A SYMMETRY BREAKING PHENOMENON FOR ANISOTROPIC HARMONIC MAPS FROM A 2D ANNULUS INTO $\mathbb{S}^1$

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ABSTRACT. In a two dimensional annulus $A_\rho = \{x \in \mathbb{R}^2 : \rho < |x| < 1\}$, $\rho \in (0,1)$, we characterize 0-homogeneous minimizers, in $H^1(A_\rho; \mathbb{S}^1)$ with respect to their own boundary conditions, of the anisotropic energy

$$E_\delta(u) = \int_{A_\rho} |\nabla u|^2 + \delta \left((\nabla \cdot u)^2 - (\nabla \times u)^2 \right) \, dx, \quad \delta \in (-1,1).$$

Even for a small anisotropy $0 < |\delta| \ll 1$, we exhibit qualitative properties very different from the isotropic case $\delta = 0$. In particular, 0-homogeneous critical points of degree $d \notin \{0,1,2\}$ are always local minimizers, but in thick annuli ($\rho \ll 1$) they are not minimizers: the 0-homogeneous symmetry is broken. One corollary is that entire solutions to the anisotropic Ginzburg-Landau system have a far-field behavior very different from the isotropic case studied by Brezis, Merle and Rivière. The tools we use include: ODE and variational arguments; asymptotic expansions, interpolation inequalities and explicit computations involving near-optimizers of these inequalities for proving that 0-homogeneous critical points are not minimizers in thick annuli.

1. Introduction

For any open set $\Omega \subset \mathbb{R}^2$ and $\mathbb{S}^1$-valued map $u \in H^1(\Omega; \mathbb{S}^1)$, and given an anisotropy parameter $\delta \in (-1,1)$, we consider the anisotropic energy

$$E_\delta(u; \Omega) = \int_\Omega |\nabla u|^2 + \delta \left((\nabla \cdot u)^2 - (\nabla \times u)^2 \right) \, dx. \tag{1}$$

The energy density admits the alternative form $(1+\delta)(\nabla \cdot u)^2 + (1-\delta)(\nabla \times u)^2$: this follows from the identity $|\nabla u|^2 = (\nabla \cdot u)^2 + (\nabla \times u)^2 + 2\det(\nabla u)$, where the last term is zero for $u \in H^1(\Omega; \mathbb{S}^1)$. This energy arises in liquid crystal models, see e.g. [10, 9, 2, 14]. The energy density is the most general positive definite quadratic form of ∇u which is compatible with frame invariance: for any angle $\alpha \in \mathbb{R}$, the transformation

$$u(x) \longrightarrow e^{-i\alpha} u(e^{i\alpha} x) \tag{2}$$

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leaves the energy invariant. Critical points of E_δ in $H^1(\Omega;\mathbb{S}^1)$, characterized by

$$\frac{d}{dt}\bigg|_{t=0} E_\delta\left(\frac{\xi_\delta + t\varphi}{|\xi_\delta+t\varphi|};\Omega\right) = 0 \qquad \forall \varphi\in C_c^1(\Omega;\mathbb{R}^3),$$

satisfy the Euler-Lagrange equation

$$\mathcal{L}_\delta u = \lambda u, \qquad \lambda = u\cdot\mathcal{L}_\delta u, \tag{3}$$

where the linear operator $\mathcal{L}_\delta$ is given by

$$\mathcal{L}_\delta u = -\Delta u - \delta\left(\nabla(\nabla\cdot u) - \nabla^\perp(\nabla\times u)\right),$$

and the function λ in (3) can be interpreted as a Lagrange multiplier for the constraint $u(x)\in\mathbb{S}^1$.

The goal of this work is to exhibit nontrivial effects of the anisotropy on certain critical points of the energy. This is made manifest in the form of a symmetry breaking for minimizers within a given class, even when the anisotropy is small. To be precise, we consider the case of an annulus

$$\Omega = A_\rho = \left\{x\in\mathbb{R}^2\colon \rho<|x|<1\right\}, \qquad \rho\in(0,1),$$

and are interested in properties of 0-homogeneous critical points: in polar coordinates $x = re^{i\theta}$, they depend only on the θ variable. The main question we ask is: are 0-homogeneous critical points minimizers with respect to their own boundary conditions?

Basic facts. In the isotropic case $\delta = 0$, the equation (3) becomes $\Delta\varphi = 0$ for $u = e^{i\varphi}$. All 0-homogeneous solutions are given by

$$u(re^{i\theta}) = e^{i\alpha}e^{id\theta}, \qquad \alpha\in\mathbb{R}, \quad d\in\mathbb{Z},$$

and they are minimizers within their own homotopy class, characterized by the degree or winding number,

$$d = \deg(u) = \frac{1}{2\pi}\int_0^{2\pi} \bar{u}(re^{i\theta})\partial_\theta u(re^{i\theta})\,d\theta \in\mathbb{Z} \qquad \forall r\in[\rho,1]. \tag{4}$$

This is well-defined and does not depend on r because the trace of $u\in H^1(A_\rho;\mathbb{S}^1)$ on ∂D_r belongs to $H^{\frac12}(\partial D_r;\mathbb{S}^1)$ for all $r\in[\rho,1]$, see e.g. [5, Appendix]. Specifically, the lower bound

$$\int_{A_\rho} |\nabla u|^2\,dx \geq 2\pi d^2|\ln\rho| \qquad \forall u\in H^1(A_\rho,\mathbb{S}^1) \text{ with } \deg(u) = d, \tag{5}$$

is attained exactly at the one-dimensional family of 0-homogeneous maps $u(re^{i\theta}) = e^{i\alpha}e^{id\theta}$, $\alpha\in\mathbb{R}$. In the anisotropic case $\delta\neq 0$, the lower bound

$$E_\delta(u;A_\rho) \geq (1-|\delta|)\int_{A_\rho}|\nabla u|^2\,dx \geq (1-|\delta|)2\pi d^2|\ln\rho|,$$

is sharp only when $d=1$, and attained at the maps

$$u(re^{i\theta}) = e^{i\alpha}e^{i\theta}, \qquad \alpha = \begin{cases} 0 \bmod \pi & \text{if } \delta \in (-1,0), \\ \frac{\pi}{2} \bmod \pi & \text{if } \delta \in (0,1). \end{cases}$$

Notice that we no longer have a one-dimensional family of minimizers. It can be checked that these maps are the only 0-homogeneous solutions of (3) in A_ρ which are critical with respect to perturbations of their boundary conditions. They are also the only 0-homogeneous solutions of (3) with degree $d=1$, as shown in § 4.1. Solutions of degree $d \neq 1$ seem largely unexplored.

Main result. The scaling invariance of the energy ensures the existence of at least one 0-homogeneous critical point of any degree d. For $d \neq 1$, our main result asserts that it is unique modulo frame invariance (2) and linearly stable, but when the hole ($\rho \ll 1$) and the anisotropy ($0 < |\delta| \ll 1$) are small, it is not a minimizer with respect to its own boundary conditions, provided $d \notin \{0,1,2\}$.

Theorem 1.1. *Let $\delta \in (-1,1)$ and $d \in \mathbb{Z} \setminus \{1\}$.*

- *All 0-homogeneous solutions of degree d of the Euler-Lagrange system (3) are given by a single one-dimensional family*
$$\xi_\delta^\alpha(x) = e^{-i\alpha}\xi_\delta(e^{i\alpha}x), \qquad \alpha \in \mathbb{R}.$$
- *The unique (modulo frame invariance) 0-homogeneous critical point ξ_δ is linearly stable in A_ρ for all $0 < \rho < 1$: there exists a constant $c > 0$ depending on δ and ρ such that*
$$\frac{d^2}{dt^2}\bigg|_{t=0} E_\delta\left(\frac{\xi_\delta + t\varphi}{|\xi_\delta + t\varphi|}; A_\rho\right) \geq c\int_{A_\rho} |\nabla\varphi|^2\,dx$$
for all $\varphi \in C_c^1(A_\rho;\mathbb{R}^2)$ such that $\varphi \cdot \xi_\delta = 0$ a.e.
- *For small enough $|\delta| > 0$ and $d \notin \{0,1,2\}$, there exists a critical value $\rho_* = \rho_*(\delta,d) \in (0,1)$ such that the 0-homogeneous critical point ξ_δ is a minimizer in A_ρ for $\rho > \rho_*$ but not a minimizer for $\rho < \rho_*$:*
$$\min_{u\lfloor_{\partial A_\rho}=\xi_\delta} E_\delta(u;A_\rho) \begin{cases} = E_\delta(\xi_\delta;A_\rho) & \text{if } 1 > \rho \geq \rho_*, \\ < E_\delta(\xi_\delta;A_\rho) & \text{if } 0 < \rho < \rho_*, \end{cases}$$
where the minimum is taken over all maps $u \in H^1(A_\rho;\mathbb{S}^1)$ such that $u = \xi_\delta$ on ∂A_ρ.

Remark 1.2. We make here a few observations about the statements in Theorem 1.1:

- The case of degree $d=-1$ is the most important from the physical point of view, since only defects of degree $d \in \{\pm 1\}$ are experimentally stable (see e.g. [3]).
- The third item requires small anisotropy $0 < |\delta| \ll 1$, but the first two items are valid for any $\delta \in (-1,1)$.

- The existence of a one-dimensional family of 0-homogeneous critical points of degree $d \neq 1$, as in the isotropic case $\delta = 0$, is in strong contrast with what happens for $d = 1$, where that one-dimensional symmetry is broken for $\delta \neq 0$. We show in § 4.1 that for $0 < |\delta| < 1$, the trivial solutions $u(re^{i\theta}) = e^{i\alpha}e^{i\theta}$, $\alpha \equiv 0$ modulo $\pi/2$, are the only 0-homogeneous solutions of degree $d = 1$.
- The uniqueness statement in the first item of Theorem 1.1 implies that 0-homogenous solutions of degree $d \neq 1$ enjoy discrete symmetry properties: the map ξ_δ satisfies

$$\xi_\delta(e^{i\frac{\pi}{|d-1|}}x) = -e^{i\frac{\pi}{|d-1|}}\xi_\delta(x). \tag{6}$$

  Indeed, this symmetry constraint is compatible with the energy as noted in [21], and $u(re^{i\theta}) = e^{id\theta}$ satisfies (6) for any $d \in \mathbb{Z} \setminus \{1\}$, hence minimizing (1) among 0-homogeneous maps of degree d with this symmetry constraint produces one symmetric solution ξ_δ^{sym}. The symmetry (6) is preserved under frame invariance (2), so the one-dimensional family generated by ξ_δ^{sym} satisfies it, and by uniqueness it agrees with the one-dimensional family generated by ξ_δ in Theorem 1.1.
- The linear stability of ξ_δ can be used to show that it is a local minimizer among maps $u \in H^1(A_\rho; \mathbb{S}^1)$ agreeing with ξ_δ on ∂A_ρ, but the neighborhood in which it is a minimizer degenerates for small values of ρ, see Proposition 3.5. The critical value ρ_* in the third item of Theorem 1.1 satisfies $e^{-C|\delta|^{-1}} \leq \rho_*(\delta) \leq e^{-(C|\delta|)^{-\frac{1}{3}}}$ for a large constant $C > 0$ depending on the degree d, as can be inferred from (33) and Proposition 3.8.
- The degree 2 case is different: the unique family of 0-homogeneous solutions is given by $\xi_\delta^\alpha(re^{i\theta}) = e^{i\alpha}e^{2i\theta}$, and it is a minimizer in A_ρ for all $0 < \rho < 1$, see § 4.2.

Comparison with minimizing maps in higher dimensions. In dimension $n \geq 3$, tangent harmonic maps $\mathbb{R}^n \to \mathcal{N}$ with values into a riemannian manifold $\mathcal{N}$, that is, blow-up limits of $\mathcal{N}$-valued maps minimizing the isotropic energy $\int |\nabla u|^2$, are 0-homogeneous [25]. This is the key reason why minimizing harmonic maps are known to have a singular set of dimension at most $n - 3$, while optimal regularity estimates for minimizers of anisotropic energies are open [15, 23, 17]. Homogeneity of the isotropic tangent maps is due to the decoupling of the energy density into radial and angular derivatives:

$$|\nabla u|^2 = |\partial_r u|^2 + \frac{1}{r^2}|\nabla_\omega u|^2.$$

In our two-dimensional setting, this is the same decoupling which provides the lower bound (5) in the isotropic case $\delta = 0$. In the absence of such decoupling, it seems hard to determine whether tangent maps are 0-homogeneous. Since tangent maps are minimizers with respect to their own boundary conditions, one way to gain insight into that question is to investigate whether 0-homogeneous maps are minimizers. Our results, by showing in a particular two-dimensional case that anisotropy prevents many 0-homogeneous maps from being minimizers, therefore suggest that tangent maps for minimizers of anisotropic energies in dimension $n \geq 3$ might fail to be 0-homogeneous in some cases.

Consequences for the anisotropic Ginzburg-Landau energy. Energy-minimizing maps from an annulus (and more generally a domain with small holes) into $\mathbb{S}^1$ are strongly relevant to the analysis of the anisotropic Ginzburg Landau energy

$$GL_{\delta,\epsilon}(u;\Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{\delta}{2}\left((\nabla\cdot u)^2 - (\nabla\times u)^2\right) + \frac{1}{4\epsilon^2}(1-|u|^2)^2\,dx, \quad (7)$$

and the corresponding anisotropic Ginzburg-Landau equation

$$\mathcal{L}_\delta u = \frac{1}{\epsilon^2}(1-|u|^2)u,$$

for maps $u\colon \Omega \to \mathbb{R}^2$. The very different nature of defects of degree $d = -1$ versus $d = 1$ unveiled by Theorem 1.1 will have repercussions on a negative degree counterpart of the analysis performed in [10] for minimizers of $GL_{\delta,\epsilon}(\cdot;\Omega)$ with boundary data $g\colon \partial\Omega \to \mathbb{S}^1$ of positive degree.

The symmetry breaking demonstrated by Theorem 1.1 also has consequences on the far-field asymptotics ($r \to \infty$) of entire solutions $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ to the anisotropic Ginzburg-Landau equation, via the scaling argument of [26]. More specifically, in the entire plane $\mathbb{R}^2$ the length-scale ϵ can be set to $\epsilon = 1$, and we consider maps $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ which solve the anisotropic Ginzburg-Landau equation

$$\mathcal{L}_\delta u = (1-|u|^2)u \qquad \text{in } \mathbb{R}^2, \quad (8)$$

associated to the energy $GL_\delta = GL_{\delta,1}$ given by

$$GL_\delta(u;\Omega) = \int_\Omega \frac{1}{2}|\nabla u|^2 + \frac{\delta}{2}\left((\nabla\cdot u)^2 - (\nabla\times u)^2\right) + \frac{1}{4}(1-|u|^2)^2\,dx, \quad (9)$$

with finite potential energy

$$\int_{\mathbb{R}^2} (1-|u|^2)^2\,dx < \infty. \quad (10)$$

Such u has a well-defined degree $d = \deg(u) = \deg(u/|u|;\partial D_R) \in \mathbb{Z}$ for $R \gg 1$. In the isotropic case $\delta = 0$, solutions of any degree d can be constructed using a radial ansatz $u(re^{i\theta}) = f_d(r)e^{id\theta}$ [8, 16], and all solutions satisfy a quantization property for their potential energy [6]. This quantization is obtained as a consequence of a Pohozaev identity and far field asymptotics $u(re^{i\theta}) \to e^{id\theta}$ as $r \to \infty$ in an appropriate sense, entailing for instance

$$\int_{\mathbb{R}^2} |\partial_r u|^2\,dx < \infty.$$

In the anisotropic case $0<|\delta|<1$, and for degrees $d\notin\{0,1\}$, the mere existence of solutions is unknown in full generality. For small anisotropy $|\delta|\leq\delta_0(d)$ and negative degree $d\leq -1$, solutions were constructed in [21] via a minimization procedure under the discrete rotational symmetry constraint mentioned in Remark 1.2,

$$u(e^{i\frac{\pi}{|d-1|}}x)=-e^{i\frac{\pi}{|d-1|}}u(x)\qquad\forall x\in\mathbb{R}^2. \tag{11}$$

Large radius asymptotics in the spirit of [6] seem unexplored, apart from formal calculations for $d=-1$ and $|\delta|\ll 1$ in [9, § IV]. As a consequence of the third point in Theorem 1.1, we obtain that these asymptotics behave very differently from the isotropic case.

Corollary 1.3. *For any $d\in\mathbb{Z}\setminus\{0,1,2\}$ there exists $\delta_0\in(0,1)$ with the following property. Let $u\in H^1_{loc}(\mathbb{R}^2;\mathbb{R}^2)$ be a solution of the anisotropic Ginzburg-Landau equation* (8) *with finite potential energy* (10) *and degree* $\deg(u)=d$. *If $0<|\delta|<\delta_0$ and u is either locally minimizing:*

$$GL_\delta(u;D_R)\leq GL_\delta(v;D_R),\qquad \forall v\in H^1(D_R;\mathbb{R}^2),\ v_{\lfloor\partial D_R}=u_{\lfloor\partial D_R},$$

or symmetric (11) *and locally minimizing with respect to symmetric competitors, then we have*

$$\int_{\mathbb{R}^2}|\partial_r u|^2\,dx=+\infty,$$

and the maps $u_R\colon\mathbb{S}^1\to\mathbb{R}^2$ given by $u_R(\theta)=u(Re^{i\theta})$ do not converge as $R\to+\infty$ (in the sense of distributions).

Remark 1.4. Using the methods in [21, § 4], one can show that a locally minimizing solution must be of degree $d\in\{-1,0,1\}$, but existence of a locally minimizing solution of degree -1 is unknown. However, Corollary 1.3 applies to the symmetric solutions of degree $d\leq -1$ constructed in [21]. More precisely, the solutions constructed in [21] satisfy an additional mirror symmetry constraint $u(\bar{x})=\alpha\bar{u}(x)$ for some $\alpha\in\{\pm 1\}$, but the same proof provides existence of solutions which are locally minimizing under the symmetry constraint (11) only. (At the level of ξ_δ, the additional mirror symmetry only has the effect of selecting a value of $\xi_\delta(0)$ in $\{\pm 1\}$ or $\{\pm i\}$.) Moreover, it will be clear from the proof that Corollary 1.3 also applies to symmetric solutions which are locally minimizing under that additional mirror symmetry constraint.

