THE CONFORMAL LIMIT FOR BIMERONS IN EASY-PLANE CHIRAL MAGNETS

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ABSTRACT. We study minimizers $m\colon \mathbb{R}^2 o \mathbb{S}^2$ of the energy functional

$$E_{\sigma}(\boldsymbol{m}) = \int_{\mathbb{R}^2} \left(\frac{1}{2} |\nabla \boldsymbol{m}|^2 + \sigma^2 \boldsymbol{m} \cdot \nabla \times \boldsymbol{m} + \sigma^2 m_3^2 \right) dx \,,$$

for $0 < \sigma \ll 1$, with prescribed topological degree

$$Q(\boldsymbol{m}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \boldsymbol{m} \cdot \partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m} \, dx = \pm 1$$

This model arises in thin ferromagnetic films with Dzyaloshinskii-Moriya interaction and easy-plane anisotropy, where these minimizers represent *bimeron* configurations. We prove their existence, and describe them precisely as perturbations of specific Möbius maps: we establish in particular that they are localized at scale of order $1/|\ln(\sigma^2)|$. The proof follows a strategy introduced by Bernand-Mantel, Muratov and Simon (Arch. Ration. Mech. Anal., 2021) for a similar model with easy-axis anisotropy, but requires several adaptations to deal with the less coercive easy-plane anisotropy and different symmetry properties.

1. INTRODUCTION

1.1. Energy functional and topological degree. For maps

$$\boldsymbol{m} = (m_1, m_2, m_3) \colon \mathbb{R}^2 \to \mathbb{S}^2 \subset \mathbb{R}^3,$$

and $\sigma > 0$, we consider the energy

$$E_{\sigma}(\boldsymbol{m}) = D(\boldsymbol{m}) + \sigma^{2} \left(A(\boldsymbol{m}) + \tilde{H}(\boldsymbol{m}) \right), \qquad (1.1)$$
$$D(\boldsymbol{m}) = \frac{1}{2} \int_{\mathbb{R}^{2}} |\nabla \boldsymbol{m}|^{2} dx, \qquad A(\boldsymbol{m}) = \int_{\mathbb{R}^{2}} m_{3}^{2} dx,$$
$$\tilde{H}(\boldsymbol{m}) = 2 \int_{\mathbb{R}^{2}} m_{3} (\partial_{1} m_{2} - \partial_{2} m_{1}) dx.$$

It arises in the description of a thin ferromagnetic film with Dzyaloshinskii-Moriya interaction (DMI) and easy-plane anisotropy (see e.g. [2, 8]). The map m represents the magnetization, the Dirichlet term D(m) corresponds

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to the exchange energy, and the term $A(\mathbf{m})$ to easy-plane anisotropy favoring the horizontal plane $\{m_3 = 0\}$. The DMI term $\tilde{H}(\mathbf{m})$ is well-defined as soon as the two other terms are finite. Moreover, for $0 < \sigma < 1/2$, the energy density ¹

$$e_{\sigma}(\boldsymbol{m}) = rac{1}{2} |
abla \boldsymbol{m}|^2 + \sigma^2 m_3^2 + 2\sigma^2 m_3 (\partial_1 m_2 - \partial_2 m_1)$$

 $\geq rac{1-2\sigma}{2} |
abla \boldsymbol{m}|^2 + \sigma^2 (1-2\sigma) m_3^2 \geq 0,$

is integrable if and only if $D(\boldsymbol{m}) + A(\boldsymbol{m}) < \infty$.

If a map $\boldsymbol{m} \colon \mathbb{R}^2 \to \mathbb{S}^2$ is continuous and has a limit as $|x| \to +\infty$, then it can be identified with a continuous map $\tilde{\boldsymbol{m}} = \boldsymbol{m} \circ \Phi^{-1} \colon \mathbb{S}^2 \to \mathbb{S}^2$, where $\Phi \colon \mathbb{R}^2 \cup \{\infty\} \approx \mathbb{C} \cup \{\infty\} \to \mathbb{S}^2$ is the inverse stereographic projection

$$\Phi(z) = \left(\frac{2z}{1+|z|^2}, \frac{|z|^2 - 1}{1+|z|^2}\right) \qquad \forall z \in \mathbb{C} \cup \{\infty\}.$$
(1.2)

The continuous map $\tilde{\boldsymbol{m}} \colon \mathbb{S}^2 \to \mathbb{S}^2$ carries a topological degree, which can also be defined if $\tilde{\boldsymbol{m}} \in H^1(\mathbb{S}^2; \mathbb{S}^2)$ (see [5]), and which characterizes the homotopy class of $\tilde{\boldsymbol{m}}$. In terms of the original map \boldsymbol{m} , this corresponds to the topological degree

$$Q(\boldsymbol{m}) = \frac{1}{4\pi} \int_{\mathbb{R}^2} \boldsymbol{m} \cdot (\partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m}) \, dx \in \mathbb{Z} \,. \tag{1.3}$$

see more details in Appendix A, in particular Corollary A.2. The purpose of this article is to describe, for $0 < \sigma \ll 1$, minimizers of E_{σ} with unit degree $Q(\boldsymbol{m}) = \pm 1$, called *bimerons* [2]. The precise functional setting will be presented in § 1.4.

An analogous question is analysed in [6, 4] for a model with easy-axis anisotropy, where the minimizes stand for *skyrmions*. The results and proofs share similarities, but also many differences and additional difficulties, and we will carefully compare them after the statement of our main result. For the easy-plane model considered here, results complementary to ours, dealing with bounded domains and a wider range of parameters, are obtained in [3]. In the context of bounded domains, boundary magnetic vortices are also studied in the presence of the DMI term, see [12].

Remark 1.1. For maps $\boldsymbol{m} \colon \mathbb{R}^2 \to \mathbb{S}^2$ such that $D(\boldsymbol{m}) + A(\boldsymbol{m}) < \infty$ and m_3 decays sufficiently fast at ∞ , the DMI term $\tilde{H}(\boldsymbol{m})$ in (1.1) coincides with the classical expression

$$H(\boldsymbol{m}) = \int_{\mathbb{R}^2} \boldsymbol{m} \cdot \nabla \times \boldsymbol{m} \, dx = \lim_{R \to \infty} \int_{B_R} \boldsymbol{m} \cdot \nabla \times \boldsymbol{m} \, dx \,,$$

¹In our results, σ will be a small positive parameter $0 < \sigma \ll 1$.

where ∇ is identified with $(\partial_1, \partial_2, 0)$, thanks to the identity

$$\boldsymbol{m} \cdot \nabla \times \boldsymbol{m} - 2m_3(\partial_1 m_2 - \partial_2 m_1) = -\partial_1(m_3 m_2) + \partial_2(m_3 m_1).$$

Under the mere condition that $D(m) + A(m) < \infty$, we do have

$$\lim_{R_k\to\infty}\int_{B_{R_k}}\boldsymbol{m}\cdot\nabla\times\boldsymbol{m}\,dx=\widetilde{H}(\boldsymbol{m})\,,$$

along a sequence $R_k \to \infty$, but $H(\boldsymbol{m})$ might not be well-defined and the full limit as $R \to +\infty$ might fail to exist.

1.2. **Symmetries.** Several groups of geometric transformations play an important role in our analysis: dilations

$$\mathfrak{D}_{\rho}\boldsymbol{m}(x) = \boldsymbol{m}\left(\frac{x}{\rho}\right), \quad \rho > 0, \qquad (1.4)$$

translations

$$\mathfrak{T}_{x_0}\boldsymbol{m}(x) = \boldsymbol{m}(x - x_0), \quad x_0 \in \mathbb{R}^2, \qquad (1.5)$$

and corotations

$$\mathfrak{R}_{\phi}\boldsymbol{m}(x) = R_{\mathbf{e}_{3},\phi}\boldsymbol{m}(e^{-i\phi}x), \quad \phi \in \mathbb{R}, \qquad (1.6)$$

where $R_{\mathbf{e}_3,\phi} \in SO(3)$ is the rotation of axis \mathbf{e}_3 and angle ϕ in \mathbb{R}^3 , and $e^{-i\phi}$ is the rotation of angle $-\phi$ in $\mathbb{R}^2 \approx \mathbb{C}$.

Dilations have different effects on each energy term in (1.1) namely, for any $\rho > 0$,

$$D(\mathfrak{D}_{\rho}\boldsymbol{m}) = D(\boldsymbol{m}),$$

$$\widetilde{H}(\mathfrak{D}_{\rho}\boldsymbol{m}) = \rho \widetilde{H}(\boldsymbol{m}),$$

$$A(\mathfrak{D}_{\rho}\boldsymbol{m}) = \rho^{2}A(\boldsymbol{m}),$$
(1.7)

and the topological degree (1.3) is invariant under dilations:

$$Q(\mathfrak{D}_{\rho}\boldsymbol{m}) = Q(\boldsymbol{m})$$
 .

As for translations and corotations, they keep both energy and topological degree invariant:

$$E_{\sigma}(\mathfrak{T}_{x_0}\boldsymbol{m}) = E_{\sigma}(\mathfrak{R}_{\phi}\boldsymbol{m}) = E_{\sigma}(\boldsymbol{m}), \qquad (1.8)$$
$$Q(\mathfrak{T}_{x_0}\boldsymbol{m}) = Q(\mathfrak{R}_{\phi}\boldsymbol{m}) = Q(\boldsymbol{m}), \quad \forall x_0 \in \mathbb{R}^2, \forall \phi \in \mathbb{R}.$$

Finally we also note the effect of the reflection $m \rightsquigarrow -m$, which keeps the energy invariant but reverses the degree:

$$E_{\sigma}(-\boldsymbol{m}) = E_{\sigma}(\boldsymbol{m}), \quad Q(-\boldsymbol{m}) = -Q(\boldsymbol{m}).$$
 (1.9)

Thanks to this property, we reduce the study of minimizers under the topological constraint $Q(\mathbf{m}) = \pm 1$, to the case $Q(\mathbf{m}) = -1$.

Remark 1.2. One consequence of the scaling properties (1.7) is that the specific choice of coefficients in front of each term in (1.1) is not restrictive: for any λ , $\rho > 0$ we have

$$\lambda E_{\sigma}(\mathfrak{D}_{\rho}\boldsymbol{m}) = \lambda D(\boldsymbol{m}) + \lambda \rho \sigma^2 \widetilde{H}(\boldsymbol{m}) + \lambda \rho^2 \sigma^2 A(\boldsymbol{m}),$$

so any result about E_{σ} can be translated into a result about an energy with arbitrary coefficients $a = \lambda$, $b = \lambda \rho \sigma^2$, $c = \lambda \rho^2 \sigma^2 > 0$ in front of the three energy terms. With these general coefficients, the regime $\sigma^2 \ll 1$ considered in this article corresponds to $b^2 \ll ac$.

1.3. The conformal limit: heuristic description. In the conformal limit $\sigma \to 0$, the energy E_{σ} in (1.1) formally reduces to the Dirichlet energy $D(\boldsymbol{m})$, whose minimizers with prescribed degree $Q(\boldsymbol{m})$ are well-known and satisfy $D(\boldsymbol{m}) = 4\pi |Q(\boldsymbol{m})|$ (see e.g. [14]). More precisely, the pointwise inequality

$$\left|\boldsymbol{m}\cdot(\partial_{1}\boldsymbol{m}\times\partial_{2}\boldsymbol{m})\right|\leqrac{1}{2}|
abla \boldsymbol{m}|^{2},$$

implies indeed that

$$D(\boldsymbol{m}) \ge 4\pi |Q(\boldsymbol{m})|,\tag{1.10}$$

with equality if and only if $\partial_1 \boldsymbol{m} \cdot \partial_2 \boldsymbol{m} = |\partial_1 \boldsymbol{m}|^2 - |\partial_2 \boldsymbol{m}|^2 = 0$, that is, \boldsymbol{m} is conformal. Conformal maps of degree $Q(\boldsymbol{m}) = -1$ are given by the Möbius group

$$\mathcal{M} = \left\{ \Phi\left(\frac{az+b}{cz+d}\right) \colon a, b, c, d \in \mathbb{C}, \, ad-bc \neq 0 \right\},\tag{1.11}$$

where $\Phi \colon \mathbb{C} \cup \{\infty\} \to \mathbb{S}^2$ is the inverse stereographic projection defined in (1.2). In the parametrization (1.11), the determinant $ad - bc \in \mathbb{C} \setminus \{0\}$ can always be fixed, and the Möbius group is six-dimensional (as a real manifold).

Formally, the constraint $A(\mathbf{m}) < \infty$ forces $m_3 \to 0$ at ∞ . For small positive σ , one therefore expects minimizers of E_{σ} of degree -1 to be close to conformal maps with |a| = |c| in the parametrization (1.11). Another real parameter can be fixed by minimizing the DMI term, and this leaves a four-real-parameter family of conformal maps. Three of these parameters come from the translational and corotational invariance of the energy (1.8). The last parameter can be interpreted as a scaling parameter $\rho > 0$, corresponding to a dilation (1.4). Performing these reductions explicitly, one expects that a minimizer \mathbf{m}_{σ} of E_{σ} with $Q(\mathbf{m}_{\sigma}) = -1$ should be close to a Möbius map

$$\Psi(z) = \Phi\left(w_*\left(\frac{z}{\rho}\right)\right), \qquad w_*(z) = i\frac{z-1}{z+1},$$

modulo translation (1.5) and corotation (1.6). The expression of $w_*(z)$ features a vortex at z = 1 and an antivortex at z = -1, characteristic of bimeron structures, as described in [2] (see [9] or [11, § 7.2] for a survey about the mathematical analysis of magnetic vortices).

The three energy terms have different scaling behaviors (1.7), so one further expects the scale of concentration $\rho > 0$ to be fixed by the competition between these three terms. However, since Möbius maps $\Psi \in \mathcal{M}$ have infinite anisotropy $A(\Psi) = \infty$, one cannot simply plug this ansatz into the energy: identifying that concentration scale ρ is a more subtle task. For skyrmions, this task was carried out in [4] by introducing several powerful new tools and ideas. Here we adapt that strategy to bimerons, and obtain that minimizers m_{σ} of E_{σ} with topological degree $Q(m_{\sigma}) = -1$ concentrate at scale

$$ho_{\sigma} = rac{1+o(1)}{\ln(1/\sigma^2)} \quad ext{as } \sigma o 0 \,,$$

that is, m_{σ} is close, in a sense to be made precise below, to the orbit of the Möbius map

$$\Psi_{\sigma}(z) = \Phi\left(w_*\left(\frac{z}{\rho_{\sigma}}\right)\right), \quad w_*(z) = i\frac{z-1}{z+1},$$

under the action of translations (1.5) and corotations (1.6).

