SHARP QUANTITATIVE STABILITY OF THE MÖBIUS GROUP AMONG SPHERE-VALUED MAPS IN ARBITRARY DIMENSION

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Abstract. In this work we prove a sharp quantitative form of Liouville's theorem, which asserts that, for all $n \geq 3$, the weakly conformal maps of $\mathbb{S}^{n-1}$ with degree ± 1 are Möbius transformations. In the case $n = 3$ this estimate was first obtained by Bernand-Mantel, Muratov and Simon (Arch. Ration. Mech. Anal. 239(1):219-299, 2021), with different proofs given later on by Topping, and by Hirsch and the third author. The higher-dimensional case $n \geq 4$ requires new arguments because it is genuinely nonlinear: the linearized version of the estimate involves quantities which cannot control the distance to Möbius transformations in the conformally invariant Sobolev norm. Our main tool to circumvent this difficulty is an inequality introduced by Figalli and Zhang in their proof of a sharp stability estimate for the Sobolev inequality.

1. Introduction

Rigidity theorems of geometric nature are ubiquitous in analysis, geometry and mathematical physics, one of the most classical, yet prominent examples being *Liouville's theorem* on the characterization of isometries and conformal transformations among domains of $\mathbb{R}^n$ ($n \geq 2$, resp. $n \geq 3$). Its qualitative as well as quantitative extensions (cf. for instance [29, 12, 16, 21, 22, 27, 28] and the references therein) have formed a very active research direction over the recent years, not only because of a pure geometric interest, but also in view of applications in mathematical models of materials science.

Our starting point is the spherical version of Liouville's theorem for conformal maps. A Sobolev map $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ is called *weakly conformal* if and only if

$$(\nabla_T u)^t \nabla_T u = \frac{|\nabla_T u|^2}{n-1} I_x \tag{1.1}$$

for $\mathcal{H}^{n-1}$-a.e. x on $\mathbb{S}^{n-1}$. Here and throughout the paper $\nabla_T u \in \mathbb{R}^{n\times(n-1)}$ denotes the extrinsic gradient of u with respect to a local orthonormal frame $\{\tau_1, \dots, \tau_{n-1}\}$ on $\mathbb{S}^{n-1}$ indicated by the unit normal, and I_x stands for the identity transformation on $T_x\mathbb{S}^{n-1}$. For maps in that class one can define the *topological degree* of u through

$$\deg u := \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \mathrm{d}\mathcal{H}^{n-1}, \tag{1.2}$$

see e.g. [5] for more information on the degree.

Let us write $\text{Möb}_+(\mathbb{S}^{n-1})$ for the group of *orientation-preserving Möbius maps* of $\mathbb{S}^{n-1}$, see Subsection 2.2 for the precise definition. It is a remarkable geometric fact that solutions of (1.1) are very rigid:

Liouville's Theorem. *Let $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ be a solution of* (1.1)*.*

(i) *if $n = 3$ and $\deg u = 1$, then $u \in \text{Möb}_+(\mathbb{S}^{n-1})$;*
(ii) *if $n \geq 4$ and $\deg u \geq 1$, then $u \in \text{Möb}_+(\mathbb{S}^{n-1})$.*

In fact, (ii) holds locally even if u solves (1.1) only on a domain of $\mathbb{S}^{n-1}$ [16], while for $n = 3$ solutions to (1.1) of general degree have also been classified [17, 26]. We refer the reader to [19, Appendix A] for a simple proof of Liouville's Theorem when $\deg u = 1$.

In this paper we prove a sharp quantitative version of Liouville's Theorem on $\mathbb{S}^{n-1}$ in all dimensions $n \geq 3$. To state the result, we consider the class of admissible maps

$$\mathcal{A}_{\mathbb{S}^{n-1}} := \left\{ u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1}) : \ \deg u = 1 \right\}, \tag{1.3}$$

and the deficit

$$\delta_{n-1}(u) := \fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1} \right)^{\frac{n-1}{2}} \mathrm{d}\mathcal{H}^{n-1} - 1 \,. \tag{1.4}$$

Note that the deficit in (1.4) is clearly a conformally-invariant quantity which, by a well-known inequality relating the $(n-1)$-Dirichlet energy and the degree of maps in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$, satisfies $\delta_{n-1}(u) \geq 0$ for all $u \in \mathcal{A}_{\mathbb{S}^{n-1}}$, with

$$\delta_{n-1}(u) = 0 \iff u \in \text{Möb}_+(\mathbb{S}^{n-1}) \,,$$

see Corollary 2.2 below.

Our main result shows that $\delta_{n-1}(u)$ measures in a sharp fashion the distance of u from a particular Möbius map in an integral sense, namely we have the following.

Theorem 1.1. *Let $n \geq 3$. There exists a constant $C_n > 0$ such that for every $u \in \mathcal{A}_{\mathbb{S}^{n-1}}$,*

$$\inf_{\phi \in \text{Möb}_+(\mathbb{S}^{n-1})} \fint_{\mathbb{S}^{n-1}} |\nabla_T u - \nabla_T \phi|^{n-1} \, \mathrm{d}\mathcal{H}^{n-1} \leq C_n \delta_{n-1}(u) \,. \tag{1.5}$$

Remark 1.2 (Sharpness)**.** Estimate (1.5) is sharp in the sense that, on the right-hand side, the deficit cannot be replaced with $\delta_{n-1}(u)^\beta$ for some $\beta > 1$. An example addressing the optimality of the exponent is discussed in Appendix A.

Remark 1.3 (Maps with other degrees)**.** By inequality (2.11), we have

$$|\deg u| > 1 \quad \implies \quad \delta_{n-1}(u) \geq |\deg u| - 1 \geq 1 \,.$$

In particular, this easily implies that (1.5) also holds for $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ with $|\deg u| > 1$, cf. Step 0 in Section 3 below. Likewise, the case of maps of degree -1 is completely analogous to the degree 1 case, and can simply be derived by the latter, by composing any map of degree -1 with the orientation-reversing orthogonal map $(x_1, \ldots, x_{n-1}, x_n) \mapsto (x_1, \ldots, x_{n-1}, -x_n)$, and replacing $\text{Möb}_+(\mathbb{S}^{n-1})$ with $\text{Möb}_-(\mathbb{S}^{n-1})$ (the group of orientation-reversing Möbius maps).

Theorem 1.1 was first proved for $n = 3$ in [3, Theorem 2.4], with simpler and different proofs having been supplied in [14, 24]. Thus, the novelty of this paper is the higher dimensional, genuinely non-linear case $n \geq 4$. We also refer to [6, 7] for related results for half-harmonic maps from $\mathbb{R}$ to $\mathbb{S}$ and a local stability result for maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ with higher degree, where the assumption of apriori closeness to a harmonic map however prevents bubbling phenomena for almost energy-minimizing maps. Without this strong assumption, an optimal quantitative improvement has been established very recently in [23], which, rougly speaking, asserts that maps from $\mathbb{S}^2$ to $\mathbb{S}^2$ of higher degree and small excess energy are close to a collection of rational maps that describe the behaviour at very different scales. The interested reader is also referred to [8] and the references therein for further quantitative stability results related to problems in conformal geometry.

Our proof of Theorem 1.1 is based on the approach from [14], which however does not extend trivially to higher dimensions, the main difficulty being the genuine nonlinearity and degenerate convexity of the conformal Dirichlet energy.

Sketch of proof. Let us give here a sketch of the argument, which can be thought of as a quantitative perturbation of the one in [19, Appendix A] for the exact case. The starting point of the proof is that it suffices to prove Theorem 1.1 for maps $u \in \mathcal{A}_{\mathbb{S}^{n-1}}$ for which

$$\delta_{n-1}(u) \ll 1\,, \qquad \fint_{\mathbb{S}^{n-1}} u = 0\,, \qquad \|u - x\|_{W^{1,n-1}(\mathbb{S}^{n-1})} \ll 1\,.$$

While the first reduction is clear, the reduction to zero-mean maps can always be achieved by precomposition with a Möbius transformation, see Lemma 2.5, while the last one is ensured by a contradiction/compactness argument, see Lemma 2.4 and Step 3 in Section 3. After these reductions, and since u is $\mathbb{S}^{n-1}$-valued, in the case $n = 3$ of [14], the deficit transforms into the deficit in the sharp Poincaré inequality on $\mathbb{S}^2$ (cf. (2.7), (2.8)), namely

$$\delta_2(u) = \frac{1}{2}\fint_{\mathbb{S}^2} |\nabla_T u|^2 - \fint_{\mathbb{S}^2} |u|^2 \gtrsim \fint_{\mathbb{S}^2} |\nabla_T (u - Ax)|^2\,,$$

where Ax is the linear part of u, i.e., its projection onto the first order spherical harmonics (cf. Subsection 2.1). Keeping in mind that u was precomposed with a Möbius map to achieve the zero-mean condition, the proof is concluded in this case by showing that A can be replaced in the above estimate by a rotation matrix, i.e., that

$$\operatorname{dist}^2(A; \mathrm{SO}(n)) \lesssim \delta_2(u)\,.$$

The last estimate is achieved by an appropriate expansion of the formula for the degree (1.2) around the linear part of u.

In the case $n \geq 4$, which is the case of study of this work, the previous argument however cannot be adapted directly. Indeed, setting (for simplicity) $w := u - x$, and naively Taylor-expanding the deficit around the identity would formally yield

$$\delta_{n-1}(u) = \frac{1}{2}\bigg(\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 - (n-1)\fint_{\mathbb{S}^{n-1}} |w|^2 + \frac{n-3}{n-1}\fint_{\mathbb{S}^{n-1}} (\operatorname{div}_{\mathbb{S}^{n-1}} w)^2\bigg) + \fint_{\mathbb{S}^{n-1}} \mathcal{O}(|\nabla_T w|^3)\,,$$

which however does not give a control on the desired optimal norm we want in the left hand side of (1.5). Moreover, even though the first part of the quadratic form appearing in the expansion gives again the deficit in the sharp Poincaré inequality on $\mathbb{S}^{n-1}$, a suboptimal estimate with respect to the L^2-norm on the left hand side of (1.5) would not apriori be possible, because of the presence of the higher-order (cubic) terms in the expansion.

