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Persistence of superconductivity in thin shells beyond H_{c_1}

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In Ginzburg–Landau theory, a strong magnetic field is responsible for the breakdown of superconductivity. This work is concerned with the identification of the region where superconductivity persists, in a thin shell superconductor modeled by a compact surface $\mathcal{M} \subset \mathbb{R}^3$, as the intensity h of the external magnetic field is raised above H_{c_1}. Using a mean field reduction approach devised by Sandier and Serfaty as the Ginzburg–Landau parameter κ goes to infinity, we are led to studying a *two-sided* obstacle problem. We show that superconductivity survives in a neighborhood of size $(H_{c_1}/h)^{1/3}$ of the zero locus of the normal component H of the field. We also describe intermediate regimes, focusing first on a symmetric model problem. In the general case, we prove that a striking phenomenon we call *freezing of the boundary* takes place: one component of the superconductivity region is insensitive to small changes in the field.

Keywords: Ginzburg–Landau; two-sided obstacle problem.

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1. Introduction

Let $\mathcal{M}$ be a compact surface homeomorphic to $\mathbb{S}^2$, embedded in $\mathbb{R}^3$. For $\kappa, h > 0$ and $\mathbf{A}$ a vector field on $\mathcal{M}$, we consider the Ginzburg–Landau functional $\mathcal{G}_{\mathcal{M},\kappa} : H^1(\mathcal{M};\mathbb{C}) \to \mathbb{R}_+$,

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_{\mathcal{M}} \left(|\nabla_{\mathcal{M}} - ih\mathbf{A}\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 \right) d\mathcal{H}^2_{\mathcal{M}}(x). \tag{1}$$

The functional $\mathcal{G}_{\mathcal{M},\kappa}$ arises as the Γ-limit (see [6]) of the full three-dimensional Ginzburg–Landau energy

$$\begin{aligned} G_{\varepsilon,\kappa}(\psi, A) = \frac{1}{\varepsilon} \Bigg[&\int_{\Omega_\varepsilon} \left(|(\nabla - iA)\psi|^2 + \frac{\kappa^2}{2}(|\psi|^2 - 1)^2 \right) dx \\ &+ \int_{\mathbb{R}^3} |\nabla \times A - \mathbf{H}_{\text{ext}}|^2 \, dx \Bigg], \end{aligned} \tag{2}$$

where for all $\varepsilon > 0$ sufficiently small, Ω_ε corresponds to a uniform tubular neighborhood of $\mathcal{M}$. In (2) $\mathbf{H}_{\text{ext}}$ is the external magnetic field. As $\varepsilon \to 0$, the field completely penetrates the sample which then implies that in the Γ-limit A is prescribed to be equal to $\mathbf{A}$, the tangential component of a divergence free vector field $\mathbf{A}^e$ such that $\nabla \times h\mathbf{A}^e = \mathbf{H}_{\text{ext}}$.

A central question in Ginzburg–Landau theory is the determination of the so-called *critical fields*. The first critical field corresponds to the appearance of zeros of ψ carrying nontrivial degree — called vortices in this context — in minimizers of the energy.

The analysis in [6] includes the computation of the first critical field of a thin shell of a surface of revolution subject to a constant vertical field which turns out to be surprisingly simple and depending only on an intrinsic quantity, in the $\kappa \to \infty$ limit:

$$H_{c_1} \sim \left(\frac{4\pi}{\text{Area of } \mathcal{M}} \right) \ln \kappa.$$

This result is extended in [5], to general surfaces and magnetic fields. For a fixed field $\mathbf{H}^e$, an external magnetic field of the form $\mathbf{H}_{\text{ext}} = h(\kappa)\mathbf{H}^e = h(\kappa)\nabla \times \mathbf{A}^e$ is considered. Then the first critical field is

$$H_{c_1} \sim \frac{1}{\max_{\mathcal{M}} *F - \min_{\mathcal{M}} *F} \ln \kappa,$$

where $d^*F = *d*F = \mathbf{A}$ and $*$ denotes the Hodge star-operator. In fact, the study shows also that, somewhat remarkably, not all fields $\mathbf{H}^e$ give rise to a first critical field. This phenomenon is related to the geometry and relative location of $\mathcal{M}$ with respect to $\mathbf{H}^e$. For $\mathbf{H}^e$ that yield a finite H_{c_1}, the topological obstruction imposed by $\mathcal{M}$ implying that the total degree of $\frac{\psi}{|\psi|}$ is zero is used in [5] to show that there is an even number of vortices in minimizers of $\mathcal{G}_{\mathcal{M},\kappa}$, half with positive degree, half with negative degree concentrating respectively on the set where $*F$ achieves its minimum and maximum. The optimal number $2n$ and location of vortices and anti-vortices in $\mathcal{M}$ is established in [5] for values of $h(k)$ slightly above H_{c_1} and in addition it is shown that if the minimum and maximum of $*F$ are attained at finitely many points then the two sets of vortices minimize, independently, a renormalized energy.

The results in [5, 6] cover only a moderate regime; in these works the intensity of the applied field is $H_{c_1} + \mathcal{O}(\ln \ln \kappa)$ and thus the number of vortices remains bounded as κ goes to infinity.

Once the value of h becomes much larger than H_{c_1}, that is there is a constant $C > 0$ such that $h - H_{c_1} \geq C \ln \kappa$, then the number of vortices in minimizers diverges as $\kappa \to \infty$. For even larger h, superconductivity persists only in a narrow region in the sample.

In the case of an infinite cylinder whose cross section is a domain $\Omega \subseteq \mathbb{R}^2$ and for constant applied fields parallel to the axis of the cylinder a reduction to a two-dimensional problem is possible. In this case it is known that as the intensity increases superconductivity is lost in the bulk and only a thin superconductivity region near $\partial\Omega$ persists (see [16, Chap. 7]). For much higher values still, superconductivity is completely lost: this value is known as H_{c_3} and is estimated by a delicate spectral analysis of the magnetic Laplacian operator as in the monograph [9].

In our setting, corresponding to the above functional $\mathcal{G}_{\mathcal{M},\kappa}$ (1) on the compact surface $\mathcal{M}$, there is no boundary, so what happens to the superconductivity region is not obvious. Another crucial difference lies in the behavior of the (normalized) magnetic field H induced on $\mathcal{M}$, which is the normal component of $\mathbf{H}^e$, or equivalently $H \, d\mathcal{H}^2_{\mathcal{M}} = d\mathbf{A}$ (viewing $\mathbf{A}$ as a 1-form). Namely, in our case, H vanishes and changes sign. The spectral analysis in [11] therefore suggests that superconductivity should persist near the set $\{H = 0\}$, where the external magnetic field is tangent to the surface $\mathcal{M}$. In [12] the authors study the case of a vanishing magnetic field in the infinite cylinder model, and observe indeed nucleation of superconductivity near the zero locus of the magnetic field, for very high values of the applied field (near the putative H_{c_3}) under the condition that the gradient of the magnetic field does not vanish on its zero locus. The problem of the determination of the upper critical field for vanishing fields remains largely open otherwise. Here, we are concerned with much lower values of the applied field: a main motivation of this work is to understand the transition from the vortexless to normal state regimes.

Another interesting difference is the fact that in the infinite cylinder model only positive vortices exist and so the location and growth of the vortex region is always ruled by the competing effects of mutual repulsion, and confinement provided by the external field. In the present setting, this is no longer the case. Vortices of positive and negative degree must coexist and so repulsion and attraction are common features of the relative placement of vortices in $\mathcal{M}$, this without taking into account the external field.

In this way, the shrinking of the superconductivity region is a multifaceted phenomenon. Moreover, the problems mentioned in the characterization of this region are present even in the most emblematic case of a constant external field $\mathbf{H}^e$: the region of persistence of superconductivity does not only depend on the field and on the topology of $\mathcal{M}$, but also on extrinsic geometric properties of the surface; the relative position of $\mathcal{M}$ with respect to $\mathbf{H}^e$ affects H and therefore the zero locus of the induced field.

In the present work we address the question of identifying the region where superconductivity persists in the $\kappa \to \infty$ limit, when

$$\frac{H_{c_1}}{h}$$

is small; we show that as this quantity gets small superconductivity persists in a small neighborhood of the place where the applied field is tangential to the sample, provided the field satisfies a generic nondegeneracy condition (see (14) below). Another thrust of this work is aimed at uncovering some new intermediate regimes only present in this setting, when the normal component of the external field changes sign multiples times. In the model problem of a surface of revolution and constant vertical field, we identify several structural transitions undergone by the superconductivity region. Furthermore, we observe a new phenomenon which we refer to as *freezing of the boundary*, where a component of the vortex region stops growing even after increasing the intensity of the external field. This phenomenon holds in great generality (not only in the surface of revolution case), as is shown at the end of Sec. 4.

To carry out our analysis we start by using a reduction to a mean field model, first derived rigorously in [15]. More precisely, if we write a critical point ψ of $\mathcal{G}_{\mathcal{M},\kappa}$ in polar form $\psi = \rho e^{i\phi}$, variations of the phase yield $d(\rho^2(d\phi - hd^*F)) = 0$, and because $H^1_{dr}(\mathcal{M}) = 0$ this implies there is a V such that $*dV = \rho^2(d\phi - hd^*F)$. Taking $V = hW$, the function W is expected to minimize

$$\int_{\mathcal{M}} |\nabla_{\mathcal{M}} W|^2 d\mathcal{H}^2_{\mathcal{M}} + \frac{\ln\kappa}{h}\int_{\mathcal{M}} |-\Delta_{\mathcal{M}} W + \Delta_{\mathcal{M}} * F| d\mathcal{H}^2_{\mathcal{M}}. \tag{3}$$

The details of this mean field reduction can be found in [15] in the case of a positive external field applied in a bounded planar domain. However, the analysis in [15] does not handle the additional restriction of total zero mass which affects the construction of an upper bound in this setting. The steps needed to extend the proof to the present case are included in the appendix.

The measure $-\Delta_{\mathcal{M}} V + \Delta_{\mathcal{M}} * F$ can be interpreted as the normalized measure generated by the vortices. On the other hand, we observe that

$$\Delta_{\mathcal{M}} * F d\mathcal{H}^2_{\mathcal{M}} = d * d * F = d\mathbf{A} = H d\mathcal{H}^2_{\mathcal{M}},$$

where the function H is the normal component of the external magnetic field $\mathbf{H}^e$ relative to $\mathcal{M}$. In what follows we refer to H simply as *the* magnetic field, and we assume that $H \in C^1(\mathcal{M})$. Moreover, we drop the explicit dependence on $\mathcal{M}$ in expressions like $\Delta_{\mathcal{M}}$, $\nabla_{\mathcal{M}}$.

Before we state our main result we make the following assumption: there exists $\beta > 0$ such that

$$\lim_{\kappa\to\infty} \frac{\ln\kappa}{h} = \beta. \tag{4}$$

Once the connection to the mean field problem (3) is established we proceed to locate very precisely the region of persistence of superconductivity, that is, the region SC_β where the vorticity measure $-\Delta V + H$ vanishes. We find that this region corresponds to a $\beta^{\frac{1}{3}}$ neighborhood of the set where H vanishes, in the $\beta \to 0$ limit. More precisely, we have the following theorem.