Sketch of proof of Theorem 1.1. To prove Theorem 1.1, we start by showing that any 0-homogeneous solution ξ is linearly stable. We achieve this using identities satisfied by the Jacobi field $w=(d/d\alpha)|_{\alpha=0}[\xi^\alpha]$ generated by the symmetry (2), $\xi^\alpha(x)=e^{-i\alpha}\xi(e^{i\alpha}x)$. (The idea of proving stability via a Jacobi-field-based decomposition is classical, see e.g. [24, 19, 22, 20, 18].) When restricting the energy to 0-homogeneous maps, this linear stability implies local minimality of the solution ξ.

Since this is valid for any 0-homogeneous solution ξ, uniqueness follows: in the presence of two distinct (modulo frame invariance) solutions, a non-locally-minimizing solution could be obtained by a classical mountain pass argument and would provide a contradiction (an earlier implementation of this kind of argument in the context of Ginzburg-Landau can be found in [1]). In that way we obtain the first two items of Theorem 1.1.

For the third item, we wish to show that, for $|\delta| \ll 1$, the 0-homogeneous solution ξ_δ is minimizing in A_ρ if ρ is not too small, but not minimizing if $\rho \ll 1$. A formal expansion of the energy for small perturbations around ξ_δ gives quadratic terms that are positive thanks to the linear stability, and remainder terms which are formally of lower order. Estimating these remainder terms to absorb them into the positive quadratic terms proves local minimality of ξ_δ. This requires adequate interpolation inequalities, but the constants involved in these inequalities behave badly as $\rho \to 0$ and the neighborhood of local minimality becomes very small.

On the one hand, when ρ is not too small this is enough to deduce minimality, using the fact that for $|\delta| \ll 1$ any minimizer must be close to the isotropic minimizer $e^{id\theta}$ and belong therefore to the neighborhood of local minimality of ξ_δ.

On the other hand, for very small ρ, identifying near-optimizers for the interpolation inequalities provides a reasonable guess of a perturbation of ξ_δ which would produce negative remainder terms that cannot be compensated by the positive quadratic terms. In order to check that this reasonable guess actually works, we determine an expansion $\xi_\delta = e^{id\theta} + \delta\zeta_1 + \delta^2\zeta_2 + \mathcal{O}(\delta^3)$, and deduce an explicit expression of the bad part of the remainder terms. Choosing appropriate values for ρ and the amplitude of the perturbation then ensures that all non-explicit terms are controlled, and eventually produces a lower energy.

Plan of the article. In Section 2 we study 0-homogenous critical points and prove the first two items of Theorem 1.1. In Section 3 we prove the third item, namely that ξ_δ is minimizing for $\rho \approx 1$ but not minimizing for small ρ, when $|\delta| \ll 1$. In Section 4 we treat the particular cases $d \in \{1, 2\}$. In Section 5 we prove Corollary 1.3.

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2. Stability and uniqueness of 0-homogeneous critical points

In this section we study 0-homogeneous solutions of the Euler-Lagrange system (3) and prove the first two items of Theorem 1.1. With a slight abuse of notation, we identify a 0-homogeneous $\mathbb{S}^1$-valued map ξ with a map depending only on the polar angle θ:

$$\xi(re^{i\theta}) = \xi(\theta), \qquad \xi \in H^1(\mathbb{S}^1;\mathbb{S}^1).$$

In that context, the equation (3) can be rewritten as

$$\widehat{\mathcal{L}}_\delta \xi = \hat{\lambda}\xi, \qquad \hat{\lambda} = \xi \cdot \widehat{\mathcal{L}}_\delta \xi, \tag{12}$$

where the reduced linear operator $\widehat{\mathcal{L}}_\delta$ is given by

$$\widehat{\mathcal{L}}_\delta \xi = -\partial_\theta^2 \xi - \delta\, \partial_\theta \left[(\partial_\theta \xi \cdot ie^{i\theta}) ie^{i\theta} - (\partial_\theta \xi \cdot e^{i\theta}) e^{i\theta} \right]$$

Solutions of (12) correspond exactly to critical points of the reduced energy

$$\widehat{E}_\delta(\xi) = \int_{\mathbb{S}^1} |\xi'|^2 + \delta \left((\xi' \cdot ie^{i\theta})^2 - (\xi' \cdot e^{i\theta})^2 \right) d\theta. \tag{13}$$

There exists at least one solution of degree d, obtained by minimizing (13) among maps of degree d, since the degree is continuous under weak convergence in $H^1(\mathbb{S}^1;\mathbb{S}^1)$.

We start by proving that any solution $\xi \in H^1(\mathbb{S}^1;\mathbb{S}^1)$ of (12) with degree $d \neq 1$ generates, via frame invariance (2), a non-vanishing Jacobi field.

Lemma 2.1. *Let $|\delta| < 1$ and $\xi \in H^1(\mathbb{S}^1;\mathbb{S}^1)$ with degree $d \in \mathbb{Z} \setminus \{1\}$ solve the reduced equation* (12). *Then $\xi \in C^\infty(\mathbb{S}^1;\mathbb{S}^1)$, and the Jacobi field $w \in C^\infty(\mathbb{S}^1;\mathbb{R}^2)$ given by*

$$w(\theta) = \frac{1}{d-1} \left.\frac{d}{d\alpha}\right|_{\alpha=0} [\xi^\alpha(\theta)], \qquad \xi^\alpha(\theta) = e^{-i\alpha}\xi(\theta+\alpha),$$

satisfies $|w| > 0$ in $\mathbb{S}^1$.

Proof of Lemma 2.1. For any $\xi \in H^1(\mathbb{S}^1;\mathbb{S}^1)$ of degree d, there is a lifting $\varphi \in H^1(\mathbb{S}^1;\mathbb{R})$ such that

$$\xi(\theta) = e^{id\theta} e^{i\varphi(\theta)} \qquad \forall \theta \in \mathbb{R}.$$

In terms of this lifting the energy is of the form

$$F_\delta(\varphi) = \widehat{E}_\delta(e^{id\theta}e^{i\varphi}) = \int_{\mathbb{S}^1} \left(1 + \delta\cos(2(d-1)\theta + 2\varphi)\right)(d+\varphi')^2\, d\theta,$$

and the Euler-Lagrange equation (12) becomes

$$\begin{aligned} &\frac{d}{d\theta}\left[\left(1 + \delta\cos(2(d-1)\theta + 2\varphi)\right)(d+\varphi') \right] \\ &= -\delta \sin(2(d-1)\theta + 2\varphi)(d+\varphi')^2. \end{aligned} \tag{14}$$

This implies that $\varphi \in C^\infty(\mathbb{S}^1;\mathbb{R})$ (see e.g. [12, Theorem 4.36]) and therefore $\xi \in C^\infty(\mathbb{S}^1;\mathbb{S}^1)$ for any solution of (12). For any $\alpha \in \mathbb{R}$, we have

$$\xi^\alpha(\theta) = e^{id\theta} e^{iT_\alpha\varphi(\theta)}, \qquad T_\alpha\varphi(\theta) = \varphi(\theta+\alpha) + (d-1)\alpha.$$

In terms of φ, the Jacobi field w can be explicitly computed and is given by

$$w = \frac{\varphi' + d - 1}{d-1} i\xi.$$

To prove that w does not vanish, we show that φ' cannot take the value $(1-d)$. To that end, we first note that, for any $\psi_0 \in \mathbb{R}$, the functions

$$\psi(\theta) = \psi_0 - d\theta, \qquad \psi(\theta) = \psi_0 + (2-d)\theta,$$

are solutions of (14), as can be checked by a direct calculation. As a consequence, φ' cannot take the values $\{-d, 2-d\}$, unless it is constant: if $\varphi'(\theta_0) \in \{-d, 2-d\}$, then φ and one of the above solutions ψ have same value and derivative at θ_0, and are therefore equal by uniqueness of the Cauchy problem for the ODE (14). The cases where φ' is constant equal to $-d$ or $2-d$ can only occur if $d \in \{0,2\}$ since φ is periodic, and then w obviously does not vanish. Otherwise, we have on the one hand $\varphi'(\mathbb{R}) \subset \mathbb{R} \setminus \{-d, 2-d\}$, and on the other hand $0 \in \varphi'(\mathbb{R})$ because φ is periodic and smooth. If $d \leq 0$ we deduce $\varphi' \leq -d < 1-d$, and if $d \geq 2$ we deduce $\varphi' \geq 2-d > 1-d$. In both cases, this implies that w does not vanish. $\square$

Remark 2.2. Since $\alpha \mapsto T_\alpha\varphi(0)$ is surjective onto $\mathbb{R}$ (because φ is bounded and $d \neq 1$), we may always choose $\alpha \in \mathbb{R}$ such that $T_\alpha\varphi(0) = 0$, or equivalently $\xi^\alpha(0) = 1$.

Next we use the fact that w does not vanish, and that it solves the linearized equation

$$\widehat{\mathcal{L}}_\delta w - \hat\lambda w = \hat\mu\xi, \qquad \hat\mu = \widehat{\mathcal{L}}_\delta w \cdot \xi, \tag{15}$$

to prove that the homogeneous critical point ξ is linearly stable for the reduced energy $\widehat{E}_\delta$. This will be enough to deduce uniqueness modulo frame invariance (the first item of Theorem 1.1), and will serve as a warm-up to the proof of linear stability for the full energy E_δ (the second item of Theorem 1.1).

Lemma 2.3. *Let $|\delta| < 1$ and $\xi \in H^1(\mathbb{S}^1;\mathbb{S}^1)$ with degree $d \in \mathbb{Z} \setminus \{1\}$ solve the reduced equation* (12)*. Then for all $\varphi \in H^1(\mathbb{S}^1;\mathbb{R}^2)$ we have*

$$\frac{1}{2}\frac{d^2}{dt^2}\bigg|_{t=0} \widehat{E}_\delta\left(\frac{\xi + t\varphi}{|\xi + t\varphi|}\right) = \widehat{Q}_\xi[\varphi - (\xi\cdot\varphi)\xi],$$

$$\widehat{Q}_\xi[v] = \int_{\mathbb{S}^1} \Big(|v'|^2 + \delta\left((v'\cdot ie^{i\theta})^2 - (v'\cdot e^{i\theta})^2 \right) - \hat\lambda |v|^2 \Big)\, d\theta,$$

with $\hat\lambda = \widehat{\mathcal{L}}_\delta\xi\cdot\xi$ as in (12)*. For any tangent field $v \in H^1(\mathbb{S}^1;\mathbb{R}^2)$ with $v\cdot\xi = 0$ a.e., there is $f \in H^1(\mathbb{S}^1;\mathbb{R})$ such that $v = fw$, where w is the smooth Jacobi field* *generated by ξ as in Lemma 2.1, and $\widehat{Q}_\xi$ satisfies the coercivity inequality*

$$\widehat{Q}_\xi[v] = \widehat{Q}_\xi[fw] \geq (1-|\delta|)\int_{\mathbb{S}^1} |f'|^2\,|w|^2\,d\theta.$$

Proof of Lemma 2.3. First we establish the expression of $\widehat{Q}_\xi$. We start with a preliminary calculation which is also of independent interest. For any $u\in H^1(\mathbb{S}^1;\mathbb{S}^1)$, letting $v=u-\xi$ and integrating by parts we find

$$\begin{aligned}\widehat{E}_\delta(u)-\widehat{E}_\delta(\xi) &= \widehat{E}_\delta(\xi+v)-\widehat{E}_\delta(\xi)\\ &= \int_{\mathbb{S}^1}\Big(|v'|^2+\delta\left((v'\cdot ie^{i\theta})^2-(v'\cdot e^{i\theta})^2\right)+2\widehat{\mathcal{L}}_\delta\xi\cdot v\Big)\,d\theta.\end{aligned}$$

Using that ξ solves (12) this becomes

$$\widehat{E}_\delta(u)-\widehat{E}_\delta(\xi) = \int_{\mathbb{S}^1}\Big(|v'|^2+\delta\left((v'\cdot ie^{i\theta})^2-(v'\cdot e^{i\theta})^2\right)+2\hat{\lambda}\,\xi\cdot v\Big)\,d\theta.$$

And recalling that $1=|u|^2=|v|^2+2v\cdot\xi+1$, we rewrite the last term using $\xi\cdot v=-|v|^2/2$ and find

$$\widehat{E}_\delta(u)-\widehat{E}_\delta(\xi)=\widehat{Q}_\xi[u-\xi], \tag{16}$$

with $\widehat{Q}_\xi$ defined as in Lemma 2.3. Noting that $\|\varphi\|_\infty\leq C\|\varphi\|_{H^1}$ and applying this to

$$u=\frac{\xi+t\varphi}{|\xi+t\varphi|}=\xi+t\left[\varphi-(\xi\cdot\varphi)\xi\right]+t^2\psi_t,\quad \|\psi_t\|_{H^1}\leq C(\xi,\varphi),$$

for $t\leq 1/(2+\|\varphi\|_\infty)$, we deduce

$$\widehat{E}_\delta\left(\frac{\xi+t\varphi}{|\xi+t\varphi|}\right)=t^2\widehat{Q}_\xi\left[\varphi-(\xi\cdot\varphi)\xi\right]+\mathcal{O}(t^3),$$

which proves the claimed expression for the second derivative at $t=0$.

Next we prove the coercivity of $\widehat{Q}_\xi$. Let $v\in H^1(\mathbb{S}^1;\mathbb{R}^2)$ such that $v\cdot\xi=0$ a.e. Since the smooth Jacobi-field also takes values orthogonal to ξ and does not vanish, this implies $v(\theta)=f(\theta)w(\theta)$ for some real-valued $f(\theta)$ and a.e. $\theta\in\mathbb{S}^1$, and $f=|w|^{-2}v\cdot w\in H^1(\mathbb{S}^1;\mathbb{R})$. Integrating by parts we find

$$\widehat{Q}_\xi[fw]=\int_{\mathbb{S}^1}\Big[\widehat{\mathcal{L}}_\delta(fw)-\hat{\lambda}\,fw\Big]\cdot fw\,d\theta.$$

To simplify that expression we compute

$$\begin{aligned}\widehat{\mathcal{L}}_\delta(fw) &= f\widehat{\mathcal{L}}_\delta w - f''\,w-2f'\,w'\\ &\quad -\delta\frac{d}{d\theta}\left[f'((w\cdot ie^{i\theta})ie^{i\theta}-(w\cdot e^{i\theta})e^{i\theta})\right]\\ &\quad -\delta f'\left((w'\cdot ie^{i\theta})ie^{i\theta}-(w'\cdot e^{i\theta})e^{i\theta}\right),\end{aligned}$$

and deduce

$$\begin{aligned}\widehat{\mathcal{L}}_\delta(fw)\cdot fw &= f^2\widehat{\mathcal{L}}_\delta w\cdot w + (f')^2\left(|w|^2+\delta(w\cdot ie^{i\theta})^2-\delta(w\cdot e^{i\theta})^2\right)\\ &\quad -\frac{d}{d\theta}\left[ff'\left(|w|^2+\delta(w\cdot ie^{i\theta})^2-\delta(w\cdot e^{i\theta})^2\right)\right]\end{aligned}$$

Coming back to the expression of $\widehat{Q}_\xi$ we find

$$\begin{aligned}\widehat{Q}_\xi[fw] = \int_{\mathbb{S}^1}\Big[&f^2\left(\widehat{\mathcal{L}}_\delta w-\hat{\lambda}w\right)\cdot w\\ &+(f')^2\left(|w|^2+\delta(w\cdot ie^{i\theta})^2-\delta(w\cdot e^{i\theta})^2\right)\Big]\,d\theta.\end{aligned}$$

Finally we use the facts that the Jacobi field w solves the linearized equation (15) and $w\cdot\xi=0$ to simplify the above to

$$\widehat{Q}_\xi[fw] = \int_{\mathbb{S}^1}(f')^2\left(|w|^2+\delta(w\cdot ie^{i\theta})^2-\delta(w\cdot e^{i\theta})^2\right)\,d\theta.$$

The coercivity inequality of Lemma 2.3 then follows from the pointwise inequality $|w|^2+\delta(w\cdot ie^{i\theta})^2-\delta(w\cdot e^{i\theta})^2\geq(1-|\delta|)|w|^2$. ☐

The next step is to turn the linear stability proved in Lemma 2.3 into a local minimality statement.

Lemma 2.4. *Let* $|\delta|<1$ *and* $\xi\in H^1(\mathbb{S}^1;\mathbb{S}^1)$ *with degree* $d\in\mathbb{Z}\setminus\{1\}$ *solve the reduced equation* (12)*. There exist* $c,\eta>0$ *such that*

$$\widehat{E}_\delta(u)\geq\widehat{E}_\delta(\xi)+c\inf_{\alpha\in\mathbb{R}}\|u-\xi^\alpha\|_{H^1}^2,$$

for all $u\in H^1(\mathbb{S}^1;\mathbb{S}^1)$ *such that* $\inf_{\alpha\in\mathbb{R}}\|u-\xi^\alpha\|_{H^1}<\eta$.

