

SINGULAR PERTURBATION OF MANIFOLD-VALUED MAPS WITH ANISOTROPIC ENERGY

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ABSTRACT. We establish small energy Hölder bounds, uniform with respect to $\varepsilon \in (0, 1)$, for minimizers u_ε of

$$E_\varepsilon(u) := \int_{\Omega} W(\nabla u) + \frac{1}{\varepsilon^2} \int_{\Omega} f(u),$$

where W is a positive definite quadratic form and the potential f constrains u to be close to a given manifold \mathcal{N} . This implies that, up to subsequence, u_ε converges locally uniformly to an \mathcal{N} -valued W -harmonic map, away from its singular set. This is the first result of its kind for general anisotropic energies covering in particular the previously open case of 3D Landau-de Gennes model for liquid crystals, with three distinct elastic constants. Similar results in the isotropic case $W(\nabla u) = |\nabla u|^2$ rely on three ingredients: a monotonicity formula for the scale-invariant energy on small balls, a uniform pointwise bound, and a Bochner equation for the energy density; all of these ingredients are absent for general anisotropic W 's. In particular, the lack of monotonicity formula is an important reason why optimal estimates on the singular set of W -harmonic maps constitute an open problem. To circumvent these difficulties we devise an argument that relies on showing appropriate decay for the energy on small balls, separately at scales smaller and larger than ε : the former is obtained from the regularity of solutions to elliptic systems while the latter is inherited from the regularity of W -harmonic maps. This also allows us to handle physically relevant boundary conditions for which, even in the isotropic case, uniform convergence up to the boundary was open.

1. INTRODUCTION

Let $\Omega \subset \mathbb{R}^n$ ($n \geq 3$) be a smooth domain and $u: \Omega \rightarrow \mathbb{R}^k$. For $\varepsilon > 0$ define:

$$E_\varepsilon(u; \Omega) := \int_{\Omega} W(x, \nabla u) + \frac{1}{\varepsilon^2} \int_{\Omega} f(u).$$

Here $f: \mathbb{R}^k \rightarrow [0, \infty)$ is a smooth potential such that $\mathcal{N} = \{f = 0\}$ is a smooth submanifold of \mathbb{R}^k , with f vanishing nondegenerately on \mathcal{N} , and $W: \Omega \times \mathbb{R}^{k \times n} \rightarrow [0, \infty)$ is an elastic energy density such that $W(x, \cdot)$ is a positive definite quadratic form on $\mathbb{R}^{k \times n}$, uniformly in x .

We are interested in the behavior, as $\varepsilon \rightarrow 0$, of minimizers of E_ε with respect to general boundary conditions: strong or weak anchoring in the terminology of liquid crystals. Strong anchoring corresponds to Dirichlet boundary conditions

$$u_\varepsilon = u_b \quad \text{on } \partial\Omega,$$

where we assume u_b to be regular and to take values into \mathcal{N} . Weak anchoring corresponds to minimizing the modified functional

$$(1.1) \quad F_\varepsilon(u; \Omega) := E_\varepsilon(u; \Omega) + \int_{\partial\Omega} g(x, u),$$

where $g : \partial\Omega \times \mathbb{R}^k \rightarrow [0, \infty)$ is a C^2 function. (This translates into Neumann-type boundary conditions but we will only use the variational formulation.) It can be checked (see e.g. [35]) that minimizers of E_ε converge, up to subsequence, strongly in H^1 , to a map $u_* : \Omega \rightarrow \mathcal{N}$ which minimizes the energy

$$E_*(u; \Omega) = \int_{\Omega} W(x, \nabla u), \quad u : \Omega \rightarrow \mathcal{N}$$

subject to the same Dirichlet boundary conditions, in the strong anchoring case. Similarly, in the weak anchoring situation, minimizers of F_ε converge to minimizers of

$$F_*(u; \Omega) = \int_{\Omega} W(x, \nabla u) + \int_{\partial\Omega} g(x, u), \quad u : \Omega \rightarrow \mathcal{N},$$

again, strongly in H^1 and up to subsequence. (In fact strong H^1 compactness of bounded energy sequences holds locally, without fixing boundary conditions, see Appendix B.) All difficulties arising in this article are already present in the case of W with constant coefficients. We are not concerned with the critical dimension $n = 2$, where the energy monotonicity formula (whose absence in dimension $n \geq 3$ constitutes, as explained below, one of the main difficulties) is automatically satisfied.

A strong motivation for the study of the class of energy functionals E_ε , comes from their connection to physical problems in material sciences. An important occurrence is the Landau-de Gennes energy for nematic liquid crystals [37], where the unknown is a map $Q : \Omega \rightarrow \mathcal{S}_0 := \{Q \in \mathbb{R}_{sym}^{3 \times 3}, \text{tr } Q = 0\} \simeq \mathbb{R}^5$, and

$$(1.2) \quad W_{LdG}(\nabla Q) = L_1 |\nabla Q|^2 + L_2 \partial_j Q_{ik} \partial_k Q_{ij} + L_3 \partial_j Q_{ij} \partial_k Q_{ik},$$

$$(1.3) \quad f_{LdG}(Q) = a^2 |Q|^2 - b^2 \text{tr}(Q^3) + c^2 |Q|^4.$$

(In (1.2) and in the rest of the article, summation over repeated indices is implicitly assumed.) The vacuum manifold is $\mathcal{N} = \{s_*(n \otimes n - I/3) : n \in \mathbb{S}^2\}$ for some $s_*(a, b, c) > 0$. In order for W_{LdG} to be positive definite, the elastic constants satisfy (see e.g. [31])

$$(1.4) \quad L_1 + L_2 > 0, \quad 2L_1 - L_2 > 0 \quad \text{and} \quad 6L_1 + L_2 + 10L_3 > 0.$$

This theory has motivated a wealth of new mathematical results in the past few years, regarding e.g. the London limit $\varepsilon \rightarrow 0$ [35, 13, 14, 9, 20, 18], the fine structure of defects [29, 30, 19, 27, 28, 13, 16, 24], colloidal suspensions [3, 2, 4] or lifting issues [5, 8, 6, 11, 26]. The isotropic case corresponds to $L_2 = L_3 = 0$, a restriction which was assumed in most of the above works. The general anisotropic case of three distinct elastic constants has remained largely unexplored, due to the many mathematical challenges involved (see e.g. [31]). In particular, our results extend the conclusions of [35, 38, 18] to any L_1, L_2 and L_3 such that W remains positive definite. Note that another, physically motivated potential $f(Q)$ was introduced in [7], to which it would be interesting to extend our analysis.

In the isotropic case

$$W_{iso}(\nabla u) = |\nabla u|^2 \quad (\text{or } |\nabla_g u|^2 \text{ for some Riemannian metric } g),$$

minimizers of E_* are \mathcal{N} -valued harmonic maps. They are smooth outside a rectifiable singular set of dimension at most $(n-3)$, and the convergence of u_ε towards u_* is locally uniform away from this singular set and from the boundary [15, 35, 18]. Moreover, for fixed \mathcal{N} -valued

Dirichlet boundary conditions, the convergence is also uniform up to the boundary [38, 18]. For the Ginzburg-Landau potential $k = 2$, $f(u) = (1 - |u|^2)^2$, uniform convergence up to the boundary is also obtained for weak anchoring in the special case $g(x, u) = |u - u_b(x)|^2$ in [10].

For more general anisotropic elastic energies, the regularity of minimizers of E_\star is not fully understood. It is known that the singular set has dimension strictly less than $n - 2$ [23, 25], but due to the failure of the energy monotonicity formula, Federer's dimension reduction argument can not be applied to show that the singular set has dimension at most $n - 3$. It is an open problem to find the optimal estimate on the dimension of the singular set for these anisotropic harmonic maps, but not only that, the uniform convergence away from the singular set has also proved to be an elusive question due to the technical limitations of the classical approaches, mainly derived from the theory of harmonic maps. Here we address this open question and extend the results in [18] to include, for the first time, anisotropic elastic energies.

Comparison with the isotropic case. To shed some light on the underlying difficulties, we mention that the available proofs of uniform convergence for the isotropic energy [35, 38, 18] follow the strategy of [15], inspired by [39] (related results can be found in [12] for $n = 2$, and [10] for higher dimensions, in the case of the Ginzburg-Landau potential). The main tool is a small energy estimate, which relies on 3 crucial ingredients:

- a uniform L^∞ bound $\|u_\varepsilon\|_{L^\infty} \leq M$,
- a ‘‘Bochner type’’ inequality $-\Delta e_\varepsilon \lesssim e_\varepsilon^2$ satisfied by $e_\varepsilon = \frac{1}{2} |\nabla u_\varepsilon|^2 + \varepsilon^{-2} f(u_\varepsilon)$,
- and a monotonicity formula for the renormalized energy $\frac{d}{dr}[r^{2-n} E_\varepsilon(u_\varepsilon; B_r)] \geq 0$.

All three of these ingredients do not seem to be available in the anisotropic case. We circumvent these difficulties by adopting a more fundamental, but flexible variational approach (in contrast with the PDE ones used in the isotropic case). Moreover, our novel strategy allows us to obtain uniform estimates at the boundary for strong and weak anchoring, while the previous methods could only deal with strong anchoring (see [18]).

Locally uniform convergence for general anisotropic energies. As pointed out before, we already know that a subsequence of minimizers u_ε of the functionals E_ε (resp. F_ε) converges in H^1 to a generalized harmonic map u_\star , i.e. a minimizer of E_\star (resp. F_\star). In order to improve this to uniform convergence away from the singular set \mathcal{S} of u_\star , we establish uniform Hölder bounds for u_ε on compact subsets of $\bar{\Omega} \setminus \mathcal{S}$. Classically, this is done by means of a small energy estimate:

$$r^{2\alpha} |u_\varepsilon|_{C^\alpha(B(x_0, r))}^2 \leq C(W, f) r^{2-n} E_\varepsilon(u_\varepsilon; B(x_0, 2r)),$$

for a constant $C(W, f)$ depending on the specified parameters (but not on ε), provided the renormalized energy $r^{2-n} E_\varepsilon(u_\varepsilon; B(x_0, 2r))$ is small enough. Granted such estimate, H^1 convergence automatically improves to uniform convergence away from \mathcal{S} , since there the renormalized energy of u_\star is small. Therefore we will concentrate on proving small energy estimates, in the interior and at the boundary.

For u_\star , an equivalent of this small energy estimate is indeed valid, and at the core of the regularity theory in [22, 34]. However its proof relies strongly on the scaling invariance of the energy E_\star . It is at this level that a big difference arises: our perturbed energy E_ε contains two terms which scale differently. This is reflected in the presence of a characteristic length scale

ε . At scales larger than ε we expect minimizers to behave like generalized harmonic maps (i.e. minimizers of u_\star). As we move to finer scales, the particular shape of the potential f plays a more prominent role, and thus our minimizer resembles less and less a harmonic map. Note that in the isotropic case this effect can be somewhat controlled, as the monotonicity formula ensures that if the energy is small at one scale, then it is automatically small at all smaller scales.

Of the three crucial ingredients for the small energy estimate, which are present in the isotropic case but not here – namely, the uniform pointwise bound, the Bochner equation satisfied by the energy density, and the energy monotonicity formula – the uniform pointwise bound $\|u_\varepsilon\|_\infty \leq M$ turns out to be the most problematic. We manage in fact to develop a general method that needs neither Bochner equation nor energy monotonicity, but, in order to avoid assuming an *a priori* L^∞ bound, we need to restrict ourselves to dimension $n = 3$ and potentials f satisfying some additional technical assumption. We use these restrictions in our proof of Lemma 2.5 which deals with the scales below ε (see also Remarks 1.1 and 1.3 for more insight into why we need them). These restrictions are satisfied in the physically relevant case of the Landau-de Gennes functional. To the best of our knowledge, this work is the first to treat this more general model.

The key elements that allow us to overcome the lack of Bochner equation and monotonicity formula can be explained as follows. Establishing a Hölder estimate amounts to proving a suitable energy decay on small balls, and we do this in two steps.

- In the first step we obtain energy decay on balls of radii much larger than ε . We rely on variational arguments inspired from the harmonic map literature: carefully constructed comparison maps lead to an energy improvement estimate showing energy decay from one fixed scale to another fixed smaller scale. For anisotropic harmonic maps (minimizers of E_\star) scale invariance then allows to iterate this estimate and prove energy decay and Hölder regularity. Here in contrast we can only iterate this as long as the scale remains much larger than ε . In fact the absence of scale invariance constitutes an obstacle not only to iterating the energy improvement estimate, but also to obtaining it in the first place. For anisotropic harmonic maps, the contradiction argument leading to energy improvement relies indeed on blowing up the image of \mathcal{N} -valued maps at scale given by the square root of their energy [22, 34]. In our case the potential term behaves very badly with respect to such blow-up. That is why we first need to perform, with controlled energy cost, a homotopy between the boundary values of our minimizers and \mathcal{N} -valued boundary values. The associated \mathcal{N} -valued minimizer could then have strictly larger energy and we blow up the original map at this possibly larger scale, while for anisotropic harmonic maps there was only one relevant blow-up scale.
- In the second step we deal with scales of order ε and below. There, the energy decay is obtained from elliptic estimates for a fixed ε_0 . We exploit the shape of the potential in a decisive manner to be able to connect these estimates to the ones in the first step. This is where the uniform L^∞ bound (A3a) plays a crucial role, and in its absence we have to resort to the technical assumption (A3b) which leads to a non-standard bootstrapping argument (whence the two growth conditions on f and ∇f).