Theorem 1.1. *Under the nondegeneracy assumption that ∇H is nowhere vanishing on $\{H = 0\}$, there exists $C > 0$ independent of β such that the superconductivity region SC_β is contained in $\{x \in \mathcal{M} : d(x, \{H = 0\}) < C\beta^{\frac{1}{3}}\}$, and contains $\{x \in \mathcal{M} : d(x, \{H = 0\}) < C^{-1}\beta^{\frac{1}{3}}\}$, for β sufficiently small.*

The nondegeneracy assumption on H implies that the set $\{H = 0\}$ is a finite union of smooth closed curves. It is the same assumption as the one made in [11, 12] for the study of the third critical field H_{c_3}.

To prove Theorem 1.1 we reformulate the mean field approximation as an obstacle problem, and construct comparison functions. We note that a construction in the same spirit was carried out in [17, Appendix A] for the planar Ginzburg–Landau model in a different context. In our case however the construction is not immediate, because our obstacle problem is two-sided and our magnetic field H changes sign. Indeed, our proof makes use of a comparison principle for two-sided obstacle problems proved in [7] which allows to compare solutions to obstacle problems corresponding to different data H. Hence the comparison functions will not be merely "super- or sub-solutions" of our problem, but actual solutions of modified problems. In particular they have to be quite regular. As a consequence, we cannot use functions of the distance to $\{H = 0\}$ as comparison functions. We have to use a particular coordinate system near each component of $\{H = 0\}$ and explicitly build local functions satisfying local obstacle problems with appropriate modifications of H. Pasting these constructions we are able to appeal to [7] to obtain the desired estimates. In so doing we note a key feature of the proof, related to the fact that the obstacle problem is two-sided: the barriers thus obtained cannot be used independently to get neither the inner nor the outer bound separately, but together they yield the conclusion of the theorem. This is explained in more detail in Sec. 3.

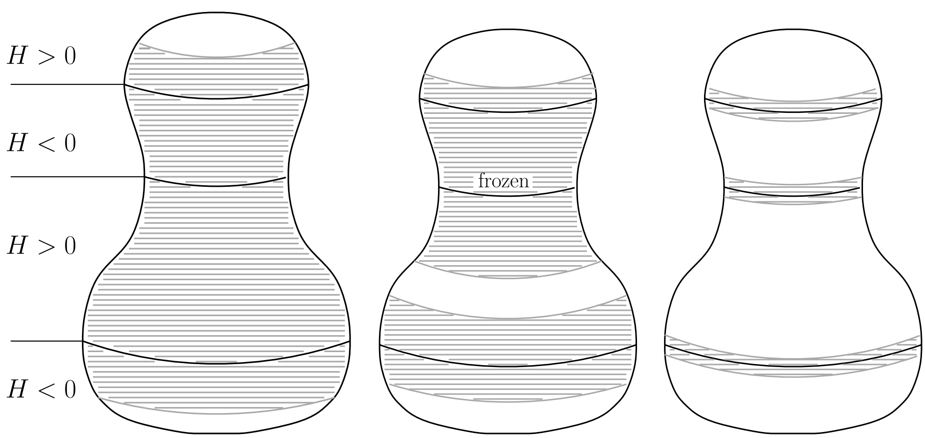
Thanks to Theorem 1.1, we have a clear picture of the superconductivity region for $\beta \to 0$: it is a union of tubular neighborhoods of the connected components of $\{H = 0\}$. In particular, the superconductivity region has at least as many connected components as $\{H = 0\}$. On the other hand, we also have a clear picture of the superconductivity region as $\beta \to \beta_c$, where positive (respectively, negative) vortices are concentrated near the points where $*F$ achieves its maximum (respectively, minimum). In particular, the superconductivity region has, generically, one connected component. In the last part of this work, we investigate the intermediate regimes. If $\{H = 0\}$ has more than one connected component, transitions *have* to occur: when β crosses some critical value, the number of connected components of SC_β changes.

Studying such transitions, and determining the values of β at which they occur, seems out of our reach in all generality. That is why we concentrate first on a simple model problem. We consider a surface of revolution around the vertical axis $\mathbf{e_z}$, and assume that the external magnetic field $\mathbf{H}^e = \mathbf{e_z}$ is vertical and constant. (In fact in Sec. 4.1, more general magnetic fields are considered.) In that case, the induced field H on $\mathcal{M}$ is just $H = \mathbf{e_z} \cdot \nu$, where ν is an outward normal vector on $\mathcal{M}$. The set $\{H = 0\}$ consists exactly of the points where $\mathbf{e_z}$ is tangent to $\mathcal{M}$, and it is a union of circles. Note that H has to change sign an odd number of times, since $H = -1$ at the "south pole" and $+1$ at the "north pole", thus there are an odd number of those circles. As explained above, interesting transitions happen when $\{H = 0\}$ has more than one connected component. Therefore we focus on the simplest nontrivial situation, which corresponds to $\{H = 0\}$ consisting of three circles. We state loosely here the result that we obtain for that simple model problem in Sec. 4.1 (see Fig. 1).

Proposition 1.2. *Let $\mathcal{M}$ be a surface of revolution of the form* (39) *with constraints specified in Sec.* 4.1 *below. Assume the induced magnetic potential is rotationally symmetric. Then there exist $\beta_c > \beta_1^* \geq \beta_2^* > 0$ such that*

- *for $\beta \in (\beta_1^*, \beta_c)$, SC_β has one connected component,*
- *for $\beta \in (\beta_2^*, \beta_1^*)$, SC_β has two connected components,*
- *for $\beta \in (0, \beta_2^*)$, SC_β has three connected components.*

Moreover, for $\beta \in (\beta_2^*, \beta_1^*)$, one connected component of SC_β remains constant.



(a) $\beta \in (\beta_1^*, \beta_c)$ (b) $\beta \in (\beta_2^*, \beta_1^*)$ (c) $\beta \in (0, \beta_2^*)$

Fig. 1. The region SC_β in the three regimes of Proposition 1.2.

The most striking part of Proposition 1.2 is the appearance of an intermediate regime in which one connected component of SC_β remains constant: one part of the free boundary is *frozen*. In [1] a similar occurrence is observed in an explicit solution to a two-sided obstacle problem arising in the study of almost planar thin films in the presence of strong parallel fields. In Sec. 4.2 we identify the features responsible for such "freezing of the boundary" phenomenon depicted in Proposition 1.2 and prove a similar "freezing property" in a general (nonsymmetric) setting (see Proposition 4.3). We note that since our proof relies on a general comparison principle, it is likely that it could be adapted to include the setting in [1].

An other interesting outcome of the precise version of Proposition 1.2 (Proposition 4.2 in Sec. 4.1) are the expressions of the critical values β_1^* and β_2^*, in terms of integral quantities involving $\mathbf{A}$ and the parametrization of $\mathcal{M}$. Transferring these conditions to a general nonsymmetric setting seems far from obvious and constitutes an interesting challenge.

The plan of the paper is as follows. In the next section we collect some basic properties of solutions to an obstacle problem that serves as the starting point in our analysis. In Sec. 3 we identify the thin region of superconductivity when β is small. In Sec. 4 we turn to the symmetric situation and identify in Proposition 4.2 the further transitions as β decreases to zero from $\beta_c = \max(*F) - \min(*F)$. We also prove the "freezing of the boundary" property at the end of Sec. 4.

2. The Obstacle Problem

This preamble is devoted to the derivation of the obstacle problem dual to the mean field approximation. We also prove some basic results we will need later on. We think it is worthwhile recording these properties because in our setting, there is a degeneracy that is not present in other classical results in the literature.

In the first part of this section we show that — as in [16, Chap. 7] — the minimizer of

$$E_\beta(V) = \int_{\mathcal{M}} |\nabla V|^2 + \beta \int_{\mathcal{M}} |-\Delta V + H| \tag{5}$$

is the solution of an obstacle problem, and then we study general properties of the contact set. There are two main differences with the obstacle problem arising in [16, Chap. 7].

- In our case there are no boundary conditions and the minimizer is well defined only up to a constant. We need to deal with this degeneracy.
- While in [16, Chap. 7] the obstacle problem is one-sided, we have to consider a two-sided obstacle problem. This is due to the fact that, in our case, the magnetic field H changes sign.

The functional E_β is, under assumption (4), the limit of the sequence of energies considered in (3). The link between E_β and the superconductivity region is, as mentioned in the introduction, proved in the appendix.

2.1. *Derivation of the obstacle problem*

Proposition 2.1. *Let $\beta > 0$. A function $V_0 \in H^1(\mathcal{M})$ minimizes E_β (5) if and only if V_0 minimizes*

$$\mathcal{F}(V) = \int_{\mathcal{M}} (|\nabla V|^2 + 2HV) \tag{6}$$

among all $V \in H^1(\mathcal{M})$ such that $(\operatorname{ess\,sup} V - \operatorname{ess\,inf} V) \leq \beta$.

Remark 2.1. Since the functional $\mathcal{F}(V)$ is translation invariant, V_0 coincides, up to a constant, with any minimizer of the two-sided obstacle problem

$$\min\left\{\int_{\mathcal{M}} (|\nabla V|^2 + 2HV) : V \in H^1(\mathcal{M}), |V| \leq \beta/2\right\}.$$

Moreover, recalling that $H = \Delta * F$, this obstacle problem can also be rephrased as

$$\min\left\{\int_{\mathcal{M}} |\nabla(V - *F)|^2 : V \in H^1(\mathcal{M}), |V| \leq \beta/2\right\}. \tag{7}$$

The fact that minimizers coincide only up to a constant does not matter, since the physically relevant object is the vorticity measure $-\Delta V + H$. Moreover, it is easy to check that, if the obstacle problem (7) admits a solution V that "touches" the obstacles, i.e. satisfies $\max V - \min V = \beta$, then this solution is unique because any other solution differs from it by a constant, which has to be zero. On the other hand, a solution satisfying $\max V - \min V < \beta$ would have to be $V = *F + \alpha$ for some constant α. Therefore, for $\beta \leq \max *F - \min *F$ the solution is unique.

The proof of Proposition 2.1 relies on the following classical result of convex analysis (easily deduced from [13] or [4, Theorem 1.12]).

Lemma 2.2. *Let $\mathcal{H}$ be a Hilbert space and $\varphi : \mathcal{H} \to \mathbb{R} \cup \{+\infty\}$ be a convex lower semi-continuous function. Then the minimizers of the problems*

$$\min_{x\in\mathcal{H}} \left(\frac{1}{2}\|x\|_{\mathcal{H}}^2 + \varphi(x)\right) \quad \textit{and} \quad \min_{y\in\mathcal{H}} \left(\frac{1}{2}\|y\|_{\mathcal{H}}^2 + \varphi^*(-y)\right)$$

coincide, where φ^* denotes the Fenchel conjugate of φ,

$$\varphi^*(y) := \sup_{z\in\mathcal{H}} \langle y, z\rangle_{\mathcal{H}} - \varphi(z).$$

Proof of Proposition 2.1. We apply Lemma 2.2 in the Hilbert space

$$\mathcal{H} := \dot{H}^1(\mathcal{M}) = \left\{ V \in H^1(\mathcal{M}) : \int_{\mathcal{M}} V = 0 \right\},$$

endowed with the norm $\|V\|^2 = \int |\nabla V|^2$, to the function

$$\varphi(V) = \varphi_\beta(V) = \frac{\beta}{2} \int_{\mathcal{M}} |-\Delta V + H|. \tag{8}$$

In formula (8), it is implicit that $\varphi(V) = +\infty$ if $\mu = -\Delta V + H$ is not a Radon measure. Note that, when μ *is* a Radon measure, it must have zero average $\int \mu = 0$, since $\mu = \Delta(*F - V)$.

