SOME LIFTING AND APPROXIMATION PROPERTIES FOR MAPS IN $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$

ANDRÉ GUERRA, XAVIER LAMY, AND KONSTANTINOS ZEMAS

Abstract. Smooth maps $u\colon \mathbb{B}^3\to\mathbb{S}^2$ can be lifted to $\hat u\colon \mathbb{B}^3\to\mathbb{S}^3$ using the Hopf fibration $h\colon \mathbb{S}^3\to\mathbb{S}^2$ via the factorization $u=h\circ\hat u$. In this note we characterize the $W^{1,2}$-maps which have this lifting property in terms of exactness of the pullback form $u^*\omega_{\mathbb{S}^2}$, and deduce a smooth approximation property preserving the constraint $u^*\omega_{\mathbb{S}^2}=d\eta$.

1. Introduction

The Hopf fibration $h\colon \mathbb{S}^3\to\mathbb{S}^2$ is a prominent example of a nontrivial $\mathbb{S}^1$-bundle of $\mathbb{S}^3$ over $\mathbb{S}^2$. It maps a unit vector $(z,w)\in\mathbb{S}^3\subset\mathbb{C}^2$ to the associated complex line $[(z,w)]\in\mathbb{CP}^1\approx\mathbb{S}^2$, where the last identification is by stereographic projection.

Any smooth map $u\colon \mathbb{B}^3\to\mathbb{S}^2$ admits a smooth Hopf lift $\hat u\colon \mathbb{B}^3\to\mathbb{S}^3$, that is, it can be factorized as $u=h\circ\hat u$, see *e.g.* [5, §2]. In this note we answer two questions recently raised in [7] about the extension of this lifting property to weakly differentiable maps, and related approximation issues.

Let us start by remarking that, if $u\in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ admits a Hopf lift $\hat u\in W^{1,2}(\mathbb{B}^3;\mathbb{S}^3)$, then the 2-form $u^*\omega_{\mathbb{S}^2}\in L^1(\mathbb{B}^3;\bigwedge^2(\mathbb{R}^3)^*)$ must be (weakly) exact. The chain rule implies indeed

$$u^*\omega_{\mathbb{S}^2}=\hat u^*(h^*\omega_{\mathbb{S}^2})=2\hat u^*d\theta\,,$$

where $\theta\in\Omega^1(\mathbb{S}^3)$ is such that $h^*\omega_{\mathbb{S}^2}=2d\theta$. One may check that $\eta=2\hat u^*\theta\in L^2(\mathbb{B}^3;\bigwedge^1(\mathbb{R}^3)^*)$ satisfies

$$d\eta=2\hat u^*d\theta$$

in the sense of distributions (*e.g.*, by mollifying $\hat u$ and extending θ to $\mathbb{R}^4$), and therefore one has $u^*\omega_{\mathbb{S}^2}=d\eta$. Our main result states that the converse is also true, thereby positively answering [7, Open Problem 8].

Theorem 1.1. *Let $u\in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ be such that there exists $\eta\in L^2(\mathbb{B}^3;\bigwedge^1(\mathbb{R}^3)^*)$ with*

$$d\eta=u^*\omega_{\mathbb{S}^2}\quad\text{in the sense of distributions}\,. \tag{1.1}$$

Then, there exists $\hat u\in W^{1,2}(\mathbb{B}^3;\mathbb{S}^3)$ such that

$$u=h\circ\hat u\,, \tag{1.2}$$

where $h\colon \mathbb{S}^3\to\mathbb{S}^2$ is the Hopf fibration.

It would be interesting to also investigate under which conditions, for $1\le p<2$, a map $u\in W^{1,p}(\mathbb{B}^3;\mathbb{S}^2)$ admits a Hopf lift $\hat u\in W^{1,p}(\mathbb{B}^3;\mathbb{S}^3)$ as in (1.2). In the different context of Riemannian coverings $\widetilde{\mathcal{N}}\to\mathcal{N}$, *e.g.*, $\mathbb{R}\to\mathbb{S}^1$ or $\mathbb{S}^2\to\mathbb{RP}^2$, the question of lifting $\mathcal{N}$-valued maps to $\widetilde{\mathcal{N}}$-valued maps in Sobolev spaces has been quite thoroughly studied, see [8] and references therein. A notable difference with the present setting is that coverings have discrete fibers, while the Hopf fibers are circles.

