REGULARITY OF SOLUTIONS TO SCALAR CONSERVATION LAWS WITH A FORCE

BENJAMIN GESS AND XAVIER LAMY

Abstract. We prove regularity estimates for entropy solutions to scalar conservation laws with a force. Based on the kinetic form of a scalar conservation law, a new decomposition of entropy solutions is introduced, by means of a decomposition in the velocity variable, adapted to the non-degeneracy properties of the flux function. This allows a finer control of the degeneracy behavior of the flux. In addition, this decomposition allows to make use of the fact that the entropy dissipation measure has locally finite singular moments. Based on these observations, improved regularity estimates for entropy solutions to (forced) scalar conservation laws are obtained.

1. Introduction

We consider the regularity of solutions to scalar conservation laws

$$\begin{aligned} \partial_t u + \operatorname{div} A(u) &= S \quad \text{on } (0,T)\times\mathbb{R}^n \\ u(0) &= u_0 \quad \text{on } \mathbb{R}^n, \end{aligned} \tag{1.1}$$

for $S \in L^1([0,T]\times\mathbb{R}^n)$, $u_0 \in L^1(\mathbb{R}^n)$ and $A \in C^2(\mathbb{R};\mathbb{R}^n)$ satisfying a non-degeneracy condition to be specified below.

In the special case, $n = 1$, $S \equiv 0$ and A convex, the one-sided Oleinik inequality for entropy solutions can be used to obtain optimal regularity estimates for (1.1). More precisely, assuming in addition that

$$\inf_{(u,v)\in\mathbb{R}^2,\, u\neq v} \frac{|A'(u)-A'(v)|}{|u-v|^\ell} > 0$$

for some $l > 0$, Bourdarias, Gisclon and Junca have shown in [4] that bounded entropy solutions for (1.1) satisfy $u(t) \in W^{\frac{1}{\ell}-\varepsilon,\ell}_{loc}(\mathbb{R})$ for all $t, \varepsilon > 0$. A typical example is $A(u) = |u|^{\ell+1}$, $\ell \geq 1$. For a flux function A that fails to be convex, $n = 1$, $S \equiv 0$, the same regularity can be obtained under some restrictive assumptions on the zeroes of A'', by combining results of Cheng [6] and Jabin [14].

In multiple dimensions, or for S non-smooth, these arguments do not apply anymore. In this case, the best known regularity estimates rely on the kinetic formulation of (1.1), as introduced by Lions, Perthame

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and Tadmor in [17]. In this work it was observed[1] that if u is an entropy solution to (1.1) then the kinetic function

$$f(t,x,v) := \mathbb{1}_{0<v<u(t,x)} - \mathbb{1}_{0>v>u(t,x)} \tag{1.2}$$

satisfies

$$\partial_t f + a(v)\cdot\nabla_x f = \partial_v m + \delta_{v=u} S, \tag{1.3}$$

for some Radon measure $m \geq 0$ and $a := A'$. Based on this and on averaging techniques, regularity estimates for bounded entropy solutions to (1.1) have been obtained in [17] assuming a non-degeneracy property for the flux A and $S \equiv 0$. For the special case of (1.1) with $A(u) = u^{\ell+1}$ this leads to

$$u \in W^{s,p}_{loc}((0,T)\times\mathbb{R}^n) \quad \forall s < \frac{1}{1+2\ell},\ p < \frac{4\ell+1}{2\ell+1}. \tag{1.4}$$

In this work, we provide improved regularity estimates in the case $l > 1$, based on a careful treatment of the degeneracy at $u = 0$.

Regularity estimates for scalar conservation laws. The improved regularity estimates for the special case $A(u) = u^{\ell+1}$ will be obtained as a consequence of a general regularity result for possibly higher dimensional fluxes $A(u)$ with a finite number of degeneracy points. To state this result precisely, we need to introduce technical assumptions satisfied by the velocity field $a(u) = A'(u)$, which quantify the "degree of nonlinearity" at these isolated points and away from them. Loosely speaking, we ask that:

- The overall nonlinearity be "higher" than some threshold represented by a number $\alpha \in (0,1]$, where $\alpha = 1$ corresponds in one dimension to the least degenerate flux $A(u) = u^2/2$ and $\alpha = 1/\ell$ to the degenerate flux $A(u) = u^{\ell+1}$. See (1.6) below.
- The flux at the degeneracy points be "flatter" than some threshold represented by an exponent $\kappa > 0$, where $\kappa = \ell - 1$ corresponds in one dimension to the degenerate flux $A(u) = u^{\ell+1}$ with $\ell > 1$. See (1.7) below.
- The nonlinearity away from the degeneracy points be "higher" than some threshhold $\beta > \alpha$ (that is, strictly higher than the overall nonlinearity), in a way quantified, as one approaches a degeneracy point, by an exponent $\tau > 0$ (the higher τ is, the faster nonlinearity is "lost" as one approaches the point). In one dimension, the flux $A(u) = u^{\ell+1}$ corresponds to $\beta = 1$ and $\tau = \ell - 1$. See (1.8) below.

More specifically, we consider a velocity field $a \in C^1(\mathbb{R};\mathbb{R}^n)$ such that the set of degeneracy points

$$Z := \{a' = 0\} \text{ is locally finite} \tag{1.5}$$

[1]In fact, [17] treated the case $S \equiv 0$ but the same applies to non-vanishing S.

and assume that there exist $\alpha < \beta \in (0,1]$ and $\kappa, \tau \geq 0$ such that for any bounded interval $I \subset \mathbb{R}_v$ and $\lambda, \delta > 0$ it holds

$$\sup_{\tau^2+|\xi|^2=1} |\{v \in I \colon |\tau + a(v)\cdot\xi| \leq \delta\}| \lesssim \delta^{\alpha}, \tag{1.6}$$

$$\sup_{v\in I,\, \mathrm{dist}(v,Z)\leq\lambda} |a'(v)| \lesssim \lambda^{\kappa}, \tag{1.7}$$

$$\sup_{\tau^2+|\xi|^2=1} |\{v \in I \colon \ \mathrm{dist}(v,Z) \geq \lambda,\ |\tau + a(v)\cdot\xi| \leq \delta\}| \lesssim \lambda^{-\tau}\delta^{\beta}. \tag{1.8}$$

Here the symbol $\lesssim$ denotes inequality up to a multiplicative constant that depends only on the interval I and the velocity field a. Note that since I is bounded, (1.6)–(1.8) are trivially satisfied for δ, λ large.

We are now in a position to state our main general result on the regularity of entropy solutions to scalar conservation laws.

Theorem 1. *Let $A \in C^2(\mathbb{R};\mathbb{R}^n)$ satisfy (1.5)-(1.8), $u_0 \in L^1(\mathbb{R}^n_x)$, $S \in L^1([0,T]\times\mathbb{R}^n_x)$ and $u(t,x)$ be an entropy solution of (1.1) with associated kinetic function f as in (1.2). Then, for all $\phi \in C_c^\infty(\mathbb{R})$,*

$$\int f(t,x,v)\phi(v)\,dv \in W^{s,r}_{loc}((0,T)\times\mathbb{R}^n_x) \quad \forall s < s_*,$$

where

$$s_* = (1-\eta)\theta_\alpha + \eta\theta_\beta,$$
$$\theta_a = \frac{a}{a+2}, \quad (a=\alpha,\beta), \quad \eta = \frac{E_1}{E_1+E_2},$$
$$E_1 = \min\left(\kappa+1, \frac{1}{\alpha}\right)\theta_\alpha,$$
$$E_2 = \max\left(\frac{2\tau}{\beta} - \kappa - 1, \frac{\tau-1}{\beta}, 0\right)\theta_\beta,$$

and the order of integrability r is given by

$$\frac{1}{r} = \frac{1-\eta}{r_\alpha} + \frac{\eta}{r_\beta}, \qquad \frac{1}{r_a} = \frac{1+\theta_a}{2}, \quad (a=\alpha,\beta).$$

In particular, if $u_0 \in L^\infty(\mathbb{R}^n)$ and $S \in L^\infty([0,T]\times\mathbb{R}^n)$ then

$$u \in W^{s,r}_{loc}((0,T)\times\mathbb{R}^n_x) \quad \forall s < s_*.$$

We next provide several examples of fluxes A satisfying the assumptions (1.5)-(1.8).

Example 2. Let $A \in C^2(I;\mathbb{R}^n)$ for some interval $I \subseteq \mathbb{R}$.

(1) Let $A \in C^\infty(I;\mathbb{R})$, $n=1$. The valuation of A at $v \in I$ is defined as $m_A(v) = \inf\{k \geq 1 : \ A^{(k+1)}(v) \neq 0\}$, the degeneracy of A on I is $m_A := \sup_{v\in I} m_A(v)$. If $0 < m_A < \infty$ we say that A is non-degenerate of order m_A. In this case (1.6) is satisfied with $\alpha = 1/m_A$ (cf. [3, Lemma 1]).

