GLOBAL MINIMALITY OF THE HOPF MAP IN THE FADDEEV–SKYRME MODEL WITH LARGE COUPLING CONSTANT

ANDRÉ GUERRA, XAVIER LAMY, AND KONSTANTINOS ZEMAS

ABSTRACT. We prove that, modulo rigid motions, the Hopf map is the unique minimizer of the Faddeev–Skyrme energy in its homotopy class, for a sufficiently large coupling constant.

1. Introduction

The Skyrme model, introduced in [29], is a classical nonlinear $O(3)$-sigma model which has proved successful in quantum field theory. The static fields in Skyrme's model, which are called *Skyrmions*, can be mathematically described as critical points of the energy

$$\mathcal{S}(u) := \int_{\mathbb{M}^3} |du|^2 + \frac{1}{4}\int_{\mathbb{M}^3} |du \wedge du|^2 , \qquad u\colon \mathbb{M}^3 \to \mathbb{S}^3 ,$$

where $\mathbb{M}^3$ is a Riemannian manifold[1]. Skyrmions are a particular example of topological solitons, and we refer the reader to the monograph [25] for further physical background. The existence of maps minimizing $\mathcal{S}$ in suitable homotopy classes has been proved rigorously in the works [8, 9], in the case $\mathbb{M}^3 = \mathbb{R}^3$. In this case, finite-energy maps exhibit rapid decay at infinity to a constant value $b \in \mathbb{S}^3$, so that they can be considered as maps from $\mathbb{R}^3 \cup \{\infty\} \cong \mathbb{S}^3 \to \mathbb{S}^3$.

A further refinement of Skyrme's model was proposed by Faddeev [11], considering Skyrme's model in the case where the maps take values in an equator $\mathbb{S}^2 \subset \mathbb{S}^3$. As in Skyrme's model, there is a satisfactory existence theory for this problem, especially in the case of compact $\mathbb{M}^3$ [1, 22, 23, 33], the case of non-compact $\mathbb{M}^3$ being more challenging [22, 23]. In the simplest non-compact example $\mathbb{M}^3 = \mathbb{R}^3$, and following Ward [33], one can consider an approximation of $\mathbb{R}^3$ by $\mathbb{S}^3_\rho$, the sphere of radius $\rho \gg 1$. Thus, considering the energy on $\mathbb{S}^3_\rho$, after a change of variables and normalization by a factor ρ^{-1}, one is led to a *perturbed version* of the Faddeev–Skyrme energy, namely

$$\mathcal{FS}_\rho(u) := \int_{\mathbb{S}^3} |du|^2 + \frac{1}{4\rho^2}\int_{\mathbb{S}^3} |du \wedge du|^2 , \qquad u\colon \mathbb{S}^3 \to \mathbb{S}^2. \tag{1.1}$$

One can rewrite the second term in the energy above as

$$\frac{1}{4}|du \wedge du|^2 = |u^*\omega_{\mathbb{S}^2}|^2 ,$$

where $\omega_{\mathbb{S}^2}$ is the volume form on $\mathbb{S}^2$, see (2.4) below. The energy (1.1) is therefore naturally defined in the space

$$\mathcal{U}_{\mathcal{FS}} := \left\{ u \in W^{1,2}(\mathbb{S}^3; \mathbb{S}^2)\colon \int_{\mathbb{S}^3} |du|^2 + \int_{\mathbb{S}^3} |u^*\omega_{\mathbb{S}^2}|^2 < +\infty \right\} . \tag{1.2}$$

[1]Here, $\wedge$ is the natural wedge product on $T^*\mathbb{M}^3 \otimes \mathbb{R}^4$ characterized by $(\alpha \otimes a) \wedge (\beta \otimes b) = (\alpha \wedge \beta) \otimes (a \wedge b)$.

Homotopy classes of smooth maps from $\mathbb{S}^3$ to $\mathbb{S}^2$, corresponding to elements of $\pi_3(\mathbb{S}^2) \cong \mathbb{Z}$ are classified by their *Hopf invariant* (or *Hopf charge* in the physics literature) [34], which is defined for $u \in C^\infty(\mathbb{S}^3; \mathbb{S}^2)$ as

$$Q(u) := \frac{1}{16\pi^2} \int_{\mathbb{S}^3} \beta \wedge d\beta \in \mathbb{Z}\,, \tag{1.3}$$

where β is any 1-form on $\mathbb{S}^3$ such that $d\beta = u^*\omega_{\mathbb{S}^2}$. This definition can be meaningfully extended to the class of finite Faddeev–Skyrme energy maps, see [2] or Section 3 below.

A basic question concerning the above models is whether one can characterize minimizers in some simple cases; the non-convexity of the problems makes this a rather difficult question. For the Skyrme model with $\mathbb{M}^3 = \mathbb{S}^3_\rho$, it is conjectured [24] that the homothety $x \mapsto x/\rho$ is a minimizer in its homotopy class whenever $0 < \rho \leq \sqrt{2}$; this map is known to be unstable whenever $\rho > \sqrt{2}$. At present minimality is only known for $0 < \rho \leq \sqrt{3/2} < \sqrt{2}$, see [28] and the references therein.

For the Faddeev–Skyrme model (1.1) the situation is more complicated, as one has *knotted* solutions to the variational equations, and the *Hopf map* h is a critical point of $\mathcal{FS}_\rho$ of particular interest from the point of view of both geometry and physics. This map gives a prominent example of a non-trivial $\mathbb{S}^1$-fiber bundle of $\mathbb{S}^3$ onto $\mathbb{S}^2$: in complex coordinates, and denoting by $\pi\colon \mathbb{C} \to \mathbb{S}^2$ the inverse stereographic projection, *i.e.*,

$$\pi\colon \mathbb{C} \cup \{\infty\} \to \mathbb{S}^2\,, \quad \pi(z) := \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1} \right)\,, \tag{1.4}$$

the Hopf map $h : \mathbb{S}^3 \subset \mathbb{C}^2 \to \mathbb{S}^2$ is given by

$$h(z,w) := \pi(z/w) = (2z\bar{w}, |z|^2 - |w|^2)\,, \tag{1.5}$$

where the last equality holds true because $|z|^2 + |w|^2 = 1$ for $(z,w) \in \mathbb{S}^3$.

For the Faddeev–Skyrme model it is also known [33] that h is unstable for $\rho > \sqrt{2}$, while it is linearly stable for $0 < \rho \leq \sqrt{2}$, cf. [17, 30], similarly to the homothety in the Skyrme model. It is therefore conjectured that, in the latter case, h is a minimizer of the energy in its homotopy class.

Actually, as pointed out[2] in [31], the above conjectured minimality property of the Hopf map was unknown for *any* value of the coupling constant $0 < \rho \leq \sqrt{2}$. In this paper we give a positive answer to the above conjecture, for sufficiently small values of ρ:

Theorem 1.1. *There exists $\rho_0 \in (0, \sqrt{2})$ such that, for every $0 < \rho \leq \rho_0$ and $u \in \mathcal{U}_{\mathcal{FS}}$ with Hopf invariant equal to 1, it holds that*

$$\mathcal{FS}_\rho(u) \geq \mathcal{FS}_\rho(h)\,. \tag{1.6}$$

Equality in (1.6) *holds if and only if $u = h \circ R$ for some $R \in \mathrm{SO}(4)$.*

The case where the energy consists only of the second term in (1.1), which formally corresponds to the limit $\rho \to 0$, was previously considered in [31], where a short proof of the minimality of the Hopf map was provided.

It is also interesting to compare our results with another conjecture, this the Hopf map should minimize, in its homotopy class, the conformally-invariant energy

$$\int_{\mathbb{S}^3} |du|^3\,, \qquad u\colon \mathbb{S}^3 \to \mathbb{S}^2\,. \tag{1.7}$$

[2]Specifically, cf. the paragraph below Theorem 1.4 in [31].

Linear stability of h in this case was already proved in [27], as well as minimality if the power 3 is replaced with 4, but otherwise this conjecture remains open; see also [7] for related results. On the one hand, there are several similarities between (1.7) and the Faddeev–Skyrme problem, cf. for instance [22, 23], and as we will see in this paper the required spectral analysis is also quite similar. On the other hand, the problems are at the same time fairly different, since (1.1) is not conformally invariant and the corresponding integrand is non-convex.

Our strategy to prove Theorem 1.1 elaborates on two already mentioned known facts about the Hopf map:

(a) it is linearly stable for $0 < \rho \leq \sqrt{2}$, cf. [17, 30];

(b) it is minimal for the second term in the Faddeev-Skyrme energy (1.1), cf. [31].

We establish quantitative versions of these two facts: loosely stated,

(a') the Hopf map is (modulo symmetries) a strict local minimizer for $0 < \rho < \sqrt{2}$;

(b') if a map is almost minimal for the second energy term, then it must be close to the Hopf map (modulo symmetries).

Since the second energy term in (1.1) is dominant for small ρ, these strengthened assertions provide a proof of Theorem 1.1 : minimizers of the full energy must be almost minimal for the second energy term, hence close to the Hopf map thanks to (b'), and therefore equal to the Hopf map (modulo rotational symmetries) thanks to (a'). Here it is of course crucial that the closeness to the Hopf map provided by (b') implies the closeness required by (a'). In fact, a version of (a') established in [17] requires closeness in a too restrictive distance and cannot be combined with (b'). Another difficulty is related to the symmetries in these statements: the second energy term in (1.1), which depends only on the closed 2-form $\alpha = u^*\omega_{\mathbb{S}^2}$, has more symmetries than the full energy. Hence, the closeness *modulo symmetries* provided by (b') cannot imply the closeness *modulo symmetries* required by (a'). But this difficulty is resolved thanks to an idea introduced by Rivière [27, 28] and used also in [17], which allows to apply the outlined strategy using a relaxed energy on closed 2-forms.

Plan of the paper: In Section 2 we gather some notation and basic properties related to the Hopf map and the Hodge Laplacian on $\mathbb{S}^3$. In Section 3 we prove several useful facts about the Hopf invariant for maps from $\mathbb{S}^3$ to $\mathbb{S}^2$ with finite Faddeev–Skyrme energy, and include a concise proof of its integrality in this lower-regularity case. In Section 4, we establish local coercivity properties for a relaxed energy functional on closed 2-forms. In Section 5, we turn this local coercivity estimate into a global stability estimate for the relaxed energy, for sufficiently large values of the coupling constant ρ^{-2}, and with this as our main ingredient, we conclude the proof of Theorem 1.1.

2. Preliminaries

For $p \in [1, +\infty)$ and for a vector bundle $E \to \mathbb{S}^3$, we denote by $L^p(\mathbb{S}^3; E)$ (respectively $W^{1,p}(\mathbb{S}^3; E)$) the space of L^p- (respectively $W^{1,p}$-) sections of $E \to \mathbb{S}^3$.