This is naturally linked with the problem of smooth approximation in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. In fact, it is known [1] that (1.1) is equivalent to the existence of $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ such that

$$u_k \to u \text{ strongly in } W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)\,,$$

and our proof of Theorem 1.1 is about choosing Hopf lifts $\hat{u}_k$ of u_k with good convergence properties. Here, it is natural to look for smooth approximations of the pair (u,η) of (1.1) preserving in addition the nonlinear constraint $d\eta = u^*\omega_{\mathbb{S}^2}$. The existence of a Hopf lift a posteriori enables us to achieve this, thus positively answering [7, Open Problem 7].

Theorem 1.2. *Under the assumptions of Theorem 1.1, there exist $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ and $(\eta_k)_{k\in\mathbb{N}} \subset \Omega^1(\mathbb{B}^3)$ such that*

$$\begin{aligned} &\text{(i)} \quad d\eta_k = u_k^*\omega_{\mathbb{S}^2}\,, \\ &\text{(ii)} \quad (u_k,\eta_k) \to (u,\eta) \textit{ strongly in } W^{1,2}(\mathbb{B}^3;\mathbb{S}^2) \times L^2\big(\mathbb{B}^3; \textstyle\bigwedge^1(\mathbb{R}^3)^*\big)\,. \end{aligned} \tag{1.3}$$

2. Preliminaries

2.1. **Some basic facts involving differential forms.** We adopt an *extrinsic viewpoint*, namely we consider

$$u := (u^1,u^2,u^3) : \mathbb{B}^3 \to \mathbb{S}^2 \subset \mathbb{R}^3\,,$$

where $\mathbb{R}^3$ is endowed with the standard Euclidean basis (e_1,e_2,e_3), so that $du(x) \in \mathbb{R}^3 \otimes \mathbb{R}^3$ is given in coordinates as

$$du(x) = \sum_{k=1}^{3} du^k(x) \otimes e_k = \begin{pmatrix} du^1 \\ du^2 \\ du^3 \end{pmatrix}.$$

The volume form on $\mathbb{S}^n \subset \mathbb{R}^{n+1}$ is given by

$$\omega_{\mathbb{S}^n} = \sum_{j=1}^{n+1} (-1)^{j-1} x^j dx^1 \wedge \cdots \wedge \widehat{dx^j} \wedge \cdots \wedge dx^{n+1}\,, \tag{2.1}$$

where $\widehat{\cdot}$ denotes that the corresponding term is omitted. Thus, for $n=2$, its pull-back by u is

$$\begin{aligned} u^*\omega_{\mathbb{S}^2} &= u^1 du^2 \wedge du^3 + u^2 du^3 \wedge du^1 + u^3 du^1 \wedge du^2 \\ &= \frac{1}{2} u \cdot du \wedge du = \sum_{1\le j<\ell\le 3} u \cdot (\partial_j u \times \partial_\ell u)\, dx^j \wedge dx^\ell\,, \end{aligned} \tag{2.2}$$

where $\cdot$ and $\times$ denote the Euclidean inner and outer products respectively. Since $|u|=1$, this implies that

$$|u^*\omega_{\mathbb{S}^2}|^2 = \sum_{1\le j<\ell\le 3} |\partial_j u \times \partial_\ell u|^2\,. \tag{2.3}$$

We infer in particular the inequality

$$|u^*\omega_{\mathbb{S}^2}| \le \frac{1}{2}|du|^2 \quad \mathcal{L}^3\text{-a.e. on } \mathbb{B}^3\,, \tag{2.4}$$

valid for any $u \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$. We also note from the expression in the second line of (2.2), that $u^*\omega_{\mathbb{S}^2}$ can be identified, via Hodge duality, with the vector field

$$D(u) := \big(u\cdot(\partial_2 u \times \partial_3 u), u\cdot(\partial_3 u \times \partial_1 u), u\cdot(\partial_1 u \times \partial_2 u)\big)\,, \tag{2.5}$$

so that, in the sense of distributions

$$d(u^*\omega_{\mathbb{S}^2}) = \operatorname{div}(D(u))\,\mathrm{d}x\,, \tag{2.6}$$

i.e., (distributional) closedness of the 2-form $u^*\omega_{\mathbb{S}^2}$ is equivalent to the vector field $D(u)$ being (distributionally) divergence-free.

