Institut de Mathématiques de Toulouse
Université Paul Sabatier

Habilitation à Diriger des Recherches

# Quelques contributions à l'analyse mathématique de modèles de cristaux liquides et d'énergies de ligne

*Some contributions to the mathematical analysis of liquid crystal and line-energy models*

Xavier Lamy

Soutenue le 15 décembre 2022,

après avis de:

**François Alouges**, Ecole Normale Supérieure Paris-Saclay
**Sylvia Serfaty**, New York University
**Didier Smets**, Sorbonne Université

devant le jury composé de:

**François Alouges**, Ecole Normale Supérieure Paris-Saclay
**Adriana Garroni**, Sapienza, University of Rome
**Radu Ignat**, Université Toulouse 3
**Petru Mironescu**, Université Lyon 1
**Félix Otto**, Max Planck Institute for Mathematics in the Sciences
**Sylvia Serfaty**, New York University
**Didier Smets**, Sorbonne Université

# Résumé

De nombreux phénomènes physiques peuvent être décrits par des modèles mathématiques similaires: cristaux liquides, supraconductivité, micromagnétisme, elasticité, formation de motifs, etc. Les états d'équilibre minimisent une certaine énergie, caractérisée par la compétition entre deux effets: un terme pénalise les déformations, mais un autre terme favorise les transitions de phases. Cette compétition encourage la formation de *singularités*: des déformations abruptes dans de petites régions où se concentrent les transitions de phases, et des déformations minimales en dehors. Les méthodes mathématiques du *calcul des variations* et des *équations aux dérivées partielles* permettent d'étudier les propriétés de ces états et de leurs singularités.

Les travaux présentés dans ce mémoire portent sur l'analyse de deux types de singularités: singularités ponctuelles ou linéaires dans des systèmes de dimension 3, et singularités linéaires dans des systèmes de dimension 2. Cette distinction reflète aussi leurs liens avec deux types d'équations aux dérivées partielles: *elliptiques* ou *hyperboliques*.

L'étude des singularités du premier type est motivée ici essentiellement par la physique des cristaux liquides, et structurée en deux axes de recherche: comprendre l'effet de l'immersion de particules étrangères, et celui d'une anisotropie dans la pénalisation des déformations. Pour le premier axe, la présence de particules impose des déformations au cristal liquide, et l'objectif est de décrire les singularités engendrées par ces déformations, ainsi que les interactions entre particules immergées. Pour le deuxième axe, l'anisotropie des déformations, physiquement plus réaliste, restreint considérablement les outils mathématiques disponibles: le défi principal est de développer de nouvelles techniques qui permettent d'analyser efficacement les modèles anisotropes.

L'étude des singularités du second type est motivée par de très divers phénomènes physiques: cristaux liquides, élasticité, micromagnétisme, formation de motifs, physique statistique. L'objectif est de comprendre les phénomènes de concentration de singularités linéaires: peuvent-elle s'accumuler au point de former des structures fractales? L'accent est mis ici sur la quantification des déformations admissibles et la stabilité de structures élémentaires bien identifiées. Du point de vue mathématique, ces questions présentent l'originalité d'aborder des équations hyperboliques par le biais du calcul des variations.

# Remerciements

Un grand merci tout d'abord à François Alouges, Sylvia Serfaty et Didier Smets pour avoir pris le temps d'écrire un rapport sur ce mémoire, et pour l'intérêt qu'ils ont exprimé pour mes travaux. Merci à Radu Ignat d'avoir parrainé cette habilitation. Merci à Adriana Garroni, Petru Mironescu et Felix Otto pour l'honneur et le plaisir qu'ils me font en participant au jury.

Merci à ceux qui ont guidé mes premiers pas de chercheur et m'ont ouvert de nombreux horizons: Petru Mironescu en dirigeant ma thèse de doctorat, Felix Otto lors de mon séjour postdoctoral.

Merci bien sûr à mes co-auteurs et co-autrice: Stan Alama, Lia Bronsard, Andrés Contreras, Andrés Contreras Hip, Benjamin Gess, Francesco Ghiraldin, Dmitry Golovaty, Radu Ignat, Mike Kowalczyk, Matthias Kurzke, Andrew Lorent, Elio Marconi, Petru Mironescu, Felix Otto, Guanying Peng, Rémy Rodiac, Panayotis Smyrnelis, Raghav Venkatraman, Andrés Zùñiga. Merci à tous les collègues avec qui, bien qu'ils ne soient pas dans cette liste, j'ai eu de nombreuses et enrichissantes discussions mathématiques.

Merci à tous les collègues de l'IMT pour l'atmosphère amicale et stimulante que j'y ai trouvée.

Merci enfin à mes amis, à ma famille, à Laury.

# Contents

# Chapter 1

# Introduction

This memoir contains a survey of my post-PhD work on singularities arising in phase transition models of Ginzburg-Landau type. I divided these models in two big categories depending on the type of singularities they feature: singularities of codimension 2 or 3, motivated mostly by the physics of liquid crystals; and line singularities in two dimensions, which can have various physical motivations (liquid crystals, but also micromagnetism, elasticity, statistical physics, etc.).

All variational models under consideration are characterized by the competition, at a typically small length-scale $\varepsilon$, of at least two terms: an elastic term penalizing deformations, and a potential term penalizing values away from a given manifold. A classical example is the Ginzburg-Landau energy

$$\int \left( \frac{1}{2}|\nabla u|^2 + \frac{1}{\varepsilon^2}(1-|u|^2)^2 \right) dx,$$

where the potential penalizes values away from $\mathbb{S}^1 \subset \mathbb{R}^2$. More generally, the potential can penalize values away from a submanifold $\mathcal{N} \subset \mathbb{R}^k$, and additional terms can be present. The limiting objects, as $\varepsilon \to 0$, are energy-minimizing manifold-valued maps, which solve an elliptic PDE. They can have singularities of codimension 2 (with infinite energy), and of codimension 3 or higher (with finite energy).

In some cases, an additional divergence constraint can be imposed (directly or via other terms in the energy), and we obtain a second type of models, where the limiting objects are divergence-free manifold-valued vector-fields. The corresponding PDE is usually hyperbolic, and they typically have singularities of codimension one.

In Chapter 2 I present results on models of the first type arising in the description of liquid crystals, and in Chapter 3 on two-dimensional models of the second type.

The type of models studied in Chapter 2 has a rich mathematical history, which includes the theory of harmonic maps – see for instance the monograph [Simon, 1996] – and the analysis of the Ginzburg-Landau energy initiated in [Bethuel et al., 1994]. I focus here on two specific lines of research motivated by the physics of liquid crystals: in § 2.2 I present results about exterior problems modelling the presence of foreign particles immersed in liquid crystals, and in § 2.3 about the effects of elastic anisotropy on the regularity and qualitative properties of minimizing configurations.

In Chapter 3, the limiting objects are weak solutions of the eikonal $|\nabla u| = 1$ (equivalently, divergence-free $\mathbb{S}^1$-valued vector fields $m = \nabla^\perp u$), or of Burgers' equation $u_t + (u^2/2)_x = 0$. These are extremely classical equations, with a well-studied class of specific solutions: viscosity solutions for the eikonal equation, entropy solutions for Burgers' equation. Here, however, the variational models under study lead to a much larger class of solutions which are still far from being well-understood. I present results on their regularity, and about rigidity and stability properties of specific solutions within that class.

# List of Publications

*Articles marked with ∘ were part of my PhD thesis or anterior.*

### Preprints

- "Stability of the vortex in micromagnetics and related models", with E. Marconi.
- "Quantitative rigidity of differential inclusions in two dimensions", with A. Lorent and G. Peng.
- "Far-field expansions for harmonic maps and the electrostatics analogy in nematic suspensions", with S. Alama, L. Bronsard and R. Venkatraman.

### Published or accepted

- "On optimal regularity estimates for finite-entropy solutions of scalar conservation laws", with A. Lorent and G. Peng.
  *C. R. Math. Acad. Sci. Paris*, to appear.
- "On a generalized Aviles-Giga functional: compactness, zero-energy states, regularity estimates and energy bounds", with A. Lorent and G. Peng.
  *Comm. Partial Differential Equations.*, 2022.
- "Singular perturbation of manifold-valued maps with anisotropic energy", with A. Contreras.
  *Anal. PDE*, 2022.
- "Entire vortex solutions of negative degree for the anisotropic Ginzburg-Landau system", with M. Kowalczyk and P. Smyrnelis.
  *Arch. Ration. Mech. Anal.*, 2022.
- "Generalized characteristics for finite entropy solutions of Burgers' equation", with A. Contreras Hip and E. Marconi.
  *Nonlinear Anal.*, 2022.
- "On the stability of radial solutions to an anisotropic Ginzburg-Landau equation", with A. Zúñiga.
  *SIAM J. Math. Anal.*, 2022.

- "Saturn ring defect around a spherical particle immersed in nematic liquid crystal", with S. Alama, L. Bronsard and D. Golovaty.
  *Calc. Var. Partial Differential Equations*, 2021.

- "On the $L^2$ stability of shock waves for finite entropy solutions of Burgers", with A. Contreras Hip.
  *J. Differential Equations*, 2021.

- "Rigidity of a non-elliptic differential inclusion related to the Aviles-Giga conjecture", with A. Lorent and G. Peng.
  *Arch. Ration. Mech. Anal.*, 2020.

- "Global uniform estimate for the modulus of 2D Ginzburg-Landau vortexless solutions with asymptotically infinite boundary energy", with R. Ignat and M. Kurzke.
  *SIAM J. Math. Anal.*, 2020.

- "Optimal Besov differentiability for entropy solutions of the eikonal equation", with F. Ghiraldin.
  *Comm. Pure Appl. Math.*, 2020.

- "Lifting of $\mathbb{RP}^{d-1}$-valued maps in $BV$ and applications to uniaxial $Q$-tensors. With an appendix on an intrinsic $BV$-energy for manifold-valued maps", with R. Ignat.
  *Calc. Var. Partial Differential Equations*, 2019.

- "On the convergence of minimizers of singular perturbation functionals", with A. Contreras and R. Rodiac.
  *Indiana Univ. Math. J.*, 2018.

- "Regularity of solutions to scalar conservation laws with a force", with B. Gess.
  *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2018.

- "On the regularity of weak solutions to Burgers' equation with finite entropy production", with F. Otto.
  *Calc. Var. Partial Differential Equations*, 2018.

- "Spherical particle in nematic liquid crystal under an external field: the Saturn ring regime", with S. Alama and L. Bronsard.
  *J. Nonlinear Sci.*, 2018.

- "Biaxial escape in nematics at low temperature", with A. Contreras.
  *J. Funct. Anal.*, 2017.

- "Minimizers of the Landau-de Gennes energy around a spherical colloid particle", with S. Alama and L. Bronsard.
  *Arch. Ration. Mech. Anal.*, 2016.

- "Analytical description of the Saturn-ring defect in nematic colloids", with S. Alama and L. Bronsard.
  *Phys. Rev. E*, 2016.

- "Boundary regularity of weakly anchored harmonic maps", with A. Contreras and R. Rodiac.
  *C. R. Math. Acad. Sci. Paris*, 2015.

- "Vortex structure in $p$-wave superconductors", with S. Alama and L. Bronsard.
  *J. Math. Phys*, 2015.

- "Persistence of superconductivity in thin shells behond $H_{c_1}$", with A. Contreras.
  *Commun. Contemp. Math.*, 2015.

- "Characterization of function spaces via low regularity mollifiers", with P. Mironescu.
  *Discrete Contin. Dyn. Syst. A*, 2015.

- "Uniaxial symmetry in nematic liquid crystals".
  *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 2015.

- "Bifurcation analysis in a frustrated nematic cell".
  *J. Nonlinear Sci.*, 2014.

- "Existence of critical points with semi-stiff boundary conditions for singular perturbation problems in simply connected planar domains", with P. Mironescu.
  *J. Math. Pures Appl.*, 2014.

- "Some properties of the nematic radial hedgehog in the Landau–de Gennes theory".
  *J. Math. Anal. Appl.*, 2013.

# Chapter 2

# Defects in liquid crystals

## 2.1 Introduction

### 2.1.1 Order parameters

Liquid crystals are a state of matter characterized by their *orientational order*. There are many different types of liquid crystals with different levels of order and symmetries. Here we focus mostly on nematic liquid crystals, which present the most basic type of orientational order : roughly speaking, molecules are elongated and tend to align in the same direction. Two types of macroscopic order parameter can be used to describe the nematic phase:

- a unit director $n \in \mathbb{S}^2$, giving the local direction of alignment of the molecules,
- or a so-called $Q$-tensor $Q \in \mathbb{R}^{3\times 3}_{sym}$, $\mathrm{tr}(Q) = 0$, whose eigendirections and eigenvalues reflect the local alignment (via the anisotropic response to external fields caused by orientational order), and which can be related to the statistics of alignement of the molecules at the microscopic level [De Gennes and Prost, 1993].

The $Q$-tensor order parameter can describe phase transitions between non-ordered and ordered states. One can distinguish three types of phases depending on their symmetry:

- Isotropic: $Q$ commutes with all rotations; all eigenvalues are equal and $Q = 0$.
- Uniaxial: $Q$ commutes with rotations of a given axis $n \in \mathbb{S}^2$; two eigenvalues are equal but nonzero, and one can write $Q$ in the form

$$Q = s\left(n \otimes n - \frac{1}{3}I\right), \quad s \in \mathbb{R} \setminus \{0\},\ n \in \mathbb{S}^2. \tag{2.1}$$

- Biaxial: $Q$ does not commute with any nontrivial rotation; its three eigenvalues are distinct.

The director description can be interpreted via (2.1) as considering only uniaxial phases with fixed eigenvalues. Because $n$ and $-n$ are equivalent in this context (in agreement with head-to-tail symmetry of nematic molecules), it is actually more accurate to consider $n \in \mathbb{RP}^2 = \mathbb{S}^2/\{\pm 1\}$. (This leads to interesting lifting issues which we will not discuss here in detail, see e.g. [Bethuel and Chiron, 2007, Ball and Zarnescu, 2011, Bedford, 2016, Ignat and Lamy, 2019, Canevari and Orlandi, 2019].)

### 2.1.2 Energy functionals

A nematic configuration in a domain $\Omega \subset \mathbb{R}^3$ will be described by a map

$$n \colon \Omega \to \mathbb{S}^2,$$

or

$$Q \colon \Omega \to \mathcal{S}_0 = \left\{ Q \in \mathbb{R}^{3\times 3}_{sym} \colon \operatorname{tr} Q = 0 \right\}.$$

Singularities of the map $n$, or of the map $Q$'s eigenvectors, are called *defects*. Equilibrium configurations are expected to be local minima of a certain energy, and we wish to exploit this variational characterization to gain insight into the physics of defects.

In a simplified setting, the energy of a director configuration is given by

$$E(n) = \int_\Omega |\nabla n|^2 \, dx,$$

and minimizing configurations (with respect to their own boundary conditions) are $\mathbb{S}^2$-valued harmonic maps. They are smooth away from a discrete set of point singularities [Schoen and Uhlenbeck, 1982, Schoen and Uhlenbeck, 1983], all modeled on the radial map $\pm x/|x|$ [Brezis et al., 1986].

Again in a simplified setting, the energy of a $Q$-tensor configuration is of the form

$$E_\varepsilon(Q) = \int_\Omega \left( \frac{1}{2}|\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q) \right) dx, \tag{2.2}$$

where $f(Q) \geq 0$ is a potential accounting for phase transitions, and $\varepsilon > 0$ is the characteristic length scale at which the two terms compete.

The potential $f(Q)$ is a quartic polynomial whose coefficients depend on material properties. Most relevant to us here is that it is frame invariant, $f(RQR^{-1}) = f(Q)$ for any isometry $R \in O(3)$, and minimized by uniaxial $Q$-tensors:

$$f^{-1}(\{0\}) = \left\{ s_* \left( n \otimes n - \frac{1}{3} I \right) : n \in \mathbb{S}^2 \right\} =: \mathcal{U}_* \approx \mathbb{RP}^2, \tag{2.3}$$

where $s_* > 0$ depends on the precise expression of $f$. Moreover that minimum is non-degenerate: the Hessian of $f$ at any element $Q_* \in \mathcal{U}_*$ is positive definite in directions orthogonal to the tangent plane $T_{Q_*}\mathcal{U}_*$.

Hence for $\varepsilon \ll 1$ one expects minimizers of the energy (2.2) to satisfy

$$Q(x) \approx s_* \left( n(x) \otimes n(x) - \frac{1}{3} I \right),$$

where $n \colon \Omega \to \mathbb{S}^2$ minimizes the Dirichlet energy. This approximation is very good away from singularities of $n$ [Majumdar and Zarnescu, 2010, Nguyen and Zarnescu, 2013], and the fine properties of singularities can be understood by zooming in at scale $\varepsilon$, which leads to the study of entire solutions of the Euler-Lagrange equations $\Delta Q = \nabla f(Q)$.

Such considerations are extremely similar to the analysis of the Ginzburg-Landau energy, pioneered in [Bethuel et al., 1994], where $\mathbb{R}^2$-valued maps approach $\mathbb{S}^1$-valued maps. Yet several differences arise, due for instance to the different topological properties of $\mathbb{S}^1$ and $\mathbb{RP}^2$, the higher codimension of the embedding $\mathbb{RP}^2 \subset \mathcal{S}_0$ versus $\mathbb{S}^1 \subset \mathbb{R}^2$, and the specific symmetries and phases of $Q$-tensors. Moreover, physical applications motivate new types of questions.

In the above simplified models, the elastic energy densities $|\nabla n|^2$ and $|\nabla Q|^2$ are highly symmetric. More realistic models must, however, take into account elastic anisotropy: $|\nabla n|^2$ may be replaced by the Oseen-Frank energy density

$$k_1(\nabla \cdot n)^2 + k_2(n \cdot \nabla \times n)^2 + k_3|n \times (\nabla \times n)|^2,$$

for some elastic constants $k_1, k_2, k_3 > 0$ respectively associated to *splay*, *twist* and *bend* deformations (configurations for which only the corresponding term is nonzero). For $k_1 = k_2 = k_3$ (the one-constant approximation) one recovers an energy density proportional to $|\nabla n|^2$, up to a null Lagrangian term. Similarly, $|\nabla Q|^2$ may be replaced by $\mathbb{A}(Q)[\nabla Q]$, where $\mathbb{A}(Q)$ is a positive definite quadratic form on gradients of $\mathcal{S}_0$-valued maps. The reduced symmetry of these more general models strongly restrict the available mathematical tools, and many questions are open.

In the rest of this chapter I will describe my and my coauthors' contributions on two types of problems about minimizers of these energy functionals:

- understanding the effects of foreign particles immersed in nematic configurations (§ 2.2),
- and understanding the effects of elastic anisotropy (§ 2.3).

## 2.2 Nematic colloids

Nematic colloids are systems composed of foreign particles immersed in nematic liquid crystal. Nematic alignment is modified by these particles via their *anchoring* properties: liquid crystal molecules are forced to align in a certain way at the particles' surfaces. Different aspects of such systems lead to different types of mathematical questions. One can for instance consider densely packed colloid particles and study them from a homogenization perspective [Berlyand et al., 2005, Calderer et al., 2014,

Canevari and Zarnescu, 2020b, Canevari and Zarnescu, 2020a]. Here we concentrate instead on the effect of a single particle, with the long-term goal of characterizing interactions between particles that are not too densely packed. This is motivated by the so-called *electrostatic analogy*, an approximation that can be found in the physics literature, where each particle is replaced by a singular forcing term in a linearized equation: the form of the singular term depends on the type of defects and long-range deformations induced by the single particle (see e.g. the review article [Muševič, 2019]).

To fix ideas, consider a spherical particle represented by the unit ball $B \subset \mathbb{R}^3$, so the liquid crystal is contained in the exterior domain $\Omega = \mathbb{R}^3 \setminus \overline{B}$, and assume that the particle's surface promotes radial anchoring: within a director description, this amounts to

$$n = \mathbf{e}_r \quad \text{on } \partial B, \quad \text{where } \mathbf{e}_r = \frac{x}{|x|}.$$

Far away from the particle, uniform alignment is favored, say along the $\mathbf{e}_3$ direction:

$$n(x) \to \mathbf{e}_3 \qquad \text{for } |x| \to +\infty.$$

The topological constraint imposed by these boundary conditions forces the appearance of singularities: the map $\mathbf{e}_r \colon \mathbb{S}^2 \to \mathbb{S}^2$ is not homotopic to a constant within $\mathbb{S}^2$-valued maps. Two types of defects are observed in experiments:

- a single point singularity forming a dipole with the particle,
- or a line singularity forming a 'Saturn ring' around the particle.

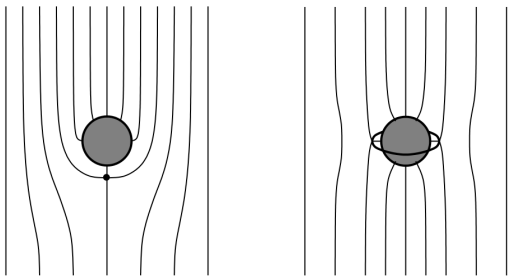

Figure 2.1: Schematic representations of a dipole (left) and a Saturn ring (right) configuration [Stark and Ventzki, 2001].

*Remark* 2.1. To understand how the topological constraint is resolved by such singularities, it is convenient to think about axisymmetric configurations, fully determined by their restriction to a planar slice containing the $\mathbf{e}_3$ axis,

$$\bar{n} \colon \mathbb{R}^2 \setminus \overline{D} \to \mathbb{S}^1,$$

where $D \subset \mathbb{R}^2$ is the unit disk. The boundary datum $n_{\partial D} = \mathbf{e}_r$ has winding number, or *degree*, equal to 1. This topological charge needs to be cancelled in $\pi_1(\mathbb{S}^1) \approx \mathbb{Z}$ in order to match the degree 0 constant map at infinity. The simplest way to do that is to create one point singularity of degree $-1$, placed on the $\mathbf{e}_3$ axis: this gives the dipole configuration; see Figure 2.1. To describe the Saturn ring configuration, recall that $n$ and $-n$ should actually be equivalent, so singularities of half-integer degree are allowed (half a turn brings $n$ to $-n$). Therefore the degree 1 boundary data can also be cancelled by two point singularities of degree $-1/2$, to the left and the right of $D$: in the three-dimensional axially symmetric picture, these are the trace of a ring singularity around the particle; see Figures 2.1 and 2.3. (Similarly, in the absence of a foreign particle, a $+1$ point defect is topologically equivalent to a $+1/2$ ring defect, and determining the occurrence of both configurations has motivated several interesting works [Dipasquale et al., 2021a, Dipasquale et al., 2020, Dipasquale et al., 2021b, Yu, 2020].)

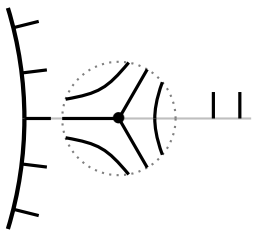

Figure 2.2: Schematic view of a $(-1/2)$ defect in a planar slice, to the right of the particle's equator. Along a horizontal ray, the director goes from $\mathbf{e}_r$ at the particle's surface, to $\mathbf{e}_3$ at infinity. Rotating this picture around the vertical axis gives a Saturn ring as in Figure 2.1.

There are two important qualitative differences between these topologically admissible configurations: their *symmetry* and *orientability* properties. First, unlike the dipole, the Saturn-ring is mirror-symmetric across the equatorial plane. Second, the dipole configuration is orientable: it can be described by an $\mathbb{S}^2$-valued map continuous outside the singular point. But the Saturn ring requires an $\mathbb{RP}^2$-valued map: an $\mathbb{S}^2$-valued map would be singular on the full equatorial plane.

In my PhD thesis, we gave, in collaboration with S. Alama and L. Bronsard, a mathematical justification of the occurrence of these two configurations depending on the particle's radius [Alama et al., 2016]: within the $Q$-tensor description, we proved that minimizers have a Saturn ring structure if the particle is much smaller than the characteristic length scale $\varepsilon$, and a dipole structure if the particle is much larger. Among many questions left open by that work, one concerns the description of Saturn ring structures around not-so-small particles, which seem stable in experiments; and another one is the asymptotic description of nematic alignment far away from the particle, which is at the basis of the aforementioned electrostatics analogy. These questions are the subject of the three works I describe in the rest of this section:

- In [Alama et al., 2018] with S. Alama and L. Bronsard we investigate the stabiliz-

ing effect of an external magnetic field on the Saturn ring structure.