Proof of Lemma 2.4. Let $u\in H^1(\mathbb{S}^1;\mathbb{S}^1)$. We assume without loss of generality (since the statement is invariant under application of the change of frame transformation (2) to ξ) that

$$\|u-\xi\|_{H^1}=\inf_{\alpha\in\mathbb{R}}\|u-\xi^\alpha\|_{H^1}<\eta, \tag{17}$$

with η to be chosen later. We define $v=u-\xi$ and write

$$v=fw+g\xi,\qquad f=\frac{v\cdot w}{|w|^2},\ g=v\cdot\xi\in H^1(\mathbb{S}^1;\mathbb{R}).$$

We first gather some estimates on f and g. The identity

$$1=|u|^2=|\xi+v|^2=1+2g+g^2+f^2|w|^2,$$

implies $g=-1\pm\sqrt{1-f^2|w|^2}$, where the sign $\pm$ may depend on θ. But by Sobolev embedding and the explicit expressions of f,g in terms of v we have

$$\|f\|_{L^\infty}+\|g\|_{L^\infty}\leq c\|v\|_{H^1}\leq c\eta,$$

for some generic constant $c>0$ depending on ξ. Hence choosing η small enough ensures that $g=\sqrt{1-f^2|w|^2}-1$ and $|g|\leq c|f|^2$. Combining this with Sobolev embedding for f we deduce

$$\|g\|_{L^\infty}+\|f\|_{L^\infty}^2\leq c\|f\|_{H^1}^2. \tag{18}$$

Further, minimality of $\alpha=0$ in (17) implies

$$\begin{aligned}0&=\int_{\mathbb{S}^1}(v\cdot w+v'\cdot w')\,d\theta\\&=\int_{\mathbb{S}^1}f(|w|^2+|w'|^2)\,d\theta+\int_{\mathbb{S}^1}(f'w\cdot w'+g'\xi\cdot w'+g\xi'\cdot w')\,d\theta,\end{aligned}$$

and combining this with the Poincaré inequality

$$\int_{\mathbb{S}^1}\phi^2\,d\theta\leq c\int_{\mathbb{S}^1}(\phi')^2\,d\theta\quad\text{ if }\int_{\mathbb{S}^1}\phi(|w|^2+|w'|^2)\,d\theta=0,$$

we infer

$$\|f\|_{H^1}^2\leq c\int_{\mathbb{S}^1}(f')^2\,d\theta+\int_{\mathbb{S}^1}(g')^2\,d\theta+\|g\|_\infty^2.$$

Estimating the last term with (18) yields

$$\|f\|_{H^1}^2\leq c\int_{\mathbb{S}^1}(f')^2\,d\theta+c\int_{\mathbb{S}^1}(g')^2\,d\theta+\|f\|_{H^1}^4.$$

Taking into account that $\|f\|_{H^1}\leq c\|v\|_{H^1}\leq c\eta$ and choosing η small enough, the last term can be absorbed into the left-hand side and we are left with

$$\|f\|_{H^1}^2\leq c\int_{\mathbb{S}^1}(f')^2\,d\theta+c\int_{\mathbb{S}^1}(g')^2\,d\theta.$$

And combining this with (18) leads to

$$\|g\|_{L^\infty}+\|f\|_{L^\infty}^2\leq c\int_{\mathbb{S}^1}(f')^2\,d\theta+c\int_{\mathbb{S}^1}(g')^2\,d\theta \tag{19}$$

Now we let $\widehat{B}_\xi$ denote the symmetric bilinear form on $H^1(\mathbb{S}^1;\mathbb{R}^2)$ associated to the quadratic form $\widehat{Q}_\xi$. Thanks to (16) we have

$$\widehat{E}_\delta(u)-\widehat{E}_\delta(\xi)=\widehat{Q}_\xi[v]=\widehat{Q}_\xi[fw]+2\widehat{B}_\xi[fw,g\xi]+\widehat{Q}_\xi[g\xi]. \tag{20}$$

Using the same calculations as in Lemma 2.3 we find

$$\begin{aligned}\widehat{Q}_\xi[g\xi]&=\int_{\mathbb{S}^1}\left[g^2\left(\widehat{\mathcal{L}}_\delta\xi-\hat{\lambda}\xi\right)\cdot\xi+(g')^2\left(|\xi|^2+\delta(\xi\cdot ie^{i\theta})^2-\delta(\xi\cdot e^{i\theta})^2\right)\right]\,d\theta\\&=\int_{\mathbb{S}^1}(g')^2\left(1+\delta(\xi\cdot ie^{i\theta})^2-\delta(\xi\cdot e^{i\theta})^2\right)\,d\theta.\end{aligned}$$

For the last equality we used $\widehat{\mathcal{L}}_\delta\xi = \hat\lambda\xi$ and $|\xi| = 1$. Plugging this and the expression of $\widehat{Q}_\xi[fw]$ obtained in Lemma 2.3 into (20) we find

$$\begin{aligned}\widehat{E}_\delta(u) - \widehat{E}_\delta(\xi) &= \int_{\mathbb{S}^1} (f')^2 \left(|w|^2 + \delta(w\cdot ie^{i\theta})^2 - \delta(w\cdot e^{i\theta})^2\right) d\theta \\ &\quad + \int_{\mathbb{S}^1} (g')^2 \left(1 + \delta(\xi\cdot ie^{i\theta})^2 - \delta(\xi\cdot e^{i\theta})^2\right) d\theta \\ &\quad + 2\widehat{B}_\xi[fw, g\xi]. \end{aligned} \tag{21}$$

The bilinear form $\widehat{B}_\xi$ is given by

$$\widehat{B}_\xi[v_1, v_2] = \int_{\mathbb{S}^1} \left[\widehat{\mathcal{L}}_\delta v_1\cdot v_2 - \hat\lambda\, v_1\cdot v_2\right] d\theta.$$

Applying it to $v_1 = fw$ and $v_2 = g\xi$, the last term disappears because $w\cdot\xi = 0$, and since

$$\begin{aligned}\widehat{\mathcal{L}}_\delta(fw) = f\widehat{\mathcal{L}}_\delta w - f''\,w - 2f'\,w' &\\ - \delta\frac{d}{d\theta}\left[f'((w\cdot ie^{i\theta})ie^{i\theta} - (w\cdot e^{i\theta})e^{i\theta})\right]&\\ - \delta f'\left((w'\cdot ie^{i\theta})ie^{i\theta} - (w'\cdot e^{i\theta})e^{i\theta}\right),&\end{aligned}$$

we find, using that $\widehat{\mathcal{L}}_\delta w = \hat\lambda w + \hat\mu\xi$ and $w\cdot\xi = 0$,

$$\begin{aligned}\widehat{\mathcal{L}}_\delta(fw)\cdot g\xi &= \hat\mu\, fg - 2f'g\,(w'\cdot\xi) \\ &\quad - \delta\frac{d}{d\theta}\left[gf'\left\{(w\cdot ie^{i\theta})(\xi\cdot ie^{i\theta}) - (w\cdot e^{i\theta})(\xi\cdot e^{i\theta})\right\}\right] \\ &\quad + \delta g'f'\left[(w\cdot ie^{i\theta})(\xi\cdot ie^{i\theta}) - (w\cdot e^{i\theta})(\xi\cdot e^{i\theta})\right] \\ &\quad + \delta gf'\left[(w\cdot ie^{i\theta})(\xi'\cdot ie^{i\theta}) - (w\cdot e^{i\theta})(\xi'\cdot e^{i\theta})\right] \\ &\quad - \delta gf'\left[(w'\cdot ie^{i\theta})(\xi\cdot ie^{i\theta}) - (w'\cdot e^{i\theta})(\xi\cdot e^{i\theta})\right], \end{aligned} \tag{22}$$

and eventually

$$\begin{aligned}\widehat{B}_\xi[fw, g\xi] &= \int_{\mathbb{S}^1} \left(\hat\mu\, fg - 2f'g\,(w'\cdot\xi)\right) d\theta \\ &\quad + \delta\int_{\mathbb{S}^1} \Big(g'f'\left[(w\cdot ie^{i\theta})(\xi\cdot ie^{i\theta}) - (w\cdot e^{i\theta})(\xi\cdot e^{i\theta})\right] \\ &\qquad\quad + gf'\left[(w\cdot ie^{i\theta})(\xi'\cdot ie^{i\theta}) - (w\cdot e^{i\theta})(\xi'\cdot e^{i\theta})\right] \\ &\qquad\quad - gf'\left[(w'\cdot ie^{i\theta})(\xi\cdot ie^{i\theta}) - (w'\cdot e^{i\theta})(\xi\cdot e^{i\theta})\right]\Big)\, d\theta. \end{aligned}$$

Plugging this into (21) we obtain

$$\begin{aligned}\widehat{E}_\delta(u) - \widehat{E}_\delta(\xi) = &\int_{\mathbb{S}^1} \left[(f')^2|w|^2 + (g')^2 + 2\hat{\mu}\, fg - 4f'g\,(w'\cdot\xi)\right]\, d\theta \\ &+ \delta \int_{\mathbb{S}^1} \Big((f')^2 \left((w\cdot ie^{i\theta})^2 - (w\cdot e^{i\theta})^2\right) \\ &\qquad + (g')^2 \left((\xi\cdot ie^{i\theta})^2 - (\xi\cdot e^{i\theta})^2\right) \\ &\qquad + 2g'f' \left[(w\cdot ie^{i\theta})(\xi\cdot ie^{i\theta}) - (w\cdot e^{i\theta})(\xi\cdot e^{i\theta})\right] \Big)\, d\theta \\ &+ 2\delta \int_{\mathbb{S}^1} gf' \Big((w\cdot ie^{i\theta})(\xi'\cdot ie^{i\theta}) - (w\cdot e^{i\theta})(\xi'\cdot e^{i\theta}) \\ &\qquad\qquad - (w'\cdot ie^{i\theta})(\xi\cdot ie^{i\theta}) + (w'\cdot e^{i\theta})(\xi\cdot e^{i\theta}) \Big)\, d\theta. \end{aligned} \tag{23}$$

The integrand in the second integral is of the form $A(f'|w|,g)\cdot(f'|w|,g)$, with a symmetric matrix A given by

$$\begin{aligned} &A = \begin{pmatrix} a_1^2 - a_2^2 & a_1b_1 - a_2b_2 \\ a_1b_1 - a_2b_2 & b_1^2 - b_2^2 \end{pmatrix}, \\ &a_1 = \frac{w}{|w|}\cdot ie^{i\theta},\ a_2 = \frac{w}{|w|}\cdot e^{i\theta},\ b_1 = \xi\cdot ie^{i\theta},\ b_2 = \xi\cdot e^{i\theta}. \end{aligned}$$

The vectors $a=(a_1,a_2)$, $b=(b_1,b_2)$ satisfy $|a|=|b|=1$ and $a\cdot b=0$, so writing $a=e^{i\alpha}$, $b=e^{i\beta}$ with $\beta=\alpha+\pi/2 \bmod \pi$, we find

$$A = \begin{pmatrix} \cos(2\alpha) & \cos(\alpha+\beta) \\ \cos(\alpha+\beta) & \cos(2\beta) \end{pmatrix} = \begin{pmatrix} \cos(2\alpha) & \pm\sin(2\alpha) \\ \pm\sin(2\alpha) & -\cos(2\alpha) \end{pmatrix},$$

hence $\det A = -1$, $\operatorname{tr} A = 0$ and A has eigenvalues ± 1. This implies that the integrand in the second integral of (23) has absolute value $\leq (f')^2|w|^2 + (g')^2$, and we deduce

$$\begin{aligned}\widehat{E}_\delta(u) - \widehat{E}_\delta(\xi) \geq &(1-|\delta|)\int_{\mathbb{S}^1} \left[(f')^2|w|^2 + (g')^2\right]\, d\theta \\ &- c_1 \int_{\mathbb{S}^1} |fg|\, d\theta - c_2 \int_{\mathbb{S}^1} |f'g|\, d\theta. \end{aligned}$$

with $c_1 = 2\|\hat{\mu}\|_\infty$ and $c_2 = 4(2\|w'\|_\infty + \|w\|_\infty\|\xi'\|_\infty)$. Recalling (19) we deduce

$$\begin{aligned}\widehat{E}_\delta(u) - \widehat{E}_\delta(\xi) \geq &(1-|\delta|)\int_{\mathbb{S}^1} \left[(f')^2|w|^2 + (g')^2\right]\, d\theta \\ &- c\left(\int_{\mathbb{S}^1} (f')^2\, d\theta\right)^{\frac{3}{2}} - c\left(\int_{\mathbb{S}^1} (g')^2\, d\theta\right)^{\frac{3}{2}} \end{aligned}$$

We deduce from this that

$$\widehat{E}_\delta(u) - \widehat{E}_\delta(\xi) \geq (1-|\delta|)\frac{\min(1, \inf |w|^2)}{2}\int_{\mathbb{S}^1} \left[(f')^2 + (g')^2\right]\, d\theta,$$

if $\|(f,g)\|_{H^1} \leq \|v\|_{H^1}$ is small enough (depending on ξ and δ). Finally we remark that $\|u-\xi\|_{H^1} = \|v\|_{H^1} \leq c\|(f,g)\|_{H^1}$, and using again (18) we have $\|(f,g)\|_{H^1} \leq c\|f'\|_{L^2} + c\|g'\|_{L^2}$, so $\widehat{E}_\delta(u) - \widehat{E}_\delta(\xi) \geq c\|u-\xi\|_{H^1}$ and this concludes the proof of Lemma 2.4. □

Remark 2.5. The constants c, η in Lemma 2.4 depend only on $M, m > 0$ such that $\|\xi\|_{C^2} \leq M$ and $|w| \geq m$, as can be checked directly from the proof. Since ξ solves (12), its C^2 norm is controlled by its H^1 norm. Moreover, the lower bound $|w| \geq m > 0$ is uniform among solutions ξ of (12) of degree $d \neq 1$ with bounded H^1 norm: otherwise one could find a sequence of solutions ξ_k bounded in H^1, hence in C^2, such that $\inf |w_k| \to 0$, and extracting a converging sequence in C^1 would produce a solution ξ with $\inf |w| = 0$, in contradiction with Lemma 2.1. Therefore the constants c, η in Lemma 2.4 depend only on $M > 0$ such that $\|\xi\|_{H^1} \leq M$.

Now we use all the preceding lemmas and a mountain pass argument to prove the first item of Theorem 1.1, namely uniqueness of 0-homogeneous critical points, modulo frame invariance (2).

Proposition 2.6. *Let $|\delta| < 1$ and $d \in \mathbb{Z}\setminus\{1\}$. If $\xi, \zeta \in H^1(\mathbb{S}^1; \mathbb{S}^1)$ are two solutions of* (12)*, then there exists $\alpha \in \mathbb{R}$ such that $\zeta = \xi^\alpha$.*

Proof of Proposition 2.6. First note that $H^1(\mathbb{S}^1;\mathbb{S}^1)$ is a smooth Hilbert submanifold of $H^1(\mathbb{S}^1;\mathbb{R}^2)$. This can be checked e.g. by noting that for any $\xi \in H^1(\mathbb{S}^1;\mathbb{S}^1)$, restricting the map $H^1(\mathbb{S}^1;\mathbb{R}) \ni \varphi \mapsto \xi e^{i\varphi}$ to a small neighborhood of 0 provides a smooth parametrization of a neighborhood of ξ in $H^1(\mathbb{S}^1;\mathbb{S}^1)$. In particular, $H^1(\mathbb{S}^1;\mathbb{S}^1)$ is a complete smooth Finsler manifold, see [27, § II.3.7]. Moreover, it can be checked rather directly that the energy $\widehat{E}_\delta$ is C^1 on $H^1(\mathbb{S}^1;\mathbb{S}^1)$ and satisfies the Palais-Smale condition [27, § II.2.], so that the deformation Lemma [27, § II.3, Theorem 3.11] is valid.

Assume now by contradiction that there are two solutions ξ_1, ξ_2 of (12) such that

$$\inf_{\alpha,\beta\in\mathbb{R}} \|\xi_1^\beta - \xi_2^\alpha\|_{H^1} > 0.$$

Thanks to Lemma 2.4, we know there are constants $c, \eta > 0$ such that, for $j = 1, 2$,

$$\widehat{E}_\delta(u) \geq \widehat{E}_\delta(\xi_j) + c\inf_{\alpha\in\mathbb{R}} \|u - \xi_j^\alpha\|_{H^1}^2, \tag{24}$$

for all $u \in H^1(\mathbb{S}^1;\mathbb{S}^1)$ such that $\inf_{\alpha\in\mathbb{R}} \|u - \xi_j^\alpha\|_{H^1} < \eta$. Choosing η small enough we may moreover assume that

$$\inf_{\alpha,\beta\in\mathbb{R}} \|\xi_1^\beta - \xi_2^\alpha\|_{H^1} > 2\eta.$$

Therefore, defining

$$P = \{p \in C^0([0,1]; H^1(\mathbb{S}^1;\mathbb{S}^1)) : p(0) = \xi_2, p(1) = \xi_2\},$$

any path $p \in P$ must intersect the sets of maps u such that $\inf_{\alpha\in\mathbb{R}} \|u - \xi_j^\alpha\|_{H^1} = \eta$, for $j = 1, 2$, and from (24) we deduce

$$\max_{u\in p} \widehat{E}_\delta(u) \geq \max_{j=1,2} \widehat{E}_\delta(\xi_j) + \bar{\epsilon},$$

for $\bar{\epsilon} = c\eta^2 > 0$ and all $p \in P$. Then a standard application of the deformation lemma gives that

$$\beta = \inf_{p\in P} \max_{u\in p} \widehat{E}_\delta(u),$$

is a critical value. Assume indeed that β is not a critical value. Since $\beta \geq \widehat{E}_\delta(\xi_j) + \bar{\epsilon}$ for $j = 1, 2$, the deformation lemma [27, § II.3, Theorem 3.11] then provides $\epsilon \in (0, \bar{\epsilon})$ and a family $\{\Phi(\cdot, t)\}_{t\geq 0}$ of continuous maps of $H^1(\mathbb{S}^1;\mathbb{S}^1)$ into itself, such that $\Phi(\xi_j, 1) = \xi_j$ for $j = 1, 2$, and $\Phi(\cdot, 1)$ maps the level set $\{\widehat{E}_\delta < \beta + \epsilon\}$ into $\{\widehat{E}_\delta < \beta - \epsilon\}$. By definition of β there exists $p \in P$ such that $\widehat{E}_\delta(u) < \beta + \epsilon$ for all $u \in p$, but then $\tilde{p} = \Phi(p, 1) \in P$ satisfies $\widehat{E}_\delta(\tilde{u}) < \beta - \epsilon$ for all $\tilde{u} \in \tilde{p}$, contradicting the definition of β.

Finally we show that the fact that β is a critical value contradicts the local minimality of all critical points established in Lemma 2.4. Let $K_\beta \subset H^1(\mathbb{S}^1;\mathbb{S}^1)$ denote the set of all critical points ξ with $\widehat{E}_\delta(\xi) = \beta$. Since K_β is bounded, we infer that there exist uniform constants $c, \eta > 0$ such that the conclusion of Lemma 2.4 is valid for all $\xi \in K_\beta$, see Remark 2.5. As a consequence, any path $p \in P$ such that $\mathrm{dist}_{H^1}(p, K_\beta) < \eta$ must satisfy $\max_{u\in p} \widehat{E}_\delta(u) \geq \beta + c\eta^2$, and we deduce that the infimum defining β can be taken over paths $p \in P$ such that $\mathrm{dist}_{H^1}(p, K_\beta) \geq \eta$.