We will also rely on the following form of the Hodge decomposition [6, Corollary 5.6]:

Proposition 2.1. *Let $1 < p < \infty$. For any $\eta \in L^p\big(\mathbb{B}^3, \bigwedge^1(\mathbb{R}^3)^*\big)$, there exists a unique representative $\eta' \in L^p\big(\mathbb{B}^3, \bigwedge^1(\mathbb{R}^3)^*\big)$ such that*

$$d\eta' = d\eta, \qquad d^*\eta' = 0, \qquad \eta'_N = 0 \tag{2.7}$$

in the weak sense. Precisely, the last two identities in (2.7) are equivalent to

$$\int_{\mathbb{B}^3} \langle \eta', d\psi\rangle = 0 \quad \textit{for all } \psi \in C^\infty(\mathbb{B}^3). \tag{2.8}$$

2.2. **The Hopf map.** Denoting by π the inverse stereographic projection

$$\pi\colon \mathbb{C}\cup\{\infty\} \to \mathbb{S}^2\,, \quad \pi(z) := \left(\frac{2z}{|z|^2+1}, \frac{|z|^2-1}{|z|^2+1}\right)\,,$$

the Hopf map $h : \mathbb{S}^3 \subset \mathbb{C}^2 \to \mathbb{S}^2$ is given by

$$h(z,w) := \pi(z/w) = (2z\bar{w}, |z|^2 - |w|^2)\,, \tag{2.9}$$

where the last equality holds true because $|z|^2+|w|^2 = 1$ for $(z,w) \in \mathbb{S}^3$. The 1-form on $\mathbb{S}^3 \subset \mathbb{R}^4$ given by

$$\theta := -x^2dx^1 + x^1dx^2 - x^4dx^3 + x^3dx^4\,, \tag{2.10}$$

satisfies

$$h^*\omega_{\mathbb{S}^2} = 2d\theta = 4(dx^1\wedge dx^2 + dx^3\wedge dx^4)\,. \tag{2.11}$$

It is convenient to choose particular coordinates on $\mathbb{S}^3$ and $\mathbb{S}^2$, namely

$$\begin{aligned}&[0,\pi/2]\times\mathbb{S}^1\times\mathbb{S}^1 \ni (t, e^{i\varphi_1}, e^{i\varphi_2}) \mapsto (e^{i\varphi_1}\sin t, e^{i\varphi_2}\cos t) \in \mathbb{S}^3\,,\\ &[0,\pi/2]\times\mathbb{S}^1 \ni (t, e^{i\varphi}) \mapsto (e^{i\varphi}\sin 2t, -\cos 2t) \in \mathbb{S}^2\,.\end{aligned} \tag{2.12}$$

In these coordinates, we have

$$\begin{aligned} g_{\mathbb{S}^3} &= (dt)^2 + \sin^2 t\,(d\varphi_1)^2 + \cos^2 t\,(d\varphi_2)^2\,,\\ g_{\mathbb{S}^2} &= 4(dt)^2 + (\sin^2 2t)\,(d\varphi)^2\,,\end{aligned}$$

and the Hopf map takes the simple form

$$h\colon (t, e^{i\varphi_1}, e^{i\varphi_2}) \mapsto (t, e^{i(\varphi_1-\varphi_2)})\,.$$