We compute the Fenchel conjugate of φ. It holds

$$\begin{aligned}
\varphi^*(V) &= \sup_{U \in \mathcal{H}} \left\{ \int_{\mathcal{M}} \nabla V \cdot \nabla U - \frac{\beta}{2} \int_{\mathcal{M}} |-\Delta U + H| \right\} \\
&= -\int_{\mathcal{M}} HV + \sup_{U \in \mathcal{H}} \left\{ \int_{\mathcal{M}} (-\Delta U + H)V - \frac{\beta}{2} \int_{\mathcal{M}} |-\Delta U + H| \right\} \\
&= -\int_{\mathcal{M}} HV + \sup_{\int P = 0} \left\{ \int_{\mathcal{M}} \left(PV - \frac{\beta}{2} |P| \right) \right\}.
\end{aligned}$$

In the last equality, the supremum may — by a density argument — be taken over all L^2 functions P with zero average.

If $(\operatorname{ess\,sup} V - \operatorname{ess\,inf} V) \leq \beta$, then $|V + \alpha| \leq \beta/2$ for some $\alpha \in \mathbb{R}$, so that

$$\int_{\mathcal{M}} \left(PV - \frac{\beta}{2} |P| \right) = \int_{\mathcal{M}} \left((V + \alpha)P - \frac{\beta}{2} |P| \right) \leq 0,$$

and in that case

$$\varphi^*(V) = -\int_{\mathcal{M}} HV.$$

On the other hand, if $(\operatorname{ess\,sup} V - \operatorname{ess\,inf} V) > \beta$, then up to translating V we may assume that $\{V > \beta/2\}$ and $\{V < -\beta/2\}$ have positive measures. It is then easy to construct a function P supported in those sets, such that $\int P = 0$, $\int |P| = 1$, and $\int PV > \beta/2$. Using λP as a test function for arbitrary $\lambda > 0$, we deduce that $\varphi^*(V) = +\infty$.

From Lemma 2.2 it follows that $V_0 \in \dot{H}^1(\mathcal{M})$ minimizes E_β if and only if V_0 minimizes

$$\frac{1}{2} \int_{\mathcal{M}} |\nabla V|^2 + \int_{\mathcal{M}} HV$$

among $V \in \dot{H}^1(\mathcal{M})$ such that $\operatorname{ess\,sup} V - \operatorname{ess\,inf} V \leq \beta$. Since both problems are invariant under addition of a constant, the restriction to the space $\dot{H}^1(\mathcal{M})$ can be relaxed to obtain Proposition 2.1. □

2.2. *Basic properties*

In this section we concentrate on the obstacle problem

$$\min\left\{\int_{\mathcal{M}} (|\nabla V|^2 + 2HV) : V \in H^1(\mathcal{M}), |V| \le \beta/2\right\}. \tag{9}$$

We recall the classical interpretation of (9) as a free boundary problem, and establish a monotonicity property of the free boundary.

The first step to these basic properties is the reformulation of the obstacle problem (9) as a variational inequality: a function $V \in H^1(\mathcal{M})$ solves (9) if and only if $|V| \le \beta/2$ and

$$\int_{\mathcal{M}} \nabla V \cdot \nabla(W - V) \ge -\int_{\mathcal{M}} H(W - V), \quad \forall\, W \in H^1(\mathcal{M}),\ |W| \le \beta/2. \tag{10}$$

The proof of this weak formulation is elementary and can be found in many textbooks on convex analysis. See for instance [14].

Next we recall the standard reformulation of (10) as a free boundary problem.

Lemma 2.3. *A function $V \in H^1(\mathcal{M})$ with $|V| \le \beta/2$ solves* (9) *or equivalently* (10) *if and only if*

$$\begin{cases} V \in W^{2,p}(\mathcal{M}), & 1 < p < \infty, \\ \Delta V = H & \text{in } \{|V| < \beta/2\}, \\ 0 \ge H & \text{in } \{V = \beta/2\}, \\ 0 \le H & \text{in } \{V = -\beta/2\}. \end{cases} \tag{11}$$

In particular $V \in C^{1,\alpha}(\mathcal{M})$, so that at every regular point of the free boundaries $\partial\{V = \pm\beta/2\}$, the function V satisfies the overdetermining boundary conditions $V = \pm\beta/2$ and $\partial V/\partial\nu = 0$.

The only nonelementary part of Lemma 2.3 is the $W^{2,p}$ regularity of the solution. For the one-sided obstacle problem, it is proven for instance in [10, Theorem 3.2]. The proof adapts easily to our two-sided obstacle problem: see, e.g., [10, Problem 2, p. 29].

Recall that in our case, $\mu = -\Delta V + H$ represents the vorticity measure. In light of Lemma 2.3, this measure is supported in $\{V = \pm\beta/2\}$. In that region, vortices are distributed with density H.

For $\beta > \beta_c$, where

$$\beta_c := \max(*F) - \min(*F), \tag{12}$$

the function $*F + \alpha$ solves the obstacle problem (9), as long as the constant α satisfies $\max(*F) - \beta/2 \le \alpha \le \min(*F) + \beta/2$, and the vorticity measure $-\Delta V + H$ is identically zero.

For $\beta \le \beta_c$, the solution $V = V_\beta$ of the obstacle problem (9) must satisfy

$$\max V_\beta - \min V_\beta = \beta,$$

and therefore is unique (see Remark 2.1). Recall that the superconductivity region SC_β is defined as the set where the vorticity measure $-\Delta V + H$ vanishes. According to Lemma 2.3, that region is exactly

$$\mathrm{SC}_\beta = \{|V_\beta| < \beta/2\}. \tag{13}$$

A first basic property of the superconductivity region SC_β is its monotonicity.

Proposition 2.4. *For any $0 < \beta_1 < \beta_2 \le \beta_c$, it holds*

$$\mathrm{SC}_{\beta_1} \subset \mathrm{SC}_{\beta_2}.$$

In other words, increasing the intensity of the applied magnetic field shrinks the region of persisting superconductivity, which consistent with physical intuition. Since we have to deal with a two-sided obstacle problem, this monotonicity property is not as obvious as in [16, Chap. 7]. To prove it, we use a comparison principle for two-sided obstacle problems [7, Lemma 2.1]. We state and prove here a particular form that will also be useful later on.

Lemma 2.5. *Let $H_1 \ge H_2$ be bounded, real-valued functions on $\mathcal{M}$. Let also $\alpha_1 \le \alpha_2$ and $\beta_1 \le \beta_2$ be real numbers. For $j = 1, 2$, let $V_j \in H^1(\mathcal{M})$ solve respectively the obstacle problems*

$$\min\left\{\int_{\mathcal{M}} (|\nabla V|^2 + 2H_j V) : \alpha_j \le V \le \beta_j\right\}.$$

Then either $V_1 - V_2$ is constant, or $V_1 \le V_2$.

Proof. For the convenience of the reader, we provide here the elementary proof, which consists in remarking that

$$W_1 = \min(V_1, V_2) \quad \text{and} \quad W_2 = \max(V_1, V_2)$$

are admissible test functions in the variational inequalities

$$\int_{\mathcal{M}} \nabla V_j \cdot \nabla(W_j - V_j) \ge -\int_{\mathcal{M}} H_j(W_j - V_j), \quad \forall\, W_j \in H^1,\ \alpha_j \le W_j \le \beta_j.$$

Subtracting the resulting inequalities, we obtain

$$\int_{\mathcal{M}} |\nabla(V_1 - V_2)_+|^2 \le \int_{\mathcal{M}} (H_2 - H_1)(V_1 - V_2)_+ \le 0,$$

where $(V_1 - V_2)_+ = \max(V_1 - V_2, 0)$. We conclude that $(V_1 - V_2)_+$ is a constant function. $\square$

With Lemma 2.5 at hand, we may prove the monotonicity of the superconductivity region.

Proof of Proposition 2.4. Let V_1 and V_2 denote the solution of the obstacle problem (9) corresponding respectively to $\beta = \beta_1$ and $\beta = \beta_2$. Let

$$\widetilde{V}_1 = V_1 + \beta_1/2, \quad \text{and} \quad \widetilde{V}_2 = V_2 + \beta_2/2,$$

so that for $j = 1, 2$, $\widetilde{V}_j$ solves the obstacle problem

$$\min\left\{ \int_{\mathcal{M}} (|\nabla V|^2 + 2HV) : 0 \leq V \leq \beta_j \right\}.$$

Therefore, applying Lemma 2.5 with $H_1 = H_2 = H$, $\alpha_1 = \alpha_2 = 0$ and $\beta_1 \leq \beta_2$, we deduce that

$$V_1 + \beta_1/2 \leq V_2 + \beta_2/2.$$

(If $\widetilde{V}_1 - \widetilde{V}_2$ is constant, then $\beta_2 = \max V_1 - \min V_1 = \beta_1$.) In particular, we obtain that

$$\{V_1 > -\beta_1/2\} \subset \{V_2 > -\beta_2/2\}.$$

By a similar argument, we show that

$$\{V_1 < \beta_1/2\} \subset \{V_2 < \beta_2/2\},$$

and conclude that $\mathrm{SC}_{\beta_1} \subset \mathrm{SC}_{\beta_2}$. □

Remark 2.2. It follows from the above proof that

$$|V_1 - V_2| \leq (\beta_2 - \beta_1)/2,$$

thus proving the continuity of $\beta \mapsto V_\beta$ for $0 \leq \beta \leq \beta_c$.

3. The Small β Limit

In this section we study what happens to the superconductivity set when the intensity of the field is high enough to confine it in a narrow region. We make the (generic) nondegeneracy assumption that

$$|H| + |\nabla H| > 0 \quad \text{in } \mathcal{M}. \tag{14}$$

In other words, $\nabla H \neq 0$ in $\{H = 0\}$. This implies in particular that the set $\Sigma := \{H = 0\}$ where the magnetic field vanishes is a finite disjoint union of smooth closed curves. We also note that condition (14) also implies that we are not in the situation where not even the first critical field is defined (see [5, Theorem 3.1]).

Let us say a few words here about the nondegeneracy assumption (14). This is the same nondegeneracy assumption that has been considered in works on the spectral analysis of the magnetic Laplacian [11] and on higher applied magnetic fields in Ginzburg–Landau [12, 2]. Moreover, we emphasize that (14) is a generic assumption, in the following sense.

Lemma 3.1. *Let $X = \{H \in C^1(\mathcal{M}) : \int_{\mathcal{M}} H = 0\}$. The functions satisfying* (14) *form and open and dense subset of X.*

Proof. The fact that (14) is an open condition is clear. For the density, it suffices to show that any $H \in X \cap C^\infty(\mathcal{M})$ can be approached by functions satisfying (14). This follows from a simple transversality argument: recall (see, e.g., [8, 3.7]) that a smooth function Φ is transverse to $\{0\}$ if and only if Φ is a submersion on $\{\Phi = 0\}$. In particular (14) is equivalent to H being transverse to $\{0\}$. Fix $H_1, H_2 \in X \cap C^\infty(\mathcal{M})$ such that $\{H_1 = H_2 = 0\}$ is void. Then the smooth function

$$\Phi : \mathbb{R}^2 \times \mathcal{M} \to \mathbb{R}, \quad (\lambda, x) \mapsto H(x) + \lambda_1 H_1(x) + \lambda_2 H_2(x),$$

is transverse to $\{0\}$, and therefore $\Phi(\lambda, \cdot) = H + \lambda_1 H_1 + \lambda_2 H_2$ is transverse to $\{0\}$ for λ arbitrarily small [8, Theorem 3.7.4]. $\square$

We are interested in the behavior, as $\beta \to 0$, of the superconductivity region SC_β (13).