- In [Alama et al., 2021] with S. Alama, L. Bronsard and D. Golovaty we obtain Saturn ring configurations around large particles as minimizers subject to a mirror symmetry constraint.
- In [Alama et al., 2022] with S. Alama, L. Bronsard and R. Venkatraman we study the far-field expansion of minimizing configurations and characterize its first coefficient as a function of the particle and its anchoring properties.

### 2.2.1 The effect of an external field

Experiments [Gu and Abbott, 2000, Loudet and Poulin, 2001] and numerical simulations [Fukuda et al., 2004a, Fukuda and Yokoyama, 2006] have demonstrated that applying a strong enough electric or magnetic field stabilizes the Saturn-ring defect around a colloid particle. A heuristic explanation is proposed in [Stark, 2002, Fukuda and Yokoyama, 2006]: the applied field favorizes uniform alignment, hence confines deformations to a thin region around the particle. Under this confinement, conjectured orders of magnitude suggest that the Saturn ring costs less energy than the dipole, provided the external field has high enough amplitude. In this section, we present energy bounds proved in [Alama et al., 2018] and showing that, although the orders of magnitude conjectured in [Stark, 2002, Fukuda and Yokoyama, 2006] seem incorrect, the minimizing configuration should indeed be a Saturn ring when the field's amplitude is well above a critical value.

Following [Fukuda et al., 2004a, Fukuda and Yokoyama, 2006] we make the simplifying assumption that the external field is constant throughout the system: very far from the challenging mathematical analysis of interactions between electromagnetic field and nematic material (see e.g. the monograph [Sandier and Serfaty, 2007] for superconductors), we simply want to identify, in the most elementary mathematical setting, the confinement mechanism described in [Stark, 2002]. This amounts to adding a symmetry-breaking term to the Landau-de Gennes energy (2.2):

$$E_{\varepsilon,\eta}(Q;\Omega) = \int_\Omega \left( \frac{1}{2}|\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q) + \frac{1}{\eta^2} g(Q) \right) dx. \tag{2.4}$$

Here the new characteristic length $\eta$ is related to the amplitude $h$ of the applied field via $\eta \sim \varepsilon/h$. The potential $g(Q) = \sqrt{2/3} - Q_{33}/|Q|$ (and $g(0) = 0$) satisfies

$$g(Q) \geq 0 \quad \text{and} \quad g^{-1}(\{0\}) = \{\lambda\, Q_\infty \colon \lambda \geq 0\}\,,$$
$$Q_\infty = \left( \mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3} I \right),$$

hence induces alignment along the direction $\mathbf{e}_3$ of the applied field. Recall that the domain $\Omega = \mathbb{R}^3 \setminus \overline{B}$ is the exterior of the unit ball representing the (rescaled) particle. The uniform alignment favored by the external field competes with radial anchoring conditions at the particle surface:

$$Q = Q_b = s_* \left( \mathbf{e}_r \otimes \mathbf{e}_r - \frac{1}{3} I \right) \quad \text{on } \partial\Omega = \partial B. \tag{2.5}$$

Here $s_* > 0$ is uniquely determined by the potential $f$ via (2.3). The confinement effect comes from the competition between field-induced alignment and boundary anchoring, which suggests a boundary layer of width $\eta$.

One way of characterizing the Saturn ring configuration is via its mirror symmetry with respect to the equatorial plane $\{x_3 = 0\}$, which is completely broken by the dipole configuration. For a high enough external field, this symmetry is satisfied asymptotically by the energy of minimizers:

**Theorem 2.2.** *If $\eta_\varepsilon > 0$ is such that*

$$0 < \eta_\varepsilon \ll \frac{1}{|\ln \varepsilon|} \qquad \text{as } \varepsilon \to 0,$$

*and $Q_\varepsilon \colon \Omega \to \mathcal{S}_0$ minimizes $E_{\varepsilon,\eta_\varepsilon}$ with radial anchoring conditions (2.5), then*

$$E_{\varepsilon,\eta_\varepsilon}(Q_\varepsilon; \Omega \cap \{x_3 > 0\}) \sim E_{\varepsilon,\eta_\varepsilon}(Q_\varepsilon; \Omega \cap \{x_3 < 0\}) \qquad \text{as } \varepsilon \to 0.$$

In terms of external field amplitude, the condition $\eta \ll 1/|\ln \varepsilon|$ corresponds to $h \gg \varepsilon |\ln \varepsilon|$, in agreement with the critical value $h_c \approx \varepsilon |\ln \varepsilon|$ conjectured in [Fukuda and Yokoyama, 2006]. Theorem 2.2 is proved in [Alama et al., 2018] as a consequence of more precise energy asymptotics: we show that the energy is concentrated in a boundary layer of size $\eta$, where a one-dimensional transition takes place, along radial rays, from the boundary value $Q_b$ to a field-aligned state proportional to $Q_\infty$.

The energy of such one-dimensional transition is given by

$$F_\lambda(Q) = \int_1^\infty \left( \left| \frac{dQ}{dr} \right|^2 + \lambda^2 f(Q) + g(Q) \right) dr,$$

where $\lambda \in [0, \infty]$ is the limit of the ratio $\eta/\varepsilon$. The case $\lambda = \infty$ should be understood as imposing that $Q$ takes values in $\mathcal{U}_* = f^{-1}(\{0\})$. The energy of the boundary layer is then described via the function $D_\lambda \colon \mathcal{S}_0 \to [0, \infty]$ given by

$$D_\lambda(Q_0) = \min \left\{ F_\lambda(Q) \colon Q \in H^1_{loc}([1, \infty); \mathcal{S}_0),\ Q(1) = Q_0 \right\}. \tag{2.6}$$

The existence of a minimizer can be shown using the direct method of the calculus of variations. Moreover the function $D_\lambda$ is continuous on its domain $\{D_\lambda < \infty\}$, and its values depend continuously on $\lambda$. (The degenerate case $\lambda = 0$ was left aside in [Alama et al., 2018] because finite-energy maps do not necessarily have a limit as $r \to \infty$, but this constitutes in fact no obstacle and that case can be included with minor modifications.)

The main result of [Alama et al., 2018] states that, to main order, the energy corresponds indeed to that boundary layer.

**Theorem 2.3.** *Let $\eta_\varepsilon > 0$ be such that*

$$0 < \eta_\varepsilon \ll \frac{1}{|\ln \varepsilon|} \quad \text{and} \quad \frac{\eta_\varepsilon}{\varepsilon} \to \lambda \in [0, \infty] \qquad \text{as } \varepsilon \to 0,$$

*and* $Q_\varepsilon \colon \Omega \to \mathcal{S}_0$ *minimize* $E_{\varepsilon,\eta_\varepsilon}$ *with radial anchoring conditions* (2.5). *For any measurable* $U \subset \mathbb{S}^2$, *the energy in the cone* $\mathcal{C}(U) = \{t\omega \colon t > 1,\, \omega \in U\} \subset \Omega$ *satisfies*

$$E_{\varepsilon,\eta_\varepsilon}(Q_\varepsilon; \mathcal{C}(U)) \sim \frac{1}{\eta_\varepsilon} \int_U D_\lambda(Q_b(\omega))\, d\mathcal{H}^2(\omega),$$

*as* $\varepsilon \to 0$.

This implies the asymptotic mirror symmetry stated in Theorem 2.2 because the boundary condition $Q_b$ is itself symmetric.

The proof of Theorem 2.3 relies on a lower bound obtained via an elementary scaling argument, and a matching upper bound, obtained via different methods for $\lambda < \infty$ and $\lambda = \infty$. We focus here on the case $\lambda = \infty$, which is more interesting because $\eta$ is closer to the critical value $1/|\ln \varepsilon|$. Moreover we have a more precise description in that case, because minimizers of $F_\infty$ can be determined explicitly:

**Proposition 2.4.** *For* $n_b \in \mathbb{S}^2$ *of spherical coordinates* $(\theta, \varphi) \in [0, \pi] \times [0, 2\pi)$ *with* $\theta \neq \pi/2$, *and* $Q_b = s_* \left(n_b \otimes n_b - I/3\right) \in \mathcal{U}_*$, *the minimum* $D_\infty(Q_b)$ *in* (2.6) *is attained exactly by a map* $r \mapsto \overline{Q}(r, \theta, \varphi)$ *of the form*

$$\overline{Q}(r, \theta, \varphi) = s_* \left( n(r, \theta, \varphi) \otimes n(r, \theta, \varphi) - \frac{1}{3} I \right),$$

*where* $n(r, \theta, \varphi) \in \mathbb{S}^2$ *is*

- *axisymmetric:* $n(r, \theta, \varphi) = R_\varphi n(r, \theta, 0)$, *for* $R_\varphi$ *the rotation of axis* $\mathbf{e}_3$ *and angle* $\varphi$,
- *and mirror-symmetric:* $n(r, \pi - \theta, \varphi) = Sn(r, \theta, \varphi)$, *for* $S$ *the reflection across* $\mathbf{e}_3^\perp$.

*Moreover, in the upper half domain* $\{r \geq 1,\, 0 \leq \theta < \pi/2\}$, *the map* $n$ *is smooth, and*

$$|n - \mathbf{e}_3|^2 + |\nabla n|^2 \leq C e^{-\kappa r},$$

*for some constants* $C, \kappa > 0$ *depending only on* $s_*$.

*Remark* 2.5. The map $n$ is given explicitly, for $0 \leq \theta < \pi/2$, by

$$n(r, \theta, \varphi) = \left(\cos\varphi \sin\Theta(r, \theta), \sin\varphi\, \sin\Theta(r, \theta), \cos\Theta(r, \theta)\right),$$
$$\Theta(r, \theta) = 2 \arctan\left( e^{\frac{\kappa}{2}(1-r)} \tan\frac{\theta}{2} \right), \qquad \kappa = \frac{(24)^{\frac{1}{4}}}{s_*}.$$

The proof follows classical ideas for the existence of heteroclinic connections, see e.g. [Sternberg, 1991]. This also provides the explicit value $D_\infty(Q_b) = s_* \kappa (1 - |\cos\theta|)$. For $\theta = \pi/2$, the problem $D_\infty(Q_b)$ admits two minimizers, described by the left and right limits $n(r, (\pi/2)^+, \varphi)$ and $n(r, (\pi/2)^-, \varphi)$.

The upper bound in Theorem 2.3 for $\lambda = \infty$ is obtained by constructing a comparison map with a boundary layer given by Proposition 2.4:

$$Q(r,\theta,\varphi) \approx \overline{Q}\left(1 + \frac{r-1}{\eta}, \theta, \varphi\right).$$

Because $\overline{Q}$ is discontinuous at the equatorial plane $\{x_3 = 0\} = \{\theta = \pi/2\}$, this needs to be modified around that plane, and replaced by a Saturn ring defect with energy cost of order $|\ln \varepsilon|$, hence the upper bound

$$\min E_{\varepsilon,\eta} \leq \frac{1}{\eta}\int_{\mathbb{S}^2} D_\infty(Q_b)\, d\mathcal{H}^2 + s_*^2\pi^2 |\ln\varepsilon| + \mathcal{O}(1). \tag{2.7}$$

The map providing this upper bound is $\mathcal{U}_*$-valued, except in the defect core of the Saturn ring, of thickness $\varepsilon$. The second term is negligible if $|\ln\varepsilon| \ll 1/\eta$, and this is where the critical value $\eta \sim 1/|\ln\varepsilon|$ comes from.

One can be slightly more precise concerning this critical value, by recalling that Saturn ring and dipole can be distinguished not only via their symmetry, but also through their orientability properties. Because in the regime $\varepsilon \ll \eta$ ($\lambda = \infty$) we expect the boundary layer to be mostly $\mathcal{U}_*$-valued, it makes sense to consider $\mathcal{U}_*$-valued competitors. We call a $\mathcal{U}_*$-valued map $Q$ *orientable* if it can be written in the form

$$Q = s_*\left(n \otimes n - \frac{1}{3}I\right),$$

where $n\colon \Omega \to \mathbb{S}^2$ has finite energy $\int_\Omega |\nabla n|^2\, dx < \infty$. In that case, the value of $n$ outside the boundary layer must choose one of the two orientations $\pm\mathbf{e}_3$ (which would be equivalent in the nonorientable $\mathbb{RP}^2$-valued setting). With this restriction, the minimization of $F_\infty$ provides a value twice as high (not four times, as erroneously stated in [Alama et al., 2018, Proposition 1.6]), and we deduce that

$$\min\{E_{\varepsilon,\eta}(Q;\Omega),\ Q\colon \Omega \to \mathcal{U}_* \text{ orientable}\} \geq \frac{2}{\eta}\int_{\mathbb{S}^2} D_\infty(Q_b)\, d\mathcal{H}^2.$$

Comparing with (2.7), this suggests that the Saturn ring configuration has lower energy whenever $\eta < \beta_c/|\ln\varepsilon|$, where $s_*^2\pi^2\beta_c = \int_{\mathbb{S}^2} D_\infty(Q_b)\, d\mathcal{H}^2$, that is, $s_*\beta_c = 2\kappa/\pi$.

Following our work [Alama et al., 2018], the critical regime $\eta \sim \beta/|\ln\varepsilon|$ has been studied in much greater detail in [Alouges et al., 2021], which the authors even generalize later on to non-spherical geometries [Alouges et al., 2022]. They perform a complete $\Gamma$-convergence analysis and are able to describe the regimes in which Saturn ring or dipole are minimizing or locally minimizing. Their conclusions confirm in particular the critical value $s_*\beta = 2\kappa/\pi$ (with $\kappa = 2c_*$ in their notations) as the value at which both configurations have the same asymptotic energy.

### 2.2.2 Saturn-ring configurations under a symmetry constraint

In the absence of an external field, we proved in [Alama et al., 2016] with S. Alama and L. Bronsard that very small particles with radial anchoring generate a Saturn ring defect

around them. Since the particle is represented by the unit ball $B$, this corresponds to the regime $\varepsilon \gg 1$ in the Landau-de Gennes energy (2.2). This approach provides a very good description of the Saturn ring, but is not entirely satisfactory because Saturn rings are observed in practice around quite larger particles [Lubensky et al., 1998]. In this section we present a description of a Saturn ring configuration for $\varepsilon \ll 1$, obtained in [Alama et al., 2021] by minimizing the Landau-de Gennes energy under a mirror symmetry constraint. The main result of [Alama et al., 2021] can be split in two parts:

- Precise energy asymptotics consistant with the presence of a single ring defect in the equatorial plane, at distance $\sim 1/|\ln \varepsilon|$ from the particle. This is essentially an adaptation of now-standard arguments from the study of the 2D Ginzburg-Landau functional [Bethuel et al., 1994, Struwe, 1994, Jerrard, 1999, Sandier, 1998], and weighted versions of it [André and Shafrir, 1998, Beaulieu and Hadiji, 1998], together with recent adjustments for the $Q$-tensor setting [Golovaty and Montero, 2014, Canevari, 2015].

- A refined study of the limit configuration ruling out the possible presence of point defects, under an extra axisymmetry assumption. This part is much less standard, as point defects are not "seen" by the energy asymptotics of the first part, and it is well known that energy-minimizing $\mathbb{S}^2$-valued maps may have many point singularities even in the absence of topological constraints [Hardt and Lin, 1986].

In [Alama et al., 2021] axisymmetry is imposed all along, but is really needed only for the second part, as we will explain below.

We first introduce some notations. As in the previous section, we work in the exterior domain $\Omega = \mathbb{R}^3 \setminus \overline{B}$. We impose radial boundary conditions at the particle's surface (2.5), and uniform alignment at infinity:

$$\int_\Omega \frac{|Q - Q_\infty|^2}{r^2}\, dx < \infty, \qquad Q_\infty = s_* \left( \mathbf{e}_3 \otimes \mathbf{e}_3 - \frac{1}{3} I \right).$$

Moreover we impose a mirror symmetry constraint:

$$Q(Sx) = SQ(x)S^{-1} \qquad \forall x \in \Omega,$$

where $S = I - \mathbf{e}_3 \otimes \mathbf{e}_3$ is the reflection across the equatorial plane $\mathbf{e}_3^\perp$. We denote by $\mathcal{H}_{mirror}$ the space of maps $Q \in H^1_{loc}(\overline{\Omega}; \mathcal{S}_0)$ satisfying these constraints. The Landau-de Gennes energy

$$E_\varepsilon(Q; \Omega) = \int_\Omega \left( \frac{1}{2} |\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q) \right) dx,$$

admits a minimizer $Q_\varepsilon$ in this space (as can be seen with the direct method of the calculus of variations, using Hardy's inequality to obtain coercivity).

**Energy asymptotics**

A first important remark is that, in the equatorial plane $\{x_3 = 0\}$, mirror symmetry forces $\mathbf{e}_3$ to be an eigenvector of $Q(x_1, x_2, 0)$. There is no continuous path in $\mathcal{U}_*$ going from $Q_b(x_1, x_2, 0)$ to $Q_\infty$ while having $\mathbf{e}_3$ as an eigenvector: therefore, at least one singularity will form in each radial direction in the equatorial plane, creating a singular ring around the particle.

The energetic cost of an interior ring defect of degree $(-1/2)$ should be of order $\ell(\pi/2)|\ln\varepsilon|$, where $\ell$ is the length of the ring. If the ring is at distance $\sigma$ from the equator, that length is $\ell = 2\pi(1+\sigma)$, so one would like to take $\sigma \to 0$, and the ring defect is no longer interior: from scale $\varepsilon$ to $\sigma$ its cost is $\ell(\pi/2)\ln(\sigma/\varepsilon)$, and from scale $\sigma$ to 1 it has the cost $\ell_0\pi|\ln\sigma|$ of a boundary ring defect of degree $(-1/2)$, with $\ell_0 = 2\pi$. The total energy cost, divided by $2\pi$, is therefore of order

$$\frac{\pi}{2}\ln\frac{\sigma}{\varepsilon} + \sigma\frac{\pi}{2}\ln\frac{1}{\varepsilon} + \pi\ln\frac{1}{\sigma}.$$

Minimizing this with respect to $\sigma$ gives $\sigma = 1/|\ln\varepsilon|$. This type of reasoning is standard in the study of weighted Ginzburg-Landau energies [André and Shafrir, 1998, Beaulieu and Hadiji, 1998]. Note that it actually does not distinguish between a ring defect of degree $-(1/2)$, which is the configuration we expect, or degree $+(1/2)$, which could be compensated by point singularities of negative degrees: the lower energy cost of point singularities would not change the main order asymptotics.

These heuristic considerations lead to the following energy bounds.

**Theorem 2.6.** *Let $Q_\varepsilon$ minimize $E_\varepsilon$ in $\mathcal{H}_{mirror}$. We have, as $\varepsilon \to 0$,*

$$\frac{1}{2\pi}E_\varepsilon(Q_\varepsilon;\Omega) = \frac{\pi}{2}|\ln\varepsilon| + \frac{\pi}{2}\ln|\ln\varepsilon| + \mathcal{O}(1),$$

*and the energy is concentrated near the equatorial circle $\mathcal{C} = \partial B \cap \{x_3 = 0\}$. Specifically, for any $\delta > 0$, its $\delta$-neighborhood $\Omega_\delta^{int} = \Omega \cap \{\mathrm{dist}(\cdot,\mathcal{C}) \le \delta\}$ satisfies the energy lower bound*

$$\frac{1}{2\pi}E_\varepsilon(Q_\varepsilon;\Omega_\delta^{int}) \ge \frac{\pi}{2}|\ln\varepsilon| + \frac{\pi}{2}\ln|\ln\varepsilon| - \pi|\ln\delta| - C,$$

*for some absolute constant $C > 0$.*

The upper bound in the energy asymptotics of Theorem 2.6 is obtained by constructing an axisymmetric configuration with a ring defect at distance $\sigma = 1/|\ln\varepsilon|$ from the particle. By axisymmetric we mean, using cylindrical coordinates $(\rho,\varphi,z)$, that

$$Q(\rho,\varphi,z) = R_\varphi Q(\rho,0,z)R_\varphi^{-1}, \tag{2.8}$$

where $R_\varphi$ is the rotation of axis $\mathbf{e}_3$ and angle $\varphi$. The proof of the lower bound starts by writing

$$\begin{aligned} E_\varepsilon(Q;\Omega) &= \int_0^{2\pi}\iint_{\rho^2+z^2>1}\left(|\nabla_{\rho,z}Q|^2 + \frac{1}{\rho^2}|\partial_\varphi Q|^2 + \frac{1}{\varepsilon^2}f(Q)\right)\rho\,d\rho dz\,d\varphi \\ &\ge \int_0^{2\pi} E_\varepsilon^{2D}(Q(\cdot,\varphi,\cdot);U)\,d\varphi, \end{aligned}$$

where $U = \big\{(\rho, z)\colon \rho > 0,\ \rho^2 + z^2 > 1\big\}$ is the 2D azimuthal slice of $\Omega$, and

$$E_\varepsilon^{2D}(Q; U) = \iint_U \left( |\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q) \right) \rho \, d\rho dz,$$

is a weighted Landau-de Gennes functional, with weight $\rho$. For almost every azimuthal angle $\varphi$, the trace $\overline{Q} = Q(\cdot, \varphi, \cdot)$ inherits the radial boundary conditions (2.5) on the half-circle $\partial U \cap \{\rho > 0\}$, uniform far-field alignment

$$\iint_U \frac{|\overline{Q} - Q_\infty|^2}{\rho^2 + z^2} \, \rho \, d\rho dz,$$

and the mirror symmetry

$$\overline{Q}(\rho, -z) = S\overline{Q}(\rho, z)S^{-1}.$$

Studying minimizers of $E_\varepsilon^{2D}$ among configurations satisfying these constraints, the heuristic arguments outlined above Theorem 2.6 can be made rigorous using ideas from [Bethuel et al., 1994, Struwe, 1994, Jerrard, 1999, Sandier, 1998, André and Shafrir, 1998, Beaulieu and Hadiji, 1998, Golovaty and Montero, 2014, Canevari, 2015], and this provides the lower bound

$$\min E_\varepsilon^{2D}(\cdot; U) \geq \frac{\pi}{2} |\ln \varepsilon| + \frac{\pi}{2} \ln |\ln \varepsilon| + \mathcal{O}(1).$$

After integrating with respect to $\varphi$ this gives the energy asymptotics in Theorem 2.6. Moreover, because of the matching upper bound, the lower bound on $E_\varepsilon^{2D}$ must be saturated on a.e. azimuthal slice, and this saturation implies that the energy is in fact concentrated near $(\rho, z) = (1, 0)$:

$$E_\varepsilon^{2D}(Q_\varepsilon(\cdot, \varphi, \cdot; U_\delta^{int}) \geq \frac{\pi}{2} |\ln \varepsilon| + \frac{\pi}{2} \ln |\ln \varepsilon| - \pi |\ln \delta| - C,$$

where $U_\delta^{int} = U \cap \{(\rho - 1)^2 + z^2 < \delta\}$. Integrating with respect to $\varphi$ concludes the proof of the last assertion in Theorem 2.6.

**Limit configuration in the axisymmetric case**

The above energy asymptotics provide the upper bound

$$\frac{1}{2\pi} E_\varepsilon(Q_\varepsilon; \Omega_\delta^{ext}) \leq \pi |\ln \delta| + \mathcal{O}(1),$$
$$\text{where } \Omega_\delta^{ext} = \Omega \setminus \overline{\Omega}_\delta^{int} = \Omega \cap \{\mathrm{dist}(\cdot, \mathcal{C}) > \delta\}$$

Here, recall that $\mathcal{C} = \partial\Omega \cap \{x_3 = 0\}$ is the equatorial circle. Classically, this allows one to extract a sequence $Q_\varepsilon \to Q_*$ in $H^1_{loc}(\overline{\Omega} \setminus \mathcal{C}; \mathcal{S}_0)$, and $Q_*$ is a $\mathcal{U}_*$-valued maps which is energy minimizing with respect to compact perturbations in $\Omega \setminus \mathcal{C}$. This implies [Schoen and Uhlenbeck, 1982, Schoen and Uhlenbeck, 1983] that $Q_*$ is smooth away from a set of point singularities isolated in $\overline{\Omega} \setminus \mathcal{C}$. We would like to show that $Q_*$

is actually smooth in $\overline{\Omega} \setminus \mathcal{C}$, but are not able to do so in that generality: we restrict ourselves to axisymmetric maps.