2.1. Some basic identities involving differential forms.

Let us first define precisely the terms in (1.1). Here, we adopt an *extrinsic viewpoint*, namely we consider

$$u := (u^1, u^2, u^3) : \mathbb{S}^3 \to \mathbb{S}^2 \subset \mathbb{R}^3 ,$$

where $\mathbb{R}^3$ is endowed with the standard Euclidean basis (e_1, e_2, e_3), so that $du(x) \in T_x^*\mathbb{S}^3 \otimes \mathbb{R}^3$ is given in coordinates as

$$du(x) = \sum_{k=1}^{3} du^k(x) \otimes e_k = \begin{pmatrix} du^1 \\ du^2 \\ du^3 \end{pmatrix}.$$

This gives rise to an alternating vector-valued 2-form $du(x) \wedge du(x) \in \bigwedge^2 T_x^*\mathbb{S}^3 \otimes \mathbb{R}^3$, which in coordinates is expressed as

$$du(x) \wedge du(x) = \left(\sum_{k=1}^{3} du^k(x) \otimes e_k\right) \wedge \left(\sum_{k'=1}^{3} du^{k'}(x) \otimes e_{k'}\right) = 2 \begin{pmatrix} du^2 \wedge du^3 \\ du^3 \wedge du^1 \\ du^1 \wedge du^2 \end{pmatrix}, \tag{2.1}$$

where we also used the relations $e_1 \wedge e_2 = e_3\,,\ e_2 \wedge e_3 = e_1\,,\ e_3 \wedge e_1 = e_2\,.$

For later reference, we note that the volume form on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is given by

$$\omega_{\mathbb{S}^n} = \sum_{j=1}^{n+1} (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1}\,, \tag{2.2}$$

where as usual $\widehat{\cdot}$ denotes that the corresponding term is ommited. Thus, when $n = 2$, its pull-back by u is

$$\begin{aligned} u^*\omega_{\mathbb{S}^2} &= u^1 du^2 \wedge du^3 + u^2 du^3 \wedge du^1 + u^3 du^1 \wedge du^2 \\ &= \frac{1}{2} u \cdot du \wedge du = \sum_{1 \le j < k \le 3} u \cdot (\partial_j u \times \partial_k u)\, dx^j \wedge dx^k\,, \end{aligned} \tag{2.3}$$

where $\cdot$ and $\times$ denote the Euclidean inner and outer product respectively. Since $|u| = 1$, (2.1) and (2.3) imply that

$$|u^*\omega_{\mathbb{S}^2}|^2 = \sum_{1 \le j < k \le 3} |\partial_j u \times \partial_k u|^2 = \frac{1}{4} |du \wedge du|^2\,. \tag{2.4}$$

Throughout this paper we will use systematically Hodge theory for differential forms with coefficients in Sobolev spaces, and we refer the reader to [18, §5] for a comprehensive discussion of this theory. Here we record the following simple lemma:

Lemma 2.1. *The following statements hold true:*

(i) *Let* $\alpha, \beta \in W^{1,2}(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3)$. *Then,*

$$\int_{\mathbb{S}^3} \alpha \wedge d\beta = \int_{\mathbb{S}^3} d\alpha \wedge \beta\,. \tag{2.5}$$

(ii) *There exists a constant* $C > 0$ *such that for every* $\varphi \in L^2(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3)$ *with* $d^*\varphi = 0$, *there holds*

$$\left| \int_{\mathbb{S}^3} \varphi \wedge d\varphi \right| \le C \|d\varphi\|^2_{L^2(\mathbb{S}^3)}\,. \tag{2.6}$$

Proof. The claim in (i) follows by the formula $d(\alpha \wedge \beta) = d\alpha \wedge \beta - \alpha \wedge d\beta$ and Stokes' Theorem. Regarding (ii), let $\overline{\varphi} := \fint_{\mathbb{S}^3} \varphi \in \bigwedge^1 T^*\mathbb{S}^3$ be the *mean value of* φ, intended component-wise.

Then,

$$\Big|\int_{\mathbb{S}^3}\varphi\wedge d\varphi\Big| = \Big|\int_{\mathbb{S}^3}(\varphi-\overline{\varphi})\wedge d\varphi\Big| \le \|\varphi-\bar{\varphi}\|_{L^2(\mathbb{S}^3)}\|d\varphi\|_{L^2(\mathbb{S}^3)}$$
$$\le C\|\nabla\varphi\|_{L^2(\mathbb{S}^3)}\|d\varphi\|_{L^2(\mathbb{S}^3)} \le C\|d\varphi\|^2_{L^2(\mathbb{S}^3)}\,,$$

where in the first line we used that, by (2.5), $\int_{\mathbb{S}^3} c\wedge d\varphi = 0$ for every constant coefficient 1-form $c\in\bigwedge^1 T^*\mathbb{S}^3$ and the Cauchy-Schwartz inequality. In the passage to the second line we used successively the Poincaré inequality on forms, that φ is co-closed and Gaffney's inequality

$$\|\nabla\varphi\|_{L^2(\mathbb{S}^3)} \le C\big(\|d\varphi\|_{L^2(\mathbb{S}^3)} + \|d^*\varphi\|_{L^2(\mathbb{S}^3)}\big)\,, \tag{2.7}$$

which the reader can find in [18, §4]. □

We will also use, often implicitly, the characterization of the De Rham cohomology groups of spheres. In particular, for $\mathbb{S}^3$ we have that

$$H^k_{\mathrm{dR}}(\mathbb{S}^3) = \begin{cases}\mathbb{R}\,, & \text{for } k=0,3\\ 0\,, & \text{for } k=1,2\,.\end{cases}$$

cf. [21, Theorem 17.21]. In particular closed 1- and 2-forms on $\mathbb{S}^3$ are exact, and the same holds for forms with Sobolev coefficients, cf. [18, §5].

2.2. **Spectral properties of the Hodge-Laplacian on closed forms.** We collect here some well-known spectral properties of the Hodge-Laplacian $\Delta := dd^* + d^*d$ on closed 2-forms on $\mathbb{S}^3$, where we recall that

$$d^* := *d*\colon \Omega^2(\mathbb{S}^3)\to\Omega^1(\mathbb{S}^3).$$

In fact, it is also useful to consider the square root of Δ which, restricted to closed 2-forms, becomes the operator $d*$. The corresponding eigenvalue equation is then

$$d*\alpha = \lambda\alpha\,,\qquad \text{with } \alpha\in\Omega^2(\mathbb{S}^3) \text{ such that } d\alpha = 0\,. \tag{2.8}$$

Note that if α solves (2.8) then $\Delta\alpha = \lambda^2\alpha$. Since the spectrum of Δ on $\Omega^2(\mathbb{S}^3)$ is the set $\{(k+2)^2\}_{k\in\mathbb{N}}$, cf. [16, 19], it follows that for (2.8) to be solvable we must have $\lambda = \pm(k+2)$. In fact (2.8) is solvable for all such values, see again [16, 19] as well as [27, Proposition IV.2], and we have the following precise description of the corresponding eigenspaces:

Proposition 2.2 (Spectrum of $d*$)**.** *Let* $\mathrm{Id}\colon \Omega^2(\mathbb{S}^3)\to\Omega^2(\mathbb{S}^3)$ *be the identity operator. The eigenspaces*

$$E_k^{\pm} := \ker\big(d*\mp(k+2)\mathrm{Id}\big)\cap\ker d\subset\Omega^2(\mathbb{S}^3) \tag{2.9}$$

are the restriction to $\mathbb{S}^3$ *of the self-dual (respectively anti-self-dual) closed and co-closed 2-forms on* $\mathbb{R}^4$ *which are homogeneous and polynomial of degree* k*. These spaces induce an* L^2*-orthogonal decomposition*

$$L^2\Big(\mathbb{S}^3;\bigwedge^2 T^*\mathbb{S}^3\Big) = \bigoplus_{k=0}^{\infty}(E_k^+\oplus E_k^-)\,. \tag{2.10}$$

We recall that a form $\alpha\in\Omega^2(\mathbb{R}^4)$ is said to be self-dual/anti-self-dual if $*\alpha = \pm\alpha$.

Remark 2.3. Since $** = \mathrm{Id}$ on $\Omega^2(\mathbb{R}^4)$ and the Hodge-star operation is an isometry, the spaces of *self-dual* and *anti-self-dual* forms give a pointwise-orthogonal decomposition of $\Omega^2(\mathbb{R}^4)$. In the case $k=0$, the elements in $E_0^{\pm}$ have a specially simple description, since they are the restrictions to $\mathbb{S}^3$ of constant-coefficient 2-forms on $\mathbb{R}^4$, and hence they can be identified with the self-dual (or anti-self-dual) constant-coefficient 2-forms on $\mathbb{R}^4$. For instance, we have

$$E_0^+ = \operatorname{span}\{\omega_{0,1}^+,\ \omega_{0,2}^+,\ \omega_{0,3}^+\}\,, \tag{2.11}$$

where

$$\omega_{0,1}^+ := dx^1\wedge dx^2 + dx^3\wedge dx^4\,,\ \omega_{0,2}^+ := dx^1\wedge dx^3 - dx^2\wedge dx^4\,,\ \omega_{0,3}^+ := dx^1\wedge dx^4 + dx^2\wedge dx^3\,,$$

with a similar basis for E_0^-, up to flipping the signs in the above basis. Note in particular that $\dim E_0^\pm = 3$.

We will also need the fact that the natural action of SO(4) on the space of self-dual (or anti-self-dual) constant-coefficient 2-forms on $\mathbb{R}^4$ is *transitive*, *i.e.*,

$$\forall \omega_1,\omega_2\in E_0^\pm \text{ with } |\omega_1| = |\omega_2| \text{ there exists } \widetilde{R}\in SO(4) \text{ such that } \omega_2 = \widetilde{R}^*\omega_1\,. \tag{2.12}$$

This follows directly from the fact that any $\omega\in E_0^+$ can be represented as $\omega = \lambda R^*\omega_{0,1}^+$ for some $\lambda\in\mathbb{R}$ and $R\in \mathrm{SO}(4)$ (and analogously for E_0^-). To check this, consider the skew-symmetric matrix $A\in\mathbb{R}^{4\times 4}$ of the 2-form ω in the canonical basis, *i.e.*, $A_{ij} := \omega(e_i,e_j)\ \forall i,j = 1,\dots,4$: there exists a rotation $R\in\mathrm{SO}(4)$ such that RAR^t is block-diagonal with 2×2 skew-symmetric blocks. Since ω is self-dual, these two blocks must be the same. This means that $RAR^t = \lambda A_{0,1}^+$ for some $\lambda\in\mathbb{R}$, where $A_{0,1}^+$ is the matrix of $\omega_{0,1}^+$, and therefore $\omega = \lambda R^*\omega_{0,1}^+$.

2.3. **The Hopf map and horizontal conformality.** In this subsection we gather some well-known facts about the Hopf map. We first recall the following definition, which the reader can find more generally in [3, Definition 2.4.2 and Lemma 2.4.4]:

Definition 2.4. A map $u\in W^{1,2}(\mathbb{S}^3;\mathbb{S}^2)$ is said to be *horizontally weakly conformal* if

$$du(x)\circ du(x)^t = \frac{1}{2}|du(x)|^2\,\mathrm{Id}_{T_{u(x)}\mathbb{S}^2} \quad \text{for } \mathcal{H}^3\text{-a.e. } x\in\mathbb{S}^2\,, \tag{2.13}$$

where $du(x)^t\colon T_{u(x)}\mathbb{S}^2\to T_x\mathbb{S}^3$ denotes the adjoint map of $du(x)\colon T_x\mathbb{S}^3\to T_{u(x)}\mathbb{S}^2$.

The following lemma gives a simple but very useful pointwise inequality related to weak horizontal conformality. Its proof is elementary and can be found for instance in [17, Proof of Lemma 2.1].

Lemma 2.5. *For a map $u\in W^{1,2}(\mathbb{S}^3;\mathbb{S}^2)$ the following inequality holds for $\mathcal{H}^3$-a.e. $x\in\mathbb{S}^3$:*

$$|u^*\omega_{\mathbb{S}^2}(x)| \le \frac{1}{2}|du(x)|^2\,. \tag{2.14}$$

Equality holds if and only if u is horizontally weakly conformal at x.

We remark that this type of estimate also plays an important role in the minimality arguments in [7].