We let $d : \mathcal{M} \to \mathbb{R}_+$ denote the distance function to the set $\Sigma = \{H = 0\}$, that is

$$d(x) = \mathrm{dist}(x, \{H = 0\}). \tag{15}$$

In this context we characterize the behavior of SC_β in terms of the function d, as follows (this is a more explicit version of Theorem 1.1).

Theorem 3.2. *Under the nondegeneracy assumption* (14) *on the magnetic field, there exists* $\beta_0 > 0$ *and* $C > 0$ *such that, for* $\beta \in (0, \beta_0)$,

$$\left\{ d \leq \frac{1}{C} \beta^{1/3} \right\} \subset \mathrm{SC}_\beta \subset \{ d \leq C \beta^{1/3} \}, \tag{16}$$

where SC_β *is the superconductivity region* (13), *and* d *denotes the distance to the zero locus of the magnetic field* (15).

In the proof we construct explicit solutions to modified obstacle problems, in order to apply the comparison principle Lemma 2.5. The comparison functions are constructed locally near each component Γ of $\{H = 0\}$, and then we need to extend and paste these functions and the associated modified obstacle problem data. Although the construction looks local, it is worth noting that we really need to make it near *every* component Γ of $\{H = 0\}$. Otherwise the pasting would not provide us with obstacle problems comparable to the original one, because a solution has to change sign near *every* curve Γ.

Remark 3.1. Another natural approach to proving Theorem 3.2 would be to construct separate comparison functions in $\{H > 0\}$ and $\{H < 0\}$. In those regions, the obstacle problem becomes one-sided, so that more standard constructions with a classical comparison principle can be made. On the other hand, there is no boundary condition in those regions, so that such a construction would only provide us with the outer bound

$$\mathrm{SC}_\beta \subset \{d \leq C\beta^{1/3}\}. \tag{17}$$

To obtain the bounds (16) which show that the superconductivity set extends to *both* sides of the zero locus of H by a $\beta^{\frac{1}{3}}$ margin, it seems that we really have to appeal to the comparison principle for two-sided obstacle problems. However, if we would just content ourselves with showing that the superconductivity set had "thickness" proportional to $\beta^{\frac{1}{3}}$, namely

$$\mathrm{dist}(\{V = \beta/2\}, \{V = -\beta/2\}) \geq c\beta^{1/3}, \tag{18}$$

there would be a simpler and elegant way. In fact (18) can be directly inferred from (17). This is a simple consequence of the interpolated elliptic estimate (see [3, Appendix A])

$$\|\nabla V\|_\infty^2 \leq C \|\Delta V\|_\infty \|V\|_\infty, \tag{19}$$

which implies, since $|V| \leq \beta$ and $|\Delta V| = |H\mathbb{1}_{\mathrm{SC}_\beta}| \leq C\beta^{1/3}$, that

$$|\nabla V| \leq C\beta^{2/3} \quad \text{in } \mathcal{M}. \tag{20}$$

Hence, for any $x_\pm \in \{V = \pm\beta/2\}$ and any arc-length parametrized curve $\gamma(s)$, $(0 \leq s \leq \ell)$ going from x_- to x_+, it holds

$$\beta = V(x_+) - V(x_-) = \int_0^\ell \nabla V(\gamma(s)) \cdot \gamma'(s) ds \leq C\beta^{2/3}\ell,$$

so that the length of γ satisfies $\ell \geq c\beta^{1/3}$, which proves (18). Let us emphasize again that (17)–(18) really is weaker than (16), since (18) does not prevent vortices from coming arbitrarily close to one side of $\{H = 0\}$.

Next we turn to the proof of Theorem 3.2.

Proof of Theorem 3.2. We will construct, for small enough β, bounded functions $H_1 \leq H \leq H_2$, and comparison functions V_1 and V_2 of regularity $W^{2,\infty}$, satisfying for $j = 1, 2$,

$$\begin{gathered} \Delta V_j = H_j \mathbb{1}_{|V_j| < \beta/2}, \\ |V_j| \leq \beta/2, \quad H_j \geq 0 \quad \text{in } \{V_j = -\beta/2\}, \quad H_j \leq 0 \quad \text{in } \{V_j = \beta/2\}, \end{gathered} \tag{21}$$

and the bounds

$$\left\{d \leq \frac{1}{C}\beta^{1/3}\right\} \subset \{|V_j| < \beta/2\} \subset \{d \leq C\beta^{1/3}\}. \tag{22}$$

By Lemma 2.3, (21) implies that V_j solves the obstacle problem (9) with $H = H_j$. Therefore we may apply the comparison principle for two-sided obstacle problems (Lemma 2.5) to conclude that $V_1 \geq V \geq V_2$. In view of the bounds (22) satisfied by V_1 and V_2, this obviously implies that the superconductivity region satisfies the bounds (16).

The rest of the proof is devoted to constructing V_1 and V_2. To this end we introduce good local coordinates in a neighborhood of $\Sigma = \{H = 0\}$. Recall that, thanks to the nondegeneracy assumption (14), Σ is a finite union of closed smooth curves. Let us fix one of them, Γ, together with an arc-length parametrization of it:

$$\Gamma = \{\gamma(x) \colon x \in \mathbb{R}/\ell\mathbb{Z}\}, \quad |\gamma'(x)| = 1.$$

Let us also fix a smooth normal vector $\nu(x)$ to Γ on $\mathcal{M}$, that is

$$\nu(x) \in T_{\gamma(x)}\mathcal{M}, \quad |\nu| = 1, \quad \nu \cdot \gamma' = 0,$$

and impose that $\nu(x)$ points in the direction of $\{H > 0\}$ (since $H < 0$ on one side of Γ and $H < 0$ on the other side). We introduce Fermi coordinates along Γ: for small enough δ, the map

$$\mathbb{R}/\ell\mathbb{Z} \times (-\delta, \delta) \to \mathcal{M}, \quad (x, y) \mapsto \exp_{\gamma(x)}(y\nu(x)),$$

is a diffeomorphism. It defines local coordinates (x, y) on $\mathcal{M}$ in a neighborhood of Γ, in which the Laplace operator has the form

$$\Delta = \frac{1}{f}(\partial_y f \partial_y + \partial_x f^{-1} \partial_x), \tag{23}$$

where $f(x, y) = 1 - y\kappa(x, y)$ for some smooth function κ. Note that y is nothing else than the signed distance to Γ, and in particular $|y| = d$ in a neighborhood of Γ. At first glance, this may seem like a good coordinate system. However, we need a parametrization that is better adapted to the Laplacian (i.e. it allows for a separation of variables). To that end let (x, z) be the local coordinates where

$$z = y + \frac{1}{2}y^2\kappa(x, y). \tag{24}$$

Clearly the map $(x, y) \mapsto (x, z)$ is a diffeomorphism for small enough y, so that (x, z) define indeed local coordinates on $\mathcal{M}$. The reason for using the coordinates (x, z) is that the Laplace operator is then approximately

$$\Delta \approx \partial_x^2 + \partial_z^2,$$

which will allow us to obtain nice bounds for functions depending only on z.

Note that, since we choose the normal vector ν to point in the direction of $\{H > 0\}$, and since $|\nabla H| \geq c > 0$ in a neighborhood of Γ thanks to the nondegeneracy assumption (14), it holds

$$\partial_z H \geq c > 0, \quad |z| < \delta.$$

On the other hand, ∇H is bounded, so that there exist $C \geq c > 0$ such that

$$Cz\mathbb{1}_{z<0} + cz\mathbb{1}_{z>0} \leq H \leq cz\mathbb{1}_{z<0} + Cz\mathbb{1}_{z>0}, \quad |z| < \delta. \tag{25}$$

Next we concentrate on the construction of V_1 (H_1 will be defined accordingly). Away from the set Σ, we simply define

$$V_1 = -\mathrm{sign}(H)\beta/2 \quad \text{in } \{d > \delta/2\}. \tag{26}$$

The interesting part is of course what happens near Σ. Near each of the smooth curves $\Gamma \subset \Sigma$, we will look for V_1 in the form $V_1 = v(z)$, where v is a $W^{2,\infty}$ function satisfying

$$v(z) = \begin{cases} \beta/2 & \text{for } z < -\eta_-, \\ -\beta/2 & \text{for } z > \eta_+, \end{cases} \tag{27}$$

for some parameters $\eta_\pm > 0$ that will depend on β. A straightforward computation using (23) and (24) shows that

$$\Delta V_1 = v''(z) + z(g_1(x,z)v''(z) + g_2(x,z)v'(z)), \tag{28}$$

where g_1 and g_2 are bounded functions. We are going to define in $(-\eta_-, \eta_+)$ the function v so that

$$v'' \le 2Cz\mathbb{1}_{z<0} + \frac{c}{2}z\mathbb{1}_{z>0}, \quad |v'| = o(\beta),\ |v''| = o(\beta). \tag{29}$$

We then define H_1 in $(-\eta_-, \eta_+)$ simply as ΔV_1. Thus, recalling (25), we will have, for small enough $\beta > 0$,

$$\Delta V_1 = H_1 \mathbb{1}_{|V_1|<\beta/2} \quad \text{with } H_1 \le H \quad \text{in } \{-\eta_- < z < \eta_+\}. \tag{30}$$

It is then straightforward to extend H_1 to a function defined on $\mathcal{M}$, such that $H_1 \le H$, and having the same sign as H outside of $\{-\eta_- < z < \eta_+\}$. The resulting H_1 and V_1 satisfy (21).