In what follows, we always make the following two generic assumptions:

- $W(x, \nabla u)$ is a positive definite quadratic form on $\mathbb{R}^{k \times n}$ with coefficients depending smoothly on x , i.e. $W(x, \nabla u) = a_{ij}^{\alpha\beta}(x) \partial_i u^\alpha \partial_j u^\beta$ with

$$(A1) \quad \lambda |\xi|^2 \leq a_{ij}^{\alpha\beta} \xi_i^\alpha \xi_j^\beta \leq \Lambda |\xi|^2 \quad \forall \xi \in \mathbb{R}^{k \times n}, \quad \text{and} \quad \left\| a_{ij}^{\alpha\beta} \right\|_{C^1} \leq \Lambda,$$

for some $\Lambda > \lambda > 0$,

- f vanishes nondegenerately on \mathcal{N} , i.e. $\nabla^2 f(z)$ restricted to $(T_z \mathcal{N})^\perp$ is positive definite for all $z \in \mathcal{N}$, and f does not vanish at infinity. This implies (see e.g. [18]) that

$$(A2) \quad f(z) \lesssim \text{dist}^2(z, \mathcal{N}) \lesssim f(z) \quad \text{for } z \text{ close enough to } \mathcal{N}, \quad \text{and} \quad \liminf_{|z| \rightarrow \infty} f(z) > 0.$$

Here and throughout the article, the symbol \lesssim will denote inequality up to a multiplicative constant that depends only on the fixed parameters (f, W, g) , unless otherwise specified.

But, in addition to these natural requirements, we will assume either that there exists $M > 0$ such that

$$(A3a) \quad \|u_\varepsilon\|_\infty \leq M \quad \forall \varepsilon > 0,$$

or that

$$(A3b) \quad n = 3, \text{ and there exist } p > \frac{3}{2} \text{ and } \frac{1}{2} \leq a \leq \min\left(\frac{4}{5}, \frac{4}{3} - \frac{1}{p}\right),$$

such that $|\nabla f(z)| \lesssim |z|^{\frac{6}{p}}$ and $|\nabla f(z)| \lesssim f(z)^a$ as $|z| \rightarrow \infty$.

Remark 1.1. We actually expect the pointwise bound (A3a) to hold true for minimizers of E_ε under rather mild conditions on f . In the isotropic case, one only needs to assume that $u \cdot \nabla f(u) \geq 0$ for $|u| \geq M$ (in fact this is valid for all critical points that satisfy the Euler-Lagrange equations [33]), but establishing this estimate in the anisotropic case turns out to be surprisingly difficult. To shed more light on this issue, note that one obvious difference between isotropic and general anisotropic W 's is that, in the isotropic case, the linear second order elliptic operator \mathcal{L} associated to critical points of $\int W(\nabla u)$ comes in the form of a scalar operator acting separately on each component, while in the anisotropic case it really couples all components. It is known that, even for minimizers of $\int W(\nabla u)$ without any constraint, i.e. solutions of $\mathcal{L}u = 0$, the maximum principle $\|u\|_{L^\infty(\Omega)} \leq \|u\|_{L^\infty(\partial\Omega)}$ holds in this sharp form if and only if \mathcal{L} is (up to a linear change of variables) isotropic [32, Theorem 2.4]. In general one only has $\|u\|_{L^\infty(\Omega)} \leq C \|u\|_{L^\infty(\partial\Omega)}$ for some $C > 1$. In other words the elastic term $W(\nabla u)$ does somehow penalize large pointwise values of u , but not as sharply as it does in the isotropic case: therefore one can not hope to directly generalize the isotropic arguments for (A3a) to the anisotropic case.

The following theorem is a corollary of our main results Theorems 2.1, 3.1 and 3.4.

Theorem 1.2. *Let (u_{ε_k}) be a sequence of minimizers of E_{ε_k} in Ω (with respect to their own boundary conditions) and assume that u_{ε_k} converges strongly in H^1 to $u_\star \in C^\alpha(\overline{\Omega} \setminus \mathcal{S})$. Assume (A1)-(A2), and moreover that either (A3a) or (A3b) holds. Then we have:*

- Interior convergence:

$$u_{\varepsilon_k} \longrightarrow u_\star \text{ locally uniformly in } \Omega \setminus \mathcal{S},$$

- Convergence up to the boundary for strong anchoring: if $u_{\varepsilon_k} = u_b$ on $\partial\Omega$ for some fixed $u_b \in C^2(\partial\Omega; \mathcal{N})$, then

$$u_{\varepsilon_k} \longrightarrow u_\star \text{ locally uniformly in } \bar{\Omega} \setminus \mathcal{S},$$

- Convergence up to the boundary for weak anchoring: if u_{ε_k} minimizes F_{ε_k} (1.1), and (A3a) holds, then

$$u_{\varepsilon_k} \longrightarrow u_\star \text{ locally uniformly in } \bar{\Omega} \setminus \mathcal{S}.$$

Remark 1.3. A few observations about this technical assumption are in order.

- Note that under assumption (A3b), we are able to obtain the uniform convergence away from \mathcal{S} , but we still do not know if the uniform bound (A3a) holds.
- Note also that in the weak anchoring case we cannot avoid assuming (A3a). We will comment more about this in § 3.2.
- Finally, we point out that (A3b) consists of two growth requirements on ∇f in dimension $n = 3$. The first growth requirement $|\nabla f(z)| \lesssim |z|^{6/p}$ is fairly natural; it allows to obtain Hölder continuity of any solution u of the Euler-Lagrange equations for a fixed $\varepsilon_0 > 0$, via classical arguments relying on Calderon-Zygmund estimates. On the other hand, the second hypothesis $|\nabla f| \lesssim f^a$ is, admittedly, less natural but it is what ultimately allows us to make the connection with the estimates at large scales obtained in the first step. Concerning the first growth requirement, alternate hypotheses – that also apply to the Landau-de Gennes energy – are available, but we choose the current presentation due to its transparency.

Remark 1.4. The convergence statements in Theorem 1.2 are consequences of local C^β bounds proved in Theorems 2.1, 3.1 and 3.4, so we actually have convergence in $C_{loc}^\gamma(\bar{\Omega} \setminus \mathcal{S})$ for all $0 < \gamma < \beta$. In fact our methods provide C^γ convergence for any $0 < \gamma < 1$, let us quickly explain why:

- Our estimates at large scales (Lemmas 2.2, 3.2 and 3.6) provide energy decay corresponding to C^β regularity for any $0 < \beta < 1$.
- Regarding small scales (Lemmas 2.5, 3.3 and 3.7), under assumption (A3a) we obtain a Lipschitz bound. The combination of the large and small scale estimates (see Section 2.3) then provides a C^β bound, and β can be arbitrarily close to 1. Under assumption (A3b) the small scale estimate only provides a fixed Hölder exponent $\beta_0 < 1$. But the local C^{β_0} bound implies a uniform L^∞ bound in any compact $K \subset \bar{\Omega} \setminus \mathcal{S}$, which allows to *a posteriori* apply the estimates obtained under assumption (A3a), and conclude that we have local C^γ convergence for any $0 < \gamma < 1$ in all cases.

Landau-de Gennes with three distinct elastic constants. In the physically relevant case of the Landau-de Gennes potential, assumption (A3b) is satisfied with $p = 2$ and $a = 3/4$, since $f_{LdG}(Q) \gtrsim |Q|^4$ and $|\nabla f_{LdG}(Q)| \lesssim |Q|^3$ as $|Q| \rightarrow \infty$. Thus Theorem 1.2 is the first to provide an unconditional result in this context. More explicitly, specializing to the Landau-de Gennes model, Theorem 1.2 says:

Theorem 1.5. *Let $\Omega \subseteq \mathbb{R}^3$ a smooth bounded domain, $(Q_\varepsilon)_{\varepsilon>0} \subseteq H^1(\Omega; \mathcal{S}_0)$ be a family of minimizers with respect to their own boundary conditions of the Landau-de Gennes energy*

$$E_\varepsilon^{LdG}(Q) := \int_{\Omega} \left(W_{LdG}(\nabla Q) + \frac{1}{\varepsilon^2} f_{LdG}(Q) \right) dx,$$

where W_{LdG} and f_{LdG} are defined in (1.2) and (1.3) respectively. Assume the elastic constants L_1, L_2 , and L_3 satisfy (1.4). Then

- a) *There exists a subsequence $\varepsilon_k \rightarrow 0$, such that the maps Q_{ε_k} converge to an \mathcal{N} -valued, W_{LdG} -harmonic map Q_\star strongly in H^1 and locally uniformly in $\Omega \setminus \mathcal{S}$, where \mathcal{S} is the singular set of Q_\star .*
- b) *If in addition we assume that there exists a C^2 map $Q_b : \partial\Omega \rightarrow \mathcal{N}$ such that $Q_\varepsilon = Q_b$ on $\partial\Omega$, then the convergence is locally uniform up to the boundary, that is, in $\bar{\Omega} \setminus \mathcal{S}$.*
- c) *Given any C^2 function $g : \partial\Omega \times \mathcal{S}_0 \rightarrow [0, \infty)$, and under the additional assumption that*

$$\sup_{\varepsilon>0} \|Q_\varepsilon\|_\infty < \infty,$$

the same conclusion holds for minimizers of the weak anchoring energy $F_\varepsilon^{LdG}(Q) := E_\varepsilon^{LdG}(Q) + \int_{\partial\Omega} g(x, Q)$, that is, the convergence is locally uniform in $\bar{\Omega} \setminus \mathcal{S}$.

Remark 1.6. A natural and interesting direction for future investigations is to extend our results to elastic energy densities of the form $W(x, u, \nabla u)$ instead of $W(x, \nabla u)$; recently, in an attempt to address some of the analytical shortcomings of the classical Landau-de Gennes model, a modification with a more general quadratic form W_{LdG} with coefficients depending on the order parameter Q has been proposed [21].

The paper is organized as follows: in the next section we prove the fundamental lemmas that imply energy decay at the different scales, and use these to prove the interior small energy estimate. In Section 3 we outline the adaptations needed to handle the boundary estimates, which follow the same general strategy as for the interior, but where technical differences make the proofs more delicate. The paper finishes with two appendices where we prove a technical boundary modification lemma, and the strong H_{loc}^1 compactness of bounded energy sequences.

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2. INTERIOR ESTIMATES

In this section we prove the interior small energy estimate.

Theorem 2.1. *Assume that f satisfies (A2), that W satisfies (A1), and moreover that either (A3a) or (A3b) holds. There exist $\delta, \varepsilon_0 > 0$ and $\alpha \in (0, 1)$ (depending on λ, Λ and f , and*

under (A3a) also on M) such that for any $r_0 \in (0, 1)$, $\varepsilon \in (0, r_0\varepsilon_0)$, and any u_ε minimizing $E_\varepsilon(\cdot; B_{2r_0})$ with respect to its own boundary conditions,

$$(2r_0)^{2-n} E_\varepsilon(u_\varepsilon; B_{2r_0}) \leq \delta^2 \implies r_0^{2\alpha} |u_\varepsilon|_{C^\alpha(B_{r_0})}^2 \lesssim (2r_0)^{2-n} E_\varepsilon(u_\varepsilon; B_{2r_0}),$$

where the constant in the last inequality depends on λ , Λ and f , and under (A3a) also on M .

We prove Theorem 2.1 by obtaining uniform bounds for the decay of the energy on small balls. Our proof reflects the fact that different scales are at stake in this problem, due to the different homogeneities of the two terms in E_ε . At scales larger than ε the decay of the Dirichlet energy is inherited from the small energy regularity of (anisotropic) ‘‘harmonic’’ maps (Lemma 2.2). At scales smaller than ε it is inherited from regularity estimates for elliptic systems (Lemma 2.5). This is where the absence of a uniform L^∞ bound (that was easy to obtain in the isotropic case) is an issue and we have to either assume it (A3a) or to require f to satisfy the technical assumption (A3b).

2.1. Energy decay at larges scales. We begin by an energy improvement lemma which implies that u_ε has the energy decay of minimizing W -harmonic maps at scales larger than ε . This is the only part of our argument where the minimality of u_ε is crucial; at smaller scales the decay comes from the regularity of solutions to a PDE. It is worth mentioning that energy decay of stationary W -harmonic maps is not known and that mere criticality is not sufficient for this property to hold even in the isotropic case. Thus, to obtain similar Hölder estimates for critical points of E_ε one would need to avoid the contradiction argument which relies on the regularity of W -harmonic maps.