Applying the deformation lemma again provides $\epsilon > 0$ and a family $\{\Phi(\cdot, t)\}_{t\geq 0}$ of continuous maps of $H^1(\mathbb{S}^1;\mathbb{S}^1)$ into itself, such that $\Phi(\xi_j, 1) = \xi_j$ for $j = 1, 2$, and $\Phi(\cdot, 1)$ maps the level set $\{\widehat{E}_\delta < \beta + \epsilon\}$ deprived of the neigborhood $N = \{\mathrm{dist}_{H^1}(\cdot, K_\beta) < \eta\}$ of K_β, into $\{\widehat{E}_\delta < \beta - \epsilon\}$. By the above, there exists $p \in P$ such that $\mathrm{dist}_{H^1}(p, K_\beta) \geq \eta$ and $\max_p \widehat{E}_\delta < \beta + \epsilon$, but then the path $\tilde{p} = \Phi(p, 1) \in P$ satisfies $\max_{\tilde{p}} \widehat{E}_\delta < \beta$. This contradicts the definition of β, and concludes the proof of Proposition 2.6. □

Finally we show that ξ is linearly stable not only with respect to 0-homogeneous perturbations (Lemma 2.3), but also with respect to all compactly supported perturbations in ∂A_ρ, as claimed in the second item of Theorem 1.1

Proposition 2.7. *Let $|\delta|<1$, $d\in\mathbb{Z}\setminus\{1\}$ and $\rho\in(0,1)$. Let $\xi\in H^1(\mathbb{S}^1;\mathbb{S}^1)$ with $\deg(\xi)=d$ solve the reduced equation* (12)*. Then for all $\varphi\in C^1_c(\mathbb{S}^1,\mathbb{R}^2)$ we have*

$$\begin{aligned}
&\frac{1}{2}\,\frac{d^2}{dt^2}\bigg|_{t=0} E_\delta\left(\frac{\xi+t\varphi}{|\xi+t\varphi|};A_\rho\right)=Q_\xi[\varphi-(\xi\cdot\varphi)\xi],\\
&Q_\xi[v]=\int_{A_\rho}\left(|\nabla v|^2+\delta\left((\nabla\cdot v)^2-(\nabla\times v)^2\right)-\frac{\hat\lambda}{r^2}|v|^2\right)dx,
\end{aligned}$$

with $\hat\lambda(\theta)=\widehat{\mathcal{L}}_\delta\xi\cdot\xi$ as in (12)*. For any tangent field $v\in H^1_0(A_\rho;\mathbb{R}^2)$ with $v\cdot\xi=0$ a.e., there is $f\in H^1_0(A_\rho;\mathbb{R})$ such that $v=fw$, where w is the smooth Jacobi field generated by ξ as in Lemma 2.1, and Q_ξ satisfies the coercivity inequality*

$$Q_\xi[v]=Q_\xi[fw]\geq(1-|\delta|)\int_{A_\rho}|\nabla f|^2|w|^2\,dx.$$

Proof of Proposition 2.7. The argument is very similar to the proof of Lemma 2.3, we only need to deal with additional radial derivative terms in the energy. Identifying as above $\xi(\theta)=\xi(re^{i\theta})$ with a function on A_ρ, we see that ξ solves the full Euler-Lagrange equation (3) with $\lambda=\hat\lambda/r^2$. For any $u\in H^1(A_\rho;\mathbb{S}^1)$ such that $u=\xi$ on ∂A_ρ, letting $v=u-\xi\in H^1_0(A_\rho;\mathbb{R}^2)$ we find, integrating by parts as in the proof of Lemma 2.3,

$$\begin{aligned}
&E_\delta(u;A_\rho)-E_\delta(\xi;A_\rho)\\
&=\int_{A_\rho}\left(|\nabla v|^2+\delta\left((\nabla\cdot v)^2-(\nabla\times v)^2\right)+2\,\mathcal{L}_\delta\xi\cdot v\right)dx,
\end{aligned}$$

and using $\mathcal{L}_\delta\xi=(\hat\lambda/r^2)\xi$ and $\xi\cdot v=-|v|^2/2$ we obtain

$$E_\delta(u;A_\rho)-E_\delta(\xi;A_\rho)=Q_\xi[u-\xi], \tag{25}$$

with Q_ξ as in Proposition 2.7. Applying this to

$$u=\frac{\xi+t\varphi}{|\xi+t\varphi|}=\xi+t\,[\varphi-(\xi\cdot\varphi)\xi]+t^2\psi_t,\quad \|\psi_t\|_{H^1}\leq C(\xi,\varphi),$$

for $t\leq 1/(2+\|\varphi\|_\infty)$, we deduce

$$E_\delta\left(\frac{\xi+t\varphi}{|\xi+t\varphi|};A_\rho\right)=t^2Q_\xi\,[\varphi-(\xi\cdot\varphi)\xi]+\mathcal{O}(t^3),$$

which proves the claimed expression for the second derivative at $t=0$.

Using polar coordinates, we rewrite Q_ξ as

$$\begin{aligned}
Q_\xi[v] &= \int_{A_\rho} \Big[|\partial_r v|^2 + |\partial_\theta v|^2 - \frac{\hat\lambda}{r^2}|v|^2 \\
&\qquad + \delta\big(\partial_r v\cdot e^{i\theta} + \frac{1}{r}\partial_\theta v\cdot ie^{i\theta}\big)^2 - \delta\big(\partial_r v\cdot ie^{i\theta} - \frac{1}{r}\partial_\theta v\cdot e^{i\theta}\big)^2\Big]\, dx \\
&= \int_{A_\rho} \Big[|\partial_r v|^2 + \delta(\partial_r v\cdot e^{i\theta})^2 - \delta(\partial_r v\cdot ie^{i\theta})^2 + \frac{1}{r^2}\big[\widehat{\mathcal{L}}_\delta v - \hat\lambda v\big]\cdot v \\
&\qquad + \frac{2\delta}{r}\big[(\partial_r v\cdot e^{i\theta})(\partial_\theta v\cdot ie^{i\theta}) + (\partial_r v\cdot ie^{i\theta})(\partial_\theta v\cdot e^{i\theta})\big]\Big]\, dx
\end{aligned}$$

Hence, for a function $f\in C^2_c(A_\rho;\mathbb{R})$ we have the explicit expression

$$\begin{aligned}
Q_\xi[fw] = \int_{A_\rho} \Big[&(\partial_r f)^2\left(|w|^2 + \delta(w\cdot e^{i\theta})^2 - \delta(w\cdot ie^{i\theta})^2\right) \\
&+ \frac{1}{r^2}\Big[\widehat{\mathcal{L}}_\delta(fw) - \hat\lambda fw\Big]\cdot fw \\
&+ 4\frac{\delta}{r}\partial_\theta f\partial_r f\,(w\cdot e^{i\theta})(w\cdot ie^{i\theta}) \\
&+ 2\frac{\delta}{r} f\partial_r f\ \left\{(w\cdot e^{i\theta})(w'\cdot ie^{i\theta}) + (w\cdot ie^{i\theta})(w'\cdot e^{i\theta})\right\}\Big]\, dx
\end{aligned}$$

To simplify the second line we compute, exactly as in the proof of Lemma 2.3,

$$\begin{aligned}
\widehat{\mathcal{L}}_\delta(fw)\cdot fw = f^2\widehat{\mathcal{L}}_\delta w\cdot w &\\
&+ (\partial_\theta f)^2\left(|w|^2 + \delta(w\cdot ie^{i\theta})^2 - \delta(w\cdot e^{i\theta})^2\right) \\
&- \partial_\theta\left[f\partial_\theta f\left(|w|^2 + \delta(w\cdot ie^{i\theta})^2 - \delta(w\cdot e^{i\theta})^2\right)\right].
\end{aligned}$$

Using the equation (15) satisfied by w to simplify the first term, and coming back to the expression of $Q_\xi[fw]$ we find

$$\begin{aligned}
Q_\xi[fw] = \int_{A_\rho} \Big[&(\partial_r f)^2\left(|w|^2 + \delta(w\cdot e^{i\theta})^2 - \delta(w\cdot ie^{i\theta})^2\right) \\
&+ \frac{(\partial_\theta f)^2}{r^2}\left(|w|^2 + \delta(w\cdot ie^{i\theta})^2 - \delta(w\cdot e^{i\theta})^2\right) \\
&+ 4\frac{\delta}{r}\partial_\theta f\partial_r f\,(w\cdot e^{i\theta})(w\cdot ie^{i\theta}) \\
&+ \frac{\delta}{r}\partial_r(f^2)\ \left\{(w\cdot e^{i\theta})(w'\cdot ie^{i\theta}) + (w\cdot ie^{i\theta})(w'\cdot e^{i\theta})\right\}\Big]\, dx.
\end{aligned}$$

Since $dx = r\,dr\,d\theta$ and $f \in C^2_c(A_\rho;\mathbb{R})$, the last line can be integrated out with respect to r, and we are left with

$$\begin{aligned}Q_\xi[fw] = \int_{A_\rho} \Big[&(\partial_r f)^2 \left(|w|^2 + \delta(w\cdot e^{i\theta})^2 - \delta(w\cdot ie^{i\theta})^2\right)\\ &+ \frac{(\partial_\theta f)^2}{r^2}\left(|w|^2 + \delta(w\cdot ie^{i\theta})^2 - \delta(w\cdot e^{i\theta})^2\right)\\ &+ 4\delta\,\partial_r f\frac{\partial_\theta f}{r}(w\cdot e^{i\theta})(w\cdot ie^{i\theta})\Big]\,dx.\end{aligned}$$

If $\delta \geq 0$ we use the inequality

$$4\delta\,\partial_r f\frac{\partial_\theta f}{r}(w\cdot e^{i\theta})(w\cdot ie^{i\theta}) \geq -2\delta(\partial_r f)^2(w\cdot e^{i\theta})^2 - 2\delta\frac{(\partial_\theta f)^2}{r^2}(w\cdot ie^{i\theta})^2,$$

to deduce that the integrand is bounded below by $(1-\delta)|\nabla f|^2|w|^2$, and if $\delta < 0$

$$4\delta\,\partial_r f\frac{\partial_\theta f}{r}(w\cdot e^{i\theta})(w\cdot ie^{i\theta}) \geq 2\delta(\partial_r f)^2(w\cdot ie^{i\theta})^2 + 2\delta\frac{(\partial_\theta f)^2}{r^2}(w\cdot e^{i\theta})^2,$$

so the integrand is bounded below by $(1+\delta)|\nabla f|^2|w|^2$. In both cases we obtain

$$Q_\xi[fw] \geq (1-|\delta|)\int_{A_\rho}|\nabla f|^2|w|^2\,dx,$$

for all $f \in C^2_c(A_\rho;\mathbb{R})$, and by density for all $f \in H^1_0(A_\rho;\mathbb{R})$. $\square$

As a consequence of Proposition 2.7, we obtain the second item of Theorem 1.1 by a contradiction argument: otherwise, there exists a sequence $\varphi_k \in C^1_c(A_\rho;\mathbb{R}^2)$ such that

$$\varphi_k\cdot\xi_\delta = 0 \text{ a.e.},\qquad Q_\xi[\varphi_k]\to 0,\qquad \int_{A_\rho}|\nabla\varphi_k|^2\,dx = 1.$$

We may extract a subsequence $\varphi_k\to\varphi$ strongly in L^2. We have $\varphi\cdot\xi_\delta = 0$ a.e., and using the estimate of Proposition 2.7 together with $Q_\xi[\varphi_k]\to 0$ we see that $\varphi = 0$. Considering that

$$Q_\xi[\varphi_k] \geq c_1\int_{A_\rho}|\nabla\varphi_k|^2\,dx - c_2\int_{A_\rho}|\varphi_k|^2\,dx = c_1 - c_2\int_{A_\rho}|\varphi_k|^2\,dx,$$

for some $c_1, c_2 > 0$ depending on δ and ρ, the fact that $\varphi_k\to 0$ strongly in L^2 gives the contradiction $0 = \lim Q_\xi[\varphi_k] \geq c_1 > 0$.

3. Small anisotropy: minimality in thin annuli and symmetry breaking

In this section we prove the third item of Theorem 1.1, valid for $0 < |\delta| \ll 1$: the unique (modulo frame invariance) 0-homogeneous critical point ξ_δ of degree $d \in \mathbb{Z}\setminus\{0,1,2\}$ is minimizing in a thin annulus, but it loses this minimality property in a very thick annulus.

3.1. **Preliminaries.** Two preliminary ingredients are common to the proofs of both statements: an energy splitting formula and expansions of ξ_δ and related quantities in terms of powers of δ. We dedicate the next two subsections to these tasks.

3.1.1. *Energy splitting.* A first ingredient, common to the proofs of both (minimality and symmetry breaking) statements, is a general energy splitting formula with respect to a 0-homogeneous critical point ξ_δ.

Lemma 3.1. *Let $d \in \mathbb{Z}\setminus\{1\}$, $0<|\delta|<1$ and ξ_δ of degree d solve* (12). *Denote by w_δ the corresponding Jacobi field defined in Lemma 2.1. For any $u \in H^1(A_\rho;\mathbb{S}^1)$ such that $u=\xi_\delta$ on ∂A_ρ, writing*

$$u = f w_\delta + (1+g)\xi_\delta, \qquad f,g \in H_0^1(A_\rho;\mathbb{R}),$$

we have

$$\begin{aligned} E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho) &= (1+\mathcal{O}(\delta))\int_{A_\rho}\left(|w_\delta|^2|\nabla f|^2 + |\nabla g|^2\right)dx \\ &\quad + 2\delta\int_{A_\rho}\left(\frac{\alpha_\delta(\theta)}{r^2}fg + \frac{\beta_\delta(\theta)}{r^2}g\partial_\theta f + \frac{\gamma_\delta(\theta)}{r}g\partial_r f\right)dx, \end{aligned}$$

where α_δ, β_δ and γ_δ are given by

$$\begin{aligned} \alpha_\delta &= \frac{\hat\mu_\delta - 2d\,\partial_\theta[|w_\delta|]}{\delta}, \\ \beta_\delta &= \frac{-2(w_\delta'\cdot\xi_\delta + d) + 2d(1-|w_\delta|)}{\delta} \\ &\quad + (w_\delta\cdot ie^{i\theta})(\xi_\delta'\cdot ie^{i\theta}) - (w_\delta\cdot e^{i\theta})(\xi_\delta'\cdot e^{i\theta}) \\ &\quad + (w_\delta'\cdot e^{i\theta})(\xi_\delta\cdot e^{i\theta}) - (w_\delta'\cdot ie^{i\theta})(\xi_\delta\cdot ie^{i\theta}), \\ \gamma_\delta &= (w_\delta\cdot e^{i\theta})(\xi_\delta'\cdot ie^{i\theta}) + (w_\delta\cdot ie^{i\theta})(\xi_\delta'\cdot e^{i\theta}) \\ &\quad - (w_\delta'\cdot ie^{i\theta})(\xi_\delta\cdot e^{i\theta}) - (w_\delta'\cdot e^{i\theta})(\xi_\delta\cdot ie^{i\theta}), \end{aligned}$$

and $\hat\mu_\delta = \widehat{\mathcal{L}}_\delta w_\delta\cdot\xi_\delta$ as in (15).

Proof of Lemma 3.1. According to (25) we have

$$E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho) = Q[fw_\delta + g\xi_\delta],$$

where the quadratic form $Q = Q_{\xi_\delta}$ is defined in Proposition 2.7. We expand this expression as

$$E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho) = Q[fw_\delta] + Q[g\xi_\delta] + 2B[fw_\delta, g\xi_\delta], \tag{26}$$

where B is the symmetric bilinear form associated to Q. We first deal with the first two terms in the right-hand side of (26), using computations similar to the proof of Proposition 2.7. We will consider without loss of generality a map $u \in C^2(A_\rho;\mathbb{S}^1)$ such that $u-\xi_\delta$ has compact support in A_ρ, and therefore functions $f,g \in C_c^2(A_\rho;\mathbb{R})$.

The general case follows by approximation: since $u\bar{\xi}_\delta = 1$ on ∂A_ρ, one can find a lifting $\varphi \in H_0^1(A_\rho;\mathbb{R})$ such that $u\bar{\xi}_\delta = e^{i\varphi}$ (this follows e.g. from [5, Appendix] or [7, Proposition 14.1]), and approximate φ with functions in $C_c^2(A_\rho)$.