The Hopf map h introduced in (1.5) is a main example of a horizontally conformal map. We now detail some of its basic properties. For the 1-form on $\mathbb{S}^3\subset\mathbb{R}^4$ given by

$$\theta := -x^2dx^1 + x^1dx^2 - x^4dx^3 + x^3dx^4\,, \tag{2.15}$$

it holds that

$$h^*\omega_{\mathbb{S}^2} = 2d\theta = 4(dx^1\wedge dx^2 + dx^3\wedge dx^4)\,, \tag{2.16}$$

and as a result of (2.15), (2.16) and (2.2) we also have

$$\theta\wedge d\theta = 2\omega_{\mathbb{S}^3}\,. \tag{2.17}$$

It is actually convenient to choose particular coordinates on $\mathbb{S}^3$ and $\mathbb{S}^2$, namely

$$\begin{aligned} &\text{(i)} \quad [0,\pi/2]\times\mathbb{S}^1\times\mathbb{S}^1 \ni (t, e^{i\varphi_1}, e^{i\varphi_2}) \mapsto (e^{i\varphi_1}\sin t, e^{i\varphi_2}\cos t) \in \mathbb{S}^3\,,\\ &\text{(ii)} \quad [0,\pi/2]\times\mathbb{S}^1 \ni (t, e^{i\varphi}) \mapsto (e^{i\varphi}\sin 2t, -\cos 2t) \in \mathbb{S}^2\,. \end{aligned} \tag{2.18}$$

In these coordinates,

$$\begin{aligned} g_{\mathbb{S}^3} &= (dt)^2 + \sin^2 t\,(d\varphi_1)^2 + \cos^2 t\,(d\varphi_2)^2\,,\\ g_{\mathbb{S}^2} &= 4(dt)^2 + (\sin^2 2t)\,(d\varphi)^2\,, \end{aligned}$$

and the Hopf map takes the simple form

$$h\colon (t, e^{i\varphi_1}, e^{i\varphi_2}) \mapsto (t, e^{i(\varphi_1-\varphi_2)})\,.$$

One can then choose an orthonormal frame $\{\tau_1,\tau_2,\tau_3\}$, defined at a point $(t, e^{i\varphi_1}, e^{i\varphi_2}) \in \mathbb{S}^3$ as

$$\tau_1 := \frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\,, \quad \tau_2 := \frac{\partial}{\partial t}\,, \quad \tau_3 := \cot t\,\frac{\partial}{\partial\varphi_1} - \tan t\,\frac{\partial}{\partial\varphi_2}\,, \tag{2.19}$$

so that τ_1 is the *fundamental vertical vector field*, *i.e.*, the generating vector field of the $\mathbb{S}^1$-action $\mathbb{S}^1\times\mathbb{S}^3 \ni (e^{it},(u,v)) \to (e^{it}u, e^{it}v) \in \mathbb{S}^3$. With this choice,

$$dh(\tau_1) = 0\,, \quad dh(\tau_2) = \frac{\partial}{\partial t}\,, \quad dh(\tau_3) = \frac{2}{\sin 2t}\frac{\partial}{\partial\varphi}\,, \tag{2.20}$$

where $\varphi := \varphi_1 - \varphi_2$. In particular, $\{f_1, f_2\} := \left\{\frac{1}{2}\frac{\partial}{\partial t}, \frac{1}{\sin 2t}\frac{\partial}{\partial\varphi}\right\}$ is an orthonormal frame on $\mathbb{S}^2$, and in these coordinates the differential of the Hopf map $dh\colon T\mathbb{S}^3 \to T\mathbb{S}^2$ has matrix representation

$$dh = \begin{pmatrix} 0 & 2 & 0\\ 0 & 0 & 2 \end{pmatrix}\,. \tag{2.21}$$

As a consequence of (2.21),(2.16) and (2.17), we obtain:

Lemma 2.6. *The Hopf map $h:\mathbb{S}^3\to\mathbb{S}^2$ is a smooth horizontally conformal map of constant dilation factor, namely*

$$|dh| = 2\sqrt{2} \quad \text{and} \quad |h^*\omega_{\mathbb{S}^2}| = 4\,, \tag{2.22}$$

and is of unit charge, i.e., $Q(h) = 1$.

3. The Hopf invariant for maps of finite Faddeev–Skyrme energy

In this section we prove some results concerning the definition and integrality of the Hopf invariant for maps with finite Faddeev–Skyrme energy, cf. (1.2). The integrality property of the Hopf invariant in this class was correctly claimed in [22, Section 3] but, as explained in [2, §2.1.4], there is a gap in the argument. In [2, Corollary 5.2] a complete proof of integrality was given, but the proof therein is fairly long and not self-contained, as it relies on the earlier results by the authors in [1]. Here we present a new, more concise proof of this integrality which, although also not self-contained, we believe to be fairly transparent, and which contains ingredients which will also be important in the sequel.

In order to discuss how the notion of *Hopf invariant* generalizes to maps with finite Faddeev–Skyrme energy, we first recall its definition and basic properties in the smooth setting, so let $u \in C^\infty(\mathbb{S}^3;\mathbb{S}^2)$. Since the volume-form $\omega_{\mathbb{S}^2}$ is closed and the exterior differential commutes with the pull-back, we have

$$d(u^*\omega_{\mathbb{S}^2}) = u^*(d\omega_{\mathbb{S}^2}) = 0\,,$$

and in view of the fact that $H^2_{\mathrm{dR}}(\mathbb{S}^3)=0$ there exists $\beta\in\Omega^1(\mathbb{S}^3)$ such that $u^*\omega_{\mathbb{S}^2}=d\beta$. The *Hopf invariant* or *Hopf charge* of u, see [34], is then given by the integral formula (1.3), and in particular $Q(u)\in\mathbb{Z}$.

Remark 3.1. The definition (1.3) can easily be seen to be independent of the choice of the representative β. Indeed, if $\beta_1,\beta_2\in\Omega^1(\mathbb{S}^3)$ are such that $d\beta_1=d\beta_2=u^*\omega_{\mathbb{S}^2}$, using the fact that $H^1_{\mathrm{dR}}(\mathbb{S}^3)=0$ there exists $\psi\in C^\infty(\mathbb{S}^3)$ such that $\beta_1-\beta_2=d\psi$, and then the assertion follows from the identity

$$\beta_1\wedge d\beta_1=\beta_2\wedge d\beta_2+d(\psi d\beta_2)\,,$$

and Stokes' theorem. Moreover, the definition is also independent of the choice of the particular 2-form on $\mathbb{S}^2$, *i.e.*, $\omega_{\mathbb{S}^2}$ can be replaced therein with any $\omega\in\Omega^2(\mathbb{S}^2)$ for which $\frac{1}{4\pi}\int_{\mathbb{S}^2}\omega=1$.

In order to generalize these results to maps with finite Faddeev–Skyrme energy, we will consider the function spaces

$$\mathcal{A}_3(\mathbb{S}^3):=\left\{v\in W^{1,2}(\mathbb{S}^3;\mathbb{S}^3)\colon \operatorname{cof}(dv)\in L^{3/2}(\mathbb{S}^3)\right\}, \tag{3.1}$$

where $\operatorname{cof}(dv)$ is identified with the pull-back map $v^*\colon\Omega^2(\mathbb{S}^3)\to\Omega^2(\mathbb{S}^3)$. These function spaces were introduced in the context of nonlinear elasticity theory in [4] and appear naturally in our problem, since we have the following generalization of the results in [10, 26, 32], due to [13, Theorems 1.3 and 4.4]:

Proposition 3.2. *If $v\in\mathcal{A}_3(\mathbb{S}^3)$, then $v^*d=dv^*\colon\Omega^\ell(\mathbb{S}^3)\to\Omega^{\ell+1}(\mathbb{S}^3)$ for every $\ell\in\{0,1,2\}$. Moreover, one can define*

$$\deg(v):=\fint_{\mathbb{S}^3}v^*\omega_{\mathbb{S}^3}\,, \tag{3.2}$$

where $\omega_{\mathbb{S}^3}$ is the volume-form on $\mathbb{S}^3$, and it holds that $\deg(v)\in\mathbb{Z}$.

We emphasize that all known proofs of Proposition 3.2 yield the result *directly* for maps in $\mathcal{A}_3(\mathbb{S}^3)$, *i.e.*, the conclusion does not follow by approximation of maps in $\mathcal{A}_3(\mathbb{S}^3)$ with smooth maps, see already Remark 3.8 below.

With these notations, and recalling the definition of $\mathcal{U}_{\mathcal{FS}}$ in (1.2), the main result of this section is the following:

Theorem 3.3. *Let $u\in\mathcal{U}_{\mathcal{FS}}$. Then $u^*\omega_{\mathbb{S}^2}$ is an exact form, that is, there exists a 1-form $\beta\in W^{1,2}(\mathbb{S}^3;\bigwedge^1T^*\mathbb{S}^3)$ so that*

$$u^*\omega_{\mathbb{S}^2}=d\beta\,,\quad d^*\beta=0\,. \tag{3.3}$$

Moreover, there exists a lift $\hat u\in\mathcal{A}_3(\mathbb{S}^3)$ such that $u=h\circ\hat u$ and

$$Q(u)=\frac{1}{16\pi^2}\int_{\mathbb{S}^3}\beta\wedge d\beta=\deg(\hat u)\in\mathbb{Z}\,. \tag{3.4}$$

In addition to Proposition 3.2, the following result due to [5] is the key ingredient in our proof of Theorem 3.3.

Theorem 3.4. *A map $u\in W^{1,2}(\mathbb{S}^3;\mathbb{S}^2)$ is in the strong $W^{1,2}$-closure of $C^\infty(\mathbb{S}^3;\mathbb{S}^2)$ if and only if $u^*\omega_{\mathbb{S}^2}$ is distributionally closed.*

Proof. By [5, Theorem 1], a map $u\colon\mathbb{S}^3\to\mathbb{S}^2$ is $W^{1,2}$-strongly approximable by smooth maps if and only if it pulls-back (distributionally) closed 2-forms on $\mathbb{S}^2$ to (distributionally) closed 2-forms on $\mathbb{S}^3$. Since any form in $\Omega^2(\mathbb{S}^2)$ can be represented as $f\omega_{\mathbb{S}^2}$ for some $f\in C^\infty(\mathbb{S}^2)$, it is then necessary and sufficient to check closedness of $u^*\omega_{\mathbb{S}^2}$. ☐

In order to apply Theorem 3.4, we will use the following:

Lemma 3.5. *If $u \in \mathcal{U}_{\mathcal{FS}}$, then $u^*\omega_{\mathbb{S}^2}$ is a distributionally closed 2-form.*

Proof. This is detailed in [2, Lemma 3.2]. Fix $\{j,k,\ell\} \subset \{1,2,3\}$. Thanks to classical approximation by (localisation and) mollification, we see that $u^k, u^\ell \in W^{1,2}(\mathbb{S}^3;\mathbb{R})$ satisfy

$$d(du^k \wedge du^\ell) = 0\,,$$

hence the 2-form $\alpha^{k\ell} :\, = du^k \wedge du^\ell$ is distributionally closed, and using again approximation by mollification of $u^j \in W^{1,2}(\mathbb{S}^3;\mathbb{R})$ and of the closed form $\alpha^{k\ell} \in L^2(\mathbb{S}^3; \bigwedge^2 T^*\mathbb{S}^3)$, we infer that

$$d(u^j \wedge \alpha^{k\ell}) = du^j \wedge \alpha^{k\ell}\,.$$

Using this and (2.3), we obtain

$$d(u^*\omega_{\mathbb{S}^2}) = 3\, du^1 \wedge du^2 \wedge du^3 \in L^1\left(\mathbb{S}^3; \bigwedge^3 T^*\mathbb{S}^3\right),$$

in the sense of distributions.

Since $u(x) \in \mathbb{S}^2$ for $\mathcal{H}^3$-a.e. $x \in \mathbb{S}^3$, the vectors $(du^i(x))_{i=1}^3$ must be linearly dependent, hence the last wedge product vanishes. □

The next lemma, due to [14], gives us a gauge for the lifts of u.