Lemma 2.2. *Assume (A1) and (A2). Let $0 < \beta < 1$. There exist $\delta_0, \varepsilon_0 > 0$ and $\theta_0 \in (0, 1/2)$ (depending on β , λ , Λ and f) such that any minimizer u_ε of $E_\varepsilon(\cdot; B_1)$ with $\varepsilon \in (0, \varepsilon_0)$ satisfies*

$$E_\varepsilon(u_\varepsilon; B_1) \leq \delta_0^2 \implies \theta_0^{2-n} E_\varepsilon(u_\varepsilon; B_{\theta_0}) \leq \theta_0^{2\beta} E_\varepsilon(u_\varepsilon; B_1).$$

The proof of Lemma 2.2 is modelled on the proof of the corresponding energy decay result for minimizing \mathcal{N} -valued maps in [22, 34]. There, it relies on the observation that a sequence of minimizers with arbitrarily small energy converges, after translating and rescaling its image, to a map minimizing energy under the *linear* constraint $v \in T_z \mathcal{N}$ a.e., for some $z \in \mathcal{N}$. For such a map, classical elliptic regularity applies and this allows to conclude. Here we are faced with the additional difficulty that the potential term in the energy does not behave well with respect to such rescaling. To circumvent this difficulty we first need to be able to modify boundary values with the help of the following lemma, whose proof we postpone to the appendix.

Lemma 2.3. *There exists $\delta_1 = \delta_1(\mathcal{N}, f) > 0$ such that for all $0 < \varepsilon \leq \lambda < 1$ and any $u \in H^1(\partial B_1; \mathbb{R}^k)$ with $E_\varepsilon(u; \partial B_1) \leq \delta_1^2 \lambda^{n-1}$, there exist*

$$w \in H^1(\partial B_1; \mathcal{N}), \quad \varphi \in H^1(B_1 \setminus B_{1-\lambda}; \mathbb{R}^k),$$

with $\varphi = u$ on ∂B_1 , $\varphi = w((1-\lambda)\cdot)$ on $\partial B_{1-\lambda}$,

satisfying the bounds

$$E_\varepsilon(\varphi; B_1 \setminus B_{1-\lambda}) \lesssim \lambda E_\varepsilon(u; \partial B_1) \quad \text{and} \quad \int_{\partial B_1} |\nabla w|^2 \lesssim E_\varepsilon(u; \partial B_1).$$

Remark 2.4. In the Landau-de Gennes setting ($k = 5$, $\mathcal{N} \approx \mathbb{RP}^2$), results similar to Lemma 2.3 are proved in [14]. There the energy $E_\varepsilon(u)$ is allowed to be of order $o(|\log \varepsilon|)$, and $\lambda \sim \varepsilon^{1/2} |\log \varepsilon|$. These results would enable us to perform the proof of Lemma 2.2 in the Landau-de Gennes case, but in our case we need only to consider small energies, which makes the proof of Lemma 2.3 much simpler, and independent of the topology of \mathcal{N} .

Granted this boundary modification lemma, we turn to the proof of Lemma 2.2.

Proof of Lemma 2.2. Thanks to the ellipticity assumption (A1), minimizers v of $\int_{B_1} W(\nabla v)$ under the linear constraint $v \in T_z \mathcal{N}$ a.e. enjoy elliptic regularity estimates, uniformly in $z \in \mathcal{N}$ (see for example [1, § 10]). In particular, there exists $\theta_0 \in (0, 1/4)$ such that for any $z \in \mathcal{N}$ and any $v \in H^1(B_1; T_z \mathcal{N})$ minimizing $\int_{B_1} W(\nabla \tilde{v})$ among all maps $\tilde{v} \in H^1(B_1; T_z \mathcal{N})$ with $\tilde{v} = v$ on ∂B_1 , it holds

$$(2.1) \quad (2\theta_0)^{2-n} \int_{B_{2\theta_0}} W(\nabla v) \leq \frac{\theta_0^{2\beta}}{2^{n-1}} \int_{B_1} W(\nabla v).$$

In order to prove the validity of Lemma 2.2 for this value of θ_0 , we assume by contradiction that there exist sequences $\delta_\ell, \varepsilon_\ell \rightarrow 0$ and u_ℓ minimizing $E_{\varepsilon_\ell}(\cdot; B_1)$ such that $E_{\varepsilon_\ell}(u_\ell; B_1) = \delta_\ell^2$, and

$$\theta_0^{2-n} E_{\varepsilon_\ell}(u_\ell; B_{\theta_0}) > \theta_0^{2\beta} E_{\varepsilon_\ell}(u_\ell; B_1).$$

By Fubini's theorem we may choose $\rho \in [1/2, 1]$ such that $E_{\varepsilon_\ell}(u_\ell; \partial B_\rho) \lesssim \delta_\ell^2$. Setting $\hat{u}_\ell(\hat{x}) = u_\ell(\rho \hat{x})$, $\hat{\varepsilon}_\ell = \varepsilon_\ell / \rho$ and $\hat{\delta}_\ell^2 = E_{\hat{\varepsilon}_\ell}(\hat{u}_\ell; B_1)$ we then have

$$(2\theta_0)^{2-n} E_{\hat{\varepsilon}_\ell}(\hat{u}_\ell; B_{2\theta_0}) > \frac{\theta_0^{2\beta}}{2^{n-2}} E_{\hat{\varepsilon}_\ell}(\hat{u}_\ell; B_1), \quad \text{and} \quad E_{\hat{\varepsilon}_\ell}(\hat{u}_\ell; \partial B_1) \lesssim \hat{\delta}_\ell^2.$$

Dropping the hats to simplify notations, we thus have sequences $\varepsilon_\ell, \delta_\ell, u_\ell$ satisfying

$$(2.2) \quad \begin{aligned} E_{\varepsilon_\ell}(u_\ell; \partial B_1) &\lesssim \delta_\ell^2 = E_{\varepsilon_\ell}(u_\ell; B_1) \rightarrow 0, \\ \text{and} \quad (2\theta_0)^{2-n} E_{\varepsilon_\ell}(u_\ell; B_{2\theta_0}) &> \frac{\theta_0^{2\beta}}{2^{n-2}} E_{\varepsilon_\ell}(u_\ell; B_1). \end{aligned}$$

The assumptions (A2) on f imply that $\text{dist}(u_\ell, \mathcal{N}) \rightarrow 0$ a.e., and we deduce that u_ℓ converges strongly in $H^1(B_1; \mathbb{R}^k)$ to a constant $z \in \mathcal{N}$. From Lemma 2.3 we obtain $\lambda_\ell \rightarrow 0$, $w_\ell: \partial B_1 \rightarrow \mathcal{N}$, $\varphi_\ell: B_1 \setminus B_{1-\lambda_\ell} \rightarrow \mathbb{R}^k$ such that

$$(2.3) \quad \begin{aligned} \varphi_\ell &= u_\ell \text{ on } \partial B_1, \quad \varphi_\ell = w_\ell((1 - \lambda_\ell) \cdot) \text{ on } \partial B_{1-\lambda_\ell}, \\ E_{\varepsilon_\ell}(\varphi_\ell; B_1 \setminus B_{1-\lambda_\ell}) &\lesssim \lambda_\ell \delta_\ell^2, \quad \int_{\partial B_1} |\nabla w_\ell|^2 \lesssim \delta_\ell^2. \end{aligned}$$

Note that, as argued in [14, Corollary 34], this implies that

$$(2.4) \quad \int_{\partial B_1} |w_\ell - u_\ell|^2 \lesssim \lambda_\ell \delta_\ell^2.$$

Since $n \geq 3$, the set of maps $w \in H^1(B_1; \mathcal{N})$ which agree with w_ℓ on ∂B_1 is not void (see e.g. [22, Lemma 1.1]), and we can choose $\bar{w}_\ell \in H^1(B_1; \mathcal{N})$ such that $\bar{w}_\ell = w_\ell$ on ∂B_1 , and

$$(2.5) \quad \alpha_\ell^2 := \int_{B_1} W(\nabla \bar{w}_\ell) = \min \left\{ \int_{B_1} W(\nabla w) : w \in H^1(B_1; \mathcal{N}), w|_{\partial B_1} = w_\ell \right\}.$$

The map

$$\tilde{w}_\ell = \begin{cases} \varphi_\ell & \text{in } B_1 \setminus B_{1-\lambda_\ell}, \\ \bar{w}_\ell((1-\lambda_\ell)\cdot) & \text{in } B_{1-\lambda_\ell}, \end{cases}$$

agrees with u_ℓ on ∂B_1 , and we infer, recalling (2.3), that

$$\delta_\ell^2 = E_{\varepsilon_\ell}(u_\ell; B_1) \leq E_{\varepsilon_\ell}(\tilde{w}_\ell; B_1) \leq (1 + o(1))\alpha_\ell^2 + o(1)\delta_\ell^2,$$

hence

$$(2.6) \quad \delta_\ell^2 \leq (1 + o(1))\alpha_\ell^2.$$

On the other hand, the minimality property (2.5) of \bar{w}_ℓ ensures, comparing its energy with the energy of the 0-homogeneous map $w_\ell(x/|x|)$, that

$$(2.7) \quad \alpha_\ell^2 \lesssim \int_{\partial B_1} |\nabla w_\ell|^2 \lesssim \delta_\ell^2.$$

This, together with (2.4) and the fact that u_ℓ converges to $z \in \mathcal{N}$, implies that $w_\ell \rightarrow z$ in $H^1(B_1; \mathbb{R}^k)$. Next we argue as in [34] and translate and rescale \bar{w}_ℓ in order to obtain a limiting map with values into $T_z \mathcal{N}$. Since $y_\ell := f w_\ell$ converges to $z \in \mathcal{N}$, for large enough ℓ we may define $z_\ell := \pi_{\mathcal{N}}(y_\ell)$, and Poincaré's inequality then ensures that

$$(2.8) \quad \int_{B_1} \text{dist}(\bar{w}_\ell - z_\ell, T_{z_\ell} \mathcal{N}) \lesssim \int_{B_1} |\bar{w}_\ell - z_\ell|^2 \lesssim \alpha_\ell^2.$$

Hence the map

$$v_\ell := \frac{1}{\alpha_\ell}(\bar{w}_\ell - z_\ell),$$

is bounded in $H^1(B_1; \mathbb{R}^k)$ and up to a subsequence (that we do not relabel) it converges weakly to a map $v \in H^1(B_1; \mathbb{R}^k)$ which, thanks to (2.8), takes a.e. values into $T_z \mathcal{N}$. In particular v_ℓ converges strongly to v in $L^2(\partial B_1; \mathbb{R}^k)$. Moreover v_ℓ is bounded in $H^1(\partial B_1; \mathbb{R}^k)$ thanks to (2.3) and (2.6). Thus we may argue exactly as in [34, Proposition 1] and construct good comparison maps to deduce that v minimizes $\int_{B_1} W(\nabla \tilde{v})$ among $T_z \mathcal{N}$ -valued maps \tilde{v} that agree with v on ∂B_1 , and that the convergence $v_\ell \rightarrow v$ is in fact strong in $H^1(B_1; \mathbb{R}^k)$. In particular, v enjoys the energy decay property (2.1). Thanks to the strong convergence $v_\ell \rightarrow v$ in H^1 , after rescaling it holds

$$(2.9) \quad (2\theta_0)^{2-n} \int_{B_{2\theta_0}} W(\nabla \bar{w}_\ell) \leq \frac{\theta_0^{2\beta} + o(1)}{2^{n-1}} \int_{B_1} W(\nabla \bar{w}_\ell).$$

The last part of the proof consists in obtaining, from this energy decay for \bar{w}_ℓ , similar energy decay for u_ℓ , thus contradicting (2.2). To that end we define

$$\tilde{v}_\ell := \frac{1}{\alpha_\ell}(u_\ell - z_\ell).$$

Note that here we do not divide by δ_ℓ but rather by the (possibly) larger α_ℓ corresponding to a minimizer among \mathcal{N} -valued maps of a closely related problem. The reason is that even though we already know that δ_ℓ and α_ℓ are of the same order, we need exact bounds in (2.10) below for the corresponding cancellations to take place.