1.4. Functional framework and precise statement. For any measurable map $m : \mathbb{R}^2 \to \mathbb{S}^2$ belonging to the homogeneous Sobolev space

$$\mathcal{H}(\mathbb{R}^2;\mathbb{S}^2) = \left\{ \boldsymbol{m} \in H^1_{\text{loc}}(\mathbb{R}^2;\mathbb{S}^2) \colon \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 \, dx < \infty \right\}, \quad (1.12)$$

composing with the stereographic projection (1.2) gives a map

$$ilde{m{m}} = m{m} \circ \Phi^{-1} \in H^1(\mathbb{S}^2;\mathbb{S}^2)\,,$$

whose topological degree is a well-defined integer [5] and equal to the topological degree Q(m) defined in (1.3). See Appendix A for more details about these claims.

For all $\sigma > 0$ we consider the energy E_{σ} defined by (1.1) on the set

$$\mathcal{W} = \left\{ \boldsymbol{m} \in \mathcal{H}(\mathbb{R}^2; \mathbb{S}^2) \colon A(\boldsymbol{m}) < \infty \right\}$$

= $\left\{ \boldsymbol{m} \in \mathcal{H}(\mathbb{R}^2; \mathbb{S}^2) \colon \int_{\mathbb{R}^2} m_3^2 dx < \infty \right\},$ (1.13)

which the value of the topological degree partitions into the subsets

$$\mathcal{W}_q = \left\{ \boldsymbol{m} \in \mathcal{W} \colon Q(\boldsymbol{m}) = q \right\}, \quad \text{for } q \in \mathbb{Z}.$$
 (1.14)

Recall that the reflection $m \rightsquigarrow -m$ provides a bijection between W_q and W_{-q} , while preserving the energy (1.9). With these notations, our main result provides precise asymptotics for minimizers of E_{σ} on W_{-1} in the limit $\sigma \to 0$.

Theorem 1.3. There exist absolute constants $\sigma_0, C > 0$ such that, for any $\sigma \in (0, \sigma_0]$, the infimum of E_{σ} over W_{-1} is attained, satisfies

$$\min_{\mathcal{W}_{-1}} E_{\sigma} = 4\pi - \frac{2\pi\sigma^2}{\ln\left((1/\sigma^2)\ln^2(1/\sigma^2)\right)} + \mathcal{O}\left(\frac{\sigma^2}{\ln^2(1/\sigma^2)}\right),$$

and, for any minimizing map $\mathbf{m}_{\sigma} \in \mathcal{W}_{-1}$, there exist $\rho_{\sigma} > 0$ and $\alpha_{\sigma} \in \mathbb{R}$ estimated by

$$\left|\ln(1/\sigma^2)\rho_{\sigma} - 1\right| + \left|\alpha_{\sigma}\right| \le \frac{C}{\sqrt{\ln(1/\sigma^2)}},$$

and a Möbius map $\Psi_{\sigma} \in \mathcal{M}$ characterized by

0

$$\mathfrak{T}_{z_{\sigma}}\mathfrak{R}_{\phi_{\sigma}}\Psi_{\sigma}(z) = \Phi\left(w_*\left(e^{-i\alpha_{\sigma}}\frac{z}{\rho_{\sigma}}\right)\right), \quad w_*(z) = i\frac{z-1}{z+1}, \quad (1.15)$$

for some translation $\mathfrak{T}_{z_{\sigma}}$ and corotation $\mathfrak{R}_{\phi_{\sigma}}$ as in (1.5)-(1.6), such that

$$\int_{\mathbb{R}^2} \left| \nabla \left(\boldsymbol{m}_{\sigma} - \Psi_{\sigma} \right) \right|^2 dx \le C \frac{\sigma^2}{\ln(1/\sigma^2)} \,.$$

Remark 1.4. A slightly more precise description of the orbit of Möbius maps closest to minimizers of \mathcal{E}_{σ} with degree Q = -1 is given in Proposition 5.4. Its proof also makes the contribution of each energy term apparent:

$$D(\boldsymbol{m}_{\sigma}) = 4\pi + \mathcal{O}\left(\frac{\sigma^2}{\ln^2(1/\sigma^2)}\right),$$

$$H(\boldsymbol{m}_{\sigma}) = -2A(\boldsymbol{m}_{\sigma}) = -\frac{4\pi}{\ln\left((1/\sigma^2)\ln^2(1/\sigma^2)\right)} + \mathcal{O}\left(\frac{1}{\ln^2(1/\sigma^2)}\right)$$

Here, the Pohozaev identity $H(\boldsymbol{m}_{\sigma}) = -2A(\boldsymbol{m}_{\sigma})$ comes from criticality of the minimizer \boldsymbol{m}_{σ} with respect to scaling.

1.5. Comparison with skyrmions. The main difference between bimerons and skyrmions is that the easy-plane anisotropy A(m) in the energy (1.1) is replaced by an easy-axis anisotropy

$$A_1^{\text{easy-axis}}(\boldsymbol{m}) = \int_{\mathbb{R}^2} (1 - m_3^2) \, dx \,, \tag{1.16}$$

r
$$A_2^{\text{easy-axis}}(\boldsymbol{m}) = \int_{\mathbb{R}^2} (1 - m_3) \, dx \,.$$

The first version is used in [4], and the second in [18, 6]. Formally, the first enforces only $m(x) \to \pm e_3$ as $|x| \to \infty$, and the second selects the

orientation $+e_3$. In practice, minimizers for the first version are obtained in a space which already selects an orientation, see e.g. [4, § 2.1], and the analysis of these two different easy-axis models is extremely similar. We will not comment on additional nonlocal energy terms which are also considered in [4].

Remark 1.5. Changing the anisotropy forces to change the DMI term. Let $m' = (m_1, m_2)$ and $\nabla^{\perp} = (-\partial_2, \partial_1)$. If $D(\boldsymbol{m}) + A_j^{\text{easy-axis}}(\boldsymbol{m}) < \infty$ for j = 1 or 2, the DMI term

$$\widetilde{H}^{\text{easy-axis}}(\boldsymbol{m}) = -2 \int_{\mathbb{R}^2} m' \cdot \nabla^{\perp} m_3 \, dx \,,$$

is well-defined, since $|m'|^2 = 1 - m_3^2$ is controlled by the easy-axis anisotropy. Similarly to Remark 1.1, it satisfies

$$\widetilde{H}^{\text{easy-axis}}(\boldsymbol{m}) = \lim_{k \to \infty} \int_{B_{R_k}} \boldsymbol{m} \cdot \nabla \times \boldsymbol{m} \, dx$$

for some sequence $R_k \to \infty$. Under the condition $D(\boldsymbol{m}) + A_2^{\text{easy-axis}}(\boldsymbol{m}) < \infty$ which selects the orientation $+\mathbf{e}_3$ at ∞ , this DMI term can also be rewritten as

$$\widetilde{H}^{\text{easy-axis}}(\boldsymbol{m}) = \int_{\mathbb{R}^2} (\boldsymbol{m} - \mathbf{e}_3) \cdot \nabla \times \boldsymbol{m} \, dx$$
$$= 2 \int_{\mathbb{R}^2} (m_3 - 1) (\partial_1 m_2 - \partial_2 m_1) \, dx \,, \qquad (1.17)$$

as used repeatedly in [18, 6, 4].

The main, obvious difference between our easy-plane anisotropy $A(\boldsymbol{m})$ and the easy-axis anisotropy $A_1^{\text{easy-axis}}(\boldsymbol{m})$ is the structure of admissible constant states, or, equivalently, admissible far-field limits: when it exists, the limit $\boldsymbol{m}_{\infty} = \lim_{|x| \to \infty} \boldsymbol{m}(x) \in \mathbb{S}^2$ must satisfy

$$egin{aligned} m{m}_\infty \in \mathbb{S}^1 imes \{0\} & ext{if } A(m{m}) < \infty, \ & ext{versus} & m{m}_\infty \in \{\pm \mathbf{e}_3\} & ext{if } A_1^{ ext{easy-axis}}(m{m}) < \infty. \end{aligned}$$

We list next several consequences of this difference.

Sign of the topological degree. The set $\mathbb{S}^1 \times \{0\}$ is connected, while $\{\pm \mathbf{e}_3\}$ has two components. They divide each homotopy class of \mathbb{S}^2 -valued maps into two distinct classes of admissible easy-axis maps, causing a distinction between positive and negative topological degrees in the skyrmion model [18, 6, 4]. This distinction is not present here.

Symmetric solutions. The corotational symmetry of the energy (1.8), also valid in the easy-axis model, makes it natural to look for critical points

which are axisymmetric: invariant under the action of corotations (1.6), that is, $\mathbf{m}(x) = R_{\mathbf{e}_3,\phi}\mathbf{m}(e^{-i\phi}x)$ for all $\phi \in \mathbb{R}$. In polar coordinates $z = re^{i\theta}$, axisymmetric maps are of the form $\mathbf{m}_{\text{sym}}(re^{i\theta}) = R_{\mathbf{e}_3,\theta}\mathbf{m}(r)$. This ansatz is compatible with $\mathbf{m}_{\infty} \in \{\pm \mathbf{e}_3\}$ in the easy-axis case, and axisymmetric skyrmions of degree Q = -1 are analysed in [15]. But there are no axisymmetric bimerons, because this ansatz is not compatible with $\mathbf{m}_{\infty} \in \mathbb{S}^1 \times \{0\}$, and more generally with $D(\mathbf{m}) + A(\mathbf{m}) < \infty$. Indeed, for such a symmetric ansatz, we have

$$\int_{1}^{\infty} (1 - m_3^2) \frac{dr}{r} = \frac{1}{2\pi} \int_{|x| \ge 1} \frac{|\partial_{\theta} \boldsymbol{m}_{\text{sym}}|^2}{r^2} \, dx \le D(\boldsymbol{m}_{\text{sym}}) \,,$$

and
$$\int_{1}^{\infty} m_3^2 \frac{dr}{r} \le \int_{1}^{\infty} m_3^2 r \, dr + \int_{1}^{\infty} m_3^2 \frac{dr}{r^3} \le A(\boldsymbol{m}_{\text{sym}}) + 1 \,,$$

hence $D(\boldsymbol{m}_{\rm sym}) + A(\boldsymbol{m}_{\rm sym}) < \infty$ would imply $\int_1^\infty dr/r < \infty$, a contradiction.

Dimension of the selected Möbius maps. In the easy-plane case described by Theorem 1.3, the selected orbit of Möbius maps (under the action of translations and corotations) is three-dimensional, while for skyrmions it is two-dimensional. This is related with the previous observation: the skyrmions' orbit is smaller because it contains an axisymmetric map, which stays fixed under the action of corotations (1.6).

Far-field behavior. The set $\mathbb{S}^1 \times \{0\}$ is one-dimensional, while $\{\pm \mathbf{e}_3\}$ is discrete. In that sense, easy-axis anisotropy is much more constraining, and this is reflected in the far-field behavior of finite-energy configurations, that is, their behavior as $|x| \to +\infty$. The assumption of finite easy-axis anisotropy $A_j^{\text{easy-axis}}(\boldsymbol{m}) < \infty$ (for j = 1 or 2) implies that ∞ is a Lebesgue point of \boldsymbol{m} , in the sense that

$$oldsymbol{m}_R := \int_{|x|\geq R} oldsymbol{m} \, rac{dx}{(1+|x|^2)^2}
ightarrow \pm \mathbf{e}_3 \quad ext{as } R
ightarrow \infty,$$

see Corollary A.3. Here instead, under the finite easy-plane assumption $A(\boldsymbol{m}) < \infty$, we only know that $\operatorname{dist}(\boldsymbol{m}_R, \mathbb{S}^1 \times \{0\}) \to 0$, and \boldsymbol{m}_R might fail to have a limit as $R \to \infty$. Consider for instance $\boldsymbol{m}(x) = \Phi(e^{i\varphi(x)}w_*(x))$ with w_* as in Theorem 1.3 and $\varphi(x) = \ln(1 + \ln(1 + |x|^2))$, then $\boldsymbol{m} \in \mathcal{W}_{-1}$ but ∞ is not a Lebesgue point of \boldsymbol{m} .

Control of the DMI term. As another consequence of the previous point, the easy-axis anisotropy provides a more efficient control on the DMI term: indeed, using the expression (1.17) we see that

$$\left|\widetilde{H}^{\text{easy-axis}}(\boldsymbol{m})\right|^2 \leq 16D(\boldsymbol{m})\int_{\mathbb{R}^2}(1-m_3)^2\,dx\,,$$

and the last integral is controlled by $A_2^{\text{easy-axis}}(\boldsymbol{m})$, but, near ∞ , its integrand is actually much smaller than the integrand $(1 - m_3)$ of $A_2^{\text{easy-axis}}(\boldsymbol{m})$. This improved control is used crucially in [18] and [4], but is absent in our case.

1.6. **Proof ideas.** As explained above, the proof of Theorem 1.3 relies primarily on adapting the strategy introduced in [4] for a similar model with easy-axis anisotropy (1.16). The main tool is a stability estimate [4, Theorem 2.4] (see [10, 23] for alternative proofs and [21] for a generalization to higher degrees) which implies that minimizers of E_{σ} with degree Q = -1must be close to the Möbius group (1.11). This information can then be used to obtain lower bounds on each energy term, which depend on the closest Möbius map. Comparing these lower bounds with energy competitors which are close to the optimal orbit of Möbius maps (1.15) then implies that the closest Möbius map must belong to a neighborhood of that optimal orbit.

Next we underline the main new elements in our proof, compared to the analysis performed in [4] (and also [18]).

- The classical parametrization (2.1) of the Möbius group (1.11), was convenient in [4] to describe the Möbius maps close to skyrmions, but is not well adapted to the orbit of Möbius maps (1.15) close to bimerons. We provide a new and more adapted parametrization in Lemma 2.1.
- As explained in the last point of § 1.5, the control on the DMI term H(m) provided by the easy-plane anisotropy A(m) is less coercive than that in the easy-axis case. This plays a role in two places, where we cannot use the arguments in [18] and [4]: to obtain a sharp lower bound on the DMI term, and to rule out 'vanishing' in the proof of existence. We introduce new arguments to circumvent that lack of efficient control: see § 5.2, and Step 3 in the proof of Proposition 6.1.
- The upper bound is obtained by a construction which relies on modifying Möbius maps to make their anisotropy A(m) finite. Here we use, as in [6], a basic cut-off construction at one scale. The construction in [4] is more elaborate and cuts tails off in an optimal way with a modified Bessel function. The two constructions turn out to provide the same accuracy, see Remark 4.2. It only affects the explicit bound on the remainder term of order σ²/ln² σ in the energy expansion.