Thus it remains to show that we can indeed define $v(z)$ in $\{-\eta_- < z < \eta_+\}$, satisfying the bounds (29). We look for v in the form

$$v(z) = \begin{cases} v_-(z) & \text{for } -\eta_- < z < 0, \\ v_+(z) & \text{for } 0 < z < \eta_+, \end{cases} \quad \text{with } v_\pm(z) \text{ polynomial.} \tag{31}$$

First of all, for v to be of class $W^{2,\infty}$ around the points $\pm\eta_\pm$, we should impose

$$v_-(-\eta_-) = \beta/2, \quad v_+(\eta_+) = -\beta/2, \quad v'_-(-\eta_-) = v'_+(\eta_+) = 0. \tag{32}$$

Thus we take $v_\pm$ to be of the form

$$\begin{aligned} v_-(z) &= (z+\eta_-)^2(A_- z + B_-) + \frac{\beta}{2} \\ &= A_- z^3 + (B_- + 2\eta_- A_-)z^2 + (2\eta_- B_- + \eta_-^2 A_-)z + \eta_-^2 B_- + \frac{\beta}{2}, \\ v_+(z) &= (z-\eta_+)^2(A_+ z + B_+) - \frac{\beta}{2} \\ &= A_+ z^3 + (B_+ - 2\eta_+ A_+)z^2 + (-2\eta_+ B_+ + \eta_+^2 A_+)z + \eta_+^2 B_+ - \frac{\beta}{2}. \end{aligned} \tag{33}$$

For v to be of class $W^{2,\infty}$ around $z = 0$, we have to impose

$$\eta_-^2 B_- + \frac{\beta}{2} = \eta_+^2 B_+ - \frac{\beta}{2}, \quad 2\eta_- B_- + \eta_-^2 A_- = -2\eta_+ B_+ + \eta_+^2 A_+. \tag{34}$$

We also need to ensure that

$$v'' \leq 2Cz\mathbb{1}_{z<0} + \frac{c}{2}z\mathbb{1}_{z>0}, \tag{35}$$

so we impose

$$6A_- = 2C, \quad 6A_+ = \frac{c}{2}, \quad B_- + 2\eta_- A_- = B_+ - 2\eta_+ A_+ = 0, \tag{36}$$

so that we even have an equality in (35). Plugging (36) into (34), we find

$$\frac{c}{6}\eta_+^3 + \frac{2C}{3}\eta_-^3 = \beta, \quad 4C\eta_-^2 = c\eta_+^2, \tag{37}$$

which leads us to choose

$$\eta_\pm = \alpha_\pm \beta^{1/3}, \tag{38}$$

where $\alpha_\pm > 0$ are the solutions of

$$4C\alpha_-^2 = c\alpha_+^2, \quad \frac{c}{6}\alpha_+^3 + \frac{2C}{3}\alpha_-^3 = 1.$$

With $A_\pm$, $B_\pm$ and $\eta_\pm$ chosen as in (36)–(38), the function v is of class $W^{2,\infty}$ and satisfies (35). Moreover, it is straightforward to check that

$$|v'| + |v''| \leq C\beta^{1/3} \quad \text{in } (-\eta_-, \eta_+),$$

so that (29) is satisfied, which concludes the construction of V_1 satisfying (21). On the other hand V_1 obviously satisfies (22) since

$$\{|V_1| < \beta/2\} = \{-\eta_- < z < \eta_+\}.$$

We omit the construction of V_2, which is completely similar to the one just performed. □

4. Intermediate Regimes

As discussed in Sec. 1, in the present section we want to understand the transitions occurring as β decreases from β_c to 0, when the set $\{H = 0\}$ has more than one connected component.

In Sec. 4.1 we study in detail a special case with rotational symmetry along a vertical axis, to provide some insight into the transition from the vortexless state to the zero solution. The reason to restrict to this setting is that it encapsules, what we believe are, the most interesting changes in the superconducting set that can occur.

On the one hand, once we drop the assumption of rotational symmetry, changes in H inside the sample could lead to arbitrarily intricate solutions to the obstacle problem for different values of β, so a general theorem is not available. On the other hand the symmetries we consider highlight many model situations with remarkable properties. One of these is the striking phenomenon that some parts of the free boundary may freeze: that is, remain constant with respect to β, for β in some interval. In Sec. 4.2 we generalize this observation to the general, nonsymmetric case.

As mentioned earlier, a generalization of the other properties is precluded due to the wide variety of solutions one could construct, having the freedom to choose both H and $\mathcal{M}$. Nevertheless, we believe that under some more restrictive assumptions, in particular fixing the topology of the level sets of H, one could extend the result on existence of the transitions observed in Proposition 4.2, however the role of the integral conditions on $I_\pm, J$ is not so easily transferable or even identifiable anymore.

4.1. *Detailed study of a symmetric case*

Here we consider a surface of revolution of the form

$$\mathcal{M} = \{(\rho(\phi)\cos\theta, \rho(\phi)\sin\theta, z(\phi)) : \phi \in [0,\pi], \theta \in [0, 2\pi]\}, \tag{39}$$

where ρ and z are smooth functions linked by the relation

$$z(\phi)\tan\phi = \rho(\phi),$$

and satisfying $\rho(0) = \rho(\pi) = 0$, $\rho > 0$ in $(0,\pi)$, $z'(0) = z'(\pi) = 0$, and

$$\gamma := \sqrt{(\rho')^2 + (z')^2} \geq c > 0.$$

The volume form on such $\mathcal{M}$ is $d\mathcal{H}^2_{\mathcal{M}} = \rho\gamma d\theta d\phi$.

The induced magnetic potential $\mathbf{A}$ on $\mathcal{M}$ is also assumed to be symmetric, of the form

$$\mathbf{A} = a(\phi)d\theta = \frac{a(\phi)}{\rho(\phi)}\hat{e}_\theta,$$

and we make the following assumptions on the functions a:

(a1): $a(0) = a(\pi) = 0$, and $a > 0$ in $(0,\pi)$.
(a2): $a' > 0$ in $(0,\phi_1)$ and (ϕ_2,ϕ_3) and $a' < 0$ in (ϕ_1,ϕ_2) and (ϕ_3,π), for some $0 < \phi_1 < \phi_2 < \phi_3 < \pi$.

The function $a(\phi)$ has two local maxima $a_1 = a(\phi_1)$ and $a_3 = a(\phi_3)$, and one local minimum $a_2 = a(\phi_2)$. To simplify notations to come, we assume in addition that $a_1 < a_3$. See Fig. 2.

Remark 4.1. The case, presented in Sec. 1, of a uniform external magnetic field $\mathbf{H}^e = \mathbf{e_z}$ corresponds to $a = \rho^2/2$.

In that setting, the functions H and $*F$ are also axially symmetric: they depend only on ϕ , and are given by

$$H = \frac{a'}{\rho\gamma}, \quad (*F)' = a\frac{\gamma}{\rho}.$$

By uniqueness (up to a possible additive constant), the solution of the obstacle problem (9) is also rotationally symmetric: it holds $V = v(\phi)$. Since $V \in C^1(\mathcal{M})$, the function v should satisfy

$$v \in C^1([0,\pi]), \quad v'(0) = v'(\pi) = 0.$$

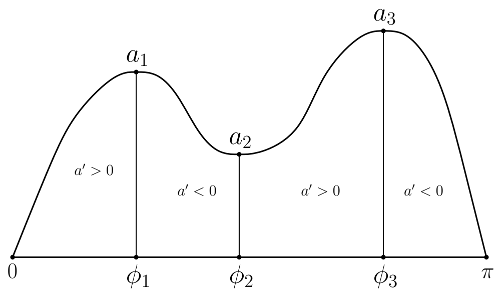


Fig. 2. The shape of $a(\phi)$.

Moreover, the free boundary problem (11) becomes

$$\begin{cases} |v| \leq \beta/2 & \text{in } [0,\pi], \\ (\rho\gamma^{-1}v' - a)' = 0 & \text{in } \{|u| < \beta/2\}, \\ a' \geq 0 & \text{in } \{v = -\beta/2\}, \\ a' \leq 0 & \text{in } \{v = \beta/2\}. \end{cases} \tag{40}$$

We investigate, for $\beta < \beta_c$, the changes in the shape of the superconducting set $\mathrm{SC}_\beta = \{|v| < \beta/2\}$. The critical values at which that shape changes depend on the values of integrals $\int a\,\gamma\rho^{-1}d\phi$ on some intervals related to the level sets of $a(\phi)$. That is why we start by fixing some notations concerning the level sets of $a(\phi)$. There are three different cases, depicted in Fig. 3:

- For $\alpha \in (0, a_2)$, $\{a = \alpha\} = \{\phi_- < \phi_+\}$.
- For $\alpha \in (a_2, a_1)$, $\{a = \alpha\} = \{\phi_- < \psi_+ < \psi_- < \phi_+\}$.
- For $\alpha \in (a_1, a_3)$, $\{a = \alpha\} = \{\psi_- < \phi_+\}$.

The functions $\phi_\pm(\alpha)$, $\psi_\pm(\alpha)$ are continuous on their intervals of definition.

For $\alpha \in (a_2, a_1)$, we define

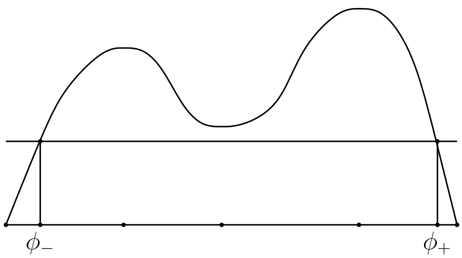
$$\begin{aligned} I_-(\alpha) &= \int_{\phi_-}^{\psi_+} (a-\alpha)\frac{\gamma}{\rho}d\phi, \quad I_+(\alpha) = \int_{\psi_-}^{\phi_+} (a-\alpha)\frac{\gamma}{\rho}d\phi, \\ J(\alpha) &= -\int_{\psi_+}^{\psi_-} (a-\alpha)\frac{\gamma}{\rho}d\phi. \end{aligned} \tag{41}$$

Those integrals correspond to "weighted" areas of the regions depicted in Fig. 4, with respect to the measure $\gamma\rho^{-1}d\phi$. Note that both the integrands and the intervals of integration depend on α.

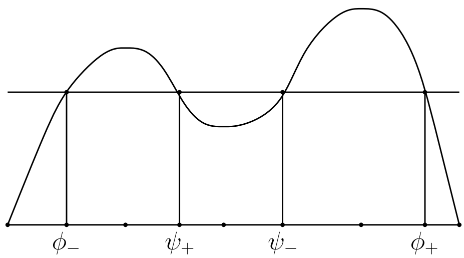
We identify a critical value of α with respect to these integrals.

Lemma 4.1. *There exists $\alpha_* \in (a_2, a_1)$ such that*:

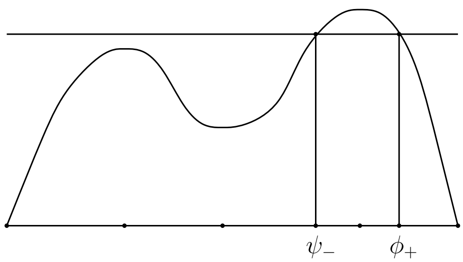
- *for $a_2 < \alpha < \alpha_*$, $J < \min(I_\pm)$;*
- *for $\alpha_* < \alpha < a_1$, $\min(I_\pm) < J$.*



(a) $\alpha \in (0, a_2)$



(b) $\alpha \in (a_2, a_1)$



(c) $\alpha \in (a_1, a_3)$

Fig. 3. Level sets $\{a = \alpha\}$.

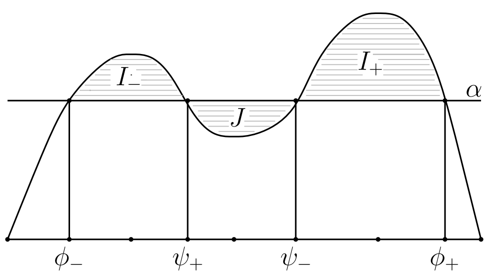


Fig. 4. The integrals $I_\pm$ and J.

Proof. It follows from the obvious facts that J is increasing, $I_\pm$ are decreasing, $J(a_2) = 0$, $I_-(a_1) = 0$, and the functions are continuous. $\square$

Now we may give the precise version of Proposition 1.2.

Proposition 4.2. *Let $\beta_c > \beta_1^* \geq \beta_2^* > 0$ be defined by*

$$\beta_1^* := \max(I_\pm(\alpha^*)), \quad \beta_2^* := \min(I_\pm(\alpha^*)).$$

Then the conclusion of Proposition 1.2 *holds*:

- *For $\beta_c > \beta > \beta_1^*$, SC_β is an interval.*

- *For $\beta_1^* > \beta > \beta_2^*$, SC_β is the union of two disjoint intervals, one of them independent of β.*
- *For $\beta_2^* > \beta > 0$, SC_β is the union of three disjoint intervals.*

Remark 4.2. It may happen that $I_-(\alpha^*) = I_+(\alpha^*)$. In that case, $\beta_1^* = \beta_2^*$ and the second regime predicted by Proposition 4.2 never happens.