Thanks to (2.6) it holds $\int_{B_1} |\nabla \tilde{v}_\ell|^2 \lesssim 1$, and therefore $\tilde{v}_\ell - f \tilde{v}_\ell$ converges (up to a subsequence) weakly in $H^1(B_1; \mathbb{R}^k)$, and strongly in $L^2(\partial B_1; \mathbb{R}^k)$. From (2.4) and (2.6) we infer that

$$\int_{\partial B_1} |\tilde{v}_\ell - v_\ell|^2 \lesssim \lambda_\ell \rightarrow 0.$$

In particular \tilde{v}_ℓ is bounded in $L^2(\partial B_1; \mathbb{R}^k)$, so that $f \tilde{v}_\ell$ must be bounded. We may assume that it converges in \mathbb{R}^k , and hence deduce that

$$\tilde{v}_\ell \longrightarrow \tilde{v} \quad \text{weakly in } H^1(B_1; \mathbb{R}^k) \text{ and a.e.},$$

for some map $\tilde{v} \in H^1(B_1; \mathbb{R}^k)$ such that $\tilde{v} = v$ on ∂B_1 . We claim that $\tilde{v} \in T_z \mathcal{N}$ a.e. This follows from the energy bound $E_\varepsilon(u_\ell; B_1) = \delta_\ell^2 \lesssim \alpha_\ell^2$, which thanks to assumption (A2) on f implies in particular that

$$\frac{1}{\alpha_\ell} \text{dist}(u_\ell, \mathcal{N}) \longrightarrow 0 \text{ a.e.}$$

Fixing $x \in B_1$ at which this convergence and $\tilde{v}_\ell(x) \rightarrow \tilde{v}(x)$ hold, we have

$$\begin{aligned} \text{dist}(u_\ell(x), \mathcal{N}) &= \text{dist}(z_\ell + \alpha_\ell \tilde{v}(x) + \alpha_\ell(\tilde{v}_\ell(x) - \tilde{v}(x)), \mathcal{N}) \\ &\geq \text{dist}(z_\ell + \alpha_\ell \tilde{v}(x), \mathcal{N}) - o(\alpha_\ell) \\ &\geq \alpha_\ell |P_\ell \tilde{v}(x)| - O(\alpha_\ell^2) - o(\alpha_\ell), \end{aligned}$$

where P_ℓ denotes the orthogonal projection onto $(T_{z_\ell} \mathcal{N})^\perp$. This shows that $|P_\ell \tilde{v}(x)| \rightarrow 0$ and therefore $\tilde{v}(x) \in T_z \mathcal{N}$ since $z_\ell \rightarrow z$.

We denote by $B(\cdot, \cdot)$ the symmetric bilinear form on $\mathbb{R}^{k \times n}$ corresponding to W , so that it holds

$$\begin{aligned} \int_{B_1} W(\nabla \bar{w}_\ell - \nabla u_\ell) + \frac{1}{\varepsilon_\ell^2} \int_{B_1} f(u_\ell) &= \int_{B_1} W(\nabla \bar{w}_\ell) + E_\varepsilon(u_\ell; B_1) - 2 \int_{B_1} B(\nabla \bar{w}_\ell, \nabla u_\ell) \\ (2.10) \qquad \qquad \qquad &\leq \alpha_\ell^2 + (1 + o(1))\alpha_\ell^2 - 2\alpha_\ell^2 \int_{B_1} B(\nabla v_\ell, \nabla \tilde{v}_\ell). \end{aligned}$$

Because $v_\ell \rightarrow v$ strongly and $\tilde{v}_\ell \rightarrow \tilde{v}$ weakly we have

$$\int_{B_1} B(\nabla v_\ell, \nabla \tilde{v}_\ell) \rightarrow \int_{B_1} B(\nabla v, \nabla \tilde{v}) = \int_{B_1} B(\nabla v, \nabla(\tilde{v} - v)) + \int_{B_1} W(\nabla v).$$

Since v is a minimizer of $\int_{B_1} W(\nabla v)$ among $T_z\mathcal{N}$ -valued maps and $(\tilde{v} - v)$ is $T_z\mathcal{N}$ -valued and vanishes on ∂B_1 , it holds

$$\int_{B_1} B(\nabla v, \nabla(\tilde{v} - v)) = 0.$$

Moreover, since $\int_{B_1} W(\nabla v_\ell) = 1$ and v_ℓ converges strongly to v it holds $\int_{B_1} W(\nabla v) = 1$, and therefore

$$\int_{B_1} B(\nabla v_\ell, \nabla \tilde{v}_\ell) = 1 + o(1).$$

Plugging this into (2.10) we find

$$\int_{B_1} W(\nabla \bar{w}_\ell - \nabla u_\ell) + \frac{1}{\varepsilon_\ell^2} \int_{B_1} f(u_\ell) \leq o(1) \alpha_\ell^2.$$

Using this estimate and (2.9) we obtain

$$\begin{aligned} (2\theta_0)^{2-n} E_{\varepsilon_\ell}(u_\ell; B_{2\theta_0}) &\leq \frac{4}{3} (2\theta_0)^{2-n} \int_{B_{2\theta_0}} W(\nabla \bar{w}_\ell) + o(1) \alpha_\ell^2 \\ &\leq \frac{2\theta_0^{2\beta} + o(1)}{3 \cdot 2^{n-2}} \int_{B_1} W(\nabla \bar{w}_\ell) + o(1) \alpha_\ell^2 \\ &\leq \frac{2\theta_0^{2\beta} + o(1)}{3 \cdot 2^{n-2}} \int_{B_1} W(\nabla u_\ell) + o(1) \alpha_\ell^2. \end{aligned}$$

Since $\alpha_\ell^2 \lesssim \delta_\ell^2 = E_{\varepsilon_\ell}(u_\ell; B_1)$, this contradicts (2.2) and concludes the proof of Lemma 2.2. \square

2.2. Regularity at small scales. Iterating Lemma 2.2 will enable us to obtain energy decay up to scales $r \geq \varepsilon/\varepsilon_0$ (see § 2.3 below). In the present subsection we prove an estimate for the energy decay of u_ε at a fixed value of ε (that we will later take to be $\varepsilon = \varepsilon_0$). What is crucial here (and not completely straightforward), is that the estimate is only in terms of the energy: we will need this in order to transfer the energy decay from large to small scales. As a consequence of this particular constraint imposed by our strategy, the constants we obtain behave very poorly with ε . One would morally expect that the constants should become better (or at least be uniform) for large values of ε – the constants we obtain here do become uniform under the L^∞ bound assumption (A3a), but not when (A3a) is replaced by (A3b).

Lemma 2.5. *Assume (A1)-(A2), and either (A3a) or (A3b). For all $\varepsilon > 0$ there exists $\delta, \alpha, C > 0$ (depending on n, λ, Λ, f and M , but also on ε) such that any minimizer u_ε of $E_\varepsilon(\cdot; B_2)$ satisfies*

$$E_\varepsilon(u_\varepsilon; B_2) \leq \delta^2 \quad \implies \quad r^{2-n} \int_{B_r} |\nabla u_\varepsilon|^2 \leq Cr^\alpha E_\varepsilon(u_\varepsilon; B_2) \quad \forall r \in (0, 1).$$

Remark 2.6. We prove in fact Lemma 2.5 with $\delta = 1$. We state it in this way to emphasize that this statement is strong enough to prove Theorem 2.1. We expect Lemma 2.5 to hold under much less restrictive assumptions: in any dimension $n \geq 3$, for an elastic energy

$W(x, u, \nabla u)$ which may be a positive definite quadratic form in ∇u with coefficients depending smoothly on x and u , and for a potential f with some radial growth at infinity.

We will prove Lemma 2.5 separately under the assumptions (A3a) and (A3b). In both cases we use elliptic estimates for the equation

$$\mathcal{L}u = \frac{1}{\varepsilon^2} \nabla f(u)$$

where \mathcal{L} is the second order elliptic operator such that $\int W(\nabla \varphi) = \int \mathcal{L} \varphi \cdot \varphi$ for all test functions φ . This operator satisfies elliptic estimates

$$\|\nabla^2 v\|_{L^p} \lesssim \|\mathcal{L}v\|_{L^p} + \|v\|_{L^p},$$

for $1 < p < \infty$ and all maps v with compact support in the unit ball B_1 . The inequality is up to a constant depending on λ , Λ , n and p (see e.g. [1, § 10] or [36, § 6.4]). For any $0 < R_1 < R_2 \leq 1$, one may apply this to $v = (u - \xi)\varphi$ for any $\xi \in \mathbb{R}^k$ and φ a cut-off function such that $\varphi \equiv 1$ in B_{R_1} , $\varphi \equiv 0$ outside of B_{R_2} , and $|\nabla^\ell \varphi| \lesssim (R_2 - R_1)^{-\ell}$, and conclude with Poincaré's inequality that

$$(2.11) \quad \|\nabla^2 u\|_{L^p(B_{R_1})} \lesssim \|\mathcal{L}u\|_{L^p(B_{R_2})} + \frac{1}{(R_2 - R_1)^2} \|\nabla u\|_{L^p(B_{R_2})}.$$

Under the uniform L^∞ bound assumption (A3a), Lemma 2.5 will follow from using (A2) to bound $\mathcal{L}u_\varepsilon$ in terms of the energy E_ε , and bootstrapping the elliptic estimate (2.11). Without the uniform L^∞ bound however, both estimating $\mathcal{L}u_\varepsilon$ in terms of E_ε , and bootstrapping, do not work directly and this is why we need assumption (A3b).

Proof of Lemma 2.5 under (A3a). In this proof we drop the subscripts ε to simplify notation. We fix a sequence of radii $R_k \in (1/2, 1]$ by letting $R_0 = 1$ and $R_k = R_{k-1} - 2^{-(k+1)}$ for $k \geq 1$.

The map u solves

$$\mathcal{L}u = \frac{1}{\varepsilon^2} \nabla f(u).$$

We start by applying (2.11) with $p = p_0 = 2$, hence

$$\|\nabla^2 u\|_{L^2(B_{R_1})} \lesssim (1 + \varepsilon^{-2}) \|\nabla f(u)\|_{L^2(B_{R_0})} + \|\nabla u\|_{L^2(B_{R_0})}.$$

Thanks to (A2) and (A3a) we have $|\nabla f(u)| \lesssim f(u)^{1/2}$, so that we deduce that

$$\|\nabla^2 u\|_{L^2(B_{R_1})} \lesssim \varepsilon^{-2} E(u; B_1)^{\frac{1}{2}}.$$

Next we set $p_1 = 2^* = 2n/(n-2)$. By Sobolev embedding and the above we obtain

$$\begin{aligned} \|\nabla u\|_{L^{p_1}(B_{R_1})} &\lesssim \|\nabla^2 u\|_{L^2(B_{R_1})} + \|\nabla u\|_{L^2(B_{R_1})} \lesssim (1 + \varepsilon^{-2}) E(u; B_1)^{\frac{1}{2}}, \\ \|\nabla f(u)\|_{L^{p_1}(B_{R_1})} &\lesssim \|\nabla^2 f(u)\|_{L^\infty} \|\nabla u\|_{L^2(B_{R_1})} + \|\nabla f(u)\|_{L^2(B_{R_1})} \lesssim (1 + \varepsilon^{-2}) E(u; B_1)^{\frac{1}{2}}. \end{aligned}$$

Applying (2.11) again we have therefore

$$\|\nabla^2 u\|_{L^{p_1}(B_{R_2})} \lesssim (1 + \varepsilon^{-4}) E(u; B_1)^{\frac{1}{2}}.$$

If $p_1 < n$, we iterate the above, and obtain a sequence of exponents $p_k < n$ such that $p_{k+1} = p_k^* = p_k n / (n - p_k)$, and

$$\begin{aligned} \|\nabla u\|_{L^{p_k}(B_{R_k})} + \|\nabla f(u)\|_{L^{p_k}(B_{R_k})} &\lesssim (1 + \varepsilon^{-2k})E(u; B_1)^{\frac{1}{2}}, \\ \|\nabla^2 u\|_{L^{p_{k+1}}(B_{R_{k+1}})} &\lesssim (1 + \varepsilon^{-2k-2})E(u; B_1)^{\frac{1}{2}}. \end{aligned}$$

The sequence (p_k) is strictly increasing and $p_{k+1} - p_k = p_k(n/(n - p_k) - 1) > p_0(n/(n - p_0) - 1)$, so that after a finite number of iterations we have $p_{k+1} \geq n$. If $p_{k+1} > n$, by Sobolev embedding this implies $\|\nabla u\|_{L^\infty(B_{R_{k+1}})} \lesssim E(u; B_1)^{1/2}$. If $p_{k+1} = n$ we may replace it by $\tilde{p}_{k+1} < n$ but arbitrarily close to n and iterate one more time. In any case we infer

$$\|\nabla u\|_{L^\infty(B_{1/2})} \lesssim (1 + \varepsilon^{-2\kappa})E(u; B_1)^{1/2},$$

for some $\kappa = \kappa(n) \in \mathbb{N}$, and this implies the conclusion of Lemma 2.5 with $\alpha = 1$. \square

Proof of Lemma 2.5 under (A3b). In this proof we drop the subscripts ε to simplify notation.