One can choose an orthonormal frame $\{\tau_1,\tau_2,\tau_3\}$, defined at a chart point $(t, e^{i\varphi_1}, e^{i\varphi_2}) \in \mathbb{S}^3$ as

$$\tau_1 := \frac{\partial}{\partial\varphi_1} + \frac{\partial}{\partial\varphi_2}\,, \quad \tau_2 := \frac{\partial}{\partial t}\,, \quad \tau_3 := \cot t\frac{\partial}{\partial\varphi_1} - \tan t\frac{\partial}{\partial\varphi_2}\,, \tag{2.13}$$

so that τ_1 is the *fundamental vertical vector field*, *i.e.*, the generating vector field of the $\mathbb{S}^1$-action $\mathbb{S}^1\times\mathbb{S}^3 \ni (e^{it},(u,v)) \to (e^{it}u, e^{it}v) \in \mathbb{S}^3$. With this choice,

$$dh(\tau_1) = 0\,, \quad dh(\tau_2) = \frac{\partial}{\partial t}\,, \quad dh(\tau_3) = \frac{2}{\sin 2t}\frac{\partial}{\partial\varphi}\,,$$

where $\varphi := \varphi_1 - \varphi_2$. The pair $\{f_1, f_2\} := \left\{\frac{1}{2}\frac{\partial}{\partial t}, \frac{1}{\sin 2t}\frac{\partial}{\partial \varphi}\right\}$ is an orthonormal frame on $\mathbb{S}^2$, and in these coordinates the differential of the Hopf map $dh\colon T\mathbb{S}^3 \to T\mathbb{S}^2$ has matrix representation

$$dh = \begin{pmatrix} 0 & 2 & 0 \\ 0 & 0 & 2 \end{pmatrix} . \tag{2.14}$$

2.3. **Known approximation and lifting results.**

Theorem 2.2. *A map $u \in W^{1,2}(\mathbb{B}^3;\mathbb{S}^2)$ is in the strong $W^{1,2}$-closure of $C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ if and only if $u^*\omega_{\mathbb{S}^2}$ is distributionally closed. Moreover, any map in $W^{1,2}(\mathbb{B}^3;\mathbb{S}^3)$ is in the strong $W^{1,2}$-closure of $C^\infty(\mathbb{B}^3;\mathbb{S}^3)$.*

Proof. The first assertion is a direct reformulation of [1, Theorem 1], via the identification of the 2-form $u^*\omega_{\mathbb{S}^2}$ with the vector-field $D(u)$ of (2.5) and the subsequent identity (2.6). The second assertion follows directly from [2, Theorem 1]. □

Thanks to the fact that any $\mathbb{S}^1$-bundle over $\mathbb{B}^3$ is trivial, every map $u \in C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ admits a Hopf lift $\hat{u} \in C^\infty(\mathbb{B}^3;\mathbb{S}^3)$ such that $u = h \circ \hat{u}$. Moreover, all possible lifts are classified according to their gauge $\eta = 2\hat{u}^*\theta$, where θ is the 1-form defined in (2.10).

Lemma 2.3 ([5, Lemma 2.1])**.** *For any $u \in C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ and $\eta \in \Omega^1(\mathbb{B}^3)$ with $u^*\omega_{\mathbb{S}^2} = d\eta$, there exists a unique (modulo $\mathbb{S}^1$) Hopf lift $\hat{u} \in C^\infty(\mathbb{B}^3;\mathbb{S}^3)$ such that*

$$u = h \circ \hat{u} \quad \text{and} \quad \eta = 2\hat{u}^*\theta . \tag{2.15}$$

Moreover, the chain rule relates the differentials du and $d\hat{u}$ to the gauge η. We note here that this relationship is valid as soon as u and $\hat{u}$ are weakly differentiable and record in particular the following formula.

Lemma 2.4. *For any $u \in W^{1,1}(\mathbb{B}^3;\mathbb{S}^2)$ and $\hat{u} \in W^{1,1}(\mathbb{B}^3;\mathbb{S}^3)$ such that $u = h \circ \hat{u}$, we have*

$$|d\hat{u}|^2 = \frac{1}{4}|\eta|^2 + \frac{1}{4}|du|^2 \quad \mathcal{L}^3\text{-a.e. on } \mathbb{B}^3 , \tag{2.16}$$

where $\eta \in L^1(\mathbb{B}^3;\bigwedge^1(\mathbb{R}^3)^*)$ is given by $\eta = 2\hat{u}^*\theta$, for $\theta \in \Omega^1(\mathbb{S}^3)$ defined in (2.10).