To overcome these issues we consider a Taylor-type lower inequality instead of the exact expansion for the conformal Dirichlet energy (cf. Lemma 3.1 below), which can be thought of as an appropriate interpolation between the first up to quadratic terms, and the desired $\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^{n-1}$-term. Such an expansion is a special case of the ones introduced in [10] to prove a sharp quantitative version of the Sobolev inequality in $\mathbb{R}^n$, cf. also [4, 9, 11] for related results in this direction. Surprisingly, such an expansion finally leads us to an estimate of the form

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T (u - Ax)|^{n-1} \lesssim \delta_{n-1}(u) + \operatorname{dist}^2(A; \mathrm{SO}(n)),$$

where again Ax is the linear part of u. The proof would again be concluded if the last term on the right hand side of the above inequality is optimally estimated by the deficit itself, which is achieved also by an appropriate expansion of the degree of u around its linear part, and L^2-L^{n-1} interpolation estimates, see Step (2b) in Section 3.

Remark 1.4. Interestingly, the observation that the group $\text{Möb}_+(\mathbb{S}^{n-1})$ can be used to fix the mean value of maps in $\mathcal{A}_{\mathbb{S}^{n-1}}$ to 0 was also used in the proof of Theorem 1.1 in the case $n = 3$ by P. Topping [24]. While in [14] and here we used it to link the problem to the stability of the sharp Poincaré inequality on $\mathbb{S}^{n-1}$, in [24] it is used in order to start the harmonic map heat flow (cf. [18]) with initial datum $u \in \mathcal{A}_{\mathbb{S}^2}$ with $\fint_{\mathbb{S}^2} u = 0$. With this centering, the flow does not produce a bubble in finite time and converges as $t \to \infty$ to a map in $\text{Möb}_+(\mathbb{S}^2)$. This limiting map turns out to be the one for which the desired stability estimate is satisfied. It would be interesting to see if such an approach can be followed to get an alternative proof of Theorem 1.1 also in the case $n \geq 4$ by using the nonlinear $(n-1)$-harmonic map heat flow [15], which will be a question for future work.

Plan of the paper. The plan of the paper is the following. After fixing some notation, in Section 2 we collect some basic facts about spherical harmonics and Möbius maps on $\mathbb{S}^{n-1}$, as well as some boundary expressions of integrals of Jacobian subdeterminants (null-Lagrangians) over the unit ball. These will be useful in the proof of Theorem 1.1, which is the content of Section 3. For the reader's convenience the proof is divided into several steps, following in spirit and method the approach of [14] for the case $n = 3$. Step 2 therein (divided into further substeps) contains the main novelty in the argumentation for the higher dimensional case $n \geq 4$, as these were sketched previously. In Appendix A we present an example showing the optimality of the estimate (1.5) (cf. Remark 1.2).

Notation. We use $\langle \cdot, \cdot \rangle$ to denote the inner product between vectors in $\mathbb{R}^n$ and $A : B$ to denote the Hilbert–Schmidt inner product between any two matrices $A, B \in \mathbb{R}^{n\times m}$. For every two vectors $a \in \mathbb{R}^n$ and $b \in \mathbb{R}^m$ we denote by $a \otimes b \in \mathbb{R}^{n\times m}$ their tensor product,

$$(a \otimes b)_{ij} := a_i b_j \quad \forall i = 1, \dots, n \text{ and } j = 1, \dots, m.$$

As mentioned above, $\nabla_T u$ denotes the extrinsic gradient of $u\colon \mathbb{S}^{n-1} \to \mathbb{S}^{n-1}$ (when viewed as a map with values in $\mathbb{R}^n$), represented in local coordinates by the $n \times (n-1)$-matrix with entries

$$(\nabla_T u)_{ij} = \langle \nabla_T u^i, \tau_j \rangle, \quad \forall i = 1, \dots, n \text{ and } j = 1, \dots, n-1.$$

Here, $\{\tau_1, \dots, \tau_{n-1}\}$ is a local orthonormal frame on $T_x\mathbb{S}^{n-1}$ indicated by the unit normal, i.e., for every $x \in \mathbb{S}^{n-1}$ the set of vectors $\{\tau_1(x), \dots, \tau_{n-1}(x), x\}$ is a positively oriented orthonormal frame of $\mathbb{R}^n$.

By $\mathrm{id}_{\mathbb{S}^{n-1}}, I_x, I_n$ we denote the identity map on $\mathbb{S}^{n-1}$, the identity transformation on $T_x\mathbb{S}^{n-1}$, and the identity matrix in $\mathbb{R}^{n\times n}$, respectively. We also set for brevity,

$$P_T := \nabla_T \mathrm{id}_{\mathbb{S}^{n-1}}\,, \text{ i.e.}\,, \ (P_T)_{ij} = \langle e_i, \tau_j\rangle\,.$$

By ω_n we denote the Euclidean volume of the unit ball in $\mathbb{R}^n$ and by $\mathrm{SO}(n)$ we label as usual the group of orientation-preserving rotation matrices, i.e.,

$$\mathrm{SO}(n) := \{R \in \mathbb{R}^{n\times n} \colon R^t R = I_n \text{ and } \det R = 1\}\,.$$

For any $A \subset \mathbb{R}^n$, we denote by $\mathbf{1}_A$ its indicator function, i.e.,

$$\mathbf{1}_A(y) := \begin{cases} 1\,, & y \in A\,, \\ 0\,, & y \notin A\,. \end{cases}$$

We consider maps in the Sobolev space

$$W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1}) := \{u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)\colon \ |u(x)| = 1 \text{ for } \mathcal{H}^{n-1}\text{-a.e. } x \in \mathbb{S}^{n-1}\}\,.$$

Note that, in the definition of the degree for such maps in (1.2), we identify $\bigwedge_{i=1}^{n-1}\partial_{\tau_i}u$ with $*\bigwedge_{i=1}^{n-1}\partial_{\tau_i}u$ through Hodge-duality.

By $\lesssim, \ll$ we mean that the corresponding inequality is valid up to a multiplicative (resp. sufficiently small) constant that depends only on the dimension, but is allowed to vary from line to line. With this notation, for any two quantities F, G we say that $F \sim G$ iff $F \lesssim G$ and $G \lesssim F$.

2. Geometry of $\mathbb{S}^{n-1}$ and null-Lagrangians

2.1. **Spherical harmonics.** We begin this section by recalling some useful facts about spherical harmonics, referring the reader to [13] for further details. For every $k \in \mathbb{N}$ let $H_{n,k}$ be the subspace of $L^2(\mathbb{S}^{n-1};\mathbb{R}^n)$ consisting of vector fields whose components are k-th order *vectorial spherical harmonics* $\{\psi_{n,k,j}\}_j$, i.e., eigenfunctions of the Laplace-Beltrami operator $-\Delta_{\mathbb{S}^{n-1}}$ corresponding to the eigenvalue

$$\lambda_{n,k} := k(k+n-2)\,, \tag{2.1}$$

normalized so that $\|\psi_{n,k,j}\|_{L^2(\mathbb{S}^{n-1})} = 1$. As it is well known, $d_{n,k} := \dim H_{n,k} < +\infty$, and we have the orthogonal decomposition

$$L^2(\mathbb{S}^{n-1};\mathbb{R}^n) = \bigoplus_{k=0}^{\infty} H_{n,k}\,. \tag{2.2}$$

In fact, the spherical harmonics are not just mutually L^2-orthogonal, but also $W^{1,2}$-orthogonal. In particular, if for $u \in W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$, we write

$$u = \sum_{k=0}^{\infty}\sum_{j=1}^{d_{n,k}} \alpha_{n,k,j}\psi_{n,k,j}\,,$$

then we have the *Parseval identities*

$$\int_{\mathbb{S}^{n-1}} |u|^2 = \sum_{k=0}^{\infty}\sum_{j=1}^{d_{n,k}} |\alpha_{n,k,j}|^2, \qquad \int_{\mathbb{S}^{n-1}} |\nabla_T u|^2 = \sum_{k=0}^{\infty}\sum_{j=1}^{d_{n,k}} \lambda_{n,k}|\alpha_{n,k,j}|^2\,. \tag{2.3}$$

We consider the orthogonal projection

$$\Pi_{n,k}\colon L^2(\mathbb{S}^{n-1};\mathbb{R}^n) \to H_{n,k}\,. \tag{2.4}$$

Clearly $H_{n,0}$ is the span of the constant function equal to 1, so

$$\Pi_{n,0}(u) = \fint_{\mathbb{S}^{n-1}} u\,, \tag{2.5}$$

while $H_{n,1}$ is the span of the coordinate functions $\{x_i\}_{i=1}^n$, and it is easy to check that

$$\Pi_{n,1}(u)(x) = \nabla u_h(0)x\,, \tag{2.6}$$

where $u_h\colon \overline{B_1} \to \mathbb{R}^n$ denotes the component-wise harmonic extension of $u \in L^2(\mathbb{S}^{n-1};\mathbb{R}^n)$ in the interior of the unit ball. From (2.1) and (2.3) we deduce several sharp forms of the Poincaré inequality, for instance

$$\fint_{\mathbb{S}^{n-1}} \left|\nabla_T u\right|^2 \geq (n-1) \fint_{\mathbb{S}^{n-1}} \left|u - \Pi_{n,0}(u)\right|^2, \tag{2.7}$$

$$\fint_{\mathbb{S}^{n-1}} \left|\nabla_T(u - \Pi_{n,1}(u))\right|^2 \geq 2n \fint_{\mathbb{S}^{n-1}} \left|u - \big(\Pi_{n,0}(u) + \Pi_{n,1}(u)\big)\right|^2. \tag{2.8}$$

Another relatively simple consequence of (2.3) is the sharp estimate

$$\fint_{B_1} |\nabla u_h|^2 \,\mathrm{d}x \leq \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^2 \,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{2.9}$$

whose proof can be found for instance in [19, Lemma C.2]).