Because the energy asymptotics of Theorem 2.6 rely on an axisymmetric upper bound and on a lower bound for azymuthal slices, it is also valid when considering minimizers subject to the axisymmetry constraint (2.8), and this is in fact the way it is stated in [Alama et al., 2021]. In that case we obtain an axisymmetric $\mathcal{U}_*$-valued limit $Q_*$ which can be written in the form

$$Q_*(\rho, \varphi, z) = s_* \left( R_\varphi n(\rho, z) \otimes R_\varphi n(\rho, z) - \frac{1}{3} I \right),$$

for a map $n \colon U \to \mathbb{S}^2$ (recall $U = \{(\rho, z) \colon \rho > 0,\ \rho^2 + z^2 > 1\}$) which minimizes the energy

$$\hat{E}(n; U_\delta^{ext}) = \int_{U_\delta^{ext}} \left( |\nabla n|^2 + \frac{n_1^2 + n_2^2}{\rho^2} \right) \rho \, d\rho dz,$$
$$U_\delta^{ext} = U \setminus \overline{U}_\delta^{int} = U \cap \left\{ (\rho - 1)^2 + z^2 > \delta \right\},$$

among maps $m \colon U_\delta^{ext} \to \mathbb{S}^2$ such that $m \otimes m$ is mirror symmetric and equal to $n \otimes n$ on $\partial U_\delta^{ext}$. The second term in the energy $\hat{E}$ comes from the azimuthal derivative $|\partial_\varphi Q_*|^2$, and makes the far-field alignment condition superfluous. The choice of an orientation ($n$ is $\mathbb{S}^2$-valued and not $\mathbb{RP}^2$-valued) is possible because $U$ is simply connected [Bethuel and Chiron, 2007, Ball and Zarnescu, 2011], and it forces the radial boundary conditions to take the following form:

$$n = \begin{cases} +\mathbf{e}_r & \text{on } \partial U \cap \{\rho > 0,\ z > 0\}, \\ -\mathbf{e}_r & \text{on } \partial U \cap \{\rho > 0,\ z < 0\}. \end{cases} \tag{2.9}$$

On the equatorial plane, the mirror symmetry imposes

$$n = \tau \mathbf{e}_3 \text{ on } U \cap \{z = 0\}, \qquad \tau \in \{\pm 1\}, \tag{2.10}$$

Similarly to the regularity of minimizing harmonic maps in two dimensions, one can show that $n$ is smooth away from the Saturn-ring at $(\rho, z) = (1, 0)$ and from the vertical axis $\{\rho = 0\}$, where the weight in the energy degenerates. The second main result of [Alama et al., 2021] states that $n$ is actually smooth also on the vertical axis.

**Theorem 2.7.** *The map $n$ is smooth in $\overline{U} \setminus \{(1, 0)\}$. Moreover it satisfies the additional symmetry property $n_2(\rho, z) = 0$.*

The symmetry property $n_2(\rho, z) = 0$ implies that $R_\varphi n(\rho, z)$ takes values in the azimuthal plane generated by $\mathbf{e}_\rho, \mathbf{e}_z$, and the azimuthal vector $\mathbf{e}_\varphi$ is an eigenvector of $Q_*(\rho, \varphi, z)$. The fact that this symmetry is valid is similar to why geodesics on $\mathbb{S}^2$ are contained in a plane: projecting the values of $n$ onto $\{n_2 = 0\}$ decreases the energy. However, a bit of work is required to make that simple argument work, because boundary conditions are not explicit on the small circle arc $\partial U_\delta^{ext} \cap U$, so one cannot be sure that

the projection preserves them. This difficulty is dealt with by taking advantage of the upper bound

$$\hat{E}(n; U_\delta^{ext}) \leq \pi |\ln \delta| + \mathcal{O}(1), \tag{2.11}$$

which is inherited from the analogous bound on $E_\varepsilon(Q_\varepsilon; \Omega_\delta^{ext})$. To explain how to exploit (2.11), it is convenient to use polar coordinates $(r, \theta)$ around the point $(1, 0)$,

$$\rho = 1 + r\cos\theta, \quad z = r\sin\theta.$$

The circle arc $\partial U_\delta^{ext} \cap U$ corresponds to $r = \delta$ and $\theta \in (-\theta_0(r), \theta_0(r))$, where $\theta_0 \approx \pi/2$. Along that arc, the boundary conditions (2.9)-(2.10) force $n$ to go from $n_0 \approx -\mathbf{e}_1$ at $\theta \approx -\pi/2$ to $n_1 \approx \mathbf{e}_1$ at $\theta \approx \pi/2$, via $\tau\mathbf{e}_3$ at $\theta = 0$; see Figure 2.3. The corresponding geodesic on $\mathbb{S}^2$ has energy $\approx \pi$, and this provides a lower bound which matches the upper bound (2.11). Hence this lower bound is saturated, and thanks to stability properties of geodesics on $\mathbb{S}^2$ this imposes

$$n(r, \theta) \approx (\sin\theta, 0, \tau\cos\theta) \qquad \text{for } r \ll 1.$$

This closeness, in an appropriate sense, can be used to make the above energy comparison argument work and show that $n_2 = 0$ everywhere.

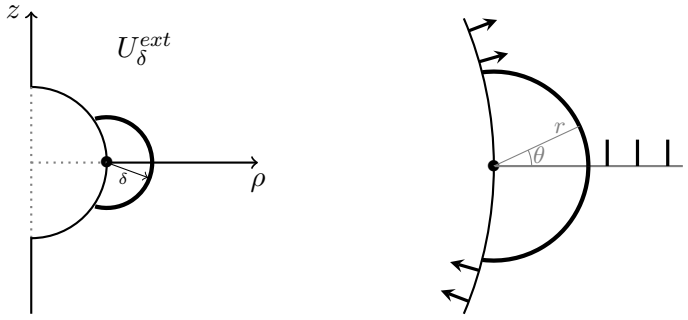


Figure 2.3: The domain $U_\delta^{ext}$ (left), and zoom near the equator (right): coordinates $(r, \theta)$ and oriented boundary conditions.

The proof that $n$ is smooth on the vertical axis is inspired by a similar statement in [Alama et al., 2016]. Since $n$ is constrained by the second energy term to be equal to $\pm\mathbf{e}_3$ on the vertical axis, singularities correspond to jump between those two values. On the other hand, if the sets $\{n_3 \geq 0\}$ or $\{n_3 \leq 0\}$ have connected components which don't touch the boundary $\partial U_\delta^{ext} \cap \{\rho > 0\}$, one can reflect $n$ at no cost of energy and make that connected component disappear. If $n_3$ does not change sign on $\partial U_\delta^{ext} \cap \{\rho > 0\}$, this is enough to rule out any sign change of $n_3$ on the vertical axis, and therefore any singularity. We have $n_3 > 0$ on $\partial U \cap \{0 < \rho < 1\}$, so it remains to understand what happens on small circle arcs surrounding the singular point $(\rho, z) = (1, 0)$. There one

can take advantage again of the rigidity provided by the energy bound (2.11), which imposes

$$n_3 \approx \tau \cos\theta \quad \text{on} \quad \partial U_\delta^{ext} \cap U = \{r = \delta,\, -\theta_0(r) < \theta < \theta_0(r)\}.$$

If $\tau = +1$ this is enough to conclude that $n_3 > 0$, by carefully studying what happens near $\theta = \pm\theta_0(r)$.

But if $\tau = -1$ we conclude that $n$ has two mirror-symmetric point singularites on the vertical axis. That configuration is perfectly allowed by topological constraints: it corresponds to a positively charged ring of degree $+1/2$ (whose two traces on a meridian plane provide a $+1$ topological charge), compensated by two $-1$ charges on the vertical axis, and the $+1$ charge of the radial boundary conditions. Moreover, it doesn't seem possible to rule out that possibility using only the minimizing properties of the map $n$. Instead we go back to the energy asymptotics of Theorem 2.6 and compute more precisely the $\mathcal{O}(1)$ remainder. This is the most technical part of the argument, achieved in [Alama et al., 2021, § 4.3.2], and it enables us to rule out the case $\tau = -1$ and to prove Theorem 2.7.

*Remark* 2.8. One may think that determining the $\mathcal{O}(1)$ remainder is very similar to the classical Ginzburg-Landau analysis [Bethuel et al., 1994], but it is not obvious that it could be directly transposed. Consider for instance that the radial scaling property of an interior vortex, reflected in the identity $|\ln(\varepsilon/r)| = |\ln\varepsilon| - |\ln r|$, is crucially used in [Bethuel et al., 1994, Chapter VIII]. Here the singularity is deformed when approaching the boundary and that nice scaling behavior is lost: the energy expansion of Theorem 2.6 does not behave well when replacing $\varepsilon$ by $\varepsilon/r$.

### 2.2.3 Far-field asymptotics

Interactions between particles suspended in nematic liquid crystals are commonly described in the physics literature using the so-called *electrostatics analogy* [Muševič, 2019]. This approximation originates in [Brochard and de Gennes, 1970] and replaces each single particle by a singular forcing term in a linearized equation, which allows to explicitly compute the interactions. The precise form of the singular forcing term depends on the properties of the corresponding particle, via the far-field deformation it generates in the nematic alignment. This relies on two implicit assumptions:

- far away from the particle, the asymptotic expansion of the order parameter solves a linearized equation (around a fixed uniform alignment),
- and the first nontrivial term of that expansion is characterized by the properties (size, symmetry, etc.) of the particle.

This legitimizes replacing the particle by a singular forcing term (derivatives of a Dirac mass) which generates the same asymptotic behavior in the linearized equation. The second assumption is the most mathematically challenging, this is where all the nonlinear nature of the original problem is retained. In this section we present results of

[Alama et al., 2022] which provide a precise statement regarding the first assumption, and a first step towards justifying the second assumption.

We represent the particle by a smooth bounded domain $G \subset \mathbb{R}^3$, and describe nematic alignment by a director field $n\colon \mathbb{R}^3 \setminus G \to \mathbb{S}^2$. Its energy is given by

$$F(n) = \int_{\mathbb{R}^3\setminus G} |\nabla n|^2 \, dx + F_s(n_{\lfloor \partial G}), \tag{2.12}$$

where

$$F_s\colon H^{\frac{1}{2}}(\partial G; \mathbb{S}^2) \to [0, \infty],$$

is a weakly lower semicontinuous surface energy, with non-empty domain $\{F_s < \infty\}$, which describes the particle's anchoring properties. For instance the domain of $F_s$ may consist of one map $n_b\colon \partial G \to \mathbb{S}^2$, which corresponds to Dirichlet boundary conditions $n_{\partial\Omega} = n_b$, but $F_s$ can be much more general.

Uniform far-field alignment is imposed by requiring

$$\int_{\mathbb{R}^3\setminus G} \frac{|n - n_0|^2}{1 + r^2} \, dx < \infty, \tag{2.13}$$

for some $n_0 \in \mathbb{S}^2$. We denote by $\mathcal{H}_{n_0}(G)$ the space of maps $n \in H^1_{loc}(\overline{\mathbb{R}^3 \setminus G}; \mathbb{S}^2)$ satisfying this far-field alignment condition. The direct method of the calculus of variations ensures the existence of a map $n$ minimizing the energy $F$ over $\mathcal{H}_{n_0}(G)$. Such map satisfies the harmonic map equation

$$-\Delta n = |\nabla n|^2 n \qquad \text{in } \mathbb{R}^3 \setminus G,$$

whose linearization around the constant $n_0$ is simply $\Delta n = 0$. The asymptotic expansion can be split into a harmonic part, which solves that linearized equation, and a non-harmonic correction.

**Theorem 2.9.** *As $r = |x| \to +\infty$, we have*

$$n = n_0 + n_{harm} + n_{corr} + \mathcal{O}\left(\frac{1}{r^4}\right),$$

*where*

$$n_{harm} = \frac{1}{r} v_0 + \sum_{j=1}^{3} p_j \partial_j \left(\frac{1}{r}\right) + \sum_{1 \le k,\ell \le 3} c_{k\ell} \partial_k \partial_\ell \left(\frac{1}{r}\right),$$

*for some coefficients $v_0, p_j, c_{k\ell} \in \mathbb{R}^3$, and*

$$n_{corr} = -\frac{|v_0|^2}{r^2} n_0 - \frac{|v_0|^2}{2r^3} v_0 - \frac{1}{3r} \sum_{j=1}^{3} v_0 \cdot p_j \partial_j \left(\frac{1}{r}\right) n_0.$$

*Moreover, the coefficients $v_0, p_j$ are orthogonal to $n_0$.*

One could of course obtain an expansion to any order, which would consist of a harmonic part, and a non-harmonic correction depending on the coefficients of the harmonic part. The interesting feature of the expansion up to $\mathcal{O}(1/r^4)$ is that, if the first term in $n_{harm}$ vanishes, $v_0 = 0$, which, as we will see, is a natural assumption, then $n_{corr} = 0$.

Consider for instance an axisymmetric situation: assume $RG = G$ for all rotations $R$ of axis $n_0$, and $n$ minimizes $F$ in $\mathcal{H}_{n_0}(G)$ subject to the axisymmetry constraint

$$n(Rx) = Rn(x) \qquad \forall x \in \mathbb{R}^3 \setminus G,$$

for all rotations $R$ of axis $n_0$. Theorem 2.9 is still valid for that constrained minimizer, and plugging the expansion into the axisymmetry constraint imposes $Rv_0 = v_0$ for all rotations of axis $n_0$, so $v_0 = 0$ because $v_0 \perp n_0$. Therefore in that setting, the physicists' first assumption, that the asymptotic expansion solves the linearized equation, is correct up to $\mathcal{O}(1/r^4)$.

The proof of Theorem 2.9 is obtained by applying iteratively a standard surjectivity property of the Laplace operator:

$$\begin{aligned} & f = \mathcal{O}\left(\frac{1}{r^\gamma}\right) \text{ for some non-integer } \gamma > 3, \\ \Rightarrow \quad & \exists u = \mathcal{O}\left(\frac{1}{r^{\gamma-2}}\right) \text{ such that } \Delta u = f. \end{aligned}$$

Applying this to $f = \Delta n$ provides a harmonic part $n - u$ which has an expansion in terms of derivatives of the fundamental solution $1/r$. That expansion can be plugged back into the equation $-\Delta n = |\nabla n|^2 n$ in order to identify the non-harmonic correction. One can then repeat this operation to improve the order of the expansion. The orthogonality conditions on $v_0, p_j$ simply come from the constraint $|n|^2 = 1$.

The only technical issue is that one needs to get the iteration started, by showing that

$$\Delta n = \mathcal{O}\left(\frac{1}{r^\gamma}\right) \quad \text{for some } \gamma > 3.$$

A simple rescaling of small energy estimates for harmonic maps [Schoen, 1984] provides the decay $\Delta n = o(1/r^3)$, which is just not enough. We present in [Alama et al., 2022] two different proofs of a starting decay estimate $\Delta n = \mathcal{O}(1/r^{4^-})$. The first proof is by adapting small energy estimates of [Hardt et al., 1986, Luckhaus, 1988], to compare the energy decay of $\mathbb{S}^2$-valued minimizing maps with the energy decay of $T_{n_0}\mathbb{S}^2$-valued minimizing maps. The latter behave well since they solve a linear equation. The second proof, inspired by [Schoen, 1983], is obtained by differentiating the Euler-Lagrange equation $-\Delta n = |\nabla n|^2 n$, which provides a linearized equation for $w = \partial_j n$. A rescaled Cacciopoli's inequality then allows to improve the initial decay $|w|^2 = o(1/r^3)$. The first proof is slightly more flexible in the sense that it adapts directly to a more general class of energies (than the Dirichlet energy).

The next question is to understand the link between the expansion provided by Theorem 2.9, and properties of the particle, given by the domain $G$ and its anchoring energy $F_s$ appearing in (2.12). We succeed to do so only for the first coefficient $v_0$:

**Theorem 2.10.** *Recall that $\mathcal{H}_{n_0}(G)$ is the space of maps $n \in H^1_{loc}(\overline{\mathbb{R}^3 \setminus G}; \mathbb{S}^2)$ aligned with $n_0$ at far-field, in the sense of (2.13). The function $\hat{F}\colon \mathbb{S}^2 \to [0, \infty)$, defined by*

$$\hat{F}(n_0) = \min \{F(n)\colon n \in \mathcal{H}_{n_0}(G)\},$$

*is Lipschitz, and for a.e. $n_0 \in \mathbb{S}^2$ we have*

$$\nabla \hat{F}(n_0) = -8\pi v_0(n_0), \tag{2.14}$$

*where $v_0(n_0) = \lim_{r\to\infty} r(n - n_0)$ for any minimizing map $n \in \mathcal{H}_{n_0}(G)$ such that $\hat{F}(n_0) = F(n)$. Moreover $\hat{F}$ is semiconcave: for all $n_0 \in \mathbb{S}^2$, any $v_0$ as above, and all $m_0 \in \mathbb{S}^2$, we have*

$$\hat{F}(m_0) \leq \hat{F}(n_0) - 8\pi v_0 \cdot (m_0 - n_0) + C|m_0 - n_0|^2,$$

*for some constant $C = C(G, F_s) > 0$.*

This shows in particular that, if $n_0$ is a differentiable point of $\hat{F}$, then the coefficient $v_0$ is unique, although minimizing maps $n$ need not be unique. At a hypothetical non-differentiable point, there may be several possible $v_0$, and they are all admissible in the semiconcavity inequality.

The validity of (2.14) is suggested by formal calculations in [Brochard and de Gennes, 1970] showing, in an axisymmetric setting, that the torque applied by particle $G$ on the nematic is proportional to $n_0 \times v_0$. These formal calculations can be interpreted as follows. Given $n_0^t \approx n_0$ for $t \approx 0$, assume that the minimization problem for $\hat{F}(n_0^t)$ is attained a map $n^t \in \mathcal{H}_{n_0^t}(G)$ depending smoothly on $t$. Then one may write

$$\left.\frac{d}{dt}\right|_{t=0} \hat{F}(n_0^t) = \left.\frac{d}{dt}\right|_{t=0} F(n^t) = DF(n^0)[\dot{n}^0],$$

where $\dot{n}^0 = (d/dt)|_{t=0} n^t$. But because $n^0$ minimizes $F$ over $\mathcal{H}_{n_0}(G)$, we have $DF(n^0)[v] = 0$ for any perturbation $v$ which preserves (infinitesimally) the space $\mathcal{H}_{n_0}(G)$. The map $v = \dot{n}^0$ does not have that property, because it moves the condition at infinity, but this tells us that the result should only depend on the far-field behavior of the minimizing map $n^0$. In fact, performing these calculations on a finite ball $B_R$, we are left only with a boundary term on $\partial B_R$, which gives (2.14) when sending $R \to +\infty$.

Here we have no hope of showing that minimizing maps in $\mathcal{H}_{n_0}(G)$ depend smoothly on $n_0$, so this argument does not apply. Instead we construct rather natural comparison maps which enable us to prove the semiconcavity inequality, and then (2.14) follows at any differentiable point $n_0$.

Theorem 2.10 has the advantage of telling us that the first asymptotic coefficient $v_0$ is uniquely determined by the far-field condition $n_0$, generically. It does express $v_0$ by a formula (2.14) depending only on $n_0$, $G$ and $F_s$, but that formula is not easy to calculate in practice. Nevertheless, it has some simple corollaries which do seem to bring valuable information.

First consider that, at equilibrium, a given particle should tend to align itself in a position that minimizes the energy. From the physical point of view, rotating the particle or the far-field alignment is totally equivalent, so we may view that equilibrium condition as requiring that $n_0$ be a local minimizer of $\hat{F}$. Moreover the semiconcavity condition ensures that $\hat{F}$ is differentiable at a local minimum, and therefore $v_0 = 0$ as a consequence of (2.14).

**Corollary 2.11.** *If $n_0 \in \mathbb{S}^2$ is locally minimizing for $\hat{F}$, then any map $n$ minimizing $F$ over $\mathcal{H}_{n_0}(G)$ has an asymptotic expansion*

$$n = n_0 + n_{harm} + \mathcal{O}\left(\frac{1}{r^4}\right), \quad \Delta n_{harm} = 0, \quad n_{harm} = \mathcal{O}\left(\frac{1}{r^2}\right),$$

*as $r = |x| \to +\infty$.*

Second, consider the case of symmetric particles. We say that $G$ is axisymmetric around $\mathbf{u} \in \mathbb{S}^2$ if, for any rotation $R$ of axis $\mathbf{u}$, we have $RG = G$ and $F_s(Rn \circ R^{-1}) = F_s(n)$ for all $n \in H^{1/2}(\partial G; \mathbb{S}^2)$. And $G$ is spherically symmetric if this is true for any rotation $R$, of any axis. The function $\hat{F}$ inherits the same invariances: $\hat{F}(Rn_0) = \hat{F}(n_0)$ for all $n_0 \in \mathbb{S}^2$ and rotation $R$ in the symmetry group of $G$. Differentiating this identity automatically gives us, via (2.14), information about $v_0$.

**Corollary 2.12.** *If $G$ is axisymmetric around $\mathbf{u} \in \mathbb{S}^2$, then for a.e. $n_0 \in \mathbb{S}^2$ we have*

$$v_0(n_0) \cdot (\mathbf{u} \times n_0) = 0.$$

*If in addition $\hat{F}$ is differentiable at $\mathbf{u}$ then $v_0(\mathbf{u}) = 0$. If $G$ is sperically symmetric, then $v_0(n_0) = 0$ for all $n_0 \in \mathbb{S}^2$.*

As in Corollary 2.11, one could also write down corresponding consequences concerning the asymptotic expansion given by Theorem 2.9. Note that we do not know whether there exist axisymmetric particles such that $\hat{F}$ would not be differentiable at $\mathbf{u}$. In that case, the semiconcavity of $\hat{F}$ would force it to have a local maximum at $\mathbf{u}$, with its graph looking locally like a cone.

*Remark* 2.13. All results presented in this section for the isotropic Dirichlet energy can be directly generalized to more general energies, for instance the anisotropic Oseen-Frank energy, and also to higher dimensions.

### 2.2.4 Perspectives

The main motivation of the works presented here about nematic colloids is to eventually gain a rigorous understanding of interactions between immersed particles. So far we have concentrated on the effect of a single particle, and I present here some further perspectives about this question, before evoking the next step of determining interactions.

**Single particle**

In § 2.2.2 we describe a Saturn-ring configuration obtained by minimizing under a mirror-symmetry constraint. The equatorial defect line arising in the limit $\varepsilon \to 0$ is a length-maximizing closed curve on the sphere, and as such we would expect the Saturn-ring configuration to be unstable. But numerical evidence [Fukuda et al., 2004b, Lubensky et al., 1998] indicates that it is stable. That apparent paradox is resolved if we consider that the Saturn ring configuration is stable only in a neighborhood that shrinks as $\varepsilon \to 0$, and this phenomenon may be due to boundary effects. Tools developed in [Serfaty, 2005, Canevari and Orlandi, 2021] are related to this stability issue and will provide a starting point to investigate it.

In § 2.2.3, Corollary 2.12 can be interpreted as a result of symmetry at infinity for minimizing maps. It is natural to expect that this can be used, at least in the sperically symmetric setting, to prove that minimizers are axisymmetric: examples were asymptotic symmetry is used to deduce full symmetry can be found in [Mironescu, 1996, Millot and Pisante, 2010]. (Note that methods of [Sandier and Shafrir, 1993] to prove axisymmetry of $\mathbb{S}^2$-valued maps will not apply here.)

Finally, it seems important in view of applications to adapt the analysis of § 2.2.3 to the $Q$-tensor description: obtain far-field asymptotics uniform with respect to $\varepsilon$, and an equivalent of Theorem 2.10. The methods of [Contreras and Lamy, 2022] will provide a technical framework to resolve the first step.

**Interactions**

In the absence of a full understanding of the next terms in the asymptotic expansion of Theorem 2.9, a reasonable way to approach the problem of estimating interactions between particles is suggested by the phenomenological model proposed in [Lubensky et al., 1998, § 5]. There, the presence of particles is modeled by additional singular terms in the energy, but without linearizing the equation. In that context, justifying the electrostatic analogy amounts to a linearization procedure where the issue of boundary anchoring and accompanying defects has been hidden in that simplified energy. A first step will be to consider two particles and obtain an asymptotic expansion of the full energy as their mutual distance grows to infinity: this amounts to a rigorous justification of formal arguments in [Lubensky et al., 1998, § 5]. It is then natural to move on to the case of many particles.