1.7. **Plan of the article.** The article is organized as follows. In § 2 we describe our tailored parametrization of the Möbius group. In § 3 we calculate the DMI energy of conformal maps. In § 4 we describe the construction

which provides the energy upper bound. In § 5 we prove the lower bound and characterize maps which almost saturate it. In § 6 we use that characterization to prove existence of minimizers for $0 < \sigma \ll 1$ and conclude the proof of Theorem 1.3.

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2. A PARAMETRIZATION OF THE MÖBIUS GROUP

The Möbius group (1.11) can be parametrized by

$$\boldsymbol{m} = S\Phi\left(\frac{z-z_0}{\rho}\right), \qquad z_0 \in \mathbb{C}, \ \rho > 0, \ S \in SO(3).$$
 (2.1)

This parametrization was convenient for the study of skyrmions in [4]. It is naturally expressed in terms of the translation operators \mathfrak{T}_{z_0} defined in (1.5), and of the dilation operators \mathfrak{D}_{ρ} defined in (1.4). The corotation operators \mathfrak{R}_{ϕ} defined in (1.6) do not appear, which is consistent with the fact that the optimal orbit of Möbius maps closest to skyrmions contains an axisymmetric map, as explained in § 1.5. But in our case we need to keep better track of corotations, and we choose therefore a different parametrization.

Lemma 2.1. The Möbius group can be parametrized by

$$\boldsymbol{m}^{\{\boldsymbol{z}_{0},\boldsymbol{\rho},\boldsymbol{\phi},\boldsymbol{\alpha},\boldsymbol{\beta}\}} = \mathfrak{T}_{\boldsymbol{z}_{0}}\mathfrak{D}_{\boldsymbol{\rho}}\mathfrak{R}_{\boldsymbol{\phi}}\boldsymbol{m}^{[\boldsymbol{\alpha},\boldsymbol{\beta}]},$$

$$\boldsymbol{z}_{0} \in \mathbb{C}, \ \boldsymbol{\rho} > 0, \ \boldsymbol{\phi}, \boldsymbol{\alpha}, \boldsymbol{\beta} \in \mathbb{R},$$
(2.2)

where

$$\boldsymbol{m}^{[\alpha,\beta]} = R_{\mathbf{e}_1,2\beta} \Phi(w_*(e^{-i\alpha}z)) = \Phi(w^{[\alpha,\beta]}(z)),$$
$$w^{[\alpha,\beta]}(z) = \frac{\cos(\beta)w_*(e^{-i\alpha}z) + i\sin(\beta)}{i\sin(\beta)w_*(e^{-i\alpha}z) + \cos(\beta)}, \quad w_*(z) = i\frac{z-1}{z+1}$$

Remark 2.2. For $\beta = \pi/4$, we have $w^{[\alpha, \pi/4]}(z) = e^{i(\frac{\pi}{2} - \alpha)}z$.

Remark 2.3. For $\beta \in \frac{\pi}{2}\mathbb{Z}$ we have $w^{[0,\beta]} = w_*$ or $1/w_*$ depending on the value of β modulo $\pi/2$, but these two maps are on the same orbit generated by corotations: for $\phi = \pi$ we have $\Re_{\pi}[\Phi(w_*)] = \Phi(1/w_*)$.

Proof of Lemma 2.1. The maps (2.2) are clearly Möbius maps, we need to check that all Möbius maps can be obtained this way.

Consider an arbitrary Möbius map \boldsymbol{m} . There exists $\phi \in \mathbb{R}$ such that $\mathbf{v} = R_{\mathbf{e}_3,-\phi}\boldsymbol{m}(\infty)$ belongs to $\mathbf{e}_1^{\perp} \cap \mathbb{S}^2$, the large circle which contains \mathbf{e}_2 and \mathbf{e}_3 . Then there exists $\beta \in \mathbb{R}$ such that $R_{\mathbf{e}_1,-2\beta}\mathbf{v} = \mathbf{e}_2$, so that

$$\tilde{\boldsymbol{m}} = R_{\mathbf{e}_1,-2\beta}\mathfrak{R}_{-\phi}\boldsymbol{m}$$

satisfies $ilde{m{m}}(\infty)={f e}_2.$ As a consequence, $ilde{m{m}}$ can be written as

$$\tilde{\boldsymbol{m}}(z) = \Phi\left(i\frac{z-a}{z+b}\right),$$

for some $a, b \in \mathbb{C}$ such that $a + b \neq 0$. We infer that

$$\boldsymbol{m} = \mathfrak{R}_{\phi} R_{\mathbf{e}_1, 2\sigma} \Phi(\tilde{w}), \quad \tilde{w}(z) = i \frac{z-a}{z+b}.$$

Using the identities

$$R_{\mathbf{e}_{1,2\theta}}\Phi(w) = \Phi\left(\frac{w\cos(\theta) + i\sin(\theta)}{iw\sin(\theta) + \cos(\theta)}\right),$$
$$R_{\mathbf{e}_{3,\theta}}\Phi(w) = \Phi(e^{i\theta}w),$$

this becomes $\boldsymbol{m} = \Phi(w)$ with

$$w(z) = e^{i\phi} \frac{\cos(\beta)\tilde{w}(e^{-i\phi}z) + i\sin(\beta)}{i\sin(\beta)\tilde{w}(e^{-i\phi}z) + \cos(\beta)}.$$

Thus, for any $z_0 \in \mathbb{C}$ and $\rho > 0$ we have

$$\mathfrak{D}_{1/\rho}\mathfrak{T}_{-z_0}\boldsymbol{m} = \mathfrak{R}_{\phi}R_{\mathbf{e}_1,2\beta}\Phi(\bar{w}),$$

with

$$\bar{w}(z) = \tilde{w}(\rho z + e^{-i\phi}z_0) = i\frac{z - (a - e^{-i\phi}z_0)/\rho}{z + (b + e^{-i\phi}z_0)/\rho}$$

Choosing

$$\rho = \frac{|a+b|}{2}, \quad z_0 = e^{i\phi} \frac{a-b}{2},$$

we obtain

$$\bar{w}(z) = i \frac{z - e^{i\alpha}}{z + e^{i\alpha}} = w_*(e^{-i\alpha}z), \qquad \alpha = \arg(a+b).$$

We conclude that

$$oldsymbol{m} = \mathfrak{T}_{z_0}\mathfrak{D}_
ho\mathfrak{R}_\phioldsymbol{m}^{[lpha,eta]},$$

with $\boldsymbol{m}^{[\alpha,\beta]}(z) = R_{\mathbf{e}_1,2\beta} \Phi(w_*(e^{-i\alpha}z))$ and the parametrization (2.2) is indeed surjective.

3. DMI ENERGY OF MÖBIUS MAPS

In this section we compute the DMI energy of Möbius maps

$$\begin{aligned} \boldsymbol{m}(z) &= \Phi(w^{[\alpha,\beta]}(z)) = R_{\mathbf{e}_{1},2\beta-\pi/2} \Phi(w^{[\alpha,\pi/4]}(z)) \\ &= R_{\mathbf{e}_{1},2\beta-\pi/2} \Phi(ie^{-i\alpha}z) \,, \end{aligned}$$

where Φ is the stereographic map defined in (1.2). The equalities follow from the definition of $w^{[\alpha,\beta]}$ in Lemma 2.1 and from Remark 2.2. For such map \boldsymbol{m} , the integrand of $\tilde{H}(\boldsymbol{m})$ is not absolutely integrable, but we will see that the integral

$$\widetilde{H}(\boldsymbol{m}; B_R) = 2 \int_{B_R} m_3(\partial_1 m_2 - \partial_2 m_1) \, dx \,, \tag{3.1}$$

admits a limit as $R \to \infty$, thus providing a meaningful definition of $\widetilde{H}(\boldsymbol{m})$. Using the notation $z = x + iy \in \mathbb{C} \approx \mathbb{R}^2$, we find

$$m_{1} = \frac{2(x\sin(\alpha) - y\cos(\alpha))}{x^{2} + y^{2} + 1},$$

$$m_{2} = \frac{\cos(2\beta)(x^{2} + y^{2} - 1)}{x^{2} + y^{2} + 1} + \frac{2(x\cos(\alpha) + y\sin(\alpha))\sin(2\beta)}{x^{2} + y^{2} + 1},$$

$$m_{3} = \frac{\sin(2\beta)(x^{2} + y^{2} - 1)}{x^{2} + y^{2} + 1} - \frac{2(x\cos(\alpha) + y\sin(\alpha))\cos(2\beta)}{x^{2} + y^{2} + 1},$$

and

$$2m_3(\partial_1 m_2 - \partial_2 m_1) = \frac{4g(x,y)h(x,y)}{(1+x^2+y^2)^3},$$

where

$$g(x,y) = -\sin(2\beta) + (x^2 + y^2)\sin(2\beta)$$

$$-2x\cos(\alpha)\cos(2\beta) - 2y\sin(\alpha)\cos(2\beta),$$

$$h(x,y) = \cos(\alpha)(1 + \sin(2\beta) + (1 - \sin(2\beta))(x^2 - y^2))$$

$$+2x\cos(2\beta) + 2xy\sin(\alpha)(1 - \sin(2\beta)).$$

Integrating on B_R , and using that a function f(x, y) has zero integral if it satisfies one of the antisymmetry properties f(x, -y) = -f(x, y) or f(-x, y) = -f(x, y) or f(y, x) = -f(x, y), we are left with

$$\frac{1}{4}\widetilde{H}(\boldsymbol{m}; B_R) = \cos(\alpha)\sin(2\beta)(1+\sin(2\beta))I_1$$
$$-4\cos\alpha\cos^2(2\beta)I_2,$$

where

$$\begin{split} I_1 &= \int_{B_R} \frac{x^2 + y^2 - 1}{(x^2 + y^2 + 1)^3} \, dx dy = -\pi \frac{R^2}{(1 + R^2)^2} \,, \\ I_2 &= \int_{B_R} \frac{x^2}{(x^2 + y^2 + 1)^3} dx dy = \frac{\pi}{4} \frac{R^4}{(1 + R^2)^2} \,, \end{split}$$

and therefore

$$\frac{(1+R^2)^2}{4\pi}\widetilde{H}(\boldsymbol{m}; B_R) = -\cos\alpha\cos^2(2\beta)R^4 -\cos(\alpha)\sin(2\beta)(1+\sin(2\beta))R^2$$

Taking the limit as $R \to \infty$, we deduce

$$\widetilde{H}(\boldsymbol{m}^{[\alpha,\beta]}) = \lim_{R \to \infty} \widetilde{H}(\boldsymbol{m}^{[\alpha,\beta]}; B_R) = -4\pi \cos(\alpha) \cos^2(2\beta) , \quad (3.2)$$

and we also obtain the estimate

$$\left|\widetilde{H}(\boldsymbol{m}^{[\alpha,\beta]}) - \widetilde{H}(\boldsymbol{m}^{[\alpha,\beta]};B_R)\right| \leq \frac{C}{R^2},$$
(3.3)

for all $R \ge 2$, where C > 0 is an absolute constant.

4. Energy upper bound

In this section we prove the upper bound on the minimal energy, by estimating the energy of explicit competitors.

Proposition 4.1. The infimum of the energy E_{σ} defined in (1.1) over the space W_{-1} defined in (1.14) is bounded by

$$\inf_{\mathcal{W}_{-1}} E_{\sigma} \le 4\pi - \frac{\pi\sigma^2}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)} + C\frac{\sigma^2}{\ln^2\sigma},\tag{4.1}$$

for some absolute constant C > 0.

Proof of Proposition 4.1. As explained in the introduction, one would like to use Möbius maps as competitors, but they have infinite anisotropy energy $A(\mathbf{m})$, so we first need to modify them. We introduce a truncation parameter L > 0 and define $w_*^L : \mathbb{C} \to \mathbb{C}$ as

$$w_*^L(z) = \chi(|z|/L)w_*(z) + (1 - \chi(|z|/L))i$$

$$= i - \chi(|z|/L)\frac{2i}{z+1},$$
(4.2)

where $w_*(z) = i(z-1)/(z+1)$ as in Lemma 2.1 and χ is a smooth cut-off function satisfying

$$\mathbf{1}_{r\leq 1}\leq \chi(r)\leq \mathbf{1}_{r\leq 2},\quad 0\geq \chi'\geq -2\,.$$

In analogy with the parametrization of Möbius maps given in Lemma 2.1, this truncated function w_*^L can be used to define general truncated Möbius maps

$$\begin{split} \boldsymbol{m}_{L}^{\{z_{0},\rho,\phi,\alpha,\beta\}} &= \mathfrak{T}_{z_{0}}\mathfrak{D}_{\rho}\mathfrak{R}_{\phi}\boldsymbol{m}_{L}^{[\alpha,\beta]}, \quad z_{0}\in\mathbb{C}, \ \rho>0, \ \phi,\sigma,\alpha,\beta\in\mathbb{R}, \\ \boldsymbol{m}_{L}^{[\alpha,\beta]} &= R_{\mathbf{e}_{1},2\beta}\Phi(w_{*}^{L}(e^{-i\alpha}z))\,. \end{split}$$

Recall that here Φ is the stereographic map defined in (1.2). Note that the invariances of the Dirichlet energy ensure

$$\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}_L^{\{z_0,\rho,\phi,\alpha,\beta\}} - \nabla \boldsymbol{m}^{\{z_0,\rho,\phi,\alpha,\beta\}}|^2 dx$$

= $\int_{\mathbb{R}^2} |\nabla [\Phi(w_*) - \Phi(w_*^L)]|^2 dz$
= $\int_{|z|\ge L} |\nabla [\Phi(w_*) - \Phi(w_*^L)]|^2 dz \le \frac{C}{L^2},$

so for large enough L these modified maps m_L must satisfy $Q(m_L) = -1$, and belong to the admissible set W_{-1} defined in (1.14).