Lemma 3.6. *Let $u \in \mathcal{U}_{\mathcal{FS}}$ and $\beta \in W^{1,2}(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3)$ as in* (3.3)*. Suppose that there is a lift $\hat{u} \in \mathcal{A}_3(\mathbb{S}^3)$ such that*

$$h \circ \hat{u} = u \quad \textit{and} \quad \beta = 2\hat{u}^*\theta\,, \tag{3.5}$$

where, in the notation of (2.15) *and* (2.16)*,*

$$h^*\omega_{\mathbb{S}^2} = 2d\theta\,. \tag{3.6}$$

Then $\mathcal{H}^3$-a.e. on $\mathbb{S}^3$ it holds that

$$|d\hat{u}|^2 = \frac{1}{4}|\beta|^2 + \frac{1}{4}|du|^2\,. \tag{3.7}$$

Moreover, if $u \in C^\infty(\mathbb{S}^3;\mathbb{S}^2)$ then such a lift exists and is unique up to multiplication by a constant in $\mathbb{S}^1$ (that is, the action of $\mathbb{S}^1$ on $\mathbb{S}^3$ given in § 2.3 by componentwise multiplication in $\mathbb{C}^2$).

Proof. The proof in the case of a smooth map u can be found in [14, Lemma 2.1], and otherwise the conclusion of (3.7) can be repeated analogously, since the calculations therein can be carried over pointwise $\mathcal{H}^3$-a.e. on $\mathbb{S}^3$. The coefficients in (3.7) are actually different in [14] due to a different choice of normalization for the metric on $\mathbb{S}^2$ (which also changes the normalization constant in the integral formula (1.3) for the Hopf degree). For the readers' convenience we reproduce the proof of (3.7) with the metric used here. In the orthonormal frame $\{\tau_1,\tau_2,\tau_3\}$ on $T\mathbb{S}^3$, the 1-form θ corresponds to the scalar product against τ_1. Thus, the condition $\beta = 2\hat{u}^*\theta$ and (2.18)–(2.19), imply

$$\beta = 2\tau_1(\hat{u}) \cdot d\hat{u} \implies |\beta|^2 = 4|\tau_1 \cdot d\hat{u}|^2\,. \tag{3.8}$$

Moreover, recalling the expression (2.21) of dh in this orthonormal frame, the chain rule gives

$$du = dh \circ d\hat{u} = 2(\tau_2 \cdot d\hat{u})f_1 + 2(\tau_3 \cdot d\hat{u})f_2 \implies |du|^2 = 4|\tau_2 \cdot d\hat{u}|^2 + 4|\tau_3 \cdot d\hat{u}|^2\,, \tag{3.9}$$

which, together with (3.8) directly implies

$$4|d\hat{u}|^2 = |\beta|^2 + |du|^2\,,$$

as wished. □

Proof of Theorem 3.3. Using Theorem 3.4 and Lemma 3.5 we can find a sequence of maps $(u_j)_{j\in\mathbb{N}} \subset C^\infty(\mathbb{S}^3;\mathbb{S}^2)$ such that

$$u_j \to u \text{ strongly in } W^{1,2}(\mathbb{S}^3;\mathbb{S}^2) \text{ as } j\to\infty\,. \tag{3.10}$$

We can also fix forms $(\beta_j)_{j\in\mathbb{N}} \subset C^\infty(\mathbb{S}^3;\bigwedge^1 T^*\mathbb{S}^3)$ such that

$$d\beta_j = u_j^*\omega_{\mathbb{S}^2}\,, \qquad d^*\beta_j = 0\,, \qquad \fint_{\mathbb{S}^3}\beta_j = 0\,. \tag{3.11}$$

The last condition can be assumed without restriction after possibly subtracting a constant coefficient 1-form. By standard elliptic estimates we have for instance the bound

$$\|\beta_j\|_{L^{5/4}(\mathbb{S}^3)} \le C\|u_j^*\omega_{\mathbb{S}^2}\|_{L^1(\mathbb{S}^3)}\,, \tag{3.12}$$

since $\frac{5}{4} < \frac{3}{2} =: 1^*$. We note that the borderline $L^{3/2}$-estimate is actually also true, see *e.g.* [20] or [6, Corollary 12], at least in $\mathbb{R}^3$ instead of $\mathbb{S}^3$. Invoking Lemma 3.6, we now fix lifts $(\hat{u}_j)_{j\in\mathbb{N}} \subset C^\infty(\mathbb{S}^3;\mathbb{S}^3)$ satisfying

$$h\circ\hat{u}_j = u_j \ \text{ and } \beta_j = 2\hat{u}_j^*\theta\,, \tag{3.13}$$

cf. (3.5). In view of (2.14) and (3.10), by passing to a non-relabelled subsequence, we obtain $\beta \in L^{5/4}\big(\mathbb{S}^3;\bigwedge^1 T^*\mathbb{S}^3\big)$ so that

$$\beta_j \rightharpoonup \beta \text{ weakly in } L^{5/4}\Big(\mathbb{S}^3;\bigwedge^1 T^*\mathbb{S}^3\Big) \text{ as } j\to\infty\,, \tag{3.14}$$

and since convergence of averages is preserved under weak convergence, also

$$\fint_{\mathbb{S}^3}\beta = 0\,. \tag{3.15}$$

By (3.7) applied to the triplet $(\hat{u}_j,\beta_j,u_j)$, together with (3.10) and (3.12), we have

$$\sup_{j\in\mathbb{N}}\|\hat{u}_j\|_{W^{1,5/4}(\mathbb{S}^3;\mathbb{S}^3)} < +\infty\,,$$

and passing to a further non-relabeled subsequence, we obtain $\hat{u} \in W^{1,5/4}(\mathbb{S}^3;\mathbb{S}^3)$ so that

$$\hat{u}_j \rightharpoonup \hat{u} \text{ weakly in } W^{1,5/4}\big(\mathbb{S}^3;\mathbb{S}^3\big) \text{ as } j\to\infty\,. \tag{3.16}$$

Using now (3.14) and (3.16), we can pass to the limit in (3.13), to infer that

$$h\circ\hat{u} = u \quad \mathcal{H}^3\text{-a.e.}\,, \qquad \beta = 2\hat{u}^*\theta\,. \tag{3.17}$$

Indeed, in view of (2.15), the forms $\hat{u}_j^*\theta$ can be written in coordinates as second-order polynomials of the form $\hat{u}_j \bullet d\hat{u}_j$, where the latter notation indicates that each monomial in the expression is of first order separately in $\hat{u}_j$ and $d\hat{u}_j$. In particular, since $\hat{u}_j \to \hat{u}$ strongly in $L^p(\mathbb{S}^3;\mathbb{S}^2)$ for every $1 \le p \le 6$ and $d\hat{u}_j \rightharpoonup d\hat{u}$ weakly in $L^{5/4}(\mathbb{S}^3;\bigwedge^1 T^*\mathbb{S}^3)$, this particular product structure of weakly convergent objects and strongly convergent ones justifies (3.17).

By passing to the limit in L^1 in (3.11), using that (2.3) and (3.10) imply

$$\|u_j^*\omega_{\mathbb{S}^2} - u^*\omega_{\mathbb{S}^2}\|_{L^1(\mathbb{S}^2)} \to 0 \ \text{ as } j\to\infty\,,$$

together with (3.14), we also obtain

$$d\beta = u^*\omega_{\mathbb{S}^2}\,, \quad d^*\beta = 0\,. \tag{3.18}$$

Note however that $u \in \mathcal{U}_{\mathcal{FS}}$, which recalling (1.2) implies that $u^*\omega_{\mathbb{S}^2} \in L^2\big(\mathbb{S}^3; \bigwedge^2 T^*\mathbb{S}^3\big)$. Thus, (3.18), Gaffney's inequality (2.7) and (3.15) actually imply that $\beta \in W^{1,2}\big(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3\big)$. The last assertion and (3.7) in turn imply that $\hat{u} \in W^{1,2}(\mathbb{S}^3; \mathbb{S}^3)$. To conclude the proof it remains to verify that $\hat{u} \in \mathcal{A}_3(\mathbb{S}^3)$ and that (3.4) holds true.

As in the proof of (3.7), recall (3.8)-(3.9), the identities $du = dh \circ d\hat{u}$ and $\beta = 2\hat{u}^*\theta$ provide explicit expressions for the components of $d\hat{u}$ in terms of du and β: in the orthonormal frame $\{\tau_1, \tau_2, \tau_3\}$ on $T\mathbb{S}^3$ given by (2.19), the matrix of $d\hat{u}$ is given by

$$d\hat{u} = \frac{1}{2}\begin{bmatrix} \beta(\tau_1) & \beta(\tau_2) & \beta(\tau_3) \\ du(\tau_1)\cdot f_1 & du(\tau_2)\cdot f_1 & du(\tau_3)\cdot f_1 \\ du(\tau_1)\cdot f_2 & du(\tau_2)\cdot f_2 & du(\tau_3)\cdot f_2 \end{bmatrix} = \frac{1}{2}\begin{bmatrix} \beta \\ du\cdot f_1 \\ du\cdot f_2 \end{bmatrix},$$

where $\{f_2, f_2\}$ is the orthonormal frame on $T\mathbb{S}^2$ defined after (2.20). Thus, using also (2.4), we have

$$\begin{aligned} 16|\mathrm{cof}(d\hat{u})|^2 &= 4|d\hat{u}\wedge d\hat{u}|^2 \\ &= |\beta\wedge(du\cdot f_1)|^2 + |\beta\wedge(du\cdot f_2)|^2 + |(du\cdot f_1)\wedge(du\cdot f_2)|^2 \\ &= |\beta\wedge du|^2 + \frac{1}{4}|du\wedge du|^2 . \end{aligned}$$

Since $u \in \mathcal{U}_{\mathcal{FS}}$, hence $u^*\omega_{\mathbb{S}^2} \in L^2$, the last term is in L^1, cf. (2.4). For the other term, we note that by (3.18), (3.15), (2.7) and the Sobolev embedding we obtain $\beta \in L^6\big(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3\big)$, thus by Hölder's inequality $\beta\wedge du \in L^{3/2}(\mathbb{S}^3; \bigwedge^2 T^*\mathbb{S}^3)$, and we deduce that $\mathrm{cof}(d\hat{u}) \in L^{3/2}(\mathbb{S}^3)$.

Since $\hat{u} \in \mathcal{A}_3(\mathbb{S}^3)$, we can apply Proposition 3.2 to infer that $\deg(\hat{u})$ is a well-defined integer and that $\hat{u}^*$ and d commute. Hence, recalling (2.16), we have

$$u^*\omega_{\mathbb{S}^2} = \hat{u}^*h^*\omega_{\mathbb{S}^2} = \hat{u}^*(2d\theta) = d(2\hat{u}^*\theta).$$

Thus, by (2.17) and the fact that $\mathcal{H}^3(\mathbb{S}^3) = 2\pi^2$, we obtain

$$\begin{aligned} Q(u) &= \frac{1}{16\pi^2}\int_{\mathbb{S}^3}(2\hat{u}^*\theta)\wedge d(2\hat{u}^*\theta) \\ &= \frac{1}{4\pi^2}\int_{\mathbb{S}^3}\hat{u}^*\theta\wedge\hat{u}^*(d\theta) = \frac{1}{4\pi^2}\int_{\mathbb{S}^3}\hat{u}^*(\theta\wedge d\theta) = \frac{1}{2\pi^2}\int_{\mathbb{S}^3}\hat{u}^*\omega_{\mathbb{S}^3} = \deg(\hat{u}), \end{aligned}$$

which concludes the proof. □

Corollary 3.7. *If $u \in \mathcal{U}_{\mathcal{FS}}$, then $du^* = u^*d\colon \Omega^1(\mathbb{S}^2) \to \Omega^2(\mathbb{S}^3)$.*

Proof. This property follows immediately from Theorem 3.3, in particular the lifting identity $u = h\circ\hat{u}$, Proposition 3.2, and the smoothness of h, which also implies that $dh^* = h^*d$. □

Remark 3.8. There is a serious obstruction to having a more direct proof of Theorem 3.3, namely that it is not known whether one can approximate in the Faddeev–Skyrme energy a general map in $\mathcal{U}_{\mathcal{FS}}$ by smooth maps. This is not known even if one ignores the image constraint on maps in $\mathcal{U}_{\mathcal{FS}}$, and we refer the reader to [12] for some partial results in the unconstrained case. We emphasize that from the above proof one cannot infer that the Hopf invariant of u is obtained as the limit of the Hopf invariants of $(u_j)_{j\in\mathbb{N}}$, as in (3.10), since the latter sequence converges to u only in too weak a sense for this purpose (and similarly for the liftings).