(2) Let a' be κ-Hölder continuous, i.e. $A \in C^{2+\kappa}(I;\mathbb{R}^n)$. Then (1.7) is satisfied.
(3) Assume $n = 1$, (1.5) and that for some $\tau \geq 0$ and all $\lambda > 0$
$$\lambda^\tau \lesssim \inf_{v\in I,\, \mathrm{dist}(v,Z)\geq\lambda} |a'(v)|.$$
Then a satisfies (1.8) with $\beta = 1$.
(4) Let $A(v) = \sin(v)$ or $A(v) = \cos(v)$. Then A satisfies (1.5)-(1.8) with $\alpha = 1/2$, $\beta = 1$ and $\kappa = \tau = 1$ (cf. Example 5 below).
(5) Our model one-dimensional velocity field $a(v) = v^\ell$ satisfies (1.5)-(1.8) with $\alpha = 1/\ell$, $\beta = 1$ and $\kappa = \tau = \ell - 1$.

The regularity for entropy solutions in the special case $A(u) = u^{\ell+1}$ is an immediate consequence of Theorem 1.

Corollary 3. *Let $\ell \geq 1$, $u_0 \in L^1(\mathbb{R})$, $S \in L^1([0,T]\times\mathbb{R})$ and $u(t,x)$ be an entropy solution of (1.1) with $n = 1$, $A(v) = |v|^{\ell+1}$ or $A(v) = \mathrm{sgn}(v)|v|^{\ell+1}$ and associated kinetic function f as in (1.2). Then, for all $\phi \in C_c^\infty(\mathbb{R})$,*
$$\int f(t,x,v)\phi(v)\,dv \in W^{s,1}_{loc}((0,T)\times\mathbb{R}^n) \quad \forall s < \min\left(\frac{1}{3}, \frac{1}{\ell+1}\right).$$
In particular, if $u_0 \in L^\infty(\mathbb{R}^n)$ and $S \in L^\infty([0,T]\times\mathbb{R}^n)$ then
$$u \in W^{s,1}_{loc}((0,T)\times\mathbb{R}^n) \quad \forall s < \min\left(\frac{1}{3}, \frac{1}{\ell+1}\right).$$

Motivated by some ideas going back to Tadmor and Tao [19], the proof of Theorem 1 relies on introducing a new decomposition of entropy solutions u which allows to make use of the fact that apart from the degeneracy at $u = 0$, the flux $A(u) = u^{\ell+1}$ has non-vanishing second derivative. Using this aspect alone we show that it is possible to improve the regularity in (1.4) to $s < \frac{1}{2+\ell}$. In the literature, a key draw-back of the methods to estimate the regularity of solutions to (1.4) based on averaging techniques is that these methods are not able to make use of the sign of the entropy dissipation measure m in (1.3). Indeed, these arguments could only use that m has locally finite mass. In contrast, we make use of the observation that for entropy solutions to (1.1) the entropy defect measure m has, thanks to its sign, locally finite singular moments, that is, $|v|^{-\gamma}m$ has locally finite mass for all $\gamma \in [0,1)$. This is, to our knowledge, the first time that a kinetic averaging lemma manages, when applied to scalar conservation laws, to take advantage of the sign of the entropy production (see also [11]). Specializing our results to the particular case $A(u) = u^{l+1}$ we obtain the following result.

Remark 4. Solutions of (1.1) for which the entropy dissipation m is only assumed to be a locally finite signed measure are sometimes called quasi-solutions [8]. For the model case in Corollary 3, the arguments in

[17] still apply to this larger class of solutions and provide the regularity (1.4). However, when ℓ is an integer and $S \equiv 0$, Crippa, Otto and Westdickenberg obtain in [7, Proposition 4.4], without using averaging lemmata, a better order of differentiability $s < 1/(2+\ell)$ which has been shown to be optimal by De Lellis and Westdickenberg [9]. In the case where A is convex, Golse and Perthame [12] provide a proof of the same regularity that could be adapted to the presence of a forcing term S. Our arguments yield this optimal order of differentiability $s < 1/(2+\ell)$ for quasi-solutions and for all A as in Corollary 3 and in the presence of the forcing term S.

Example 5. Consider (1.1) with flux $A(v) = \sin(v)$ or $A(v) = \cos(v)$, $u_0 \in L^\infty(\mathbb{R})$ and $S \in L^\infty([0,T] \times \mathbb{R})$. Then

$$u \in W^{s,r}_{loc}((0,T) \times \mathbb{R}) \quad \forall s < \frac{1}{3},\, r \leq \frac{3}{2},$$

despite the existence of degeneracy points, i.e. $\{v \in \mathbb{R} :\, A''(v) = 0\} \neq \emptyset$. This improves the previously known regularity of $s < \frac{1}{5}$ [17].

Our estimates are based on the strategy introduced by Lions, Perthame and Tadmor in [17], namely applying averaging lemmas to the kinetic formulation (1.3). Accordingly, a general regularity estimate for solutions to kinetic equations will be given in the following section.

Averaging lemmas for kinetic equations. It is a well-known phenomenon that under suitable nonlinearity assumptions on the velocity field $a(v)$, velocity averages of f solving (1.3) are more regular than f. In [17], Lions, Perthame and Tadmor use the following assumption: there exists an $\alpha \in (0,1]$ such that for every bounded interval $I \subset \mathbb{R}_v$ and all $\delta > 0$,

$$\sup_{\tau^2+|\xi|^2=1} |\{v \in I \colon |\tau + a(v) \cdot \xi| < \delta\}| \lesssim \delta^\alpha. \tag{1.9}$$

Note that this is exactly our assumption (1.6). They prove that if (1.9) holds and $f \in L^p$ solves (1.3) with $m \in L^q$ for some $p, q \in (1,2]$, then for any bump function $\phi \in C_c^\infty(I)$, the velocity averages

$$\bar{f}(t,x) := \int f(t,x,v)\phi(v)\, dv,$$

satisfy

$$\bar{f} \in W^{s,r}_{loc}((0,T) \times \mathbb{R}^n), \quad \forall s < \theta = \frac{\alpha/p'}{\alpha(1/p' - 1/q') + 2},\ \frac{1}{r} = \frac{1-\theta}{p} + \frac{\theta}{q}.$$

In [19], Tadmor and Tao introduce the additional assumption

$$\sup\left\{|a'(v) \cdot \xi| \colon v \in I,\, \tau^2 + \xi^2 = 1,\, |\tau + a(v) \cdot \xi| < \delta\right\} \lesssim \delta^\mu. \tag{1.10}$$

They prove that if (1.9)-(1.10) hold, then the velocity averages satisfy

$$\bar{f} \in W^{s,r}_{loc}, \quad \forall s < \theta' = \frac{\alpha/p'}{\alpha(1/p' - 1/q') + 2 - \mu}, \ \frac{1}{r} = \frac{1-\theta'}{p} + \frac{\theta'}{q}.$$

However, in many cases of interest this additional assumption is not satisfied. As an example let us consider the velocity field $a(v) = v^\ell$ for some $\ell \geq 1$. Then (1.9) holds with $\alpha = \frac{1}{\ell}$ and this is used in [17] to obtain that entropy solutions of (1.1) enjoy differentiability of order $s = 1/(1+2\ell)$. On the other hand, choosing $\xi = -\tau = 1/\sqrt{2}$ and $v = 1$ in (1.10) shows that one cannot do better than $\mu = 0$. Hence, for $a(v) = v^\ell$ the result in [19] can not provide any improvement on [17].

Theorem 1 will be obtained as a corollary of a general averaging lemma for the kinetic equation

$$\partial_t f + a(v) \cdot \nabla_x f = \partial_v g + h \quad \text{on } \mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}_v. \tag{1.11}$$

As outlined above, our argument is based on the idea underlying the assumption (1.10) in [19] but requires a finer decomposition, relying on our technical assumptions (1.5)-(1.8).

Theorem 6. *Let $a \in C^1(\mathbb{R};\mathbb{R}^n)$ satisfy (1.5)-(1.8). Let $p, q \in [1,2]$ with $p \geq q$, $\gamma \in [0,1]$ and $\sigma \in [0,1)$. Assume that $f \in L^p_{loc}(\mathbb{R}_t \times \mathbb{R}^n_x; W^{\sigma,p}_{loc}(\mathbb{R}_v))$ solves the kinetic equation (1.11) with*

$$h, (1 + \operatorname{dist}(v,Z)^{-\gamma})g \in \begin{cases} L^q_{loc}(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}_v) & \text{if } q \in (1,p], \\ \mathcal{M}_{loc}(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}_v) & \text{if } q = 1. \end{cases} \tag{1.12}$$

Then, for any $\phi \in C^\infty_c(\mathbb{R})$, the average $\bar{f}(t,x) = \int f(t,x,v)\phi(v)\,dv$ satisfies

$$\bar{f} \in W^{s,r}_{loc}(\mathbb{R}_t \times \mathbb{R}^n_x) \qquad \forall s \in [0, s_*),$$

where the order of differentiability s_* is given by

$$\begin{aligned}
&s_* = (1-\eta)\theta_\alpha + \eta\theta_\beta, \\
&\theta_a = \frac{a/\bar{p}}{a(1/\bar{p} - 1/q') + 2} \quad (a = \alpha, \beta), \qquad \eta = \frac{E_1}{E_1 + E_2}, \\
&\bar{p} \in [\frac{p'}{1+\sigma p'}, p'] \cap (1, \infty), \\
&E_1 = \min\left((\kappa + \gamma), \frac{1}{\alpha} - (1-\gamma) \right) \theta_\alpha, \\
&E_2 = \max\left(\frac{2\tau}{\beta} - \kappa - \gamma, \frac{\tau - 1}{\beta} + 1 - \gamma, 0 \right) \theta_\beta,
\end{aligned}$$

and the order of integrability r is given by

$$\frac{1}{r} = \frac{1-\eta}{r_\alpha} + \frac{\eta}{r_\beta}, \qquad \frac{1}{r_a} = \frac{1-\theta_a}{p} + \frac{\theta_a}{q} \quad (a = \alpha, \beta).$$

The proof of Theorem 6 consists in splitting the velocity average into velocities which are close to the degeneracy set $\{v \in \mathbb{R} : \operatorname{dist}(v, Z) \leq \lambda\}$ and far away from it $\{v \in \mathbb{R} : \operatorname{dist}(v, Z) \geq \lambda\}$. Close to Z, assumption (1.6) only allows us to obtain a differentiability of order θ_α by arguing as in [17], but assumption (1.7) allows us (in the spirit of [19]) to estimate the corresponding norms with λ^{E_1}. Away from Z, assumption (1.8) allows us to obtain differentiability of the better order θ_β, with a corresponding estimate in λ^{-E_2}. Then optimizing the choice of λ yields the conclusion.