If $\beta \neq 0$ modulo $\pi/2$, these maps have infinite anisotropy $A(\mathbf{m})$, and all values of $\beta = 0$ modulo $\pi/2$ give the same energy, so we assume $\beta = 0$. The invariances (1.8) allow us to assume without loss of generality $z_0 = 0$ and $\phi = 0$. We are therefore left with three free parameters and denote

$$\boldsymbol{m}_{\alpha,\rho,L}(z) = \mathfrak{D}_{\rho} \boldsymbol{m}_{L}^{[\alpha,0]}(z) = \Phi\left(w_{*}^{L}\left(e^{-i\alpha}\frac{z}{\rho}\right)\right).$$

Using the invariances of each energy term, the properties

$$\begin{aligned} (\boldsymbol{m}_{\alpha,1,L})_3^2 &= |\nabla m_{\alpha,1,L}|^2 = 0 \quad \text{in } \mathbb{R}^2 \setminus B_{2L} \,, \\ (\boldsymbol{m}_{\alpha,1,L})_3^2 &\leq \frac{C}{L^2} \quad \text{and } |\nabla m_{\alpha,1,L}|^2 \leq \frac{C}{L^4} \quad \text{in } B_{2L} \setminus B_L \,, \end{aligned}$$

and $w_*^L = w_*$ in B_L , we find

$$D(\boldsymbol{m}_{\alpha,\rho,L}) = D(\boldsymbol{m}_{0,1,L}) = \int_{B_L} \left| \nabla [\Phi(w_*)] \right|^2 dz + \mathcal{O}(1/L^2),$$

$$\widetilde{H}(\boldsymbol{m}_{\alpha,\rho,L}) = \rho \widetilde{H}(\boldsymbol{m}_{\alpha,1,L}) = \rho \widetilde{H}(\boldsymbol{m}^{[\alpha,0]}; B_L) + \mathcal{O}(\rho/L),$$

$$A(\boldsymbol{m}_{\alpha,\rho,L}) = \rho^2 A(\boldsymbol{m}_{0,1,L}) = \rho^2 \int_{B_L} \Phi_3^2(w_*) dz + \mathcal{O}(\rho^2).$$

The expressions involving $\Phi(w_*)$ and $\boldsymbol{m}^{[\alpha,0]}$ in the right-hand sides can be calculated explicitly. For the Dirichlet energy we know that the limit is 4π as $L \to \infty$, and the decay $|\nabla[\Phi(w_*)]|^2 \leq C/|z|^4$ as $|z| \to \infty$ gives an error

of order $1/L^2$. For the DMI energy we use (3.2) and (3.3). And for the anisotropy term we have

$$\begin{split} &\int_{B_L} \Phi_3^2(w_*) \, dz = \int_{B_L} \frac{(|w_*|^2 - 1)^2}{(|w_*|^2 + 1)^2} \, dz \\ &= \int_{B_L} \frac{(|z - 1|^2 - |z + 1|)^2}{(|z - 1|^2 + |z + 1|)^2} \, dz = 4 \int_0^{2\pi} \int_0^L \frac{\cos^2 \theta \, r^2}{(1 + r^2)^2} \, r dr \, d\theta \\ &= 2\pi \int_0^{L^2} \frac{t}{(1 + t)^2} \, dt = 2\pi \ln(1 + L^2) - \frac{2\pi L^2}{1 + L^2} \end{split}$$

Thus we find

$$D(\boldsymbol{m}_{\alpha,\rho,L}) = 4\pi + \mathcal{O}(1/L^2),$$

$$\widetilde{H}(\boldsymbol{m}_{\alpha,\rho,L}) = -4\pi\rho\,\cos(\alpha) + \mathcal{O}(\rho/L),$$

$$A(\boldsymbol{m}_{\alpha,\rho,L}) = 4\pi\rho^2\ln L + \mathcal{O}(\rho^2),$$

and deduce

$$E_{\sigma}(\boldsymbol{m}_{\alpha,\rho,L}) \leq 4\pi + \mathcal{E}_{\sigma}(\alpha,\rho,L) + \mathcal{O}(\sigma^{2}\rho/L + \sigma^{2}\rho^{2}), \qquad (4.3)$$

where $\mathcal{E}_{\sigma}(\alpha,\rho,L) = \frac{C_{1}}{L^{2}} + 4\pi\sigma^{2}\left(\rho^{2}\ln L - \rho\cos(\alpha)\right),$

for some absolute constant $C_1 > 0$. Minimizing \mathcal{E}_{σ} over α gives $\alpha = 0$ and

$$\mathcal{E}_{\sigma}(0,\rho,L) = \frac{C_1}{L^2} + 4\pi\sigma^2 \left(\rho^2 \ln L - \rho\right).$$

Minimizing over ρ gives $\rho_L = 1/(2 \ln L)$ and

$$\mathcal{E}_{\sigma}(0,\rho_L,L) = \frac{C_1}{L^2} - \frac{\pi\sigma^2}{\ln L}$$

Minimizing over L leads to, at main order for $\sigma \to 0$,

$$L_{\sigma} = \sqrt{\frac{2C_1}{\pi}} \frac{\ln(1/\sigma)}{\sigma},$$

$$\rho_{\sigma} = \rho_{L_{\sigma}} = \frac{1}{2\ln(1/\sigma)} \left(1 + \mathcal{O}\left(\frac{\ln\ln(1/\sigma)}{\ln(1/\sigma)}\right)\right),$$

and

$$\mathcal{E}_{\sigma}(0,\rho_{\sigma},L_{\sigma}) = -\frac{\pi\sigma^2}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)} + \mathcal{O}\left(\frac{\sigma^2}{\ln^2\sigma}\right) \,.$$

Plugging this into (4.3), we deduce the upper bound (4.1).

Remark 4.2. The constant C_1 in the above proof depends on the truncation (4.2) that we used to transform Möbius maps into maps with finite anisotropy energy A(m). In [4], this modification of Möbius maps is done

in a much more refined way, in order to obtain an optimal constant C_1 . However, we see in the above proof that the precise value of C_1 does not affect the upper bound at main order when $\sigma \to 0$. That refinement is therefore superfluous here, and it seems to us that the results in [4] could also be obtained without that refinement.

5. Energy lower bound

In this section we consider a map m in the space W_{-1} defined in (1.14), that is, $m \in \mathcal{H}(\mathbb{R}^2; \mathbb{S}^2)$ such that $A(m) < \infty$ and Q(m) = -1, and prove a sharp lower bound on its energy, following quite closely the strategy in [4], with some necessary adaptations.

The most important tool in that strategy is a stability estimate for the Möbius group (1.11) as minimizers of the Dirichlet energy $D(\boldsymbol{m})$, proved in [4, Theorem 2.4]. That theorem provides an absolute constant $c_* > 0$ (which could be made explicit) such that, for any $\boldsymbol{m} \in H_c^1(\mathbb{R}^2; \mathbb{S}^2)$, there exists a Möbius map $\Psi \in \mathcal{M}$ satisfying

$$\int_{\mathbb{R}^2} \left| \nabla(\boldsymbol{m} - \Psi) \right|^2 dx \le c_* \Big(\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 dx - 8\pi \Big).$$
(5.1)

Moreover, it is apparent from the alternative proofs of this result in [10, 23] that the map Ψ can be chosen so that

$$u = \boldsymbol{m} \circ \Psi^{-1} - \mathrm{id}_{\mathbb{S}^2} = (\boldsymbol{m} - \Psi) \circ \Psi^{-1},$$

has zero average on \mathbb{S}^2 . Applying the Moser-Trudinger inequality on \mathbb{S}^2 [19] and changing variables, this implies

$$\int_{\mathbb{R}^2} |\nabla \Psi|^2 \exp\left(\frac{|\boldsymbol{m} - \Psi|^2}{\int_{\mathbb{R}^2} |\nabla(\boldsymbol{m} - \Psi)|^2 \, dx}\right) dx$$
$$= 2 \int_{\mathbb{S}^2} \exp\left(\frac{|u|^2}{\int_{\mathbb{S}^2} |\nabla u|^2 d\mathcal{H}^2}\right) d\mathcal{H}^2 \le c_{\mathrm{MT}},$$

for some explicit absolute constant $c_{\rm MT} > 0$. Defining

$$L = \left(\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 dx - 8\pi\right)^{-1/2},$$
(5.2)

and setting

$$oldsymbol{v} = oldsymbol{m} - \Psi\,, \qquad \Psi = oldsymbol{m}^{\{z_0,
ho,\phi,lpha,eta\}}\,.$$

for some $z_0 \in \mathbb{C}$, $\rho > 0$ and $\phi, \alpha, \beta \in \mathbb{R}$ (according to the parametrization of \mathcal{M} provided by Lemma 2.1) we have therefore

$$\int_{\mathbb{R}^2} |\nabla \boldsymbol{v}|^2 \, dx \le \frac{c_*}{L^2},\tag{5.3}$$

and
$$\int_{\mathbb{R}^2} \exp\left(\frac{v_3^2}{\int_{\mathbb{R}^2} |\nabla v_3|^2}\right) |\nabla \Psi|^2 dx \le c_{\mathrm{MT}}.$$
 (5.4)

In the next two subsections we use these stability estimates to provide lower bounds on the anisotropy and DMI energy in terms of Ψ and L.

5.1. Lower bounds for the anisotropy term. In this section we prove two lower bounds on the anisotropy term

$$A(\boldsymbol{m}) = \int_{\mathbb{R}^2} m_3^2 \, dx \, .$$

The first lower bound serves to show that the angle β must be close to 0 modulo $\pi/2$, which then makes the second lower bound quite sharp. The proofs are natural modifications of the two lower bounds in [4, Lemma 6.1 & Lemma 6.4].

Lemma 5.1. There exist $L_0, C > 0$ depending on c_{MT} and c_* such that, if L defined in (5.2) satisfies $L \ge L_0$, then

$$\frac{A(\boldsymbol{m})}{\rho^2} \ge \frac{1}{C} \sin^2(2\beta) L^2, \qquad (5.5)$$

and
$$\frac{A(m)}{\rho^2} \ge 4\pi \cos^2(2\beta) \ln L - C$$
, (5.6)

where $\rho > 0$ and $\beta \in \mathbb{R}$ are such that (5.3) and (5.4) are satisfied.

Remark 5.2. From (5.5) we infer

$$\cos^2(2\beta) = 1 - \sin^2(2\beta) \ge 1 - \frac{C}{L^2} \frac{A(\mathbf{m})}{\rho^2}.$$

Plugging this into (5.6), we deduce

$$\left(1+4\pi C\frac{\ln L}{L^2}\right)\frac{A(\boldsymbol{m})}{\rho^2} \ge 4\pi \ln L - C\,,$$

and therefore

$$\frac{A(\boldsymbol{m})}{\rho^2} \ge 4\pi \ln L - C, \qquad (5.7)$$

for all $L \ge L_0$ and a possibly larger constant C > 0.

Proof of Lemma 5.1. Using the invariances, we assume without loss of generality that $\rho = 1$, $z_0 = 0$, $\phi = 0$ and $\alpha = \pi/2$, so by (5.3) we have

$$oldsymbol{m} = oldsymbol{m}^{[\pi/2,eta]} + oldsymbol{v}, \qquad \int_{\mathbb{R}^2} |
abla oldsymbol{v}|^2 \, dx \leq rac{c_*}{L^2} \, .$$

Recalling Remark 2.2, we have

$$oldsymbol{m}^{[\pi/2,eta]}=R_{\mathbf{e}_1,2eta-\pi/2}\Phi\,,$$

and we deduce that the third component of m is given by

$$m_3 = -\cos(2\beta)\Phi_2 + \sin(2\beta)\Phi_3 + v_3.$$
(5.8)

From this identity, the proofs of the first and second lower bound (5.5) and (5.6) follow different strategies. For the first, we fix $R \ge 1$, square (5.8), use the elementary inequality $(a + b)^2 \ge a^2/2 - 2b^2$ and integrate on B_R , which gives

$$\int_{B_R} m_3^2 dx \ge \frac{1}{2} \int_{B_R} \left(\sin(2\beta) \Phi_3 - \cos(2\beta) \Phi_2 \right)^2 dx - 2 \int_{B_R} v_3^2 dx.$$

Using polar coordinates $x = re^{i\theta}$ and the explicit expression (1.2) of the stereographic map Φ we can explicitly calculate the first integral in the right-hand side,

$$\begin{split} &\int_{B_R} \left(\sin(2\beta) \Phi_3 - \cos(2\beta) \Phi_2 \right)^2 dx \\ &= 2\pi \sin^2(2\beta) \int_0^R \frac{(r^2 - 1)^2}{(r^2 + 1)^2} r dr + 4\pi \cos^2(2\beta) \int_0^R \frac{r^2}{(r^2 + 1)^2} r dr \\ &= \pi \sin^2(2\beta) \left(R^2 - 6\ln(1 + R^2) + \frac{6}{1 + R^2} \right) \\ &\quad + 2\pi \left(\ln(1 + R^2) + \frac{1}{1 + R^2} \right) \\ &\geq \frac{\pi}{2} \sin^2(2\beta) R^2 + 2\pi \ln R \,, \end{split}$$

if $R \ge R_0$ for a large enough absolute constant $R_0 \ge 1$. Next we estimate the integral of v_3^2 , relying as in [4, Lemma 6.1] on the Moser-Trudinger inequality (5.4). First we apply the inequality $xy \le e^x + y \ln(y/e)$ with $x = v_3^2 / \int |\nabla v_3|^2 dx$ and $y = |\nabla \Phi|^{-2}$, to write

$$\frac{\int_{B_R} v_3^2 \, dx}{\int_{\mathbb{R}^2} |\nabla v_3|^2 \, dx} = \int_{B_R} \frac{v_3^2}{\int_{\mathbb{R}^2} |\nabla v_3|^2 \, dx} |\nabla \Phi|^{-2} |\nabla \Phi|^2 \, dx$$
$$\leq \int_{B_R} \exp\left(\frac{v_3^2}{\int_{\mathbb{R}^2} |\nabla v_3|^2 \, dx}\right) |\nabla \Phi|^2 \, dx + \int_{B_R} \ln\left(\frac{1}{e \, |\nabla \Phi|^2}\right) \, dx.$$

Noting that $|\nabla \Phi|^2 = 8/(1+|x|^2)^2 = |\nabla \boldsymbol{m}^{[\pi/2,\beta]}|^2$ and recalling the Moser-Trudinger inequality (5.4) where $\Psi = \boldsymbol{m}^{[\pi/2,\beta]}$, we deduce

$$\frac{\int_{B_R} v_3^2 \, dx}{\int_{\mathbb{R}^2} |\nabla v_3|^2 \, dx} \le c_{\rm MT} + 16\pi R^2 \ln R \,,$$

and combining this with the stability estimate (5.3) gives

$$\int_{B_R} v_3^2 \, dx \le c_{\rm MT} \frac{c_*}{L^2} + 16\pi c_* \frac{R^2}{L^2} \ln R \,. \tag{5.9}$$

Gathering the above inequalities, we infer

$$\int_{\mathbb{R}^2} m_3^2 dx \ge \frac{\pi}{4} \sin^2(2\beta) R^2 + \pi \ln R - 2c_{\rm MT} \frac{c_*}{L^2} - 16\pi c_* \frac{R^2}{L^2} \ln R.$$

Choosing $R = L/(4\sqrt{2c_*})$, this becomes

$$\int_{\mathbb{R}^2} m_3^2 dx \ge \frac{\pi}{2^7 c_*} \sin^2(2\beta) L^2 + \frac{\pi}{2} \ln L - 2c_{\rm MT} \frac{c_*}{L^2} - \frac{\pi}{2} \ln(4\sqrt{2c_*})$$
$$\ge \frac{\pi}{2^7 c_*} \sin^2(2\beta) L^2 \,,$$

provided $L \ge L_0$ for a large enough $L_0 \ge 1$, and proves the first lower bound (5.5).