We conclude this section with some easy but helpful consequences of Theorem 3.3. The first one is that for maps of finite Faddeev–Skyrme energy the integral definition of the Hopf invariant is independent of the particular choice of the 2-form on $\mathbb{S}^2$, as long as its induced oriented area is fixed.

Lemma 3.9. *If $\omega_1, \omega_2 \in L^2(\mathbb{S}^2; \bigwedge^2 T^*\mathbb{S}^2)$ are such that*

$$\int_{\mathbb{S}^2} \omega_1 = \int_{\mathbb{S}^2} \omega_2 \tag{3.19}$$

and $v \in \mathcal{U}_{\mathcal{FS}}$ (cf. (1.2)) is such that $v^*\omega_1, v^*\omega_2 \in L^2(\mathbb{S}^3; \bigwedge^2 T^*\mathbb{S}^3)$, then

$$\int_{\mathbb{S}^3} \beta_1 \wedge d\beta_1 = \int_{\mathbb{S}^3} \beta_2 \wedge d\beta_2 \,, \tag{3.20}$$

for any $\beta_1, \beta_2 \in W^{1,2}(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3)$ satisfying $d\beta_i = v^*\omega_i$ for $i = 1, 2$.

Proof. Since the map $H^2_{\mathrm{dR}}(\mathbb{S}^2) \ni [\omega] \mapsto \int_{\mathbb{S}^2} \omega \in \mathbb{R}$ is a linear isomorphism, our assumption in particular implies that

$$[\omega_1] = [\omega_2] \in H^2_{\mathrm{dR}}(\mathbb{S}^2) \,,$$

thus there exists $\eta \in W^{1,2}(\mathbb{S}^2; \bigwedge^1 T^*\mathbb{S}^2)$ such that

$$\omega_1 = \omega_2 + d\eta \,. \tag{3.21}$$

Using Corollary 3.7, we can also calculate

$$\begin{aligned} d(\beta_1 - \beta_2 - v^*\eta) &= d\beta_1 - d\beta_2 - dv^*\eta = v^*\omega_1 - v^*\omega_2 - v^*d\eta \\ &= v^*\omega_1 - v^*\omega_2 - v^*(\omega_1 - \omega_2) = 0 \,, \end{aligned}$$

and since $H^1_{\mathrm{dR}}(\mathbb{S}^3) = 0$, for the 1-form $\beta_1 - \beta_2 - v^*\eta$ we can find $\gamma \in W^{1,2}(\mathbb{S}^3)$ such that

$$\beta_1 = (\beta_2 + v^*\eta) + d\gamma \,. \tag{3.22}$$

In view of (3.22), Remark 3.1 (which holds true even in this less regular setting) and (2.5), we have

$$\begin{aligned} \int_{\mathbb{S}^3} \beta_1 \wedge d\beta_1 &= \int_{\mathbb{S}^3} (\beta_2 + v^*\eta) \wedge \big(d\beta_2 + d(v^*\eta)\big) \\ &= \int_{\mathbb{S}^3} \beta_2 \wedge d\beta_2 + 2\int_{\mathbb{S}^3} d\beta_2 \wedge v^*\eta + \int_{\mathbb{S}^3} v^*\eta \wedge v^*d\eta \\ &= \int_{\mathbb{S}^3} \beta_2 \wedge d\beta_2 + 2\int_{\mathbb{S}^3} v^*(\omega_2 \wedge \eta) + \int_{\mathbb{S}^3} v^*(\eta \wedge d\eta) \\ &= \int_{\mathbb{S}^3} \beta_2 \wedge d\beta_2 \,, \end{aligned}$$

where in the last step we used the trivial fact that

$$\omega_2 \wedge \eta, \eta \wedge d\eta \in L^1\big(\mathbb{S}^2; \bigwedge^3 T^*\mathbb{S}^2\big) = \{0\} \,,$$

hence the integrands in the last two summands vanish identically. This proves (3.20). □

As an immediate consequence of the previous lemma, we obtain the following *degree formula*, for which we recall that for a map $\psi \in W^{1,2}(\mathbb{S}^2; \mathbb{S}^2)$ its *degree* is defined, analogously to (3.2), as the integral

$$\deg(\psi) := \fint_{\mathbb{S}^2} \psi^*\omega_{\mathbb{S}^2} \in \mathbb{Z} \,, \tag{3.23}$$

which in particular is continuous with respect to the strong-$W^{1,2}(\mathbb{S}^2; \mathbb{S}^2)$ topology.

Lemma 3.10. *Let $u \in \mathcal{U}_{\mathcal{FS}}$ and $\psi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ be such that $\psi^*\omega_{\mathbb{S}^2} \in L^2\big(\mathbb{S}^2;\bigwedge^2 T^*\mathbb{S}^2\big)$ and $\psi \circ u \in \mathcal{U}_{\mathcal{FS}}$. Then*

$$Q(\psi \circ u) = \deg(\psi)^2\, Q(u)\,. \tag{3.24}$$

Proof. By (1.3) we have that

$$16\pi^2\, Q(\psi \circ u) = \int_{\mathbb{S}^3} \beta_1 \wedge d\beta_1\,, \text{ where } d\beta_1 = (\psi \circ u)^*\omega_{\mathbb{S}^2} = u^*(\psi^*\omega_{\mathbb{S}^2}) = u^*\omega_1\,, \tag{3.25}$$

where $\omega_1 := \psi^*\omega_{\mathbb{S}^2} \in L^2(\mathbb{S}^2;\bigwedge^2 T^*\mathbb{S}^2)$. By (3.23) we observe that

$$\int_{\mathbb{S}^2} \omega_1 = \int_{\mathbb{S}^2} \psi^*\omega_{\mathbb{S}^2} = 4\pi \deg(\psi)\,.$$

Thus, setting $\omega_2 := \deg(\psi)\,\omega_{\mathbb{S}^2} \in \Omega^2(\mathbb{S}^2)$, we automatically have that

$$\int_{\mathbb{S}^2} \omega_1 = \int_{\mathbb{S}^2} \omega_2\,. \tag{3.26}$$

Let now $\beta \in W^{1,2}\big(\mathbb{S}^3;\bigwedge^1 T^*\mathbb{S}^3\big)$ be such that $d\beta = u^*\omega_{\mathbb{S}^2}$. Setting $\beta_2 := \deg(\psi)\,\beta$, we have

$$d\beta_2 = \deg(\psi)\, u^*\omega_{\mathbb{S}^2} = u^*(\deg(\psi)\,\omega_{\mathbb{S}^2}) = u^*\omega_2\,. \tag{3.27}$$

Thus, by (3.25)-(3.27) and Lemma 3.9, we infer that

$$16\pi^2\, Q(\psi \circ u) = \int_{\mathbb{S}^3} \beta_1 \wedge d\beta_1 = \int_{\mathbb{S}^3} \beta_2 \wedge d\beta_2 = \deg(\psi)^2 \int_{\mathbb{S}^3} \beta \wedge d\beta = 16\pi^2 \deg(\psi)^2\, Q(u)\,,$$

which shows (3.24). □

4. Relaxed form of the perturbed Faddeev–Skyrme energy

In order to prove Theorem 1.1, and following the approach in [17, 27], we consider a relaxation of $\mathcal{FS}_\rho$ to the space of closed differential forms. Let us first define, for a closed 2-form α on $\mathbb{S}^3$,

$$Q(\alpha) := \frac{1}{16\pi^2} \int_{\mathbb{S}^3} \beta \wedge d\beta\,, \quad \alpha = d\beta\,. \tag{4.1}$$

By Stokes' Theorem this quantity, referred to as the *Hopf invariant of the form* α, is well-defined, *i.e.*, Q depends on β only through $d\beta$, see also the corresponding discussion in Remark 3.1 for the Hopf invariant of a map u. Introducing the space of admissible forms

$$\mathcal{A}_{\mathcal{FS}} := \left\{\alpha \in L^2\Big(\mathbb{S}^3;\bigwedge^2 T^*\mathbb{S}^3\Big) \colon d\alpha = 0\,, Q(\alpha) > 0\right\}\,, \tag{4.2}$$

we consider the relaxed energy $\mathcal{E}_\rho \colon \mathcal{A}_{\mathcal{FS}} \to [0,+\infty)$, defined by

$$\mathcal{E}_\rho(\alpha) := \frac{2}{Q(\alpha)^{\frac{1}{2}}} \int_{\mathbb{S}^3} |\alpha| + \frac{\rho^{-2}}{Q(\alpha)} \int_{\mathbb{S}^3} |\alpha|^2\,. \tag{4.3}$$

As an immediate consequence of the definitions of the energies and Lemma 2.5, one obtains:

Corollary 4.1. *For every $\rho > 0$ and every $u \in \mathcal{U}_{\mathcal{FS}}$ it holds that*

$$\mathcal{FS}_\rho(u) \geq \mathcal{E}_\rho(u^*\omega)\,, \tag{4.4}$$

with equality if and only if u is horizontally weakly conformal.

Recalling Proposition 2.2, we further introduce the following notation:

$$E_{0,1}^{+} := \{\alpha \in E_0^{+} \colon Q(\alpha) = 1\}\,. \tag{4.5}$$

Using (2.11), it is straightforward to verify that

$$E_{0,1}^{+} = \left\{c_1\omega_{0,1}^{+} + c_2\omega_{0,2}^{+} + c_3\omega_{0,3}^{+} \colon (c_1,c_2,c_3) \in \mathbb{R}^3 \text{ with } c_1^2+c_2^2+c_3^2 = 16\right\}. \tag{4.6}$$

An important property of the relaxed energy $\mathcal{E}_\rho$ is its *local L^2-coercivity* at $E_{0,1}^{+}$, as asserted in the following proposition, which is similar to [17, Lemma 3.2] but is valid in an L^2-neighborhood of $E_{0,1}^{+}$, instead of a C^0-neighborhood as in [17].

Proposition 4.2. *For every $\rho_0 \in (0,\sqrt{2})$ there exist $\varepsilon_0 := \varepsilon_0(\rho_0) > 0$ and $c := c(\rho_0) > 0$, such that the following statement holds. If $\rho \in (0,\rho_0]$ and $\alpha \in \mathcal{A}_{\mathcal{FS}}$ is such that $Q(\alpha) = 1$ and*

$$\|\alpha - \alpha_{0,1}^{+}\|_{L^2(\mathbb{S}^3)} = \operatorname{dist}_{L^2}(\alpha, E_{0,1}^{+}) \le \varepsilon_0\,, \tag{4.7}$$

where $\alpha_{0,1}^{+} \in E_{0,1}^{+}$, then

$$\mathcal{E}_\rho(\alpha) - \mathcal{E}_\rho(\alpha_{0,1}^{+}) \ge c\big\|\alpha - \alpha_{0,1}^{+}\big\|_{L^2(\mathbb{S}^3)}^2\,. \tag{4.8}$$

The rest of the section is devoted to the proof of Proposition 4.2. Before doing so, we present some auxiliary lemmata that will be essential for the proof. For $q = 1, 2$, let us introduce the auxiliary functionals $\mathcal{I}_q : \mathcal{A}_{\mathcal{FS}} \to [0,+\infty)$, defined as

$$\mathcal{I}_q(\alpha) := \int_{\mathbb{S}^3} |\alpha|^q\,. \tag{4.9}$$

Observe that for $\alpha_{0,1}^{+} \in E_{0,1}^{+}$, setting

$$\beta_0 := \frac{1}{2}(*\alpha_{0,1}^{+}), \text{ we have } d\beta_0 = \alpha_{0,1}^{+}\,. \tag{4.10}$$

The normalization condition for the Hopf invariant in (4.5), together with the facts that $\alpha_{0,1}^{+}$ is a constant coefficient 2-form (see (4.6) and Remark 2.3), that

$$\beta_0 \wedge d\beta_0 = \tfrac{1}{2}(*\alpha_{0,1}^{+}) \wedge \alpha_{0,1}^{+} = \tfrac{1}{2}|\alpha_{0,1}^{+}|^2\,,$$

and that $\mathcal{H}^3(\mathbb{S}^3) = 2\pi^2$, imply that

$$|\alpha_{0,1}^{+}| = 4\,, \quad \text{hence} \quad \mathcal{I}_q(\alpha_{0,1}^{+}) = 2\pi^2 4^q \quad \text{for every } \alpha_{0,1}^{+} \in E_{0,1}^{+}\,. \tag{4.11}$$

The next lemma gives us a precise expansion for the energy $\mathcal{E}_\rho$ around elements of that first eigenspace.