Proof of Proposition 4.2. By uniqueness (see Remark 2.1), it suffices to exhibit, for each regime listed in Proposition 4.2, a solution of (40) satisfying the desired properties.

Case 1: $\beta \in (\beta_1^*, \beta_c)$. The function

$$I(\alpha) := \int_{\phi_-}^{\phi_+} (a - \alpha)\frac{\gamma}{\rho} d\phi, \quad \alpha \in (0, a_1),$$

is continuous, decreasing and satisfies $I(0) = \beta_c$ and $I(\alpha^*) = \beta_1^*$. Therefore there exists a unique $\alpha \in (0, \alpha^*)$ such that $I(\alpha) = \beta$. We define

$$v(\phi) = \begin{cases} -\beta/2 & \text{for } \phi \in (0, \phi_-), \\ -\beta/2 + \displaystyle\int_{\phi_-}^{\phi} (a - \alpha)\frac{\gamma}{\rho} d\tilde{\phi} & \text{for } \phi \in (\phi_-, \phi_+), \\ \beta/2 & \text{for } \phi \in (\phi_+, \pi). \end{cases}$$

The shape of the function v is sketched in Fig. 5.

The function v is clearly continuous since β has been chosen accordingly. Moreover, it holds

$$v'(\phi_+) = (a(\phi_+) - \alpha)\frac{\gamma}{\rho} = v'(\phi_-) = 0,$$

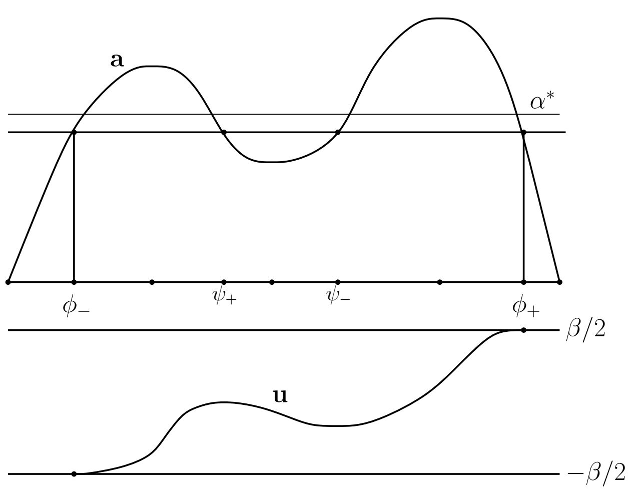


Fig. 5. The shape of v for $\beta \in (\beta_1^*, \beta_c)$.

since by definition $a(\phi_+) = a(\phi_-) = \alpha$. Hence v is in fact C^1 in $[0, \pi]$. Also by definition, $a' \geq 0$ in $(0, \phi_-)$ and $a' \leq 0$ in (ϕ_+, π). In addition, we clearly have $(\rho\gamma^{-1}v' - a)' = 0$ in (ϕ_-, ϕ_+). To prove that v solves (40), it only remains to show that $|v| < \beta/2$ in (ϕ_+, ϕ_-). We consider two different cases, depending on whether $\alpha \in (0, a_2]$ or $\alpha \in (a_2, \alpha^*)$.

If $\alpha \in (0, a_2)$, then (see Fig. 3(a))

$$v' = (a - \alpha)\frac{\gamma}{\rho} > 0 \quad \text{in } (\phi_-, \phi_+),$$

so that v is increasing on (ϕ_-, ϕ_+) and it clearly holds $|v| < \beta/2$. For $\alpha = a_2$ the derivative v' only vanishes at one point and the same conclusion is valid.

If, on the other hand $\alpha \in (a_2, \alpha^*)$, then (see Fig. 5)

$$v' = (a - \alpha)\frac{\gamma}{\rho} \begin{cases} > 0 & \text{in } (\phi_-, \psi_+), \\ < 0 & \text{in } (\psi_+, \psi_-), \\ > 0 & \text{in } (\psi_-, \phi_+). \end{cases}$$

Therefore it suffices to check that $v(\psi_+) < \beta/2$ and $v(\psi_-) > -\beta/2$. We have, since $I(\alpha) = \beta$ and by definition of $I_\pm$ and J (see Fig. 4),

$$\begin{aligned} v(\psi_+) - \beta/2 &= I_-(\alpha) - \beta = I_-(\alpha) - I(\alpha) = J(\alpha) - I_+(\alpha), \\ v(\psi_-) + \beta/2 &= I_-(\alpha) - J(\alpha). \end{aligned}$$

Since $\alpha < \alpha^*$ we find indeed (by definition of α^*) that $v(\psi_+) < \beta/2$ and $v(\psi_-) > -\beta/2$, and in that case also we conclude that v solves the free boundary problem (40).

Case 2: $\beta \in (\beta_2^*, \beta_1^*)$. We treat the case where $\min(I_\pm(\alpha^*)) = I_-(\alpha^*)$. Thus $\beta_1^* = I_+(\alpha^*)$ and $\beta_2^* = I_-(\alpha^*)$. The other case can be dealt with similarly.

The function $I_+(\alpha)$ is continuous and decreasing on (a_2, a_3) and satisfies $I_+(\alpha^*) = \beta_1^*$ and $I_+(a_3) = 0 < \beta_2^*$ (see Fig. 4). Therefore there exists $\alpha > \alpha^*$ such that $I_+(\beta) = \alpha$. We denote by ψ_- and ϕ_+ the two points of $\{a = \alpha\} \cap (\phi_2, \pi)$, and by $\phi_-^* < \psi_+^* < \psi_-^*$ the three points of $\{a = \alpha^*\} \cap (0, \phi_3)$ (as in Fig. 6). Note that, since $\alpha > \alpha^*$, $\psi_-^* < \psi_-$. Next we define

$$v(\phi) = \begin{cases} -\beta/2 & \text{for } \phi \in (0, \phi_-^*), \\ -\beta/2 + \displaystyle\int_{\phi_-^*}^{\phi} (a - \alpha^*)\frac{\gamma}{\rho} d\tilde{\phi} & \text{for } \phi \in (\phi_-^*, \psi_-^*), \\ -\beta/2 & \text{for } \phi \in (\psi_-^*, \psi_-), \\ -\beta/2 + \displaystyle\int_{\psi_-}^{\phi} (a - \alpha)\frac{\gamma}{\rho} d\tilde{\phi} & \text{for } \phi \in (\psi_-, \phi_+), \\ \beta/2 & \text{for } \phi \in (\phi_+, \pi). \end{cases}$$

The shape of the function v is sketched in Fig. 6.

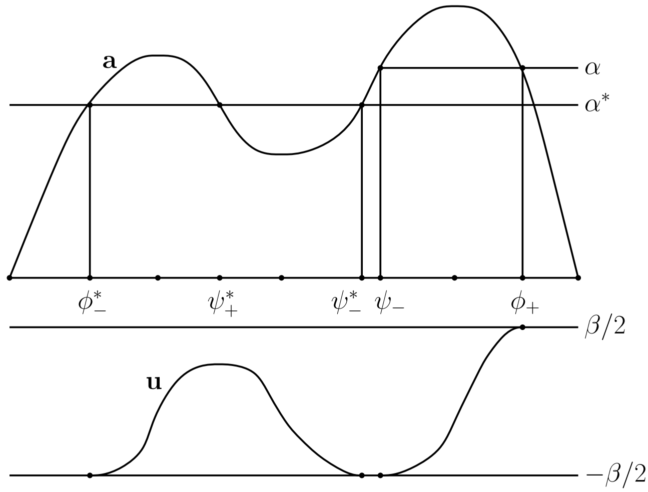


Fig. 6. The shape of v for $\beta \in (\beta_2^*, \beta_1^*)$.

Continuity of v at ψ_-^* is ensured by the fact that $I_-(\alpha^*) = J(\alpha^*)$. Continuity at ϕ_+ by $I_+(\alpha) = \beta$. The function v is C^1 because the facts that $a(\phi_-^*) = a(\psi_-^*) = \alpha^*$ and $a(\psi_-) = a(\phi_+) = \alpha$ guarantee that $v'(\phi_-^*) = v'(\psi_-^*) = v'(\psi_-) = v'(\phi_+) = 0$. The sign of a' is positive in $(0, \phi_-^*)$ and (ψ_-^*, ψ_-) and negative in (ϕ_+, π). In the two intervals (ϕ_-^*, ψ_-^*) and (ψ_-, ϕ_+), the equation $(\rho\gamma^{-1}v' - a)' = 0$ is obviously satisfied, and it remains to check that $|v| < \beta/2$ in those intervals.

Since $v' = (a - \alpha)\gamma\rho^{-1} > 0$ in (ψ_-, ϕ_+), it clearly holds $|v| < \beta/2$ in (ψ_-, ϕ_+).

In the interval (ϕ_-^*, ψ_-^*), the sign of v' shows that v attains its minimum at the boundary and its maximum at ψ_+^*, and it holds

$$v(\psi_+^*) - \beta/2 = -\beta + I_-(\alpha^*) = -\beta + \beta_2^* < 0.$$

We conclude that v solves the free boundary problem (40). Moreover, the interval (ϕ_-^*, ψ_-^*) clearly does not depend on β.

Case 3: $\beta \in (0, \beta_2^*)$. Since I_- is continuous and decreasing, $I_-(\alpha^*) > \beta_2^*$ and $I_-(a_1) = 0$, there exists $\alpha_1 > \alpha^*$ such that $I_-(\alpha_1) = \beta$. Similarly, there exist $\alpha_2 < \alpha^*$ and $\alpha_3 > \alpha^*$ such that $J(\alpha_2) = I_+(\alpha_3) = \beta$. We denote by

$$0 < \phi_-^1 < \psi_+^1 < \psi_+^2 < \psi_-^2 < \psi_-^3 < \phi_+^3 < \pi$$

the points such that (see Fig. 7)

$$\{a = \alpha_1\} \cap (0, \phi_2) = \{\phi_-^1, \psi_+^1\},$$

$$\{a = \alpha_2\} \cap (\phi_1, \phi_3) = \{\psi_+^2, \psi_-^2\},$$

$$\{a = \alpha_3\} \cap (\phi_2, \pi) = \{\psi_-^3, \phi_+^3\}.$$

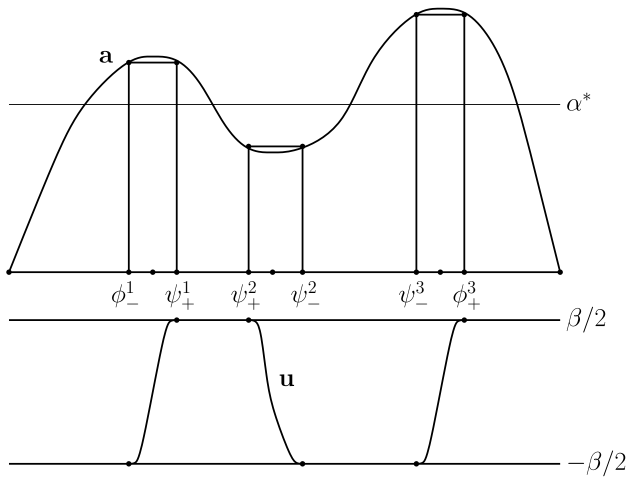


Fig. 7. The shape of v for $\beta \in (0, \beta_2^*)$.