Proof. The coefficients in (2.16) are actually different in [5] due to a different choice of normalization for the metric on $\mathbb{S}^2$, see [4, Lemma 3.6]. For the readers' convenience we reproduce the proof here. In the orthonormal frame $\{\tau_1, \tau_2, \tau_3\}$ on $T\mathbb{S}^3$, the 1-form θ of (2.10) corresponds to the scalar product against τ_1, cf. (2.13). Thus, the condition $\eta = 2\hat{u}^*\theta$ and (2.12), (2.13), imply $\eta = 2\tau_1(\hat{u}) \cdot d\hat{u}$, hence

$$|\eta|^2 = 4|\tau_1 \cdot d\hat{u}|^2 . \tag{2.17}$$

Moreover, recalling the expression (2.14) of dh in this orthonormal frame, the chain rule gives $du = dh \circ d\hat{u} = 2(\tau_2 \cdot d\hat{u})f_1 + 2(\tau_3 \cdot d\hat{u})f_2$, hence

$$|du|^2 = 4|\tau_2 \cdot d\hat{u}|^2 + 4|\tau_3 \cdot d\hat{u}|^2 ,$$

which, together with (2.17), implies (2.16). □

3. Proofs of the main theorems

Proof of Theorem 1.1. The proof follows in a sense the lines of proof of [4, Theorem 3.3]. Using the assumption (1.1) and Theorem 2.2 we can find a sequence of maps $(u_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{B}^3;\mathbb{S}^2)$ such that

$$u_k \to u \text{ strongly in } W^{1,2}(\mathbb{B}^3;\mathbb{S}^2) \text{ and } \mathcal{L}^3\text{-a.e., as } k\to\infty\,. \tag{3.1}$$

Since (by the smoothness of u_k), we have

$$du_k^*\,\omega_{\mathbb{S}^2} = u_k^*(d\omega_{\mathbb{S}^2}) = 0\,,$$

and $\mathbb{B}^3$ is simply connected, we can also fix smooth 1-forms $(\eta_k)_{k\in\mathbb{N}} \subset \Omega^1(\mathbb{B}^3)$ such that

$$d\eta_k = u_k^*\,\omega_{\mathbb{S}^2}\,, \qquad d^*\eta_k = 0\,, \qquad (\eta_k)_N = 0\,, \tag{3.2}$$

cf. Proposition 2.1. By the Bourgain–Brezis elliptic estimates, see for instance [3, Theorem 4.1], we have

$$\|\eta_k\|_{L^{3/2}(\mathbb{B}^3)} \le C\|u_k^*\,\omega_{\mathbb{S}^2}\|_{L^1(\mathbb{B}^3)}\,, \tag{3.3}$$

for a universal constant $C>0$. We remark that, nevertheless, the critical exponent $3/2 = 1^*$ on the right-hand side is not needed for the argument below, and hence one could also appeal to more standard elliptic estimates.

Invoking Lemma 2.3, we now fix lifts $(\hat{u}_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{B}^3;\mathbb{S}^3)$ satisfying

$$h\circ\hat{u}_k = u_k \text{ and } \eta_k = 2\hat{u}_k^*\theta\,. \tag{3.4}$$

In view of (2.4) and (3.1)-(3.3), by passing to a non-relabeled subsequence, we have

$$\eta_k \rightharpoonup \widetilde{\eta} \text{ weakly in } L^{3/2}\big(\mathbb{B}^3;\textstyle\bigwedge^1(\mathbb{R}^3)^*\big) \text{ as } k\to\infty\,, \tag{3.5}$$