The proof of the second lower bound (5.6) follows the strategy of [4, Lemma 6.4] relying on the Fourier transform

$$\mathcal{F}\varphi(\xi) = \int_{\mathbb{R}^2} e^{-ix\cdot\xi}\varphi(x) \, dx \, .$$

Differentiating the identity (5.8) and taking Fourier transforms we have, for $\ell = 1, 2$,

$$i\xi_{\ell}\mathcal{F}m_3 = -\cos(2\beta)\mathcal{F}(\partial_{\ell}\Phi_2) + \sin(2\beta)\mathcal{F}(\partial_{\ell}\Phi_3) + \mathcal{F}(\partial_{\ell}v_3).$$

Since Φ_3 and Φ_2/x_2 are radial and real-valued, a direct calculation using polar coordinates shows that

$$\mathfrak{Re}\,\mathcal{F}(\partial_\ell\Phi_3)=\mathfrak{Im}\,\mathcal{F}(\partial_\ell\Phi_2)=0\,,$$

and we deduce

$$\frac{\xi_{\ell}}{|\xi|} \operatorname{\mathfrak{Im}} \mathcal{F} m_3 = \cos(2\beta) \frac{\mathcal{F}(\partial_{\ell} \Phi_2)}{|\xi|} - \frac{\operatorname{\mathfrak{Re}} \mathcal{F}(\partial_{\ell} v_3)}{|\xi|}$$

We fix $\mu \ge 0$, to be chosen later. Applying the identity

$$\begin{split} &\int_{\mathbb{R}^2} |f|^2 \, d\xi - \int_{\mathbb{R}^2} \frac{\mu |\xi|^2}{1 + \mu |\xi|^2} |g|^2 \, d\xi + \mu \int_{\mathbb{R}^2} |\xi|^2 |f - g|^2 \, d\xi \\ &= \int_{\mathbb{R}^2} (1 + \mu |\xi|^2) \Big| f - \frac{\mu |\xi|^2}{1 + \mu |\xi|^2} g \Big|^2 \, d\xi \ge 0 \,, \end{split}$$

valid for any $f \in L^2(\mathbb{R}^2; (1+|\xi|^2)d\xi)$ and $g \in L^2(\mathbb{R}^2; |\xi|^2d\xi)$, to $f = \xi_\ell \operatorname{\mathfrak{Im}} \mathcal{F}m_3/|\xi|$ and $g = \cos(2\beta)\mathcal{F}(\partial_\ell \Phi_2)/|\xi|$, and summing over $\ell = 1, 2$ we obtain

$$\int_{\mathbb{R}^2} (\Im \mathfrak{m} \,\mathcal{F} m_3)^2 \,d\xi \ge \cos^2(2\beta) \int_{\mathbb{R}^2} \frac{\mu |\xi|^2}{1+\mu |\xi|^2} \frac{|\mathcal{F}(\nabla \Phi_2)|^2}{|\xi|^2} d\xi -\mu \int_{\mathbb{R}^2} |\mathcal{F}(\nabla v_3)|^2 \,d\xi \,.$$

Using Plancherel's identity

$$4\pi^2 \int_{\mathbb{R}^2} |\varphi|^2 \, dx = \int_{\mathbb{R}^2} |\mathcal{F}\varphi|^2 \, d\xi \,,$$

this implies

$$\int_{\mathbb{R}^2} m_3^2 dx \ge \cos^2(2\beta) \int_{\mathbb{R}^2} \frac{\mu |\xi|^2}{1 + \mu |\xi|^2} \frac{|\mathcal{F}(\nabla \Phi_2)|^2}{4\pi^2 |\xi|^2} d\xi - \mu \int_{\mathbb{R}^2} |\nabla v_3|^2 dx \,.$$
(5.10)

As in [4], the first integral in the right-hand side can be explicitly calculated. It is shown in [4, Lemma A.5] that

$$\mathcal{F}(\nabla \Phi_2) = -4\pi K_1(|\xi|)\xi_2 \frac{\xi}{|\xi|} \,,$$

where K_1 is a modified Bessel function [1, § 9.6]. Thus we have

$$\int_{\mathbb{R}^2} \frac{\mu |\xi|^2}{1+\mu |\xi|^2} \frac{|\mathcal{F}(\nabla \Phi_2)|^2}{4\pi^2 |\xi|^2} d\xi = 4\pi \int_0^\infty \frac{\mu r^2}{1+\mu r^2} K_1(r)^2 r dr \,,$$

and, using the known asymptotics of Bessel functions [1, § 9.6.11],

$$K_1(r) = \frac{1}{r} + \mathcal{O}(r \ln r) \quad \text{as } r \to 0 \,,$$

we infer

$$\int_0^\infty \frac{\mu r^2}{1 + \mu r^2} K_1(r)^2 r dr \ge \int_0^1 \frac{\mu r^2}{1 + \mu r^2} K_1(r)^2 r dr$$
$$\ge \int_0^1 \frac{\mu r}{1 + \mu r^2} dr - c_1 \int_0^1 r |\ln r| dr$$

for some absolute constant $c_1 > 0$, and therefore

$$\int_{\mathbb{R}^2} \frac{\mu |\xi|^2}{1+\mu |\xi|^2} \frac{|\mathcal{F}(\nabla \Phi_2)|^2}{4\pi^2 |\xi|^2} d\xi$$

$$\geq 4\pi \int_0^1 \frac{\mu r}{1+\mu r^2} dr - c_1 = 2\pi \ln(1+\mu) - c_1.$$

Coming back to the lower bound (5.10) on $\int m_3^2 dx$ and using also the stability estimate (5.3), this gives

$$\int_{\mathbb{R}^2} m_3^2 \, dx \ge 2\pi \cos^2(2\beta) \ln(1+\mu) - c_1 - \frac{c_*\mu}{L^2}$$

Choosing $\mu = c_1 L^2 / c_*$ we infer

$$\int_{\mathbb{R}^2} m_3^2 dx \ge 4\pi \cos^2(2\beta) \ln L - 2c_1 - |\ln(c_1/c_*)|$$

which proves the second lower bound (5.5).

5.2. Lower bound for the DMI term. In this section we prove a lower bound on the DMI term

$$\widetilde{H}(\boldsymbol{m}) = 2 \int_{\mathbb{R}^2} m_3(\partial_1 m_2 - \partial_2 m_1) \, dx \, .$$

Similar to Step 2 of [4, Lemma 6.5], that lower bound is in term of the DMI energy of the Möbius map Ψ and a small error term. Here the error term is not as good as in [4], this is due to the different form of our DMI term, and the fact that the components of the stereographic map Φ given by (1.2) have different integrability properties: $(\Phi_3 + 1)^2$ is integrable, but Φ_1^2 and Φ_2^2 are not. This forces us to use a slightly more involved argument to control the error term.

Lemma 5.3. There exist $L_0, C > 0$ depending on c_{MT} and c_* such that, if L defined in (5.2) satisfies $L \ge L_0$, then

$$\frac{\widetilde{H}(\boldsymbol{m})}{\rho} \ge -4\pi \cos(\alpha) \cos^2(2\beta) - C\sqrt{\frac{\ln L}{L}} - \frac{C}{\sqrt{L}} \sqrt{\frac{A(\boldsymbol{m})}{\rho^2}},$$

where $\rho > 0$ and $\alpha, \beta \in \mathbb{R}$ are such that (5.3) and (5.4) are satisfied.

Proof of Lemma 5.3. Using the invariances, we assume without loss of generality that $\rho = 1$, $z_0 = 0$ and $\phi = 0$. Taking Remark 2.2 into account, we are therefore left with

$$\boldsymbol{m} = \Psi + \boldsymbol{v}, \quad \Psi = \boldsymbol{m}^{[\alpha,\beta]} = R_{\mathbf{e}_1,\beta-\pi/2} \widetilde{\Phi}, \quad \widetilde{\Phi}(z) = \Phi(e^{-i\alpha}iz),$$

where Φ is the stereographic map defined in (1.2), and v satisfies (5.3) and (5.4). The integrand of $\widetilde{H}(m)$ satisfies the identities

$$m_3(\partial_1 m_2 - \partial_2 m_1) - m_3(\partial_1 v_2 - \partial_2 v_1)$$

= $\Psi_3(\partial_1 \Psi_2 - \partial_2 \Psi_1) + v_3(\partial_1 \Psi_2 - \partial_2 \Psi_1)$
= $m_3(\partial_1 \Psi_2 - \partial_2 \Psi_1)$,

from which we infer

$$\widetilde{H}(\boldsymbol{m}) - 2 \int_{\mathbb{R}^2} m_3(\partial_1 v_2 - \partial_2 v_1) \, dx$$

= $\widetilde{H}(\Psi; B_{\sqrt{L}}) + 2 \int_{B_{\sqrt{L}}} v_3(\partial_1 \Psi_2 - \partial_2 \Psi_1) \, dx$
+ $2 \int_{|x| \ge \sqrt{L}} m_3(\partial_1 \Psi_2 - \partial_2 \Psi_1) \, dx$.

This implies

$$\begin{split} &\frac{1}{2} \left(\widetilde{H}(\boldsymbol{m}) - \widetilde{H}(\boldsymbol{\Psi}; B_{\sqrt{L}}) \right) \\ &= \int_{\mathbb{R}^2} m_3(\partial_1 v_2 - \partial_2 v_1) \, dx + \int_{B_{\sqrt{L}}} v_3(\partial_1 \Psi_2 - \partial_2 \Psi_1) \, dx \\ &+ \int_{|x| \ge \sqrt{L}} m_3(\partial_1 \Psi_2 - \partial_2 \Psi_1) \, dx \,, \end{split}$$

and, using the Cauchy-Schwarz inequality,

$$\begin{split} &\frac{1}{8} \Big(\widetilde{H}(\boldsymbol{m}) - \widetilde{H}(\Psi; B_{\sqrt{L}}) \Big)^2 \\ &\leq A(\boldsymbol{m}) \int_{\mathbb{R}^2} |\nabla \boldsymbol{v}|^2 \, dx + \int_{B_{\sqrt{L}}} v_3^2 \, dx \int_{\mathbb{R}^2} |\nabla \Psi|^2 \, dx \\ &+ A(\boldsymbol{m}) \int_{|x| \geq \sqrt{L}} |\nabla \Psi|^2 \, dx \, . \end{split}$$

Using the fact that $|\nabla \Psi|^2 = |\nabla \Phi|^2 = 8/(1+|x|^2)^2$, the stability estimate (5.3) on $\int |\nabla v|^2 dx$, and the estimate (5.9) on integrals of v_3^2 , this implies

$$\begin{split} &\frac{1}{4} \Big(\widetilde{H}(\boldsymbol{m}) - \widetilde{H}(\boldsymbol{\Psi}; B_{\sqrt{L}}) \Big)^2 \\ &\leq A(\boldsymbol{m}) \frac{c_* + 8\pi}{L} + 8\pi \Big(c_{\mathrm{MT}} \frac{c_*}{L} + 8\pi c_* \frac{\ln L}{L} \Big) \end{split}$$

Finally, recalling the explicit expression (3.2) of $\widetilde{H}(\Psi)$ and its error (3.3) from $\widetilde{H}(\Psi; B_{\sqrt{L}})$, we have

$$\widetilde{H}(\Psi; B_{\sqrt{L}}) \ge -4\pi \cos(\alpha) \cos^2(2\beta) - \frac{C}{L} \,,$$

and combining this with the previous estimate gives the conclusion.

5.3. Lower bound for the full energy. In this section we combine the lower bounds of Lemma 5.1 and Lemma 5.3 with elementary calculations to deduce a sharp energy lower bound and characterize the case of near equality.

Proposition 5.4. There exists $C_0 > 0$ and $\sigma_0 \in (0, 1/4]$ depending explicitly on c_{MT} and c_* such that, for any map $\mathbf{m} \in \mathcal{W}_{-1}$ defined in (1.14) and $0 < \sigma < \sigma_0$, the energy $E_{\sigma}(\mathbf{m})$ defined in (1.1) is bounded below by

$$E_{\sigma}(\boldsymbol{m}) \ge 4\pi - \frac{\pi\sigma^2}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)} - C_0 \frac{\sigma^2}{\ln^2 \sigma}.$$
(5.11)

Moreover, if $m = m_{\sigma}$ saturates that lower bound, in the sense that

$$E_{\sigma}(\boldsymbol{m}_{\sigma}) \leq 4\pi - \frac{\pi\sigma^2}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)} + K \frac{\sigma^2}{\ln^2\sigma}, \qquad (5.12)$$

for some $K \ge C_0$, then there exists a constant C(K) > 0 depending explicitly on K and a Möbius map $\Psi = m^{\{z_0, \rho, \phi, \alpha, \beta\}}$ as in (2.2) such that

$$\frac{1}{C(K)}\frac{\sigma^2}{\ln^2\sigma} \le \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}_{\sigma} - \nabla \Psi|^2 \, dx \le C(K)\frac{\sigma^2}{\ln^2\sigma} \,,$$

and the parameters $\rho > 0$, $\alpha, \beta \in \mathbb{R}$ satisfy

$$\begin{split} \left| \rho - \frac{1}{2\ln(1/\sigma)} \right| &\leq \frac{C(K)}{\ln^{\frac{3}{2}}(1/\sigma)} \,, \quad |\alpha| \leq \frac{C(K)}{\sqrt{\ln(1/\sigma)}} \,, \\ \text{and} \quad |\beta| \leq \frac{\sigma}{\sqrt{\ln(1/\sigma)}} \,. \end{split}$$

Proof of Proposition 5.4. We first note, for $0 < \sigma < 1/4$, the basic lower bound

$$E_{\sigma}(\boldsymbol{m}) \geq \frac{\sigma^2}{2} A(\boldsymbol{m}) + (1 - 8\sigma^2) D(\boldsymbol{m})$$

$$\geq \frac{\sigma^2}{2} A(\boldsymbol{m}) + (1 - 8\sigma^2) 4\pi |Q(\boldsymbol{m})|, \qquad (5.13)$$

where the last inequality follows from (1.10) and the first inequality from

$$\begin{aligned} \left| \widetilde{H}(\boldsymbol{m}) \right| &= 2 \left| \int_{\mathbb{R}^2} m_3(\partial_1 m_2 - \partial_2 m_1) \, dx \right| \\ &\leq \frac{1}{2} \int_{\mathbb{R}^2} |m_3|^2 \, dx + 2 \int_{\mathbb{R}^2} (\partial_1 m_2 - \partial_2 m_1)^2 \, dx \\ &\leq \frac{1}{2} A(\boldsymbol{m}) + 8D(\boldsymbol{m}) \, . \end{aligned}$$

We may assume without loss of generality that the map m satisfies

$$E_{\sigma}(\boldsymbol{m}) \leq 4\pi$$
,

since otherwise (5.11) is automatically true. From the basic lower bound (5.13) and the definition (5.2) of L, this implies, for $0 < \sigma < \sigma_0 \le 1/4$,

$$A(\boldsymbol{m}) \le 64\pi$$
 and $L \ge \frac{1}{4\sigma}$. (5.14)

If σ_0 is small enough, then $L \ge L_0$, for L_0 as in Lemma 5.1 and Lemma 5.3. During the rest of the proof we will assume that L_0 is as large as we need, and will denote by C a generic constant which may change from line to line, but whose explicit dependence on c_* and $c_{\rm MT}$ can be kept explicit track of.