Notation. All along the proofs of the main results Theorem 6 and Theorem 1 in Section 2 below, the functions f, g, h will be fixed, as well as the cut-off function ϕ, and we will systematically denote by C a constant that may depend on these functions and may change from line to line, but that will be independent of the interpolation parameters λ and δ to be introduced.

Further, $\mathcal{F} = \mathcal{F}_{t,x}$ denotes the Fourier transform in the (t, x) variables and for $(\tau, \xi) \in \mathbb{R}^{n+1}$ let

$$(\tau', \xi') := \frac{1}{\sqrt{\tau^2 + |\xi|^2}}(\tau, \xi),$$

so that $(\tau')^2 + |\xi'|^2 = 1$. For $p \geq 1$ we let p' be the conjugate exponent, that is, $\frac{1}{p} + \frac{1}{p'} = 1$. For $Z \subseteq \mathbb{R}$, $\operatorname{dist}(v, Z) := \inf_{z \in Z} |v - z|$.

Structure of the paper. The plan of the paper is as follows. In Section 2 we present the proof of the main results Theorem 6 and Theorem 1. Some background material on scalar conservation laws with an L^1-force is presented in Appendix A. In Appendix B we recall a basic L^p estimate for Fourier multipliers.

2. Proofs of the main results

Reduction to $Z \cap \operatorname{supp} \phi = \{0\}$ and localization. If $Z \cap \operatorname{supp} \phi = \emptyset$ then Theorem 6 does not improve on [17], so we may assume that $Z \cap \operatorname{supp} \phi$ contains at least one element. If $Z \cap \operatorname{supp} \phi = \{v_1, \ldots, v_N\}$, we may choose a smooth partition of unity $\phi_1(v) + \cdots + \phi_N(v) = 1$ such that $Z \cap \operatorname{supp} \phi_j = \{v_j\}$ for all $j \in \{1, \ldots, N\}$. Since $\bar{f} = \bar{f}_1 + \cdots + \bar{f}_N$ with $\bar{f}_j = \int f(t, x, v)\phi(v)\phi_j(v)\, dv$, it suffices to prove Theorem 6 in the case where $Z \cap \operatorname{supp} \phi$ contains exactly one element. Translating v, we may moreover assume that this element is 0.

Note that we may moreover assume that f, g, h have compact support: for $\phi(t, x, v)$ smooth and compactly supported, the function $\tilde{f} = \phi f$ is compactly supported and satisfies

$$\partial_t \tilde{f} + a(v) \cdot \nabla_x \tilde{f} = \partial_v \tilde{g} + \tilde{h}, \tag{2.1}$$

with

$$\tilde{h} = f\partial_t\phi + a(v)f\nabla_x\phi - (\partial_v\phi)g + h\phi$$
$$\tilde{g} = \phi g.$$

We note that $\tilde{h}, \tilde{g}$ are compactly supported and satisfy (1.12) since $q \le p$.

Hence, the assumptions (1.7)-(1.12) become

$$\sup_{v\in I,\, |v|\le\lambda} |a'(v)| \lesssim \lambda^\kappa, \tag{2.2}$$

$$\sup_{\tau^2+|\xi|^2=1} |\{v \in I\colon |v| \ge \lambda,\ |\tau + a(v)\cdot\xi| \le \delta\}| \lesssim \lambda^{-\tau}\delta^\beta, \tag{2.3}$$

$$h, (1+|v|^{-\gamma})g \in \begin{cases} L^q(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}_v) & \text{if } q \in (1,2], \\ \mathcal{M}(\mathbb{R}_t \times \mathbb{R}^n_x \times \mathbb{R}_v) & \text{if } q = 1. \end{cases} \tag{2.4}$$

Separating small and large velocities. We fix a bounded interval $I \subset [-\Lambda, \Lambda] \subset \mathbb{R}$ and a bump function $\phi \in C_c^\infty(I)$. We further fix a cut-off function $\eta_1 \in C_c^\infty(\mathbb{R})$ satisfying

$$\eta_1(v) \in [0,1] \text{ for all } v \in \mathbb{R},$$
$$\eta_1(v) \equiv 1 \text{ for } |v| \le 1, \quad \eta_1(v) \equiv 0 \text{ for } |v| \ge 2.$$

Then we set $\eta_2 := 1 - \eta_1$, so that for any $\lambda > 0$ it holds

$$\begin{aligned} \bar{f}(t,x) &= \int f(t,x,v)\phi(v)\eta_1(\frac{v}{\lambda})\, dv + \int f(t,x,v)\phi(v)\eta_2(\frac{v}{\lambda})\, dv \\ &=: A_1^\lambda f + A_2^\lambda f. \end{aligned} \tag{2.5}$$

Note that for all $\lambda \ge \Lambda$ we have $A_1^\lambda f = \bar{f}$ and $A_2^\lambda f = 0$ so that in the sequel we will only need to consider $\lambda \le \Lambda$.

Since $A_2^\lambda f$ does not see small velocities, we could use assumption (2.3) and obtain from [17] that $A_2^\lambda f$ has differentiability of order $s = \theta_\beta$. In contrast, for $A_1^\lambda f$ we can only use (1.6) to see that it has differentiability of order $s = \theta_\alpha < \theta_\beta$. But our assumptions allow us to take advantage of the fact that $A_1^\lambda f$ only sees small velocities in two ways: first, by using that $a'(v)$ is small thanks to (2.2) – along the idea that led to introducing the assumption (1.10) in [19]; and second, by using the finite singular moment assumption (2.4) on g. That way we find that the estimate for $A_1^\lambda f$ comes with a constant that goes to zero when λ approaches zero (cf. Lemma 7 below). On the other hand, the estimate for $A_2^\lambda f$ comes with a constant that blows up when λ approaches zero (cf. Lemma 8 below).

Lemma 7. *For all $s \in [0, \theta_\alpha)$ there exists a constant $C > 0$ such that for any $\lambda \le \Lambda$ it holds*

$$\left\| A_1^\lambda f \right\|_{W^{s,r_\alpha}} \le C\lambda^{E_1},$$

where E_1 is given by

$$E_1 = \min\left((\kappa+\gamma), \frac{1}{\alpha} - (1-\gamma)\right)\theta_\alpha.$$

Proof. The proof will follow the strategy of [19, Averaging Lemma 2.1], the main difference residing in the fact that we want to keep track of the dependence on λ of all the estimates.

We fix $\psi_0(z)$ supported in $|z| \le 2$ and $\psi_1(z)$ supported in $1/2 \le |z| \le 2$ such that

$$1 \equiv \psi_0(z) + \sum_{j\ge1} \psi_1(2^{-j}z), \quad \forall z \in \mathbb{C}.$$

For any $\delta > 0$ we decompose f as

$$f = f^0 + f^1, \tag{2.6}$$

where

$$f^0 = \mathcal{F}^{-1}\psi_0\left(\frac{i\tau' + ia(v)\cdot\xi'}{\delta}\right)\mathcal{F}f, \tag{2.7}$$

$$f^1 = \sum_{j\ge1} \mathcal{F}^{-1}\psi_1\left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right)\mathcal{F}f = \sum_{j\ge1} f^{(j)}.$$

Then we estimate the L^p norm of $A_1^\lambda f^0$ and the $\dot W^{1,q}$ norm of $A_1^\lambda f^1$ and conclude using real interpolation.