for some $\widetilde{\eta} \in L^{3/2}\big(\mathbb{B}^3;\bigwedge^1(\mathbb{R}^3)^*\big)$. By (3.5), (3.2) and (2.8) (applied to η_k), we also see that $\widetilde{\eta}_N = 0$. By (2.16) applied to the triplet $(\hat{u}_k,\eta_k,u_k)$, together with (3.1) and (3.3), we have

$$\sup_{k\in\mathbb{N}} \|\hat{u}_k\|_{W^{1,3/2}(\mathbb{B}^3)} < +\infty\,,$$

and passing to a further non-relabeled subsequence, we obtain $\hat{u} \in W^{1,3/2}(\mathbb{B}^3;\mathbb{S}^3)$ so that

$$\hat{u}_k \rightharpoonup \hat{u} \text{ weakly in } W^{1,3/2}\big(\mathbb{B}^3;\mathbb{R}^4\big) \text{ and pointwise } \mathcal{L}^3\text{-a.e. in } \mathbb{B}^3\,, \text{ as } k\to\infty\,. \tag{3.6}$$

Using now (3.5) and (3.6), we can pass to the limit in (3.4), to infer that

$$h\circ\hat{u} = u\,, \quad \widetilde{\eta} = 2\hat{u}^*\theta \quad \mathcal{L}^3\text{- a.e. in } \mathbb{B}^3\,. \tag{3.7}$$

Indeed, the first identity in (3.7) follows from the $\mathcal{L}^3$-a.e. convergence of $(\hat{u}_k,u_k)$ to $(\hat{u},u)$, while for the second one therein, we observe the following. In view of (2.10), the forms $\hat{u}_k^*\theta$ can be written in coordinates as second-order polynomials of the form $\hat{u}_k \bullet d\hat{u}_k$, where the latter notation indicates that each monomial in the expression is of first order separately in $\hat{u}_k$ and $d\hat{u}_k$. In particular, since

$$\hat{u}_k \to \hat{u} \;\text{ strongly in } L^3(\mathbb{B}^3;\mathbb{R}^3) \;\text{ and } d\hat{u}_k \rightharpoonup d\hat{u} \text{ weakly in } L^{3/2}(\mathbb{B}^3;\mathbb{R}^3\otimes\mathbb{R}^3)\,, \text{ as } k\to\infty\,,$$

this product structure of weakly convergent against strongly convergent objects justifies the second identity in (3.7), by taking the limit of the second identity of (3.4).

Further, for any $j,\ell\in\{1,2,3\}$, we have

$$u_k\cdot(\partial_j u_k\times\partial_\ell u_k) = u_k\cdot(\partial_j u_k\times\partial_\ell u_k - \partial_j u\times\partial_\ell u) + u_k\cdot(\partial_j u\times\partial_\ell u)\,.$$

Since $|u_k| = 1$, by (3.1) the first term in the right-hand side above converges to 0 in $L^1(\mathbb{B}^3)$, and the second term converges to $u \cdot (\partial_j u \times \partial_\ell u)$ in $L^1(\mathbb{B}^3)$ by dominated convergence. Recalling the expression (2.2) of $u^*\omega_{\mathbb{S}^2}$, we deduce the convergence

$$u_k^*\omega_{\mathbb{S}^2} \to u^*\omega_{\mathbb{S}^2} \quad \text{in } L^1\big(\mathbb{B}^3; \bigwedge\nolimits^2 (\mathbb{R}^3)^*\big) . \tag{3.8}$$

Therefore, taking also (3.5) into account, we may pass to the limit in (3.2) and deduce

$$d\widetilde{\eta} = u^*\omega_{\mathbb{S}^2} , \quad d^*\widetilde{\eta} = 0 , \quad \widetilde{\eta}_N = 0 ,$$

in the sense of distributions. We now note that, as $d\eta = u^*\omega_{\mathbb{S}^2}$ and $\eta \in L^2$, by Proposition 2.1 we have also $\widetilde{\eta} \in L^2$. Hence, by (3.7) and (2.16), we conclude that $\hat{u} \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^3)$. □