## 2.3 The effects of elastic anisotropy

As mentioned in the introduction of this chapter, within the director description, the isotropic Dirichlet energy

$$E(n) = \int_\Omega |\nabla n|^2 \, dx, \quad n \colon \Omega \to \mathbb{S}^2,$$

is a simplification of the more general Oseen-Frank model,

$$E^{OF}(n) = \int_\Omega \left( k_1 |\nabla \cdot n|^2 + k_2 (n \cdot \nabla \times n)^2 + k_3 |n \times (\nabla \times n)|^2 \right) \, dx, \tag{2.15}$$

which we call anisotropic whenever $k_1, k_2, k_3 > 0$ are distinct. In contrast with isotropic harmonic maps, minimizers of the anisotropic Oseen-Frank energy are not that well understood: their singular set has dimension strictly less than one [Hardt et al., 1986, Hardt et al., 1988], but is not known whether it is discrete (unless the constants are almost equal [Almgren and Lieb, 1988]), and it is not known for all values of $k_1, k_2, k_3$ whether the map $x/|x|$ is minimizing [Hélein, 1987, Lin, 1987, Cohen and Taylor, 1990, Kinderlehrer and Ou, 1992, Ou, 1992, Alouges and Ghidaglia, 1997].

More generally, one may consider anisotropic energies (with elliptic integrands) for maps with values into a given submanifold. In addition to the unknown dimension of a minimizing map's singular set, there is a long list of questions that are solved in the isotropic case and open for anisotropic integrands: smoothness of critical maps in a two-dimensional domain [Hélein, 2002, Rivière, 2007], regularity of stationary critical maps [Evans, 1991, Bethuel, 1993], regularity of anisotropic minimal surfaces [De Lellis et al., 2021, De Rosa and Tione, 2022], etc.

These difficulties are due to important structural properties that are absent in the anisotropic case. Let us illustrate this with two basic observations:

- The isotropic Euler-Lagrange system

  $$-\Delta n = |\nabla n|^2 n,$$

  is, at main order, determined by the scalar operator $-\Delta$ which acts diagonally on $n$. This specific diagonal structure is absent in the anisotropic case: all that remains is a quasilinear strongly elliptic system with a right-hand side quadratic in $\nabla n$.

- The isotropic energy density enjoys a decoupling

  $$|\nabla n|^2 = |\partial_r n|^2 + \frac{1}{r^2} |\nabla_\omega n|^2,$$

  into radial and angular derivatives. This ensures that tangent maps (limits of rescaled maps $n_r(x) = n(rx)$ along sequences $r \to 0$) are 0-homogeneous, by providing a *monotonicity formula* for the energy of rescaled maps. Such decoupling is absent in the anisotropic case (see [Allard, 1973] in the case of anisotropic minimal surfaces).

Another interesting observation is that, like geodesics, isotropic harmonic maps are intrinsic geometric objects: the energy density

$$|\nabla n|^2 = \sum_j |\partial_j n|^2$$

depends only on the riemannian structure of $\mathbb{S}^2$ (to define the norm of the tangent vectors $\partial_j n$), but not on the isometric embedding $\mathbb{S}^2 \subset \mathbb{R}^3$. In contrast, the anisotropic Oseen-Frank energy does depend on that embedding: applying a rotation changes in general the value of the energy [Kinderlehrer et al., 1993].

These structural properties also play an important role in the analysis of minimizers and critical points of the Landau-de Gennes energy (2.2). In the anisotropic case, new tools are needed to circumvent the non-diagonal form of the Euler-Lagrange system and the lack of monotonicity formula. In the rest of this section I describe three works where such new tools are developed, and anisotropy-related phenomena are unveiled:

- in [Contreras and Lamy, 2022] with A. Contreras we prove uniform convergence of minimizers of an anisotropic version of (2.2) towards minimizers of the Oseen-Frank energy, away from singularities;
- in [Lamy and Zùñiga, 2022] with A. Zùñiga and in [Kowalczyk et al., 2022] with M. Kowalczyk and P. Smyrnelis we study two-dimensional entire solutions (which describe the fine structure of defects) of an anisotropic Ginzburg-Landau system.

### 2.3.1 Small energy estimates for Landau-de Gennes

In [Majumdar and Zarnescu, 2010, Nguyen and Zarnescu, 2013] (see also [Contreras et al., 2018]) the authors consider minimizers of the Landau-de Gennes energy

$$E_\varepsilon(Q;\Omega) = \int_\Omega \left( \frac{1}{2}|\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q) \right) dx$$

in a bounded domain $\Omega \subset \mathbb{R}^3$, with fixed smooth $\mathcal{U}_*$-valued boundary conditions on $\partial\Omega$. They prove that, along a sequence $\varepsilon \to 0$, these converge uniformly away from a finite number of points, to a $\mathcal{U}_*$-valued map $Q_* = s_*(n \otimes n - \frac{1}{3}I)$. The map $n\colon \Omega \to \mathbb{S}^2$ is a minimizing harmonic map, which is smooth away from said finite set of point.

The main tool of the proof is a small-energy estimate for smooth solutions of the Euler-Lagrange equation

$$\Delta Q = \frac{1}{\varepsilon^2} \nabla f(Q), \tag{2.16}$$

which asserts the existence of $\delta > 0$ such that

$$r^2 \sup_{B_{r/2}} |\nabla Q|^2 \leq C \frac{1}{r} E_\varepsilon(Q; B_r) \qquad \text{if } \frac{1}{r} E_\varepsilon(Q; B_r) \leq \delta^2, \tag{2.17}$$

for any ball $B_r \subset \Omega$ and $\varepsilon \ll 1$. Inspired by similar arguments in [Schoen, 1984, Chen and Struwe, 1989, Bethuel et al., 1993, Chen and Lin, 1993], this small energy estimate is obtained as a consequence of a Bochner inequality for the energy density:

$$-\Delta[e_\varepsilon(Q)] \leq C e_\varepsilon(Q)^2, \qquad e_\varepsilon(Q) = \frac{1}{2}|\nabla Q|^2 + \frac{1}{\varepsilon^2} f(Q).$$

Note that here the operator $-\Delta$ acting on scalar-valued functions is related to the specific diagonal form of the Euler-Lagrange system (2.16), and this argument seems therefore constrained to the isotropic setting. That Bochner inequality can then be combined with the energy monotonicity formula (another specifically isotropic ingredient) as in [Chen and Lin, 1993, Majumdar and Zarnescu, 2010, Nguyen and Zarnescu, 2013], or with a Moser iteration as in [Haslhofer and Müller, 2011, Lemma 3.3], to prove the small-energy estimate (2.17).

*Remark* 2.14. Methods in [Bethuel et al., 2001] also use the isotropic structure, and are moreover tailored for $\mathbb{C}$-valued maps converging to $\mathbb{S}^1$-valued maps. In that case, the smallness requirement in (2.17) can be relaxed to a logarithmic bound with small factor, see e.g. [Bethuel et al., 2005]. For the isotropic Landau-de Gennes energy, this is achieved in [Canevari, 2017].

In the anisotropic setting where $|\nabla Q|^2$ is replaced by a general positive definite quadratic form $\mathbb{A}(Q)[\nabla Q]$, there is no known equivalent of a Bochner inequality. In a two-dimensional domain, one can take advantage of Sobolev embeddings to avoid using that inequality [Bauman et al., 2012], but this does not seem easily generalizable to higher dimensions. In [Contreras and Lamy, 2022] we propose a different argument leading to analogous estimates and uniform convergence results, in the presence of elastic anisotropy.

For any positive definite quadratic form $\mathbb{A}$ on $(\mathcal{S}_0)^3$, we consider maps $Q\colon \Omega \to \mathcal{S}_0$ minimizing (with respect to their own boundary conditions) the anisotropic energy

$$E_\varepsilon^{\mathbb{A}}(Q;\Omega) = \int_\Omega \left( \mathbb{A}[\nabla Q] + \frac{1}{\varepsilon^2} f(Q) \right) dx.$$

We may allow the coefficients of $\mathbb{A}$ to depend smoothly on $x$, but not on $Q$ at this stage (although it would be desirable in view of applications, see e.g. [Golovaty et al., 2021]). A widely used example is the quadratic form

$$\mathbb{A}[\nabla Q] = L_1 |\nabla Q|^2 + L_2 \, |\nabla \cdot Q|^2,$$

where the divergence $\nabla \cdot Q$ is taken row-wise, and $3L_1 + 5L_2 > 0$.

Any sequence $Q_\varepsilon$ of minimizing maps with bounded energy $E_\varepsilon^{\mathbb{A}}(Q_\varepsilon;\Omega) \le C$ as $\varepsilon \to 0$, is precompact in $H^1_{loc}(\Omega;\mathcal{S}^0)$ (see e.g. [Canevari, 2017]) and converges, along a subsequence, to a minimizing $\mathcal{U}_*$-valued map. Thanks to the regularity theory of anisotropic harmonic maps [Hardt et al., 1986, Hardt et al., 1988, Luckhaus, 1988], that $\mathcal{U}_*$-valued map is smooth away from a closed singular set of dimension strictly less than one. We show that the convergence is uniform away from that singular set, as a consequence of the following small-energy estimate.

**Theorem 2.15.** *For any $\alpha \in (0,1)$, there exist $\delta, \varepsilon_0 > 0$ such that, for any map $Q\colon B_r \to \mathcal{S}_0$ minimizing the anisotropic energy $E_\varepsilon^{\mathbb{A}}(\cdot;B_r)$, we have*

$$r^{2\alpha} |Q|_{C^\alpha(B_{r/2})} \le C \frac{1}{r} E_\varepsilon^{\mathbb{A}}(Q;B_r) \qquad \text{if } \frac{1}{r} E_\varepsilon^{\mathbb{A}}(Q;B_r) \le \delta^2, \tag{2.18}$$

*provided $0 < r \le 1$ and $0 < \varepsilon < \varepsilon_0 r$.*

The proof of Theorem 2.15 starts from the observation that it is true in the limit, that is, for $\mathcal{U}_*$-valued energy-minimizing maps [Hardt et al., 1986, Hardt et al., 1988, Luckhaus, 1988]. But, in contrast with the limit problem, the energy $E_\varepsilon^{\mathbb{A}}$ is not scale-invariant, due to the presence of the potential term and the characteristic length scale $\varepsilon > 0$. Roughly speaking, at large scales $r \gg \varepsilon$ one expects a minimizing map $Q$ to "look like" a $\mathcal{U}_*$-valued map, but at small scales $r \leq \varepsilon$ it should simply look like a solution of the rescaled Euler-Lagrange system $\mathcal{L}_{\mathbb{A}} Q = \nabla f(Q)$, where $\mathcal{L}_{\mathbb{A}}$ is the elliptic operator naturally associated to $\mathbb{A}$. In both cases, there is a regularity theory, and the proof of Theorem 2.15 follows that dichotomy: we obtain energy decay estimates at large scales by adapting the regularity theory of manifold-valued maps, and at small scales using classical elliptic regularity.

An unexpected difficulty concerns the small scales, which boil down to regularity estimates for the semilinear elliptic system $\mathcal{L}_{\mathbb{A}} Q = \nabla f(Q)$. One needs an initial bound in order to start bootstrapping. In the isotropic case it is very easy to obtain $L^\infty$ bounds under reasonable boundary conditions, but in general this seems surprisingly hard. We rely instead on *ad hoc* estimates exploiting the low dimension and the specific form of the potential (a quartic polynomial).

*Remark* 2.16. Let us describe more specifically the issue of $L^\infty$ bounds. In the isotropic case $\mathbb{A}[\nabla Q] = |\nabla Q|^2$, if a potential $f \geq 0$ has the coercivity property that

$$\sup_{|Q| \leq R_0} f(Q) \leq \inf_{|Q| \geq R_0} f(Q) \quad \text{for some } R_0 > 0,$$

then any map $Q$ minimizing $E_\varepsilon(\cdot; \Omega)$ in a bounded domain $\Omega$ satisfies the maximum principle

$$\|Q\|_{L^\infty}(\Omega) \leq \max(R_0, \|Q\|_{L^\infty(\partial\Omega)}).$$

The reason is simple: projecting the values of $Q$ orthogonally onto the ball $\{|Q| \leq R_0\}$ decreases both terms in the energy. This is no longer true when the elastic term is anisotropic. In fact, a maximum principle in this form cannot be valid for a truly anisotropic $\mathbb{A}$, because the homogeneous linear system $\mathcal{L}_{\mathbb{A}} Q = 0$ does not even enjoy the sharp maximum principle $\|Q\|_{L^\infty(\Omega)} \leq \|Q\|_{L^\infty(\partial\Omega)}$ [Kresin and Maz'ya, 2012]. Only a weaker form with a multiplicative constant $C > 1$ is valid: this indicates that the projection argument will not be easily adaptable to the anisotropic case. Nevertheless, it seems reasonable to conjecture that for a potential $f$ which is "coercive enough", any minimizer $Q$ of $E_\varepsilon^{\mathbb{A}}(\cdot, \Omega)$ will satisfy

$$\|Q\|_{L^\infty}(\Omega) \leq C \max(R_0, \|Q\|_{L^\infty(\partial\Omega)}),$$

for some constants $C > 1$, $R_0 > 0$ depending on $\mathbb{A}$ and $f$. We were, however, not able to prove that conjecture.

The method used to prove the interior estimate of Theorem 2.15 can also be applied to obtain boundary estimates. We consider in [Contreras and Lamy, 2022] two types of boundary conditions:

- Dirichlet boundary conditions

$$Q = Q_b \qquad \text{on } \partial\Omega,$$

for a fixed smooth map $Q_b\colon \partial\Omega \to \mathcal{U}_*$. In that case, small-energy estimates at the boundary are valid, similar to Theorem 2.15.

- Weak anchoring conditions described by a surface energy: the energy to minimize becomes

$$E_\varepsilon^{\mathbb{A}}(Q;\Omega) + \int_{\partial\Omega} g(x,Q)\,dx,$$

for some smooth map $g\colon \Omega \times \mathcal{S}_0 \to [0,\infty[$. In that case, we are only able to prove small-energy estimates under the additional assumption of an *a priori* uniform $L^\infty$ bound. This assumption is verified for a large class of anchoring energy densities $g(x,Q)$ in the isotropic setting. Even in that restrictive case, the result seems new, because the method based on Bochner's inequality is not adapted to treat weak anchoring conditions.

### 2.3.2 Two-dimensional vortex profiles

In this section we concentrate on a simplified two-dimensional model, in order to obtain a more precise understanding of the qualitative differences introduced by elastic anistropy. For a domain $\Omega \subset \mathbb{R}^2$ and maps $u\colon \Omega \to \mathbb{R}^2$, the anisotropic Ginzburg-Landau energy is given by

$$\begin{aligned} E_\varepsilon^\delta(u;\Omega) &= \int_\Omega \left( \mathbb{A}_\delta[\nabla u] + \frac{1}{4\varepsilon^2}(1-|u|^2)^2 \right) dx, \\ \mathbb{A}_\delta[\nabla u] &= \frac{1}{2}|\nabla u|^2 + \frac{\delta}{2}\left( (\nabla\cdot u)^2 - (\nabla \times u)^2 \right), \qquad |\delta| < 1. \end{aligned} \tag{2.19}$$

The parameter $\delta \in (-1,1)$ quantifies the anisotropy, the isotropic case $\delta = 0$ corresponds to the classical Ginzburg-Landau functional. The identity

$$\mathbb{A}_\delta[\nabla u] = (1+\delta)(\nabla\cdot u)^2 + (1-\delta)(\nabla\times u)^2 - 2\det(\nabla u),$$

where the last term is a null Lagrangian, allows to interpret this elastic energy as a two-dimensional version of the anisotropic Oseen-Frank energy (2.15). It arises naturally as a model for two-dimensional liquid crystal configurations [Lee et al., 2006, Barboza et al., 2016, Barboza et al., 2013, Clerc et al., 2014], and various mathematical aspects have been studied in [Colbert-Kelly and Phillips, 2013, Colbert-Kelly et al., 2017, Golovaty et al., 2020, Golovaty et al., 2019]. In particular, an equivalent of the analysis of minimizers of the classical Ginzburg-Landau energy [Bethuel et al., 1994] has been obtained in [Colbert-Kelly and Phillips, 2013] when the boundary condition has positive degree, but is open for negative degrees. The reason for this is that anisotropic vortices of degree +1 behave very differently from vortices of other degrees (in particular, from

vortices of degree $-1$). This difference will also be apparent in the results presented in this section about vortex profiles: entire solutions of the Euler-Lagrange equation with prescribed degree at infinity.

Entire vortex solutions arise when zooming in at scale $\varepsilon$ around a singularity, which has the effect of setting $\varepsilon = 1$, hence we consider solutions of the Euler-Lagrange system

$$\mathcal{L}_\delta u = -\left(1 - |u|^2\right) u \qquad \text{in } \mathbb{R}^2, \tag{2.20}$$

where the operator $\mathcal{L}_\delta$ is defined by

$$\mathcal{L}_\delta u = \Delta u + \delta \left( \nabla(\nabla \cdot u) - \nabla^\perp(\nabla \times u) \right),$$

with the notation $\nabla^\perp = (-\partial_2, \partial_1)$. Given a weak solution $u \in H^1_{loc}(\mathbb{R}^2; \mathbb{R}^2)$ of (2.20), if it has finite potential energy

$$\int_{\mathbb{R}^2} (1 - |u|^2)^2 \, dx < \infty, \tag{2.21}$$

then one can show (see e.g. [Kowalczyk et al., 2022, § 3]) that $u$ is smooth and $|u(x)| \to 1$ as $|x| \to \infty$. This allows to define its degree, or winding number,

$$\deg(u) = \deg\left( \frac{u}{|u|}, \partial D_R \right) \in \mathbb{Z} \qquad \forall R \gg 1,$$

and we are interested here in solutions with a prescribed degree $d \in \mathbb{Z}$.

In the isotropic case $\delta = 0$, the classical Ginzburg-Landau equation

$$\Delta u = -(1 - |u|^2)u \qquad \text{in } \mathbb{R}^2,$$

admits, for any given degree $d \in \mathbb{Z} \setminus \{0\}$, a radial solution

$$u_d(re^{i\theta}) = f_d(r) e^{id\theta}, \qquad f_d \geq 0, \quad f_d(0) = 0, \quad \lim_{+\infty} f_d = 1,$$

and the radial profile $f_d$ is uniquely determined via the ordinary differential equation that it must satisfy [Hervé and Hervé, 1994, Chen et al., 1994]. Each solution $u_d$ generates, by invariance under translations and rotations, a three-dimensional family of solutions

$$e^{i\alpha} u_d(\cdot + a), \qquad \alpha \in \mathbb{R},\ a \in \mathbb{R}^2.$$

There are no other entire solutions of degree $d = \pm 1$ [Mironescu, 1996], but for higher degrees it is open whether there exist nonradial solutions.

In the anisotropic case $\delta \neq 0$, the situation is different. The only solutions of (2.20)-(2.21) of the form $u(re^{i\theta}) = f(r)e^{i\alpha}e^{i\theta}$ with a real-valued radial profile $f(r)$ are of degree $d = 1$, and phase shift $\alpha \equiv 0 \bmod \pi/2$ [Clerc et al., 2014]. This can be partially interpreted in terms of symmetries of the equation: radial functions of the form $f(r)e^{id\theta}$ can be characterized as invariant under the transformations $u(z) \to e^{-id\alpha} u(e^{i\alpha} z)$ for all $\alpha \in \mathbb{R}$, but these transformations preserve the equation (2.20) if and only if $\delta = 0$ or $d = 1$.

*Remark* 2.17. More specifically, it is instructive to make a list of the elementary symmetries of (2.20), which are the transformations

$$u(z) \longrightarrow \tau e^{-i\alpha} u(e^{i\alpha} z), \qquad \alpha \in \mathbb{R}, \tau \in \{\pm 1\},$$
$$u(z) \longrightarrow \overline{u(\bar{z})},$$
$$u(z) \longrightarrow u(z + a), \qquad a \in \mathbb{R}^2,$$

associated to rotation equivariance, reflection equivariance, and translation invariance. In the isotropic case $\delta = 0$, the equation is, in addition, invariant under rotation and reflection, separately in the variable and in the target. For instance, $u(z) \to e^{i\alpha} u(z)$ preserves the isotropic equation for any $\alpha \in \mathbb{R}$, but for $\delta \neq 0$ this is only true if $\alpha \equiv 0 \mod \pi$.

From the physical point of view, the most important vortex profiles are those of degree $\pm 1$, since they correspond to defects that are stable in experiments (see e.g. [Lee et al., 2006, Barboza et al., 2013, Clerc et al., 2014]). In the isotropic case $\delta = 0$, the two degrees $d = \pm 1$ are essentially equivalent since the transformation $u \to \bar{u}$ maps one to the other, and their uniqueness [Mironescu, 1996] and stability properties [Mironescu, 1995] are well understood. In the anisotropic case $\delta \neq 0$, the two cases $d = \pm 1$ are quite different:

- For $d = 1$ there is a radial solution, but its stability and uniqueness properties are not fully understood. In [Lamy and Zùñiga, 2022] we show that it is linearly stable for small anisotropy but loses stability when the anisotropy is higher.
- For $d = -1$ there is no radial solution, and the mere existence of an entire solution is already an issue. In [Kowalczyk et al., 2022] we prove such existence for small anisotropy $|\delta| < \delta_0$.

**Degree $+1$: linear stability analysis**

In the isotropic case $\delta = 0$, nondegenerate linear stability of the radial solution $u = f(r)e^{i\theta}$ has been proved in [Mironescu, 1995, Del Pino et al., 2004] and recently upgraded to a nonlinear stability estimate [Gravejat et al., 2021]. Here we consider the radial solution of the anisotropic equation (2.20) given by

$$u^\delta_{rad}(re^{i\theta}) = f^\delta(r)e^{i\theta}, \qquad f^\delta(r) = f\left(\frac{r}{\sqrt{1+\delta}}\right),$$

where $f(r) = f^0(r)$ is the radial profile of the isotropic vortex.

The second variation of the energy $E^\delta_1$ at $u^\delta_{rad}$ is the quadratic form

$$Q^\delta_{rad}[v] = \int_{\mathbb{R}^2} \left( \mathbb{A}_\delta[\nabla v] - (1 - (f^\delta)^2)|v|^2 + 2(f^\delta)^2 (e^{i\theta} \cdot v)^2 \right) dx,$$

associated to the linear operator obtained by linearizing the equation (2.20) around $u^\delta_{rad}$. Taking into account the expansion $f(r) = 1 + \mathcal{O}(r^{-2})$ ar $r \to \infty$, the quadratic form $Q^\delta_{rad}$ is well defined on the space

$$\mathcal{H}_{lin} = \left\{ v \in H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2) \colon \int_{\mathbb{R}^2} \left( |\nabla v|^2 + \frac{|v|^2}{r^2} + (e^{i\theta}\cdot v)^2 \right) dx < \infty \right\}.$$

Invariance by translation provides two elements of the kernel of $Q_{rad}$, namely $\partial_j u^\delta_{rad}$ ($j = 1, 2$), and we denote by $K$ the linear subspace of $\mathcal{H}_{lin}$ generated by these two functions.

**Theorem 2.18.** *There exists $\delta_1 \in (0,1)$ such that $u^\delta_{rad}$ is*

- *nondegenerately stable if $\delta \in (-\delta_1, 0]$: $Q^\delta_{rad}[v] > 0$ for all $v \in \mathcal{H}_{lin} \setminus K$.*
- *unstable if $\delta \in (-1, -\delta_1) \cup (0,1)$: $Q^\delta_{rad}[v] < 0$ for some $v \in \mathcal{H}_{lin}$.*

*Remark* 2.19. The asymmetry between $\delta > 0$ and $\delta < 0$ is due to the fact that $e^{i\theta}$ is energy-minimizing (among $\mathbb{S}^1$-valued maps depending only on $\theta$) if and only if $\delta \le 0$. There is another radial solution, $\tilde{u}^\delta_{rad}(re^{i\theta}) = f^{-\delta}(r) ie^{i\theta}$, which is stable for $0 \le \delta < \delta_1$ and unstable otherwise. This completes at the end a symmetric picture: for $|\delta| < \delta_1$ exactly one of the two radial solutions is stable, and for $\delta_1 < |\delta| < 1$ they are both unstable.

The proof of Theorem 2.18 is very close to the stability analysis of the isotropic case [Mironescu, 1995] (see also [Del Pino et al., 2004, Ignat et al., 2016a, Ignat et al., 2016b]), based on separation of variables: the Fourier decomposition

$$v = e^{i\theta} \sum_{n \ge 0} \left( w_n(r) e^{in\theta} + w_{-n}(t) e^{-in\theta} \right),$$

turns out to be orthogonal for the quadratic form $Q^\delta_{rad}$, hence it suffices to study each mode separately. The lower modes $n = 0, 1$ play a special role and can be studied via a decomposition introduced in [Mironescu, 1995], using the equation satisfied by the radial profile $f$. They are both stable for $\delta \in (-1, 0]$, and the $n = 0$ mode is unstable for $\delta > 0$.