Lemma 4.3. *Given $\varphi \in W^{1,2}(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3)$, let $\alpha := \alpha_{0,1}^{+} + d\varphi$, where $\alpha_{0,1}^{+} \in E_{0,1}^{+}$. Then,*

$$\begin{aligned}\mathcal{E}_\rho(\alpha) - \mathcal{E}_\rho(\alpha_{0,1}^{+}) = \rho^{-2}\left(\int_{\mathbb{S}^3} |d\varphi|^2 - 2\int_{\mathbb{S}^3} \varphi \wedge d\varphi\right) - \frac{1}{2}\int_{\mathbb{S}^3} \varphi \wedge d\varphi + \frac{1}{128\pi^2}\left(\int_{\mathbb{S}^3} \langle \alpha_{0,1}^{+}, d\varphi\rangle\right)^2 \\ + \left(2 - \frac{1}{16\pi^2}\int_{\mathbb{S}^3} \langle \alpha_{0,1}^{+}, d\varphi\rangle\right)\mathcal{G}_1(\alpha_{0,1}^{+}, d\varphi) + \big(2+\rho^{-2}\big)\mathcal{O}\left(\|d\varphi\|_{L^2(\mathbb{S}^3)}^3\right),\end{aligned} \tag{4.12}$$

where we set

$$\mathcal{G}_1(\alpha_{0,1}^{+}, d\varphi) := \int_{\mathbb{S}^3} \left\{|\alpha_{0,1}^{+} + d\varphi| - |\alpha_{0,1}^{+}| - \left\langle \frac{\alpha_{0,1}^{+}}{|\alpha_{0,1}^{+}|}, d\varphi\right\rangle\right\} \ge 0\,. \tag{4.13}$$

Proof. Since $H^2_{\mathrm{dR}}(\mathbb{S}^3) = 0$, by the L^2-Hodge decomposition we can assume that φ satisfies

$$d^*\varphi = 0\,. \tag{4.14}$$

Using the notation in (4.9), the relaxed energy in (4.3) can be written as

$$\mathcal{E}_\rho(\alpha) = 2Q(\alpha)^{-1/2}\mathcal{I}_1(\alpha) + \rho^{-2}Q(\alpha)^{-1}\mathcal{I}_2(\alpha)\,. \tag{4.15}$$

We first expand the three quantities appearing in (4.15) separately, obtaining:

$$\begin{aligned}
&\text{(i)} \quad \mathcal{I}_1(\alpha) = \mathcal{I}_1(\alpha^+_{0,1}) + \mathcal{I}_1'(\alpha^+_{0,1})[d\varphi] + \mathcal{G}_1(\alpha^+_{0,1}, d\varphi)\,,\\
&\text{(ii)} \quad \mathcal{I}_2(\alpha) = \mathcal{I}_2(\alpha^+_{0,1}) + \mathcal{I}_2'(\alpha^+_{0,1})[d\varphi] + \int_{\mathbb{S}^3} |d\varphi|^2\,,\\
&\text{(iii)} \quad Q(\alpha)^{-q/2} = \left(1 + Q'(\alpha^+_{0,1})[d\varphi] + \frac{1}{16\pi^2}\int_{\mathbb{S}^3}\varphi\wedge d\varphi\right)^{-q/2}\\
&\qquad = 1 - \frac{q}{2}Q'(\alpha^+_{0,1})[d\varphi] - \frac{q}{32\pi^2}\int_{\mathbb{S}^3}\varphi\wedge d\varphi + \frac{q(q+2)}{8}\big(Q'(\alpha^+_{0,1})[d\varphi]\big)^2\\
&\qquad\quad + \mathcal{O}(\|d\varphi\|^3_{L^2(\mathbb{S}^3)})\,,
\end{aligned} \tag{4.16}$$

where we have used the standard notation

$$\mathcal{I}_1'(\alpha^+_{0,1})[d\varphi] := \frac{d}{dt}\Big|_{t=0}\mathcal{I}_1(\alpha^+_{0,1} + td\varphi)\,,$$

and similarly for $\mathcal{I}_2'(\alpha^+_{0,1})[d\varphi]$ and $Q'(\alpha^+_{0,1})[d\varphi]$. Note that (4.16)(i) follows simply by the definition of $\mathcal{G}_1$ in (4.13), and (4.16)(ii) by a simple expansion of the quadratic energy $\mathcal{I}_2$. For the first line in the third expansion we have used (4.10), the integral formula (4.1), together with $Q(\alpha^+_{0,1}) = 1$, cf. the definition of $E^+_{0,1}$ in (4.5). Finally, for the last line in (4.16)(iii) we used a Taylor expansion for $t \mapsto (1+t)^{-q/2}$ around 0, together with the estimate

$$\Big|\int_{\mathbb{S}^3}\varphi\wedge d\varphi\Big| = \mathcal{O}(\|d\varphi\|^2_{L^2(\mathbb{S}^3)})\,,$$

coming from Lemma 2.1(ii).

To proceed further we note that, by the definition of β_0 in (4.10) and Lemma 2.1(i) we have

$$16\pi^2 Q'(\alpha^+_{0,1})[d\varphi] = \int_{\mathbb{S}^3}\beta_0\wedge d\varphi + \int_{\mathbb{S}^3}\varphi\wedge d\beta_0 = 2\int_{\mathbb{S}^3}\beta_0\wedge d\varphi = \int_{\mathbb{S}^3}\langle\alpha^+_{0,1}, d\varphi\rangle\,. \tag{4.17}$$

Then, a simple calculation using (4.11) and (4.17) (with a general test form in the place of $d\varphi$) reveals that, for $q = 1, 2$,

$$\mathcal{I}_q'(\alpha^+_{0,1}) = \frac{q}{2}\mathcal{I}_q(\alpha^+_{0,1})Q'(\alpha^+_{0,1})\,; \tag{4.18}$$

this last equation, together with $Q(\alpha^+_{0,1}) = 1$, expresses the fact that $\alpha^+_{0,1}$ is a critical point of the functionals $\alpha \mapsto Q^{-q/2}(\alpha)\mathcal{I}_q(\alpha)$. We now analyze each of these two functionals separately: using (4.11), (4.16), (4.18) and the fact that by the definition in (4.13),

$$0 \le \mathcal{G}_1(\alpha, d\varphi) \le 2\|d\varphi\|_{L^1(\mathbb{S}^3)} \le 2\sqrt{2}\pi\|d\varphi\|_{L^2(\mathbb{S}^3)}\,,$$

we find that

$$\begin{aligned}
\text{(i)}\quad \frac{\mathcal{I}_1(\alpha)}{Q(\alpha)^{\frac{1}{2}}} - \mathcal{I}_1(\alpha_{0,1}^+) &= -\frac{1}{4}\int_{\mathbb{S}^3} \varphi\wedge d\varphi + \pi^2\big(Q'(\alpha_{0,1}^+)[d\varphi]\big)^2 \\
&\quad + \left(1 - \frac{1}{2}Q'(\alpha_{0,1}^+)[d\varphi]\right)\mathcal{G}_1(\alpha_{0,1}^+, d\varphi) + \mathcal{O}(\|d\varphi\|^3_{L^2(\mathbb{S}^3)})\,, \\
\text{(ii)}\quad \frac{\mathcal{I}_2(\alpha)}{Q(\alpha)} - \mathcal{I}_2(\alpha_{0,1}^+) &= \int_{\mathbb{S}^3} |d\varphi|^2 - 2\int_{\mathbb{S}^3}\varphi\wedge d\varphi + \mathcal{O}(\|d\varphi\|^3_{L^2(\mathbb{S}^3)})\,.
\end{aligned} \tag{4.19}$$

Collecting (4.3), the expansions (4.19)(i)-(ii), and using also (4.17), we find precisely (4.12). □

Proof of Proposition 4.2. Let

$$\alpha_{0,1}^+ \in \underset{\widetilde{\alpha}\in E_{0,1}^+}{\operatorname{argmin}} \|\alpha - \widetilde{\alpha}\|_{L^2(\mathbb{S}^3)}\,. \tag{4.20}$$

In view of (2.11) and (4.6), we can identify $E_0^+ \cong \mathbb{R}^3$ and $E_{0,1}^+ \cong \mathbb{S}^2_4$. Using (2.10) it is immediate that (4.20) also implies

$$\alpha_{0,1}^+ \in \underset{\widetilde{\alpha}\in E_{0,1}^+}{\operatorname{argmin}} \left\|P_{E_0^+}\alpha - \widetilde{\alpha}\right\|_{L^2(\mathbb{S}^3)}\,, \tag{4.21}$$

where $P_{E_0^+}\alpha$ is the L^2-orthogonal projection of α onto E_0^+. Thus, $(P_{E_0^+}\alpha - \alpha_{0,1}^+)$ is in the normal space to $E_{0,1}^+ \cong \mathbb{S}^2_4 \subset \mathbb{R}^3$ at $\alpha_{0,1}^+$, *i.e.*,

$$P_{E_0^+}\alpha = (1+t)\alpha_{0,1}^+ \ \text{ for some } t\in\mathbb{R}\,. \tag{4.22}$$

Actually, since $d(\alpha - \alpha_{0,1}^+) = 0$ and $H^2_{\mathrm{dR}}(\mathbb{S}^3) = 0$, there exists $\varphi \in W^{1,2}\big(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3\big)$ with

$$\alpha = \alpha_{0,1}^+ + d\varphi \quad\text{and}\quad d^*\varphi = 0\,. \tag{4.23}$$

Recalling (4.10), we can write $\alpha = \alpha_{0,1}^+ + d\varphi = (1+t)\alpha_{0,1}^+ + (d\varphi - t\,d\beta_0)$, or equivalently,

$$\frac{\alpha}{1+t} = \alpha_{0,1}^+ + d\psi\,, \quad \psi := \frac{\varphi - t\beta_0}{1+t}\,, \tag{4.24}$$

where we note that, by (4.22), $d\psi \in (E_0^+)^\perp$, the latter intended in the L^2-sense: hence

$$\int_{\mathbb{S}^3} \langle \alpha_{0,1}^+, d\psi\rangle = 0\,. \tag{4.25}$$

We have $\mathcal{E}_\rho(\frac{\alpha}{1+t}) = \mathcal{E}_\rho(\alpha)$, and applying Lemma 4.3 to $\frac{\alpha}{1+t}$ and using (4.13) and (4.25), we find

$$\begin{aligned}
\mathcal{E}_\rho(\alpha) - \mathcal{E}_\rho(\alpha_{0,1}^+) &= \rho^{-2}\left(\int_{\mathbb{S}^3}|d\psi|^2 - 2\int_{\mathbb{S}^3}\psi\wedge d\psi\right) - \frac{1}{2}\int_{\mathbb{S}^3}\psi\wedge d\psi \\
&\quad + 2\mathcal{G}_1(\alpha_{0,1}^+, d\psi) + (2+\rho^{-2})\,\mathcal{O}\left(\|d\psi\|^3_{L^2(\mathbb{S}^3)}\right) \\
&\geq \rho^{-2}\left(\int_{\mathbb{S}^3}|d\psi|^2 - 2\int_{\mathbb{S}^3}\psi\wedge d\psi\right) - \frac{1}{2}\int_{\mathbb{S}^3}\psi\wedge d\psi \\
&\quad + (2+\rho^{-2})\,\mathcal{O}\left(\|d\psi\|^3_{L^2(\mathbb{S}^3)}\right)\,.
\end{aligned} \tag{4.26}$$