For the convenience of the reader we recall here assumption (A3b):

$$\begin{aligned} n = 3, \text{ and there exist } p > \frac{3}{2} \text{ and } \frac{1}{2} \leq a \leq \min\left(\frac{4}{5}, \frac{4}{3} - \frac{1}{p}\right), \\ \text{such that } |\nabla f(z)| \lesssim |z|^{\frac{6}{p}} \text{ and } |\nabla f(z)| \lesssim f(z)^a \text{ as } |z| \rightarrow \infty. \end{aligned}$$

There is no loss of generality in assuming that

$$(2.12) \quad \frac{3}{2} < p \leq \frac{15}{8},$$

since for $p \geq 15/8$ we have $4/3 - 1/p \geq 4/5$.

We let $\delta = 1$, and consider a map u minimizing $E(\cdot; B_2)$ and satisfying $E(u; B_2) \leq 1$. The map u solves

$$\mathcal{L}u = \frac{1}{\varepsilon^2} \nabla f(u).$$

Since, by Sobolev embedding $u \in L^6(B_2)$, we deduce from the first growth assumption $|\nabla f(z)| \lesssim |z|^{6/p}$ in (A3b) that $\nabla f(u) \in L^p(B_2)$. Hence applying the elliptic estimates (2.11) yields that $\nabla^2 u \in L_{loc}^p(B_2)$. Again by Sobolev embedding, we deduce that $\nabla u \in L_{loc}^{p_*}(B_2)$, where $p_* = 3p/(3 - p)$. Note that the condition $p > 3/2$ implies that $p_* > 3$. This is enough to deduce that

$$\frac{1}{r} \int_{B_r} |\nabla u|^2 \lesssim r^\alpha \quad \text{for some } \alpha \in (0, 1).$$

However Lemma 2.5 claims a bound in terms of the energy $E(u; B_2)$, which is not provided by the above argument. This is why we need the second growth assumption in (A3b), namely $|\nabla f| \lesssim f^a$.

Note that since $f \geq 0$, and since the nondegeneracy assumption (A2) implies $|\nabla f(z)| \lesssim f(z)^{1/2}$ for $|z| \lesssim 1$, using (A3b) we deduce

$$|\nabla f(z)| \lesssim f(z)^{1/2} + f(z)^a \quad \forall z \in \mathbb{R}^k.$$

Because $1/2 \leq a \leq 4/3 - 1/p$, this implies

$$|\nabla f(z)| \lesssim f(z)^{1/2} + f(z)^A \quad \forall z \in \mathbb{R}^k, \quad A := \frac{4}{3} - \frac{1}{p}.$$

Applying (2.11) we obtain, for any $1 \leq R_1 < R_2 \leq 2$,

$$\int_{B_{R_1}} |\nabla^2 u|^p \lesssim \frac{1}{(R_2 - R_1)^{2p}} \int_{B_{R_2}} |\nabla u|^p + \varepsilon^{-2p} \int_{B_{R_2}} |\nabla f(u)|^p.$$

On the other hand, by Sobolev embedding $W^{1,p} \subset L^{p^*}$, it holds

$$\left(\int_{B_{R_1}} |\nabla u|^{p^*} dx \right)^{\frac{2}{p^*}} \lesssim \left(\int_{B_{R_1}} |\nabla u|^p dx \right)^{\frac{2}{p}} + \left(\int_{B_{R_1}} |\nabla^2 u|^p dx \right)^{\frac{2}{p}}.$$

Gathering the above, we have

$$\begin{aligned} \left(\int_{B_{R_1}} |\nabla u|^{p^*} dx \right)^{\frac{2}{p^*}} &\lesssim \frac{1}{(R_2 - R_1)^4} \left(\int_{B_{R_2}} |\nabla u|^p dx \right)^{\frac{2}{p}} + \varepsilon^{-4} \left(\int_{B_{R_2}} |\nabla f(u)|^p dx \right)^{\frac{2}{p}} \\ &\lesssim \frac{1}{(R_2 - R_1)^4} \left(\int_{B_{R_2}} |\nabla u|^p dx \right)^{\frac{2}{p}} \\ &\quad + \varepsilon^{-4} \left(\int_{B_{R_2}} f(u)^{p/2} dx \right)^{\frac{2}{p}} + \varepsilon^{-4} \left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}}, \end{aligned}$$

and by Jensen's inequality, since $2/p \geq 1$,

$$(2.13) \quad \left(\int_{B_{R_1}} |\nabla u|^{p^*} dx \right)^{\frac{2}{p^*}} \lesssim \frac{1}{(R_2 - R_1)^4} \int_{B_{R_2}} |\nabla u|^2 dx + \varepsilon^{-4} \int_{B_{R_2}} f(u) dx + \varepsilon^{-4} \left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}}$$

$$(2.14) \quad \lesssim \frac{1 + \varepsilon^{-2}}{(R_2 - R_1)^4} E(u; B_{R_2}) + \varepsilon^{-4} \left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}}$$

Then we use the fact that

$$Ap = \frac{4}{3}p - 1 \in \left[1, \frac{3}{2} \right],$$

so that by Sobolev embedding $W^{1,1} \subset L^{Ap}$. Hence we find

$$\begin{aligned} \left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}} &\lesssim \left(\int_{B_{R_2}} f(u) dx \right)^{2A} + \left(\int_{B_{R_2}} |\nabla u| |\nabla f(u)| dx \right)^{2A} \\ &\lesssim \varepsilon^{4A} E(u; B_{R_2})^{2A} \\ &\quad + \left(\int_{B_{R_2}} |\nabla u| f(u)^{1/2} dx \right)^{2A} + \left(\int_{B_{R_2}} |\nabla u| f(u)^A dx \right)^{2A}. \end{aligned}$$

The second term in the above right-hand side is $\lesssim \varepsilon^{2A} E(u; B_{R_2})^{2A}$. To estimate the third term we apply Hölder's inequality to see that

$$\int_{B_{R_2}} |\nabla u| f(u)^A dx \lesssim \left(\int_{B_{R_2}} |\nabla u|^{p^*} dx \right)^{\frac{1}{p^*}} \left(\int_{B_{R_2}} f(u)^{Ap^*/(p^*-1)} dx \right)^{\frac{p^*-1}{p^*}}.$$

Finally, since $E(u; B_{R_2}) \leq 1$ (and $2A \geq 1$), we conclude

$$\begin{aligned} \left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}} &\lesssim (\varepsilon^{2A} + \varepsilon^{4A}) E(u; B_{R_2}) \\ &\quad + \left(\int_{B_{R_2}} |\nabla u|^{p^*} dx \right)^{\frac{2A}{p^*}} \left(\int_{B_{R_2}} f(u)^{Ap^*/(p^*-1)} dx \right)^{\frac{2A(p^*-1)}{p^*}}. \end{aligned}$$

Recall that

$$A = \frac{4}{3} - \frac{1}{p} = \frac{p^* - 1}{p^*},$$

so that the above implies

$$\begin{aligned} \left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}} &\lesssim (\varepsilon^{2A} + \varepsilon^{4A}) E(u; B_{R_2}) \\ &\quad + \varepsilon^{4A^2} \left(\int_{B_{R_2}} |\nabla u|^{p^*} dx \right)^{\frac{2A}{p^*}} E(u; B_{R_2})^{2A^2}. \end{aligned}$$

Since $A < 1$ we may invoke Young's inequality $xy \lesssim x^{1/A} + y^{1/(1-A)}$ for all $x, y \geq 0$, and deduce

$$\begin{aligned} \left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}} &\lesssim (\varepsilon^{2A} + \varepsilon^{4A}) E(u; B_{R_2}) \\ &\quad + \eta^{1/A} \left(\int_{B_{R_2}} |\nabla u|^{p^*} dx \right)^{\frac{2}{p^*}} + \frac{\varepsilon^{\frac{4A^2}{1-A}}}{\eta^{1/(1-A)}} E(u; B_{R_2})^{\frac{2A^2}{1-A}}, \end{aligned}$$

for any $\eta > 0$. Since $2A^2/(1-A) \geq 1$ (because $A \geq 1/2$) and $E(u; B_{R_2}) \leq 1$ this implies

$$\left(\int_{B_{R_2}} f(u)^{Ap} dx \right)^{\frac{2}{p}} \lesssim \frac{\varepsilon^{2A} + \varepsilon^{\frac{4A^2}{1-A}}}{\eta^{1/(1-A)}} E(u; B_{R_2}) + \eta^{1/A} \left(\int_{B_{R_2}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}},$$

and plugging this into (2.14) we get

$$\begin{aligned} \left(\int_{B_{R_1}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}} &\lesssim \left(\frac{1 + \varepsilon^{-2}}{(R_2 - R_1)^4} + \frac{\varepsilon^{2A-4} + \varepsilon^{\frac{4A^2+A-1}{1-A}}}{\eta^{1/(1-A)}} \right) E(u; B_{R_2}) \\ &\quad + \eta^{1/A} \left(\int_{B_{R_2}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}}. \end{aligned}$$

Choosing η small enough, we infer

$$\left(\int_{B_{R_1}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}} \leq \frac{1}{2} \left(\int_{B_{R_2}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}} + \frac{\varepsilon^{2A-4} + \varepsilon^{\frac{4A^2+A-1}{1-A}}}{(R_2 - R_1)^4} E(u; B_{R_2}),$$

for some constant $C > 0$. Setting $\rho_j = 3/2 - 1/(2K^j)$ for some $K \in (1, 2^{1/4})$ and iterating the above estimate applied to $R_1 = \rho_j$ and $R_2 = \rho_{j+1}$ we obtain

$$\begin{aligned} \left(\int_{B_{\rho_0}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}} &\leq \frac{1}{2^j} \left(\int_{B_{\rho_j}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}} \\ &\quad + C (\varepsilon^{2A-4} + \varepsilon^{\frac{4A^2+A-1}{1-A}}) \left(\frac{K-1}{2K} \right)^{2p} \left(\sum_{\ell=0}^{j-1} \left(\frac{K^{2p}}{2} \right)^\ell \right) E(u; B_{\rho_j}) \\ &\leq \frac{1}{2^j} \left(\int_{B_{3/2}} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}} + C' (\varepsilon^{2A-4} + \varepsilon^{\frac{4A^2+A-1}{1-A}}) E(u; B_{3/2}). \end{aligned}$$

Letting $j \rightarrow \infty$ and recalling that we already know that $\nabla u \in L^{p_*}(B_{3/2})$, we deduce that

$$\left(\int_{B_1} |\nabla u|^{p_*} dx \right)^{\frac{2}{p_*}} \lesssim (\varepsilon^{2A-4} + \varepsilon^{\frac{4A^2+A-1}{1-A}}) E(u; B_{3/2}).$$

It is directly checked that $p_* > 3$, Hence Hölder's inequality implies

$$\begin{aligned} r^{-1} \int_{B_r} |\nabla u|^2 &\leq r^{2(1-3/p_*)} \left(\int_{B_r} |\nabla u|^{p_*} \right)^{\frac{2}{p_*}} \\ &\lesssim (\varepsilon^{2A-4} + \varepsilon^{\frac{4A^2+A-1}{1-A}}) r^{2(1-3/p_*)} E(u; B_2), \end{aligned}$$

for all $r \in (0, 1)$. □

2.3. Interior regularity. We are ready to prove our main result.

Proof of Theorem 2.1. In this proof we are going to rescale repeatedly in the x variable, and should accordingly define a new quadratic form W at each step, unless W has constant coefficients. The quadratic form W with rescaled coefficients will still satisfy (A1), hence it will not affect the implicit constants in the conclusions of Lemmas 2.2 and 2.5, which are uniform with respect to quadratic forms satisfying (A1). Therefore we will, for the sake of clarity, assume that W has constant coefficients: this does not change the strategy of the proof, but it does simplify a lot the notations (since it avoids redefining W at each step).

Let u_ε minimize $E_\varepsilon(\cdot; B_{2r_0})$ and satisfy

$$(2r_0)^{2-n} E_\varepsilon(u_\varepsilon; B_{2r_0}) \leq \delta^2,$$

for some $\delta \in (0, 1]$ to be fixed later. Fix $0 < \beta < 1$ and let θ_0 and δ_0 be as in Lemma 2.2.