We treat first the case $q > 1$. Invoking Lemma 13 and using (1.6) we have

$$\left\|A_1^\lambda f^0\right\|_{L^p} \le C \sup_{\tau^2+|\xi|^2=1} \left|\{v\in I : |v| \le 2\lambda,\ |\tau + a(v)\cdot\xi| \le 2\delta\}\right|^{1/\bar p}$$

$$\le C\min(\delta^\alpha, \lambda)^{1/\bar p}. \tag{2.8}$$

Using (1.3) in Fourier variables yields, for all $(\tau,\xi,v)\in\mathbb{R}^{2+n}$ such that $\tau' + i\xi'\cdot a(v) \ne 0$,

$$\begin{aligned}\mathcal{F}f &= \frac{1}{|(\tau,\xi)|}\frac{1}{i\tau' + i\xi'\cdot a(v)}\mathcal{F}(\partial_v g + h)\\ &= \mathcal{F}(-\Delta_{t,x})^{-1/2}\mathcal{F}^{-1}\frac{1}{i\tau' + i\xi'\cdot a(v)}\mathcal{F}(\partial_v g + h).\end{aligned}$$

Hence, setting $\widetilde{\psi}_1(z) := \psi_1(z)/z$ we find that for $j \ge 1$ we have

$$\begin{aligned}\mathcal{F}(-\Delta_{t,x})^{1/2}A_1^\lambda f^{(j)} = \frac{1}{2^j\delta}\int \widetilde{\psi}_1\left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right)\mathcal{F}\partial_v g\,\phi(v)\eta_1\left(\frac{v}{\lambda}\right)dv\\ + \frac{1}{2^j\delta}\int \widetilde{\psi}_1\left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right)\mathcal{F}h\,\phi(v)\eta_1\left(\frac{v}{\lambda}\right)dv.\end{aligned}$$

Integrating by parts thus yields

$\mathcal{F}(-\Delta_{t,x})^{1/2}A_1^\lambda f^{(j)}$

$$
\begin{aligned}
&= -\frac{1}{(2^j\delta)^2}\int \widetilde{\psi}_1'\left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right) ia'(v)|v|^\gamma\cdot\xi'\mathcal{F}|v|^{-\gamma}g\,\phi(v)\eta_1\left(\frac{v}{\lambda}\right)\,dv\\
&\quad -\frac{1}{2^j\delta\lambda}\int \widetilde{\psi}_1\left(\frac{i\tau' + ia(v)\xi'}{2^j\delta}\right)|v|^\gamma\mathcal{F}|v|^{-\gamma}g\,\phi(v)\eta_1'\left(\frac{v}{\lambda}\right)\,dv\\
&\quad -\frac{1}{2^j\delta}\int \widetilde{\psi}_1\left(\frac{i\tau' + ia(v)\xi'}{2^j\delta}\right)|v|^\gamma\mathcal{F}|v|^{-\gamma}g\,\phi'(v)\eta_1\left(\frac{v}{\lambda}\right)\,dv\\
&\quad +\frac{1}{2^j\delta}\int \widetilde{\psi}_1\left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right)\mathcal{F}h\,\phi(v)\eta_1\left(\frac{v}{\lambda}\right)\,dv.
\end{aligned}
$$

Invoking Lemma 13 with $p = q, \sigma = 0, r = q'$, recalling that ξ' is a bounded L^q multiplier, that $|v|^{-\gamma}g \in L^q$ and using (2.2), we deduce

$$
\begin{aligned}
\left\|A_1^\lambda f^{(j)}\right\|_{\dot W^{1,q}} &\le C\Big(2^{-2j}\delta^{-2}\lambda^{\kappa+\gamma}(2^j\delta)^{\frac{\alpha}{q'}} + 2^{-j}\delta^{-1}\lambda^{\gamma-1}(2^j\delta)^{\frac{\alpha}{q'}}\\
&\qquad + 2^{-j}\delta^{-1}\lambda^{\gamma}(2^j\delta)^{\frac{\alpha}{q'}} + 2^{-j}\delta^{-1}(2^j\delta)^{\frac{\alpha}{q'}}\Big)\\
&\le C\big(2^{-2j}\delta^{-2}\lambda^{\kappa+\gamma}(2^j\delta)^{\frac{\alpha}{q'}} + 2^{-j}\delta^{-1}\lambda^{\gamma-1}(2^j\delta)^{\frac{\alpha}{q'}}\big).
\end{aligned}
$$

In the second inequality we were able to discard the two last terms in the previous line because $\lambda \le \Lambda$ and $\gamma \le 1$. Since $\alpha < q'$, summing over $j \ge 1$ yields

$$
\left\|A_1^\lambda f^1\right\|_{\dot W^{1,q}} \le C\Big(\delta^{-2+\frac{\alpha}{q'}}\lambda^{\kappa+\gamma} + \delta^{-1+\frac{\alpha}{q'}}\lambda^{\gamma-1}\Big). \tag{2.9}
$$

From (2.8)-(2.9) we obtain for all $t > 0$ that

$$
\begin{aligned}
K(t, A_1^\lambda f) &:= \inf_{A_1^\lambda f = \tilde f^0 + \tilde f^1}\left(\left\|\tilde f^0\right\|_{L^p} + t\left\|\tilde f^1\right\|_{\dot W^{1,q}}\right)\\
&\le C\Big(\delta^{\frac{\alpha}{\bar p}} + t\delta^{\frac{\alpha}{q'}-2}\lambda^{\kappa+\gamma} + t\delta^{\frac{\alpha}{q'}-1}\lambda^{\gamma-1}\Big).
\end{aligned}
$$

Next we optimize in δ. We choose it of the form $\delta = t^a\lambda^b$, where b will be chosen later and a is determined by balancing the powers of t in the first two terms:

$$
a\frac{\alpha}{\bar p} = 1 + a\left(\frac{\alpha}{q'} - 2\right) \quad \text{i.e.} \quad a = \frac{\bar p}{\alpha}\theta_\alpha.
$$

This gives

$$
t^{-\theta_\alpha}K(t, A_1^\lambda f) \le C\Big(\lambda^{b\frac{\alpha}{\bar p}} + \lambda^{b\left(\frac{\alpha}{q'}-2\right)+\kappa+\gamma} + t^{\frac{\bar p}{\alpha}\theta_\alpha}\lambda^{b\left(\frac{\alpha}{q'}-1\right)+\gamma-1}\Big).
$$

Note that the last term is small for small t. On the other hand for large t we can use the fact (obtained from (2.8) by sending $\delta \to \infty$) that

$$
\left\|A_1^\lambda f\right\|_{L^p} \le C\lambda^{\frac{1}{p}},
$$

to deduce, for any $\mu > 0$,

$$
t^{-\theta_\alpha}K(t, A_1^\lambda f) \le C\begin{cases}\lambda^{b\frac{\alpha}{\bar p}} + \lambda^{b\left(\frac{\alpha}{q'}-2\right)+\kappa+\gamma} + \mu^{\frac{\bar p}{\alpha}\theta_\alpha}\lambda^{b\left(\frac{\alpha}{q'}-1\right)+\gamma-1} & \text{for } t \le \mu,\\ \mu^{-\theta_\alpha}\lambda^{\frac{1}{p}} & \text{for } t \ge \mu.\end{cases}
$$

Next we choose μ in order to balance the last terms of the above two lines, i.e.

$$\mu = \lambda^{\frac{\alpha}{\alpha+\bar{p}}\frac{1}{\theta_\alpha}\left(\frac{1}{\bar{p}}+b\left(1-\frac{\alpha}{q'}\right)+1-\gamma\right)},$$

and conclude that

$$t^{-\theta_\alpha}K(t,A_1^\lambda f) \leq C\left(\lambda^{b\frac{\alpha}{\bar{p}}} + \lambda^{b\left(\frac{\alpha}{q'}-2\right)+\kappa+\gamma} + \lambda^{\frac{1}{\alpha+\bar{p}}\left(1-\alpha(1-\gamma)-b\alpha\left(1-\frac{\alpha}{q'}\right)\right)}\right).$$

Finally we want to choose b to optimize the above powers of λ : set

$$E_1 := \sup_{b\in\mathbb{R}} \min \left\{ \begin{array}{c} \dfrac{\alpha}{\bar{p}}b \\ \kappa+\gamma-\left(2-\dfrac{\alpha}{q'}\right)b \\ \dfrac{1}{\alpha+\bar{p}}\left(1-\alpha(1-\gamma)-\alpha\left(1-\dfrac{\alpha}{q'}\right)b\right) \end{array} \right\}.$$

We denote by $L_1(b), L_2(b), L_3(b)$ the three affine functions of b appearing in the definition of E_1. Since L_1 is increasing and L_2, L_3 are decreasing, the function $\min(L_1, L_2, L_3)$ is bounded from above, thus $E_1 < +\infty$. Moreover E_1 is given by

$$\begin{aligned} E_1 &= \min(L_1(L_1 = L_2), L_1(L_1 = L_3)) \\ &= \min\left((\kappa+\gamma), \frac{1}{\alpha} - (1-\gamma)\right)\theta_\alpha. \end{aligned}$$

Then, denoting by $\|\cdot\|_\theta$ the norm in the real interpolation space $[L^p, \dot{W}^{1,q}]_{\theta,\infty}$ (see e.g. [2] for definition and properties), we have

$$\left\|A_1^\lambda f\right\|_{\theta_\alpha} \leq C\lambda^{E_1},$$

which implies the conclusion of Lemma 7.

In the case $q=1$, we obtain the same estimates, but the space $\dot{W}^{1,q} = (-\Delta_{t,x})^{-1/2}L^q$ has to be replaced with $(-\Delta_{t,x})^{-1/2}\mathcal{M}$. Since this space contains $\dot{W}^{s,1}$ for all $s<1$ we still obtain the conclusion. □

Lemma 8. *For all $s \in [0, \theta_\beta)$ there exists $C > 0$ such that for any $\lambda \leq \Lambda$ it holds*

$$\left\|A_2^\lambda f\right\|_{W^{s,r_\beta}} \leq C\lambda^{-E_2},$$

where E_2 is given by

$$E_2 = \max\left(\frac{2\tau}{\beta} - \kappa - \gamma, \frac{\tau-1}{\beta} + 1 - \gamma, 0\right)\theta_\beta.$$

Proof. As in the proof of Lemma 7 we consider the decomposition (2.6) and treat first the case $q > 1$.