2.2. **Conformal maps.** We now turn to conformal maps of the sphere. We define $\text{Möb}_+(\mathbb{S}^{n-1})$ to be the group of orientation-preserving Möbius transformations of $\mathbb{S}^{n-1}$, that is,

$$\text{Möb}_+(\mathbb{S}^{n-1}) := \{R\phi_{\xi,\lambda} : R \in \text{SO}(n), \xi \in \mathbb{S}^{n-1}, \lambda > 0\}\,, \tag{2.10}$$

where

$$\phi_{\xi,\lambda} := \sigma_\xi^{-1} \circ i_\lambda \circ \sigma_\xi\,,$$

$\sigma_\xi\colon \mathbb{S}^{n-1} \to T_\xi\mathbb{S}^{n-1} \cup \{\infty\}$ being the stereographic projection from $-\xi \in \mathbb{S}^{n-1}$, and $i_\lambda : T_\xi\mathbb{S}^{n-1} \mapsto T_\xi\mathbb{S}^{n-1}$ the dilation in $T_\xi\mathbb{S}^{n-1}$ by factor $\lambda > 0$. Explicitly, we have

$$\sigma_\xi(x) = -\frac{1}{1+\langle x,\xi\rangle}(x - \langle x,\xi\rangle\xi)\,, \quad \sigma_\xi^{-1}(y) = -\Big(\frac{2}{1+|y|^2}\Big)y + \Big(\frac{1-|y|^2}{1+|y|^2}\Big)\xi,$$

$$\phi_{\xi,\lambda}(x) = \frac{-\lambda^2(1-\langle x,\xi\rangle)\xi + 2\lambda(x - \langle x,\xi\rangle\xi) + (1+\langle x,\xi\rangle)\xi}{\lambda^2(1-\langle x,\xi\rangle) + (1+\langle x,\xi\rangle)}\,,$$

for all $x \in \mathbb{S}^{n-1}$ and $y \in T_\xi\mathbb{S}^{n-1}$.

We next recall a well-known inequality between the conformally invariant $(n-1)$-Dirichlet energy and the degree of maps on the sphere.

Lemma 2.1. *For every $u \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ there holds*

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} \mathrm{d}\mathcal{H}^{n-1} \geq |\deg u|\,, \tag{2.11}$$

with equality if and only if u is weakly conformal.

Proof. Applying the arithmetic mean-geometric mean inequality to the eigenvalues of $\sqrt{\nabla_T u^t \nabla_T u}$, together with the fact that $|u| = 1$ $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$, from (1.2) we obtain

$$\begin{aligned}\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} &\geq \fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)} \\ &= \fint_{\mathbb{S}^{n-1}} \left|\bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right| \geq \fint_{\mathbb{S}^{n-1}} \left|\left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle\right| \geq |\deg(u)|\,.\end{aligned} \tag{2.12}$$

If equality holds then we must have equality in the arithmetic mean-geometric mean inequality, which means that all of the eigenvalues of $\sqrt{\nabla_T u^t \nabla_T u}$ must be the same $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$. In other words, if equality holds, u is weakly conformal. □

As an immediate consequence of Lemma 2.1 and (1.3), we have the following.

Corollary 2.2. *For $u \in \mathcal{A}_{\mathbb{S}^{n-1}}$, the deficit in* (1.4) *satisfies $\delta_{n-1}(u) \geq 0$. We have $\delta_{n-1}(u) = 0$ if and only if u is weakly conformal, which by Liouville's Theorem holds if and only if $u \in \mathrm{M\ddot{o}b}_+(\mathbb{S}^{n-1})$.*

Remark 2.3. For maps $u \in (W^{1,n-1} \cap L^\infty)(\mathbb{S}^{n-1}; \mathbb{R}^n)$ the quantity

$$V_n(u) := \fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u \right\rangle \mathrm{d}\mathcal{H}^{n-1}$$

gives the *relative extrinsic volume* of the image of u: for instance, if u is a smooth embedding, then $V_n(u)$ is the signed volume of the open set in $\mathbb{R}^n$ whose boundary is $u(\mathbb{S}^{n-1})$, divided by the volume of the unit ball. Of course, if $u \in W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$, we have $V_n(u) = \deg u$, cf. (1.2). In fact, $V_n(u)$ extends to an analytic functional in $W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{R}^n)$, for which *Wente's isoperimetric inequality* holds:

$$\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}} \geq \fint_{\mathbb{S}^{n-1}} \sqrt{\det(\nabla_T u^t \nabla_T u)} \geq \left|\fint_{\mathbb{S}^{n-1}} \left\langle u, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} u\right\rangle\right|^{\frac{n-1}{n}}, \tag{2.13}$$

the first inequality being again a consequence of the arithmetic mean-geometric mean inequality and the second one being the functional form of the isoperimetric inequality, see e.g. [1], [19, equation (1.9)], [20, Lemma 1.3], or [25]. Equality holds in (2.13) if and only if the image $u(\mathbb{S}^{n-1})$ is another round sphere and, in addition, u is weakly conformal, that is, if and only if u is a Möbius transformation of $\mathbb{S}^{n-1}$ up to translation and scaling.

Sequences with zero-mean and vanishing deficit exhibit strong compactness properties, see e.g. [19, Lemma A.3]:

Lemma 2.4 (Compactness)**.** *Let $(u_j)_{j\in\mathbb{N}} \subset W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ be a sequence such that*

$$\fint_{\mathbb{S}^{n-1}} u_j = 0, \qquad \deg u_j = 1, \qquad \lim_{j\to\infty} \delta_{n-1}(u_j) = 0\,.$$

Then there exists $R \in \mathrm{SO}(n)$ such that, up to a not-relabeled subsequence,

$$u_j \to R\,\mathrm{id}_{\mathbb{S}^{n-1}} \text{ strongly in } W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})\,.$$

Proof. The condition $\lim_{j\to\infty}\delta_{n-1}(u_j)=0$ can be rewritten as

$$\lim_{j\to\infty}\fint_{\mathbb{S}^{n-1}}\left(\frac{|\nabla_T u_j|^2}{n-1}\right)^{\frac{n-1}{2}}\mathrm{d}\mathcal{H}^{n-1}=1\,.$$

Thus, up to a subsequence we have $u_j\rightharpoonup u$ weakly in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$; in particular, this convergence is also strong in $L^{n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$, and so

$$\Pi_{n,0}(u)=\fint_{\mathbb{S}^{n-1}}u=0\,.$$

By the sequential weak lower semicontinuity of the conformal Dirichlet energy, Jensen's inequality, and the Poincaré inequality (2.7), we have

$$\begin{aligned}1=\lim_{j\to\infty}\fint_{\mathbb{S}^{n-1}}\left(\frac{|\nabla_T u_j|^2}{n-1}\right)^{\frac{n-1}{2}}&\geq\fint_{\mathbb{S}^{n-1}}\left(\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}}\\&\geq\left(\fint_{\mathbb{S}^{n-1}}\frac{|\nabla_T u|^2}{n-1}\right)^{\frac{n-1}{2}}\geq\left(\fint_{\mathbb{S}^{n-1}}|u|^2\right)^{\frac{n-1}{2}}=1\,,\end{aligned}$$

hence equality holds throughout. This already shows the strong convergence $u_j\to u$ in $W^{1,n-1}$, so $\deg u=1$ as well. Since u satisfies (2.7) with equality, by (2.3) we see that u must be linear, i.e. $u(x)=Rx$ for some $R\in\mathbb{R}^{n\times n}$. Finally, since $u\colon\mathbb{S}^{n-1}\to\mathbb{S}^{n-1}$ has degree one, it must be that $R\in\mathrm{SO}(n)$. □

A basic topological fact is that, up to a Möbius map, the zero-mean condition of Lemma 2.4 can always be achieved:

Lemma 2.5. *Given $u\in\mathcal{A}_{\mathbb{S}^{n-1}}$ there exists $\psi\in\mathrm{M\ddot{o}b}_+(\mathbb{S}^{n-1})$ so that*

$$\fint_{\mathbb{S}^{n-1}}u\circ\psi=0\,. \tag{2.14}$$

This lemma is well-known, cf. the proof of Liouville's theorem in [19, Appendix A] as well as Step 1 in the proof of [14, Theorem 1.1] for the case $n=3$. However, to make the presentation reasonably self-contained, we also revise the argument here.

Proof. Assume first that $u\in\mathcal{A}_{\mathbb{S}^{n-1}}$ is smooth, so in particular u is surjective. If

$$b_u:=\fint_{\mathbb{S}^{n-1}}u=0\,,$$

then (2.14) is trivially satisfied for $\psi:=\mathrm{id}_{\mathbb{S}^{n-1}}$. If $b_u\neq 0$, we claim that there exists $\xi_0\in\mathbb{S}^{n-1}$ and $\lambda_0>0$ such that (2.14) holds true for $\psi:=\phi_{\xi_0,\lambda_0}$. Indeed, consider the map $F:\mathbb{S}^{n-1}\times[0,1]\mapsto\overline{B}_1$ defined as

$$F(\xi,\lambda):=\fint_{\mathbb{S}^{n-1}}u\circ\phi_{\xi,\lambda}\ \text{ for }\lambda\in(0,1]\,,\quad\text{and }F(\xi,0):=u(\xi)=\lim_{\lambda\searrow 0}F(\xi,\lambda)\,.$$

The map F is continuous with $F(\mathbb{S}^{n-1},0)=u(\mathbb{S}^{n-1})=\mathbb{S}^{n-1}$ and $F(\mathbb{S}^{n-1},1)=\{b_u\}$, hence F is a continuous homotopy between $\mathbb{S}^{n-1}$ and the point $b_u\in\overline{B}_1\setminus\{0\}$. Therefore there exists $\lambda_0\in(0,1)$ and $\xi_0\in\mathbb{S}^{n-1}$ such that $F(\xi_0,\lambda_0)=0$, as wished.

In the general case of a map $u \in \mathcal{A}_{\mathbb{S}^{n-1}}$, by the approximation property given in [5, Lemma 7, Section I.4.], there exists a sequence $(u_j)_{j\in\mathbb{N}} \subset C^\infty(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$ with the property that

$$u_j \underset{j\to\infty}{\longrightarrow} u \text{ strongly in } W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1}) \quad \text{and} \quad \deg u_j = \deg u = 1 \quad \forall j \in \mathbb{N}\,.$$

Up to passing to a not-relabeled subsequence, we can without loss of generality also suppose that $u_j \to u$ and $\nabla_T u_j \to \nabla_T u$ pointwise $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$, as $j \to \infty$. Since the maps u_j are smooth and surjective, by the previous argument there exist $(\xi_j)_{j\in\mathbb{N}} \subset \mathbb{S}^{n-1}$ and $(\lambda_j)_{j\in\mathbb{N}} \subset (0,1]$ so that for every $j \in \mathbb{N}$,

$$\fint_{\mathbb{S}^{n-1}} u_j \circ \phi_{\xi_j,\lambda_j} = 0\,.$$

Up to subsequences we can suppose further that $\xi_j \to \xi_0 \in \mathbb{S}^{n-1}$ and $\lambda_j \to \lambda_0 \in [0,1]$ as $j \to \infty$, thus $\phi_{\xi_j,\lambda_j} \to \phi_{\xi_0,\lambda_0}$ pointwise $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$ and also weakly in $W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{S}^{n-1})$.