Combining the definitions (1.1) of the energy E_{σ} and (5.2) of L with the lower bound for the DMI term $\widetilde{H}(\boldsymbol{m})$, we find

$$E_{\sigma}(\boldsymbol{m}) \ge 4\pi + \frac{1}{2L^2} + \sigma^2 A(\boldsymbol{m}) - 4\pi \sigma^2 \rho \cos(\alpha) \cos^2(2\beta) - C\sigma^2 \frac{\sqrt{A(\boldsymbol{m})} + \rho \sqrt{\ln L}}{\sqrt{L}},$$

and, using

$$2\rho \frac{\sqrt{\ln L}}{\sqrt{L}} \le \rho^2 + \frac{\ln L}{L} \le \rho^2 + \frac{1}{\sqrt{L}}$$

and (5.14) to estimate the last term,

$$E_{\sigma}(\boldsymbol{m}) - 4\pi + C\sigma^{2}(\rho^{2} + \sqrt{\sigma})$$

$$\geq \frac{1}{2L^{2}} + \sigma^{2}A(\boldsymbol{m}) - 4\pi\sigma^{2}\rho\cos(\alpha)\cos^{2}(2\beta). \qquad (5.15)$$

Plugging in the lower bound (5.7) of Remark 5.2 for the anisotropy term $A(\mathbf{m})$ we find

$$E_{\sigma}(\boldsymbol{m}) - 4\pi + C\sigma^{2}(\rho^{2} + \sqrt{\sigma})$$

$$\geq \frac{1}{2L^{2}} + 4\pi\sigma^{2}\rho^{2}\ln L - 4\pi\sigma^{2}\rho + 4\pi\sigma^{2}\rho(1 - \cos\alpha) + 4\pi\sigma^{2}\rho\cos(\alpha)\sin^{2}(2\beta)$$

$$\geq \frac{1}{2L^{2}} + 4\pi\sigma^{2}\rho^{2}\ln L - 4\pi\sigma^{2}\rho + 4\pi\sigma^{2}\rho(1 - \cos\alpha). \quad (5.16)$$

Using that the last term is nonnegative, that $E_{\sigma}(m) \leq 4\pi$, and that $4\pi\sigma^2 \rho \leq 2\pi\sigma^2\rho^2 + 4\pi\sigma^2$, we deduce in particular that

$$4\pi\sigma^2\rho^2 \ln L \le C\sigma^2\rho^2 + C\sigma^{5/2} + 4\pi\sigma^2$$
.

If L_0 is large enough and σ_0 is small enough we can absorb the first term of the right-hand side into the first term of the left-hand side, and the second term of the right-hand side into the last, to deduce

$$2\pi\sigma^2\rho^2\ln L \le 8\pi\sigma^2\,,$$

and therefore, recalling that $L \ge 1/4\sigma$,

$$\rho \le \frac{8}{\ln(1/\sigma)} \,. \tag{5.17}$$

Using this to estimate the third term in the first line of (5.16), we obtain

$$E_{\sigma}(\boldsymbol{m}) - 4\pi + C \frac{\sigma^2}{\ln^2 \sigma}$$

$$\geq f_{\sigma}\left(\rho, \frac{1}{L}\right) + 4\pi \sigma^2 \rho \left(1 - \cos \alpha\right), \qquad (5.18)$$

where $f_{\sigma}(\rho, t) = \frac{t^2}{2} - 4\pi \sigma^2 \rho^2 \ln t - 4\pi \sigma^2 \rho.$

The function $t \mapsto f_{\sigma}(\rho, t)$ is convex, with zero derivative at $t_*(\rho) = 2\sqrt{\pi}\sigma\rho$, and we have the identity

$$f_{\sigma}(\rho, t) - f_{\sigma}(\rho, t_{*}(\rho)) = \frac{1}{2}(t - t_{*}(\rho))^{2} + 4\pi\sigma^{2}\rho^{2}g_{*}(t_{*}(\rho)/t), \qquad (5.19)$$

where $g_{*}(x) = \ln x - 1 + \frac{1}{x} \ge 0 \qquad \forall x > 0$.

Moreover, its minimal value at t_* is given by

$$f_{\sigma}(\rho, t_{*}(\rho)) = 2\pi\sigma^{2}\left(\rho^{2} + 2\rho^{2}\ln\left(\frac{1}{2\sqrt{\pi}\sigma\rho}\right) - 2\rho\right)$$

$$\geq 4\pi\sigma^{2}\left(\rho^{2}\ln\left(\frac{1}{2\sqrt{\pi}\sigma\rho}\right) - \rho\right)$$

$$= -\frac{\pi\sigma^{2}}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)}$$

$$+ 4\pi\sigma^{2}\ln\left(\frac{1}{2\sqrt{\pi}\sigma\rho}\right)\left(\rho - \frac{1}{2\ln\left(1/(2\sqrt{\pi}\sigma\rho)\right)}\right)^{2}$$

$$+ \frac{\pi\sigma^{2}\ln\left(1/(2\sqrt{\pi}\rho\ln(1/\sigma))\right)}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)\ln\left(1/(2\sqrt{\pi}\sigma\rho)\right)}.$$

If σ_0 is small enough, then thanks to (5.17) we have $2\sqrt{\pi\rho} \leq 1$ and the denominator of the last term is $\geq \ln^2 \sigma$. Combining this with (5.19) and (5.17), we deduce

$$f_{\sigma}(\rho, t) \geq -\frac{\pi \sigma^2}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)} + 4\pi \sigma^2 \ln\left(\frac{\ln(1/\sigma)}{\sigma}\right) \left(\rho - \frac{1}{2\ln\left(1/(2\sqrt{\pi}\sigma\rho)\right)}\right)^2$$

$$+ 4\pi\sigma^2 \rho^2 g_* \left(\frac{2\sqrt{\pi}\sigma\rho}{t}\right) \\ + \frac{\pi\sigma^2 \ln\left(4/(\rho \ln(1/\sigma))\right)}{\ln^2 \sigma} - C \frac{\sigma^2}{\ln^2 \sigma}.$$

Plugging this inequality into the lower bound (5.18) we obtain

$$\frac{E_{\sigma}(\boldsymbol{m}) - 4\pi}{\sigma^{2}} + \frac{\pi}{\ln\left(\sigma^{-1}\ln(1/\sigma)\right)} + \frac{C_{0}}{\ln^{2}\sigma} \\
\geq 4\pi \ln\left(\frac{\ln(1/\sigma)}{\sigma}\right) \left(\rho - \frac{1}{2\ln\left(1/(2\sqrt{\pi}\sigma\rho)\right)}\right)^{2} \\
+ \frac{\pi \ln\left(4/(\rho\ln(1/\sigma))\right)}{\ln^{2}\sigma} + 4\pi\rho^{2}g_{*}\left(2L\sqrt{\pi}\sigma\rho\right) \\
+ 4\pi\rho\left(1 - \cos\alpha\right),$$
(5.20)

for some absolute constant $C_0 > 0$ depending explicitly on c_* and $c_{\rm MT}$. Since all terms in the last three lines of (5.20) are nonnegative, this implies the lower bound (5.11).

Now we assume that the map $m = m_{\sigma}$ saturates that lower bound, that is, it satisfies the upper bound (5.12) for some $K \ge C_0$. In the rest of the proof, we will be denote by C(K) a generic positive constant depending on K and the previous absolute constants, whose value may change from line to line.

Under the assumption (5.12), all the nonnegative terms in the right-hand side of (5.20) are bounded by $2K/\ln^2 \sigma$. For the second term, this implies

$$\rho \ge \frac{4e^{-2K/\pi}}{\ln(1/\sigma)} \,,$$

and we deduce

$$\left|\frac{1}{2\ln\left(1/(2\sqrt{\pi}\sigma\rho)\right)} - \frac{1}{2\ln(1/\sigma)}\right| = \frac{\ln\left(1/(2\sqrt{\pi}\rho)\right)}{2\ln(1/\sigma)\ln\left(1/(2\sqrt{\pi}\sigma\rho)\right)}$$
$$\leq \frac{\ln\ln(1/\sigma) + CK}{2\ln^2\sigma}.$$

This and the fact that the first term in the right-hand side of (5.20) is $\leq 2K/\ln^2 \sigma$ imply

$$\left| \rho - \frac{1}{2\ln(1/\sigma)} \right| \le \frac{C(K)}{\ln^{\frac{3}{2}}(1/\sigma)},$$
(5.21)

which is the claimed estimate on ρ .

Next we use that the third term in the right-hand side of (5.20) is $\leq 2K/\ln^2 \sigma$ and that $C(K)\rho \geq \ln(1/\sigma)$, to obtain

 $g_*(2L\sqrt{\pi}\sigma\rho) \le C(K),$

where we recall that $g_*(x) = \ln x - 1 + 1/x$. Since $g_*(x) \to +\infty$ as $x \to 0$ and $x \to \infty$, we infer

$$\frac{1}{C(K)} \le L\sigma\rho \le C(K),$$

and using (5.21) this gives

$$\frac{1}{C(K)} \frac{\ln(1/\sigma)}{\sigma} \le L \le C(K) \frac{\ln(1/\sigma)}{\sigma}.$$
(5.22)

The fact that the fourth term in the right-hand side of (5.20) is $\leq 2K/\ln^2 \sigma$, implies

$$\frac{1}{C}\operatorname{dist}^2(\alpha, 2\pi\mathbb{Z}) \le 1 - \cos\alpha \le \frac{C(K)}{\ln(1/\sigma)}.$$

Since α can be chosen in $[-\pi, \pi]$, this implies the claimed estimate on α .

Finally, plugging the assumption (5.12) and the estimate (5.21) of ρ back into the inequality (5.15) implies

$$A(\boldsymbol{m}) \leq \frac{C(K)}{\ln(1/\sigma)}$$
.

Combining this with the first lower bound (5.5) on A(m) and with the estimates (5.21) and (5.22) of ρ and L gives

$$\sin^2(2\beta) \le \frac{C(K)\sigma^2}{\ln(1/\sigma)} \,,$$

and therefore

$$\operatorname{dist}^{2}\left(\beta, \frac{\pi}{2}\mathbb{Z}\right) \leq \frac{\pi^{2}}{4}\sin^{2}(2\beta) \leq C(K)\frac{\sigma^{2}}{\ln(1/\sigma)}$$

Taking into account Remark 2.3, after possibly redefining the angle ϕ this implies the claimed estimate on β .

Recalling the definition (5.2) of L, the above bounds (5.22) turn into

$$\frac{1}{C(K)}\frac{\sigma^2}{\ln^2\sigma} \le \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}_{\sigma}|^2 \, dx - 8\pi \le C(K)\frac{\sigma^2}{\ln^2\sigma}$$

Combining this with the stability estimate (5.1) and the classical identity

$$\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}_{\sigma}|^2 \, dx - 8\pi = \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}_{\sigma} - \nabla \Psi|^2 \, dx$$
$$- \int_{\mathbb{R}^2} |\nabla \Psi|^2 |\boldsymbol{m}_{\sigma} - \Psi|^2 \, dx$$

$$\leq \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}_{\sigma} - \nabla \Psi|^2 \, dx \, ,$$

which follows from the harmonic map equation $-\Delta \Psi = |\nabla \Psi|^2 \Psi$ and the identity $|\boldsymbol{m}_{\sigma} - \Psi|^2 = 2(\Psi - \boldsymbol{m}_{\sigma}) \cdot \Psi$, we deduce

$$\frac{1}{C(K)} \frac{\sigma^2}{\ln^2 \sigma} \le \int_{\mathbb{R}^2} |\nabla \boldsymbol{m}_{\sigma} - \nabla \Psi|^2 \, dx \le C(K) \frac{\sigma^2}{\ln^2 \sigma} \,,$$

thus concluding the proof of Proposition 5.4.

6. EXISTENCE OF MINIMIZERS AND PROOF OF THEOREM 1.3

In this section we rely on the characterization of near-minimizers in Proposition 5.4 to show that the infimum of E_{σ} on W_{-1} is attained, provided σ is small enough. This follows the standard concentration-compactness strategy applied also in [7, 16, 18, 6], but we need a different argument to rule out "vanishing".

Proposition 6.1. There exists an absolute constant $\sigma_0 > 0$, depending explicitly on c_{MT} and c_* , such that

$$\inf_{\mathcal{W}_{-1}} E_{\sigma} = \min_{\mathcal{W}_{-1}} E_{\sigma} \,,$$

for $0 < \sigma < \sigma_0$.

Thanks to this existence result we may now prove our main theorem.

Proof of Theorem 1.3. The existence of a minimizer $m_{\sigma} \in W_{-1}$ is provided by Proposition 6.1. The upper bound (4.1) in the energy expansion is provided by the construction in § 4. The lower bound and the description of the minimizer in terms of Möbius maps is provided by Proposition 5.4, taking into account the parametrization of Möbius maps provided by Lemma 2.1.

Finally we prove that E_{σ} attains its infimum on \mathcal{W}_{-1} .

Proof of Proposition 6.1. The proof is divided in three steps: some basic observations on minimizing sequences, the conclusion under a tightness assumption, and finally a proof of that tightness assumption. All these steps follow well-known arguments, but we provide a fair amount of details to convince the reader that they do adapt to our case.

Step 1: Basic observations.