The main novelty, with respect to the isotropic case, is the mechanism of instability for $\delta$ close to $-1$. In the isotropic case, the higher modes $n \ge 2$ are automatically stable as a consequence of the stability of lower modes, but here this principle is broken: for $-1 < \delta < -\delta_1$, the lower modes are stable and instability is caused by high modes. Note that stability of the high modes for $\delta$ close to 0 can be obtained by a perturbation argument from the isotropic case $\delta = 0$, but direct careful estimates of the anisotropic quadratic form provide a quantitative range of stability: we know that $\delta_1 \ge 1/\sqrt{5}$.

### Degree $-1$: existence

When trying to prove existence of an entire solution with prescribed degree $d$, a natural strategy is to obtain solutions on finite disks $D_R$ by minimizing the energy with some

boundary condition of degree $d$, and then let $R \to \infty$. The hard part is to check that the degree constraint is conserved in that limit. For radial solutions $f(r)e^{id\theta}$ this is not an issue, since the symmetry constraint passes easily to the limit. But in the anisotropic case $\delta = 0$, one cannot look for radial solutions unless $d = 1$. To obtain the existence of solutions of degree $-1$, we use instead a discrete symmetry constraint: we impose

$$u(e^{i\frac{\pi}{2}}z) = -e^{i\frac{\pi}{2}}u(z) \qquad \forall z \in \mathbb{R}^2. \tag{2.22}$$

Since it is compatible with the symmetries of (2.20) (see Remark 2.17), minimizing under this constraint does provide solutions of (2.20), which are energy-minimizing with respect to compact modifications. Moreover, one can check that, if $u$ has a well-defined degree, then the constraint (2.22) implies

$$\deg(u; \partial D_r) \equiv -1 \bmod 4.$$

This is not enough to fix the degree, but ensures at least that it is nontrivial. Moreover, in the equivalence class $-1 + 4\mathbb{Z}$, the degree with the lowest modulus is precisely $-1$, hence energy-minimizing configurations are likely to have degree $-1$, at least if the anisotropy is not too large.

**Theorem 2.20.** *There exists $\delta_0 > 0$ such that, for $|\delta| < \delta_0$, the anisotropic Ginzburg-Landau equation* (2.20) *admits an entire solution $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ with finite potential energy* (2.21) *and* $\deg(u) = -1$.

*Remark* 2.21. The same ideas apply to construct solutions of any negative degree $d \leq -1$ (but not for $d \geq 2$). Moreover, imposing in addition one of the two different symmetry constraints $u(\bar{z}) = \pm\overline{u(z)}$, we can obtain two distinct solutions (modulo the elementary symmetries of Remark 2.17). These two solutions converge, as $\delta \to 0$, to $u_d$ and $iu_d$, where $u_d = f_d(r)e^{id\theta}$ is the isotropic radial solution of degree $d$.

The proof of Theorem 2.20 follows the two steps described above:

- The first step is to obtain an entire solution of (2.20)-(2.21) which minimizes the energy under the symmetry constraint (2.22). This follows by imposing well-chosen boundary conditions on $\partial D_R$ and using a Pohozaev identity to bound the potential energy uniformly in $R$. This does not require any restriction on $\delta \in (-1, 1)$.
- The second step is to take advantage of the minimal modulus property of $-1$ in the equivalence class $-1 + 4\mathbb{Z}$ in order to show that this solution actually has degree $-1$. This is the most technical part of the proof, and where we need the smallness assumption on $|\delta|$.

Let us describe more precisely the second step. The main idea comes from the proof in [Shafrir, 1994] that nontrivial minimizing entire solutions of the isotropic Ginzburg-Landau equation must have degree $\pm 1$, relying on the quantization effect proved in [Brezis et al., 1994]: a solution of degree $d$ has energy of order $\pi d^2 \ln R$ on a large disk $D_R$. But one can modify such solution inside $D_R$ to obtain a configuration of energy

$\pi|d| \ln R$, with one central vortex of degree $\pm 1$ and $(|d| - 1)$ escaping vortices of degree $\pm 1$ (say on a circle of radius $R/4$). The minimizing property therefore implies $d^2 \leq |d|$, so $d = \pm 1$.

One difficulty we face in our case is that it is not obvious at all that a quantization effect could be valid in the anisotropic case. In fact in degree $d = +1$ there are two solutions with different energy asymptotics. However we can obtain non-sharp upper and lower bound by simply using the fact that $\mathbb{A}_\delta[\nabla u] = (1 + \mathcal{O}(\delta))|\nabla u|^2$. Accordingly, we will only be able to prove that $d^2 \leq (1+\mathcal{O}(\delta))|d|$, and this implies $|d| = 1$ only under a smallness assumption on $\delta$.

Another difficulty is of a more technical nature: in order to perform a construction similar to [Shafrir, 1994], one must modify the original solution in a boundary layer $D_R \setminus D_{R/2}$ with low energy cost. For this, it is enough to have a logarithmic bound on the energy, which in the isotropic case is a byproduct of the quantization result of [Brezis et al., 1994]. Here we don't know if such bound could be proved for all solutions with finite potential energy (2.21), but we obtain it using again the minimizing property and constructing a competitor based on harmonic extensions of phase and modulus. This argument also makes use of $\mathbb{A}_\delta[\nabla u] = (1 + \mathcal{O}(\delta))|\nabla u|^2$ and therefore requires a smallness assumption on $\delta$. Note however that the threshold $\delta_0$ in Theorem 2.20 is completely explicit.

### 2.3.3 Perspectives

One central open problem about anisotropic energies is the regularity of anisotropic harmonic maps: could the singular set of anisotropic Oseen-Frank minimizers be larger than locally finite? An answer to this question seems out of reach at this stage, and the stand point adopted in this chapter is to explore a variety of physically motivated questions about anisotropic energies in order to develop new methods and understand anisotropy-specific phenomena. I present here some perspectives concerning the two-dimensional Ginzburg-Landau energy (2.19) of § 2.3.2, and the radial defect $x/|x|$ in three dimensions.

#### Two-dimensional Ginzburg-Landau

Concerning the radial vortex profile of degree one, a natural follow-up to the linear stability study in Theorem 2.18, is to establish a nonlinear stability estimate analog to [Gravejat et al., 2021], in the linearly stable regime. An interesting feature is that the rotational symmetry which plays an important role in [Gravejat et al., 2021] is broken as soon as $\delta \neq 0$: a nonlinear stability estimate will provide a quantitative insight into that phenomenon.

Concerning entire vortex profiles of degree -1, an outstanding question left unresolved by Theorem 2.20 is their existence for all values $|\delta| < 1$. We plan to achieve this by applying continuation methods in the spirit of [Kowalczyk et al., 2012]. To that end, the first step is to obtain, for $|\delta| \ll 1$, good asymptotic estimates of the solution provided by Theorem 2.20. This is related to another important issue: the asymptotic analysis

of negative degre minimizers of the anisotropic Ginzburg-Landau energy $E_\varepsilon^\delta$ (2.19).

To gain insight into that question, an even more basic problem is that of $\mathbb{S}^1$-valued minimizers of $E_\varepsilon^\delta$ in an annulus. In a work in preparation with A. Contreras, we investigate all 0-homogeneous (that is, $r$-independent) $\mathbb{S}^1$-valued critical points of degree -1: we find that, in a annulus with small hole, they are not minimizing with respect to their own boundary conditions for $0 < |\delta| \ll 1$. This seems connected to the regularity issue for general anisotropic harmonic maps in dimension $n \geq 3$: improved estimates in the isotropic case are due to the fact that tangent maps must be 0-homogeneous, but here we have a situation where 0-homogeneous maps can not be minimizing.

Still regarding the asymptotic analysis of $E_\varepsilon^\delta$, for positive degrees a renormalized energy is obtained in [Colbert-Kelly and Phillips, 2013], but it is not explicit enough to understand how the classical logarithmic Coulomb interaction is modified. In another work in preparation with A. Contreras, we obtain a description of the anisotropic interaction for small $|\delta|$, and it will be interesting to investigate how the corresponding crystallization phenomena are modified.

**The radial singularity in three dimensions**

There is a range of values of the elastic constants in the anisotropic Oseen-Frank energy (2.15) for which the radial map $x/|x|$ is known to be linearly stable, but whether it is minimizing (even locally) with respect to its own boundary conditions is an open question. One reason why the linear stability does not provide enough information is that, for perturbations which move the central singularity, the linear stability does not control the next nonlinear terms [Cohen and Taylor, 1990]. This motivates studying the effect of inner variations, which we do in a work in preparation with P. Bousquet and R. Rodiac. Another interesting direction is to study the stability of the radial profile in a Ginzburg-Landau approximation (there, stability does imply local minimality), which should be attainable by combining methods of [Cohen and Taylor, 1990, Kinderlehrer and Ou, 1992] and of [Ignat et al., 2015].

A related question is to obtain, in the range where $x/|x|$ is known to be minimizing, nonlinear stability estimates with respect to boundary perturbations, in the spirit of [Hardt and Lin, 1989]. That work relies very much on the monotonicity formula, and generalizations to an anisotropic setting are therefore all the more interesting. Here there is a hope to obtain such generalization because the linearized equation is completely explicit, and bad solutions that are usually ruled out by the monotonicity formula are simply not present.

# Chapter 3

# Line-energy models in 2D

## 3.1 Introduction

### 3.1.1 Eikonal equation and energy concentration

Several two-dimensional physical models have in common to exhibit concentration phenomena on one-dimensional subsets, corresponding to singularities of the eikonal equation

$$|\nabla u|^2 = 1 \qquad \text{in } \Omega \subset \mathbb{R}^2,$$

or, for the two-dimensional vector-field $m = \nabla^\perp u$,

$$\nabla \cdot m = 0 \text{ in } \mathcal{D}'(\Omega), \qquad |m| = 1 \text{ a.e. in } \Omega. \tag{3.1}$$

Subject to the boundary conditions $u = 0$ on $\partial\Omega$, the gradient of the viscosity solution $u(x) = \operatorname{dist}(x, \partial\Omega)$ has bounded variation, which is enough to ensure energy concentration on a one-dimensional subset (in a sense precised below). However, for the physical models under consideration here, the relevant class of solutions is larger, and energy concentration is not fully understood.

Let us focus here on one model, the Aviles-Giga functional

$$AG_\varepsilon(m;\Omega) = \int_\Omega \left( \frac{\varepsilon}{2}|\nabla m|^2 + \frac{1}{2\varepsilon}(1-|m|^2)^2 \right) dx,$$
$$m\colon \Omega \to \mathbb{R}^2, \quad \nabla \cdot m = 0.$$

Via the identification $m = \nabla^\perp u$, which can be made in any simply connected domain, this functional can also be seen as a second-order Ginzburg-Landau energy. Proposed in [Aviles and Giga, 1987] as a model for smectic liquid crystals, it is related to several physical applications (thin-film elasticity, micromagnetism, pattern formation), and we refer to the introduction of [Jin and Kohn, 2000] for a review of these applications. When imposing tangential boundary conditions $m \cdot n_{\partial\Omega} = 0$, it is conjectured in [Ortiz and Gioia, 1994] that minimizers of $AG_\varepsilon(\cdot;\Omega)$ converge to the viscosity solution

$m = \nabla^\perp \operatorname{dist}_{\partial\Omega}$. That conjecture is proved only in the case of an ellipse or very similar domains [Marconi, 2021a]; see also [Ignat and Merlet, 2012], where counterexamples in nonconvex domains are given.

Given a sequence of bounded energy $AG_\varepsilon(m_\varepsilon) \le C$, it is shown in [Ambrosio et al., 1999, DeSimone et al., 2001, Jabin and Perthame, 2001] that $(m_\varepsilon)$ is precompact in $L^1(\Omega)$, and therefore any limit $m = \lim m_\varepsilon$ solves the eikonal equation (3.1). Moreover, the calibration method developed in [Jin and Kohn, 2000] provides the lower bound

$$\liminf AG_\varepsilon(m_\varepsilon;\Omega) \ge \frac{1}{6}\int_{J_m} |m^+ - m^-|^3 \, d\mathcal{H}^1, \tag{3.2}$$

where $J_m \subset \Omega$ is the 1-rectifiable jump set of $m$, with traces $m^\pm$ on each side of it. This makes sense if $m$ is of bounded variation ($BV$), but also for any limit $m = \lim m_\varepsilon$ of a bounded energy sequence thanks to results of [De Lellis and Otto, 2003]. If $m \in BV(\Omega)$, a matching upper bound (in the sense of $\Gamma$-convergence) is available [Conti and De Lellis, 2007, Poliakovsky, 2007], thus showing energy concentration on the 1D set $J_m$. The construction providing that upper bound is done by replacing the sharp jumps between $m^+$ and $m^-$ with one-dimensional transitions at scale $\varepsilon$ in the direction normal to the jump set. Performing this modification and estimating its energy requires fine information about the structure of $m$, provided in this case by the theory of $BV$ functions.

If $m \notin BV(\Omega)$, validity of a matching upper bound is a long-standing open question: this, and related open questions, are the central theme of the present chapter. The main difficulty is to understand the fine structure, ideally $BV$-like, of solutions of the eikonal equation (3.1) which are limits $m = \lim m_\varepsilon$ of bounded energy sequences.

### 3.1.2 Entropy productions

To understand that class of solutions, an important concept, introduced in [DeSimone et al., 2001] and borrowed from scalar conservation laws, is that of *entropy*: a smooth map $\Phi\colon \mathbb{S}^1 \to \mathbb{R}^2$ is an entropy for the eikonal equation (3.1) if $\nabla \cdot \Phi(m) = 0$ for any smooth solution $m$ of (3.1). This is equivalent to the condition

$$e^{i\theta} \cdot \frac{d}{d\theta}\Phi(e^{i\theta}) = 0 \qquad \forall \theta \in \mathbb{R}.$$

For a general weak solution $m$, the *entropy production* $\nabla \cdot \Phi(m)$ is a distribution, which should "detect" singularities. Here we are interested in solutions $m = \lim m_\varepsilon$ which are limits of bounded energy sequences. This imposes a finite-entropy condition: entropy productions are finite real-valued Radon measures [DeSimone et al., 2001]. Moreover, their absolute variation is bounded by the energy:

$$|\nabla \cdot \Phi(m)|(U) \le C\|\Phi\|_{C^2} \liminf_{\varepsilon\to 0} AG_\varepsilon(m_\varepsilon;U) \qquad \forall U \subset \Omega. \tag{3.3}$$

In fact, the lower bound (3.2) is obtained by the same principle, using a specific family of entropies.

A first outcome of the bound (3.3) is that *zero-energy states* (limits $m = \lim m_\varepsilon$ of vanishing energy sequences $E_\varepsilon(m_\varepsilon;\Omega) \to 0$) have zero entropy production. As a consequence, they satisfy a a kinetic formulation

$$e^{is} \cdot \nabla \mathbf{1}_{m\cdot e^{is}>0} = 0 \qquad \text{in } \mathcal{D}'(\Omega), \quad \forall s \in \mathbb{R}. \tag{3.4}$$

This follows from the fact that $\Phi_s(z) = e^{is}\mathbf{1}_{z\cdot e^{is}>0}$ can be approximated by smooth entropies [DeSimone et al., 2001, Jabin et al., 2002]. The kinetic equation (3.4) can be interpreted as a weak way of expressing that $m$ is constant along characteristics of direction $m^\perp$, and this is used in [Jabin et al., 2002] to obtain a rigid characterization of zero-energy states:

- in $\Omega = \mathbb{R}^2$, $m$ must be a constant, or any translation of a vortex $m(x) = \pm ix/|x|$;
- in a smooth bounded simply connected domain $\Omega$, with the natural boundary condition $m \cdot n_{\partial\Omega} = 0$ ($m$ tangent at the boundary), $\Omega$ must be a disk and $m$ a centered vortex.

As another consequence of the bound (3.3), concentration properties of the energy imply concentration properties of the entropy production measures. Therefore, it becomes crucial to understand the following question concerning the structure of finite-entropy solutions: are entropy productions supported on a 1-rectifiable subset ?

A first result in that direction, proved in [De Lellis and Otto, 2003], is that the one-dimensional part of the entropy production is countably rectifiable. More precisely, the Radon-Nikodym decomposition of the measure $\nabla \cdot \Phi(m)$ with respect to $\mathcal{H}^1$ is given by

$$\nabla \cdot \Phi(m) = (\Phi(m^+) - \Phi(m^-)) \cdot \nu \, \mathcal{H}^1\lfloor_{J_m} + \eta_\Phi, \qquad \eta_\Phi \perp \mathcal{H}^1,$$

where $J_m$ is a countably 1-rectifiable subset of $\Omega$, with unit normal $\nu$, on each side of which $m$ admits strong $L^1$ traces $m^\pm$. A more precise description of the higher-dimensional part $\eta_\Phi$ has been given recently in [Marconi, 2021a], but it is still not enough to conclude that $\eta_\Phi = 0$.

All these results are relevant for the Aviles-Giga functional presented here, but also for micromagnetics models studied in [Rivière and Serfaty, 2001, Rivière and Serfaty, 2003, Alouges et al., 2002]. In the case of [Rivière and Serfaty, 2001, Rivière and Serfaty, 2003] the setting can be slightly simplified, an equivalent of the rectifiability results of [De Lellis and Otto, 2003] can be found in [Ambrosio et al., 2002], and it is shown in [Marconi, 2021b] that $\eta_\Phi = 0$.

### 3.1.3 Burgers' equation

Questions very similar to the ones presented above arise when the eikonal equation (3.1) is replaced by Burgers' equation

$$\partial_t u + \partial_x \left(\frac{u^2}{2}\right) = 0. \tag{3.5}$$

Formally, the connection with the eikonal equation (3.1) can be seen by considering $u = -m_1$, $m_2 = \sqrt{1-u^2}$, so that $\nabla \cdot m = 0$ becomes

$$\partial_1 u - \partial_2 f(u) = 0,$$

where $f(u) = \sqrt{1-u^2} \approx -u^2/2$ for $|u| \ll 1$.

Weak solutions of Burgers' equation (3.5) arise in limits of several variational models: large deviation principles for some stochastic processes [Varadhan, 2004, Mariani, 2010, Bellettini et al., 2010], variational characterizations of vanishing viscosity solutions [Poliakovsky, 2008] (see also [Blaser and Rivière, 2010]), smectic liquid crystals [Novack and Yan, 2022]. Again, the relevant solutions are those whose entropy productions

$$\partial_t \eta(u) + \partial_x q(u), \qquad \eta \in C^2(\mathbb{R}), \quad q'(v) = v\eta'(v),$$

are finite measures. This is a strictly larger class than that of Kruzkov's entropy solutions [Kružkov, 1970], whose entropy productions are nonpositive measures for convex entropies $\eta$ (and which constitute an analog of viscosity solutions of the eikonal equation).

Also there, 1-rectifiability of the entropy productions plays a central role. This rectifiablity has been established recently in [Marconi, 2022b], which constitutes major progress on the type of questions considered in this chapter. However, completing the $\Gamma$-convergence of the corresponding variational models does still seem to require more information on the structure of finite-entropy solutions.

### 3.1.4 Outline

In the rest of this chapter I present several results which help understand the structure of finite-entropy solutions to the above equations:

- In [Ghiraldin and Lamy, 2020] with F. Ghiraldin we characterize finite-entropy solutions of the eikonal equation in terms of Besov regularity.
- In [Lamy et al., 2020] with A. Lorent and G. Peng, we characterize zero-entropy solutions of the eikonal equation as solutions of a differential inclusion.
- In [Lamy and Otto, 2018] with F. Otto, in [Contreras Hip and Lamy, 2021, Contreras Hip et al., 2022] with A. Contreras Hip and E. Marconi, in [Lamy and Marconi, 2022] with E. Marconi, we prove various stability results for solutions of Burgers and the eikonal equation.

## 3.2 Optimal regularity estimates

It has been remarked in [Jabin and Perthame, 2001] that limits $m = \lim m_\varepsilon$ of bounded energy sequences $AG_\varepsilon(m_\varepsilon;\Omega) \leq C$ satisfy a kinetic formulation similar to (3.4), but with a forcing term:

$$e^{is} \cdot \nabla_x \mathbf{1}_{m(x)\cdot e^{is}>0} = \partial_s \sigma \qquad \text{in } \mathcal{D}'(\Omega \times \mathbb{S}^1), \tag{3.6}$$

where $\sigma$ is a finite Radon measure on $\Omega \times \mathbb{S}^1$. Here and in the sequel we systematically identify $\mathbb{S}^1$ and $\mathbb{R}/2\pi\mathbb{Z}$. This kinetic formulation can be interpreted in terms of entropy productions: for any $g \in C^\infty(\mathbb{S}^1)$, the map

$$\Phi_g(z) = \int_{\mathbb{S}^1} g(s) e^{is} \mathbf{1}_{z \cdot e^{is} > 0} \, ds, \tag{3.7}$$

is an entropy, and according to (3.6) its entropy production is the finite measure

$$\nabla \cdot \Phi_g(m) = - \int_{\mathbb{S}^1} g'(s) \, d\sigma(\cdot, s).$$

Kinetic equations of the form (3.6) have been long known to enjoy regularity properties via averaging lemmas (see e.g. [Lions et al., 1994]), which are usually obtained by arguments of harmonic analysis. As a consequence, solutions of (3.6) belong to $W^{(1/3)-,(3/2)-}_{loc}$ [Jabin and Perthame, 2002]. Another type of averaging lemmas, based on identities reminiscent of compensated compactness, has been used in [Golse and Perthame, 2013] to show that finite-entropy solutions of Burgers' equation have the Besov regularity $B^{1/3}_{3,\infty}$. This, and examples in [De Lellis and Westdickenberg, 2003], suggests that the optimal regularity of finite-entropy solutions of the eikonal equation should be $m \in B^{1/3}_{3,\infty}$, that is,

$$h \mapsto \frac{\|m(\cdot + h) - m\|_{L^3}}{|h|^{\frac{1}{3}}} \in L^\infty.$$

This is indeed what we prove in [Ghiraldin and Lamy, 2020], locally.

**Theorem 3.1.** *Let $m \colon \Omega \to \mathbb{R}^2$ a weak solution of the eikonal equation* (3.1)*. Then the following are equivalent:*

*(i)* $\nabla \cdot \Phi(m) \in \mathcal{M}_{loc}(\Omega)$ *for all* $C^2$ *entropies* $\Phi$*,*

*(ii)* $m$ *satisfies the kinetic formulation* (3.6) *with* $\sigma \in \mathcal{M}_{loc}(\Omega \times \mathbb{S}^1)$*,*

*(iii)* $m \in B^{1/3}_{3,\infty,loc}(\Omega)$*.*

Note that the finite-entropy condition $(i)$ is satisfied by limits of bounded energy sequences thanks to (3.3). Its link with the kinetic formulation is explained above via the family of entropies (3.7), which makes it clear that $(ii)$ implies $(i)$. The reverse implication is crucial in our argument and follows from a beautiful application of Banach-Steinhaus' uniform boundedness principle that we learnt from [De Lellis et al., 2003], and a generalized Riesz representation theorem for Radon measures (see [Lorent and Peng, 2021, Appendix B] for details to the originally very succinct proof of [Ghiraldin and Lamy, 2020, Lemma 3.4]).

The fact that $(iii)$ implies $(i)$ follows from a commutator argument similar to the proof of energy conservation for regular enough solutions of Euler's equation [Constantin et al., 1994] (note that our regularity exponent 1/3 is the same as in Onsager's conjecture). That

commutator argument has already been used in the context of the eikonal equation to prove rigidity (vanishing of all entropy productions) for regular enough solutions [De Lellis and Ignat, 2015]. In [Ghiraldin and Lamy, 2020] we simply remark that the threshold regularity $B^{1/3}_{3,\infty}$ implies that entropy productions are finite measures. The idea is to use a regularization $m_\varepsilon$ of $m$ at scale $\varepsilon$, and to estimate $\nabla\cdot\Phi(m_\varepsilon)$ for an appropriate extension of the entropy $\Phi$ to $\mathbb{R}^2$. The map $m_\varepsilon$ is smooth and divergence free, but it is not $\mathbb{S}^1$-valued, so $\nabla\cdot\Phi(m_\varepsilon)$ is not zero (as it would be for a smooth solution of the eikonal equation), but can be estimated in terms of the error $1-|m_\varepsilon|$ from solving the equation, which can be viewed as a commutator $|m|_\varepsilon - |m_\varepsilon|$.