Proof of Theorem 1.2. By Theorem 1.1 there exists $\hat{u} \in W^{1,2}(\mathbb{B}^3; \mathbb{S}^3)$ satisfying (1.1) and (1.2). Applying Theorem 2.2, we find $(\hat{u}_k)_{k\in\mathbb{N}} \subset C^\infty(\mathbb{B}^3; \mathbb{S}^3)$ such that

$$\hat{u}_k \to \hat{u} \ \text{ strongly in } W^{1,2}(\mathbb{B}^3; \mathbb{S}^3) \text{ and } \mathcal{L}^3\text{-a.e. in } \mathbb{B}^3 . \tag{3.9}$$

Let us now set

$$u_k := h \circ \hat{u}_k \in C^\infty(\mathbb{B}^3; \mathbb{S}^2) , \qquad \zeta_k := 2\hat{u}_k^*\theta \in \Omega^1(\mathbb{B}^3) . \tag{3.10}$$

Since h is smooth on $\mathbb{S}^3$, the convergence (3.9) and the lifting identity (1.2) imply

$$u_k = h \circ \hat{u}_k \to h \circ \hat{u} = u \ \text{ strongly in } W^{1,2}(\mathbb{B}^3; \mathbb{S}^2) . \tag{3.11}$$

The explicit expression (2.10) of θ gives

$$\zeta_k = 2\hat{u}_k^*\theta = 2\big(- \hat{u}_k^2 d\hat{u}_k^1 + \hat{u}_k^1 d\hat{u}_k^2 - \hat{u}_k^4 d\hat{u}_k^3 + \hat{u}_k^3 d\hat{u}_k^4 \big) =: \hat{u}_k \bullet d\hat{u}_k , \tag{3.12}$$

where, as in the proof of Theorem 1.1, the latter notation indicates that each monomial in the expression (3.12) is of first order separately in $\hat{u}_k$ and $d\hat{u}_k$. As for the proof of (3.8), we write

$$\hat{u}_k \bullet d\hat{u}_k = \hat{u}_k \bullet (d\hat{u}_k - d\hat{u}) + \hat{u}_k \bullet d\hat{u} .$$

The fact that $|u_k| = 1$ and the convergence (3.9) ensure that the first term in the right-hand side converges to 0 in L^2, and the second term converges in L^2 to $\hat{u} \bullet d\hat{u}$ by dominated convergence. Hence letting $k \to \infty$ in (3.12) gives

$$\zeta_k \to 2\hat{u}^*\theta \quad \text{strongly in } L^2\big(\mathbb{B}^3; \bigwedge\nolimits^1 (\mathbb{R}^3)^*\big) . \tag{3.13}$$

Since

$$d(\eta - 2\hat{u}^*\theta) = 0 ,$$

cf. again (2.15), (2.11), and where η is as in (1.1), by a standard mollification argument we can also find $(\widetilde{\zeta}_k)_{k\in\mathbb{N}} \subset \Omega^1(\mathbb{B}^3)$ such that

$$d\widetilde{\zeta}_k = 0 , \quad \widetilde{\zeta}_k \to \eta - 2\hat{u}^*\theta \quad \text{strongly in } L^2\big(\mathbb{B}^3; \bigwedge\nolimits^1 (\mathbb{R}^3)^*\big) . \tag{3.14}$$

Finally, setting $\eta_k := \zeta_k + \widetilde{\zeta}_k$, (1.3)(i) follows from (3.10) and the first property in (3.14), while (1.3)(ii) from (3.13) and (3.14). □

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(André Guerra) Department of Pure Mathematics and Mathematical Statistics, University of Cambridge, Wilberforce Rd, Cambridge CB3 0WB, UK

Email address: adblg2@cam.ac.uk

(Xavier Lamy) Institut de Mathématiques de Toulouse, Université de Toulouse, 118 route de Narbonne, F-31062 Toulouse Cedex 8, France

Email address: Xavier.Lamy@math.univ-toulouse.fr

(Konstantinos Zemas) Institute for Applied Mathematics, University of Bonn, Endenicher Allee 60, 53115 Bonn, Germany

Email address: zemas@iam.uni-bonn.de