For any function $h \in C_c^2(A_\rho;\mathbb{R})$ and map $\zeta = \zeta(\theta) \in C^2(\mathbb{S}^1;\mathbb{R}^2)$, we have, using polar coordinates as in the proof of Proposition 2.7,

$$
\begin{aligned}
Q[h\zeta] = \int_{A_\rho} \Big[& (\partial_r h)^2 \left(|\zeta|^2 + \delta(\zeta\cdot e^{i\theta})^2 - \delta(\zeta \cdot ie^{i\theta})^2\right) \\
& + \frac{1}{r^2}\left[\widehat{\mathcal{L}}_\delta(h\zeta) - \hat{\lambda}_\delta h\zeta\right]\cdot h\zeta \\
& + 4\frac{\delta}{r}\partial_\theta h \partial_r h\,(\zeta\cdot e^{i\theta})(\zeta\cdot ie^{i\theta}) \\
& + \frac{\delta}{r}\partial_r(h^2)\left\{(\zeta\cdot e^{i\theta})(\zeta'\cdot ie^{i\theta}) + (\zeta\cdot ie^{i\theta})(\zeta'\cdot e^{i\theta})\right\}\Big]\, dx,
\end{aligned}
$$

where $\hat{\lambda}_\delta = \widehat{\mathcal{L}}_\delta\xi_\delta\cdot\xi_\delta$. As in the proof of Proposition 2.7, the last line can be integrated with respect to r since $dx = r dr d\theta$ and $h \in C_c^2(A_\rho;\mathbb{R})$, and the second line can be simplified by computing

$$
\begin{aligned}
\widehat{\mathcal{L}}_\delta(h\zeta)\cdot h\zeta = h^2 & \widehat{\mathcal{L}}_\delta\zeta\cdot\zeta \\
& + (\partial_\theta h)^2\left(|\zeta|^2 + \delta(\zeta\cdot ie^{i\theta})^2 - \delta(\zeta\cdot e^{i\theta})^2\right) \\
& - \partial_\theta\left[h\partial_\theta h\left(|\zeta|^2 + \delta(\zeta\cdot ie^{i\theta})^2 - \delta(\zeta\cdot e^{i\theta})^2\right)\right],
\end{aligned}
$$

and we deduce

$$
\begin{aligned}
Q[h\zeta] = \int_{A_\rho} \Big[& (\partial_r h)^2 \left(|\zeta|^2 + \delta(\zeta\cdot e^{i\theta})^2 - \delta(\zeta \cdot ie^{i\theta})^2\right) \\
& + \frac{(\partial_\theta h)^2}{r^2}\left(|\zeta|^2 + \delta(\zeta\cdot ie^{i\theta})^2 - \delta(\zeta\cdot e^{i\theta})^2\right) \\
& + \frac{h^2}{r^2}\left[\widehat{\mathcal{L}}_\delta\zeta - \hat{\lambda}_\delta\zeta\right]\cdot\zeta \\
& + 4\frac{\delta}{r}\partial_\theta h\partial_r h\,(\zeta\cdot e^{i\theta})(\zeta\cdot ie^{i\theta})\Big]\, dx.
\end{aligned}
$$

Applying this to $(h,\zeta) = (f, w_\delta)$ and (g,ξ_δ) and using the equations (15) and (12) satisfied by w_δ and ξ_δ, we deduce

$$
Q[fw_\delta] + Q[g\xi_\delta] = (1+\mathcal{O}(\delta))\int_{A_\rho}\left(|w_\delta|^2|\nabla f|^2 + |\nabla g|^2\right)\, dx. \tag{27}
$$

Next we turn to the last term in (26). In polar coordinates, the bilinear form B has the expression

$$\begin{aligned}
B[u,v] = \int_{A_\rho} \Big[& \partial_r u \cdot \partial_r v + \frac{1}{r^2}(\widehat{\mathcal{L}}_\delta u - \hat{\lambda} u)\cdot v \\
& + \delta \left((\partial_r u \cdot e^{i\theta})(\partial_r v \cdot e^{i\theta}) - (\partial_r u \cdot ie^{i\theta})(\partial_r v \cdot ie^{i\theta}) \right) \\
& + \frac{\delta}{r}\Big((\partial_r u \cdot e^{i\theta})(\partial_\theta v \cdot ie^{i\theta}) + (\partial_r u \cdot ie^{i\theta})(\partial_\theta v \cdot e^{i\theta}) \\
& \quad + (\partial_\theta u \cdot ie^{i\theta})(\partial_r v \cdot e^{i\theta}) + (\partial_\theta u \cdot e^{i\theta})(\partial_r v \cdot ie^{i\theta}) \Big) \Big] \, dx.
\end{aligned}$$

Applying this to $u = fw_\delta$, $v = g\xi_\delta$ we obtain

$$\begin{aligned}
B[fw_\delta, g\xi_\delta] = {} & \mathcal{O}(\delta) \int_{A_\rho} \left(|w_\delta|^2 |\nabla f|^2 + |\nabla g|^2 \right) dx \\
& + \int_{A_\rho} \Big[\frac{1}{r^2}\widehat{\mathcal{L}}_\delta(fw_\delta)\cdot g\xi_\delta \\
& \quad + \frac{\delta}{r} g\partial_r f \left((w_\delta \cdot e^{i\theta})(\xi_\delta' \cdot ie^{i\theta}) + (w_\delta \cdot ie^{i\theta})(\xi_\delta' \cdot e^{i\theta}) \right) \\
& \quad + \frac{\delta}{r} f\partial_r g \left((w_\delta' \cdot ie^{i\theta})(\xi_\delta \cdot e^{i\theta}) + (w_\delta' \cdot e^{i\theta})(\xi_\delta \cdot ie^{i\theta}) \right) \Big] \, dx.
\end{aligned}$$

Integrating by parts with respect to r in the last line, we find

$$\begin{aligned}
B[fw_\delta, g\xi_\delta] = {} & \mathcal{O}(\delta) \int_{A_\rho} \left(|w_\delta|^2 |\nabla f|^2 + |\nabla g|^2 \right) dx + \delta \int_{A_\rho} \frac{\gamma_\delta(\theta)}{r} g\partial_r f \, dx \\
& + \int_{A_\rho} \frac{1}{r^2}\widehat{\mathcal{L}}_\delta(fw_\delta)\cdot g\xi_\delta \, dx,
\end{aligned}$$

with γ_δ as in the statement of Lemma 3.1. Using the equation (15) satisfied by w_δ and the fact that $w_\delta \cdot \xi_\delta = 0$, exactly as in (22) in the proof of Lemma 2.4, we have

$$\begin{aligned}
\widehat{\mathcal{L}}_\delta(fw_\delta)\cdot g\xi_\delta = {} & \hat{\mu}_\delta fg - 2g\partial_\theta f \, w_\delta' \cdot \xi_\delta + \mathcal{O}(\delta |w_\delta| |\partial_\theta f| |\partial_\theta g|) \\
& + \delta \, g\partial_\theta f \big[(w_\delta \cdot ie^{i\theta})(\xi_\delta' \cdot ie^{i\theta}) - (w_\delta \cdot e^{i\theta})(\xi_\delta' \cdot e^{i\theta}) \\
& \qquad - (w_\delta' \cdot ie^{i\theta})(\xi_\delta \cdot ie^{i\theta}) + (w_\delta' \cdot e^{i\theta})(\xi_\delta \cdot e^{i\theta}) \big] \\
& - \delta \, \partial_\theta \left[\partial_\theta f((w_\delta \cdot ie^{i\theta})ie^{i\theta} - (w_\delta \cdot e^{i\theta})e^{i\theta}) \cdot g\xi_\delta \right].
\end{aligned}$$

Plugging this into the above expression for $B[fw_\delta, g\xi_\delta]$ we deduce

$$\begin{aligned}
B[fw_\delta, g\xi_\delta] &= \mathcal{O}(\delta)\int_{A_\rho}\left(|w_\delta|^2|\nabla f|^2 + |\nabla g|^2\right)\,dx \\
&\quad + \delta\int_{A_\rho}\left[\frac{\alpha_\delta(\theta)}{r^2}fg + \frac{\beta_\delta(\theta)}{r^2}g\partial_\theta f + \frac{\gamma_\delta(\theta)}{r}g\partial_r f\right]\,dx \\
&\quad + 2d\int_{A_\rho}\frac{1}{r^2}g\partial_\theta[f|w_\delta|]\,dx, \qquad (28)
\end{aligned}$$

where α_δ and β_δ are defined in the statement of Lemma 3.1. To simplify the last term in (28) we use the fact that u is $\mathbb{S}^1$-valued and $u\bar{\xi}_\delta$ has degree zero, so there exists a lifting $\varphi \in C_c^2(A_\rho;\mathbb{R})$ such that $u = e^{i\varphi}\xi_\delta$. By definition of f, g, and possibly multiplying φ by ± 1 (depending on the constant sign of $i\xi_\delta \cdot w_\delta$), this implies

$$f|w_\delta| = \sin\varphi, \quad 1 + g = \cos\varphi,$$

so

$$\begin{aligned}
2g\partial_\theta[f|w_\delta|] &= \partial_\theta[fg|w_\delta|] + g\partial_\theta[f|w_\delta|] - f|w_\delta|\partial_\theta g \\
&= \partial_\theta[fg|w_\delta|] + (\cos\varphi - 1)\cos\varphi\,\partial_\theta\varphi + \sin^2\varphi\,\partial_\theta\varphi \\
&= \partial_\theta[fg|w_\delta| + \varphi - \sin\varphi].
\end{aligned}$$

Therefore the last term in (28) integrates to zero, and we deduce

$$\begin{aligned}
B[fw_\delta, g\xi_\delta] &= \mathcal{O}(\delta)\int_{A_\rho}\left(|w_\delta|^2|\nabla f|^2 + |\nabla g|^2\right)\,dx \\
&\quad + \delta\int_{A_\rho}\left[\frac{\alpha_\delta(\theta)}{r^2}fg + \frac{\beta_\delta(\theta)}{r^2}g\partial_\theta f + \frac{\gamma_\delta(\theta)}{r}g\partial_r f\right]\,dx,
\end{aligned}$$

which, combined with (27), proves Lemma 3.1. □

3.1.2. *Small anisotropy expansions.* In order to make efficient use of the energy splitting with respect to ξ_δ provided by Lemma 3.1, we will need expansions of the coefficients in powers of δ. We start by expanding ξ_δ.

Lemma 3.2. *Let $d \in \mathbb{Z}\backslash\{1\}$. There exists $\delta_0 > 0$ such that for $|\delta| < \delta_0$, the equation* (12) *has a unique solution ξ_δ of degree d such that $\xi_\delta(0) = 1$, and it satisfies*

$$\xi_\delta(\theta) = e^{id\theta}e^{i\varphi_\delta(\theta)}, \qquad \varphi_\delta = \delta\psi_1 + \frac{\delta^2}{2}\psi_2 + \delta^3\mathcal{O}(1),$$

where

$$\begin{aligned}
\psi_1(\theta) &= a_1\sin(2(d-1)\theta), \qquad a_1 = \frac{d(2-d)}{4(d-1)^2} \\
\psi_2(\theta) &= a_2\sin(4(d-1)\theta), \qquad a_2 = a_2(d) \in \mathbb{R},
\end{aligned}$$

and $\mathcal{O}(1)$ is bounded in $C^k(\mathbb{S}^1;\mathbb{R})$ as $\delta \to 0$ for all $k \geq 0$.

Proof of Lemma 3.2. First recall that Remark 2.2 ensures the existence of φ_δ such that $\varphi_\delta(0) = 0$ and $\xi_\delta = e^{id\theta}e^{i\varphi_\delta}$ solves (12) and minimizes $\widehat{E}_\delta$ among $\mathbb{S}^1$-valued maps of degree d. The inequality $\widehat{E}_\delta(\xi_\delta) \leq \widehat{E}_\delta(e^{id\theta})$ implies

$$\int_{\mathbb{S}^1} (\varphi_\delta')^2\, d\theta \leq C\delta d^2,$$

for some absolute constant $C > 0$, so $\varphi_\delta \to 0$ in $H^1(\mathbb{S}^1;\mathbb{R})$ as $\delta \to 0$. This bound is valid for any solution φ_δ with $\varphi_\delta(0) = 0$, since they are all minimizing by Proposition 2.6.

Next we show, by an implicit function argument, that φ_δ is unique and depends smoothly on δ for small δ. Consider, for any $k \geq 0$ and $d \neq 1$, the map

$$\begin{aligned}
&\Psi\colon X \times \mathbb{R} \times (-1,1) \to Y,\\
&X = \left\{\varphi \in H^{k+1}(\mathbb{S}^1;\mathbb{R})\colon \varphi(0) = 0\right\}, \qquad Y = H^{k-1}(\mathbb{S}^1;\mathbb{R}),
\end{aligned}$$

given by

$$\begin{aligned}
\Psi(\varphi,t,\delta) &= -D_\varphi \widehat{E}_\delta(e^{id\theta}e^{i\varphi}) + t\\
&= \frac{d}{d\theta}\left[(1+\delta\cos(2(d-1)\theta + 2\varphi))\,(d+\varphi')\right]\\
&\quad + \delta\sin(2(d-1)\theta+2\varphi)(d+\varphi')^2 + t.
\end{aligned}$$

This map Ψ is smooth and satisfies

$$\Psi(0,0,0) = 0, \quad D_{(\varphi,t)}\Psi(0,0,0)[\eta,s] = \eta'' + s \quad \forall(\eta,\mu) \in X\times\mathbb{R}.$$

The differential $D_{(\varphi,\lambda)}\Psi(0,0,0)$ is an isomorphism from $X\times\mathbb{R}$ to Y, so by the implicit function theorem there exist $(\bar\varphi_\delta, t_\delta) \in X\times\mathbb{R}$ depending smoothly on $\delta \in (-\delta_0,\delta_0)$, the unique solution of $\Psi(\bar\varphi,t,\delta) = 0$ in a neighborhood of $(0,0)$. By uniqueness this solution does not depend on k, and because $\Psi(\varphi_\delta,0,\delta) = 0$ and $\varphi_\delta \to 0$ in $H^1(\mathbb{S}^1;\mathbb{R})$ as $\delta \to 0$, for small enough δ we must have $t_\delta = 0$, $\varphi_\delta = \bar\varphi_\delta$ is the unique solution of

$$\begin{aligned}
&\frac{d}{d\theta}\left[(1+\delta\cos(2(d-1)\theta+2\varphi_\delta))\,(d+\varphi_\delta')\right] \\
&= -\delta\sin(2(d-1)\theta+2\varphi_\delta)(d+\varphi_\delta')^2,
\end{aligned} \tag{29}$$

satisfying $\varphi_\delta(0) = 0$, and $\delta \mapsto \varphi_\delta \in H^{k+1}(\mathbb{S}^1;\mathbb{R})$ is smooth for all $k \geq 0$. We have $\varphi_\delta|_{\delta=0} \equiv 0$, and considering

$$\psi_1 = \left.\frac{d}{d\delta}\right|_{\delta=0}\varphi_\delta, \qquad \psi_2 = \left.\frac{d^2}{d\delta^2}\right|_{\delta=0}\varphi_\delta,$$

provides the expansion in Lemma 3.2. It remains to explicitly determine ψ_1 and ψ_2. Derivating the Euler-Lagrange equation (29) with respect to δ we see that ψ_1 solves

$$\frac{d}{d\theta}\left[\psi_1' + d\cos(2(d-1)\theta)\right] = -d^2\sin(2(d-1)\theta),$$

that is

$$\psi_1'' = d(d-2)\sin(2(d-1)\theta).$$

Since ψ_1 is 2π-periodic with $\psi_1(0)=0$, this implies

$$\psi_1(\theta) = \frac{d(2-d)}{4(d-1)^2}\sin(2(d-1)\theta).$$

Derivating twice the Euler-Lagrange equation (29) with respect to δ we see that ψ_2 solves

$$\begin{aligned}&\frac{d}{d\theta}\left[\psi_2' - 4d\sin(2(d-1)\theta)\psi_1 + 2\cos(2(d-1)\theta)\psi_1'\right]\\ &= -4d^2\cos(2(d-1)\theta)\psi_1 - 4d\sin(2(d-1)\theta)\psi_1',\end{aligned}$$

that is

$$\psi_2'' = a(d)\sin(4(d-1)\theta)),$$

for some $a(d)\in\mathbb{R}$, and since ψ_2 is 2π-periodic with $\psi_2(0)=0$ this gives

$$\psi_2(\theta) = a_2(d)\sin(4(d-1)\theta)),$$

for some $a_2(d)\in\mathbb{R}$. □

As a consequence of Lemma 3.2 we obtain expansions of the coefficients α_δ, β_δ appearing in Lemma 3.1.

Lemma 3.3. *As $\delta\to 0$ we have*

$$\begin{aligned}\alpha_\delta &= \alpha^0 + \delta\alpha^1 + \delta^2\mathcal{O}(1),\\ \beta_\delta &= \beta^0 + \delta\beta^1 + \delta^2\mathcal{O}(1),\end{aligned}$$

where $\mathcal{O}(1)$ is bounded in $C^k(\mathbb{S}^1;\mathbb{R})$ as $\delta\to 0$ for any $k\geq 0$, and, denoting $C_n(\theta) = \cos(n\theta)$, $S_n(\theta)=\sin(n\theta)$, the coefficients α^j, β^j, satisfy

$$\begin{aligned}&\alpha_0 = 2d(d-2)S_{2(d-1)}, \quad \beta^0 = \frac{d(2-d)}{d-1}C_{2(d-1)}\\ &\alpha^1\in\operatorname{span}(S_{4(d-1)}), \quad \beta^1\in\operatorname{span}(1, C_{4(d-1)}).\end{aligned}$$

Proof of Lemma 3.3. Recall from Lemma 3.1 that α_δ and β_δ are given by

$$\begin{aligned}\alpha_\delta &= \frac{\hat\mu_\delta - 2d\,\partial_\theta[|w_\delta|]}{\delta},\\ \beta_\delta &= \frac{-2(w_\delta'\cdot\xi_\delta + d) + 2d(1-|w_\delta|)}{\delta}\\ &\quad + (w_\delta\cdot ie^{i\theta})(\xi_\delta'\cdot ie^{i\theta}) - (w_\delta\cdot e^{i\theta})(\xi_\delta'\cdot e^{i\theta})\\ &\quad + (w_\delta'\cdot e^{i\theta})(\xi_\delta\cdot e^{i\theta}) - (w_\delta'\cdot ie^{i\theta})(\xi_\delta\cdot ie^{i\theta}).\end{aligned}$$

We first obtain expressions in terms of φ_δ. Using

$$\begin{aligned}
&\xi_\delta = e^{i(d\theta+\varphi_\delta)}, \quad \xi_\delta' = (d+\varphi_\delta')i\xi_\delta,\\
&w_\delta = \left(1+\frac{\varphi_\delta'}{d-1}\right)i\xi_\delta, \quad w_\delta' = \frac{\varphi_\delta''}{d-1}i\xi_\delta - \left(d+\frac{2d-1}{d-1}\varphi_\delta' + \frac{1}{d-1}(\varphi_\delta')^2\right)\xi_\delta,
\end{aligned}$$

we find

$$\begin{aligned}
&w_\delta'\cdot\xi_\delta + d = -\frac{2d-1}{d-1}\varphi_\delta' - \frac{1}{d-1}(\varphi_\delta')^2\\
&1-|w_\delta| = -\frac{\varphi_\delta'}{d-1},
\end{aligned}$$

and

$$\begin{aligned}
&(w_\delta\cdot ie^{i\theta})(\xi_\delta'\cdot ie^{i\theta}) - (w_\delta\cdot e^{i\theta})(\xi_\delta'\cdot e^{i\theta})\\
&+(w_\delta'\cdot e^{i\theta})(\xi_\delta\cdot e^{i\theta}) - (w_\delta'\cdot ie^{i\theta})(\xi_\delta\cdot ie^{i\theta})\\
&= -\frac{\varphi_\delta''}{d-1}\sin(2(d-1)\theta+2\varphi_\delta).
\end{aligned}$$