Let $\widetilde{\eta}(v) = \eta_1(v/2) - \eta_1(v)$, so that $\widetilde{\eta}$ is supported inside $\{1 \leq |v| \leq 4\}$ and $\eta_2(v) = \sum_{k\geq 0} \widetilde{\eta}(v/2^k)$. Hence, it holds

$$A_2^\lambda f = \sum_{k\geq 0} A_2^{(k)} f,$$
$$A_2^{(k)} f = \int f(t,x,v)\phi(v)\widetilde{\eta}\left(\frac{v}{2^k\lambda}\right) dv.$$

Next we estimate $\left\|A_2^{(k)} f\right\|_{\theta_\beta}$. Fix $k \geq 0$ and let $\mu := 2^k\lambda$, so that

$$\begin{aligned}
A_2^{(k)} f &= \int f(t,x,v)\phi(v)\widetilde{\eta}(v/\mu)\, dv \\
&= \sum_{j\geq 1} \int f^{(j)}(t,x,v)\phi(v)\widetilde{\eta}(v/\mu)\, dv =: \sum_{j\geq 1} A_2^{(k)} f^{(j)},
\end{aligned}$$

with $f^{(j)}$ defined as in (2.7). Analogously,

$$A_2^{(k)} f^0 = \int f^0(t,x,v)\phi(v)\widetilde{\eta}(v/\mu)\, dv.$$

Note that $A_2^{(k)} f$ is nonzero only for k such that $\mu = 2^k\lambda \leq \Lambda$, since ϕ is supported in $[-\Lambda, \Lambda]$ and $\widetilde{\eta}(v/\mu)$ vanishes for $|v| \leq \mu$. By Lemma 13 and assumption (2.3) it holds

$$\begin{aligned}
\left\|A_2^{(k)} f^0\right\|_{L^p} &\leq C \sup_{\tau^2+\xi^2=1} |\{v \in I\colon 4\mu \geq |v| \geq \mu,\ |\tau + a(v)\xi| \leq 2\delta\}|^{1/\bar{p}} \\
&\leq C \min(\mu^{-\tau}\delta^\beta, \mu)^{\frac{1}{\bar{p}}}. \qquad (2.10)
\end{aligned}$$

As in the proof of Lemma 7 we have

$$\begin{aligned}
&\mathcal{F}(-\Delta_{t,x})^{1/2} A_2^{(k)} f^{(j)} \\
&= -\frac{1}{(2^j\delta)^2} \int \widetilde{\psi}_1' \left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right) ia'(v)|v|^\gamma \cdot \xi' \mathcal{F}|v|^{-\gamma} g\, \phi(v)\widetilde{\eta}\left(\frac{v}{\mu}\right) dv \\
&\quad - \frac{1}{2^j\delta\mu} \int \widetilde{\psi}_1 \left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right) |v|^\gamma \mathcal{F}|v|^{-\gamma} g\, \phi(v)\widetilde{\eta}'\left(\frac{v}{\mu}\right) dv \\
&\quad - \frac{1}{2^j\delta} \int \widetilde{\psi}_1 \left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right) |v|^\gamma \mathcal{F}|v|^{-\gamma} g\, \phi'(v)\widetilde{\eta}\left(\frac{v}{\mu}\right) dv \\
&\quad + \frac{1}{2^j\delta} \int \widetilde{\psi}_1 \left(\frac{i\tau' + ia(v)\cdot\xi'}{2^j\delta}\right) \mathcal{F}h\, \phi(v)\widetilde{\eta}\left(\frac{v}{\mu}\right) dv
\end{aligned}$$

which yields, using assumptions (2.2)-(2.4),

$$\left\|A_2^{(k)} f^1\right\|_{\dot{W}^{1,q}} \leq C\Big(\delta^{-2+\frac{\beta}{q'}}\mu^{-\frac{\tau}{q'}+\kappa+\gamma} + \delta^{-1+\frac{\beta}{q'}}\mu^{-1-\frac{\tau}{q'}+\gamma}\Big). \qquad (2.11)$$

Here as in the proof of Lemma 7 we estimated the third and fourth term by the second term on the right hand side since they come with higher powers of $\mu \lesssim 1$. The estimates (2.10)-(2.11) then imply

$$\begin{aligned} K(t, A_2^{(k)} f) &:= \inf_{A_2^{(k)} f = \tilde{f}^0 + \tilde{f}^1} \left(\left\| \tilde{f}^0 \right\|_{L^p} + t \left\| \tilde{f}^1 \right\|_{\dot{W}^{1,q}} \right) \\ &\leq C \Big(\mu^{-\frac{\tau}{\bar{p}}} \delta^{\frac{\beta}{\bar{p}}} + t \delta^{\frac{\beta}{q'} - 2} \mu^{-\frac{\tau}{q'} + \kappa + \gamma} + t \delta^{\frac{\beta}{q'} - 1} \mu^{-1 - \frac{\tau}{q'} + \gamma} \Big). \end{aligned}$$

Equilibrating the first and the second term yields the choice $\delta = t^{\frac{\bar{p}}{\beta}\theta_\beta} \mu^b$. Since $\|A_2^{(k)} f\|_{L^p} \leq \mu^{\frac{1}{\bar{p}}}$ we also have $K(t, A_2^{(k)} f) \lesssim 1$ for all $t \geq 0$. We thus obtain

$$\begin{aligned} & t^{-\theta_\beta} K(t, A_2^{(k)} f) \\ & \leq C \begin{cases} \mu^{-\frac{\tau}{\bar{p}} + b\frac{\beta}{\bar{p}}} + \mu^{-\frac{\tau}{q'} + \kappa + \gamma - b\left(2 - \frac{\beta}{q'}\right)} + \nu^{\frac{\bar{p}}{\beta}\theta_\beta} \mu^{-1 - \frac{\tau}{q'} + \gamma - b\left(1 - \frac{\beta}{q'}\right)} & \text{for } t \leq \nu, \\ \nu^{-\theta_\beta} \mu^{\frac{1}{\bar{p}}} & \text{for } t \geq \nu. \end{cases} \end{aligned}$$

We choose $\nu = \mu^{\frac{1}{\theta_\beta} \frac{\beta}{\beta + \bar{p}} \left(1 - \gamma + \frac{\tau}{q'} + b\left(1 - \frac{\beta}{q'}\right) + \frac{1}{\bar{p}}\right)}$ to deduce

$$\begin{aligned} t^{-\theta_\beta} K(t, A_2^{(k)} f) \leq C \Big(& \mu^{-\frac{\tau}{\bar{p}} + b\frac{\beta}{\bar{p}}} + \mu^{-\frac{\tau}{q'} + \kappa + \gamma - b\left(2 - \frac{\beta}{q'}\right)} \\ & + \mu^{-\frac{1}{\beta + \bar{p}}\left(-1 + \beta\left(1 - \gamma + \frac{\tau}{q'}\right) + \beta\left(1 - \frac{\beta}{q'}\right) b\right)} \Big). \end{aligned}$$

Then optimizing in b we set

$$E = \inf_{b \in \mathbb{R}} \max \left\{ \begin{array}{c} \frac{\tau}{\bar{p}} - \frac{\beta}{\bar{p}} b \\ \frac{\tau}{q'} - \kappa - \gamma + \left(2 - \frac{\beta}{q'}\right) b \\ \frac{1}{\beta + \bar{p}} \left(-1 + \beta\left(1 - \gamma + \frac{\tau}{q'}\right) + \beta\left(1 - \frac{\beta}{q'}\right) b\right) \end{array} \right\},$$

and obtain (recall $\mu \leq \Lambda$) that

$$\left\| A_2^{(k)} f \right\|_{\theta_\beta} \leq C \mu^{-E} = 2^{-kE} \lambda^{-E}. \tag{2.12}$$

We denote by $L_1(b), L_2(b), L_3(b)$ the three affine functions of b appearing in the definition of E. Since L_1 is increasing while L_2, L_3 are increasing, the function $\max(L_1, L_2, L_3)$ is bounded from below, thus $E > -\infty$. Moreover it holds

$$\begin{aligned} E &= \max(L_1(L_1 = L_2), L_1(L_1 = L_3)) \\ &= \max\left(\frac{2\tau}{\beta} - \kappa - \gamma, \frac{\tau - 1}{\beta} + 1 - \gamma \right) \theta_\beta. \end{aligned}$$

If $E > 0$, then summing (2.12) over $k \geq 0$ yields $\left\| A_2^\lambda f \right\|_{\theta_\beta} \leq C \lambda^{-E}$. If $E \leq 0$, then summing (2.12) over those k satisfying $\mu = 2^k \lambda \leq \Lambda$

yields

$$\left\|A_\beta^\lambda f\right\|_{\theta_2} \le C\lambda^{-E} \sum_{0\le k\le \log(\Lambda/\lambda)} (2^{-E})^k \le C\lambda^{-E} 2^{E\log(\lambda/\Lambda)} \le C.$$

Hence we conclude that $\left\|A_2^\lambda f\right\|_{\theta_\beta} \le C\lambda^{-\max(E,0)}$.

To treat the case $q = 1$ we argue as in the proof of Lemma 7. □

Proofs of Theorem 6 and Theorem 1.