Fixing $x_0 \in B_{r_0}$ and setting $\bar{u}(\bar{x}) = u_\varepsilon(x_0 + r_0\bar{x})$ we have that \bar{u} minimizes $E_{\bar{\varepsilon}}(\cdot; B_1)$ for $\bar{\varepsilon} = \varepsilon/r_0 < \varepsilon_0$, and

$$E_{\bar{\varepsilon}}(\bar{u}; B_1) \leq \delta^2.$$

Hence we are in a situation to apply Lemma 2.2 which implies that $\tilde{u}(\tilde{x}) = \bar{u}(\theta_0\tilde{x})$ satisfies, with $\tilde{\varepsilon} = \theta_0^{-1}\bar{\varepsilon}$,

$$E_{\tilde{\varepsilon}}(\tilde{u}; B_1) = (\theta_0)^{2-n} E_{\bar{\varepsilon}}(\bar{u}; B_{\theta_0}) \leq \theta_0^{2\beta} E_{\bar{\varepsilon}}(\bar{u}; B_1) \leq \delta_0^2.$$

By induction we may in fact apply Lemma 2.2 to $\tilde{u}(\tilde{x}) = \bar{u}(\theta_0^{j+1}\tilde{x})$ and $\tilde{\varepsilon} = \theta_0^{-j-1}\bar{\varepsilon}$ for all $j \in \mathbb{N}$ such that $\theta_0^j > \bar{\varepsilon}/\varepsilon_0$ and infer

$$(\theta_0^{j+1})^{2-n} E_{\tilde{\varepsilon}}(\tilde{u}; B_{(\theta_0)^{j+1}}) \leq (\theta_0^{j+1})^{2\beta} E_{\bar{\varepsilon}}(\bar{u}; B_1).$$

This implies

$$(2.15) \quad \bar{r}^{2-n} E_{\tilde{\varepsilon}}(\tilde{u}; B_{\bar{r}}) \lesssim \bar{r}^{2\beta} E_{\bar{\varepsilon}}(\bar{u}; B_1) \quad \forall \bar{r} \in [\bar{\varepsilon}/\varepsilon_0, 1).$$

Next we set

$$(2.16) \quad r_1 = \bar{\varepsilon}/\varepsilon_0 \text{ and } \hat{u}(\hat{x}) = \bar{u}(r_1\hat{x}),$$

so that \hat{u} minimizes $E_{\hat{\varepsilon}}(\cdot; B_2)$ for $\hat{\varepsilon} = \bar{\varepsilon}/r_1 = \varepsilon_0$, and

$$E_{\hat{\varepsilon}}(\hat{u}; B_2) = r_1^{2-n} E_{\tilde{\varepsilon}}(\tilde{u}; B_{2r_1}) \lesssim r_1^{2\beta} E_{\bar{\varepsilon}}(\bar{u}; B_1) \lesssim \delta^2.$$

Lemma 2.5 ensures that if δ is small enough (depending on n, λ, Λ, f – and ε_0 which depends itself only on n, λ, Λ and f) there exists $\hat{\alpha} > 0$ such that

$$\hat{r}^{2-n} \int_{B_{\hat{r}}} |\nabla \hat{u}|^2 \lesssim \hat{r}^{2\hat{\alpha}} E_{\hat{\varepsilon}}(\hat{u}; B_2) \quad \forall \hat{r} \in (0, 1).$$

Set $\alpha = \min(\hat{\alpha}, \beta)$. Recalling (2.16), we can apply the previous inequality, with $\hat{r} = \bar{r}/r_1$, to obtain

$$\bar{r}^{2-n} \int_{B_{\bar{r}}} |\nabla \bar{u}|^2 \lesssim \left(\frac{\bar{r}}{r_1}\right)^{2\alpha} E_{\hat{\varepsilon}}(\hat{u}; B_2) \lesssim \bar{r}^{2\alpha} E_{\bar{\varepsilon}}(\bar{u}; B_1) \quad \forall \bar{r} \in (0, r_1).$$

By the Campanato-Morrey characterization of Hölder spaces this implies

$$r_0^{2\alpha} |u_\varepsilon|_{C^\alpha(B_{r_0})}^2 \lesssim (2r_0)^{2-n} E_\varepsilon(u_\varepsilon; B_{2r_0}).$$

□

3. BOUNDARY ESTIMATES

In this section, we extend our previous results in the interior to both strong and weak anchoring settings. As it will become apparent, there are technical differences between the two cases; while the proof for strong anchoring works basically along the same lines as the interior case, under either assumption (A3a) or (A3b), a proof not relying on (A3a) is, at the moment, out of reach for weak anchoring. The reason for this is that we are unable to modify Luckhaus' construction in a way that allows us to control the boundary term. We elaborate more on this in § 3.2.

3.1. The strong anchoring case. If the boundary $\partial\Omega$ is of class C^2 , we can cover it with small balls where it can be flattened, and after rescaling we are led to defining modified energy functionals of the form

$$F_\varepsilon(u; B_2^+) = \int_{B_2^+} \left(W(x, u) + \frac{1}{\varepsilon^2} f(u) \right) a(x) dx,$$

where B_2^+ denotes the half ball $B_2 \cap \{x_n > 0\} \subset \mathbb{R}^n$, the quadratic form W satisfies (A1), and the weight $a(x)$ satisfies

$$(B1) \quad \|1 - a\|_{C^1} \leq \frac{1}{2}.$$

We will denote by F_\star the corresponding limiting energy functional for \mathcal{N} -valued maps.

To obtain boundary estimates for the original energy on Ω it suffices to consider maps u_ε which minimize F_ε in B_2^+ among maps u such that $u = u_\varepsilon$ on $(\partial B_2)^+ := \partial B_2 \cap \{x_n > 0\}$ and satisfying fixed Dirichlet conditions

$$u = u_b \quad \text{on } B_2' := B_2 \cap \{x_n = 0\},$$

for some \mathcal{N} -valued map u_b of C^2 regularity.

Theorem 3.1. *Assume that W satisfies (A1) and a satisfies (B1). Also, assume that f satisfies (A2), and that either (A3a) or (A3b) holds. Then, there exist $\delta, \varepsilon_0 > 0$ and $\alpha \in (0, 1)$ (depending on n, λ, Λ, f and M) such that for any $r_0 \in (0, 1)$, $\varepsilon \in (0, r_0 \varepsilon_0)$, and any u_ε minimizing $F_\varepsilon(\cdot; B_{2r_0}^+)$ with respect to its own boundary conditions and with $u_\varepsilon = u_b$ on B_{2r_0}' ,*

$$\begin{aligned} (2r_0)^{2-n} F_\varepsilon(u_\varepsilon; B_{2r_0}^+) + N(u_b; B_{2r_0}') &\leq \delta^2 \\ \implies r_0^{2\alpha} |u_\varepsilon|_{C^\alpha(B_{r_0}^+)}^2 &\lesssim (2r_0)^{2-n} F_\varepsilon(u_\varepsilon; B_{2r_0}^+) + N(u_b; B_{2r_0}'), \end{aligned}$$

where

$$N(u_b; B_r') = r^2 \|\nabla u_b\|_{L^\infty(B_r')}^2 + r^4 \|\nabla^2 u_b\|_{L^\infty(B_r')},$$

and the constant in the above inequality depends on λ, Λ, f and M .

Lemma 3.2. *Assume (A1), (B1) and (A2). Let $0 < \beta < 1$. There exist $\delta_0, \varepsilon_0, \eta_0 > 0$ and $\theta_0 \in (0, 1/2)$ (depending on $n, \beta, \lambda, \Lambda$ and f) such that any minimizer u_ε of $F_\varepsilon(\cdot; B_1^+)$ with $\varepsilon \in (0, \varepsilon_0)$, $u_\varepsilon = u_b$ on B_1' satisfies*

$$\begin{aligned} F_\varepsilon(u_\varepsilon; B_1^+) &\leq \delta_0^2 \\ \implies \theta_0^{2-n} F_\varepsilon(u_\varepsilon, B_{\theta_0}^+) &\leq \theta_0^{2\beta} \max \left(F_\varepsilon(u_\varepsilon; B_1^+), \eta_0 \|\nabla u_b\|_{L^\infty(B_1')}^2 \right). \end{aligned}$$

Proof. As for Lemma 2.2, the proof is by contradiction, assuming the existence of sequences $\varepsilon_\ell, u_\ell, W_\ell, a_\ell, u_b^\ell$ with $F_{\varepsilon_\ell}(u_\ell; B_1^+) \rightarrow 0$ and $\|\nabla u_b^\ell\|_\infty \ll F_{\varepsilon_\ell}(u_\ell; B_1^+)$, but such that the energy decay fails. Then one needs three ingredients:

- the boundary modification Lemma 2.3 to construct a sequence of \mathcal{N} -valued minimizing maps \bar{w}_ℓ ,
- the H^1 -compactness of $v_\ell = \alpha_\ell^{-1}(\bar{w}_\ell - z_\ell)$, where $\alpha_\ell^2 = \int_{B_1^+} W(\bar{w}_\ell)$ and $z_\ell \in \mathcal{N}$ is appropriately chosen,
- and the equivalent energy decay estimate for minimizers of $\int_{B_1^+} W(\nabla v)$ under the linear constraint $v \in T_z \mathcal{N}$ a.e., with constant boundary data on B_1' ,

Lemma 2.3 can be applied here without modification, since B_1^+ is bilipschitz equivalent to B_1 and $u_\varepsilon = u_b$ is already \mathcal{N} -valued on B_1' . The compactness of v_ℓ follows as in Lemma 2.2 from the argument in [34, Proposition 1], where the extension lemma [34, Lemma 1] can also be applied without modification thanks to the bilipschitz homeomorphism between B_1 and B_1^+ . The energy decay for $T_z \mathcal{N}$ minimizers comes from standard elliptic estimates. The rest of the proof is as in Lemma 2.2. \square

The following lemma gives the decay estimate at finer scales; the difference between this and the corresponding estimate for the interior is a boundary term that, as it will be seen, behaves well under rescaling because it only involves derivatives of the boundary data u_b .

Lemma 3.3. *Assume (A1),(B1),(A2), and (A3a) or (A3b). There exists $\alpha > 0$ (depending on n, λ, Λ, f , but also on ε), such that the following holds. For all $\varepsilon > 0$, any minimizer u_ε of $F_\varepsilon(\cdot; B_2^+)$ with $u_\varepsilon = u_b$ on B_2' and $F_\varepsilon(u_\varepsilon; B_2^+) \leq 1$ satisfies*

$$\frac{1}{r^{2\alpha}} \frac{1}{r^{n-2}} \int_{B_r^+} |\nabla u_\varepsilon|^2 \lesssim E_\varepsilon(u_\varepsilon; B_2^+) + \|\nabla u_b\|_{L^\infty(B_2')}^2 + \|\nabla^2 u_b\|_{L^\infty(B_2')}^2 \quad \forall r \in (0, 1),$$

where the inequality is up to a constant depending on n, λ, Λ, f , but also on ε .

Proof. The proof can be carried out as the proof of Lemma 2.5, replacing the interior elliptic estimates (2.11) with boundary elliptic estimates. More precisely, still denoting by \mathcal{L} the elliptic operator such that $\int W(\nabla \varphi) a(x) dx = \int \mathcal{L} \varphi \cdot \varphi a(x) dx$ for all test functions φ , solutions of the Dirichlet problem

$$\mathcal{L}u = f \text{ in } B_1, \quad u = g \text{ on } B_1',$$

satisfy, for all $0 < R_1 < R_2 \leq 1$,

$$\|\nabla^2 u\|_{L^p(B_{R_1})} \lesssim \|f\|_{L^p(B_{R_2})} + \frac{1}{(R_2 - R_1)^2} \left(\|\nabla u\|_{L^p(B_{R_2})} + \|\nabla g\|_{L^\infty(B_{R_2}')} + \|\nabla^2 g\|_{L^\infty(B_{R_2}')} \right),$$

where the inequality is up to a constant depending on $n, \lambda, \Lambda, p, R_1$ and R_2 (this follows from the estimates in [1, § 10]). This is enough to reproduce the proof of Lemma 2.5, under either (A3a) or (A3b). \square

Proof of Theorem 3.1. For $x'_0 \in B'_{r_0}$ we set $\bar{u}(\bar{x}) = u_\varepsilon(x'_0 + r_0 \bar{x})$, $\bar{u}_b(\bar{x}') = u_b(x'_0 + r_0 \bar{x}')$ and $\bar{\varepsilon} = \varepsilon/\varepsilon_0$. For any $0 < \beta < 1$, provided δ is small enough we can argue as in the proof of

Theorem 2.1 and iterate Lemma 3.2 to obtain

$$\bar{r}^{2-n} F_{\bar{\varepsilon}}(\bar{u}; B_{\bar{r}}^+) \lesssim \bar{r}^{2\beta} \left(\int_{B_1^+} W(\nabla \bar{u}) a(x) dx + \|\nabla \bar{u}_b\|_{L^\infty(B_1')} \right) \quad \forall \bar{r} \in [\bar{\varepsilon}/\varepsilon_0, 1).$$

Then we set $r_1 = \bar{\varepsilon}/\varepsilon_0$ and $\hat{u}(\hat{x}) = \bar{u}(r_1 \hat{x})$, $\hat{u}_b(\hat{x}') = \bar{u}_b(r_1 \hat{x}')$, $\hat{\varepsilon} = \bar{\varepsilon}/r_1 = \varepsilon_0$ and apply Lemma 3.3 to deduce for all $\hat{r} \in (0, 1)$,

$$\begin{aligned} \frac{1}{\hat{r}^{2\hat{\alpha}}} \frac{1}{\hat{r}^{n-2}} \int_{B_{\hat{r}}^+} |\nabla \hat{u}|^2 &\lesssim F_{\hat{\varepsilon}}(\hat{u}; B_2^+) + \|\nabla \hat{u}\|_{L^\infty(B_2')}^2 + \|\nabla^2 \hat{u}\|_{L^\infty(B_2')}^2 \\ &= r_1^{2-n} F_{\bar{\varepsilon}}(\bar{u}; B_{2r_1}^+) + r_1^2 \|\nabla \bar{u}_b\|_{L^\infty(B_{2r_1}')} + r_1^4 \|\nabla^2 \bar{u}_b\|_{L^\infty(B_{2r_1}')} \\ &\lesssim r_1^{2\beta} \left(\int_{B_1^+} W(\nabla \bar{u}) a(x) dx + N(\bar{u}_b; B_1') \right). \end{aligned}$$

Rescaling and setting $\alpha = \min(\hat{\alpha}, \beta)$ we infer

$$\frac{1}{\bar{r}^{n-2}} \int_{B_{\bar{r}}^+} |\nabla \bar{u}|^2 \lesssim r^{2\alpha} \left(\int_{B_1^+} W(\nabla \bar{u}) a(x) dx + N(\bar{u}_b; B_1') \right).$$

Coming back to the original map u , the estimate above implies that

$$\frac{1}{r^{n-2}} \int_{B_r(x'_0) \cap B_{2r_0}^+} |\nabla u|^2 \lesssim \left(\frac{r}{r_0} \right)^{2\alpha} \left(\int_{B_{2r_0}^+} W(\nabla \bar{u}) a(x) dx + N(\bar{u}_b; B_{2r_0}') \right) \quad \forall r \in (0, r_0),$$

provided $x'_0 \in B_{r_0}'$.