We now proceed to give a lower bound on the last line of (4.26). Using Proposition 2.2 and that $d\psi \in (E_0^+)^\perp$, since $H^2_{\mathrm{dR}}(\mathbb{S}^3) = 0$ we can write

$$d\psi = \sum_{k=0}^{\infty} (\eta_k^+ + \eta_k^-)\,, \text{ where } \eta_0^+ = 0\,,\ \eta_k^\pm = d\psi_k^\pm \in E_k^\pm\,, \tag{4.27}$$

where $\eta_k^\pm$ is the L^2-orthogonal projection of $d\psi$ onto $E_k^\pm$. Since for every $\ell \in \mathbb{N}$ the 2-form $d\psi_\ell^\pm$ solves (2.8) with eigenvalue $\lambda_\ell^\pm := \pm(\ell+2)$, using also Lemma 2.1 we have that, for $k, \ell \in \mathbb{N}$,

$$\begin{aligned}\int_{\mathbb{S}^3} \psi_k^\pm \wedge d\psi_\ell^\pm &= \frac{1}{\lambda_\ell^\pm} \int_{\mathbb{S}^3} \psi_k^\pm \wedge d * d\psi_\ell^\pm = \frac{1}{\lambda_\ell^\pm} \int_{\mathbb{S}^3} (*d\psi_\ell^\pm) \wedge d\psi_k^\pm \\ &= \frac{1}{\lambda_\ell^\pm} \int_{\mathbb{S}^3} \langle d\psi_\ell^\pm, d\psi_k^\pm \rangle = \frac{\delta^{k^\pm \ell^\pm}}{\lambda_k^\pm} \int_{\mathbb{S}^3} |d\psi_k^\pm|^2\,.\end{aligned} \tag{4.28}$$

In particular, (4.28) implies that for every $k \geq 1$,

$$\begin{aligned}\rho^{-2}\Big(\int_{\mathbb{S}^3} |d\psi_0^-|^2 - 2\int_{\mathbb{S}^3} \psi_0^- \wedge d\psi_0^-\Big) - \frac{1}{2}\int_{\mathbb{S}^3} \psi_0^- \wedge d\psi_0^- &= \frac{8\rho^{-2}+1}{4} \int_{\mathbb{S}^3} |d\psi_0^-|^2\,, \\ \rho^{-2}\Big(\int_{\mathbb{S}^3} |d\psi_k^\pm|^2 - 2\int_{\mathbb{S}^3} \psi_k^\pm \wedge d\psi_k^\pm\Big) - \frac{1}{2}\int_{\mathbb{S}^3} \psi_k^\pm \wedge d\psi_k^\pm &= \Big(\rho^{-2}\big(1 - \frac{2}{\lambda_k^\pm}\big) - \frac{1}{2\lambda_k^\pm}\Big) \int_{\mathbb{S}^3} |d\psi_k^\pm|^2 \\ &\geq \frac{2\rho^{-2}-1}{6} \int_{\mathbb{S}^3} |d\psi_k^\pm|^2\,,\end{aligned} \tag{4.29}$$

with equality holding in the last line for ψ_1^+. Hence, in view of the pairwise L^2-orthogonality of $(\eta_k^\pm)_{k\in\mathbb{N}}$ and (4.28), we see that the quadratic form in the second line in (4.26) splits orthogonally with respect to the decomposition (4.27). Thus, using (4.29) we find

$$\begin{aligned}\mathcal{E}_\rho(\alpha) - \mathcal{E}_\rho(\alpha_{0,1}^+) &\geq \frac{2\rho^{-2}-1}{6} \int_{\mathbb{S}^3} |d\psi|^2 + \big(2+\rho^{-2}\big)\, \mathcal{O}\left(\|d\psi\|^3_{L^2(\mathbb{S}^3)}\right) \\ &\geq \left(\frac{2\rho^{-2}-1}{6} - C\,\varepsilon_0\,\big(2+\rho^{-2}\big)\right) \int_{\mathbb{S}^3} |d\psi|^2\,.\end{aligned} \tag{4.30}$$

To complete the proof it suffices to show that ψ can be replaced by φ in (4.30), as then we can choose

$$0 < \varepsilon_0 \leq \frac{2\rho^{-2}-1}{12C(2+\rho^{-2})}\,.$$

From the definition (4.24) of ψ, (4.10), (4.11) and (4.25), we have

$$\int_{\mathbb{S}^3} |d\varphi|^2 = (1+t)^2 \int_{\mathbb{S}^3} |d\psi|^2 + 32\pi^2 t^2\,, \tag{4.31}$$

and as we show next

$$|t| = \mathcal{O}\big(\|d\varphi\|^2_{L^2(\mathbb{S}^3)}\big)\,. \tag{4.32}$$

Indeed, since $Q(\alpha) = Q(\alpha_{0,1}^+) = 1$, by (2.5) we have

$$1 = Q(\alpha_{0,1}^+ + d\varphi) = Q(\alpha_{0,1}^+) + Q(d\varphi) + \frac{1}{8\pi^2}\int_{\mathbb{S}^3} \beta_0 \wedge d\varphi = 1 + Q(d\varphi) + 2t\,,$$

where in the last equality we used again (4.10), (4.11), (4.24), and (4.25). Thus, $t = -\frac{1}{2}Q(d\varphi)$ and so using also (2.6), (4.32) follows. The latter, together with the assumption (4.7), which can be rewritten as $\|d\varphi\|_{L^2(\mathbb{S}^3)} \leq \varepsilon_0$, imply that we can assume $|t| \leq \frac{1}{2}$, provided that $\varepsilon_0 > 0$ is chosen sufficiently small, which already shows that $\mathcal{O}(\|d\psi\|^3_{L^2(\mathbb{S}^3)})$ can be replaced by $\mathcal{O}(\|d\varphi\|^3_{L^2(\mathbb{S}^3)})$ in (4.30). Finally, again by (4.31) and (4.32), we can estimate

$$\int_{\mathbb{S}^3} |d\psi|^2 \geq \frac{\int_{\mathbb{S}^3} |d\varphi|^2 - \frac{C^2}{32\pi^2}\big(\int_{\mathbb{S}^3} |d\varphi|^2\big)^2}{(1+t)^2} \geq \frac{1}{3}\int_{\mathbb{S}^3} |d\varphi|^2 ,$$

where $C > 0$ is the constant of (2.6), and ε_0 is small enough so that $0 < \varepsilon_0 \leq \frac{2\sqrt{2}\pi}{C}$. Hence (4.30) also holds with $d\psi$ replaced by $d\varphi$, which completes the proof of the proposition. □

5. Proof of Theorem 1.1

In [31] it is shown that elements in $E^+_{0,1}$ minimize the functional $\alpha \mapsto \mathcal{I}_2(\alpha)/Q(\alpha)$ on $\Omega^2(\mathbb{S}^3)$. Actually, we can obtain an associated quantitative estimate, as follows.

Proposition 5.1. *Let $\alpha \in L^2\big(\mathbb{S}^3; \bigwedge^2 T^*\mathbb{S}^3\big)$ be a distributionally closed 2-form. Then*

$$\mathcal{I}_2(\alpha) - 32\pi^2 Q(\alpha) \geq \frac{1}{4} \min_{\widetilde{\alpha}^+_0 \in E^+_0} \int_{\mathbb{S}^3} |\alpha - \widetilde{\alpha}^+_0|^2 . \tag{5.1}$$

Proof. Consider a closed 2-form α, thus $\alpha = d\beta$ for some $\beta \in W^{1,2}(\mathbb{S}^3; \bigwedge^1 T^*\mathbb{S}^3)$ with $d^*\beta = 0$. The lowest eigenvalue of the Hodge Laplacian $\Delta := dd^* + d^*d$ on closed 2-forms is 4, and the corresponding eigenspace is given by

$$\ker d \cap \ker(\Delta - 4) = E^+_0 \oplus E^-_0 ,$$

cf. [17, Section 3] and (2.9). Therefore, we can (L^2-orthogonally) decompose

$$*\beta = \alpha^+_0 + \alpha^-_0 + d\varphi , \text{ where } \; d * \alpha^\pm_0 = \pm 2\alpha^\pm_0 , \quad d\varphi \perp_{L^2} E^+_0 \oplus E^-_0 , \tag{5.2}$$

which, by taking the Hodge-dual and then the exterior differential results in

$$\alpha = 2\alpha^+_0 - 2\alpha^-_0 + d\psi, \text{ where } \; \psi := *d\varphi , \quad d\psi \perp_{L^2} E^+_0 \oplus E^-_0 . \tag{5.3}$$

Using (5.2) and (5.3), we find

$$\mathcal{I}_2(\alpha) = 4\int_{\mathbb{S}^3} |\alpha^+_0|^2 + 4\int_{\mathbb{S}^3} |\alpha^-_0|^2 + \int_{\mathbb{S}^3} |d\psi|^2 ,$$

and

$$Q(\alpha) = \frac{1}{16\pi^2}\int_{\mathbb{S}^3} \langle *\beta, \alpha\rangle = \frac{1}{8\pi^2}\int_{\mathbb{S}^3} |\alpha^+_0|^2 - \frac{1}{8\pi^2}\int_{\mathbb{S}^3} |\alpha^-_0|^2 + \frac{1}{16\pi^2}\int_{\mathbb{S}^3} \psi \wedge d\psi .$$

Combining the two expressions above, we deduce

$$\begin{aligned} \mathcal{I}_2(\alpha) - 32\pi^2 Q(\alpha) &= 8\int_{\mathbb{S}^3} |\alpha^-_0|^2 + \int_{\mathbb{S}^3} |d\psi|^2 - 2\int_{\mathbb{S}^3} \psi \wedge d\psi \\ &\geq 8\int_{\mathbb{S}^3} |\alpha^-_0|^2 + \int_{\mathbb{S}^3} |d\psi|^2 - \frac{1}{4}\int_{\mathbb{S}^3} |d\psi|^2 - 4\int_{\mathbb{S}^3} |\psi|^2 . \end{aligned} \tag{5.4}$$

Note moreover that the next eigenvalue of Δ on closed 2-forms is 9, hence (5.3) (from which we deduce that $d^*\psi = 0$ and $\int_{\mathbb{S}^3} \psi = 0$ component-wise), implies

$$\int_{\mathbb{S}^3} |d\psi|^2 \geq 9\int_{\mathbb{S}^3} |\psi|^2 .$$

This, together with (5.4) and again (5.3) yields

$$\begin{aligned}\mathcal{I}_2(\alpha) - 32\pi^2 Q(\alpha) &\geq 8\int_{\mathbb{S}^3} |\alpha_0^-|^2 + \left(1 - \frac{1}{4} - \frac{4}{9}\right)\int_{\mathbb{S}^3} |d\psi|^2 \\ &\geq \frac{11}{36}\int_{\mathbb{S}^3} |\alpha - 2\alpha_0^+|^2 \geq \frac{1}{4}\min_{\widetilde{\alpha}_0^+ \in E_0^+}\int_{\mathbb{S}^3} |\alpha - \widetilde{\alpha}_0^+|^2 ,\end{aligned}$$

proving (5.1). □

Corollary 5.2. *There exists $\rho_0 \in (0, \sqrt{2})$ and $c > 0$ such that the following holds: for any closed form $\alpha \in L^2\big(\mathbb{S}^3; \bigwedge^2 T^*\mathbb{S}^3\big)$ with $Q(\alpha) > 0$, if α_0^+ is the L^2-orthogonal projection of α onto E_0^+, we have*

$$\mathcal{E}_\rho(\alpha) - \mathcal{E}_\rho(\alpha_0^+) \geq c\big\|Q(\alpha)^{-\frac{1}{2}}\alpha - Q(\alpha_0^+)^{-\frac{1}{2}}\alpha_0^+\big\|^2_{L^2(\mathbb{S}^3)} , \tag{5.5}$$

whenever $0 < \rho \leq \rho_0$.