Proof of Theorem 6. By (2.5), Lemma 7 and Lemma 8, for $\lambda \lesssim 1$ and $t \ge 0$,

$$\begin{aligned} K(t,\bar f) &:= \inf_{\bar f = \tilde f^0 + \tilde f^1} \left(\left\|\tilde f^0\right\|_{\theta_\alpha} + t\left\|\tilde f^1\right\|_{\theta_\beta}\right) \\ &\le \left\|A_1^\lambda f\right\|_{\theta_\alpha} + t\left\|A_2^\lambda f\right\|_{\theta_\beta} \\ &\le C\Big(\lambda^{E_1} + t\lambda^{-E_2}\Big). \end{aligned}$$

Choosing, $\lambda = t^{\frac{1}{E_1+E_2}}$ yields

$$K(t,\bar f) \le C t^{\frac{E_1}{E_1+E_2}} = t^\eta \qquad \forall t \lesssim 1.$$

Since $\left\|\bar f\right\|_{\theta_\alpha} \le C$ (as can be seen e.g. by choosing $\lambda = \Lambda$ in Lemma 7) we have

$$K(t,\bar f) \le C\left\|\bar f\right\|_{\theta_\alpha} \le C \quad \forall t \ge 0.$$

Hence, $\bar f$ belongs to the real interpolation space

$$\Big[[L^p, \dot W^{1,q}]_{\theta_1,\infty}, [L^p, \dot W^{1,q}]_{\theta_2,\infty}\Big]_{\eta,\infty} = [L^p, \dot W^{1,q}]_{\theta,\infty},$$

where $\theta = (1-\eta)\theta_\alpha + \eta\theta_\beta$ and the equality follows from the reiteration Theorem of real interpolation. We further note that this space contains $W^{s,r}$ for all $s < s_* = \theta$. This argument works for $q > 1$ and for $q = 1$ we may adapt it as in the proof of Lemma 7. □

Proof of Theorem 1. We apply the kinetic formulation for (1.1) (cf. Appendix A), that is,

$$f = \mathbb{1}_{0<v<u(t,x)} - \mathbb{1}_{0>v>u(t,x)},$$

satisfies, in the sense of distributions,

$$\partial_t f + a(v)\cdot\nabla_x f = \partial_v m + \delta_{v=u} S \quad \text{on } [0,T]\times\mathbb{R}^n_x\times\mathbb{R}_v \tag{2.13}$$

for some Radon measure $m \ge 0$. We further note that

$$f \in L^1([0,T]\times\mathbb{R}^n_x\times\mathbb{R}_v) \cap L^\infty([0,T]\times\mathbb{R}^n_x\times\mathbb{R}_v)$$

and $f \in L^\infty([0,T]\times\mathbb{R}^n_x; BV(\mathbb{R}_v))$. Hence, by interpolation,

$$f \in L^2_{loc}([0,T]\times\mathbb{R}^n_x; W^{\sigma,2}(\mathbb{R}_v))$$

for all $\sigma \in [0,\frac{1}{2})$.

For a bounded interval $I \subseteq \mathbb{R}_v$ let $Z \cap I = \{z_1, \dots, z_N\}$. By Proposition 12 below, $|v - z_i|^{\alpha-1} m$ has locally finite mass for every $\alpha \in (0,1)$ and $i \in 1, \dots, N$. It follows that $\mathrm{dist}(v, Z)^{-\gamma} m$ has locally finite mass for any $\gamma \in (0,1)$.

Let $\eta \in C_c^\infty(0,T)$. Then $\tilde{f} := f\eta$ satisfies (1.11) with $g = m\eta$, $h = \delta_{v=u} S\eta + \dot{\eta} f$ and $\tilde{f} \in L^2_{loc}(\mathbb{R}_t \times \mathbb{R}^n_x; W^{\sigma,2}(\mathbb{R}_v))$ for all $\sigma \in [0, \frac{1}{2})$.

We now apply Theorem 6 with $p = 2$, $q = 1$, any $\sigma \in [0, \frac{1}{2})$ and any $\gamma \in [0,1)$, to obtain, for all $\phi \in C_c^\infty(\mathbb{R})$,

$$\int \tilde{f} \phi \, dv \in W^{s,r}_{loc}$$

for all $s < s_*$, where r is as in the conclusion of Theorem 1, and the value of s_* can be choosen, depending on σ and γ, in the way described by the conclusion of Theorem 6. It can be checked directly that taking σ arbitrarily close to $\frac{1}{2}$ and γ arbitrarily close to 1 allows to take s_* arbitrarily close to the value given by the conclusion of Theorem 1. □

Appendix A. Kinetic solutions for scalar conservation laws with a force

In this section we present some brief comments on the extension of the concept of kinetic solutions and their well-posedness for scalar conservation laws with an L^1-force (1.1). This proceeds along the lines of [5, 18]. We will refer to kinetic solutions also as entropy solutions.

Definition 9. A kinetic/entropy solution to (1.1) is a function $u \in C([0,T]; L^1(\mathbb{R}^n))$ such that the corresponding kinetic function

$$f(t,x,v) = \chi(v, u(t,x)) = \mathbb{1}_{0<v<u(t,x)} - \mathbb{1}_{0>v>u(t,x)}$$

satisfies, in the sense of distributions,

$$\begin{aligned} \partial_t f + a(v) \cdot \nabla_x f &= \partial_v m + \delta_{u=v} S \quad \text{on } (0,T) \times \mathbb{R}^n \\ f_{|t=0} &= \chi(v, u_0) \quad \text{on } \mathbb{R}^n, \end{aligned} \tag{A.1}$$

where $a := A'$, m is a non-negative Radon measure and

$$\int m(t,x,v) \, dt dx \le \mu(v) \in L^\infty_0(\mathbb{R}),$$

where $L^\infty_0(\mathbb{R})$ denotes the space of all essentially bounded functions decaying to zero for $|v| \to \infty$.

Remark 10. For a comparison of the concept of renormalized entropy solutions introduced in [1] and kinetic solutions we refer to [13]. A proof that entropy solutions give rise to kinetic solutions is contained in the proof of Theorem 11 below.

Theorem 11. *Let $u_0 \in L^1(\mathbb{R}^n)$, $S \in L^1([0,T]\times\mathbb{R}^n)$. Then there is a unique kinetic solution u to (1.1). For two kinetic solutions u^1, u^2 with initial conditions u_0^1, u_0^2 and forces S^1, S^2 respectively we have*

$$\begin{aligned}\sup_{t\in[0,T]} \|(u^1(t)-u^2(t))_+\|_{L^1(\mathbb{R}^n)} &\le \|(u_0^1-u_0^2)_+\|_{L^1(\mathbb{R}^n)} \\ &\quad + \|S^1-S^2\|_{L^1([0,T]\times\mathbb{R}^n)}.\end{aligned} \tag{A.2}$$

Proof. Contraction: We first note that the function $g(t,x,v) = \mathbb{1}_{v<u(t,x)}$ satisfies the same kinetic equation as f, since

$$\begin{aligned}&f(t,x,v) - g(t,x,v) = -\mathbb{1}_{v>0} - \mathbb{1}_{v=0}\mathbb{1}_{u(t,x)\ge 0} - \mathbb{1}_{v<0}\mathbb{1}_{u(t,x)=v},\\ &(\partial_t + a(v)\cdot\nabla_x)\mathbb{1}_{v<0} = 0 \qquad \text{in } \mathcal{D}'_{t,x,v},\\ &\qquad\text{and } \mathbb{1}_{v=0} = \mathbb{1}_{u(t,x)=v} = 0 \qquad \text{for a.e. } (t,x,v).\end{aligned}$$

The proof of the contraction inequality (A.2) relies on the identity

$$\int g^1(1-g^2)dv = (u^1-u^2)_+.$$

We introduce nonnegative mollifiers $\Phi_\varepsilon(t,x)$ and let the subscript ε denote the convolution in (t,x) with Φ_ε. In particular, we have

$$(\partial_t + a(v)\cdot\nabla_x)g_\varepsilon = \partial_v m_\varepsilon + \left(\delta_{v=u(t,x)}S(t,x)\right)_\varepsilon,$$

where $\left(\delta_{v=u(t,x)}S(t,x)\right)_\varepsilon$ is the distribution given by

$$\begin{aligned}&\left\langle \left(\delta_{v=u(t,x)}S(t,x)\right)_\varepsilon, \theta(t,x,v)\right\rangle\\ &= \int S(t,x)\int \theta(s,y,u(t,x))\Phi_\varepsilon(s-t,y-x)dsdydxdt.\end{aligned}$$