Using also

$$w_\delta'' = -\left(\left(1+\frac{\varphi_\delta'}{d-1}\right)(d+\varphi_\delta')^2 - \frac{\varphi_\delta^{(3)}}{d-1}\right)i\xi_\delta - \frac{\varphi_\delta''}{d-1}\left(3d-1+3\varphi_\delta'\right)\xi_\delta,$$

and recalling the definition (15) of $\hat\mu_\delta$, we obtain

$$\begin{aligned}
\hat\mu_\delta &= -w_\delta''\cdot\xi_\delta - \delta\,\left[(w_\delta'\cdot ie^{i\theta})ie^{i\theta} - (w_\delta'\cdot e^{i\theta})e^{i\theta}\right]'\cdot\xi_\delta\\
&= \frac{\varphi_\delta''}{d-1}\left(3d-1+3\varphi_\delta'\right)\\
&\quad - 3\delta\varphi_\delta''\left(1+\frac{\varphi_\delta'}{d-1}\right)\cos(2(d-1)\theta+2\varphi_\delta)\\
&\quad + \frac{\delta}{d-1}\left((d+\varphi_\delta')(d-1+\varphi_\delta')(d-2+\varphi_\delta') - \varphi_\delta^{(3)}\right)\sin(2(d-1)\theta+2\varphi_\delta)
\end{aligned}$$

Then we plug into these expressions the expansions

$$\begin{aligned}
\varphi_\delta &= \delta\psi_1 + \frac{\delta^2}{2}\psi_2 + \mathcal{O}(\delta^3),\\
\cos(2(d-1)\theta+2\varphi_\delta) &= C_{2(d-1)} - 2\delta S_{2(d-1)}\psi_1 + \mathcal{O}(\delta^2)\\
\sin(2(d-1)\theta+2\varphi_\delta) &= S_{2(d-1)} + 2\delta C_{2(d-1)}\psi_1 + \mathcal{O}(\delta^2),
\end{aligned}$$

where we recall the notation $C_n(\theta) = \cos(n\theta)$, $S_n(\theta) = \sin(n\theta)$. We find

$$
\begin{aligned}
&w_\delta' \cdot \xi_\delta + d = -\delta \frac{2d-1}{d-1}\psi_1' - \frac{\delta^2}{d-1}\left(\frac{2d-1}{2}\psi_2' + (\psi_1')^2\right) + \mathcal{O}(\delta^3), \\
&1 - |w_\delta| = -\frac{\delta}{d-1}\psi_1' - \frac{\delta^2}{2(d-1)}\psi_2' + \mathcal{O}(\delta^3), \\
&(w_\delta \cdot ie^{i\theta})(\xi_\delta' \cdot ie^{i\theta}) - (w_\delta \cdot e^{i\theta})(\xi_\delta' \cdot e^{i\theta}) \\
&+ (w_\delta' \cdot e^{i\theta})(\xi_\delta \cdot e^{i\theta}) - (w_\delta' \cdot ie^{i\theta})(\xi_\delta \cdot ie^{i\theta}) \\
&= -\delta \frac{\psi_1''}{d-1} S_{2(d-1)} + \mathcal{O}(\delta^2),
\end{aligned}
$$

and

$$
\begin{aligned}
\hat{\mu}_\delta = \delta &\left(\frac{3d-1}{d-1}\psi_1'' + d(d-2)S_{2(d-1)}\right) \\
&+ \delta^2 \Bigg(\frac{3d-1}{2(d-1)}\psi_2'' + \frac{3}{d-1}\psi_1'\psi_1'' - 3\psi_1'' C_{2(d-1)} \\
&\qquad + \frac{3d^2-6d+2}{d-1} S_{2(d-1)}\psi_1' - \frac{\psi_1^{(3)}}{d-1} S_{2(d-1)} \\
&\qquad\qquad + 2d(d-2)C_{2(d-1)}\psi_1\Bigg) + \mathcal{O}(\delta^3)
\end{aligned}
$$

Gathering the above and recalling

$$
\psi_1 = a_1 S_{2(d-1)}, \qquad \psi_2 = a_2 S_{4(d-1)},
$$

for some constants $a_1 = a_1(d)$, $a_2 = a_2(d)$, we obtain the desired expansions for α_δ and β_δ. Explicitly, we have

$$
\begin{aligned}
&w_\delta' \cdot \xi_\delta + d = -2\delta(2d-1)a_1 C_{2(d-1)} \\
&\qquad\qquad - \delta^2\left(2(2d-1)a_2 C_{4(d-1)} + 2(d-1)a_1^2(1 + C_{4(d-1)})\right) + \mathcal{O}(\delta^3), \\
&1 - |w_\delta| = -2\delta a_1 C_{2(d-1)} - 2\delta^2 a_2 C_{4(d-1)} + \mathcal{O}(\delta^3), \\
&(w_\delta \cdot ie^{i\theta})(\xi_\delta' \cdot ie^{i\theta}) - (w_\delta \cdot e^{i\theta})(\xi_\delta' \cdot e^{i\theta}) \\
&+ (w_\delta' \cdot e^{i\theta})(\xi_\delta \cdot e^{i\theta}) - (w_\delta' \cdot ie^{i\theta})(\xi_\delta \cdot ie^{i\theta}) \\
&= 2\delta a_1(d-1)(1 - C_{4(d-1)}) + \mathcal{O}(\delta^2),
\end{aligned}
$$

and

$$\begin{aligned}\hat{\mu}_\delta &= \delta\left(d(d-2) - 4(3d-1)(d-1)a_1\right) S_{2(d-1)} \\ &\quad + \delta^2\Big(-8(3d-1)(d-1)a_2 - 12(d-1)^2 a_1^2 \\ &\qquad\qquad + (14d^2 - 28d + 12)a_1\Big) S_{4(d-1)} + \mathcal{O}(\delta^3),\end{aligned}$$

and we infer that the coefficients α^j, β^j are given by

$$\begin{aligned}\alpha^0 &= \left(d(d-2) - 4(d-1)^2 a_1\right) S_{2(d-1)} = 2d(d-2) S_{2(d-1)}, \\ \alpha^1 &= \left(-8(d-1)^2 a_2 - 12(d-1)^2 a_1^2 + (14d^2 - 28d + 12)a_1\right) S_{4(d-1)}, \\ \beta^0 &= 4(d-1)a_1 C_{2(d-1)} = \frac{d(2-d)}{d-1} C_{2(d-1)}, \\ \beta^1 &= 2(d-1)a_1(1+2a_1) - 2\left((d-1)a_1(1-2a_1) + 2(d-1)a_2\right) C_{4(d-1)}.\end{aligned}$$

□

3.2. **Minimality in not-too-thick annuli.** In this section we prove that if $|\delta| \ll 1$ and the annulus A_ρ is not too thick, the unique (modulo frame invariance) 0-homogeneous solution ξ_δ of degree $d \in \mathbb{Z} \setminus \{1\}$ is minimizing.

Proposition 3.4. *Let $d \in \mathbb{Z} \setminus \{1\}$. There exists a small constant $c > 0$, depending only on d, such that if $|\delta| < c$, ξ_δ of degree d solves* (12)*, and $e^{-c/\delta^{\frac{1}{3}}} \leq \rho < 1$, then ξ_δ is minimizing in A_ρ with respect to its own boundary conditions.*

We obtain Proposition 3.4 as a consequence of two observations on maps $u \in H^1(A_\rho; \mathbb{S}^1)$ agreeing with ξ_δ on ∂A_ρ :

- on the one hand, the linear stability result of Proposition 2.7 can be enhanced to a local minimality statement: any map u close enough to ξ_δ has higher energy than ξ_δ unless $u = \xi_\delta$,
- on the other hand, as $\delta \to 0$, a minimizing map u must converge to $\xi_0(\theta) = e^{id\theta}$ (modulo frame invariance), and is therefore close to ξ_δ.

In Proposition 3.5 and Lemma 3.7 below we quantify these statements, which can then directly be combined into a proof of Proposition 3.4.

Proposition 3.5. *Let $d \in \mathbb{Z} \setminus \{1\}$. There exists a small constant $c > 0$, depending only on d, such that if $|\delta| < c$, ξ_δ of degree d solves* (12)*, and $u \in H^1(A_\rho; \mathbb{S}^1)$ is such that $u = \xi_\delta$ on ∂A_ρ for some $0 < \rho < 1$, we have*

$$\int_{A_\rho} |\nabla u - \nabla \xi_\delta|^2 dx \leq \frac{c}{\delta^2 (1 + \ln^2 \rho)^3} \qquad \Longrightarrow \qquad E_\delta(u; A_\rho) \geq E_\delta(\xi_\delta; A_\rho),$$

and the last inequality is strict unless $u = \xi_\delta$.

Proof of Proposition 3.5. We consider without loss of generality a solution ξ_δ such that $\xi_\delta(0) = 1$, as in Lemma 3.2, and write

$$u - \xi_\delta = f w_\delta + g\xi_\delta.$$

It follows from the energy splitting in Lemma 3.1 and the expansions in Lemma 3.2 and Lemma 3.3 that, if $|\delta| < c$ for small enough $c > 0$, we have

$$\begin{aligned} E_\delta(u; A_\rho) - E_\delta(\xi_\delta; A_\rho) \geq \frac{3}{4} \int_{A_\rho} \left(|\nabla f|^2 + |\nabla g|^2\right) \, dx \\ - C\delta \int_{A_\rho} \left(\frac{|fg|}{r^2} + |\nabla f| \frac{|g|}{r} \right) \, dx. \end{aligned}$$

Here $C > 0$ denotes a large constant depending only on the degree d, and may change from line to line in the rest of this proof. Using $|\nabla f||g|/r \leq |\nabla f|^2 + g^2/r^2$, we deduce

$$E_\delta(u; A_\rho) - E_\delta(\xi_\delta; A_\rho) \geq \frac{1}{2} \int_{A_\rho} \left(|\nabla f|^2 + |\nabla g|^2\right) \, dx - C\delta \int_{A_\rho} \frac{|fg| + g^2}{r^2} \, dx.$$

Next we use the fact that $|u| = 1$, i.e. $f^2|w_\delta|^2 + (1 + g)^2 = 1$, or equivalently $g = -(f^2|w_\delta|^2 + g^2)/2$, to infer that $|f|, |g| \leq 2$ and

$$|fg| + g^2 \leq C \left(|f|^3 + |g|^3\right).$$

Plugging this inequality into the previous estimate we obtain

$$E_\delta(u; A_\rho) - E_\delta(\xi_\delta; A_\rho) \geq \frac{1}{2} \int_{A_\rho} \left(|\nabla f|^2 + |\nabla g|^2\right) \, dx - C\delta \int_{A_\rho} \frac{|f|^3 + |g|^3}{r^2} \, dx.$$

The last term can be estimated using the interpolation inequality of Lemma 3.6 below, and we deduce

$$\begin{aligned} E_\delta(u; A_\rho) - E_\delta(\xi_\delta; A_\rho) \geq \left(\frac{1}{2} - C\delta(1 + \ln^2 \rho) \left(\int_{A_\rho} \frac{f^2 + g^2}{r^2} \, dx \right)^{\frac{1}{2}} \right) \\ \times \int_{A_\rho} \left(|\nabla f|^2 + |\nabla g|^2\right) \, dx. \end{aligned}$$

Since $f^2 + g^2 \leq 2|u - \xi_\delta|^2$, this implies $E_\delta(u; A_\rho) > E_\delta(\xi_\delta; A_\rho)$ whenever $u \neq \xi_\delta$ and

$$\int_{A_\rho} \frac{|u - \xi_\delta|^2}{r^2} \, dx \leq \frac{c}{\delta^2(1 + \ln^2 \rho)^2}.$$

Combining this with the Hardy-type inequality (see (31) below)

$$\frac{1}{\ln^2 \rho} \int_{A_\rho} \frac{|u - \xi_\delta|^2}{r^2} \, dx \leq C \int_{A_\rho} |\nabla u - \nabla \xi_\delta|^2 \, dx,$$

concludes the proof of Proposition 3.5 $\square$

Next we prove the interpolation inequality used in the proof of Proposition 3.5.

Lemma 3.6. *For all* $0<\rho<1$ *and* $\varphi\in H_0^1(A_\rho)$ *we have*

$$\int_{A_\rho}\frac{|\varphi|^3}{r^2}\,dx\leq C(1+\ln^2\rho)\left(\int_{A_\rho}\frac{\varphi^2}{r^2}\,dx\right)^{\frac12}\int_{A_\rho}|\nabla\varphi|^2\,dx,$$

for some absolute constant $C>0$.

Proof. First we show that, for all $\varphi\in C_c^\infty(A_\rho)$,

$$\int_{A_\rho}\frac{\varphi^4}{r^2}\,dx\leq C(1+\ln^2\rho)\int_{A_\rho}\frac{\varphi^2}{r^2}\,dx\int_{A_\rho}|\nabla\varphi|^2\,dx. \tag{30}$$

For $1/4\leq\rho\leq 1$ this is a consequence of the classical Ladyzhenskaya interpolation inequality in the domain $A_{1/4}$, which simply follows from applying to $\varphi^2\in C_c^\infty(A_{1/4})$ the Poincaré-Sobolev inequality of the embedding $W^{1,1}(A_{1/4})\subset L^2(A_{1/4})$:

$$\|\varphi\|^4_{L^4(A_{1/4})}=\|\varphi^2\|^2_{L^2(A_{1/4})}\leq C\|\nabla(\varphi^2)\|^2_{L^1(A_{1/4})}\leq C\|\varphi\|^2_{L^2(A_{1/4})}\|\nabla\varphi\|^2_{L^2(A_{1/4})}.$$

To obtain (30) for $0<\rho<1/2$, we consider the rescaled annuli $\mathbb{A}_j=2^{-j}A_{1/4}$ and decompose $\varphi=\sum_{j\geq 0}\varphi_j$, with $\varphi_j\in C_c^\infty(\mathbb{A}_j)$ such that, for any $p\geq 1$,

$$\begin{aligned}&\int_{\mathbb{A}_j}\varphi_j^p\,dx\leq\int_{\mathbb{A}_j}\varphi^p\,dx\leq\int_{\mathbb{A}_j}\varphi_j^p\,dx+\int_{\mathbb{A}_{j+1}}\varphi_{j+1}^p\,dx,\\&\int_{\mathbb{A}_j}|\nabla\varphi_j|^2\,dx\leq\int_{\mathbb{A}_j}|\nabla\varphi|^2\,dx+C\int_{\mathbb{A}_j}\frac{\varphi^2}{r^2}\,dx.\end{aligned}$$

This decomposition can be obtained for instance by fixing a smooth cut-off function $\mathbf{1}_{|x|\leq 1/2}\leq\chi(x)\leq\mathbf{1}_{|x|\leq 1}$ and setting

$$\varphi_0(x)=\chi(x)\varphi(x),\quad \varphi_j(x)=(\chi(2^{j+1}x)-\chi(2^jx))\varphi(x)\text{ for }j\geq 1.$$

Rescaling Ladyzhenskaya's inequality we have

$$\int_{\mathbb{A}_j}\frac{\varphi_j^4}{r^2}\,dx\leq C\int_{\mathbb{A}_j}|\nabla\varphi_j|^2\,dx\int_{\mathbb{A}_j}\frac{\varphi_j^2}{r^2}\,dx.$$

Summing these estimates and using the properties of φ_j we obtain

$$\int_{A_\rho}\frac{\varphi^4}{r^2}\,dx\leq C\int_{A_\rho}\frac{\varphi^2}{r^2}\,dx\left(\int_{A_\rho}|\nabla\varphi|^2\,dx+\int_{A_\rho}\frac{\varphi^2}{r^2}\,dx\right).$$

This, together with the Hardy-type inequality

$$\int_{A_\rho}\frac{\varphi^2}{r^2}\,dx\leq\frac{\ln^2\rho}{\pi^2}\int_{A_\rho}|\nabla\varphi|^2\,dx\qquad\forall\varphi\in H_0^1(A_\rho), \tag{31}$$

proves (30). (Inequality (31) follows e.g. from [13, Theorem 1.5.12] and the fact that the function $\varphi_*(x)=\sin(\pi\ln|x|/|\ln\rho|)$ solves $-\Delta\varphi_*=(\pi^2/\ln^2\rho)\varphi_*/r^2$ and is positive in A_ρ.) To conclude the proof of Lemma 3.6, we write $|\varphi|^3 \leq \lambda\varphi^2 + \lambda^{-1}\varphi^4$ for any $\lambda > 0$, apply (30) and Hardy's inequality (31) to deduce

$$\frac{1}{1+\ln^2\rho}\int_{A_\rho}\frac{|\varphi|^3}{r^2}\,dx \leq C\lambda\int_{A_\rho}|\nabla\varphi|^2\,dx + \frac{C}{\lambda}\int_{A_\rho}\frac{\varphi^2}{r^2}\,dx\int_{A_\rho}|\nabla\varphi|^2\,dx,$$

and choose $\lambda = (\int \varphi^2/r^2\,dx)^{\frac{1}{2}}$. □

As explained above, the second ingredient to prove Proposition 3.4 is the convergence towards $\xi_0(\theta) = e^{id\theta}$, as $\delta \to 0$, of any minimizing map $u \in H^1(A_\rho;\mathbb{S}^1)$ agreeing with ξ_δ on ∂A_ρ. Since for $|\delta| \ll 1$ the 0-homogeneous solution ξ_δ is also close to ξ_0 (see Lemma 3.2), this implies that u must be close to ξ_δ. The next lemma makes that statement quantitative.

Lemma 3.7. *Let $d \in \mathbb{Z}\setminus\{1\}$, $|\delta| < c$ ($c > 0$ small depending on d) and ξ_δ of degree d solve* (12) *and $\xi_\delta(0) = 1$. Assume $0 < \rho < 1$ and $u \in H^1(A_\rho;\mathbb{S}^1)$ satisfies $u = \xi_\delta$ on ∂A_ρ and $E_\delta(u;A_\rho) \leq E_\delta(\xi_\delta;A_\rho)$. Then*

$$\int_{A_\rho}|\nabla u - \nabla\xi_\delta|^2\,dx \leq C\delta(1+\ln^2\rho)\ln\frac{1}{\rho},$$

for some $C > 0$ depending only on the degree d.