The fact that $(i)$-$(ii)$ implies $(iii)$ follows from adapting smart calculations of [Golse and Perthame, 2013], and the way they are revisited in [Goldman et al., 2015]. There, specific identities allow to express integrals of the increment $|D^h m|^3 = |m(\cdot+h)-m|^3$ in terms of the kinetic measure $\sigma$ in (3.6) and lead to the Besov estimate $(iii)$. Let us explain here in more details the link with compensated compactness: the starting point is the div-curl estimate

$$\int X_1 \times X_2 \, dx \lesssim pp' \|X\|_{L^p} \|\nabla\cdot X\|_{W^{-1,p'}}, \quad 1<p<\infty,$$

valid for smooth vector fields $X_1, X_2 \in C^\infty_c(\mathbb{R}^2;\mathbb{R}^2)$, where $\nabla\cdot X$ denotes the vector $(\nabla\cdot X_1, \nabla\cdot X_2)$. Applying this to the increments

$$X_j = \chi D^h \Phi_j(m) = \chi\left(\Phi_j(m(\cdot+h)) - \Phi_j(m)\right)$$

for some entropies $\Phi_1, \Phi_2$ and a smooth cut-off $\chi$, and using the finite-entropy property, leads to the bound

$$\int \chi^2 D^h\Phi_1(m) \times D^h\Phi_2(m)\, dx \lesssim pp' \|\Phi\|_{L^\infty}(1+\|\Phi\|_{C^2})|h|^{1-2/p},$$

for all $p>2$. Taking linear combinations of such bounds, for different entropies $\Phi_1, \Phi_2$, it turns out that one can obtain a left-hand side where the integrand is bounded below by $|D^h m|^{3^+}$, leading to the regularity $B^{(1/3)-}_{3+,\infty}$. This type of argument, already present in [Golse, 2010], is used in [Goldman et al., 2021] to prove regularity of zero-energy states of an unoriented Aviles-Giga energy. The borderline regularity $B^{1/3}_{3,\infty}$ in Theorem 3.1 is attained replacing that argument with integration by parts reminiscent of the same div-curl structure, but directly in the kinetic formulation.

*Remark* 3.2. In [Lamy et al., 2022a, Lamy et al., 2022b] we prove generalizations of Theorem 3.1 to finite-entropy solutions of degenerate equations. For entropy solutions, we also obtain new averaging lemmas in [Gess and Lamy, 2019].

## 3.3 Zero-energy states as solutions of a differential inclusion

A particular set of two entropies plays a distinguished role for the Aviles-Giga energy: the Jin-Kohn entropies

$$\Sigma_1(z) = \frac{2}{3}(z_2^3, z_1^3), \quad \Sigma_2(z) = e^{i\frac{\pi}{4}}\Sigma_1(e^{-i\frac{\pi}{4}}z),$$

introduced in [Jin and Kohn, 2000] as calibrations to prove the lower bound (3.2), which is conjectured to be sharp. Their link with that lower bound is made apparent by the identity

$$\frac{1}{6}|m^+ - m^-|^3 \, \mathcal{H}^1_{\lfloor J_m} = |\nabla \cdot \Sigma(m)| \qquad \text{if } m \in BV,$$

where $|\nabla \cdot \Sigma(m)|$ is the absolute variation of the vector-valued measure $(\nabla \cdot \Sigma_1(m), \nabla \cdot \Sigma_2(m))$. Note that this can be used to give a sense to the right-hand side of (3.2) when $m \notin BV$.

*Remark* 3.3. The absolute variation $|\nabla \cdot \Sigma(m)|$ can also be viewed as the supremum measure of the entropy productions $\nabla \cdot \Sigma^\theta(m)$ of all rotated entropies $\Sigma^\theta(z) = e^{i\theta}\Sigma_1(e^{-i\theta}z) = \cos(2\theta)\,\Sigma_1(z) + \sin(2\theta)\,\Sigma_2(z)$.

If the total variation $|\nabla \cdot \Sigma(m)|(\Omega)$ is indeed the $\Gamma$-limit of the Aviles-Giga energy $AG_\varepsilon(\cdot;\Omega)$, it should be possible to characterize zero-energy states by imposing only $\nabla \cdot \Sigma(m) = 0$. This is indeed what is proved in [Lorent and Peng, 2018]: if a solution of (3.1) is such that its Jin-Kohn entropy productions vanish, then all its entropy productions vanish. One of the main ideas in [Lorent and Peng, 2018] is to view the condition $\nabla \cdot \Sigma(m) = 0$ as a differential inclusion: there exists $w = (w_1, w_2)$ such that $\nabla w_k = i\Sigma_k(m)$, and therefore

$$\nabla w \in K = \Sigma^\perp(\mathbb{S}^1) \subset \mathbb{R}^{2\times 2}, \tag{3.8}$$

where $\Sigma^\perp(z)$ is the matrix whose two rows are $i\Sigma_1(z), i\Sigma_2(z)$.

The set $K$ has no rank-one connections, which implies that the differential inclusion (3.8) has no obvious irregular solutions where $\nabla w$ would jump across a line. But tangent lines to $K$ do have rank-one connections, and it is therefore not elliptic (in the sense of [Šverák, 1993]), so classical methods to obtain regularity of $w$ are not available. This is of course no surprise, since we know that the zero-energy state $m = ix/|x|$ provides a solution $w$ which is singular at the origin. Nevertheless, the differential inclusion (3.8) retains a degenerate form of ellipticity:

$$\det(A - B) \gtrsim |A - B|^4 \qquad \forall A, B \in K, \tag{3.9}$$

and this can be used to obtain some starting Besov regularity. This can also be formulated via the div-curl arguments of Section 3.2, since (3.9) implies

$$D^h\Sigma_1(m) \times D^h\Sigma_2(m) \gtrsim |D^h m|^4.$$

Compared to Section 3.2, here we know that $\nabla \cdot \Sigma(m) = 0$, so the div-curl estimate can be improved to obtain $m \in B^{1/3}_{4,\infty}$. This is then smartly combined with algebraic identities satisfied by entropies, to conclude that all entropy productions vanish.

A natural question to ask is: does the differential inclusion, by itself, enjoy the same rigidity as zero-energy states? We answer this positively in [Lamy et al., 2020].

**Theorem 3.4.** *Let $\Omega \subset \mathbb{R}^2$ and $w\colon \Omega \to \mathbb{R}^2$ satisfy the differential inclusion* (3.8) *almost everywhere. Then $\nabla w = \Sigma^\perp(m)$, with $m$ is a zero-energy state.*

In particular, the characterization of zero-energy states in [Jabin et al., 2002] implies that $\nabla w$ is Lipschitz outside a discrete set of singular points, and in any convex neighborhood of a singular point $\nabla w(x) = \Sigma^\perp(\pm ix/|x|)$ (in coordinates centered at the singular point). This seems to be the first rigidity result of that kind for a non-elliptic differential inclusion.

For the study of the Aviles-Giga functional, one interesting consequence of Theorem 3.4 is that stability estimates for zero-energy states may now be approached from the point of view of differential inclusion, where a large variety of tools have been developed for that purpose: see for instance [John, 1961, Reshetnyak, 1994, Friesecke et al., 2002, Faraco and Zhong, 2005, Lewicka and Müller, 2016, Luckhaus and Zemas, 2022]. In that perspective, the non-elliptic nature of our differential inclusion is a major obstacle to overcome, and we will comment more on that in Section 3.5.

Let us now comment on the proof of Theorem 3.4. The result of [Lorent and Peng, 2018] can be phrased as

$$(|m| = 1, \quad \nabla \cdot m = 0, \quad \nabla \cdot \Sigma(m) = 0) \quad \Rightarrow \quad \nabla \cdot \Phi(m) = 0,$$

for any entropy $\Phi$, and Theorem 3.4 reaches the same conclusion without the assumption that $m$ is divergence-free. Therefore it boils down to the implication

$$(|m| = 1, \quad \nabla \cdot \Sigma(m) = 0) \quad \Rightarrow \quad \nabla \cdot m = 0. \tag{3.10}$$

The very starting point is that this implication is true if $m$ is a smooth vector field, thanks to the identity

$$\nabla \cdot m = -2m_1 m_2 \nabla \cdot \Sigma_1(m) + (m_1^2 - m_2^2)\nabla \cdot \Sigma_2(m),$$

valid for any smooth $m \colon \Omega \to \mathbb{S}^1$. When $m$ is not smooth, it is natural to consider its regularization $m_\varepsilon$ at scale $\varepsilon$, and perform similar calculations. But the map $m_\varepsilon$ is not $\mathbb{S}^1$-valued anymore, so extra error terms appear, related to the commutator $1 - |m_\varepsilon| = |m|_\varepsilon - |m_\varepsilon|$. One would like to show that these error terms tend to zero: this requires quite a lengthy process, in which we involve all the entropy productions $\nabla \cdot \Phi(m)$.

We explain here the main underlying idea. Information about $\nabla \cdot \Phi(m)$ is obtained by studying the approximation $\nabla \cdot \overline{\Phi}(m_\varepsilon)$, where $\overline{\Phi}$ is an extension of $\Phi \in C^2(\mathbb{S}^1; \mathbb{R}^2)$ to the unit disk $\overline{D}$, where $m_\varepsilon$ takes its values. The crucial observation we make is that *different extensions* may provide different information. Accordingly, we consider two types of extensions: either we simply extend radially, or by solving an appropriate elliptic boundary value problem (which already played an important role in [Lorent and Peng, 2018]). It turns out that the only way to reconcile the different informations provided by these two types of extension is that $\nabla \cdot m = 0$. Interestingly, to reach that conclusion we make use of the fact that the Hilbert transform is not bounded from $C^0(\mathbb{S}^1)$ to $L^\infty(\mathbb{S}^1)$ [Zygmund, 2002, § VII].

## 3.4 Stability estimates

One way to gain insight into the structure of general finite-entropy solutions is to establish quantitative stability estimates which measure, in terms of the energy, how far

a finite-entropy solution may be from specific well-understood solutions: zero-energy states, elementary jump solutions, entropy or viscosity solutions, etc.

Let us discuss first the case of Burgers' equation

$$\partial_t u + \partial_x \left( \frac{u^2}{2} \right) = 0.$$

Entropy solutions are bounded weak solutions such that their entropy productions

$$\mu_\eta = \partial_t \eta(u) + \partial_x q(u), \qquad \eta'' \geq 0, \quad q'(v) = v\eta'(v),$$

are nonpositive measures $\mu_\eta \leq 0$. Natural initial-value problems are well-posed in the class of entropy solutions [Kružkov, 1970]. It turns out that a single convex entropy is sufficient to characterize entropy solutions: if a bounded weak solution $u$ of Burgers' equation is such that

$$\mu = \partial_t \frac{u^2}{2} + \partial_x \frac{u^3}{3}, \tag{3.11}$$

is a nonpositive measure, then $u$ is an entropy solution [Panov, 1994, De Lellis et al., 2004]. This leads to the natural stability question: if the positive part of the measure $\mu$ is small, can we quantify how close $u$ is to an entropy solution? We provide a first answer in [Lamy and Otto, 2018].

**Theorem 3.5.** *Let $u$ be a bounded weak solution of Burgers' equation (3.5) in $Q = (-1,1)_t \times (-1,1)_x$, such that the entropy production $\mu$ (3.11) is a measure. Then there exists a bounded entropy solution $u^{ent}$ of (3.5) in $Q$, such that*

$$\int_{\frac{1}{2}Q} \left| u - u^{ent} \right|^4 \, dtdx \leq C \, \mu_+(Q)^{\frac{4}{35}},$$

*where $C > 0$ depends only on $\|u\|_\infty$.*

In [Lamy and Otto, 2018] we use this stability estimate to obtain information about Lebesgue points of $u$: using the fact that entropy solutions are one-sided Lipschitz, together with scaling arguments, we deduce from Theorem 3.5 that the set of non-Lebesgue points of a finite-entropy solution has Hausdorff dimension at most one. More precisely, any point $z_0 = (t_0, x_0)$ at which the entropy production satisfies $|\mu|(B_r(z_0)) = \mathcal{O}(r^{1+\alpha})$ as $r \to 0^+$, for some $\alpha > 0$, is a Lebesgue point, and a standard covering argument implies the dimension estimate. This is reminiscent of what a BV solution satisfies: $\mathcal{H}^1$-a.e. point $z_0$ at which $|\mu|(B_r(z_0)) = o(r)$ is a Lebesgue point. Here we miss the points at which $|\mu|(B_r)/r$ decays to 0 more slowly than algebraically. Since then, another proof of that Lebesgue point property has been given in [Marconi, 2022a], which applies to more general scalar conservation laws, but a complete analog of the Lebesgue point properties of BV solutions is still missing.

The proof of Theorem 3.5 is in two main steps: first we obtain a weaker estimate (essentially a $W^{-1,1}$ estimate), then we improve it to an $L^4$ estimate using the regularity

provided by the finite-entropy condition. The first step is the most original and relies on the correspondence between Burgers equation and the Hamilton-Jacobi equation

$$\partial_t h + \frac{1}{2}(\partial_x h)^2 = 0, \qquad u = \partial_x h.$$

The strategy consists in refining estimates of [De Lellis et al., 2004], to obtain a quantitative version of the fact that a weak solution $h$ is a viscosity solution if $\mu_+ = 0$.

Besides obtaining information about Lebesgue points, another motivation of such stability estimate is even more directly related to the open question of proving upper bounds for the associated variational models: this requires constructing approximations of $u$ whose energy is very precisely controlled. An approach proposed in [Bellettini et al., 2010] is to separate regions were $\mu$ is mostly positive or mostly negative, and use a backward or forward (depending on the sign of $\mu$) vanishing viscosity approximation. Then one needs to quantify how close $u$ is to an entropy (or a backward entropy) solution in the corresponding regions, and to do so in a way that all errors can be summed. This motivates looking for sharp stability estimates (the exponent 4/35 in Theorem 3.5 is most probably not sharp).

It is not clear which norm is the most adapted to sharp estimates: for instance, the classical theory of entropy solutions suggests using $L^1$ distances, and the arguments of [Marconi, 2022a] employ Wasserstein distances. Relative entropy methods, which first arose in [Dafermos, 1979, DiPerna, 1979] to prove weak-strong stability results, provide an $L^2$-based framework. In [Contreras Hip and Lamy, 2021, Contreras Hip et al., 2022] we adapt the relative entropy methods of [Leger, 2011, Leger and Vasseur, 2011, Krupa and Vasseur, 2019, Krupa and Vasseur, 2020] to study the stability of shock waves, which are particular entropy solutions of Burgers' equation of the form

$$\begin{aligned} S(t,x) &= S_0(x - vt), \\ S_0(x) &= u_\ell \mathbf{1}_{x<0} + u_r \mathbf{1}_{x>0}, \quad v = \frac{u_\ell + u_r}{2}, \end{aligned}$$

for some $u_\ell > u_r$. We prove that the $L^2$ distance of $u(t) = u(t,\cdot)$ to a small drift of a shock wave $\tau_{h(t)}S(t) = S(t, \cdot - h(t))$, is estimated by the initial distance, plus the positive part of the entropy production.

**Theorem 3.6.** *Let $u$ a bounded finite-entropy solution of Burgers' equation* (3.5) *on $(0,T)\times\mathbb{R}$. For any shock wave $S$, any $t \in [0,T]$ and $R > 0$ we have*

$$\begin{aligned} &\int_{-R}^{R} |u(t) - \tau_{h(t)}S(t)|^2 \, dx \\ &\leq \int_{-R-tV}^{R+tV} |u_0 - S_0|^2 \, dx + C\,\mu_+([0,t]\times[-R-tV, R+tV]), \end{aligned}$$

*where $V = \max\{|u_\ell|, |u_r|, \|u\|_\infty\}$, $C > 0$ is an absolute constant, and $h(t)$ is a Lipschitz*

*drift controlled by*

$$\frac{1}{C}(u_\ell - u_r)\int_0^t h'(\tau)^2 d\tau$$
$$\leq \int_{-2Vt}^{2Vt} |u_0 - S_0|^2\, dx + \mu_+([0,t]\times[-2Vt, 2Vt]).$$

This generalizes results of [Krupa and Vasseur, 2019], where it is assumed that $\mu \leq 0$ ($\mu_+ = 0$) and that $u$ satisfies an additional trace assumption. The necessity of introducing a drift, at least in the case $\mu \leq 0$, is proved in [Vasseur, 2016].

The relative entropy method is based on the identity

$$\frac{1}{2}\partial_t(u - v_0)^2 = \mu - \partial_x q(u; v_0),$$

for any constant $v_0$ and some polynomial $q(u; v_0)$, whenever $u$ is a weak solution of Burgers' equation. This automatically provides estimates on the $L^2$ distance to a constant solution. When dealing with shocks, applying this on both sides of the shock provides an estimate with additional boundary terms, and it was realized in [Leger, 2011] that a well-chosen drift can make these extra terms nonpositive, when $u$ is an entropy solution. To prove Theorem 3.6 we check that the positive part of these extra terms can be controlled by the entropy production.

Moreover, an extra trace assumption on $u$ is needed in [Leger, 2011] in order to construct the drift $h(t)$. We are able to remove that trace assumption by showing in [Contreras Hip et al., 2022] that any finite-entropy solution admits generalized characteristic curves. These generalized characteristics are obtained via a Lagrangian representation introduced in [Marconi, 2022a], and which is the main tool in the proof of rectifiability of the entropy production [Marconi, 2022b]. Roughly speaking, the Lagrangian representation describes the evolution of the hypograph $H_t = \{(x, v)\colon v < u(t, x)\}$ by decomposing it into particles $(\gamma_x(t), \gamma_v(t))$ which evolve according to the characteristic equation $\dot\gamma_x = \gamma_v$. Generalized characteristics should do the same for the evolution of the graph $G_t = \{(x, v)\colon v = u(t, x)\}$. The Lagrangian representation provides fairly natural candidates (essentially "infima" of Lagrangian curves), and the difficulty is to show that these candidates do satisfy the characteristic equation $\dot\gamma = u(\gamma)$ (which needs to be interpreted correctly since $u$ is not continuous). To achieve this, our main new ingredient is a formula expressing the entropy flux across a curve (or a hypersurface for higher dimensional scalar conservation laws) in terms of the Lagrangian representation.

*Remark* 3.7. Combining the two estimates of Theorem 3.6 gives a drift-less estimate of the form

$$\int_{-R}^{R} |u(t) - u^{sh}(t)|^2\, dx \leq \int_{-R-St}^{R+St} |u_0 - u_0^{sh}|^2\, dx + C\|\mu_+\|$$
$$+ CS^{\frac{3}{2}}\sqrt{t}\sqrt{\int_{-2St}^{2St} |u_0 - u_0^{sh}|^2\, dx + C\|\mu_+\|}.$$

Approximating initial conditions by piecewise constant functions, we also prove similar estimates on the $L^2$-distance to entropy solutions with $BV$ initial data. Next, one could use the regularity provided by the finite-entropy condition to approximate $u_0$ with $BV$ functions, iterate the above estimates on small time-intervals, and deduce for instance

$$\int_{-1}^{1} |u(t) - u^{ent}(t)|^2 \, dx \leq C\sqrt{\mu_+([-1,t] \times [-1-2t, 1+2t])},$$

if $\|u\|_\infty \leq 1$, where $u^{ent}$ is the entropy solution equal to $u$ at $t = 0$, and $C > 0$ an absolute constant. This gives a strong improvement of Theorem 3.5, but a really useful estimate should not have a square root in the right-hand side.

We turn now to the eikonal equation, and to the question of making quantitative the rigidity statement of [Jabin et al., 2002]: if $m$ is a zero-energy state in a smooth bounded simply connected domain $\Omega$, with tangent boundary conditions $m \cdot n_{\partial\Omega} = 0$, then $\Omega$ must be a disk, and $m$ a vortex.

The first (non-sharp) quantitative version of that rigidity result was obtained in [Lorent, 2014] (see also [Lorent, 2012]), under the extra assumptions that $\Omega$ is convex, and that the trace $m_{\partial\Omega}$ does not jump (between the two opposite unit tangents). In [Lamy and Marconi, 2022] we obtain a sharp stability estimate and remove the convexity and extra trace assumptions:

**Theorem 3.8.** *Let $\Omega \subset \mathbb{R}^2$ a $C^{1,1}$ simply connected domain with $\mathcal{H}^1(\partial\Omega) = 2\pi$, and $m \colon \Omega \to \mathbb{R}^2$ a finite-entropy solution of the eikonal equation such that $m \cdot n_{\partial\Omega} = 0$ on $\partial\Omega$. Then, up to translating $\Omega$, we have*

$$\int_{\partial\Omega} \left| n_{\partial\Omega}(x) - \frac{x}{|x|} \right|^2 d\mathcal{H}^1(x) \leq C\|\sigma\|,$$

*where $C > 0$ depends on the maximal absolute curvature of $\partial\Omega$, and $\|\sigma\|$ is the total variation on $\Omega \times \mathbb{S}^1$ of the kinetic defect measure* (3.6).

Recall that $\|\sigma\| \leq c \liminf AG_\varepsilon(m_\varepsilon; \Omega)$ if $m = \lim m_\varepsilon$ thanks to (3.3) and the interpretation of $\sigma$ in terms of the entropies (3.7), so Theorem 3.8 really implies a bound in terms of the energy. Moreover, this is valid for other types of energies, as the one considered in [Alouges et al., 2002]. This estimate is sharp in the sense that, if $\Omega$ is a smoothed out $N$-gon and $m = \nabla^\perp \operatorname{dist}_{\partial\Omega}$, then both sides of the inequality are of order $N^{-2}$. As a corollary, we obtain other estimates which are, however, probably not sharp:

$$\operatorname{dist}(\partial\Omega; \partial D_1) \leq C\|\sigma\|^{\frac{1}{2}}, \qquad \int_\Omega \left| m - \alpha i \frac{x}{|x|} \right|^4 dx \leq C\|\sigma\|^{\frac{2}{3}},$$

for some $\alpha \in \{\pm 1\}$, up to translating $\Omega$.

The main tool that allows us to prove Theorem 3.8 is a Lagrangian representation constructed in [Marconi, 2021a], inspired by the superposition principle for transport equations [Ambrosio and Crippa, 2014]. That tool has already been very successfully

applied to solve a variety of questions related to the present chapter [Marconi, 2022b, Marconi, 2021b]. It allows to disintegrate the kinetic dissipation measure $\sigma$ along Lagrangian trajectories $(\gamma_x(t), \gamma_s(t))$ taking values in $\{m(x) \cdot e^{is} > 0\} \subset \Omega \times \mathbb{S}^1$, and solving the characteristic equation $\dot{\gamma}_x = e^{i\gamma_s}$. Along each curve $\gamma$, the disintegrated dissipation corresponds to the total variation of $\gamma_s$: how much the curve $\gamma_x$ turns.

In the absence of dissipation ($\sigma = 0$), the Lagrangian curves would be straight lines. Following them from the boundary, elementary geometric arguments provide the rigidity of zero-energy states, in the spirit of [Jabin et al., 2002]. If the domain is not a disk, the same elementary geometric arguments show that the curves must turn, and this creates dissipation. The main technical achievement in [Lamy and Marconi, 2022] is to quantify that dissipation along well-chosen packets of Lagrangian curves, whose contributions can then be summed to bound from below the total dissipation $\|\sigma\|$, in terms of how much the unit normal $n_{\partial\Omega}$ deviates from being radial.

## 3.5 Perspectives

As explained in this chapter's introduction, the central question is to complete the $\Gamma$-convergence of the Aviles-Giga energy or similar models [Rivière and Serfaty, 2001, Rivière and Serfaty, 2003, Alouges et al., 2002, Varadhan, 2004, Bellettini et al., 2010], that is, construct approximating sequence with an energy upper bound matching the lower bound (3.2) – or similar lower bounds for other models.

In the literature, the focus has been placed mostly on proving $BV$-like properties of finite-entropy solutions: rectifiability of the jump set, rectifiability of the entropy productions, Lebesgue points outside the jump set. The state of the art depends on the model, and this program is more advanced for Burgers equation or solutions of the eikonal equation coming from the micromagnetics model [Rivière and Serfaty, 2001, Rivière and Serfaty, 2003], than for the Aviles-Giga energy or [Alouges et al., 2002] – where rectifiability of the entropy productions and one-dimensionality of non-Lebesgue point are still open. The main perspectives I propose here concern sharp stability estimates and techniques related to E. Marconi's Lagrangian representation.