Proof. Due to the homogeneity of the estimate we can assume that $Q(\alpha) = Q(\alpha_0^+) = 1$. It is also enough to find $\rho_0 \in (0, \sqrt{2})$ for which (5.5) holds, as then by the fact that

$$\mathcal{E}_\rho(\alpha) - \mathcal{E}_\rho(\alpha_0^+) = \mathcal{E}_{\rho_0}(\alpha) - \mathcal{E}_{\rho_0}(\alpha_0^+) + (\rho^{-2} - \rho_0^{-2})\left(\mathcal{I}_2(\alpha) - \mathcal{I}_2(\alpha_0^+)\right) \geq \mathcal{E}_{\rho_0}(\alpha) - \mathcal{E}_{\rho_0}(\alpha_0^+) ,$$

see Proposition 5.1, the same estimate holds for all $0 < \rho \leq \rho_0$.

Let $\varepsilon_0 := \varepsilon_0(1) > 0$ be the parameter provided by Proposition 4.2 with the choice $\widetilde{\rho}_0 := 1$; we make this choice purely for convenience and there is nothing particular about it. Proposition 4.2 shows that if

$$\big\|\alpha - \alpha_0^+\big\|_{L^2(\mathbb{S}^3)} \leq \varepsilon_0 ,$$

and if we choose $\rho_0 \in (0, 1]$, then we have the desired inequality, thus we can suppose that $\|\alpha - \alpha_0^+\|_{L^2(\mathbb{S}^3)} \geq \varepsilon_0$. By the triangle inequality and the Cauchy-Schwarz inequality, we have

$$\big|\mathcal{I}_1(\alpha) - \mathcal{I}_1(\alpha_0^+)\big| \leq \int_{\mathbb{S}^3} |\alpha - \alpha_0^+| \leq \sqrt{2}\pi\|\alpha - \alpha_0^+\|_{L^2(\mathbb{S}^3)} ,$$

so that by Proposition 5.1 we find

$$\begin{aligned}\mathcal{E}_{\rho_0}(\alpha) - \mathcal{E}_{\rho_0}(\alpha_0^+) &\geq -2\sqrt{2}\pi\|\alpha - \alpha_0^+\|_{L^2(\mathbb{S}^3)} + \frac{\rho_0^{-2}}{4}\|\alpha - \alpha_0^+\|^2_{L^2(\mathbb{S}^3)} \\ &\geq \left(\frac{\rho_0^{-2}}{4} - \frac{2\sqrt{2}\pi}{\varepsilon_0}\right)\|\alpha - \alpha_0^+\|^2_{L^2(\mathbb{S}^3)} .\end{aligned}$$

Hence the assertion of the corollary follows if we choose, *e.g.*, $\rho_0 := \frac{1}{4}\big(\frac{\varepsilon_0}{\sqrt{2}\pi}\big)^{\frac{1}{2}}$. □

To obtain the uniqueness statement in Theorem 1.1 we will also need the following:

Proposition 5.3. *Let $u \in \mathcal{U}_{\mathcal{FS}}$ be a horizontally weakly conformal map such that $Q(u) = 1$ and $u^*\omega_{\mathbb{S}^2} = h^*\omega_{\mathbb{S}^2}$. Then $u = h \circ R$ for some $R \in SO(4)$.*

Proof. We first claim that

$$du(\tau_1) = 0 \quad \mathcal{H}^3\text{-a.e. on } \mathbb{S}^3 , \tag{5.6}$$

where τ_1 is the fundamental vertical vector field for the Hopf fibration defined in (2.19). By the assumptions, $\mathcal{H}^3$-a.e. on $\mathbb{S}^3$ the frame $\{\tau_1, \tau_2, \tau_3\}$ satisfies for $i, j \in \{1, 2, 3\}$

$$\omega_{\mathbb{S}^2}(du(\tau_i), du(\tau_j)) = u^*\omega_{\mathbb{S}^2}(\tau_i, \tau_j) = h^*\omega_{\mathbb{S}^2}(\tau_i, \tau_j) = \omega_{\mathbb{S}^2}(dh(\tau_i), dh(\tau_j)) . \tag{5.7}$$

Since $\{dh(\tau_2), dh(\tau_3)\}$ span $T\mathbb{S}^2$, cf. (2.20), (5.7) implies that

$$\{du(\tau_2), du(\tau_3)\} \text{ are linearly independent and span } T\mathbb{S}^2\,. \tag{5.8}$$

Using again (2.20) and (5.7), this time for $j = 2, 3$, we have

$$\omega_{\mathbb{S}^2}(du(\tau_1), du(\tau_j)) = \omega_{\mathbb{S}^2}(dh(\tau_1), dh(\tau_j)) = 0\,,$$

which by (5.8) and the non-degeneracy of $\omega_{\mathbb{S}^2}$ implies (5.6).

We can now define $\psi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ with the property that

$$u = \psi \circ h\,. \tag{5.9}$$

Indeed, since $h\colon \mathbb{S}^3 \to \mathbb{S}^2$ is onto, for every $y \in \mathbb{S}^2$ we can find $x \in \mathbb{S}^3$ such that $h(x) = y$ and then we set $\psi(y) := u(x)$. This is well-defined, *i.e.*, for $\mathcal{H}^3$-a.e. $x \in \mathbb{S}^3$, if $x' \in \mathbb{S}^3$ is such that $h(x) = h(x')$, so that x, x' belong to the same $\mathbb{S}^1$-fiber of h, then $u(x) = u(x')$. For this, observe that by the general slicing properties of Sobolev functions, u is absolutely continuous on a generic $\mathbb{S}^1$-fiber, and the claim follows from (5.6). That $\psi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ follows trivially by the fact that $u \in W^{1,2}(\mathbb{S}^3;\mathbb{S}^2)$ and that $h\colon \mathbb{S}^3 \to \mathbb{S}^2$ is a C^∞-submersion, cf. [21, Theorem 4.29] for the smooth case. In particular, we can pick smooth local coordinates $(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$ on $\mathbb{S}^3$ such that $h(\widetilde{x}) = (\widetilde{x}_1, \widetilde{x}_2) \in \mathbb{S}^2$ and we get the local expression $\psi(\widetilde{x}_1, \widetilde{x}_2) = u(\widetilde{x}_1, \widetilde{x}_2, \widetilde{x}_3)$, hence ψ has the same regularity as u.

Since h is smooth, by the chain rule for the weak derivatives we have

$$du(x) = d\psi(h(x)) \circ dh(x) \text{ for } \mathcal{H}^3\text{-a.e. } x \in \mathbb{S}^3\,,$$

which together with the weak horizontal conformality (2.13) of u and h gives

$$d\psi(h(x)) \circ dh(x) \circ (dh(x))^t \circ (d\psi(h(x)))^t = \frac{1}{4}|d\psi(h(x))|^2 |dh(x)|^2 \operatorname{Id}_{T_{\psi(h(x))}\mathbb{S}^2}$$

or, equivalently,

$$d\psi(h(x)) \circ (d\psi(h(x)))^t = \frac{1}{2}|d\psi(h(x))|^2 \operatorname{Id}_{T_{\psi(h(x))}\mathbb{S}^2}\,,$$

i.e., $\psi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2)$ is a weakly conformal map. Moreover, our assumptions, (2.22) and the weak conformality of ψ directly imply that

$$4 = |h^*\omega_{\mathbb{S}^2}| = |u^*\omega_{\mathbb{S}^2}| = |h^*\psi^*\omega_{\mathbb{S}^2}| = \frac{|d\psi|^2}{2}|h^*\omega_{\mathbb{S}^2}| = 2|d\psi|^2\,,$$

i.e., $d\psi$ is $\mathcal{H}^2$-a.e. an isometry with $|d\psi|^2 = 2$. Using Lemma 3.10, which is applicable since $|\psi^*\omega_{\mathbb{S}^2}| = \frac{|d\psi|^2}{2} = 1$ and $\psi \circ h = u \in \mathcal{U}_{\mathcal{FS}}$, we also obtain

$$1 = Q(u) = Q(\psi \circ h) = \deg(\psi)^2 Q(h) = \deg(\psi)^2\,,$$

i.e., we conclude that

$$\psi \in W^{1,2}(\mathbb{S}^2;\mathbb{S}^2) \text{ is such that } d\psi^t d\psi = \operatorname{Id}_{T\mathbb{S}^2} \text{ with } \deg(\psi) = \pm 1\,. \tag{5.10}$$

It follows that $\psi \equiv O\mathrm{id}_{\mathbb{S}^2}$ for some $O \in O(3)$, as can easily be inferred from Liouville's theorem, see for instance [15, Theorem 1.1]. Finally, using the equivariance of the Hopf map exactly as at the end of the proof of [17, Proposition 4.2], there exists $R \in \mathrm{O}(4)$ such that

$$u = \psi \circ h = h \circ R\,, \tag{5.11}$$

and since $Q(u) = Q(h) = 1$, by applying Theorem 3.3 we actually deduce that $R \in \mathrm{SO}(4)$, as desired. □

Proof of Theorem 1.1. For $\rho_0 > 0$ as in Corollary 5.2 and any $\rho \in (0, \rho_0]$, Corollary 4.1, (2.16) and the scaling-invariant nature of (5.5) imply that

$$\mathcal{FS}_\rho(u) - \mathcal{FS}_\rho(h) \geq \mathcal{E}_\rho(u^*\omega_{\mathbb{S}^2}) - \mathcal{E}_\rho(h^*\omega_{\mathbb{S}^2}) \geq c \min_{\widetilde{\alpha}^+_{0,1} \in E^+_{0,1}} \|u^*\omega_{\mathbb{S}^2} - \widetilde{\alpha}^+_{0,1}\|^2_{L^2(\mathbb{S}^2)} \geq 0\,, \tag{5.12}$$

proving the minimality of h among maps in $\mathcal{U}_{\mathcal{FS}}$. If $\widetilde{u}$ is another minimizer in this homotopy class, then equality holds in (5.12), *i.e.*, $\widetilde{u}$ is horizontally weakly conformal by Corollary 4.1 and $\widetilde{u}^*\omega_{\mathbb{S}^2} \in E^+_{0,1}$, hence $|\widetilde{u}^*\omega_{\mathbb{S}^2}| = 4$, cf. (4.11). By the invariance of the energy under rotations, using (2.12), (2.16) and (2.11), we can find $\widetilde{R} \in SO(4)$ so that the map $v := \widetilde{u} \circ \widetilde{R} \in \mathcal{U}_{\mathcal{FS}}$, which is also a minimizer and horizontally weakly conformal, is such that $Q(v) = 1$ and $v^*\omega_{\mathbb{S}^2} = h^*\omega_{\mathbb{S}^2}$. The conclusion then follows from Proposition 5.3. □

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(André Guerra) Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Rd, Cambridge CB3 0WB, UK

Email address: adblg2@cam.ac.uk

(Xavier Lamy) Institut de Mathématiques de Toulouse, Université de Toulouse, 118 route de Narbonne, F-31062 Toulouse Cedex 8, France

Email address: Xavier.Lamy@math.univ-toulouse.fr

(Konstantinos Zemas) Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Email address: zemas@iam.uni-bonn.de