Proof of Lemma 3.7. From the expansion of ξ_δ in Lemma 3.2 we infer

$$E_\delta(\xi_\delta;A_\rho) \leq (1+C\delta)2\pi d^2\ln\frac{1}{\rho},$$

and the bound $E_\delta(u;A_\rho) \leq E_\delta(\xi_\delta;A_\rho)$ therefore implies, for $|\delta| \leq c$,

$$\int_{A_\rho}|\nabla u|^2 \leq (1+C\delta)2\pi d^2\ln\frac{1}{\rho}. \tag{32}$$

Since $u = \xi_\delta$ on ∂A_ρ, it is of degree d, and we can write $u = \xi_\delta e^{i\psi}$, with $\psi \in H_0^1(A_\rho;\mathbb{R})$. We further rewrite this as $u = e^{id\theta}e^{i\varphi_\delta(\theta)}e^{i\psi}$, where $\|\varphi_\delta\|_{C^1} \leq C\delta$ by Lemma 3.2. Then we have

$$\begin{aligned}|\nabla u|^2 &= (\partial_r\psi)^2 + \frac{1}{r^2}(d+\partial_\theta\psi+\partial_\theta\varphi_\delta)^2\\ &= \frac{d^2+\mathcal{O}(\delta)}{r^2} + \frac{2d}{r^2}\partial_\theta\psi + (1+\mathcal{O}(\delta))|\nabla\psi|^2.\end{aligned}$$

Integrating on A_ρ, we deduce

$$\int_{A_\rho}|\nabla u|^2\,dx = (1+\mathcal{O}(\delta))2\pi d^2\ln\frac{1}{\rho} + (1+\mathcal{O}(\delta))\int_{A_\rho}|\nabla\psi|^2\,dx,$$

and combining this with (32) gives

$$\int_{A_\rho}|\nabla\psi|^2\,dx \leq C\delta\ln\frac{1}{\rho}.$$

Therefore we find

$$\int_{A_\rho} |\nabla u - \nabla \xi_\delta|^2\,dx \le C \int_{A_\rho} \frac{|\psi|^2}{r^2} + |\nabla\psi|^2\,dx$$
$$\le C(1+\ln^2\rho) \int_{A_\rho} |\nabla\psi|^2\,dx \le C\delta(1+\ln^2\rho)\ln\frac{1}{\rho}.$$

For the penultimate inequality we used Hardy's inequality (31). □

Combining Proposition 3.5 with Lemma 3.7 provides a proof of Proposition 3.4. Specifically, according to Lemma 3.7 and Proposition 3.5, there exist constants $C, c > 0$ depending only on d such that, if $|\delta| < c$ and

$$C\delta(1+\ln^2\rho)\ln\frac{1}{\rho} \le \frac{c}{\delta^2(1+\ln^2\rho)^3}, \tag{33}$$

then any minimizer u with $u_{\lfloor\partial A_\rho} = \xi_\delta$ must be equal to ξ_δ. If $1/2 \le \rho < 1$ then (33) is satisfied for all small enough δ, so we may assume $0 < \rho \le 1/2$, in which case $|\ln\rho| = \ln(1/\rho) \ge \ln 2 > 0$ and (33) is implied by

$$\delta\ln^3\frac{1}{\rho} \le \frac{c}{\delta^2\ln^6\frac{1}{\rho}} \quad\Leftrightarrow\quad \ln^9\frac{1}{\rho} \le \frac{c}{\delta^3} \quad\Leftrightarrow\quad \ln\frac{1}{\rho} \le \frac{c}{\delta^{\frac{1}{3}}},$$

for some generic small constant $c > 0$ depending only on d. Therefore ξ_δ is a minimizer if $\rho \ge e^{-c/\delta^{\frac{1}{3}}}$, and this proves Proposition 3.4.

3.3. **Symmetry breaking.** In this section we construct, for small δ and ρ, a non-0-homogeneous map that agrees with ξ_δ on ∂A_ρ and has strictly lower energy than ξ_δ, proving in particular the third item of Theorem 1.1.

The basic idea is to use a competitor that saturates (or almost saturates) the interpolation inequality of Lemma 3.6, which we used in Proposition 3.5 to control the non-quadratic terms for ρ not too small. We use that Hardy's inequality (31) is saturated by

$$\varphi_*(x) = \sin\left(\frac{\pi\ln|x|}{|\ln\rho|}\right),$$

since $\varphi_* \in H_0^1(A_\rho)$ satisfies $-\Delta\varphi_* = (\pi^2/\ln^2\rho)\varphi_*/r^2$ and therefore

$$\int_{A_\rho} |\nabla\varphi_*|^2\,dx = \frac{\pi^2}{\ln^2\rho}\int_{A_\rho}\frac{\varphi_*^2}{r^2}\,dx.$$

Regarding the interpolation inequality of Lemma 3.6, we have

$$\frac{\int_{A_\rho}|\varphi_*|^3/r^2\,dx}{(\int_{A_\rho}\varphi_*^2/r^2\,dx)^{\frac{1}{2}}\int_{A_\rho}|\nabla\varphi_*|^2\,dx} = c_*|\ln\rho|^{\frac{3}{2}} \quad \text{for some } c_* > 0,$$

which is enough for our purposes, even though it does not completely saturate the interpolation inequality.

Using this function φ_* we construct a competitor $u_{\epsilon,\rho,\delta}$ for ξ_δ in A_ρ, for which we can expand the energy in terms of the small parameters ϵ,ρ,δ, and eventually find that it is lower than the energy of ξ_δ for appropriate choices of $\epsilon=\epsilon(\delta)$ and $\rho=\rho(\delta)$.

Proposition 3.8. *Let $0<|\delta|<1$, $d\in\mathbb{Z}\setminus\{0,1,2\}$ and $\xi_\delta\in H^1(\mathbb{S}^1;\mathbb{S}^1)$ of degree d, a solution of (12) with $\xi_\delta(0)=1$ (see Remark 2.2). Denote by w_δ the corresponding Jacobi field defined in Lemma 2.1. For any $\rho\in(0,1/e)$, let $h_\rho\in H^1_0(A_\rho;\mathbb{R})$ be given by*

$$h_\rho(re^{i\theta})=\left(1+\frac{\sin(2(d-1)\theta)}{|\ln\rho|}\right)\sin\left(\frac{\pi\ln r}{\ln\rho}\right),$$

and, for $0<|\epsilon|<1/(2\|w_\delta\|_\infty)$, define $u_{\epsilon,\rho,\delta}\in H^1(A_\rho;\mathbb{S}^1)$ by

$$u_{\epsilon,\rho,\delta}=\sqrt{1-\epsilon^2h_\rho^2|w_\delta|^2}\,\xi_\delta+\epsilon h_\rho\,w_\delta.$$

Then there exist a large constant $\lambda>0$ and a small constant $\delta_0>0$ depending only on the degree d, and a value of ϵ depending only on the degree d and the sign of δ, such that $E_\delta(u_{\epsilon,\rho,\delta};A_\rho)<E_\delta(\xi_\delta;A_\rho)$ for $|\delta|<\delta_0$ and $\rho=e^{-\frac{\lambda}{|\delta|}}$.

Proof of Proposition 3.8. The map $u=u_{\epsilon,\rho,\delta}$ is of the form $u=\xi_\delta+fw_\delta+g\xi_\delta$, with

$$f=\epsilon h,\qquad g=\sqrt{1-\epsilon^2h^2|w_\delta|^2}-1.$$

The function $h\in H^1_0(A_\rho;\mathbb{R})$ satisfies $|h|\le 2$ and $|\partial_\theta h|\le 2|d-1|/|\ln\rho|$, so we have

$$\begin{aligned}
g&=-\frac{\epsilon^2}{2}h^2|w_\delta|^2-\frac{\epsilon^4}{8}h^4|w_\delta|^4+\mathcal{O}(\epsilon^6)\\
\nabla g&=-\frac{\epsilon^2}{2}\nabla[h^2|w_\delta|^2](1+\mathcal{O}(\epsilon^2))\\
fg&=-\frac{\epsilon^3}{2}h^3|w_\delta|^2-\frac{\epsilon^5}{8}h^5|w_\delta|^4+\mathcal{O}(\epsilon^7)\\
g\partial_\theta f&=-\frac{\epsilon^3}{6}|w_\delta|^2\partial_\theta[h^3]-\frac{\epsilon^5}{40}|w_\delta|^4\partial_\theta[h^5]+\mathcal{O}\left(\frac{\epsilon^7}{|\ln\rho|}\right)\\
g\partial_r f&=\partial_r\left[\epsilon\int_0^h(\sqrt{1-\epsilon^2t^2|w_\delta|^2}-1)\,dt\right].
\end{aligned}$$

Moreover, using also that $\partial_\theta[|w_\delta|^2]=2\delta\psi_1'/(d-1)+\mathcal{O}(\delta^2)$ thanks to Lemma 3.2, we see that

$$|\nabla g|^2=\frac{\delta^2\epsilon^4}{(d-1)^2}\frac{h^4}{r^2}(\psi_1')^2+\mathcal{O}(\epsilon^4)|\nabla h|^2+\mathcal{O}\left(\frac{\delta\epsilon^4}{|\ln\rho|}+\delta^3\epsilon^4+\delta^2\epsilon^6\right)\frac{1}{r^2}.$$

Plugging all this into the expansion obtained in Lemma 3.1, we find

$$\begin{aligned}
&E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho)\\
&= \epsilon^2 \int_{A_\rho} |\nabla h|^2\,dx + \frac{\delta^2\epsilon^4}{(d-1)^2}\int_{A_\rho} \frac{h^4}{r^2}(\psi_1')^2\,dx\\
&\quad + \delta\epsilon^3 \int_{A_\rho} \frac{\eta_\delta(\theta)}{r^2} h^3\,dx + \delta\epsilon^5 \int_{A_\rho} \frac{\nu_\delta(\theta)}{r^2} h^5\,dx\\
&\quad + \mathcal{O}(\delta\epsilon^2)\int_{A_\rho} |\nabla h|^2 dx + \mathcal{O}(\delta\epsilon^4) + \mathcal{O}(\delta^3\epsilon^4 + \delta^2\epsilon^6)|\ln\rho|, \qquad (34)
\end{aligned}$$

where

$$\eta_\delta = -\alpha_\delta |w_\delta|^2 + \partial_\theta\left[\beta_\delta \frac{|w_\delta|^2}{3}\right], \qquad \nu_\delta = -\alpha_\delta \frac{|w_\delta|^4}{4} + \partial_\theta\left[\beta_\delta \frac{|w_\delta|^4}{20}\right]$$

From the expansion of ξ_δ in Lemma 3.2, we have

$$|w_\delta|^2 = \left(1 + \frac{\varphi_\delta'}{d-1}\right)^2 = 1 + \delta\frac{2}{d-1}\psi_1' + \mathcal{O}(\delta^2),$$

and using also the expansions of α_β and β_δ in Lemma 3.3, it can be checked that the coefficients η_δ and ν_δ have expansions of the form

$$\begin{aligned}
\eta_\delta &= \eta^0 + \delta\eta^1 + \mathcal{O}(\delta^2),\\
\eta^0 &= -\alpha^0 + \frac{1}{3}\partial_\theta\beta^0 \in \mathrm{span}(S_{2(d-1)})\\
\eta^1 &= -\alpha^1 - \frac{2}{d-1}\alpha^0\psi_1' + \frac{1}{3}\partial_\theta\left[\beta^1 + \frac{2}{d-1}\beta^0\psi_1'\right] \in \mathrm{span}(S_{4(d-1)}),\\
\nu^\delta &= \nu^0 + \mathcal{O}(\delta),\\
\nu^0 &= -\frac{1}{4}\alpha^0 + \frac{1}{20}\partial_\theta\beta^0 \in \mathrm{span}(S_{2(d-1)}).
\end{aligned}$$

Here we use again the notation $S_n(\theta) = \sin(n\theta)$. Next recall that

$$\begin{aligned}
h &= \left(1 + \frac{S_{2(d-1)}(\theta)}{|\ln\rho|}\right) h_0(r), \quad h_0(r) = \sin\left(\frac{\pi\ln r}{\ln\rho}\right)\mathbf{1}_{\rho\le r\le 1},\\
\psi_1' &= \frac{d(2-d)}{2(d-1)}\cos(2(d-1)\theta).
\end{aligned}$$

We can directly compute

$$
\begin{aligned}
\int_{A_\rho} \frac{|h|^3}{r^2} &= \mathcal{O}(|\ln\rho|), \qquad \int_{A_\rho} \frac{|h|^5}{r^2} = \mathcal{O}(|\ln\rho|),\\
\int_{A_\rho} |\partial_r h|^2 dx &= \left(1+\mathcal{O}\left(\frac{1}{|\ln\rho|}\right)\right) 2\pi \frac{\pi^2}{\ln^2\rho}\int_\rho^1 \cos^2\left(\frac{\pi\ln r}{\ln\rho}\right)\frac{dr}{r}\\
&= \frac{\pi^3}{|\ln\rho|} + \mathcal{O}\left(\frac{1}{\ln^2\rho}\right),\\
\int_{A_\rho} \frac{|\partial_\theta h|^2}{r^2} dx &= \frac{4(d-1)^2}{\ln^2\rho}\int_0^{2\pi}\cos^2(2(d-1)\theta))\,d\theta \int_\rho^1 \sin^2\left(\frac{\pi\ln r}{\ln\rho}\right)\frac{dr}{r}\\
&= \frac{2\pi(d-1)^2}{|\ln\rho|}
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{A_\rho} \frac{h^4}{r^2}(\psi_1')^2\,dx &= \left(1+\mathcal{O}\left(\frac{1}{|\ln\rho|}\right)\right)\\
&\quad\cdot \frac{d^2(2-d)^2}{4(d-1)^2}\int_0^{2\pi}\cos^2(2(d-1)\theta)\,d\theta\int_\rho^1 \sin^4\left(\frac{\pi\ln r}{\ln\rho}\right)\frac{dr}{r}\\
&= \frac{3\pi}{32}\frac{d^2(2-d)^2}{(d-1)^2}|\ln\rho| + \mathcal{O}(1).
\end{aligned}
$$

So, taking the expansions of η_δ and ν_δ into account, (34) implies

$$
\begin{aligned}
&E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho)\\
&= (\pi^3 + 2\pi(d-1)^2)\frac{\epsilon^2}{|\ln\rho|} + \frac{3\pi}{32}\frac{d^2(2-d)^2}{(d-1)^4}\delta^2\epsilon^4|\ln\rho|\\
&\quad + \delta\epsilon^3\int_{A_\rho}\eta^0\frac{h^3}{r^2}\,dx + \delta^2\epsilon^3\int_{A_\rho}\eta^1\frac{h^3}{r^2}\,dx + \delta\epsilon^5\int_{A_\rho}\nu^0\frac{h^5}{r^2}\,dx\\
&\quad + \mathcal{O}\left(\frac{\delta\epsilon^2}{|\ln\rho|}\right) + \mathcal{O}\left(\frac{\epsilon^2}{|\ln\rho|^2}\right) + \mathcal{O}(\delta\epsilon^4) + \mathcal{O}(\delta^3\epsilon^3 + \delta^2\epsilon^5)|\ln\rho|. \qquad (35)
\end{aligned}
$$

Moreover, we have

$$
h^3 = h_0^3\sum_{k=0}^{3}\left(\begin{array}{c}3\\k\end{array}\right)\frac{1}{|\ln\rho|^k}S^k_{2(d-1)}, \qquad h^5 = h_0^5\sum_{k=0}^{5}\left(\begin{array}{c}5\\k\end{array}\right)\frac{1}{|\ln\rho|^k}S^k_{2(d-1)},
$$

and deduce, using that $\int_0^{2\pi} S_{2(d-1)}d\theta = \int_0^{2\pi} S_{4(d-1)}d\theta = \int_0^{2\pi} S_{2(d-1)}S_{4(d-1)}d\theta = 0$,

$$\begin{aligned}
\int_{A_\rho} \frac{h^3}{r^2} S_{2(d-1)} &= \left(3\int_0^{2\pi} S_{2(d-1)}^2\, d\theta + \mathcal{O}\left(\frac{1}{|\ln\rho|}\right)\right)\frac{1}{|\ln\rho|}\int_\rho^1 h_0^3\frac{dr}{r} \\
&= 4\pi + \mathcal{O}\left(\frac{1}{|\ln\rho|}\right) \\
\int \frac{h^5}{r^2} S_{2(d-1)} &= \mathcal{O}\left(1\right), \qquad \int \frac{h^3}{r^2} S_{4(d-1)} = \mathcal{O}\left(\frac{1}{|\ln\rho|}\right).
\end{aligned}$$

Recalling that $\eta^0, \nu^0 \in \mathrm{span}(S_{2(d-1)})$, $\eta^1 \in \mathrm{span}(S_{4(d-1)})$ and plugging these into (35) we obtain

$$\begin{aligned}
&E_\delta(u; A_\rho) - E_\delta(\xi_\delta; A_\rho) \\
&= (\pi^3 + 2\pi(d-1)^2)\frac{\epsilon^2}{|\ln\rho|} + \frac{3\pi}{32}\frac{d^2(2-d)^2}{(d-1)^4}\delta^2\epsilon^4|\ln\rho| + \delta\epsilon^3\mathfrak{a} \\
&\quad + \mathcal{O}\left(\frac{\delta\epsilon^2}{|\ln\rho|}\right) + \mathcal{O}\left(\frac{\epsilon^2}{|\ln\rho|^2}\right) + \mathcal{O}(\delta\epsilon^4) + \mathcal{O}(\delta^3\epsilon^3 + \delta^2\epsilon^5)|\ln\rho|,
\end{aligned}$$

where $\mathfrak{a} = 4\int_0^{2\pi}\eta^0 S_{2(d-1)}\, d\theta$. Using the expressions of α^0 and β^0 in Lemma 3.3, we find

$$\eta^0 = -\alpha^0 + \frac{1}{3}\partial_\theta\beta^0 = -\frac{4}{3}d(d-2)S_{2(d-1)}, \qquad \text{so } \mathfrak{a} = -\frac{16\pi}{3}d(d-2).$$

Letting $\rho = e^{-\lambda/|\delta|}$ with $\lambda \gg 1$ and optimizing with respect to ϵ, we choose

$$\epsilon = -\mathrm{sign}(\delta)\frac{2^9}{9}\frac{(d-1)^4}{d(d-2)}\frac{1}{\lambda}.$$

For this choice of ϵ, the above expansion becomes

$$\begin{aligned}
&E_\delta(u_{\epsilon,\rho,\delta}; A_\rho) - E_\delta(\xi_\delta; A_\rho) \\
&= \frac{|\delta|\epsilon^2}{\lambda}\left(\pi^3 + 2\pi(d-1)^2 - \frac{2^{12}}{27}\pi(d-1)^4 + \mathcal{O}\left(\frac{1}{\lambda} + \delta\lambda\right)\right).
\end{aligned}$$

Since $|d-1| \geq 2$, we have

$$\begin{aligned}
\frac{2^{12}}{27}\pi(d-1)^4 \geq \frac{2^{14}}{27}\pi(d-1)^2 &\geq \left(\frac{2^{14}}{27} - 2\right)\cdot 4\pi + 2\pi(d-1)^2 \\
&> \pi^3 + 2\pi(d-1)^2 + \pi^3,
\end{aligned}$$

and may therefore fix a large enough $\lambda > 0$ such that $E_\delta(u_{\epsilon,\rho,\delta}; A_\rho) < E_\delta(\xi_\delta; A_\rho)$ for all small enough $|\delta| > 0$. $\square$

3.4. **Proof of the third item of Theorem 1.1.** Let $d \in \mathbb{Z} \setminus \{0,1,2\}$ and $|\delta|$ small enough that Proposition 3.4, and the conclusion of Proposition 3.8, are valid. Define, for $0<\rho<1$,

$$G(\rho) = \inf_{u_{\lfloor \partial A_\rho}=\xi_\delta} E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho).$$

The function G is monotone nondecreasing, since for $0<\rho<\rho'<1$, any admissible competitor $u' \in H^1(A_{\rho'};\mathbb{S}^1)$ with $u'=\xi_\delta$ on $\partial A_{\rho'}$ can be extended by ξ_δ in $A_\rho \setminus A_{\rho'}$ to become an admissible competitor in A_ρ, which implies

$$\inf_{u_{\lfloor \partial A_\rho}=\xi_\delta} E_\delta(u;A_\rho) \le E_\delta(u';A_{\rho'}) + E_\delta(\xi_\delta;A_\rho) - E_\delta(\xi_\delta;A_{\rho'}).$$

Substracting $E_\delta(\xi_\delta;A_\rho)$ and taking the infimum over all admissible u' gives $G(\rho) \le G(\rho')$.