The first observation is that the energy can be rewritten as

$$E_{\sigma}(\boldsymbol{m}) = \int_{\mathbb{R}^2} e_{\sigma}(\boldsymbol{m}) \, dx, \qquad (6.1)$$

$$e_{\sigma}(\boldsymbol{m}) = \frac{1}{2}(\partial_2 m_1 - 2\sigma^2 m_3)^2 + \frac{1}{2}(\partial_1 m_2 + 2\sigma^2 m_3)^2 + \frac{1}{2}(\partial_1 m_1)^2 + \frac{1}{2}(\partial_2 m_2)^2 + \sigma^2(1 - 4\sigma^2)m_3^2$$

For $0 < \sigma < 1/2$, this identity implies

$$\int_{\mathbb{R}^2} m_3^2 dx \leq \frac{1}{\sigma^2 (1 - 4\sigma^2)} E_{\sigma}(\boldsymbol{m}) ,$$

and
$$\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}|^2 dx \leq 4E_{\sigma}(\boldsymbol{m}) + 8\sigma^4 \int_{\mathbb{R}^2} m_3^2 dx ,$$

so any minimizing sequence $m{m}^{(k)} \in \mathcal{W}_{-1}$ satisfying

$$E_{\sigma}(\boldsymbol{m}^{(k)}) \to \inf_{\mathcal{W}_{-1}} E_{\sigma},$$

admits a subsequence, still denoted $m^{(k)}$, such that

$$\nabla \boldsymbol{m}^{(k)} \rightharpoonup \nabla \boldsymbol{m}$$
 and $m_3^{(k)} \rightharpoonup m_3$ weakly in $L^2(\mathbb{R}^2)$,

for some $m \in W$, where the space W is defined in (1.13). Moreover, the identity (6.1) also implies that E_{σ} is a convex function of m, and therefore satisfies the lower semicontinuity property

$$E_{\sigma}(\boldsymbol{m}) \leq \liminf E_{\sigma}(\boldsymbol{m}^{(k)}) = \inf_{\mathcal{W}_{-1}} E_{\sigma}.$$
 (6.2)

In order to conclude that m minimizes E_{σ} in \mathcal{W}_{-1} , it remains to show that the weak limit m actually belongs to \mathcal{W}_{-1} . This is not directly obvious because the topological degree Q(m) defined in (1.3) is not continuous with respect to that weak convergence.

Note that, if $0 < \sigma < \sigma_0$ for a small enough σ_0 , the upper bound (4.1) together with the basic lower bound (5.13) imply

$$\inf_{\mathcal{W}_{-1}} E_{\sigma} < \inf_{\mathcal{W}_{q}} E_{\sigma} \qquad \text{if } |q| \ge 2.$$
(6.3)

From this and (6.2) it follows that $Q(\mathbf{m}) \in \{0, \pm 1\}$. In order to conclude that $Q(\mathbf{m}) = -1$, we therefore only need to show that $Q(\mathbf{m}) < 0$.

Step 2. Conclusion under a tightness condition.

As in [18] the proof that Q(m) < 0 relies on a concentration-compactness argument. That argument shows that, along a non-relabeled subsequence and modulo translations \mathfrak{T}_{z_k} which keep the energy invariant (1.8), the L^1 sequence

$$f_k = |\nabla \boldsymbol{m}^{(k)}|^2 + (m_3^{(k)})^2 \quad \text{is uniformly tight, that is,}$$
$$\sup_{k \ge 1} \int_{|x| \ge R} f_k \, dx \longrightarrow 0 \qquad \text{as } R \to +\infty \,. \tag{6.4}$$

That tightness, together with the strong convergence of $m^{(k)}$ in $L^2(B_R)$ for any R > 0 ensured by Rellich-Kondratchov's theorem, implies, as in [18, Lemma 4.1],

$$A(\boldsymbol{m}^{(k)}) + \widetilde{H}(\boldsymbol{m}^{(k)}) \longrightarrow A(\boldsymbol{m}) + \widetilde{H}(\boldsymbol{m}).$$

Moreover, classical arguments, see e.g. [22, Theorem 1.6], show that the functional $D(\boldsymbol{m}) + 4\pi Q(\boldsymbol{m})$ is lower semicontinuous on $H_c^1(\mathbb{R}^2; \mathbb{S}^2)$ with respect to the weak convergence $\nabla \boldsymbol{m}^{(k)} \rightharpoonup \nabla \boldsymbol{m}$ in $L^2(\mathbb{R}^2)$, because it can be written as

$$D(\boldsymbol{m}) + 4\pi Q(\boldsymbol{m}) = \int_{\mathbb{R}^2} W(\nabla \boldsymbol{m}, \boldsymbol{m}) \, dx \,,$$
$$W(\nabla \boldsymbol{m}, \boldsymbol{m}) = \frac{1}{2} |\nabla \boldsymbol{m}|^2 - \boldsymbol{m} \cdot \partial_1 \boldsymbol{m} \times \partial_2 \boldsymbol{m} \ge 0,$$

and $W(\boldsymbol{m}, \cdot)$ is a nonnegative, hence convex, quadratic form. Since $E_{\sigma} = D + \sigma^2(A + \tilde{H})$, we infer the lower semicontinuity property

$$E_{\sigma}(\boldsymbol{m}) + 4\pi Q(\boldsymbol{m}) \leq \liminf_{k \to \infty} \left(E_{\sigma}(\boldsymbol{m}^{(k)}) + 4\pi Q(\boldsymbol{m}^{(k)}) \right).$$

Recalling that $m^{(k)} \in W_{-1}$ is a minimizing sequence, we deduce

$$E_{\sigma}(\boldsymbol{m}) + 4\pi Q(\boldsymbol{m}) \leq \inf_{\mathcal{W}_{-1}} E_{\sigma} - 4\pi$$
.

If $0 < \sigma < \sigma_0$ for a small enough σ_0 , the right-hand side is negative due to the upper bound (4.1), and $E_{\sigma}(\boldsymbol{m}) \geq 0$ due to (5.13), so this implies $Q(\boldsymbol{m}) < 0$, hence $Q(\boldsymbol{m}) = -1$ and $\boldsymbol{m} \in \mathcal{W}_{-1}$. Together with (6.2) this concludes the proof that the infimum of E_{σ} on \mathcal{W}_{-1} is attained, provided we show the tightness property (6.4).

Step 3. Proof of the tightness property.

As in [18], the tightness property (6.4) is obtained by ruling out, along a subsequence, the two other cases in the alternative established in [17, Lemma I.1]: *vanishing*, that is,

$$\sup_{z_0 \in \mathbb{R}^2} \int_{|z-z_0| \le R} f_k \, dz \to 0 \,, \quad \forall R > 0 \,, \tag{6.5}$$

and *dichotomy*, which implies, modulo translations \mathfrak{T}_{z_k} , the existence of $R_k > 1$ such that

$$\liminf \int_{|z| \le R_k} f_k \, dz > 0, \quad \liminf \int_{|z| \ge 2R_k} f_k \, dz > 0, \tag{6.6}$$

and
$$\int_{R_k \le |z| \le 2R_k} f_k \, dz \to 0.$$

Contrary to [18, Lemma 4.2] for the easy-axis anisotropy, in our case vanishing (6.5) does not seem to directly imply that $\tilde{H}(\boldsymbol{m}^{(k)}) \rightarrow 0$. We rely instead on the upper bound (4.1) and the characterization of near-minimizers in Proposition 5.4 to discard vanishing. Thanks to (4.1) and the fact that $\boldsymbol{m}^{(k)}$ is a minimizing sequence, we know indeed that $\boldsymbol{m}^{(k)}$ satisfies (5.12) for some absolute constant K > 0 and large enough k. According to Proposition 5.4, there exists therefore a Möbius map $\Psi^{(k)}$ and $z_k \in \mathbb{R}^2$ such that

$$\int_{\mathbb{R}^2} |\nabla \boldsymbol{m}^{(k)} - \nabla \Psi^{(k)}|^2 \, dz \le \frac{1}{4} \,,$$

and
$$\int_{|z-z_k| \le 1} |\nabla \Psi^{(k)}|^2 \, dz \ge 1 \,.$$

The second inequality follows from the fact that the concentration scale $\rho > 0$ of the Möbius map Ψ provided by Proposition 5.4 is arbitrarily small if σ_0 is small enough. These two inequalities imply that

$$\int_{|z-z_k|\leq 1} f_k \, dz \geq \int_{|z-z_k|\leq 1} |\nabla \boldsymbol{m}^{(k)}|^2 \, dz \geq \frac{1}{4} \, ,$$

thus ruling out vanishing (6.5).

The dichotomy case (6.6) can be ruled out as in [18], by using the construction of [6, Lemma 8] adapted to our setting. It provides two maps $m^{(k,1)}, m^{(k,2)} \in \mathcal{W}$ such that

$$\boldsymbol{m}^{(k)} = \boldsymbol{m}^{(k,1)} \text{ in } B_{R_k}, \quad \boldsymbol{m}^{(k)} = \boldsymbol{m}^{(k,2)} \text{ outside } B_{2R_k},$$

$$\int_{|z| \ge R_k} |\nabla \boldsymbol{m}^{(k,1)}|^2 + (m_3^{(k,1)})^2 \, dx \longrightarrow 0,$$

$$\int_{|z| \le 2R_k} |\nabla \boldsymbol{m}^{(k,2)}|^2 + (m_3^{(k,2)})^2 \, dx \longrightarrow 0.$$
(6.7)

We briefly sketch that construction. We select a radius $\rho_k \in [R_k, 2R_k]$ such that

$$R_k \int_{\partial B_{\rho_k}} f_k \, d\mathcal{H}^1 \le \int_{R_k < |z| < 2R_k} f_k \, dz = \delta_k \to 0$$

This implies that the map $heta\mapsto\hat{m{m}}^{(k)}(heta)=m{m}^{(k)}(
ho_k e^{i heta})$ satisfies

$$\frac{1}{2} \int_0^{2\pi} |\partial_\theta \hat{\boldsymbol{m}}^{(k)}|^2 \, d\theta + R_k^2 \int_0^{2\pi} (\hat{m}_3^{(k)})^2 \, d\theta \le \delta_k \, .$$

In particular, $\hat{\boldsymbol{m}}^{(k)}$ has small oscillation over \mathbb{S}^1 , its average must be close to $\mathbb{S}^1 \times \{0\} \subset \mathbb{S}^2$, and we may find $\xi^{(k)} \in \mathbb{S}^1 \times \{0\}$ such that

$$\sup_{\theta \in \mathbb{S}^1} |\hat{\boldsymbol{m}}^{(k)}(\theta) - \boldsymbol{\xi}^{(k)}|^2 \le C\delta_k$$

Then we define, using polar coordinates $x = re^{i\theta}$,

$$\boldsymbol{m}^{(k,1)} = \begin{cases} \boldsymbol{m}^{(k)} & \text{in } B_{\rho_k}, \\ \Pi \left(\xi^{(k)} + \chi_1(r/\rho_k) \left(\hat{\boldsymbol{m}}^{(k)}(\theta) - \xi^{(k)} \right) \right) & \text{for } r \ge \rho_k \,, \\ \boldsymbol{m}^{(k,2)} = \begin{cases} \boldsymbol{m}^{(k)} & \text{outside } B_{\rho_k}, \\ \Pi \left(\xi^{(k)} + \chi_2(r/\rho_k) \left(\hat{\boldsymbol{m}}^{(k)}(\theta) - \xi^{(k)} \right) \right) & \text{for } 0 < r < \rho_k \end{cases}$$

where $\Pi(X) = X/|X|$ is the projection onto \mathbb{S}^2 and χ_1, χ_2 are smooth cut-off functions satisfying

$$\mathbf{1}_{r\leq 1} \leq \chi_1(r) \leq \mathbf{1}_{r\leq 2}, \quad \mathbf{1}_{r\geq 1} \leq \chi_2(r) \leq \mathbf{1}_{r\geq 1/2},$$

and it can be checked that these maps satisfy (6.7).

The properties (6.7) of $m^{(k,1)}$, $m^{(k,2)}$ imply that their integer-valued topological degrees satisfy

$$Q(\boldsymbol{m}^{(k,1)}) + Q(\boldsymbol{m}^{(k,2)}) = Q(\boldsymbol{m}^{(k)}) = -1$$

and their energies satisfy

$$E_{\sigma}(\boldsymbol{m}^{(k)}) = E_{\sigma}(\boldsymbol{m}^{(k,1)}) + E_{\sigma}(\boldsymbol{m}^{(k,2)}) + o(1).$$
(6.8)

Since $\mathbf{m}^{(k)}$ is a minimizing sequence for E_{σ} in \mathcal{W}_{-1} , and $E_{\sigma} \geq 0$, this together with the inequality (6.3) implies, along a subsequence and for large enough k,

$$Q(\boldsymbol{m}^{(k,1)}) = q_1 \in \{0,\pm 1\}, \qquad Q(\boldsymbol{m}^{(k,2)}) = q_2 \in \{0,\pm 1\}.$$

Since $q_1 + q_2 = -1$, we have either $q_1 = -1$ or $q_2 = -1$. Moreover, the first line in (6.6), the first inequality in (5.13), and the properties (6.7) of $m^{(k,1)}, m^{(k,2)}$ imply, along a subsequence,

$$E(\boldsymbol{m}^{(k,1)}) \to \mu_1 > 0, \quad E(\boldsymbol{m}^{(k,2)}) \to \mu_2 > 0.$$

If $q_1 = -1$, we also have $\mu_1 \ge \inf_{\mathcal{W}_{-1}} E_{\sigma} = \lim_{\sigma \to \infty} E_{\sigma}(\boldsymbol{m}^{(k)})$, thus contradicting (6.8). And if $q_2 = -1$ we have $\mu_2 \ge \lim_{\sigma \to \infty} E_{\sigma}(\boldsymbol{m}^{(k)})$ and again a contradiction. This shows that the dichotomy case (6.6) cannot occur and concludes the proof of the tightness property (6.4).