We also introduce a nonnegative cut-off function $\chi(v)$. By dominated differentiation, for any $\varepsilon_1,\varepsilon_2>0$ we have

$$\begin{aligned}T &:= \partial_t \int g^1_{\varepsilon_1}(1-g^2_{\varepsilon_2})\chi(v)dv + \nabla_x\cdot\int g^1_{\varepsilon_1}(1-g^2_{\varepsilon_2})\chi(v)a(v)dv\\ &= \int \chi(v)(1-g^2_{\varepsilon_2})(\partial_t + a(v)\cdot\nabla_x)g^1_{\varepsilon_1}dv\\ &\quad + \int \chi(v)g^1_{\varepsilon_1}(\partial_t + a(v)\cdot\nabla_x)(1-g^2_{\varepsilon_2})dv\\ &= \lim_{\delta\to 0}\left(T^1_\delta + T^2_\delta\right),\end{aligned}$$

where

$$\begin{aligned}T^1_\delta(t,x) &= \int \chi(v)(1-g^2_{\varepsilon_2}(t,x,w))(\partial_t + a(v)\cdot\nabla_x)g^1_{\varepsilon_1}(t,x,v)\rho_\delta(v-w)dvdw,\\ T^2_\delta(t,x) &= \int \chi(v)g^1_{\varepsilon_1}(t,x,w)(\partial_t + a(v)\cdot\nabla_x)(1-g^2_{\varepsilon_2}(t,x,v))\rho_\delta(v-w)dvdw,\end{aligned}$$

and $\rho_\delta(v)$ is an even nonnegative mollifier. Using the equation satisfied by g^1 we have, for any nonnegative test function $\theta(t,x)$,

$$\begin{aligned}&\left\langle T^1_\delta,\theta\right\rangle\\&=-\int m^1_{\varepsilon_1}(dt,dx,dv)\theta(t,x)\chi'(v)\int(1-g^2_{\varepsilon_2}(t,x,w))\rho_\delta(v-w)dvdw\\&\quad-\int m^1_{\varepsilon_1}(dt,dx,dv)\theta(t,x)\chi(v)\int(1-g^2_{\varepsilon_2}(t,x,w))(\rho_\delta)'(v-w)dvdw\\&\quad+\int\theta(t,x)\int S^1(s,y)\Phi_{\varepsilon_1}(t-s,x-y)\chi(u^1(s,y))\\&\qquad\cdot\int(1-g^2_{\varepsilon_2}(t,x,w))\rho_\delta(u^1(s,y)-w)dwdsdydtdx.\end{aligned}$$

The second term on right-hand side is nonpositive since $w\mapsto(1-g^2_{\varepsilon_2}(t,x,w))=(\mathbb{1}_{w\geq u^2(t,x)})_{\varepsilon_2}$ is nondecreasing. Moreover, since ρ_δ is even, for any (t,x,v) we have as $\delta\to 0$,

$$\begin{aligned}&\int(1-g^2_{\varepsilon_2}(t,x,w))\rho_\delta(v-w)\,dw\\&=\int\Phi_{\varepsilon_2}(t-s,x-y)\int\mathbb{1}_{w\geq u^2(s,y)}\rho_\delta(v-w)\,dwdsdy\\&\to\int\Phi_{\varepsilon_2}(t-s,x-y)\operatorname{sgn}^+_{\frac12}(v-u^2(s,y))\,dsdy\\&=[\operatorname{sgn}^+_{\frac12}(v-u^2)]_{\varepsilon_2}(t,x),\end{aligned}$$

where $\operatorname{sgn}^+_{\frac12}(z)=\mathbb{1}_{(0,\infty)}(z)+\frac12\mathbb{1}_{\{0\}}(z)$. Hence, we find

$$\begin{aligned}\limsup_{\delta\to0}\left\langle T^1_\delta,\theta\right\rangle\leq&\int\theta(t,x)|\chi'(v)|m^1_{\varepsilon_1}(dt,dx,dv)\\&+\int\theta(t,x)\int S^1(s,y)\Phi_{\varepsilon_1}(t-s,x-y)\chi(u^1(s,y))\\&\quad\cdot[\operatorname{sgn}^+_{\frac12}(u^1(s,y)-u^2)]_{\varepsilon_2}(t,x)\,dsdydtdx.\end{aligned}$$

A similar computation shows

$$\begin{aligned}\limsup_{\delta\to0}\left\langle T^2_\delta,\theta\right\rangle\leq&\int\theta(t,x)|\chi'(v)|m^2_{\varepsilon_2}(dt,dx,dv)\\&-\int\theta(t,x)\int S^2(s,y)\Phi_{\varepsilon_2}(t-s,x-y)\chi(u^2(s,y))\\&\quad\cdot[\operatorname{sgn}^+_{\frac12}(u^1-u^2(s,y))]_{\varepsilon_1}(t,x)\,dsdydtdx.\end{aligned}$$

By Fatou's lemma these inequalities imply

$$\begin{aligned}\langle T,\theta\rangle\leq&\Big\langle\int|\chi'(v)|m^1_{\varepsilon_1}(\cdot,\cdot,dv),\theta\Big\rangle+\Big\langle\int|\chi'(v)|m^2_{\varepsilon_2}(\cdot,\cdot,dv),\theta\Big\rangle\\&+\Big\langle\big(S^1\chi(u^1)\operatorname{sgn}^+_{\frac12}(u^1-u^2)_{\varepsilon_2}\big)_{\varepsilon_1},\theta\Big\rangle\end{aligned}$$

$$- \left\langle \left(S^2 \chi(u^2) \operatorname{sgn}^+_{\frac{1}{2}}(u^1 - u^2)_{\varepsilon_1} \right)_{\varepsilon_2}, \theta \right\rangle.$$

Next we "integrate" this inequality in x, that is, we apply it to a test function $\theta(t,x) = \zeta(t)K(x) \geq 0$ and let $K(x)$ approach $K \equiv 1$. Note that since

$$\int |g^1(s,y,v)| \cdot |1 - g^2(s',y',v)| \, dv = (u^1(s,y) - u^2(s',y'))_+,$$

for any $\theta(t,x) \in L^\infty$ and $\psi(v) \in L^\infty_{loc}$ we have

$$\begin{aligned} &\int \left| g^1_{\varepsilon_1}(1 - g^2_{\varepsilon_2})\theta(t,x)\psi(v)\chi(v) \right| \, dt dx dv \\ &\leq \|\psi\|_{L^\infty(\operatorname{supp}\chi)} \|\theta\|_{L^\infty} \left(\|u^1\|_{L^1_{t,x}} + \|u^2\|_{L^1_{t,x}} \right). \end{aligned}$$

Using this together with $S^j \in L^1$ and $\int m^j(dt,dx,dv)|\chi'(v)| < \infty$, and letting $K(x)$ approach $K \equiv 1$ nicely enough, we obtain

$$\begin{aligned} &\partial_t \int g^1_{\varepsilon_1}(1 - g^2_{\varepsilon_2})\chi(v) \, dx dv \\ &\leq \int |\chi'(v)| m^1_{\varepsilon_1}(\cdot, dx, dv) + \int |\chi'(v)| m^2_{\varepsilon_2}(\cdot, dx, dv) \\ &\quad + \int \left(S^1 \chi(u^1) \operatorname{sgn}^+_{\frac{1}{2}}(u^1 - u^2)_{\varepsilon_2} \right)_{\varepsilon_1} dx \\ &\quad - \int \left(S^2 \chi(u^2) \operatorname{sgn}^+_{\frac{1}{2}}(u^1 - u^2)_{\varepsilon_1} \right)_{\varepsilon_2} dx. \end{aligned}$$

The same integrability properties also allow to let $\varepsilon_1, \varepsilon_2 \to 0$ and to find

$$\begin{aligned} &\partial_t \int g^1(1 - g^2)\chi(v) \, dx dv \\ &\leq \int |\chi'(v)| m^1(\cdot, dx, dv) + \int |\chi'(v)| m^2(\cdot, dx, dv) \\ &\quad + \int S^1 \chi(u^1) \operatorname{sgn}^+_{\frac{1}{2}}(u^1 - u^2) \, dx - \int S^2 \chi(u^2) \operatorname{sgn}^+_{\frac{1}{2}}(u^1 - u^2) \, dx. \end{aligned}$$

We apply this inequality to a nonnegative test function $\zeta(t)$ and choose $\chi = \chi_n$ for a sequence $\chi_n \to 1$ a.e. with $\chi_n'(v) \equiv 0$ for $|v| \leq n$ and $|\chi_n'| \leq 1$. Then the first two terms in the right-hand side are estimated by

$$\|\zeta\|_{L^\infty} \cdot \sup_{|v|>n} \left(\mu^1(v) + \mu^2(v) \right),$$

which tends to 0 as $n \to \infty$ since $\mu^j \in L^\infty_0$. The two last terms converge by dominated convergence, which yields

$$\partial_t \int g^1(1 - g^2) \, dx dv \leq \int (S^1 - S^2) \operatorname{sgn}^+_{\frac{1}{2}}(u^1 - u^2) \, dx.$$

Applying this to a nonnegative test function ζ approaching $\zeta = \mathbb{1}_{[0,t]}$ and using that $u^j \in C([0,T], L^1(\mathbb{R}^n_x))$, we conclude that

$$\int (u^1(t) - u^2(t))_+ \, dx \le \int (u^1_0 - u^2_0)_+ \, dx + \|S^1 - S^2\|_{L^1((0,t)\times\mathbb{R}^n)}.$$

Interchanging the roles of u^1, u^2 thus yields

$$\begin{aligned} \sup_{t\in[0,T]} \|u^1(t) - u^2(t)\|_{L^1(\mathbb{R}^n)} \le{}& \|u^1_0 - u^2_0\|_{L^1(\mathbb{R}^n)} \\ &+ \|S^1 - S^2\|_{L^1([0,T]\times\mathbb{R}^n)}. \end{aligned} \tag{A.3}$$

Existence: Let $u^\varepsilon_0 \in W^{1,\infty}(\mathbb{R}^n)$, $S^\varepsilon \in W^{1,\infty}([0,T]\times\mathbb{R}^n)$ with $u^\varepsilon_0 \to u_0$ in $L^1(\mathbb{R}^n)$ and $S^\varepsilon \to S$ in $L^1([0,T]\times\mathbb{R}^n)$. By [15] and [16, Corollary 2.5] there is a unique entropy solution $u^\varepsilon \in C([0,T]; L^1(\mathbb{R}^n))$ to (1.1) with initial condition u^ε_0 and force S^ε.