The function G is also continuous, since for $0<\rho<\rho'<1$, any admissible competitor $u\in H^1(A_\rho;\mathbb{S}^1)$ with $u=\xi_\delta$ on A_ρ can be dilated as follows,

$$u'(r'e^{i\theta}) = u(re^{i\theta}), \quad r = \frac{1-\rho}{1-\rho'}r' - \frac{\rho'-\rho}{1-\rho'} \text{ for } r' \in (\rho',1),$$

so that u' is an admissible competitor in $A_{\rho'}$, which implies

$$\inf_{u_{\lfloor \partial A_{\rho'}}=\xi_\delta} E_\delta(u;A_{\rho'}) \le E_\delta(u';A_{\rho'}) \le E_\delta(u;A_\rho) + \mathcal{O}\left(\frac{\rho'-\rho}{1-\rho'}\right) E_\delta(u;A_\rho)$$

and we deduce

$$0 \le G(\rho') - G(\rho) \le \mathcal{O}\left(\frac{\rho'-\rho}{1-\rho'}\ln\frac{1}{\rho} + \ln\frac{\rho'}{\rho}\right).$$

So G is continuous and monotone nondecreasing on $(0,1)$. Combining this with Propositions 3.4 and 3.8 implies the existence of $\rho_* \in (0,1)$ such that $G\equiv 0$ on $[\rho_*,1)$ and $G<0$ on $(0,\rho_*)$, which proves the third item of Theorem 1.1. $\square$

4. The cases of degree 1 and 2

4.1. **The degree 1 case.** In this section we show that, for $0<|\delta|<1$, the only 0-homogenous solutions of (3) which have degree 1 are the trivial solutions $\xi(\theta) = e^{i\alpha}e^{i\theta}$, $\alpha \equiv 0$ modulo $\pi/2$.

To that end we write an arbitrary 0-homogeneous solution of (3) of degree 1, in the form $\xi(\theta) = e^{i\theta}e^{i\varphi(\theta)}$. Then we have the equation

$$\frac{d}{d\theta}\left[(1+\delta\cos(2\varphi))(1+\varphi')\right] = -\delta\sin(2\varphi)(1+\varphi')^2, \tag{36}$$

that is,

$$(1+\delta\cos(2\varphi))\varphi'' = \delta\sin(2\varphi)\left((\varphi')^2-1\right),$$

which we may also rewrite as the 1st order system

$$\dot{x} = y$$
$$\dot{y} = \frac{\delta \sin(2x)}{1+\delta\cos(2x)}(y^2-1).$$

Our goal is to find all 2π-periodic solutions of that ODE. Note that we have the conserved quantity $\frac{d}{d\theta}H(\varphi,\varphi')=0$, where

$$H(x,y) = (1+\delta\cos(2x))\left(1-y^2\right).$$

We assume from now on that $\delta>0$ (the case $\delta<0$ can be recovered using the symmetries), so we have $H\leq 1+\delta$.

For $H_0<1-\delta$, the level set $\{H=H_0\}$ is the union of two unbounded curves

$$y=\pm\sqrt{1-\frac{H_0}{1+\delta\cos(2x)}},$$

hence $H_0<1-\delta$ cannot correspond to a periodic solution φ.

For $1-\delta<H_0<1+\delta$, the level set $\{H=H_0\}$, intersected with $\{|x|<\pi/2\}$, is a closed curve, which crosses the x-axis at

$$x_\pm = \pm\frac{1}{2}\arccos\left(\frac{H_0-1}{\delta}\right).$$

It corresponds to a periodic trajectory of the differential system, whose half-period is the time needed to go from x_- to x_+ along the curve

$$\dot{x}=y=\sqrt{1-\frac{H_0}{1+\delta\cos(2x)}},$$

so the corresponding period $T=T_\delta(H_0)$ is given by

$$\begin{aligned}
T_\delta(H_0) &= 2\int_{x_-}^{x^+}\frac{dx}{\sqrt{1-\frac{H_0}{1+\delta\cos(2x)}}}\\
&= 2\int_0^{\arccos\left(\frac{H_0-1}{\delta}\right)}\frac{dx}{\sqrt{1-\frac{H_0}{1+\delta\cos(x)}}}\\
&= \frac{2}{\sqrt{\delta}}\int_\sigma^1\frac{\sqrt{1+\delta t}}{\sqrt{(1-t^2)(t-\sigma)}}dt, \qquad \sigma=\frac{H_0-1}{\delta}\in(-1,1)
\end{aligned}$$

For any $\sigma \in (-1,1)$ and $\delta \in (0,1)$ we have

$$\begin{aligned}\frac{d}{d\delta}\left[T_\delta(1+\delta\sigma)\right] &= \frac{d}{d\delta}\left[\frac{2}{\sqrt{\delta}}\int_\sigma^1 \frac{\sqrt{1+\delta t}}{\sqrt{(1-t^2)(t-\sigma)}}dt\right] \\ &= -\frac{1}{\delta^{\frac{3}{2}}}\int_\sigma^1 \frac{dt}{\sqrt{(1+\delta t)(1-t^2)(t-\sigma)}} < 0,\end{aligned}$$

so we infer

$$\begin{aligned}T_\delta(1+\delta\sigma) > T_1(1+\sigma) &= 2\int_\sigma^1 \frac{dt}{\sqrt{(1-t)(t-\hat{\sigma})}}dt \\ &= 2\left[\arcsin\left(\frac{2}{1-\sigma}t - \frac{1+\sigma}{1-\sigma}\right)\right]_\sigma^1 \\ &= 2\pi\end{aligned}$$

Therefore, all periodic solutions of (36) corresponding to values of $H_0 \in (1-\delta, 1+\delta)$ have a period strictly larger than 2π. Hence the only 2π-periodic solutions of (36) must corresponds to values $H_0 \in \{1 \pm \delta\}$.

For $H_0 = 1-\delta$, the level set $\{H = H_0\}$, intersected with $\{|x| \le \pi/2\}$, is also a closed curve, but its intersections $x = x_\pm$ with the x-axis correspond to the constant solutions $x \equiv \pm\pi/2$, so all other solutions corresponding to $H_0 = 1-\delta$ are monotone and cannot be periodic. Hence the only periodic solutions corresponding to $H_0 = 1-\delta$ are constants $\varphi \equiv \pi/2$ modulo π.

The only remaining value of H_0 is $H_0 = 1+\delta$, which corresponds to constant solutions $\varphi \equiv 0$ modulo π.

Gathering all cases, we conclude that the only 2π-periodic solutions of (36) are $\varphi \equiv 0$ modulo $\pi/2$.

4.2. **The degree 2 case.** In this Section we show that, for $d = 2$, the unique solution (modulo frame invariance) provided by the first item in Theorem 1.1 is minimizing.

In fact it turns out that $\xi_\delta(\theta) = e^{2i\theta}$ solves (12), so this is the unique solution (modulo frame invariance). Taking a competitor $u = e^{2i\theta}e^{i\varphi}$ with $\varphi \in C^2_c(A_\rho;\mathbb{R})$, and using

$$(\nabla\cdot u)^2 - (\nabla\times u)^2 = \mathfrak{Re}((\partial_\eta \bar{u})^2), \quad \partial_\eta = \partial_x + i\partial_y = e^{i\theta}\left(\partial_r + \frac{i}{r}\partial_\theta\right),$$

we obtain

$$\begin{aligned}
&|\nabla u|^2 + \delta\left((\nabla\cdot u)^2 - (\nabla\times u)^2\right) \\
&= |\nabla\varphi|^2 + \frac{4}{r^2} + \frac{4}{r^2}\partial_\theta\varphi + \delta\,\mathfrak{Re}\left(\left(-i\partial_r\varphi + \frac{2+\partial_\theta\varphi}{r}\right)^2 e^{-2i(\theta+\varphi)}\right) \\
&= |\nabla\varphi|^2 + \frac{4}{r^2} + \frac{4}{r^2}\partial_\theta\varphi \\
&\quad + \delta\left(\frac{(2+\partial_\theta\varphi)^2}{r^2} - (\partial_r\varphi)^2\right)\cos(2\theta+2\varphi) \\
&\quad - 2\delta\partial_r\varphi\frac{2+\partial_\theta\varphi}{r}\sin(2\theta+2\varphi),
\end{aligned}$$

and therefore, substracting the energy density of ξ_δ (which corresponds to $\varphi = 0$) and integrating over A_ρ, we have

$$\begin{aligned}
&E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho) \\
&= \int_{A_\rho}\Big[|\nabla\varphi|^2 + \delta\left(\frac{(\partial_\theta\varphi)^2 + 4\partial_\theta\varphi}{r^2} - (\partial_r\varphi)^2\right)\cos(2\theta+2\varphi) \\
&\qquad\qquad + \frac{4\delta}{r^2}\cos(2\theta+2\varphi) - 2\delta\partial_r\varphi\frac{2+\partial_\theta\varphi}{r}\sin(2\theta+2\varphi)\Big]\,dx.
\end{aligned}$$

Noting that

$$\begin{aligned}
&\frac{1+\partial_\theta\varphi}{r^2}\cos(2\theta+2\varphi) - \frac{1}{r}\partial_r\varphi\sin(2\theta+2\varphi) \\
&= \frac{1}{2r^2}\partial_\theta[\sin(2\theta+2\varphi)] + \frac{1}{2r}\partial_r[\cos(2\theta+2\varphi)],
\end{aligned}$$

this simplifies to

$$\begin{aligned}
&E_\delta(u;A_\rho) - E_\delta(\xi_\delta;A_\rho) \\
&= \int_{A_\rho}\Big[(1-\delta\cos(2\theta+2\varphi))(\partial_r\varphi)^2 + (1+\delta\cos(2\theta+2\varphi))\frac{(\partial_\theta\varphi)^2}{r^2} \\
&\qquad\qquad - 2\delta\partial_r\varphi\frac{\partial_\theta\varphi}{r}\sin(2\theta+2\varphi)\Big]\,dx
\end{aligned}$$

For any $\delta\in(-1,1)$ and $C,S\in[-1,1]$ such that $C^2+S^2=1$, the quadratic form

$$q(X,Y) = (1-\delta C)X^2 + (1+\delta C)Y^2 - 2\delta SXY,$$

has determinant $\det(q) = 1-\delta^2C^2-\delta^2S^2 = 1-\delta^2 > 0$ and is therefore positive definite, so $\xi_\delta(\theta) = e^{2i\theta}$ is minimizing in A_ρ, for all $\delta\in(-1,1)$.

5. Entire solutions of the anisotropic Ginzburg-Landau equation

In this section we prove Corollary 1.3. So we consider an entire solution $u\colon \mathbb{R}^2 \to \mathbb{R}^2$ of the anisotropic Ginzburg-Landau equation (8) with finite potential energy (10) and degree $\deg(u) = d \in \mathbb{Z} \setminus \{0,1,2\}$. The anisotropy satisfies $0 < |\delta| < \delta_0$, for a small enough $\delta_0 \in (0,1)$ to be adjusted in the course of the proof.

We assume that u is either locally minimizing, or symmetric (11) and locally minimizing with respect to symmetric competitors. Under these assumptions, the methods in [21, Lemma 4.3] provide a logarithmic bound for the energy (9) of u,

$$\liminf_{R\to+\infty} \frac{GL_\delta(u;D_R)}{\ln R} < \infty. \tag{37}$$

The statement of [21, Lemma 4.3] considers symmetric solutions (11) with an additional mirror symmetry constraint, but the same proof applies for non-symmetric or less symmetric solutions, as long as they are locally minimizing in their admissible class.

If δ_0 is small enough, the third point of Theorem 1.1 ensures the existence of $\rho \in (0,1/2)$ such that any map $u_* \in H^1(A_{2\rho};\mathbb{S}^1)$ which minimizes E_δ among $\mathbb{S}^1$-valued maps agreeing with u_* on $\partial A_{2\rho}$, cannot be 0-homogeneous:

$$\int_{A_{2\rho}} |\partial_r u_*|^2 dx > 0. \tag{38}$$

The same conclusion is valid if u_* is symmetric and minimizing only among symmetric maps, because the competitor in Proposition 3.8 is symmetric.

As a first step to prove Corollary 1.3, we claim that the logarithmic bound (37) implies

$$\liminf_{R\to+\infty} GL_\delta(u;D_{2R}\setminus D_{\rho R}) < \infty. \tag{39}$$

Otherwise, for any $M>0$ we have the existence of $R_0>0$ such that

$$GL_\delta(u;D_{2R}\setminus D_{\rho R}) = GL_\delta(u;D_{2R}) - GL_\delta(u;D_{\rho R}) \geq M \qquad \forall R\geq R_0.$$

Applying this to $R=(2/\rho)^j R_0$ and summing over $j=1,\ldots,k$, we deduce

$$GL_\delta(u;D_{(2/\rho)^k R_0}) \geq kM = \frac{M}{\ln(2/\rho)} \ln\frac{(2/\rho)^k R_0}{R_0},$$

which implies

$$\liminf_{R\to+\infty} \frac{GL_\delta(u;D_R)}{\ln R} \geq \frac{M}{\ln(2/\rho)},$$

in contradiction with (37) since $\rho\in(0,1/2)$ is fixed and M is arbitrary. So (39) is established, and there exists a sequence $R_k\to+\infty$ such that

$$GL_\delta(u;D_{2R_k}\setminus D_{\rho R_k}) \leq C,$$

for some constant $C > 0$. Following [26] we define the rescaled map

$$u_k(x) = u(R_k x),$$

and $\epsilon_k = 1/R_k$, so the above energy bound in $D_{2R_k} \setminus D_{\rho R_k}$ translates into a bound on the energy GL_{δ,ϵ_k} (7) of u_k in $D_2 \setminus D_\rho$, namely

$$GL_{\delta,\epsilon_k}(u_k; D_2 \setminus D_\rho) \leq C.$$

Since u_k is minimizing with respect to its own boundary conditions, with or without the symmetry constraint, standard methods (see e.g. [4, § 3] combined with an appropriate selection of traces as for instance in [11, Appendix B]) imply that, up to a non-relabeled subsequence, $u_k \to u_*$ in $H^1(A_{2\rho}; \mathbb{R}^2)$, and $u_* \in H^1(A_{2\rho}; \mathbb{S}^1)$ minimizes E_δ among $\mathbb{S}^1$-valued maps agreeing with u_* on $\partial A_{2\rho}$. Therefore u_* is not 0-homogeneous (38). This implies that $\partial_r u_k$ has a nontrivial limit in L^2 and, scaling back to the originial variable,

$$\liminf_{k\to\infty} \int_{D_{R_k} \setminus D_{2\rho R_k}} |\partial_r u|^2 \, dx > 0.$$

Along a subsequence, the annuli $D_{R_k} \setminus D_{2\rho R_k}$ are all disjoint, and we deduce the first conclusion of Corollary 1.3, that is,

$$\int_{\mathbb{R}^2} |\partial_r u|^2 \, dx = +\infty.$$

Further, by continuity of the trace embedding, for all $r \in [2\rho, 1]$ we have $u_k \to u_*$ in $L^2(\partial D_r; \mathbb{R}^2)$. Since u_* is not 0-homogeneous we may find $r_1, r_2 \in [2\rho, 1]$ such that $u_*(r_1 e^{i\theta}) \neq u_*(r_2 e^{i\theta})$ for a non-negligible set of $\theta \in \mathbb{S}^1$. As a consequence, $u_R(e^{i\theta}) = u(Re^{i\theta})$ converges to different limits in $L^2(\mathbb{S}^1; \mathbb{R}^2)$ along the sequences $R = r_1 R_k \to +\infty$ and $R = r_2 R_k \to +\infty$, which implies the last assertion of Corollary 1.3, that u_R is not convergent as $R \to +\infty$ (in $L^2(\mathbb{S}^1; \mathbb{R}^2)$, nor in the sense of distributions). $\square$

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