Then we define

$$v(\phi) = \begin{cases} -\beta/2 & \text{for } \phi \in (0, \phi^1_-) \text{ or } \phi \in (\psi^2_-, \psi^3_-), \\ -\beta/2 + \displaystyle\int_{\phi^1_-}^{\phi} (a - \alpha_1)\frac{\gamma}{\rho} d\tilde{\phi} & \text{for } \phi \in (\phi^1_-, \psi^1_+), \\ \beta/2 & \text{for } \phi \in (\psi^1_+, \psi^2_+) \text{ or } \phi \in (\phi^3_+, \pi), \\ \beta/2 + \displaystyle\int_{\psi^2_+}^{\phi} (a - \alpha_2)\frac{\gamma}{\rho} d\tilde{\phi} & \text{for } \phi \in (\psi^2_+, \psi^2_-), \\ -\beta/2 + \displaystyle\int_{\psi^3_-}^{\phi} (a - \alpha_3)\frac{\gamma}{\rho} d\tilde{\phi} & \text{for } \phi \in (\psi^3_-, \phi^3_+). \end{cases}$$

The shape of the function v is sketched in Fig. 7.

As above the C^1 regularity of v follows from the definitions of α_1, α_2 and α_3. The sign of a' is positive in $(0, \phi^1_-) \cup (\psi^2_-, \psi^3_-)$ and negative in $(\psi^1_+, \psi^2_+) \cup (\phi^3_+, \pi)$. The equation $(\rho\gamma^{-1}v' - a)' = 0$ is satisfied in the three intervals (ϕ^1_-, ψ^1_+), (ψ^2_+, ψ^2_-) and (ψ^3_-, ϕ^3_+). Moreover in those intervals, the function v is monotone, hence $|v| < \beta/2$. Therefore v solves the free boundary problem (40). $\square$

4.2. "*Freezing*" *of the free boundary*

Proposition 4.3. *Assume that, for some $\beta_0 \in (0, \beta_c)$, one connected component ω of the superconductivity set SC_{β_0} is such that V_{β_0} takes the same value on each connected component of $\partial\omega$. Then there exists $\delta > 0$ such that*

$$\mathrm{SC}_\beta \cap \overline{\omega} = \mathrm{SC}_{\beta_0} \cap \overline{\omega} = \omega, \tag{42}$$

for all $\beta \in (\beta_0 - \delta, \beta_0]$.

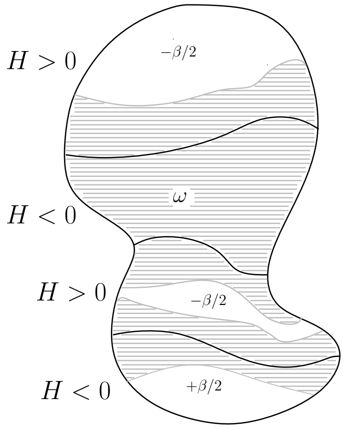


Fig. 8. An example of the situation of Proposition 4.3.

In Fig. 8 we show a situation corresponding to Proposition 4.3, with $V = -\beta/2$ on every connected component of $\partial\omega$.

Remark 4.3. The assumption on β_0 in Proposition 4.3 corresponds exactly to what happens in the symmetric case (Proposition 4.2) in the regime $\beta_1^* > \beta > \beta_2^*$, where $v(\phi_-^*) = v(\psi_-^*) = -\beta/2$ (Fig. 6).

Proof of Proposition 4.3. We present the proof in the case where $V = -\beta_0/2$ on every connected component of $\partial\omega$. The case $V = \beta_0/2$ on $\partial\omega$ can be dealt with similarly.

Since $V < \beta_0/2$ in ω and $V = -\beta_0/2$ on $\partial\omega$, it holds

$$m := \max_{\overline{\omega}} V < \beta_0/2,$$

and we define

$$\delta := \frac{1}{2}\beta_0 - m > 0.$$

Let $\beta \in (\beta_0 - \delta, \beta_0]$, and define

$$\widetilde{V}_0 := V_{\beta_0} + \frac{1}{2}(\beta_0 - \beta). \tag{43}$$

The definitions of m and δ ensure that it holds

$$-\beta/2 \leq \widetilde{V}_0 \leq \frac{1}{2}\beta_0 - \delta + \frac{1}{2}(\beta_0 - \beta) < \beta/2 \quad \text{in } \overline{\omega}. \tag{44}$$

We claim that

$$V_\beta = \widetilde{V}_0 \quad \text{in } \overline{\omega}, \tag{45}$$

which obviously implies (42).

Note that the proof of Proposition 2.4 implies that it always holds

$$V_\beta \leq \widetilde{V}_0 \quad \text{in } \mathcal{M}. \tag{46}$$

Let $\omega_\beta = SC_\beta \cap \omega$, and

$$U := \widetilde{V}_0 - V_\beta \geq 0. \tag{47}$$

Note that $U \in C^{1,\alpha}(\overline{\omega})$, and $U = 0$ on $\partial\omega$ (since, by definition of ω, $\widetilde{V}_0 = 0$ on $\partial\omega$).

Let $\omega' := \omega \cap SC_\beta$. It holds

$$\Delta U = H \mathbb{1}_{\omega\setminus\omega'} \quad \text{in } \omega. \tag{48}$$

From (44) and (46) it follows that

$$V_\beta < \beta/2 \quad \text{in } \overline{\omega}.$$

Therefore, recalling the free boundary formulation (11), we have $H \geq 0$ in $\omega\setminus\omega'$. In particular (48) implies that

$$\Delta U \geq 0 \quad \text{in } \omega.$$

Let $\varepsilon > 0$ and consider

$$\varphi := \max(U - \varepsilon, 0) \in H^1(\omega).$$

Recalling that $U \in C(\overline{\omega})$ and $U = 0$ on $\partial\omega$, we know that φ has compact support inside ω. Thus we may integrate by part (without knowing anything about the regularity of $\partial\omega$) to obtain

$$\int_\omega |\nabla\varphi|^2 = \int_\omega \nabla\varphi \cdot \nabla U = -\int_\omega \varphi \Delta U \leq 0,$$

and we deduce that $\varphi \equiv 0$ in ω, which implies that $U \leq \varepsilon$ in ω. Letting $\varepsilon \to 0$, we conclude that $U \leq 0$ in ω, which, together with (47), shows that (45) holds. □

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Appendix. The Mean Field Approximation

Recall we assume $\mathcal{M} \subset \mathbb{R}^3$ is a closed compact surface homeomorphic to a sphere, $\mathbf{A}$ a 1-form on $\mathcal{M}$ such that $\mathbf{A} = d^*F = *d*F$ for some smooth nonconstant

2-form F, and $\mathcal{G}_{\mathcal{M},\kappa}$ the Ginzburg–Landau energy

$$\mathcal{G}_{\mathcal{M},\kappa}(\psi) = \int_{\mathcal{M}} |(\nabla - ih\mathbf{A})\psi|^2 + \frac{\kappa^2}{2}\int_{\mathcal{M}} (|\psi|^2 - 1)^2.$$

The parameter $\kappa > 0$ is going to tend to $+\infty$, as is the strength of the applied field $h(\kappa) > 0$.

If ψ is a critical point of $\mathcal{G}_{\mathcal{M},\kappa}$, written locally as $\psi = \rho e^{i\varphi}$, then it holds

$$d(\rho^2(d\varphi - hd^*F)) = 0.$$

We deduce that there exists a function V such that

$$*dV = \rho^2(hd^*F - d\varphi).$$

The function V is uniquely defined up to an additive constant, which we may fix by imposing $\int_{\mathcal{M}} V = 0$. The function

$$\mu = -\Delta(V - h * F) = -\Delta V + H$$

is the vortex density.

In this paper we appeal to a mean field approximation result proved by Sandier and Serfaty in [15]. In our case we also have to handle positive and negative measures μ_+, μ_- with total zero mass $\mu_+(\mathcal{M}) - \mu_-(\mathcal{M}) = 0$. In this appendix we verify that under the additional constraints present in our context, we still have such a reduction. For an intensity $h(\kappa)$ comparable to $\ln\kappa$, the mean field approximation consists in approximating the problem of minimizing $\mathcal{G}_{\mathcal{M},\kappa}$ by a limiting problem on the vorticity measure. The result also relates the following proposition.

Proposition A.1. *Assume that* $\beta := \lim_{\kappa\to\infty} \frac{\ln\kappa}{h(\kappa)} \geq 0$ *and* $h(\kappa) = o(\kappa^2)$. *Let* ψ_κ *be a minimizer of* $\mathcal{G}_{\mathcal{M},\kappa}$, *and the corresponding* V_κ *be defined as above. Then, up to a subsequence, as* $\kappa \to \infty$,

$$\frac{V_\kappa}{h(\kappa)} \text{ converges to } W_*,$$

weakly in H^1 (*and strongly in* $W^{1,q}$ *for* $q < 2$) *where* W_* *minimizes the energy*

$$E_\beta(W) = \frac{1}{2}\int_{\mathcal{M}} |\nabla W|^2 d\mathcal{H}^2 + \frac{\beta}{2}\|-\Delta W + H\|_{TV},$$

over the set of all $W \in H^1(\mathcal{M})$ *such that* $(-\Delta W + H)$ *is a Radon measure. Here* $\|\mu\|_{TV} = |\mu|(\mathcal{M})$ *denotes the total variation norm of the Radon measure* μ.

Moreover, it holds $\mathcal{G}_{\mathcal{M},\kappa}(\psi_\kappa) = h(\kappa)^2 E_\beta(W_*) + o(h(\kappa)^2)$.

We impose the normalization conditions $\int_{\mathcal{M}} W d\mathcal{H}^2 = \int_{\mathcal{M}} *F d\mathcal{H}^2 = 0$. Then with some slight abuse of notation $E_\beta(W)$ can be expressed in terms of $\mu = -\Delta W + H$, as

$$E_\beta(W) = E_\beta(\mu) = \frac{\beta}{2}\|\mu\|_{TV} + \frac{1}{2}\int_{\mathcal{M}} G(x,y)d(\mu - H)(x)d(\mu - H)(y),$$

where $G(x, y)$ is the Green's function satisfying

$$-\Delta_{\mathcal{M}} G(\cdot, y) = \delta_y - \frac{1}{\mathcal{H}^2(\mathcal{M})}.$$

Here μ has to be a Radon measure of zero average since it comes from $\mu = -\Delta(W - *F)$, hence $\int_{\mathcal{M}} d\mu = 0$.

Note that $E_\beta(\mu)$ may not be well defined for every measure μ, but at the end we will only need it to be well defined for the particular μ_* associated to W_* solving the obstacle problem (9), and this follows from the regularity theory for the obstacle problem (see Lemma 2.3).

Sketch of the proof of the upper bound in Proposition A.1. The proof of the lower bound and compactness for minimizers follows directly from [15, Theorems 7.1 and 7.2]. We note that a by-product of the analysis in [15] is that $\frac{2\pi \sum_{i\in I} d_i \delta_{a_i}}{h}$ converges to $-\Delta\left(\frac{V_\kappa}{h} - *F\right)$ in the sense of measures and in $W^{1,p}$, for $p < 2$.