Next we consider $x_0 \in B_{r_0}^+$ and write $x_0 = (x'_0, \rho)$ for some $\rho \in (0, r_0)$ and $x'_0 = (x'_0, 0) \in B_{r_0}'$. For all $r \in [\rho, r_0]$, we have $B_r(x_0) \cap B_{2r_0}^+ \subset B_{2r}(x'_0)$ and therefore by the above,

$$\begin{aligned} \frac{1}{r^{n-2}} \int_{B_r(x_0) \cap B_{2r_0}^+} |\nabla u|^2 &\lesssim \frac{1}{(2r)^{n-2}} \int_{B_{2r}(x'_0) \cap B_{2r_0}^+} |\nabla u|^2 \\ &\lesssim r^{2\alpha} \left(\int_{B_{2r_0}^+} W(\nabla \bar{u}) a(x) dx + N(\bar{u}_b; B_{2r_0}') \right), \end{aligned}$$

whence in particular

$$\frac{1}{(\rho/2)^{n-2}} \int_{B_{\rho/2}(x_0) \cap B_{2r_0}^+} |\nabla u|^2 \lesssim \delta^2.$$

Since $B_\rho(x_0) \subset B_{2r_0}^+$, provided δ is small enough we may therefore apply the interior estimates (Theorem 2.1) in $B_\rho(x_0)$, and conclude that

$$\frac{1}{r^{n-2}} \int_{B_r(x_0) \cap B_{2r_0}^+} |\nabla u|^2 \lesssim r^{2\alpha} \left(\int_{B_{2r_0}^+} W(\nabla \bar{u}) a(x) dx + N(\bar{u}_b; B_{2r_0}') \right)$$

holds for all $r \in (0, r_0]$ and all $x_0 \in B_{r_0}^+$. This implies the desired C^α Hölder estimate. \square

3.2. Weak anchoring. We will denote by F_ε^{wa} the energy

$$F_\varepsilon^{wa}(u; B_2^+) = F_\varepsilon(u; B_2^+) + \int_{B_2'} g(x', u) dx',$$

where $g: B_2' \times \mathbb{R}^k \rightarrow [0, \infty)$ is a smooth anchoring energy density. Here we will always work under the assumption of a uniform L^∞ bound (A3a), and may therefore assume

$$(G) \quad \|g\|_{C^2(B_2' \times B_M)} \leq G,$$

for some $G > 0$. We will consider minimizers u_ε of F_ε^{wa} with respect to their own boundary conditions on $(\partial B_2)^+$. We will denote by F_\star^{wa} the corresponding limiting energy, i.e. the same energy restricted to \mathcal{N} -valued maps.

Theorem 3.4. *Assume that f satisfies (A2), that W satisfies (A1) and a satisfies (B1), and moreover that (A3a) and (G) hold. There exist $\delta, \varepsilon_0 > 0$ and $\alpha \in (0, 1)$ (depending on $n, \lambda, \Lambda, f, M$ and G) such that for any $r_0 \in (0, 1)$, $\varepsilon \in (0, r_0 \varepsilon_0)$, and any u_ε minimizing $F_\varepsilon^{wa}(\cdot; B_{2r_0}^+)$ with respect to its own boundary conditions on $(\partial B_{2r_0})^+$,*

$$\begin{aligned} (2r_0)^{2-n} F_\varepsilon(u_\varepsilon; B_{2r_0}^+) + r_0 \|g\|_{L^\infty(B_{2r_0}')} &\leq \delta^2 \\ \implies r_0^{2\alpha} |u_\varepsilon|_{C^\alpha(B_{r_0}^+)}^2 &\lesssim (2r_0)^{2-n} F_\varepsilon^{wa}(u_\varepsilon; B_{2r_0}^+), \end{aligned}$$

where the constant in the above inequality depends on $n, \lambda, \Lambda, f, M$ and G .

Proof. As above, this small energy estimate is a consequence of an energy improvement result ensuring regularity at large scales (Lemma 3.6 below), and one establishing a corresponding property at small scales (Lemma 3.7 below). The proof is a straightforward adaptation of Theorem 3.1, given the following crucial scaling property: if u minimizes F_ε^{wa} in B_r , then $\tilde{u}(\tilde{x}) := u(r\tilde{x})$ minimizes $\tilde{F}_\varepsilon^{wa}$ in B_1 where $\tilde{\varepsilon} = \varepsilon/r$ and $\tilde{F}_\varepsilon^{wa}$ corresponds to $\tilde{W}(\tilde{x}, \xi) = W(r\tilde{x}, \xi)$, $\tilde{a}(\tilde{x}) = a(r\tilde{x})$ and, most importantly, $\tilde{g}(\tilde{x}, u) = rg(r\tilde{x}, u)$. Hence as we rescale $\|g\|_{L^\infty}$ keeps getting smaller and this is what makes the iteration work. \square

Remark 3.5. In the special case of Ginzburg-Landau functionals where $W = |\nabla u|^2$, $f(u) = (1 - |u|^2)^2$ and $g(x', u) = |u - u_b(x')|^2$, uniform convergence up to the boundary is proved for critical points in the recent work [10]. For more general anchoring energies however, and even in the isotropic case $W = |\nabla u|^2$, this was not known before the present work. To prove this result, we need to assume (A3a), a property that holds in the isotropic case under rather mild assumptions, e.g. $u \cdot f(u) \geq 0$ and $u \cdot \nabla_u g(u) \geq 0$ for $|u| \geq M$ (in fact in that case (A3a) holds for all critical points provided the Euler-Lagrange equations are satisfied, see e.g. [33]). Here, the main reason for not being able to drop (A3a) is that otherwise we are unable to construct an extension φ as in Lemma 2.3, that satisfies in addition a bound on $\int g(x', \varphi(x'))$. This impedes obtaining an equivalent of Lemma 2.5 or 3.3, which is essential to deal with “large” scales $r \geq \varepsilon$. On the other hand, regarding small scales (i.e. an equivalent of Lemma 2.2 or 3.3), requiring (A3b) together with some physically motivated restrictions on $g(x', u)$, is enough to ensure the desired estimate, even in the absence of (A3a).

Lemma 3.6. *Assume (A1), (B1), (A2) and (A3a). Let $0 < \beta < 1$. There exist $\delta_0, \varepsilon_0, \eta_0 > 0$ and $\theta_0 \in (0, 1/2)$ (depending on $n, \beta, \lambda, \Lambda$ and f) such that any minimizer u_ε of $F_\varepsilon^{wa}(\cdot; B_1^+)$*

with $\varepsilon \in (0, \varepsilon_0)$, satisfies

$$\begin{aligned} F_\varepsilon^{wa}(u_\varepsilon; B_1^+) &\leq \delta_0^2 \\ \implies \theta_0^{2-n} F_\varepsilon(u_\varepsilon, B_{\theta_0}^+) &\leq \theta_0^{2\beta} \max\left(F_\varepsilon(u_\varepsilon; B_1^+), \eta_0 \|g\|_{L^\infty(B_1' \times B_M)}\right). \end{aligned}$$

Proof. As for Lemmas 2.2 and 3.2, the proof is by contradiction, assuming the existence of sequences ε_ℓ , u_ℓ , W_ℓ , a_ℓ , g_ℓ with $F_{\varepsilon_\ell}(u_\ell; B_1^+) \rightarrow 0$ and $\|g_\ell\|_\infty \ll F_{\varepsilon_\ell}(u_\ell; B_1^+)$, but such that the energy decay fails. Extending u_ℓ by symmetry to B_1 , one may apply the boundary modification Lemma 2.3, the weak anchoring energy of φ simply being controlled by $\int_{B_1' \setminus B_{1-\lambda}'} g(\varphi) \lesssim \lambda \|g_\ell\|_{L^\infty}$. Thus one obtains a sequence of minimizing \mathcal{N} -valued maps \bar{w}_ℓ with weak anchoring. The strong H^1 compactness of $v_\ell = \alpha_\ell^{-1}(w_\ell - z_\ell)$ is then obtained as in [34, Proposition 1], adapting the extension lemma [34, Lemma 1] by first extending the maps by symmetry, as also explained in [17]. The limit v is then a minimizer of $\int_{B_1^+} W(v)$ with free boundary conditions on B_1' , and under the linear constraint $v \in T_z \mathcal{N}$ a.e., and enjoys good energy decay thanks to classical elliptic estimates. \square

Lemma 3.7. *Assume (A1),(B1),(A2), (A3b) and (G). For all $\varepsilon > 0$, any minimizer u_ε of $F_\varepsilon^{wa}(\cdot; B_2^+)$ satisfies*

$$\|\nabla u_\varepsilon\|_{L^\infty(B_1^+)}^2 \lesssim F_\varepsilon(u_\varepsilon; B_2^+) + \|g\|_{C^2(B_2' \times B_M)}^2 \quad \forall r \in (0, 1),$$

where the inequality is up to a constant depending on n , λ , Λ , f , M , G but also on ε .

Proof. As in Lemmas 2.5 and 3.3, the proof relies on elliptic estimates for the equation satisfied by $u = u_\varepsilon$, namely

$$\begin{cases} \mathcal{L}u = \frac{1}{\varepsilon^2} \nabla f(u) & \text{in } B_2^+, \\ \mathcal{B}u = \nabla_u g(x', u) & \text{on } B_2', \end{cases}$$

where

$$(\mathcal{L}u)^\beta = a^{-1} \partial_j (a \cdot a_{ij}^{\alpha\beta} \partial_i u^\alpha), \quad (\mathcal{B}u)^\beta = -a \cdot a_{in}^{\alpha\beta} \partial_i u^\alpha.$$

We appeal to classical L^p estimates for elliptic systems [1, § 10], which ensure

$$\|\nabla^2 v\|_{L^p(B_1^+)} \lesssim \|\nabla v\|_{L^p(B_2^+)} + \|\mathcal{L}v\|_{L^p(B_2^+)} + \|\nabla \Phi\|_{L^p(B_2^+)} \quad \forall \Phi \text{ such that } \mathcal{B}v = \text{tr } \Phi,$$

where the inequality is up to a constant depending on λ , Λ , n and $p \in (1, \infty)$. To apply this to our map u we consider an extension G of g given by $G(x, u) = \chi(x_n)g(x', u)$ where $x = (x', x_n)$ and χ is a fixed smooth function with $\chi(0) = 1$ and $\chi \equiv 0$ on $(1, \infty)$. That way we can use $\Phi = \nabla_u G(x, u)$ in the above and estimate

$$\|\nabla[\nabla_u G(x, u)]\|_{L^p(B_2^+)} \lesssim \|g\|_{C^2(B_2' \times B_M)} + \|\nabla u\|_{L^p(B_2^+)}.$$

We deduce that u satisfies

$$\|\nabla^2 u\|_{L^p(B_1^+)} \lesssim \|\nabla u\|_{L^p(B_2^+)} + \|\nabla f(u)\|_{L^p(B_2^+)} + \|g\|_{C^2(B_2' \times B_M)},$$

and this estimate can be bootstrapped exactly as in the proof of Lemma 2.5. \square