### Lebesgue points

In a work in preparation with E. Marconi, we will extend the results of [Lamy and Otto, 2018, Marconi, 2022a] on Lebesgue points for Burgers' equation, to finite-entropy solutions of the eikonal equation. We do this by building on the strategy of [Marconi, 2022a]. That strategy does not directly work for finite-entropy solutions of the eikonal equation, basically because $\mathbb{S}^1$ is not ordered, while $\mathbb{R}$ is. As a consequence, configurations that are impossible in the case of Burgers are not obviously ruled out here: essentially, oscillations between two opposite directions $\pm m_0 \in \mathbb{S}^1$. This is also why the rectifiability results of [Marconi, 2022b, Marconi, 2021b] are not known for finite-entropy solutions of the eikonal equation (as made apparent in the expression of the kinetic measure in [Marconi, 2021a]).

We will complement the arguments of [Marconi, 2022a] by proving that such oscillations, if they happen, can be quantified in a way to contradict the vanishing mean oscillation property which is satisfied at $\mathcal{H}^1$-a.e. non-jump point thanks to the structure result of [De Lellis and Otto, 2003]. We achieve this quantitative oscillation property by exploiting the Lagrangian formulation, much in the spirit of Theorem 3.8's proof.

**Stability and differential inclusions**

We would like to extend the stability of zero-energy states of Theorem 3.8 to a case without any boundary conditions. Theorem 3.4 suggests approaching that question from the point of view of differential inclusions: show for instance that

$$\inf_{\nabla\varphi\in K}\int_{B_{1/2}}|\nabla w-\nabla\varphi|^2\,dx\leq C\left(\int_{B_1}\operatorname{dist}^2(\nabla w,K)\,dx\right)^{\alpha},$$

for some $\alpha\in(0,1)$. What we gain here, compared to solutions of the eikonal equation, is the possibility of starting out by looking at smooth perturbations of exact solutions $\nabla\varphi\in K$ (which are in correspondence with zero-energy states) – since here we do not have the constraint to satisfy the eikonal equation.

Because the set $K\subset\mathbb{R}^2$ is not elliptic, we know that, unlike what happens for $K=SO(2)$ [Friesecke et al., 2002, Faraco, 2004], the exponent $\alpha$ cannot be equal to 1. Instead, it seems reasonable to conjecture that $\alpha=1/2$, and to follow the general strategy of [John, 1961, Reshetnyak, 1994, Friesecke et al., 2002, Faraco, 2004], which is in two main steps: first, prove an infinitesimal version for smooth perturbations of exact solutions; second, deduce the general case.

As a training ground to gain insight into these issues, in [Lamy et al., 2022c] we prove a general stability result for arbitrary closed curves $K\subset\mathbb{R}^{2\times2}$ which are elliptic, going beyond the explicit case of $K=SO(2)$ [Friesecke et al., 2002]. The first step of the above strategy follows directly from general Korn-type inequalities of [John, 1961], and the second step uses ideas related to a well-studied correspondence between quasiconformal mappings and quasilinear elliptic equations in two dimensions [Astala and Faraco, 2002, Faraco, 2004, Faraco and Székelyhidi, 2008, Astala et al., 2009, Astala et al., 2020, Astala et al., 2012, Astala et al., 2017, Astala et al., 2019].

To apply that strategy for the nonelliptic differential inclusion (3.8), the first step will not reduce directly to the linear estimates of [John, 1961] (or their recent generalizations in [Arroyo-Rabasa, 2021]), but we will probably need to combine these with some interpolation inequalities. The second step should be significantly harder, and require new regularity estimates for degenerate quasilinear elliptic equations, in the spirit of [Santambrogio and Vespri, 2010, Colombo and Figalli, 2014a, Colombo and Figalli, 2014b, Bousquet and Brasco, 2018].

**Stability estimates for Burgers**

Theorem 3.6 about the stability of elementary shock waves is not entirely satisfactory because of the presence of the drift. It is proved in [Vasseur, 2016] that this drift is

necessary, in the case where $\mu_+ = 0$. It could very well be that the extra freedom provided by the $\mu_+$ term removes the necessity of that drift: there is a hope of proving a drift-less version of Theorem 3.6. To do so will definitely require new ingredients. One promising approach is to relax the $L^2$-norm setting to allow for more general quantities which are bounded above and below by multiples of the $L^2$-norm, as done e.g. in [Kang, 2021] with a weighted relative entropy method.

Another, completely different approach to optimal stability estimates for Burgers is suggested by [Marconi, 2022a]: use Wasserstein distances which arise naturally in connection with the kinetic formulation of finite-entropy solutions. In that setting, one thing that would have to be understood before considering stability estimates among finite-entropy solutions, is whether such distances behave well among entropy solutions, an issue possibly related to the methods in [Brenier, 1984, Gigli and Otto, 2013, Esselborn et al., 2016].

# Bibliography

[Alama et al., 2021] Alama, S., Bronsard, L., Golovaty, D., and Lamy, X. (2021). Saturn ring defect around a spherical particle immersed in a nematic liquid crystal. *Calc. Var. Partial Differential Equations*, 60(6):Paper No. 225, 50.

[Alama et al., 2016] Alama, S., Bronsard, L., and Lamy, X. (2016). Minimizers of the Landau–de Gennes energy around a spherical colloid particle. *Arch. Ration. Mech. Anal.*, 222(1):427–450.

[Alama et al., 2018] Alama, S., Bronsard, L., and Lamy, X. (2018). Spherical particle in nematic liquid crystal under an external field: the Saturn ring regime. *J. Nonlinear Sci.*, 28(4):1443–1465.

[Alama et al., 2022] Alama, S., Bronsard, L., Lamy, X., and Venkatraman, R. (2022). Far-field expansions for harmonic maps and the electrostatics analogy in nematic suspensions. *arXiv:2202.12794*.

[Allard, 1973] Allard, W. K. (1973). On the first variation of area and generalized mean curvature. C.I.M.E., III Ciclo Varenna 1972, geom. Measure Theory minimal Surf., 1-30 (1973).

[Almgren and Lieb, 1988] Almgren, F. and Lieb, E. H. (1988). Singularities of energy minimizing maps from the ball to the sphere: Examples, counterexamples, and bounds. *Ann. Math. (2)*, 128(3):483–530.

[Alouges et al., 2021] Alouges, F., Chambolle, A., and Stantejsky, D. (2021). The Saturn ring effect in nematic liquid crystals with external field: effective energy and hysteresis. *Arch. Ration. Mech. Anal.*, 241(3):1403–1457.

[Alouges et al., 2022] Alouges, F., Chambolle, A., and Stantejsky, D. (2022). Convergence to line and surface energies in nematic liquid crystal colloids with external magnetic field. *arXiv:2202.10703*.

[Alouges and Ghidaglia, 1997] Alouges, F. and Ghidaglia, J. M. (1997). Minimizing Oseen-Frank energy for nematic liquid crystals: Algorithms and numerical results. *Ann. Inst. Henri Poincaré, Phys. Théor.*, 66(4):411–447.

[Alouges et al., 2002] Alouges, F., Rivière, T., and Serfaty, S. (2002). Néel and cross-tie wall energies for planar micromagnetic configurations. *ESAIM Control Optim. Calc. Var.*, 8:31–68. A tribute to J. L. Lions.

[Ambrosio and Crippa, 2014] Ambrosio, L. and Crippa, G. (2014). Continuity equations and ODE flows with non-smooth velocity. *Proc. R. Soc. Edinb., Sect. A, Math.*, 144(6):1191–1244.

[Ambrosio et al., 1999] Ambrosio, L., De Lellis, C., and Mantegazza, C. (1999). Line energies for gradient vector fields in the plane. *Calc. Var. Partial Differential Equations*, 9(4):327–255.

[Ambrosio et al., 2002] Ambrosio, L., Kirchheim, B., Lecumberry, M., and Rivière, T. (2002). On the rectifiability of defect measures arising in a micromagnetics model. In *Nonlinear problems in mathematical physics and related topics II. In honour of Professor O. A. Ladyzhenskaya*, pages 29–60. New York, NY: Kluwer Academic Publishers.

[André and Shafrir, 1998] André, N. and Shafrir, I. (1998). Asymptotic behavior of minimizers for the Ginzburg-Landau functional with weight. I, II. *Arch. Rational Mech. Anal.*, 142(1):45–73, 75–98.

[Arroyo-Rabasa, 2021] Arroyo-Rabasa, A. (2021). New projection and Korn estimates for a class of constant-rank operators on domains. *arXiv:2109.14602*.

[Astala et al., 2019] Astala, K., Clop, A., Faraco, D., and Jääskeläinen, J. (2019). Manifolds of quasiconformal mappings and the nonlinear Beltrami equation. *J. Anal. Math.*, 139(1):207–238.

[Astala et al., 2017] Astala, K., Clop, A., Faraco, D., Jääskeläinen, J., and Koski, A. (2017). Nonlinear Beltrami operators, Schauder estimates and bounds for the Jacobian. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 34(6):1543–1559.

[Astala et al., 2020] Astala, K., Clop, A., Faraco, D., Jääskeläinen, J., and Koski, A. (2020). Improved Hölder regularity for strongly elliptic PDEs. *J. Math. Pures Appl. (9)*, 140:230–258.

[Astala et al., 2012] Astala, K., Clop, A., Faraco, D., Jääskeläinen, J., and Székelyhidi, L. j. (2012). Uniqueness of normalized homeomorphic solutions to nonlinear Beltrami equations. *Int. Math. Res. Not.*, 2012(18):4101–4119.

[Astala and Faraco, 2002] Astala, K. and Faraco, D. (2002). Quasiregular mappings and Young measures. *Proc. Roy. Soc. Edinburgh Sect. A*, 132(5):1045–1056.

[Astala et al., 2009] Astala, K., Iwaniec, T., and Martin, G. (2009). *Elliptic Partial Differential Equations and Quasiconformal Mappings in the Plane*, volume 48 of *Princeton Mathematical Series*. Princeton University Press.

[Aviles and Giga, 1987] Aviles, P. and Giga, Y. (1987). A mathematical problem related to the physical theory of liquid crystal configurations. In *Miniconference on geometry and partial differential equations, 2 (Canberra, 1986)*, volume 12 of *Proc. Centre Math. Anal. Austral. Nat. Univ.*, pages 1–16. Austral. Nat. Univ., Canberra.

[Ball and Zarnescu, 2011] Ball, J. M. and Zarnescu, A. (2011). Orientability and energy minimization in liquid crystal models. *Arch. Ration. Mech. Anal.*, 202(2):493–535.

[Barboza et al., 2013] Barboza, R., Bortolozzo, U., Assanto, G., Vidal-Henriquez, E., Clerc, M. G., and Residori, S. (2013). Harnessing optical vortex lattices in nematic liquid crystals. *Phys. Rev. Lett.*, 111:093902.

[Barboza et al., 2016] Barboza, R., Bortolozzo, U., Clerc, M. G., Davila, J. D., Kowalczyk, M., Residori, S., and Vidal-Henriquez, E. (2016). Light-matter interaction induces a shadow vortex. *Phys. Rev. E*, 93:050201.

[Bauman et al., 2012] Bauman, P., Park, J., and Phillips, D. (2012). Analysis of nematic liquid crystals with disclination lines. *Arch. Ration. Mech. Anal.*, 205(3):795–826.

[Beaulieu and Hadiji, 1998] Beaulieu, A. and Hadiji, R. (1998). A Ginzburg-Landau problem with weight having minima on the boundary. *Proc. Roy. Soc. Edinburgh Sect. A*, 128(6):1181–1215.

[Bedford, 2016] Bedford, S. (2016). Function spaces for liquid crystals. *Arch. Ration. Mech. Anal.*, 219(2):937–984.

[Bellettini et al., 2010] Bellettini, G., Bertini, L., Mariani, M., and Novaga, M. (2010). Γ-entropy cost for scalar conservation laws. *Arch. Ration. Mech. Anal.*, 195(1):261–309.

[Berlyand et al., 2005] Berlyand, L., Cioranescu, D., and Golovaty, D. (2005). Homogenization of a Ginzburg-Landau model for a nematic liquid crystal with inclusions. *J. Math. Pures Appl. (9)*, 84(1):97–136.

[Bethuel, 1993] Bethuel, F. (1993). On the singular set of stationary harmonic maps. *Manuscr. Math.*, 78(4):417–443.

[Bethuel et al., 1993] Bethuel, F., Brezis, H., and Hélein, F. (1993). Asymptotics for the minimization of a Ginzburg-Landau functional. *Calc. Var. Partial Differential Equations*, 1(2):123–148.

[Bethuel et al., 1994] Bethuel, F., Brezis, H., and Hélein, F. (1994). *Ginzburg-Landau vortices*. Progress in Nonlinear Differential Equations and their Applications, 13. Birkhäuser.

[Bethuel et al., 2001] Bethuel, F., Brezis, H., and Orlandi, G. (2001). Asymptotics for the Ginzburg-Landau equation in arbitrary dimensions. *J. Funct. Anal.*, 186(2):432–520.

[Bethuel and Chiron, 2007] Bethuel, F. and Chiron, D. (2007). Some questions related to the lifting problem in Sobolev spaces. In *Perspectives in nonlinear partial differential equations*, volume 446 of *Contemp. Math.*, pages 125–152. Amer. Math. Soc., Providence, RI.

[Bethuel et al., 2005] Bethuel, F., Orlandi, G., and Smets, D. (2005). Improved estimates for the Ginzburg-Landau equation: the elliptic case. *Ann. Sc. Norm. Super. Pisa, Cl. Sci. (5)*, 4(2):319–355.

[Blaser and Rivière, 2010] Blaser, M. and Rivière, T. (2010). A minimality property for entropic solutions to scalar conservation laws in $1 + 1$ dimensions. *Commun. Partial Differ. Equations*, 35(10-12):1763–1801.

[Bousquet and Brasco, 2018] Bousquet, P. and Brasco, L. (2018). $C^1$ regularity of orthotropic $p$-harmonic functions in the plane. *Anal. PDE*, 11(4):813–854.

[Brenier, 1984] Brenier, Y. (1984). Averaged multivalued solutions for scalar conservation laws. *SIAM J. Numer. Anal.*, 21:1013–1037.

[Brezis et al., 1986] Brezis, H., Coron, J.-M., and Lieb, E. H. (1986). Harmonic maps with defects. *Comm. Math. Phys.*, 107(4):649–705.

[Brezis et al., 1994] Brezis, H., Merle, F., and Rivière, T. (1994). Quantization effects for $-\Delta u = u(1 - |u|^2)$ in $\mathbf{R}^2$. *Arch. Rational Mech. Anal.*, 126(1):35–58.

[Brochard and de Gennes, 1970] Brochard, F. and de Gennes, P. (1970). Theory of magnetic suspensions in liquid crystals. *J. Phys. France*, 31(7):691–708.

[Calderer et al., 2014] Calderer, M. C., DeSimone, A., Golovaty, D., and Panchenko, A. (2014). An effective model for nematic liquid crystal composites with ferromagnetic inclusions. *SIAM J. Appl. Math.*, 74(2):237–262.

[Canevari, 2015] Canevari, G. (2015). Biaxiality in the asymptotic analysis of a 2D Landau–de Gennes model for liquid crystals. *ESAIM Control Optim. Calc. Var.*, 21(1):101–137.

[Canevari, 2017] Canevari, G. (2017). Line defects in the small elastic constant limit of a three-dimensional Landau–de Gennes model. *Arch. Ration. Mech. Anal.*, 223(2):591–676.

[Canevari and Orlandi, 2019] Canevari, G. and Orlandi, G. (2019). Topological singular set of vector-valued maps. I: Applications to manifold-constrained Sobolev and BV spaces. *Calc. Var. Partial Differ. Equ.*, 58(2):40. Id/No 72.

[Canevari and Orlandi, 2021] Canevari, G. and Orlandi, G. (2021). Topological singular set of vector-valued maps. II: $\Gamma$-convergence for Ginzburg-Landau type functionals. *Arch. Ration. Mech. Anal.*, 241(2):1065–1135.

[Canevari and Zarnescu, 2020a] Canevari, G. and Zarnescu, A. (2020a). Design of effective bulk potentials for nematic liquid crystals via colloidal homogenisation. *Math. Models Methods Appl. Sci.*, 30(2):309–342.

[Canevari and Zarnescu, 2020b] Canevari, G. and Zarnescu, A. (2020b). Polydispersity and surface energy strength in nematic colloids. *Math. Eng.*, 2(2):290–312.

[Chen et al., 1994] Chen, X., Elliott, C. M., and Qi, T. (1994). Shooting method for vortex solutions of a complex-valued Ginzburg-Landau equation. *Proc. Roy. Soc. Edinburgh Sect. A*, 124(6):1075–1088.

[Chen and Lin, 1993] Chen, Y. and Lin, F. (1993). Evolution of harmonic maps with Dirichlet boundary conditions. *Comm. Anal. Geom*, 1(3-4):327–346.

[Chen and Struwe, 1989] Chen, Y. and Struwe, M. (1989). Existence and partial regularity results for the heat flow for harmonic maps. *Math. Z.*, 201(1):83–103.

[Clerc et al., 2014] Clerc, M. G., Vidal-Henriquez, E., Davila, J. D., and Kowalczyk, M. (2014). Symmetry breaking of nematic umbilical defects through an amplitude equation. *Phys. Rev. E*, 90:012507.

[Cohen and Taylor, 1990] Cohen, R. and Taylor, M. (1990). Weak stability of the map $x/|x|$ for liquid crystal functionals. *Commun. Partial Differ. Equations*, 15(5):675–692.

[Colbert-Kelly et al., 2017] Colbert-Kelly, S., McFadden, G. B., Phillips, D., and Shen, J. (2017). Numerical analysis and simulation for a generalized planar Ginzburg-Landau equation in a circular geometry. *Commun. Math. Sci.*, 15(2):329–357.

[Colbert-Kelly and Phillips, 2013] Colbert-Kelly, S. and Phillips, D. (2013). Analysis of a Ginzburg-Landau type energy model for smectic $C^*$ liquid crystals with defects. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 30(6):1009–1026.

[Colombo and Figalli, 2014a] Colombo, M. and Figalli, A. (2014a). An excess-decay result for a class of degenerate elliptic equations. *Discrete Contin. Dyn. Syst., Ser. S*, 7(4):631–652.

[Colombo and Figalli, 2014b] Colombo, M. and Figalli, A. (2014b). Regularity results for very degenerate elliptic equations. *J. Math. Pures Appl. (9)*, 101(1):94–117.

[Constantin et al., 1994] Constantin, P., E, W., and Titi, E. S. (1994). Onsager's conjecture on the energy conservation for solutions of Euler's equation. *Comm. Math. Phys.*, 165(1):207–209.

[Conti and De Lellis, 2007] Conti, S. and De Lellis, C. (2007). Sharp upper bounds for a variational problem with singular perturbation. *Math. Ann.*, 338(1):119–146.

[Contreras and Lamy, 2022] Contreras, A. and Lamy, X. (2022). Singular perturbation of manifold-valued maps with anisotropic energy. *Anal. PDE*.

[Contreras et al., 2018] Contreras, A., Lamy, X., and Rodiac, R. (2018). On the convergence of minimizers of singular perturbation functionals. *Indiana Univ. Math. J.*, 67(4):1665–1682.

[Contreras Hip and Lamy, 2021] Contreras Hip, A. A. and Lamy, X. (2021). On the $L^2$ stability of shock waves for finite-entropy solutions of Burgers. *J. Differ. Equations*, 301:236–265.

[Contreras Hip et al., 2022] Contreras Hip, A. A., Lamy, X., and Marconi, E. (2022). Generalized characteristics for finite entropy solutions of Burgers' equation. *Nonlinear Anal.*, 219:Paper No. 112804.

[Dafermos, 1979] Dafermos, C. M. (1979). The second law of thermodynamics and stability. *Arch. Rational Mech. Anal.*, 70(2):167–179.

[De Gennes and Prost, 1993] De Gennes, P.-G. and Prost, J. (1993). *The physics of liquid crystals*. Oxford university press.

[De Lellis et al., 2021] De Lellis, C., De Philippis, G., Kirchheim, B., and Tione, R. (2021). Geometric measure theory and differential inclusions. *Ann. Fac. Sci. Toulouse, Math. (6)*, 30(4):899–960.

[De Lellis and Ignat, 2015] De Lellis, C. and Ignat, R. (2015). A regularizing property of the $2D$-eikonal equation. *Comm. Partial Differential Equations*, 40(8):1543–1557.

[De Lellis and Otto, 2003] De Lellis, C. and Otto, F. (2003). Structure of entropy solutions to the eikonal equation. *J. Eur. Math. Soc. (JEMS)*, 5(2):107–145.

[De Lellis et al., 2003] De Lellis, C., Otto, F., and Westdickenberg, M. (2003). Structure of entropy solutions for multi-dimensional scalar conservation laws. *Arch. Ration. Mech. Anal.*, 170(2):137–184.

[De Lellis et al., 2004] De Lellis, C., Otto, F., and Westdickenberg, M. (2004). Minimal entropy conditions for Burgers equation. *Q. Appl. Math.*, 62(4):687–700.

[De Lellis and Westdickenberg, 2003] De Lellis, C. and Westdickenberg, M. (2003). On the optimality of velocity averaging lemmas. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 20(6):1075–1085.

[De Rosa and Tione, 2022] De Rosa, A. and Tione, R. (2022). Regularity for graphs with bounded anisotropic mean curvature. *Invent. Math.*, pages 1–45.

[Del Pino et al., 2004] Del Pino, M., Felmer, P., and Kowalczyk, M. (2004). Minimality and nondegeneracy of degree-one Ginzburg-Landau vortex as a Hardy's type inequality. *Int. Math. Res. Not.*, 2004(30):1511–1527.

[DeSimone et al., 2001] DeSimone, A., Kohn, R. V., Müller, S., and Otto, F. (2001). A compactness result in the gradient theory of phase transitions. *Proc. Roy. Soc. Edinburgh Sect. A*, 131(4):833–844.

[Dipasquale et al., 2020] Dipasquale, F., Millot, V., and Pisante, A. (2020). Torus-like solutions for the Landau-de Gennes model. Part II: Topology of $\mathbb{S}^1$-equivariant minimizers. *arXiv:2008.13676*.

[Dipasquale et al., 2021a] Dipasquale, F., Millot, V., and Pisante, A. (2021a). Torus-like solutions for the Landau-de Gennes model. I: The Lyuksyutov regime. *Arch. Ration. Mech. Anal.*, 239(2):599–678.

[Dipasquale et al., 2021b] Dipasquale, F., Millot, V., and Pisante, A. (2021b). Torus-like solutions for the Landau-de Gennes model. Part III: torus vs split minimizers. *arXiv:2109.15178*.

[DiPerna, 1979] DiPerna, R. J. (1979). Uniqueness of solutions to hyperbolic conservation laws. *Indiana Univ. Math. J.*, 28(1):137–188.

[Esselborn et al., 2016] Esselborn, E., Gigli, N., and Otto, F. (2016). Algebraic contraction rate for distance between entropy solutions of scalar conservation laws. *J. Math. Anal. Appl.*, 435(2):1525–1551.

[Evans, 1991] Evans, L. C. (1991). Partial regularity for stationary harmonic maps into spheres. *Arch. Ration. Mech. Anal.*, 116(2):101–113.

[Faraco, 2004] Faraco, D. (2004). Tartar conjecture and Beltrami operators. *Mich. Math. J.*, 52(1):83–104.

[Faraco and Székelyhidi, 2008] Faraco, D. and Székelyhidi, L. (2008). Tartar's conjecture and localization of the quasiconvex hull in $\mathbb{R}^{2\times 2}$. *Acta Math.*, 200(2):279–305.

[Faraco and Zhong, 2005] Faraco, D. and Zhong, X. (2005). Geometric rigidity of conformal matrices. *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 5(4):557–585.

[Friesecke et al., 2002] Friesecke, G., James, R. D., and Müller, S. (2002). A theorem on geometric rigidity and the derivation of nonlinear plate theory from three-dimensional elasticity. *Comm. Pure Appl. Math.*, 55(11).

[Fukuda et al., 2004a] Fukuda, J., Stark, H., Yoneya, M., and Yokoyama, H. (2004a). Dynamics of a nematic liquid crystal around a spherical particle. *J. Physics: Condens. Matter*, 16(19):S1957.

[Fukuda and Yokoyama, 2006] Fukuda, J. and Yokoyama, H. (2006). Stability of the director profile of a nematic liquid crystal around a spherical particle under an external field. *Eur. Phys. J. E*, 21(4):341–347.