The upper bound on the other hand is a little more delicate to adapt. Next we provide the details. The main tool to derive the upper bound in [15] is a construction of measures μ_κ which approximate the measure μ_* minimizing I_β, and which are concentrated in balls of size κ^{-1} each carrying a weight 2π. Before stating the precise result, we introduce the functional $J = J_\beta$

$$J(\mu) := \beta \|\mu\|_{TV} + \int_{\mathcal{M}\times\mathcal{M}} G(x, y) d\mu(x) d\mu(y). \tag{A.1}$$

The following result then corresponds to [15, Proposition 2.2].

Proposition A.2. *Let $\mu = \mu_+ - \mu_-$ be the minimizer of I_β. Then, for κ large enough, there exist points $a^\kappa_{j,\pm}$, $1 \le j \le n_\pm(\kappa)$, such that*

$$n_\pm(\kappa) \sim \frac{h(\kappa)\mu_\pm(\mathcal{M})}{2\pi}, \quad d(a^\kappa_{j,\pm}, a^\kappa_{\ell,\pm}) > 4\kappa^{-1},$$

and, letting $\mu^{j,\pm}_\kappa$ be the uniform measure on $\partial B(a_{j,\pm}, \kappa^{-1})$ of mass 2π, the measure

$$\mu_\kappa := \frac{1}{h(\kappa)} \sum_{j=1}^{n_+(\kappa)} \mu^{j,+}_\kappa - \frac{1}{h(\kappa)} \sum_{j=1}^{n_-(\kappa)} \mu^{j,-}_\kappa \text{ converges to } \mu,$$

in the sense of measures as $\kappa \to +\infty$. Moreover it holds $\int_{\mathcal{M}} d\mu_\kappa = 0$, and

$$\limsup_{\kappa\to\infty} \int_{\mathcal{M}\times\mathcal{M}} G(x, y) d\mu_\kappa(x) d\mu_\kappa(y) \le J(\mu), \tag{A.2}$$

where $J = J_\beta$ is defined in (A.1).

Above, d denotes geodesic distance and ∂B denotes a geodesic circle accordingly. The zero average property $\int d\mu_\kappa = 0$ is needed later to solve $-\Delta(V - *F) = \mu_\kappa$. It actually amounts to asking $n_+(\kappa) = n_-(\kappa)$. The upper bound (A.2) is crucial to estimate the energy of the testing configuration constructed with help of the measures μ_κ, and requires great care in the way the points a^κ_j are distributed.

In [15], the authors consider non-negative measures defined on a domain in the plane, with no average constraint. Here we are dealing with measures on a surface having positive and negative parts, and, more importantly, satisfying the zero average constraint.

Next we state a lemma that can be directly adapted from [15, Proposition 2.2], which deals only with positive measures with support inside a coordinate neighborhood. Then we will explain how to use this lemma to obtain Proposition A.2 above.

Lemma A.3 ([15, Proposition 2.2]). *Assume that μ is a non-negative Radon measure on $\mathcal{M}$, absolutely continuous with respect to the two-dimensional measure on $\mathcal{M}$, and with support contained inside a coordinate neighborhood. Then, there exist points a_j^κ, $1 \le j \le n(\kappa)$, with*

$$n(\kappa) \sim \frac{h(\kappa)\mu(\mathcal{M})}{2\pi} \quad \text{and} \quad d(a_j^\kappa, a_\ell^\kappa) > 4\kappa^{-1},$$

such that, with μ_κ^j the uniform measure of mass 2π on $\partial B(a_j^\kappa, \kappa^{-1})$, it holds

$$\mu_\kappa = \frac{1}{h(\kappa)} \sum_{j=1}^{n(\kappa)} \mu_\kappa^j \text{ converges to } \mu,$$

and the upper bound (A.2) *is satisfied.*

The proof of Lemma A.3 is just a straightforward adaptation of [15, Proposition 2.2], using the coordinate chart to transport their construction from the plane to our surface and general properties of the Green's function of the Laplacian on a compact surface.

Next we explain how to deal with non-negative measures whose support does not lie inside a coordinate neighborhood.

Lemma A.4. *Assume that μ is a non-negative Radon measure on $\mathcal{M}$, absolutely continuous with respect to the two-dimensional measure on $\mathcal{M}$. Then the conclusion of Lemma* A.3 *holds.*

Proof. *Step* 1: We reduce to the case where the support of μ is a finite disjoint union of compact coordinate neighborhoods. Assume indeed that the conclusion of Lemma A.4 holds for such measures. It is possible to construct a sequence μ_n of such measures, such that $0 \le \mu_n \le \mu$ and μ_n converges to μ. Indeed, just define $\mu_n = \mathbb{1}_{K_n}\mu$, where K_n is a finite disjoint union of compact subsets of coordinate neighborhoods, and $\mu(\mathcal{M}\backslash K_n) \to 0$. Such a sequence K_n exists because $\mathcal{M}$ is compact and the measure μ is inner regular. For each μ_n we obtain a sequence μ_n^κ tending to μ_n with the good properties. After a diagonal process, we obtain a sequence μ_κ converging to μ, such that

$$\limsup_{\kappa\to\infty} \int_{\mathcal{M}\times\mathcal{M}} G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) \le \liminf J(\mu_n).$$

It remains to show that the right-hand side is less than $J(\mu)$, which follows from $0 \leq \mu_n \leq \mu$ and $G \geq 0$.

Step 2: We prove Lemma A.4 for μ that can be decomposed in the form

$$\mu = \mu_1 + \cdots + \mu_N,$$

where the supports of the μ_j are inside disjoint compact coordinate neighborhoods, and each μ_j is non-negative and absolutely continuous with respect to $\mathcal{H}^2_{\mathcal{M}}$. Then one can apply Lemma A.3 to each μ_j to obtain sequences $\mu_{j,\kappa}$ with the good properties. Then, defining $\mu_\kappa = \mu_{1,\kappa} + \cdots + \mu_{N,\kappa}$, one obtains

$$\begin{aligned}\limsup \int & G(x,y) d\mu_\kappa(x) d\mu_\kappa(y) \\ & \leq \sum_j J(\mu_j) + \limsup \sum_{j \neq \ell} \int G(x,y) d\mu_{j,\kappa}(x) d\mu_{\ell,\kappa}(y).\end{aligned}$$

Since the supports of distinct μ_j are disjoint and $G(x,y)$ is continuous outside the diagonal $\{x = y\}$, it holds

$$\int G(x,y) d\mu_{j,\kappa}(x) d\mu_{\ell,\kappa}(y) \to \int G(x,y) d\mu_j(x) d\mu_\ell(y) \quad \text{for } j \neq \ell,$$

and we conclude that

$$\limsup \int G(x,y) d\mu_\kappa(x) d\mu_\kappa(y) \leq \sum_j J(\mu_j) + \sum_{j \neq \ell} \int G(x,y) d\mu_j(x) d\mu_\ell(y) = J(\mu).$$

The proof is complete. □

Finally we deal with measures having positive and negative parts, and satisfying the zero average constraint.

Lemma A.5. *Let μ be a zero-average Radon measure on $\mathcal{M}$, absolutely continuous with respect to $\mathcal{H}^2_{\mathcal{M}}$. Then the conclusions of Proposition* A.2 *hold.*

Proof. *Step* 1: It suffices to construct measures μ_κ satisfying all the conclusions of Proposition A.2, except for the zero average constraint. Assume indeed that we have such a sequence. Since μ satisfies the zero average constraint, it holds $\mu_+(\mathcal{M}) = \mu_-(\mathcal{M})$ and we deduce that $n_+(\kappa) - n_-(\kappa) = o(h(\kappa))$. Up to considering a subsequence, we may assume that either $n_+(\kappa) \geq n_-(\kappa)$ for every κ (or the opposite, but this is completely symmetric). We fix a compact K such that $\mu_+(K) > 0$ and K is disjoint from the support of μ_-. Since $\mu^+_\kappa(K)$ converges to $\mu_+(K)$, the number of points $a^\kappa_{j,+}$ that are contained in K for large κ is larger than $c \cdot h(\kappa)$ for $c > 0$. In particular it is larger than $n_+ - n_-$, and we may define a measure $\tilde{\mu}^+_\kappa$ obtained from μ^+_κ by removing $(n_+ - n_-)$ points $a^\kappa_{j,+}$ that lie inside K. The measure $\tilde{\mu}_\kappa = \tilde{\mu}^+_\kappa - \mu^-_\kappa$ now satisfies the zero average condition, and since $n_+ - n_- = o(h)$ the convergence

$\tilde{\mu}_\kappa \to \mu$ still holds. It remains to prove that the upper bound (A.2) is satisfied also by $\tilde{\mu}_\kappa$. Since $G \geq 0$ and $0 \leq \tilde{\mu}_\kappa^+ \leq \mu_\kappa^+$, it holds

$$\int G(x,y)d\tilde{\mu}_\kappa(x)d\tilde{\mu}_\kappa(y) \leq \int G(x,y)d\mu_\kappa(x)d\mu_\kappa(y)$$
$$+ 2\int G(x,y)d(\mu_\kappa^+ - \tilde{\mu}_\kappa^+)(x)d\mu_\kappa^-(y).$$

The last term converges to zero since G is continuous outside the diagonal and $\mu_\kappa^+ - \tilde{\mu}_\kappa^+$ converges to zero and has support inside K which is disjoint from the support of μ_-. Hence we conclude that (A.2) holds.

Step 2: As in Step 1 of Lemma A.4, we reduce to the case of a measure μ such that μ_+ and μ_- have disjoint compact supports. Assume indeed that Lemma A.5 holds for such measures, and consider, by truncating, monotone approximations $\mu_n^\pm$ of $\mu_\pm$, with disjoints compact supports and such that $0 \leq \mu_n^\pm \leq \mu_\pm$. For each n there exist measures μ_κ^n with the good properties, converging to $\mu_n := \mu_n^+ - \mu_n^-$. After a diagonal process, one obtains a sequence μ_κ such that

$$\limsup_{\kappa\to\infty} \int_{\mathcal{M}\times\mathcal{M}} G(x,y)d\mu_\kappa(x)d\mu_\kappa(y) \leq \liminf J(\mu_n).$$

Since $G \geq 0$, by monotone convergence (or dominated convergence) terms of the form $\int G d\mu_n^\pm d\mu_n^\pm$ converge to $\int G d\mu^\pm d\mu^\pm$, so that

$$\int G(x,y)d\mu_n(x)d\mu_n(y) \to \int G(x,y)d\mu(x)d\mu(y),$$

and we also have $\|\mu_n\| \to \|\mu\|$, so that $J(\mu_n) \to J(\mu)$ and we conclude that (A.2) holds.

Step 3: We assume now that μ_+ and μ_- have disjoint compact supports. Applying Lemma A.4 to each of these non-negative measures, we can proceed exactly as in Step 2 of Lemma A.4 to obtain the conclusion. $\square$

With Lemma A.5 at hand, the proof of Proposition A.2 simply follows from the regularity theory for the obstacle problem (see Lemma 2.3), which ensures in particular that the minimizing measure μ_* is absolutely continuous with respect to $\mathcal{H}^2_{\mathcal{M}}$.

Then the upper bound is obtained by constructing test configurations with vortices at the $a_{j,\pm}^\kappa$ as in the proof of [15, Proposition 2.1]. Those test configurations are obtained by solving $-\Delta(V_\kappa - h * F) = h\mu_\kappa$ and constructing the corresponding ψ_κ which has modulus 1 outside the balls $B(a_{j,\pm}^\kappa, 2\kappa^{-1})$'s, and phase given by $d\varphi_\kappa = hd^*F - *dV_\kappa$. $\square$

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