin-film micromagnetics. *Ann. Inst. H. Poincaré Anal. Non Linéaire 28*, 2 (2011), 247–282.

[12] Jerison, D., and Kenig, C. E. The inhomogeneous Dirichlet problem in Lipschitz domains. *J. Funct. Anal. 130*, 1 (1995), 161–219.

[13] Jonsson, A., and Wallin, H. Function spaces on subsets of $\mathbf{R}^n$. *Math. Rep. 2*, 1 (1984), xiv+221.

[14] Kurzke, M. Boundary vortices in thin magnetic films. *Calc. Var. Partial Differential Equations 26*, 1 (2006), 1–28.

[15] Lamy, X. Bifurcation analysis in a frustrated nematic cell. *J. Nonlinear Sci. 24*, 6 (2014), 1197–1230.

[16] Sandier, E., and Serfaty, S. *Vortices in the magnetic Ginzburg-Landau model*, vol. 70 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 2007.

[17] Savaré, G. Regularity results for elliptic equations in Lipschitz domains. *J. Funct. Anal. 152*, 1 (1998), 176–201.