[Fukuda et al., 2004b] Fukuda, J.-I., Yoneya, M., and Yokoyama, H. (2004b). Nematic liquid crystal around a spherical particle: Investigation of the defect structure and its stability using adaptive mesh refinement. *The European Physical Journal E*, 13(1):87–98.

[Gess and Lamy, 2019] Gess, B. and Lamy, X. (2019). Regularity of solutions to scalar conservation laws with a force. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 36(2):505–521.

[Ghiraldin and Lamy, 2020] Ghiraldin, F. and Lamy, X. (2020). Optimal Besov differentiability for entropy solutions of the eikonal equation. *Comm. Pure Appl. Math.*, 73(2):317–349.

[Gigli and Otto, 2013] Gigli, N. and Otto, F. (2013). Entropic Burgers' equation via a minimizing movement scheme based on the Wasserstein metric. *Calc. Var. Partial Differ. Equ.*, 47(1-2):181–206.

[Goldman et al., 2015] Goldman, M., Josien, M., and Otto, F. (2015). New bounds for the inhomogenous Burgers and the Kuramoto-Sivashinsky equations. *Comm. Partial Differential Equations*, 40(12):2237–2265.

[Goldman et al., 2021] Goldman, M., Merlet, B., Pegon, M., and Serfaty, S. (2021). Compactness and structure of zero-states for unoriented aviles-giga functionals. *arXiv:2112.04959*.

[Golovaty and Montero, 2014] Golovaty, D. and Montero, J. A. (2014). On minimizers of a Landau–de Gennes energy functional on planar domains. *Arch. Ration. Mech. Anal.*, 213(2):447–490.

[Golovaty et al., 2021] Golovaty, D., Novack, M., and Sternberg, P. (2021). A novel Landau–de Gennes model with quartic elastic terms. *European J. Appl. Math.*, 32(1):177–198.

[Golovaty et al., 2020] Golovaty, D., Novack, M., Sternberg, P., and Venkatraman, R. (2020). A model problem for nematic-isotropic transitions with highly disparate elastic constants. *Arch. Ration. Mech. Anal.*, 236(3):1739–1805.

[Golovaty et al., 2019] Golovaty, D., Sternberg, P., and Venkatraman, R. (2019). A Ginzburg-Landau-type problem for highly anisotropic nematic liquid crystals. *SIAM J. Math. Anal.*, 51(1):276–320.

[Golse, 2010] Golse, F. (2010). Nonlinear regularizing effect for hyperbolic partial differential equations. In *XVIth International Congress on Mathematical Physics*, pages 433–437. World Sci. Publ., Hackensack, NJ.

[Golse and Perthame, 2013] Golse, F. and Perthame, B. (2013). Optimal regularizing effect for scalar conservation laws. *Rev. Mat. Iberoam.*, 29(4):1477–1504.

[Gravejat et al., 2021] Gravejat, P., Pacherie, E., and Smets, D. (2021). On the stability of the Ginzburg-Landau vortex. *arXiv:2106.02511*.

[Gu and Abbott, 2000] Gu, Y. and Abbott, N. (2000). Observation of Saturn-ring defects around solid microspheres in nematic liquid crystals. *Phys. Rev. Lett.*, 85:4719–4722.

[Hardt et al., 1986] Hardt, R., Kinderlehrer, D., and Lin, F.-H. (1986). Existence and partial regularity of static liquid crystal configurations. *Comm. Math. Phys.*, 105(4):547–570.

[Hardt et al., 1988] Hardt, R., Kinderlehrer, D., and Lin, F.-H. (1988). Stable defects of minimizers of constrained variational principles. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 5(4):297–322.

[Hardt and Lin, 1989] Hardt, R. and Lin, F. (1989). Stability of singularities of minimizing harmonic maps. *J. Differ. Geom.*, 29(1):113–123.

[Hardt and Lin, 1986] Hardt, R. and Lin, F.-H. (1986). A remark on $H^1$ mappings. *Manuscripta Math.*, 56(1):1–10.

[Haslhofer and Müller, 2011] Haslhofer, R. and Müller, R. (2011). A compactness theorem for complete Ricci shrinkers. *Geom. Funct. Anal.*, 21(5):1091–1116.

[Hélein, 1987] Hélein, F. (1987). Minima de la fonctionnelle énergie libre des cristaux liquides. (Minima of the free energy functional of liquid crystals). *C. R. Acad. Sci., Paris, Sér. I*, 305:565–568.

[Hélein, 2002] Hélein, F. (2002). *Harmonic maps, conservation laws and moving frames. Transl. from the French.*, volume 150 of *Camb. Tracts Math.* Cambridge: Cambridge University Press, 2nd ed. edition.

[Hervé and Hervé, 1994] Hervé, R.-M. and Hervé, M. (1994). Étude qualitative des solutions réelles d'une équation différentielle liée à l'équation de Ginzburg-Landau. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 11(4):427–440.

[Ignat and Lamy, 2019] Ignat, R. and Lamy, X. (2019). Lifting of $\mathbb{RP}^{d-1}$-valued maps in $BV$ and applications to uniaxial $Q$-tensors. with an appendix on an intrinsic $BV$-energy for manifold-valued maps. *Calc. Var. Partial Differ. Equ.*, 58(2):26. Id/No 68.

[Ignat and Merlet, 2012] Ignat, R. and Merlet, B. (2012). Entropy method for line-energies. *Calc. Var. Partial Differential Equations*, 44(3-4):375–418.

[Ignat et al., 2015] Ignat, R., Nguyen, L., Slastikov, V., and Zarnescu, A. (2015). Stability of the melting hedgehog in the Landau–de Gennes theory of nematic liquid crystals. *Arch. Ration. Mech. Anal.*, 215(2):633–673.

[Ignat et al., 2016a] Ignat, R., Nguyen, L., Slastikov, V., and Zarnescu, A. (2016a). Instability of point defects in a two-dimensional nematic liquid crystal model. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 33(4):1131–1152.

[Ignat et al., 2016b] Ignat, R., Nguyen, L., Slastikov, V., and Zarnescu, A. (2016b). Stability of point defects of degree $\pm\frac{1}{2}$ in a two-dimensional nematic liquid crystal model. *Calc. Var. Partial Differential Equations*, 55(5):Paper No. 119, 33.

[Jabin et al., 2002] Jabin, P.-E., Otto, F., and Perthame, B. (2002). Line-energy Ginzburg-Landau models: zero-energy states. *Ann. Sc. Norm. Super. Pisa Cl. Sci. (5)*, 1(1):187–202.

[Jabin and Perthame, 2001] Jabin, P.-E. and Perthame, B. (2001). Compactness in Ginzburg-Landau energy by kinetic averaging. *Comm. Pure Appl. Math.*, 54(9):1096–1109.

[Jabin and Perthame, 2002] Jabin, P.-E. and Perthame, B. (2002). Regularity in kinetic formulations via averaging lemmas. *ESAIM Control Optim. Calc. Var.*, 8:761–774. A tribute to J. L. Lions.

[Jerrard, 1999] Jerrard, R. L. (1999). Lower bounds for generalized Ginzburg-Landau functionals. *SIAM J. Math. Anal.*, 30(4):721–746.

[Jin and Kohn, 2000] Jin, W. and Kohn, R. V. (2000). Singular perturbation and the energy of folds. *J. Nonlinear Sci.*, 10(3):355–390.

[John, 1961] John, F. (1961). Rotation and strain. *Comm. Pure Appl. Math.*, 14(1):391–413.

[Kang, 2021] Kang, M.-J. (2021). $L^2$-type contraction for shocks of scalar viscous conservation laws with strictly convex flux. *J. Math. Pures Appl. (9)*, 145:1–43.

[Kinderlehrer and Ou, 1992] Kinderlehrer, D. and Ou, B. (1992). Second variation of liquid crystal energy at $x/|x|$. *Proc. R. Soc. Lond., Ser. A*, 437(1900):475–487.

[Kinderlehrer et al., 1993] Kinderlehrer, D., Ou, B., and Walkington, N. (1993). The elementary defects of the Oseen-Frank energy for a liquid crystal. *C. R. Acad. Sci., Paris, Sér. I*, 316(5):465–470.

[Kowalczyk et al., 2022] Kowalczyk, M., Lamy, X., and Smyrnelis, P. (2022). Entire vortex solutions of negative degree for the anisotropic Ginzburg-Landau system. *Arch. Ration. Mech. Anal.*, 245(1):565–586.

[Kowalczyk et al., 2012] Kowalczyk, M., Liu, Y., and Pacard, F. (2012). The space of 4-ended solutions to the Allen-Cahn equation in the plane. *Ann. Inst. Henri Poincaré, Anal. Non Linéaire*, 29(5):761–781.

[Kresin and Maz'ya, 2012] Kresin, G. and Maz'ya, V. (2012). *Maximum principles and sharp constants for solutions of elliptic and parabolic systems*, volume 183 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI.

[Krupa and Vasseur, 2019] Krupa, S. G. and Vasseur, A. F. (2019). On uniqueness of solutions to conservation laws verifying a single entropy condition. *J. Hyperbolic Differ. Equ.*, 16(1):157–191.

[Krupa and Vasseur, 2020] Krupa, S. G. and Vasseur, A. F. (2020). Stability and uniqueness for piecewise smooth solutions to a nonlocal scalar conservation law with applications to Burgers-Hilbert equation. *SIAM J. Math. Anal.*, 52(3):2491–2530.

[Kružkov, 1970] Kružkov, S. N. (1970). First order quasilinear equations with several independent variables. *Mat. Sb. (N.S.)*, 81 (123):228–255.

[Lamy et al., 2020] Lamy, X., Lorent, A., and Peng, G. (2020). Rigidity of a non-elliptic differential inclusion related to the Aviles-Giga conjecture. *Arch. Ration. Mech. Anal.*, 238(1):383–413.

[Lamy et al., 2022a] Lamy, X., Lorent, A., and Peng, G. (2022a). On a generalized Aviles-Giga functional: compactness, zero-energy states, regularity estimates and energy bounds. *Comm. Partial Differential Equations*.

[Lamy et al., 2022b] Lamy, X., Lorent, A., and Peng, G. (2022b). On optimal regularity estimates for finite-entropy solutions of scalar conservation laws. *arXiv:2207.12979*.

[Lamy et al., 2022c] Lamy, X., Lorent, A., and Peng, G. (2022c). Quantitative rigidity of differential inclusions in two dimensions. *arXiv:2208.08526*.

[Lamy and Marconi, 2022] Lamy, X. and Marconi, E. (2022). Stability of the vortex in micromagnetics and related models. *arXiv:2209.09662*.

[Lamy and Otto, 2018] Lamy, X. and Otto, F. (2018). On the regularity of weak solutions to burgers' equation with finite entropy production. *Calc. Var. Partial Differential Equations*, 57(4).

[Lamy and Zùñiga, 2022] Lamy, X. and Zùñiga, A. (2022). On the stability of radial solutions to an anisotropic Ginzburg-Landau equation. *SIAM J. Math. Anal.*, 54(1):723–736.

[Lee et al., 2006] Lee, J.-B., Konovalov, D., and Meyer, R. B. (2006). Textural transformations in islands on free standing smectic-C* liquid crystal films. *Phys. Rev. E*, 73:051705.

[Leger, 2011] Leger, N. (2011). $L^2$ stability estimates for shock solutions of scalar conservation laws using the relative entropy method. *Arch. Ration. Mech. Anal.*, 199(3):761–778.

[Leger and Vasseur, 2011] Leger, N. and Vasseur, A. (2011). Relative entropy and the stability of shocks and contact discontinuities for systems of conservation laws with non-BV perturbations. *Arch. Ration. Mech. Anal.*, 201(1):271–302.

[Lewicka and Müller, 2016] Lewicka, M. and Müller, S. (2016). On the optimal constants in Korn's and geometric rigidity estimates in bounded and unbounded domains, under Neumann boundary conditions. *Indiana Univ. Math. J.*, 65(2):377–397.

[Lin, 1987] Lin, F.-H. (1987). Une remarque sur l'application $x/|x|$. (A remark on the map $x/|x|$). *C. R. Acad. Sci., Paris, Sér. I*, 305:529–531.

[Lions et al., 1994] Lions, P.-L., Perthame, B., and Tadmor, E. (1994). A kinetic formulation of multidimensional scalar conservation laws and related equations. *J. Amer. Math. Soc.*, 7(1):169–191.

[Lorent, 2012] Lorent, A. (2012). A simple proof of the characterization of functions of low Aviles Giga energy on a ball via regularity. *ESAIM, Control Optim. Calc. Var.*, 18(2):383–400.

[Lorent, 2014] Lorent, A. (2014). A quantitative characterisation of functions with low Aviles Giga energy on convex domains . *Ann. Sc. Norm. Super. Pisa Cl. Sci.*, 13(5).

[Lorent and Peng, 2018] Lorent, A. and Peng, G. (2018). Regularity of the eikonal equation with two vanishing entropies. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 35(2):481–516.

[Lorent and Peng, 2021] Lorent, A. and Peng, G. (2021). Factorization for entropy production of the eikonal equation and regularity. *arXiv:2104.01467*.

[Loudet and Poulin, 2001] Loudet, J. C. and Poulin, P. (2001). Application of an electric field to colloidal particles suspended in a liquid-crystal solvent. *Phys. Rev. Lett.*, 87:165503.

[Lubensky et al., 1998] Lubensky, T. C., Pettey, D., Currier, N., and Stark, H. (1998). Topological defects and interactions in nematic emulsions. *Phys. Rev. E*, 57:610–625.

[Luckhaus, 1988] Luckhaus, S. (1988). Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold. *Indiana Univ. Math. J.*, 37(2):349–367.

[Luckhaus and Zemas, 2022] Luckhaus, S. and Zemas, K. (2022). Rigidity estimates for isometric and conformal maps from $\mathbb{S}^{n-1}$ to $\mathbb{R}^n$. *Invent. Math.*

[Majumdar and Zarnescu, 2010] Majumdar, A. and Zarnescu, A. (2010). Landau-De Gennes theory of nematic liquid crystals: the Oseen-Frank limit and beyond. *Arch. Ration. Mech. Anal.*, 196(1):227–280.

[Marconi, 2021a] Marconi, E. (2021a). Characterization of minimizers of Aviles-Giga functionals in special domains. *Arch. Ration. Mech. Anal.*, 242(2):1289–1316.

[Marconi, 2021b] Marconi, E. (2021b). Rectifiability of entropy defect measures in a micromagnetics model. *Adv. Calc. Var.*

[Marconi, 2022a] Marconi, E. (2022a). On the structure of weak solutions to scalar conservation laws with finite entropy production. *Calc. Var. Partial Differ. Equ.*, 61(1):30.

[Marconi, 2022b] Marconi, E. (2022b). The rectifiability of the entropy defect measure for Burgers equation. *J. Funct. Anal.*, 283(6):109568.

[Mariani, 2010] Mariani, M. (2010). Large deviations principles for stochastic scalar conservation laws. *Probab. Theory Related Fields*, 147(3-4):607–648.

[Millot and Pisante, 2010] Millot, V. and Pisante, A. (2010). Symmetry of local minimizers for the three-dimensional Ginzburg-Landau functional. *J. Eur. Math. Soc. (JEMS)*, 12(5):1069–1096.

[Mironescu, 1995] Mironescu, P. (1995). On the stability of radial solutions of the Ginzburg-Landau equation. *J. Funct. Anal.*, 130(2):334–344.

[Mironescu, 1996] Mironescu, P. (1996). Les minimiseurs locaux pour l'équation de Ginzburg-Landau sont à symétrie radiale. *C. R. Acad. Sci. Paris Sér. I Math.*, 323(6):593–598.

[Muševič, 2019] Muševič, I. (2019). Interactions, topology and photonic properties of liquid crystal colloids and dispersions. *Eur. Phys. J. Special Topics*, 227(17):2455–2485.

[Nguyen and Zarnescu, 2013] Nguyen, L. and Zarnescu, A. (2013). Refined approximation for minimizers of a Landau-de Gennes energy functional. *Calc. Var. Partial Differential Equations*, 47(1-2):383–432.

[Novack and Yan, 2022] Novack, M. and Yan, X. (2022). A smectic liquid crystal model in the periodic setting. *arXiv:2205.01872*.

[Ortiz and Gioia, 1994] Ortiz, M. and Gioia, G. (1994). The morphology and folding patterns of buckling-driven thin-film blisters. *J. Mech. Phys. Solids*, 42(3):531–559.

[Ou, 1992] Ou, B. (1992). Uniqueness of $x/|x|$ as a stable configuration in liquid crystals. *J. Geom. Anal.*, 2(2):183–194.

[Panov, 1994] Panov, E. Y. (1994). Uniqueness of the solution of the Cauchy problem for a first order quasilinear equation with one admissible strictly convex entropy. *Math. Notes*, 55(5):1.

[Poliakovsky, 2007] Poliakovsky, A. (2007). Upper bounds for singular perturbation problems involving gradient fields. *J. Eur. Math. Soc. (JEMS)*, 9(1):1–43.

[Poliakovsky, 2008] Poliakovsky, A. (2008). On a variational approach to the methods of vanishing viscosity for conservation laws. *Adv. Math. Sci. Appl.*, 18(2):429–451.

[Reshetnyak, 1994] Reshetnyak, Y. G. (1994). *Stability theorems in geometry and analysis. Translated from the Russian by N. S. Dairbekov and V. N. Dyatlov. Revised and updated translation*, volume 304 of *Math. Appl., Dordr.* Dordrecht: Kluwer Academic Publishers, rev. and updated transl. edition.

[Rivière, 2007] Rivière, T. (2007). Conservation laws for conformally invariant variational problems. *Invent. Math.*, 168(1):1–22.

[Rivière and Serfaty, 2001] Rivière, T. and Serfaty, S. (2001). Limiting domain wall energy for a problem related to micromagnetics. *Comm. Pure Appl. Math.*, 54(3):294–338.

[Rivière and Serfaty, 2003] Rivière, T. and Serfaty, S. (2003). Compactness, kinetic formulation, and entropies for a problem related to micromagnetics. *Comm. Partial Differential Equations*, 28(1-2):249–269.

[Sandier, 1998] Sandier, E. (1998). Lower bounds for the energy of unit vector fields and applications. *J. Funct. Anal.*, 152(2):379–403.

[Sandier and Serfaty, 2007] Sandier, E. and Serfaty, S. (2007). *Vortices in the magnetic Ginzburg-Landau model*, volume 70 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA.

[Sandier and Shafrir, 1993] Sandier, E. and Shafrir, I. (1993). On the symmetry of minimizing harmonic maps in $N$ dimensions. *Differential Integral Equations*, 6(6):1531–1541.

[Santambrogio and Vespri, 2010] Santambrogio, F. and Vespri, V. (2010). Continuity in two dimensions for a very degenerate elliptic equation. *Nonlinear Anal., Theory Methods Appl., Ser. A, Theory Methods*, 73(12):3832–3841.

[Schoen, 1984] Schoen, R. (1984). Analytic aspects of the harmonic map problem. In *Seminar on nonlinear partial differential equations (Berkeley, Calif., 1983)*, volume 2 of *Math. Sci. Res. Inst. Publ.*, pages 321–358. Springer, New York.

[Schoen and Uhlenbeck, 1982] Schoen, R. and Uhlenbeck, K. (1982). A regularity theory for harmonic maps. *J. Differential Geom.*, 17(2):307–335.

[Schoen and Uhlenbeck, 1983] Schoen, R. and Uhlenbeck, K. (1983). Boundary regularity and the Dirichlet problem for harmonic maps. *J. Differential Geom.*, 18(2):253–268.

[Schoen, 1983] Schoen, R. M. (1983). Uniqueness, symmetry, and embeddedness of minimal surfaces. *J. Differential Geom.*, 18(4):791–809 (1984).

[Serfaty, 2005] Serfaty, S. (2005). Stability in 2d Ginzburg-Landau passes to the limit. *Indiana Univ. Math. J.*, 54(1):199–221.

[Shafrir, 1994] Shafrir, I. (1994). Remarks on solutions of $-\Delta u = (1 - |u|^2)u$ in $\mathbf{R}^2$. *C. R. Acad. Sci. Paris Sér. I Math.*, 318(4):327–331.

[Simon, 1996] Simon, L. (1996). *Theorems on regularity and singularity of energy minimizing maps*. Lectures in Mathematics ETH Zürich. Birkhäuser Verlag, Basel. Based on lecture notes by Norbert Hungerbühler.

[Stark, 2002] Stark, H. (2002). Saturn-ring defects around microspheres suspended in nematic liquid crystals: An analogy between confined geometries and magnetic fields. *Phys. Rev. E*, 66:032701.

[Stark and Ventzki, 2001] Stark, H. and Ventzki, D. (2001). Stokes drag of spherical particles in a nematic environment at low Ericksen numbers. *Phys. Rev. E*, 64(3):031711.

[Sternberg, 1991] Sternberg, P. (1991). Vector-valued local minimizers of nonconvex variational problems. *Rocky Mtn. J. Mathematics*, 21:799–807.

[Struwe, 1994] Struwe, M. (1994). On the asymptotic behavior of minimizers of the Ginzburg-Landau model in 2 dimensions. *Differential Integral Equations*, 7(5-6):1613–1624.

[Varadhan, 2004] Varadhan, S. (2004). Large deviations for the asymmetric simple exclusion process. In *Stochastic analysis on large scale interacting systems*, volume 39 of *Adv. Stud. Pure Math.*, pages 1–27. Math. Soc. Japan, Tokyo.

[Vasseur, 2016] Vasseur, A. F. (2016). Relative entropy and contraction for extremal shocks of conservation laws up to a shift. In *Recent advances in partial differential equations and applications*, volume 666 of *Contemp. Math.*, pages 385–404. Amer. Math. Soc., Providence, RI.

[Šverák, 1993] Šverák, V. (1993). On Tartar's conjecture. *Ann. Inst. H. Poincaré Anal. Non Linéaire*, 10(4):405–412.

[Yu, 2020] Yu, Y. (2020). Disclinations in limiting Landau–de Gennes theory. *Arch. Ration. Mech. Anal.*, 237(1):147–200.

[Zygmund, 2002] Zygmund, A. (2002). *Trigonometric series. Vol. I, II.* Cambridge Mathematical Library. Cambridge University Press, Cambridge, third edition. With a foreword by Robert A. Fefferman.

## Résumé

De nombreux phénomènes physiques peuvent être décrits par des modèles mathématiques similaires: cristaux liquides, supraconductivité, micromagnétisme, elasticité, formation de motifs, etc. Les états d'équilibre minimisent une certaine énergie, caractérisée par la compétition entre deux effets: un terme pénalise les déformations, mais un autre terme favorise les transitions de phases. Cette compétition encourage la formation de *singularités*: des déformations abruptes dans de petites régions où se concentrent les transitions de phases, et des déformations minimales en dehors. Les méthodes mathématiques du *calcul des variations* et des *équations aux dérivées partielles* permettent d'étudier les propriétés de ces états et de leurs singularités.

Les travaux présentés dans ce mémoire portent sur l'analyse de deux types de singularités: singularités ponctuelles ou linéaires dans des systèmes de dimension 3, et singularités linéaires dans des systèmes de dimension 2. Cette distinction reflète aussi leurs liens avec deux types d'équations aux dérivées partielles: *elliptiques* ou *hyperboliques*.

L'étude des singularités du premier type est motivée ici essentiellement par la physique des cristaux liquides, et structurée en deux axes de recherche: comprendre l'effet de l'immersion de particules étrangères, et celui d'une anisotropie dans la pénalisation des déformations. Pour le premier axe, la présence de particules impose des déformations au cristal liquide, et l'objectif est de décrire les singularités engendrées par ces déformations, ainsi que les interactions entre particules immergées. Pour le deuxième axe, l'anisotropie des déformations, physiquement plus réaliste, restreint considérablement les outils mathématiques disponibles: le défi principal est de développer de nouvelles techniques qui permettent d'analyser efficacement les modèles anisotropes.

L'étude des singularités du second type est motivée par de très divers phénomènes physiques: cristaux liquides, élasticité, micromagnétisme, formation de motifs, physique statistique. L'objectif est de comprendre les phénomènes de concentration de singularités linéaires: peuvent-elle s'accumuler au point de former des structures fractales? L'accent est mis ici sur la quantification des déformations admissibles et la stabilité de structures élémentaires bien identifiées. Du point de vue mathématique, ces questions présentent l'originalité d'aborder des équations hyperboliques par le biais du calcul des variations.