In fact $\lambda_0 \in (0,1]$, i.e. the Möbius transformations $(\phi_{\xi_j,\lambda_j})_{j\in\mathbb{N}}$ do not converge to the trivial map $\phi_{\xi_0,0}(x) \equiv \xi_0$. Indeed, suppose for the sake of contradiction that this were the case. Then $u_j \circ \phi_{\xi_j,\lambda_j} \to u(\xi_0)$ pointwise $\mathcal{H}^{n-1}$-a.e. and $|u_j \circ \phi_{\xi_j,\lambda_j}| \equiv 1$, so we could use the Dominated Convergence Theorem to infer that

$$u(\xi_0) = \fint_{\mathbb{S}^{n-1}} u(\xi_0) = \lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} u_j \circ \phi_{\xi_j,\lambda_j} = 0\,,$$
$$|u(\xi_0)| = \fint_{\mathbb{S}^{n-1}} |u(\xi_0)| = \lim_{j\to\infty} \fint_{\mathbb{S}^{n-1}} |u_j \circ \phi_{\xi_j,\lambda_j}| = 1\,,$$

which is a contradiction. Having justified that $\lambda_0 \in (0,1]$, what we actually obtain by the Dominated Convergence Theorem is that

$$\fint_{\mathbb{S}^{n-1}} u \circ \phi_{\xi_0,\lambda_0} = 0\,,$$

which gives (2.14) in the general case. ☐

2.3. **Boundary integrals related to null-Lagrangians.** In this subsection we recall some useful facts about Jacobian subdeterminants and the boundary expressions of their integrals. Before doing so, however, we introduce some notation from linear algebra.

Let $M \in \mathbb{R}^{n\times n}$ with its set of eigenvalues being labeled as $\{\mu_1, \dots, \mu_n\}$ (be them real or complex). We then have

$$\det(I_n + M) = 1 + \sum_{k=1}^{n} \sigma_k(M)\,, \tag{2.15}$$

where $\sigma_k(M)$ denotes the k-th elementary symmetric polynomial in the eigenvalues of M:

$$\sigma_k(M) := \sum_{1\le i_1<\dots<i_k\le n} \mu_{i_1} \dots \mu_{i_k}\,. \tag{2.16}$$

Note that the k-homogeneity of σ_k implies the Euler identity

$$\sigma_k(M) = \frac{1}{k}\sigma_k'(M) : M\,, \tag{2.17}$$

where $\sigma_k'(M) \in \mathbb{R}^{n\times n}$ is the gradient of σ_k (with respect to the M-variable). With this notation, we have the following.

Lemma 2.6. *Let $w \in (W^{1,n-1}\cap L^\infty)(\mathbb{S}^{n-1};\mathbb{R}^n)$ and $W : \overline{B}_1 \to \mathbb{R}^n$ be any regular extension such that $W|_{\mathbb{S}^{n-1}} = w$. Then,*

$$\fint_{B_1} \det(I_n + \nabla W)\,\mathrm{d}x = 1 + \sum_{k=1}^{n} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle w, [\sigma_k'(\nabla_T w P_T^t)]^t x\rangle \,\mathrm{d}\mathcal{H}^{n-1}\,, \tag{2.18}$$

where σ_k are as in (2.16)*. In particular,* (2.18) *holds true for w_h, the component-wise harmonic extension of w in B_1 .*

Proof. By a standard approximation argument we can without restriction assume for the proof that $u \in C^\infty(\mathbb{S}^{n-1};\mathbb{R}^n)$. Since for every $k = 1,\dots,n$, σ_k is a linear combination of $k\times k$ minors, σ_k is a null Lagrangian, that is

$$\operatorname{div}\sigma_k'(\nabla U) = 0 \quad \text{ for all } U \in C^\infty(B_1;\mathbb{R}^n)\,, \tag{2.19}$$

see e.g. [2]. We now consider a particular smooth extension $\overline{W}\colon \overline{B}_1 \to \mathbb{R}^n$ of w which is constant in the radial direction. Explicitly, one may set

$$\overline{W}(y) = \begin{cases} \eta(y) w(\frac{y}{|y|}) & \text{for } y \neq 0,\\ 0 & \text{for } y = 0,\end{cases}$$

where $\eta \in C^\infty(\overline{B}_1;\mathbb{R}_+)$ is any smooth cut-off satisfying $\mathbf{1}_{\{|y|\geq 1/2\}} \leq \eta(y) \leq \mathbf{1}_{\{|y|\geq 1/4\}}$. Since the radial derivative of $\overline{W}$ vanishes on $\mathbb{S}^{n-1}$, for every $i,j = 1,\dots,n$ we have

$$(\nabla\overline{W})_{ij} = \sum_{m=1}^{n-1} \langle \nabla_T w^i, \tau_m\rangle\langle e_j,\tau_m\rangle = (\nabla_T w P_T^t)_{ij} \quad \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{S}^{n-1}. \tag{2.20}$$

Using the fact that the Jacobian determinant is a null-Lagrangian, (2.15), (2.17), (2.19) and (2.20), for every extension $W : \overline{B}_1 \to \mathbb{R}^n$ of w, we obtain

$$\begin{aligned}
\fint_{B_1} \det(I_n + \nabla W)\,\mathrm{d}x &= \fint_{B_1} \det(I_n + \nabla \overline{W})\,\mathrm{d}x \\
&= 1 + \sum_{k=1}^{n} \fint_{B_1} \sigma_k(\nabla\overline{W})\,\mathrm{d}x \\
&= 1 + \sum_{k=1}^{n} \frac{1}{k} \fint_{B_1} \sigma_k'(\nabla\overline{W})\colon \nabla\overline{W}\,\mathrm{d}x \\
&= 1 + \sum_{k=1}^{n} \frac{1}{k} \fint_{B_1} \operatorname{div}(\sigma_k'(\nabla\overline{W})\overline{W})\,\mathrm{d}x \\
&= 1 + \sum_{k=1}^{n} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma_k'(\nabla\overline{W})\overline{W}, x\rangle\,\mathrm{d}\mathcal{H}^{n-1} \\
&= 1 + \sum_{k=1}^{n} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma_k'(\nabla_T w P_T^t) w, x\rangle\,\mathrm{d}\mathcal{H}^{n-1}\,,
\end{aligned}$$

which proves exactly (2.18). □

Remark 2.7. We observe that $\sigma_1'(M) = I_n\ \forall M \in \mathbb{R}^{n\times n}$, so the term corresponding to $k=1$ in the right hand side of (2.18) is equal to $n \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle$. Similarly, for $k=n$ it is clear that

$$\fint_{\mathbb{S}^{n-1}} \langle w, [\sigma_n'(\nabla_T w P_T^t)]^t x\rangle \, \mathrm{d}\mathcal{H}^{n-1} = \fint_{B_1} \det\nabla W = \fint_{\mathbb{S}^{n-1}} \left\langle w, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} w \right\rangle \mathrm{d}\mathcal{H}^{n-1}\,, \quad (2.21)$$

while for every $k \in \{2,\dots,n-1\}$, since $\sigma_k'(M)$ is a $(k-1)$-homogeneous polynomial in the entries of M, for every $w \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n)$ we have the trivial estimate

$$\frac{n}{k}\left| \fint_{\mathbb{S}^{n-1}} \langle w, [\sigma_k'(\nabla_T w P_T^t)]^t x\rangle \, \mathrm{d}\mathcal{H}^{n-1} \right| \le C_{n,k} \fint_{\mathbb{S}^{n-1}} |w||\nabla_T w|^{k-1} \, \mathrm{d}\mathcal{H}^{n-1}\,, \quad (2.22)$$

where $C_{n,k} > 0$ is a constant depending only on n and k .

3. Proof of Theorem 1.1.

In this section we prove Theorem 1.1. As we noted in the Introduction, only the case $n \ge 4$ is new, and is the case of study here, the proof being substantially simpler for $n=3$ (cf. [14]). For the reader's convenience we split the proof in several steps.

Step 0. Reduction to maps with small deficit. We start with the standard observation that, for every $u \in \mathcal{A}_{\mathbb{S}^{n-1}}$ (recall (1.3)) , we have

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u - P_T|^{n-1} \lesssim \fint_{\mathbb{S}^{n-1}} |\nabla_T u|^{n-1} + 1 \lesssim \delta_{n-1}(u) + 1\,, \quad (3.1)$$

and therefore without restriction we may assume that

$$0 \le \delta_{n-1}(u) \le \delta_0\,. \quad (3.2)$$

Indeed, if $\delta_{n-1}(u) > \delta_0$, where $\delta_0 := \delta_0(n) > 0$ is a small but fixed dimensional constant that will be suitably chosen later, in view of (3.1) we see that (1.5) trivially holds with $\phi := \mathrm{id}_{\mathbb{S}^{n-1}}$.

Step 1. Reduction to maps with zero mean value. Letting $\psi \in \text{Möb}_+(\mathbb{S}^{n-1})$ be the map provided by Lemma 2.5, we take $\widetilde{u} := u \circ \psi$. Since both the deficit and the topological degree are conformally invariant, we see that $\widetilde{u} \in \mathcal{A}_{\mathbb{S}^{n-1}}$ and

$$\delta_{n-1}(\widetilde{u}) = \delta_{n-1}(u)\,, \quad \deg \widetilde{u} = \deg u = 1\,, \text{ and } \fint_{\mathbb{S}^{n-1}} \widetilde{u} = 0\,. \quad (3.3)$$

Hence, if we prove (1.5) for $\widetilde{u}$, then it automatically also holds for u.

Step 2(a). Local estimate in a $W^{1,n-1}$-neighbourhood of the identity. In this step we prove (1.5) for $\widetilde{u}$, under the additional assumption that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T \widetilde{u} - P_T|^{n-1} \le \theta\,, \quad (3.4)$$

where $\theta := \theta(n) > 0$ is a sufficiently small but fixed dimensional constant that will be suitably chosen later.