APPENDIX A. CRITICAL SOBOLEV SPACES ON THE PLANE AND ON THE SPHERE

In this appendix we recall, for the readers' convenience, the identification between functions with finite Dirichlet energy on the plane and on the sphere, via stereographic projection. And, more generally, functions on \mathbb{R}^n and \mathbb{S}^n with finite *n*-energy for any $n \ge 2$. That is, we compare the homogeneous Sobolev space

$$\mathcal{H}(\mathbb{R}^n) = \left\{ v \in W^{1,1}_{\mathrm{loc}}(\mathbb{R}^n) \colon \int_{\mathbb{R}^n} |\nabla v|^n \, dx < \infty \right\},\tag{A.1}$$

and the Sobolev space $W^{1,n}(\mathbb{S}^n)$. As in (1.2) we define

$$\Phi \colon \mathbb{R}^n \cup \{\infty\} \to \mathbb{S}^n \subset \mathbb{R}^{n+1},$$
$$x \mapsto \left(\frac{2x}{1+|x|^2}, \frac{|x|^2 - 1}{1+|x|^2}\right) \quad \text{if } x \in \mathbb{R}^n$$
$$\infty \mapsto \mathbf{e}_{n+1} = (0, \dots, 0, 1),$$

whose inverse is the stereographic projection

$$\Phi^{-1} \colon \mathbb{S}^n \to \mathbb{R}^n,$$

$$\xi \mapsto \frac{1}{1 - \xi_{n+1}} (\xi_1, \dots, \xi_n) \quad \text{if } \xi \neq \mathbf{e}_{n+1},$$

$$\mathbf{e}_{n+1} \mapsto \infty.$$

A crucial property of Φ is its conformality:

$$\langle \partial_i \Phi, \partial_j \Phi \rangle = \frac{4}{(1+|x|^2)^2} \delta_{ij} \quad \text{for } i, j \in \{1, \dots, n\}.$$

(In general, a diffeomorphism between riemannian manifold is conformal if it induces a conformal change of metric, see e.g. [20, Chapter 7].) Given any $w \in C^1(\mathbb{S}^n)$, we have $v = w \circ \Phi \in C^1(\mathbb{R}^n)$, and the *n*-energy has the conformal invariance property

$$\int_{\mathbb{S}^n} |\nabla_T w|^n \, d\mathcal{H}^n = \int_{\mathbb{R}^n} |\nabla v|^n \, dx \,, \tag{A.2}$$

where ∇_T denotes the tangential gradient on \mathbb{S}^n . Note that, given a measurable function $v \colon \mathbb{R}^n \to \mathbb{R}$, the function $w = v \circ \Phi^{-1}$ is defined almost everywhere and measurable on \mathbb{S}^n . The following result is folklore, but we could not find a precise reference so we include a proof below.

Proposition A.1. For any measurable $v \colon \mathbb{R}^n \to \mathbb{R}$, we have $v \in \mathcal{H}(\mathbb{R}^n)$ if and only if $w = v \circ \Phi^{-1} \in W^{1,n}(\mathbb{S}^n)$, and the identity (A.2) is satisfied.

Before proving proving Proposition A.1 we gather some consequences that are relevant to the setting of this article. First, if $v \in \mathcal{H}(\mathbb{R}^n; \mathbb{S}^n)$, then $w = v \circ \Phi^{-1} \in W^{1,n}(\mathbb{S}^n, \mathbb{S}^n)$ has a well-defined topological degree $Q \in \mathbb{Z}$, see e.g. [5]. Using the classical expression of that degree for C^1 maps and arguing by approximation gives the formula

$$Q = \oint_{\mathbb{S}^n} \det(w, \partial_{\tau_1} w, \dots, \partial_{\tau_n} w) \, d\mathcal{H}^n \,,$$

where (τ_1, \ldots, τ_n) is any choice of direct orthonormal frame of the tangent space to \mathbb{S}^n . Coming back to v and using again the conformality of Φ we obtain

Corollary A.2. For a sphere-valued map $v \in \mathcal{H}(\mathbb{R}^n; \mathbb{S}^n)$, the quantity

$$Q(v) = \frac{1}{\mathcal{H}^n(\mathbb{S}^n)} \int_{\mathbb{R}^n} \det(v, \partial_1 v, \cdots, \partial_n v) \, dx \,,$$

is a well-defined integer, which corresponds to the topological degree of $w = v \circ \Phi^{-1} \in W^{1,n}(\mathbb{S}^n; \mathbb{S}^n).$

Another consequence of Proposition A.1 concerns the vanishing mean oscillation property: functions in $W^{1,n}(\mathbb{S}^n)$ have vanishing mean oscillation at every point [5, § I.2]. Applying this to the function w at \mathbf{e}_{n+1} translates, after a change of variable, into vanishing mean oscillation "at ∞ " for any function $v \in \mathcal{H}(\mathbb{R}^n)$, namely

$$\int_{|x|\ge R} |v-v_R| \, \frac{dx}{(1+|x|^2)^n} \to 0 \qquad \text{as } R \to \infty \, ,$$

where the average is taken with respect to the measure $dx/(1+|x|^2)^n$, and v_R is the mean value

$$v_R = \int_{|x| \ge R} v \, \frac{dx}{(1+|x|^2)^n} \, .$$

In general, that mean value need not have a limit as $R \to \infty$. In fact, we have $v_R \to v_\infty$ as $R \to \infty$ if and only if \mathbf{e}_{n+1} is a Lebesgue point of w, with value $w(\mathbf{e}_{n+1}) = v_\infty$. Note that the function $R \mapsto v_R$ is continuous, so the set of possible limits of v_{R_k} along subsequence $R_k \to \infty$ is connected. Moreover, if we happen to have the additional information that

$$\int_{\mathbb{R}^n} f(v) \, \frac{dx}{(1+|x|^2)^{\frac{n}{2}}} < \infty \,,$$

for some *L*-Lipschitz function f, then we deduce that $f(v_R) \to 0$ as $R \to \infty$, thanks to the inequalities

$$\begin{aligned} |f(v_R)| &= \int_{|x| \ge R} |f(v_R)| \frac{dx}{(1+|x|^2)^n} \\ &\leq L \oint_{|x| \ge R} |v - v_R| \frac{dx}{(1+|x|^2)^n} + \int_{|x| \ge R} |f(v)| \frac{dx}{(1+|x|^2)^n} \\ &\leq L \oint_{|x| \ge R} |v - v_R| \frac{dx}{(1+|x|^2)^n} + C \int_{|x| \ge R} |f(v)| \frac{dx}{(1+|x|^2)^{\frac{n}{2}}} \,, \end{aligned}$$

where we used that

$$\frac{1}{(1+R^2)^{\frac{n}{2}}} \le C \int_{|x| \ge R} \frac{dx}{(1+|x|^2)^n}$$

for some constant C > 0 depending on n. If v takes values into a closed set $X \subset \mathbb{R}^k$, applying this to $f = \text{dist}(\cdot, X)$ shows that $\text{dist}(v_R, X) \to 0$ as $R \to \infty$. And if we know in addition that g(v) is integrable on \mathbb{R}^n for some Lipschitz function g, then all possible limits of subsequences v_{R_k} must belong to a single connected component of $X \cap g^{-1}(\{0\})$. If all such connected components are points, then v_R converges and \mathbf{e}_{n+1} is a Lebesgue point of w. Applying this to n = 2, $X = \mathbb{S}^2 \subset \mathbb{R}^3$ and $g(x) = 1 - x_3$ or $1 - x_3^2$ for $x \in \mathbb{S}^2$, we deduce the following.

Corollary A.3. For a map $m \in \mathcal{H}(\mathbb{R}^2; \mathbb{S}^2)$, if either $1 - m_3^2$ or $1 - m_3$ is integrable on \mathbb{R}^2 , then \mathbf{e}_3 is a Lebesgue point of $w = \mathbf{m} \circ \Phi^{-1} \in W^{1,2}(\mathbb{S}^2; \mathbb{S}^2)$.

We finally turn to the proof of Proposition A.1.

Proof of Proposition A.1. First we show that, if $w \in W^{1,n}(\mathbb{S}^n)$, then $v = w \circ \Phi$ belongs to $\mathcal{H}(\mathbb{R}^n)$ and (A.2) is satisfied. This follows by approximating w with smooth functions $w_{\epsilon} \in C^1(\mathbb{S}^n)$ to which we apply (A.2) and pass to the limit. To make sure that $v_{\epsilon} = w_{\epsilon} \circ \Phi$ converges in $L^1_{\text{loc}}(\mathbb{R}^n)$ to $v = w \circ \Phi$ we use for instance the identity

$$\int_{\mathbb{S}^n} u(y) \, d\mathcal{H}^n(y) = \int_{\mathbb{R}^n} u \circ \Phi(x) \frac{2^n dx}{(1+|x|^2)^n} \,, \tag{A.3}$$

valid for any measurable $u \colon \mathbb{S}^n \to [0, \infty]$, applied to $u = |w - w_{\epsilon}|$.

Reciprocally, let us assume that $v \in \mathcal{H}(\mathbb{R}^n)$, and prove that $w = v \circ \Phi^{-1}$ belongs to $W^{1,n}(\mathbb{S}^n)$. From localized versions of the identities (A.2)-(A.3) we see that

$$w \in W^{1,n}_{\operatorname{loc}}(\mathbb{S}^n \setminus \{\mathbf{e}_{n+1}\})$$
 and $\nabla_T w \in L^n(\mathbb{S}^n \setminus \{\mathbf{e}_{n+1}\}),$

where $\nabla_T w$ is the distributional gradient of w in $\mathbb{S}^n \setminus \{\mathbf{e}_{n+1}\}$. In order to conclude that $w \in W^{1,n}(\mathbb{S}^n)$, it suffices to show that $w \in L^n(\mathbb{S}^n)$. This implies indeed that $w \in W^{1,n}(\mathbb{S}^n \setminus \{\mathbf{e}_{n+1}\})$ and therefore $w \in W^{1,n}(\mathbb{S}^n)$ since, using the terminology of [13], points are removable for Sobolev spaces in dimension $n \ge 2$. We recall here the proof of that fact: working in a local chart, this amounts to showing that, if $\Omega = (-1,1)^n$ and $u \in W^{1,p}(\Omega \setminus \{0\})$ for some $p \ge 1$, then the distributional gradient of u on Ω is equal to its distributional gradient on $\Omega \setminus \{0\}$. To check this, note that for almost every $x' = (x_2, \ldots, x_n) \in \Omega' = (-1, 1)^{n-1} \setminus \{0\}$ we have $u(\cdot, x') \in W^{1,p}((-1, 1))$ with distributional derivative given by $\partial_1 u(\cdot, x') \in L^p((-1, 1)), \text{ so for any } \varphi \in C_c^{\infty}(\Omega) \text{ we can compute}$ $\langle \partial_1 u, \varphi \rangle = -\int_{\Omega'} \int_{-1}^1 u \, \partial_1 \varphi \, dx_1 dx' = \int_{\Omega'} \int_{-1}^1 \varphi \, \partial_1 u \, dx_1 dx'$ $= \int_{\Omega \setminus \{0\}} \varphi \, \partial_1 u \, dx \,,$

and similarly for the other derivatives. In view of (A.3), the fact that $w \in L^n(\mathbb{S}^n)$ is ensured by Lemma A.5 below.

Remark A.4. It would have been tempting to prove the second implication $(v \in \mathcal{H} \Rightarrow w \in W^{1,n})$ by approximation with smooth functions, as for the first implication. But there is an additional difficulty: while $w \in C^1(\mathbb{S}^n)$ implies $v = w \circ \Phi \in C^1 \cap \mathcal{H}(\mathbb{R}^n)$, the converse implication it not true. For instance, the function $v(x) = \ln(1 + \ln(1 + |x|^2))$ belongs to $C^1 \cap \mathcal{H}(\mathbb{R}^n)$, but $w = v \circ \Phi^{-1}$ is not continuous at $\mathbf{e}_{n+1} \in \mathbb{S}^n$. (In fact \mathbf{e}_{n+1} is not even a Lebesgue point of w.) That is why we argued differently.

Lemma A.5. For all $v \in \mathcal{H}(\mathbb{R}^n)$ we have

$$\int_{\mathbb{R}^n} |v|^n \frac{dx}{(1+|x|^2)^n} \le c \int_{\mathbb{R}^n} |\nabla v|^n \, dx + c \int_{B_1} |v|^n \, dx$$

for some constant c > 0 depending on n.

Proof of Lemma A.5. We introduce the notation

$$A(v) = \int_{\mathbb{R}^n} |\nabla v|^n \, dx + c \int_{B_1} |v|^n \, dx \, ,$$

and denote by c > 0 a generic constant depending on n, which may change from line to line. By Sobolev inequality in $B_1 \subset \mathbb{R}^n$ we have

$$\int_{B_1} |v|^n \, dx \le cA(v)$$

so it suffices to show that

$$\int_{|x|\ge 1} |v|^n \frac{dx}{|x|^{2n}} \le cA(v) \, .$$

To that end we consider the function

$$f(r) = \int_{\mathbb{S}^{n-1}} v(r\omega) \, d\mathcal{H}^{n-1}(\omega) \quad \forall r \ge 1 \, .$$

We fix $r_0 \in (1/2, 1)$ such that

$$|f(r_0)| \le c \int_{B_1} |v| \, dx$$

For all $R \ge 1$ we have

$$\begin{split} |f(R) - f(r_0)| &\leq \int_{r_0}^R \oint_{\mathbb{S}^{n-1}} |\partial_r v| (r\omega) \, d\mathcal{H}^{n-1}(\omega) \, dr \\ &\leq \int_{B_R \setminus B_{r_0}} |\nabla v| \, \frac{dx}{|x|^{n-1}} \\ &\leq \left(\int_{\mathbb{R}^n} |\nabla v|^n \, dx \right)^{\frac{1}{n}} \left(\int_{B_R \setminus B_{r_0}} \frac{dx}{|x|^n} \right)^{1-\frac{1}{n}} \\ &\leq cA(v)^{\frac{1}{n}} \ln(2R/r_0) \,, \end{split}$$

and therefore

$$|f(R)|^n \le cA(v)\ln^n(4R), \quad \forall R \ge 1.$$

By Sobolev embedding $W^{1,n}(\mathbb{S}^{n-1}) \subset L^{\infty}(\mathbb{S}^{n-1})$ and definition of f(r), for all $r \geq 1$ and $\omega \in \mathbb{S}^{n-1}$ we have

$$|v(r\omega)|^n \le c|f(r)|^n + cr^n \int_{\mathbb{S}^{n-1}} |\nabla v|^n(r\tilde{\omega}) \, d\mathcal{H}^{n-1}(\tilde{\omega}) \, .$$

Using the two last inequalities we deduce

$$\int_{|x|\geq 1} |v|^n \frac{dx}{|x|^{2n}} = \int_{\mathbb{S}^{n-1}} \int_1^\infty |v(r\omega)|^n \frac{dr}{r^{n+1}} d\mathcal{H}^{n-1}(\omega)$$

$$\leq c \int_1^\infty |f(r)|^n \frac{dr}{r^{n+1}} + c \int_{|x|\geq 1} |\nabla v|^n \frac{dx}{|x|^n}$$

$$\leq cA(v) \,,$$

and this concludes the proof.

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