To see that u^ε is also a kinetic solution we follow the arguments given in [18]: Let

$$f^\varepsilon(t,x,v) = \chi(v, u^\varepsilon(t,x)) = \mathbb{1}_{0<v<u^\varepsilon(t,x)} - \mathbb{1}_{0>v>u^\varepsilon(t,x)}$$

and define the distribution m by

$$m^\varepsilon := \int_0^v \partial_t f^\varepsilon \, dw + \int_0^v a(w)\cdot\nabla_x f^\varepsilon \, dw - \int_0^v \delta_{u^\varepsilon=w} S^\varepsilon \, dw.$$

Hence, f^ε satisfies (A.1) and it remains to show that m^ε is a non-negative measure. Given $\eta \in C^\infty_c(\mathbb{R})$ convex we obtain, in the sense of distributions,

$$\begin{aligned} -\int m^\varepsilon \eta''(v)\, dv &= \int \partial_t f^\varepsilon \eta'(v)\, dv + \int \eta'(v) a(v)\cdot\nabla_x f^\varepsilon \, dv \\ &\quad - \int \eta'(v)\delta_{u^\varepsilon=v} S^\varepsilon \, dv \\ &= \partial_t \eta(u^\varepsilon) + \mathrm{div} A^\eta(u^\varepsilon) - \eta'(u^\varepsilon) S^\varepsilon. \end{aligned} \tag{A.4}$$

Since u^ε is an entropy solution, the right hand side is non-positive. This implies that m^ε is a non-negative distribution and thus a measure.

A standard approximation argument (cf. e.g. [18, Proposition 3.2.3]) allows to choose $\eta'(v) = \mathrm{sgn}_+(v-k)$ in (A.4), which yields

$$\begin{aligned} &\int (u^\varepsilon(t) - k)_+ \, dx + \int m^\varepsilon(t,x,k)\, dtdx \\ &= \int (u^\varepsilon(0) - k)_+ \, dx + \int \mathrm{sgn}_+(u^\varepsilon - k) S^\varepsilon \, dtdx. \end{aligned} \tag{A.5}$$

An analogous equality is satisfied for $(u^\varepsilon(t) - k)_-$. Hence, with

$$\begin{aligned} \mu^\varepsilon(k) :={}& \mathbb{1}_{k\ge 0}(\|u^\varepsilon(0) - k)_+\|_{L^1_x} + \|\mathrm{sgn}_+(u^\varepsilon - k)S^\varepsilon\|_{L^1_{t,x}}) \\ &+ \mathbb{1}_{k\le 0}(\|u^\varepsilon(0) - k)_-\|_{L^1_x} + \|\mathrm{sgn}_-(u^\varepsilon - k)S^\varepsilon\|_{L^1_{t,x}}) \end{aligned}$$

we have $\mu^\varepsilon \in L^\infty_0(\mathbb{R})$ by dominated convergence, and

$$\int m^\varepsilon(t,x,k)\,dtdx \le \mu^\varepsilon(k).$$

Hence, u^ε is a kinetic solution.

By (A.3) we have

$$\|u^\varepsilon - u^\delta\|_{C([0,T];L^1(\mathbb{R}^n))} \le \|u_0^\varepsilon - u_0^\delta\|_{L^1(\mathbb{R}^n)} + 2\|S^\varepsilon - S^\delta\|_{L^1([0,T]\times\mathbb{R}^n)}.$$

Since u^ε and S^ε were chosen to converge in $L^1(\mathbb{R}^n)$ and $L^1([0,T]\times\mathbb{R}^n)$, this implies that u^ε is a Cauchy sequence in $C([0,T];L^1(\mathbb{R}^n))$. Hence, there is a $u \in C([0,T];L^1(\mathbb{R}^n))$ and a sequence ε_k converging to zero such that $u^{\varepsilon_k} \to u$ in $C([0,T];L^1(\mathbb{R}^n))$ and almost everywhere in $[0,T]\times\mathbb{R}^n$. Moreover, by (A.5) we obtain that m^{ε_k} has locally uniformly bounded mass. Thus, choosing a diagonal sequence we obtain a subsequence (again denoted by ε_k) such that $m^{\varepsilon_k} \rightharpoonup^* m$ in the space of measures on $\mathbb{R}_+\times\mathbb{R}^n\times\mathbb{R}$. It is then easy to see that u is a kinetic solution to (1.1). □

Proposition 12. *Let $u_0 \in L^1(\mathbb{R}^n)$, $S \in L^1([0,T]\times\mathbb{R}^n)$ and u be the corresponding entropy solution to (1.1). For each $\alpha \in (0,1)$ and each $v_0 \in \mathbb{R}$ the measure $|v-v_0|^{\alpha-1}m$ has locally finite mass.*

Proof. Let $\alpha \in (0,1)$. Let ϕ be a non-negative smooth compactly supported function in $(0,T)\times\mathbb{R}^n\times\mathbb{R}$. Then (A.1) implies, with $\tilde f := \phi f$, $\tilde m := \phi m$,

$$\partial_t \tilde f + a(v)\cdot\nabla_x \tilde f = \partial_v \tilde m - (\partial_v\phi)m + \phi\delta_{u=v}S + \partial_t\phi f + (a(v)\cdot\nabla_x\phi)f. \tag{A.6}$$

By translation we may assume $v_0 = 0$. We next choose a sequence of smooth, compactly supported functions η^ε such that $\mathrm{sgn}(\eta^\varepsilon(v)) = \mathrm{sgn}(v)$, $\eta^\varepsilon(v) \le \frac{1}{\alpha}|v|^\alpha$ and $(\eta^\varepsilon)' \uparrow |v|^{\alpha-1}$ pointwise. Multiplying (A.6) by η^ε and integrating yields

$$\int (\eta^\varepsilon \tilde f)(t)\,dxdv + \int_0^t\int (\eta^\varepsilon)'\tilde m\,dxdv\,dr = \int_{x,v}(\eta^\varepsilon\tilde f)(0)$$
$$+\int_0^t\int \big(-\eta^\varepsilon(\partial_v\phi)m + \eta^\varepsilon\phi\delta_{u=v}S + \eta^\varepsilon\partial_t\phi f + \eta^\varepsilon(a(v)\cdot\nabla_x\phi)f\big)\,dxdv\,dr.$$

Since $m \ge 0$, by Fatou's Lemma we may pass to the limit $\varepsilon \to 0$ to obtain

$$\int_0^t\int |v|^{\alpha-1}\tilde m\,dxdv\,dr \le C < \infty$$

for some constant C depending on $\|u_0\|_{L^1_{loc}}, \|m\|_{\mathcal{M}_{loc}}, \|S\|_{L^1_{loc}}$. □

Appendix B. A basic estimate

From [11] we recall the following basic L^p-estimate for certain Fourier multipliers. This result generalizes [19, Lemma 2.2] by taking into account possible v-regularity of f. This allows to avoid bootstrapping arguments in the application to scalar conservation laws. This is crucial in the case of scalar conservation laws with L^1-forcing, since in this case bootstrapping arguments do not apply.

Lemma 13. *Let $m(\tau',\xi',v) := i\tau' + ia(v)\cdot\xi'$, φ, ϕ be bounded, smooth functions, ψ be a smooth cut-off function and M_ψ be the Fourier multiplier with symbol $\varphi(\tau',\xi')\psi\left(\frac{m(\tau',\xi',v)}{\delta}\right)$. Then, for all $1 < p \le 2$, $\sigma \ge 0$, $r \in [\frac{p'}{1+\sigma p'}, p'] \cap (1,\infty)$,*

$$\| \int M_\psi f\phi \, dv\|_{L^p(\mathbb{R}_t\times\mathbb{R}^n_x)} \lesssim \|f\phi\|_{L^p(\mathbb{R}_t\times\mathbb{R}^n_x; W^{\sigma,p}(\mathbb{R}_v))} \sup_{\tau',\xi' \in \operatorname{supp}\varphi} |\Omega_m(\tau',\xi',\delta)|^{\frac{1}{r}},$$

where $\Omega_m(\tau',\xi',\delta) = \{v \in \operatorname{supp}\phi : \, |m(\tau',\xi',v)| \le \delta\}$. Moreover,

$$\| \int M_\psi f\phi \, dv\|_{\mathcal{M}(\mathbb{R}_t\times\mathbb{R}^n_x)} \lesssim \|f\phi\|_{\mathcal{M}(\mathbb{R}_t\times\mathbb{R}^n_x)}.$$

Lemma 13 relies on the fact that $\psi\left(\frac{i\tau'+ia(v)\cdot\xi'}{\delta}\right)$ is a bounded L^p (and $\mathcal{M}$) multiplier uniformly in $v \in I$ and $\delta > 0$ (the *truncation property* in [19]). This can be deduced, arguing as in [10], from the invariance of the L^p multiplier norm under partial dilations and the Marcinkiewicz multiplier theorem. For details we refer to [11, Lemma A.3].

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Max-Planck Institute for Mathematics in the Sciences, Inselstrasse 22, 04103 Leipzig, Germany

E-mail address: bgess@mis.mpg.de

Institut de Mathématiques de Toulouse; UMR5219, Université de Toulouse; CNRS, UPS IMT, F-31062 Toulouse Cedex 9, France

E-mail address: xlamy@math.univ-toulouse.fr