For $R \in \mathrm{SO}(n)$ arbitrary, consider the map $v_R \in (W^{1,n-1} \cap L^\infty)(\mathbb{S}^{n-1};\mathbb{R}^n)$ defined via

$$v_R(x) := \widetilde{u}(x) - Rx\,. \tag{3.5}$$

In particular, since $\widetilde{u}$ is $\mathbb{S}^{n-1}$-valued, (3.5) implies that for $\mathcal{H}^{n-1}$-a.e. $x \in \mathbb{S}^{n-1}$ we have

$$1 = |\widetilde{u}(x)|^2 = |v_R(x) + Rx|^2 = |v_R(x)|^2 + 2\langle v_R(x), Rx\rangle + 1\,,$$

or, equivalently,

$$\langle v_R, Rx\rangle = -\frac{1}{2}|v_R|^2 \quad \mathcal{H}^{n-1}\text{-a.e. on } \mathbb{S}^{n-1}\,. \tag{3.6}$$

The next ingredient we need is a pointwise inequality for vectors derived in [10, Lemma 2.1(ii)]. For the reader's convenience, we repeat its statement here in a weaker form than the original one, that is more adequate for our purposes.

Lemma 3.1. *Let $m \in \mathbb{N}$, $p \geq 2$. For every $\kappa \in (0,1)$ there exists a $c_0 := c_0(p,\kappa) > 0$ so that, for every two vectors $X, Y \in \mathbb{R}^m$, we have*

$$|X+Y|^p \geq |X|^p + p|X|^{p-2}\langle X, Y\rangle + \frac{(1-\kappa)p}{2}|X|^{p-2}|Y|^2 + c_0|Y|^p\,. \tag{3.7}$$

We apply (3.7) with

$$m := n(n-1)\,, \quad p := n-1 \geq 2\,, \quad X := RP_T\,, \quad Y := \nabla_T v_R\,.$$

Using (3.5) and the fact that $|RP_T|^2 = |P_T|^2 = n-1$, we deduce that $\mathcal{H}^{n-1}$-a.e. on $\mathbb{S}^{n-1}$ we have

$$\begin{aligned}
|\nabla_T \widetilde{u}|^{n-1} &\geq |RP_T|^{n-1} + (n-1)|RP_T|^{n-3} RP_T \colon \nabla_T v_R + c_0|\nabla_T v_R|^{n-1} \\
&\qquad + \frac{1-\kappa}{2}(n-1)|RP_T|^{n-3}|\nabla_T v_R|^2 \\
&= (n-1)^{\frac{n-1}{2}}\left(1 + RP_T \colon \nabla_T v_R + \frac{1-\kappa}{2}|\nabla_T v_R|^2\right) + c_0|\nabla_T v_R|^{n-1}\,.
\end{aligned}$$

Integrating the last inequality over $\mathbb{S}^{n-1}$, and after some simple algebraic manipulations (and redefining the value of the constant $c_0 > 0$), we find

$$\delta_{n-1}(\widetilde{u}) \geq \frac{1-\kappa}{2} \fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^2 + \fint_{\mathbb{S}^{n-1}} RP_T \colon \nabla_T v_R + c_0 \fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^{n-1}\,. \tag{3.8}$$

Writing in the ambient space coordinates $v_R = (v_R^1, \ldots, v_R^n)$, an integration by parts on $\mathbb{S}^{n-1}$, together with the fact that $\lambda_{n,1} = n-1$ (cf. (2.1)), yields

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} RP_T \colon \nabla_T v_R &= \sum_{k=1}^n \fint_{\mathbb{S}^{n-1}} \langle \nabla_T (Rx)^k, \nabla_T v_R^k\rangle \\
&= \sum_{k,\ell=1}^n R_{k\ell} \fint_{\mathbb{S}^{n-1}} \langle \nabla_T x^\ell, \nabla_T v_R^k\rangle \\
&= \sum_{k,\ell=1}^n R_{k\ell} \fint_{\mathbb{S}^{n-1}} (-\Delta_{\mathbb{S}^{n-1}} x^\ell) v_R^k \\
&= (n-1) \fint_{\mathbb{S}^{n-1}} \langle Rx, v_R\rangle = -\frac{n-1}{2} \fint_{\mathbb{S}^{n-1}} |v_R|^2\,,
\end{aligned} \tag{3.9}$$

where in the last equality we used (3.6). Plugging (3.9) in (3.8), we obtain

$$\delta_{n-1}(\widetilde{u}) \geq \frac{1-\kappa}{2}\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^2 - \frac{n-1}{1-\kappa}\fint_{\mathbb{S}^{n-1}} |v_R|^2\right) + c_0 \fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^{n-1}. \tag{3.10}$$

Noting that $v_R \in W^{1,n-1}(\mathbb{S}^{n-1};\mathbb{R}^n) \subset W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$, we can consider the decomposition of v_R in spherical harmonics. Recalling (2.4), by (3.3), the fact that $\fint_{\mathbb{S}^{n-1}} Rx = 0$ and (2.5), we have

$$\Pi_{n,0}(v_R) = \fint_{\mathbb{S}^{n-1}} v_R = 0\,, \tag{3.11}$$

while (2.6) yields

$$\Pi_{n,1}(v_R) = \nabla (v_R)_h(0)x = (\nabla \widetilde{u}_h(0) - R)x\,.$$

By the orthogonality of the decomposition (2.2) in $W^{1,2}(\mathbb{S}^{n-1};\mathbb{R}^n)$, the improved Poincaré inequality (2.8) and (3.11), we have:

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^2 &= \fint_{\mathbb{S}^{n-1}} |\nabla_T (v_R - \Pi_{n,1}(v_R))|^2 + \fint_{\mathbb{S}^{n-1}} |\nabla_T \Pi_{n,1}(v_R)|^2 \\
&\geq 2n \fint_{\mathbb{S}^{n-1}} |v_R - \Pi_{n,1}(v_R)|^2 + \fint_{\mathbb{S}^{n-1}} |(\nabla \widetilde{u}_h(0) - R)P_T|^2 \\
&= 2n \fint_{\mathbb{S}^{n-1}} |v_R|^2 + \fint_{\mathbb{S}^{n-1}} |(\nabla \widetilde{u}_h(0) - R)P_T|^2 - 2n \fint_{\mathbb{S}^{n-1}} |(\nabla \widetilde{u}_h(0) - R)x|^2\,.
\end{aligned} \tag{3.12}$$

Note that

$$\fint_{\mathbb{S}^{n-1}} x_k x_{k'} = \frac{\delta^{kk'}}{n} \quad \forall k,k' = 1,\dots,n\,;$$

thus, for every $A \in \mathbb{R}^{n\times n}$, we can compute

$$\begin{aligned}
\fint_{\mathbb{S}^{n-1}} |AP_T|^2 &= \sum_{i,j=1}^{n,n-1} \fint_{\mathbb{S}^{n-1}} \left(\sum_{k=1}^{n} A_{ik}(P_T)_{kj}\right)^2 \\
&= \sum_{i,j=1}^{n,n-1} \sum_{k,k'=1}^{n} A_{ik}A_{ik'} \fint_{\mathbb{S}^{n-1}} \langle e_k,\tau_j\rangle\langle e_{k'},\tau_j\rangle \\
&= \sum_{i,k,k'=1}^{n} A_{ik}A_{ik'} \fint_{\mathbb{S}^{n-1}} \left\langle \sum_{j=1}^{n-1} \langle e_k,\tau_j\rangle \tau_j, e_{k'}\right\rangle \\
&= \sum_{i,k,k'=1}^{n} A_{ik}A_{ik'} \fint_{\mathbb{S}^{n-1}} \langle e_k - x_k x, e_{k'}\rangle \\
&= \left(1-\tfrac{1}{n}\right) \sum_{i,k,k'=1}^{n} A_{ik}A_{ik'}\delta^{kk'} = \frac{n-1}{n}|A|^2\,,
\end{aligned} \tag{3.13}$$

and

$$\begin{aligned}\fint_{\mathbb{S}^{n-1}} |Ax|^2 &= \sum_{i=1}^{n} \fint_{\mathbb{S}^{n-1}} \left(\sum_{k=1}^{n} A_{ik}x_k\right)^2 \\ &= \sum_{i,k,k'=1}^{n} A_{ik}A_{ik'} \fint_{\mathbb{S}^{n-1}} x_k x_{k'} = \sum_{i,k,k'=1}^{n} A_{ik}A_{ik'} \frac{\delta^{kk'}}{n} = \frac{1}{n}|A|^2 \,.\end{aligned} \tag{3.14}$$

In view of (3.13) and (3.14) (for $A := \nabla\widetilde{u}_h(0) - R$), (3.12) can be rewritten as

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^2 \geq 2n \fint_{\mathbb{S}^{n-1}} |v_R|^2 - \frac{n+1}{n}|\nabla\widetilde{u}_h(0) - R|^2 \,. \tag{3.15}$$

In particular, plugging (3.15) in (3.10), we get

$$\delta_{n-1}(\widetilde{u}) \geq \frac{1-\kappa}{2}\left(c_{n,\kappa} \fint_{\mathbb{S}^{n-1}} |v_R|^2 - \frac{n+1}{n}|\nabla\widetilde{u}_h(0) - R|^2\right) + c_0 \fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^{n-1} \,,$$

where $c_{n,\kappa} := 2n - \frac{n-1}{1-\kappa}$. Choosing $\kappa := \kappa_n = \frac{n+1}{2n} \in (0,1)$ we have $c_{n,\kappa} = 0$, and so

$$\delta_{n-1}(\widetilde{u}) \geq -\frac{(n^2-1)}{4n^2}|\nabla\widetilde{u}_h(0) - R|^2 + c_0 \fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^{n-1} \,.$$

Recalling (3.5), we deduce that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T \widetilde{u} - RP_T|^{n-1} \leq c_n\left(\delta_{n-1}(\widetilde{u}) + |\nabla\widetilde{u}_h(0) - R|^2\right), \tag{3.16}$$

where $c_n > 0$ is a dimensional constant. Recall that, in the above argument, $R \in \mathrm{SO}(n)$ is arbitrary. Therefore, (3.16) and (3.3) directly imply (1.5) provided that we have

$$\mathrm{dist}^2(\nabla\widetilde{u}_h(0); \mathrm{SO}(n)) \lesssim \delta_{n-1}(\widetilde{u}) \,. \tag{3.17}$$

Indeed, if this is the case, then we could apply (3.16) for $R \in \mathrm{SO}(n)$ such that

$$|\nabla\widetilde{u}_h(0) - R| = \mathrm{dist}(\nabla\widetilde{u}_h(0); \mathrm{SO}(n)) \,,$$

and obtain (1.5) with $\phi = R\psi^{-1} \in \text{Möb}_+(\mathbb{S}^{n-1})$. The verification of (3.17) will be provided in the next (sub)step.

Step 2(b). Verification of (3.17). For brevity, let us here set

$$A := \nabla\widetilde{u}_h(0) \,. \tag{3.18}$$

By the mean-value property of harmonic functions, (2.9), (3.4) and Jensen's inequality,

$$\begin{aligned}|A - I_n|^2 = \left|\fint_{B_1} \nabla\widetilde{u}_h - I_n\right|^2 &\leq \fint_{B_1} |\nabla\widetilde{u}_h - I_n|^2 \\ &\leq \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T \widetilde{u} - P_T|^2 \\ &\leq \frac{n}{n-1}\left(\fint_{\mathbb{S}^{n-1}} |\nabla_T \widetilde{u} - P_T|^{n-1}\right)^{\frac{2}{n-1}} \lesssim \theta^{\frac{2}{n-1}} \ll 1 \,,\end{aligned} \tag{3.19}$$

and by choosing $\theta \in (0,1)$ sufficiently small, we can take A to be invertible and such that

$$|A|^2, |A^{-1}|^2 \ \in [n-1, n+1] \quad \text{and} \quad \det A \in \left[\tfrac{1}{2}, \tfrac{3}{2}\right] . \tag{3.20}$$

By the polar decomposition, $A = R_0\, U_A$ with $R_0 \in \mathrm{SO}(n)$ and $U_A := \sqrt{A^t A}$ symmetric positive-definite. If we label the eigenvalues of U_A as $0 < \alpha_1 \leq \cdots \leq \alpha_n$ and set

$$\lambda_i := \alpha_i - 1\,, \qquad \Lambda^2 := \sum_{i=1}^{n} \lambda_i^2 \quad (\Lambda > 0)\,, \tag{3.21}$$

we have

$$\Lambda^2 = |U_A - I_n|^2 = |A - R_0|^2 = \mathrm{dist}^2\big(A; \mathrm{SO}(n)\big) \leq |A - I_n|^2 \lesssim \theta^{\frac{2}{n-1}} \ll 1\,. \tag{3.22}$$

Next we define $w(x) := A^{-1}(\widetilde{u}(x) - Ax)$, so that

$$\widetilde{u}(x) = A(x + w(x))\,. \tag{3.23}$$

Note that (3.16) and (3.20) imply

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^{n-1} \lesssim \fint_{\mathbb{S}^{n-1}} |\nabla_T \widetilde{u} - R_0 P_T|^{n-1} + |R_0 - A|^{n-1} \lesssim \delta_{n-1}(\widetilde{u}) + \Lambda^2\,. \tag{3.24}$$

Moreover, dropping the last nonnegative term in (3.10) and letting $\kappa \to 0$ therein, we have

$$\begin{aligned} 2\delta_{n-1}(\widetilde{u}) &\geq \fint_{\mathbb{S}^{n-1}} |\nabla_T v_R|^2 - (n-1) \fint_{\mathbb{S}^{n-1}} |v_R|^2 \\ &= \fint_{\mathbb{S}^{n-1}} |\nabla_T (v_R - \Pi_{n,1}(v_R))|^2 - (n-1) \fint_{\mathbb{S}^{n-1}} |v_R - \Pi_{n,1}(v_R)|^2 \\ &\geq \left(1 - \frac{n-1}{2n}\right) \fint_{\mathbb{S}^{n-1}} |\nabla_T (v_R - \Pi_{n,1}(v_R))|^2, \end{aligned}$$

where the last inequality follows again from (2.8). In terms of w, by (3.20), and the fact that $\Pi_{n,1}(\widetilde{u})(x) = Ax$ (cf. (2.6) and (3.18)) and $\Pi_{n,1}(R_0 x) = R_0 x$, the last inequality implies

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 \leq c_n \delta_{n-1}(\widetilde{u})\,. \tag{3.25}$$

Next, we wish to expand the identity

$$1 = \deg(\widetilde{u}) = \fint_{B_1} \det(\nabla \widetilde{u}_h)\, \mathrm{d}x = \det A \fint_{B_1} \det(I_n + \nabla w_h)\, \mathrm{d}x\,, \tag{3.26}$$

in terms of w. By Lemma 2.6, we obtain

$$1 = \det A \fint_{B_1} \det(I_n + \nabla w_h) = \det A \left(1 + \sum_{k=1}^{n} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma_k'(\nabla_T w P_T^t) w, x\rangle \right). \tag{3.27}$$

By Remark 2.7, the term corresponding to $k = 1$ in the right hand side of (3.27) equals

$$n \fint_{\mathbb{S}^{n-1}} \langle w, x\rangle = n \fint_{\mathbb{S}^{n-1}} \langle (\widetilde{u}(x) - Ax), A^{-t} x\rangle = 0\,,$$

where we used again (3.18) and (2.6). Hence (3.27) becomes

$$1 = \det A + \det A \sum_{k=2}^{n} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma_k'(\nabla_T w P_T^t) w, x\rangle \,. \tag{3.28}$$

Recalling the notation $U_A = \sqrt{A^t A}$, setting for notational convenience

$$\sigma_k(\lambda) := \sigma_k(U_A - I_n) \quad \forall k = 1, \dots, n \,, \tag{3.29}$$

and using also the algebraic identity

$$\det A = \det U_A = \prod_{k=1}^{n} (1+\lambda_k) = 1 + \sum_{k=1}^{n} \sigma_k(\lambda) \,,$$

(cf. (3.21) and (2.15)), identity (3.28) can be rewritten as

$$0 = \sum_{k=1}^{n} \sigma_k(\lambda) + \det A \sum_{k=2}^{n} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma_k'(\nabla_T w P_T^t) w, x\rangle \,.$$

Recalling that, by (3.21), $\Lambda^2 = \sum_{k=1}^n \lambda_k^2 = \sigma_1(\lambda)^2 - 2\sigma_2(\lambda)$, we deduce

$$\frac{1}{2}\Lambda^2 = \sigma_1(\lambda) + \frac{\sigma_1(\lambda)^2}{2} + \sum_{k=3}^{n} \sigma_k(\lambda) + \det A \sum_{k=2}^{n} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma_k'(\nabla_T w P_T^t) w, x\rangle \,. \tag{3.30}$$

Next we estimate all terms in the right-hand side of (3.30). First note that

$$\sigma_1(\lambda) + \frac{\sigma_1(\lambda)^2}{2} = \frac{|A|^2 - n}{2} \,. \tag{3.31}$$

By the mean value property of harmonic functions, Jensen's inequality and (2.9), we have

$$|A|^2 = |\nabla \widetilde{u}_h(0)|^2 = \left| \fint_{B_1} \nabla \widetilde{u}_h \right|^2 \le \fint_{B_1} |\nabla \widetilde{u}_h|^2 \le \frac{n}{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T \widetilde{u}|^2,$$

and again by Jensen's inequality,

$$|A|^2 \le n \left[\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T \widetilde{u}|^2}{n-1} \right)^{\frac{n-1}{2}} \right]^{\frac{2}{n-1}} = n \Big(1 + \delta_{n-1}(\widetilde{u}) \Big)^{\frac{2}{n-1}} \le n + \frac{2n}{n-1} \delta_{n-1}(\widetilde{u}) \,,$$

where we used the inequality $(1+t)^\alpha \le 1 + \alpha t$, which holds true for every $t \ge 0$ and $0 \le \alpha \le 1$. In particular, (3.31) and the above inequality yield

$$\sigma_1(\lambda) + \frac{\sigma_1(\lambda)^2}{2} = \frac{|A|^2 - n}{2} \le \frac{n}{n-1} \delta_{n-1}(\widetilde{u}) \,, \tag{3.32}$$

which takes care of the first two terms in the right-hand side of (3.30). Since σ_k is k-homogeneous and Λ is small by (3.22), the third term in the right-hand side of (3.30) is estimated by (recall (3.29)),

$$\sum_{k=3}^{n} \sigma_k(\lambda) \lesssim |U_A - I|^3 = \Lambda^3 \lesssim \theta^{\frac{1}{n-1}} \Lambda^2 \,. \tag{3.33}$$

For the last sum in (3.30), we consider first all terms but the last. Using that σ'_k is $(k-1)$-homogeneous, by (2.22) we have

$$\sum_{k=2}^{n-1} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma'_k(\nabla_T w P_T^t) w, x \rangle \lesssim \sum_{k=2}^{n-1} \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^{k-1} |w| \lesssim \fint_{\mathbb{S}^{n-1}} |w| \left(|\nabla_T w| + |\nabla_T w|^{n-2} \right) , \tag{3.34}$$

where in the last inequality we used the fact that

$$1 \le k-1 \le n-2 \implies |Z|^{k-1} \le |Z| + |Z|^{n-2} \quad \forall Z \in \mathbb{R}^m .$$

Note that $|w||\nabla_T w| \le |w|^2 + |\nabla_T w|^2$ and $\|w\|_{L^\infty(\mathbb{S}^{n-1})} \lesssim 1$ by (3.20) and (3.23). Moreover, since we are here considering the case $n \ge 4$, for any $\varepsilon > 0$ there holds

$$|\nabla_T w|^{n-2} \le \varepsilon^{4-n} |\nabla_T w|^2 + \varepsilon |\nabla_T w|^{n-1} ,$$

as can be checked by distinguishing the cases $|\nabla_T w| \le \frac{1}{\varepsilon}$ and $|\nabla_T w| \ge \frac{1}{\varepsilon}$. Combining these pointwise inequalities with (3.34) results in the estimate

$$\sum_{k=2}^{n-1} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma'_k(\nabla_T w P_T^t) w, x \rangle \lesssim \fint_{\mathbb{S}^{n-1}} |w|^2 + C(\varepsilon) \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^2 + \varepsilon \fint_{\mathbb{S}^{n-1}} |\nabla_T w|^{n-1} ,$$

where $C(\varepsilon) := 1 + \varepsilon^{4-n} > 0$. Combining Poincaré's inequality (2.7) for the zero-mean map w and the estimates (3.25) and (3.24) on the L^2 and L^{n-1} norms of $\nabla_T w$, this implies

$$\sum_{k=2}^{n-1} \frac{n}{k} \fint_{\mathbb{S}^{n-1}} \langle \sigma'_k(\nabla_T w P_T^t) w, x \rangle \le C(n,\varepsilon) \delta_{n-1}(\widetilde{u}) + \varepsilon \Lambda^2, \tag{3.35}$$

for any $\varepsilon > 0$ and some $C(n,\varepsilon) > 0$. It remains to estimate the very last summand in the last term in (3.30). For this we invoke first (2.21) and Wente's isoperimetric inequality, cf. (2.13), which gives

$$\fint_{\mathbb{S}^{n-1}} \langle \sigma'_n(\nabla_T w P_T^t) w, x \rangle = \fint_{\mathbb{S}^{n-1}} \left\langle w, \bigwedge_{i=1}^{n-1} \partial_{\tau_i} w \right\rangle \le \left[\fint_{\mathbb{S}^{n-1}} \left(\frac{|\nabla_T w|^2}{n-1} \right)^{\frac{n-1}{2}} \right]^{\frac{n}{n-1}} .$$

Using the L^{n-1} estimate (3.24) for $\nabla_T w$, the convexity of the function $t \mapsto t^{n-1}$ ($t > 0$), (3.2) and (3.22), we deduce

$$\fint_{\mathbb{S}^{n-1}} \langle \sigma'_n(\nabla_T w P_T^t) w, x \rangle \lesssim \delta_{n-1}(\widetilde{u})^{\frac{n}{n-1}} + \Lambda^{\frac{2n}{n-1}} \lesssim \delta_{n-1}(\widetilde{u}) + \theta^{\frac{2}{(n-1)^2}} \Lambda^2 , \tag{3.36}$$

with $\theta \in (0,1)$ as in (3.4). Combining (3.32), (3.33), (3.35) and (3.36) to estimate the right-hand side of (3.30) we obtain,

$$\frac{1}{2} \Lambda^2 \lesssim C(n,\varepsilon) \delta_{n-1}(\widetilde{u}) + \left(\theta^{\frac{2}{(n-1)^2}} + \varepsilon \right) \Lambda^2 ,$$

and by choosing $\theta \in (0,1)$ and $\varepsilon \in (0,1)$ small enough, we obtain

$$\frac{1}{2} \Lambda^2 \le C(n,\varepsilon) \delta_{n-1}(\widetilde{u}) + \frac{1}{4} \Lambda^2,$$

Absorbing the last term in the left-hand side and recalling that $\Lambda = \operatorname{dist}(\nabla\widetilde{u}_h(0); \mathrm{SO}(n))$, this implies (3.17) and concludes Step 2(b).

Step 3. (From a local to a global estimate via $W^{1,n-1}$-compactness). Arguing by contradiction, suppose that the statement of Theorem 1.1 is false. Then, for every $k \in \mathbb{N}$ there exists a map $u_k \in \mathcal{A}_{\mathbb{S}^{n-1}}$ with $\delta_{n-1}(u_k) > 0$ such that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - \nabla_T \phi|^{n-1} \geq k\delta_{n-1}(u_k) \quad \text{for all } \phi \in \text{Möb}_+(\mathbb{S}^{n-1})\,. \tag{3.37}$$

In particular, for $\phi := \mathrm{id}_{\mathbb{S}^{n-1}} \in \text{Möb}_+(\mathbb{S}^{n-1})$, by the convexity of the function $t \mapsto t^{n-1}$ $(t > 0)$ we have

$$\begin{aligned} k\delta_{n-1}(u_k) &\leq \fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - P_T|^{n-1} \\ &\leq 2^{n-2} \fint_{\mathbb{S}^{n-1}} \left(|\nabla_T u_k|^{n-1} + |P_T|^{n-1}\right) = 2^{n-2}(n-1)^{\frac{n-1}{2}} [\delta_{n-1}(u_k) + 2]\,, \end{aligned}$$

which, for $k > \beta_n := 2^{n-2}(n-1)^{\frac{n-1}{2}}$, can be rewritten as $\delta_{n-1}(u_k) \leq \frac{2\beta_n}{k-\beta_n}$. By letting $k \to \infty$ we obtain

$$\lim_{k\to\infty} \delta_{n-1}(u_k) = 0\,.$$

We can now use the compactness result from Lemma 2.4 to obtain a contradiction: by Lemma 2.5, we find $\psi_k \in \text{Möb}_+(\mathbb{S}^{n-1})$ and $R \in \mathrm{SO}(n)$ so that the $v_k := u_k \circ \psi_k \in \mathcal{A}_{\mathbb{S}^{n-1}}$ satisfy

$$\fint_{\mathbb{S}^{n-1}} v_k = 0\,. \tag{3.38}$$

Hence, by Lemma 2.4, we have $v_k \to R\,\mathrm{id}_{\mathbb{S}^{n-1}}$ strongly in $W^{1,n-1}(\mathbb{S}^{n-1}; \mathbb{S}^{n-1})$ as $j \to \infty$, up to a not-relabeled subsequence. Without loss of generality (up to considering $R^{-1}v_k$ instead of v_k if necessary) we can also suppose that $R = I_n$. Then, for the constant $\theta \in (0,1)$ chosen in Step 2, we can find $k_0 := k_0(\theta) \in \mathbb{N}$ such that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T v_k - P_T|^{n-1} \leq \theta \ll 1 \quad \text{for all } k \geq k_0\,.$$

The sequence $(v_k)_{k\geq k_0}$ satisfies both (3.38) and the assumption (3.4) of Step 2: hence, by this step, we deduce that there exist $(\phi_k)_{k\geq k_0} \subset \text{Möb}_+(\mathbb{S}^{n-1})$ such that

$$\fint_{\mathbb{S}^{n-1}} |\nabla_T u_k - \nabla_T(\phi_k \circ \psi_k^{-1})|^{n-1} = \fint_{\mathbb{S}^{n-1}} |\nabla_T v_k - \nabla_T \phi_k|^{n-1} \leq C_n \delta_{n-1}(v_k) \quad \text{for all } k \geq k_0\,,$$

which clearly contradicts (3.37).

Appendix A. On the optimality of Theorem 1.1

We explain here Remark 1.2, by considering a detailed example in the same spirit as the second example in [10, Remark 1.2]. For the sake of making the calculations simpler, we equivalently consider maps in the class

$$\mathcal{A}_{\mathbb{R}^{n-1}} := \left\{ u \in \dot{W}^{1,n-1}(\mathbb{R}^{n-1}; \mathbb{S}^{n-1})\,,\ \frac{1}{n\omega_n} \int_{\mathbb{R}^{n-1}} \langle u, \bigwedge_{i=1}^{n-1} \partial_{x_i} u \rangle = 1 \right\}, \tag{A.1}$$

where $(\partial_{x_i})_{i=1,\dots,n-1}$ denote the standard partial derivatives in $\mathbb{R}^{n-1}$ and

$$\dot{W}^{1,n-1}(\mathbb{R}^{n-1};\mathbb{S}^{n-1}) := \left\{ v \in W^{1,n-1}_{\mathrm{loc}}(\mathbb{R}^{n-1};\mathbb{S}^{n-1}) \colon \int_{\mathbb{R}^{n-1}} |\nabla v|^{n-1}\,, \mathrm{d}x < +\infty \right\}$$

is the corresponding homogeneous Sobolev space. The above class of maps can be identified with the class $\mathcal{A}_{\mathbb{S}^{n-1}}$, defined in (1.3), via the inverse stereographic projection through the south pole $-e_n$, given by the map $\phi : \mathbb{R}^{n-1} \cup \{\infty\} \to \mathbb{S}^{n-1}$ defined by

$$\phi(x) = \left(-\frac{2x_1}{1+|x|^2}, \dots, -\frac{2x_{n-1}}{1+|x|^2}, \frac{1-|x|^2}{1+|x|^2} \right). \tag{A.2}$$

We note that ϕ is orientation-preserving and that

$$\int_{\mathbb{R}^{n-1}} |\nabla \phi|^{n-1}\,\mathrm{d}x = \gamma_n := (n-1)^{\frac{n-1}{2}} n\omega_n\,. \tag{A.3}$$

Analogously, the group $\text{Möb}_+(\mathbb{S}^{n-1})$ defined in (2.10) can be identified with

$$\Psi := \{ R\phi(\rho(\cdot - x_0)) \colon \; R \in \mathrm{SO}(n)\,,\; x_0 \in \mathbb{R}^{n-1}\,,\; \rho > 0 \}\,, \tag{A.4}$$

since clearly $\Psi\phi^{-1}$ is a Lie subgroup of $\text{Möb}_+(\mathbb{S}^{n-1})$ with the same dimension (as can be easily checked by considering their corresponding Lie algebras).

With these identifications, (1.5) is equivalent to showing that for every $u \in \mathcal{A}_{\mathbb{R}^{n-1}}$,

$$\inf_{\psi\in\Psi} \int_{\mathbb{R}^{n-1}} |\nabla u - \nabla\psi|^{n-1}\,\mathrm{d}x \le C_n \left(\int_{\mathbb{R}^{n-1}} |\nabla u|^{n-1}\,\mathrm{d}x - \gamma_n \right), \tag{A.5}$$

where γ_n is as in (A.3), for a possibly different dimensional constant $C_n > 0$.

Let now ζ be a non-trivial cut-off function such that

$$\zeta \in C^\infty_c(D_1;\mathbb{R}_+)\,, \quad \zeta|_{D_{1/2}} \equiv 1\,, \quad 0 \le \zeta \le 1\,, \tag{A.6}$$

where for $\rho > 0$ we denote

$$D_\rho := \{ x \in \mathbb{R}^{n-1} \colon |x| \le \rho \}\,.$$

For a sequence of numbers $(\varepsilon_k)_{k\in\mathbb{N}} \subset \mathbb{R}_+$ with $\varepsilon_k \searrow 0$ and points $(x_k)_{k\in\mathbb{N}} \subset \mathbb{R}^{n-1}$ with $|x_k| \to \infty$, consider the maps $(u_k)_{k\in\mathbb{N}} \subset \mathcal{A}_{\mathbb{R}^{n-1}}$, defined via

$$u_k(x) := \frac{\phi(x) + \varepsilon_k \zeta(x - x_k)e_1}{|\phi(x) + \varepsilon_k \zeta(x - x_k)e_1|}\,, \tag{A.7}$$

for which by the definition of ζ and the fact that $|\phi| \equiv 1$, we have that

$$u_k \equiv \phi \;\text{ in }\; \mathbb{R}^{n-1} \setminus D_1(x_k)\,. \tag{A.8}$$

The sequence $(u_k)_{k\in\mathbb{N}}$ gives the optimality claimed in Remark 1.2:

Proposition A.1. *Choosing the sequences $x_k \nearrow \infty$ and $\varepsilon_k \searrow 0$ appropriately, for all $k \in \mathbb{N}$ large enough, we have*

$$\int_{\mathbb{R}^{n-1}} |\nabla u_k|^{n-1}\,\mathrm{d}x - \gamma_n \sim \varepsilon_k^{n-1} + o(\varepsilon_k^{n-1})\,, \tag{A.9}$$

$$\inf_{\psi\in\Psi} \int_{\mathbb{R}^{n-1}} |\nabla u_k - \nabla\psi|^{n-1}\,\mathrm{d}x \sim \varepsilon_k^{n-1} + o(\varepsilon_k^{n-1})\,. \tag{A.10}$$

Proof. Before proceeding with the main part of the proof, we begin with some preliminary computations. Consider the maps $\hat{v}:\mathbb{R}^n\setminus\{0\}\to\mathbb{S}^{n-1}$ and $w_k:\mathbb{R}^{n-1}\to\mathbb{R}^n$ defined by

$$\hat{v}(y):=\frac{y}{|y|}\quad\text{ and }\quad w_k(x):=\phi(x)+\varepsilon_k\zeta(x-x_k)e_1\,. \tag{A.11}$$

Since

$$\nabla\hat{v}(y)=\frac{I_n}{|y|}-\frac{y\otimes y}{|y|^3}\,,$$

for every $x\in D_1(x_k):=x_k+D_1$ we can compute

$$\begin{aligned}\nabla u_k(x)&=\nabla(\hat{v}\circ w_k)(x)\\&=\left(\frac{I_n}{|w_k(x)|}-\frac{w_k(x)\otimes w_k(x)}{|w_k(x)|^3}\right)\nabla w_k(x)\\&=\frac{\nabla\phi(x)}{|w_k(x)|}+\frac{\varepsilon_k e_1\otimes\nabla\zeta(x-x_k)}{|w_k(x)|}-\frac{w_k(x)\otimes w_k(x)}{|w_k(x)|^3}\nabla w_k(x)\,.\end{aligned} \tag{A.12}$$

Note that

$$\begin{aligned}|w_k(x)|&=\sqrt{1+2\varepsilon_k\phi^1(x)\zeta(x-x_k)+\varepsilon_k^2\zeta^2(x-x_k)}\\&=1+\varepsilon_k\phi^1(x)\zeta(x-x_k)+\varepsilon_k^2 r_{k,0}(x)\,,\end{aligned} \tag{A.13}$$

for some remainder term $r_{k,0}:\mathbb{R}^{n-1}\to\mathbb{R}$ with $\|r_{k,0}\|_{L^\infty(\mathbb{R}^{n-1})}\lesssim 1$ (since $|\phi|\equiv 1$ and $0\le\zeta\le 1$). Actually, since

$$\phi^j(x)\to 0 \text{ for all } j=1,\dots,n-1 \text{ and } \nabla\phi(x)\to 0\,,\ \text{as } |x|\to\infty\,,$$

the points $(x_k)_{k\in\mathbb{N}}\subset\mathbb{R}^{n-1}$ can be chosen such that

$$\|\nabla\phi\|_{L^\infty(D_1(x_k))}+\max_{j\in\{1,\dots,n-1\}}\|\phi^j\|_{L^\infty(D_1(x_k))}\lesssim\varepsilon_k^2\,, \tag{A.14}$$

so that (A.13) yields

$$|w_k(x)|=1+\varepsilon_k^2 r_{k,1}(x)\,, \tag{A.15}$$

for some new $r_{k,1}:\mathbb{R}^{n-1}\to\mathbb{R}$ with $r_{k,1}=0$ in $\mathbb{R}^{n-1}\setminus D_1(x_k)$ and $\|r_{k,1}\|_{L^\infty(\mathbb{R}^{n-1})}\lesssim 1$. Regarding the last term in the last line of (A.12), using the fact that

$$(\phi\otimes\phi)\nabla\phi=0\,,$$

since $|\phi|\equiv 1$, we can calculate

$$\begin{aligned}(w_k\otimes w_k)\nabla w_k&=\varepsilon_k\phi^1\big(\phi\otimes\nabla\zeta(\cdot-x_k)\big)+\varepsilon_k\zeta(\cdot-x_k)\big(\phi\otimes\nabla\phi^1\big)\\&\quad+\varepsilon_k^2\zeta^2(\cdot-x_k)e_1\otimes\nabla\phi^1\\&\quad+\varepsilon_k^2\zeta(\cdot-x_k)\left(\phi\otimes\nabla\zeta(\cdot-x_k)+\phi^1 e_1\otimes\nabla\zeta(\cdot-x_k)\right)\\&\quad+\varepsilon_k^3\zeta(\cdot-x_k)\big(e_1\otimes\nabla\zeta(x-x_k)\big)\\&=\varepsilon_k^2 Z_{k,0}(x)\,,\end{aligned} \tag{A.16}$$

for some $Z_{k,0} : \mathbb{R}^{n-1} \to \mathbb{R}^{n\times(n-1)}$ with $Z_{k,0} \equiv 0$ in $\mathbb{R}^{n-1} \setminus D_1(x_k)$ and $\|Z_{k,0}\|_{L^\infty(\mathbb{R}^{n-1})} \lesssim 1$. In the last line of (A.16) we used (A.6) and (A.14). Therefore, by (A.12), (A.15), (A.16) and another Taylor expansion we deduce that

$$\nabla u_k(x) = \nabla\phi(x) + \varepsilon_k\big(e_1 \otimes \nabla\zeta(x - x_k)\big) + \varepsilon_k^2 Z_{k,1}(x)\,, \tag{A.17}$$

for a map $Z_{k,1} : \mathbb{R}^{n-1} \to \mathbb{R}^{n\times(n-1)}$ with $\|Z_{k,1}\|_{L^\infty(\mathbb{R}^{n-1})} \lesssim 1$. Again, in view of (A.6) and (A.8), we can take $Z_{k,1} \equiv 0$ in $\mathbb{R}^{n-1} \setminus D_1(x_k)$.

We are now ready to prove our assertions. Note that, by Theorem 1.1, it suffices to prove the upper bound in (A.9) and the lower bound in (A.10).

We begin by proving (A.9). From (A.17) we have

$$|\nabla u_k(x)|^{n-1} \le \big(|\nabla\phi(x)| + \varepsilon_k|\nabla\zeta(x - x_k)| + \varepsilon_k^2|Z_{k,1}(x)|\big)^{n-1}.$$

Expanding the right-hand side and using (A.14) to estimate the mixed terms, we deduce the pointwise estimate

$$|\nabla u_k|^{n-1} \le |\nabla\phi|^{n-1} + \varepsilon_k^{n-1}|\nabla\zeta(\cdot - x_k)|^{n-1} + \varepsilon_k^n r_{k,1}\,, \tag{A.18}$$

for a new function $r_{k,1} : \mathbb{R}^{n-1} \to [0, +\infty)$ with $r_{k,1}|_{\mathbb{R}^{n-1}\setminus D_1(x_k)} = 0$ and $\|r_{k,1}\|_{L^\infty(\mathbb{R}^{n-1})} \lesssim 1$. Thus, by (A.3) we get

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |\nabla u_k|^{n-1}\,\mathrm{d}x - \gamma_n &= \int_{\mathbb{R}^{n-1}} \big(|\nabla u_k|^{n-1} - |\nabla\phi|^{n-1}\big)\,\mathrm{d}x\\ &\le \varepsilon_k^{n-1}\int_{D_1} |\nabla\zeta|^{n-1} + o(\varepsilon_k^{n-1})\,,\end{aligned}$$

proving the upper bound in (A.9).

Instead of proving (A.10) directly, we first prove it with ϕ in place of ψ. Indeed, by (A.8) and (A.17),

$$\int_{\mathbb{R}^{n-1}} |\nabla u_k - \nabla\phi|^{n-1} = \varepsilon_k^{n-1}\int_{D_1(x_k)} |e_1 \otimes \nabla\zeta(\cdot - x_k) + \varepsilon_k Z_{k,1}|^{n-1}$$

and, by a Taylor expansion, we deduce that for $k \in \mathbb{N}$ large enough,

$$\int_{\mathbb{R}^{n-1}} |\nabla u_k - \nabla\phi|^{n-1} \ge \varepsilon_k^{n-1}\int_{D_1} |\nabla\zeta|^{n-1} + o(\varepsilon_k^{n-1})\,. \tag{A.19}$$

At this point, it remains to verify that the infimum in (A.10) is of the same order as the value of the integral at $\psi := \phi$. To see this, let $(\psi_k)_{k\in\mathbb{N}} \subset \Psi$ be such that

$$\int_{\mathbb{R}^{n-1}} |\nabla u_k - \nabla\psi_k|^{n-1}\,\mathrm{d}x = \inf_{\psi\in\Psi}\int_{\mathbb{R}^{n-1}} |\nabla u_k - \nabla\psi|^{n-1}\,\mathrm{d}x + o(\varepsilon_k^{n-1})\,, \tag{A.20}$$

and note that we must have $\psi_k \to \phi$ for instance in $\dot W^{1,n-1}(\mathbb{R}^{n-1})$; however, as Ψ is a finite dimensional Lie group , in fact $\psi_k \to \phi$ in any norm. In particular we have $\psi_k \to \phi$ in $C^1(\mathbb{R}^{n-1})$, which implies

$$\int_{D_1(x_k)} |\nabla\psi_k - \nabla\phi|^{n-1}\,\mathrm{d}x = o(\varepsilon_k^{n-1})\,,$$

since $\|\nabla\phi\|_{L^\infty(D_1(x_k))} \lesssim \varepsilon_k^2$ by (A.14). Therefore,

$$\begin{aligned}\int_{\mathbb{R}^{n-1}} |\nabla u_k - \nabla\psi_k|^{n-1}\,\mathrm{d}x &\geq \int_{D_1(x_k)} |\nabla u_k - \nabla\psi_k|^{n-1}\,\mathrm{d}x \\ &\geq \int_{D_1(x_k)} |\nabla u_k - \nabla\phi|^{n-1}\,\mathrm{d}x - o(\varepsilon_k^{n-1}) \\ &= \int_{\mathbb{R}^{n-1}} |\nabla u_k - \nabla\phi|^{n-1}\,\mathrm{d}x - o(\varepsilon_k^{n-1})\,.\end{aligned}$$

The last equality is valid because $u_k = \phi$ outside $D_1(x_k)$. The lower bound in (A.10) now follows immediately from (A.19) and (A.20) , and the proof is complete. □

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