APPENDIX A. PROOF OF THE BOUNDARY MODIFICATION LEMMA

In this section we prove Lemma 2.3. For the reader's convenience we recall here its statement:

Lemma 2.3. *There exists $\delta_1 = \delta_1(\mathcal{N}, f) > 0$ such that for all $0 < \varepsilon \leq \lambda < 1$ and any $u \in H^1(\partial B_1; \mathbb{R}^k)$ with $E_\varepsilon(u; \partial B_1) \leq \delta_1^2 \lambda^{n-3}$, there exist*

$$\begin{aligned} w \in H^1(\partial B_1; \mathcal{N}), \quad \varphi \in H^1(B_1 \setminus B_{1-\lambda}; \mathbb{R}^k), \\ \text{with } \varphi = u \text{ on } \partial B_1, \quad \varphi = w((1-\lambda)\cdot) \text{ on } \partial B_{1-\lambda}, \end{aligned}$$

satisfying the bounds

$$E_\varepsilon(\varphi; B_1 \setminus B_{1-\lambda}) \lesssim \lambda E_\varepsilon(u; \partial B_1) \quad \text{and} \quad \int_{\partial B_1} |\nabla w|^2 \lesssim E_\varepsilon(u; \partial B_1).$$

Proof of Lemma 2.3. The strategy is very similar to Luckhaus' extension lemma [34, Lemma 1]. For the reader's convenience we sketch the full argument, and will go into details only at points where we need to depart from [34]. We assume $\lambda = 2^{-\nu}$ for some $\nu \in \mathbb{N}$ and, using the bilipschitz equivalence of B_1 with the open unit cube, obtain a partition of ∂B_1 as

$$\partial B_1 = \bigsqcup_{j=0}^{n-1} Q_j, \quad Q_j = \bigsqcup_{i=1}^{k_j} e_i^j,$$

where each j -cell e_i^j is bilipschitz equivalent to B_λ^j , the j -dimensional open ball of radius λ . This decomposition of ∂B_1 induces a partition of $B_1 \setminus B_{1-\lambda}$ as

$$B_1 \setminus B_{1-\lambda} = \bigsqcup_{j=0}^{n-1} \widehat{Q}_j, \quad \widehat{Q}_j = \bigsqcup_{i=1}^{k_j} \widehat{e}_i^j, \quad \widehat{e}_i^j = \left\{ x \in B_1 \setminus B_{1-\lambda} : \frac{x}{|x|} \in e_i^j \right\}.$$

Moreover by Fubini's theorem we may assume that

$$\begin{aligned} \int_{Q_j} |\nabla u|^2 d\mathcal{H}^j &\lesssim \lambda^{j+1-n} \int_{\partial B_1} |\nabla u|^2 d\mathcal{H}^{n-1}, \\ \int_{Q_j} f(u) d\mathcal{H}^j &\lesssim \lambda^{j+1-n} \int_{\partial B_1} f(u) d\mathcal{H}^{n-1}. \end{aligned}$$

On the boundary of each two-dimensional cell e_i^2 (which is composed of 4 one-dimensional cells) there holds

$$\begin{aligned} (\text{osc}_{\partial e_i^2} u)^2 &\leq \lambda \int_{\partial e_i^2} |\nabla u|^2 \lesssim \lambda^{3-n} E_\varepsilon(u; \partial B_1) \lesssim \delta_1^2, \\ \int_{\partial e_i^2} f(u) &\lesssim \varepsilon^2 \lambda^{1-n} E_\varepsilon(u; \partial B_1) \lesssim \lambda^{3-n} E_\varepsilon(u; B_1) \lesssim \delta_1^2. \end{aligned}$$

If δ_1 is small enough, this implies thanks to (A2) that

$$\sup_{\partial e_i^2} |u - \int_{\partial e_i^2} u|^2 + \text{dist}^2\left(\int_{\partial e_i^2} u, \mathcal{N}\right) \lesssim \lambda^{3-n} E_\varepsilon(u; \partial B_1) \lesssim \delta_1^2.$$

Therefore, the harmonic extension \bar{u} of $u|_{\partial e_i^2}$ to e_i^2 , i.e. $\Delta \bar{u} = 0$ in e_i^2 and $\bar{u} = u$ on ∂e_i^2 , satisfies

$$\sup_{e_i^2} \text{dist}^2(\bar{u}, \mathcal{N}) \lesssim \lambda^{3-n} E_\varepsilon(u; \partial B_1) \lesssim \delta_1^2.$$

In particular, provided δ_1 is small enough, $w := \pi_{\mathcal{N}}(\bar{u})$ is well defined in e_i^2 . Note for later use that $\bar{u} = u$ on Q_1 , and thus

$$w = \pi_{\mathcal{N}}(u) \text{ on } Q_1.$$

The harmonic extension \bar{u} satisfies

$$\int_{e_i^2} |\nabla \bar{u}|^2 d\mathcal{H}^2 \lesssim \lambda \int_{\partial e_i^2} |\nabla u|^2 d\mathcal{H}^1,$$

and therefore

$$\int_{Q_2} |\nabla w|^2 d\mathcal{H}^2 \lesssim \int_{Q_2} |\nabla \bar{u}|^2 d\mathcal{H}^2 \lesssim \lambda \int_{Q_1} |\nabla u|^2 d\mathcal{H}^1 \lesssim \lambda^{3-n} E_\varepsilon(u_\varepsilon; \partial B_1).$$

On the higher dimensional skeletons Q_j ($j \geq 3$) we define the \mathcal{N} -valued map w by induction, via 0-homogeneous extensions: identifying e_i^j with B_λ^j (through a bilipschitz homeomorphism that we omit to write here), one may set

$$w(x) = w\left(\lambda \frac{x}{|x|}\right)$$

thus defining w on e_i^j through its boundary values on ∂e_i^j , which correspond to previously defined values of w on Q_{j-1} . Since $j \geq 3$, such 0-homogenous extension has finite energy, and we obtain by induction the estimates

$$\int_{Q_j} |\nabla w|^2 d\mathcal{H}^j \lesssim \lambda^{j+1-n} E_\varepsilon(u_\varepsilon; \partial B_1).$$

For $j = n - 1$ this shows that

$$\int_{\partial B_1} |\nabla w|^2 d\mathcal{H}^{n-1} \lesssim E_\varepsilon(u_\varepsilon; \partial B_1).$$

It remains to define the map φ on $B_1 \setminus B_{1-\lambda}$. We do it on each skeleton \widehat{Q}_j by induction, similarly to what is done in [34, Lemma 1]. On each cell \hat{e}_i^1 we set

$$\varphi(x) = u\left(\frac{x}{|x|}\right) + \frac{1-|x|}{\lambda} \left(w\left(\frac{x}{|x|}\right) - u\left(\frac{x}{|x|}\right) \right).$$

Since $\text{dist}(u, \mathcal{N}) \lesssim \delta_1$ and $w = \pi_{\mathcal{N}}(u)$ on Q_1 , we have

$$\begin{aligned} f(\varphi) &\lesssim \text{dist}^2(\varphi, \mathcal{N}) = |u - \pi_{\mathcal{N}}(u)|^2 \\ &\lesssim f(u) \quad \text{a.e. on } \hat{e}_i^1, \end{aligned}$$

and therefore

$$\int_{\widehat{Q}_1} f(\varphi) d\mathcal{H}^2 \lesssim \lambda \int_{Q_1} f(u) d\mathcal{H}^1 \lesssim \lambda^{3-n} \int_{\partial B_1} f(u) d\mathcal{H}^{n-1}.$$

We also have, recalling that $w = \pi_{\mathcal{N}}(u)$ on Q_1 and $\varepsilon \leq \lambda$,

$$\begin{aligned} \int_{\widehat{Q}_1} |\nabla \varphi|^2 d\mathcal{H}^2 &\lesssim \lambda \left(\int_{Q_1} |\nabla u|^2 d\mathcal{H}^1 + \int_{Q_1} |\nabla w|^2 d\mathcal{H}^1 + \frac{1}{\lambda^2} \int_{Q_1} |u - w|^2 d\mathcal{H}^1 \right) \\ &\lesssim \lambda \int_{Q_1} |\nabla u|^2 d\mathcal{H}^1 + \frac{1}{\lambda} \int_{Q_1} f(u) d\mathcal{H}^1 \\ &\lesssim \lambda E_\varepsilon(u; Q_1) \lesssim \lambda^{3-n} E_\varepsilon(u; \partial B_1). \end{aligned}$$

On the higher dimensional skeletons \widehat{Q}_j ($j \geq 2$) we extend φ by induction, via 0-homogenous extensions: identifying \widehat{e}_i^j with B_λ^{j+1} (through a bilipschitz homeomorphism that we omit to write here), one may set

$$\varphi(x) = \varphi \left(\lambda \frac{x}{|x|} \right),$$

thus defining φ on \widehat{e}_i^j through its boundary values on $\partial \widehat{e}_i^j$, which either correspond to values of u and w on Q_j or to previously defined values of φ on \widehat{Q}_{j-1} . Since $j \geq 2$, such 0-homogenous extension has finite energy, and we obtain by induction the estimates

$$E_\varepsilon(\varphi; \widehat{Q}_j) \lesssim \lambda^{j+2-n} E_\varepsilon(u; \partial B_1).$$

For $j = n - 1$ this concludes the proof. \square

APPENDIX B. LOCAL COMPACTNESS

In this section we prove the following local compactness property of sequences of minimizers with bounded energy.

Proposition B.1. *Assume that f satisfies (A2) and W satisfies (A1). For $\varepsilon > 0$, let u_ε minimize $E_\varepsilon(\cdot; B_1)$ with respect to its own boundary conditions, such that $\liminf_{\varepsilon \rightarrow 0} E_\varepsilon(u_\varepsilon; B_1) < \infty$. Then there is a subsequence $\varepsilon_\ell \rightarrow 0$ such that u_{ε_ℓ} converges strongly in $H_{loc}^1(B_1; \mathbb{R}^k)$ to a map $u_\star \in H_{loc}^1(B_1; \mathcal{N})$ which minimizes $E_\star(\cdot; B_\rho)$ for any $\rho \in (0, 1)$ (among \mathcal{N} -valued maps, and with respect to its own boundary conditions).*

The crucial ingredient is the following variant of Luckhaus' extension lemma (see also [14] for related results in the Landau-de Gennes setting).

Lemma B.2. *There exists $\eta \in (0, 1)$ such that for all $\lambda \in (0, 1)$ and any $u \in H^1(\partial B_1; \mathbb{R}^k)$, $v_\star \in H^1(\partial B_1; \mathcal{N})$ with*

$$\int_{\partial B_1} |\nabla u|^2 d\mathcal{H}^{n-1} + \int_{\partial B_1} |\nabla v_\star|^2 d\mathcal{H}^{n-1} \leq 1 \quad \text{and} \quad \int_{\partial B_1} |u - v_\star|^2 d\mathcal{H}^{n-1} \leq \eta^2 \lambda^{2n-4},$$

there exists $\varphi \in H^1(B_1 \setminus B_{1-\lambda})$ such that

$$\varphi = \begin{cases} u & \text{on } \partial B_1, \\ v_\star \left(\frac{\cdot}{1-\lambda}\right) & \text{on } \partial B_{1-\lambda}, \end{cases}$$

$$\int_{B_1 \setminus B_{1-\lambda}} |\nabla \varphi|^2 dx \lesssim \lambda \left(\int_{\partial B_1} |\nabla u|^2 d\mathcal{H}^{n-1} + \int_{\partial B_1} |\nabla v_\star|^2 d\mathcal{H}^{n-1} + \frac{1}{\lambda^2} \int_{\partial B_1} |u - v_\star|^2 d\mathcal{H}^{n-1} \right),$$

$$\int_{B_1 \setminus B_{1-\lambda}} f(\varphi) dx \lesssim \lambda \int_{\partial B_1} f(u) d\mathcal{H}^{n-1}.$$

Proof of Lemma B.2. The proof is very similar to [34, Lemma 1]. For the reader's convenience we sketch the full argument, and will go into details only at points where we need to depart from [34]. We assume $\lambda = 2^{-\nu}$ for some $\nu \in \mathbb{N}$ and, using the bilipschitz equivalence of B_1 with the open unit cube, obtain a partition of ∂B_1 as

$$\partial B_1 = \bigsqcup_{j=0}^{n-1} Q_j, \quad Q_j = \bigsqcup_{i=1}^{k_j} e_i^j,$$

where each j -cell e_i^j is bilipschitz equivalent to B_λ^j , the j -dimensional open ball of radius λ . This decomposition of ∂B_1 induces a partition of $B_1 \setminus B_{1-\lambda}$ as

$$B_1 \setminus B_{1-\lambda} = \bigsqcup_{j=0}^{n-1} \widehat{Q}_j, \quad \widehat{Q}_j = \bigsqcup_{i=1}^{k_j} \widehat{e}_i^j, \quad \widehat{e}_i^j = \left\{ x \in B_1 \setminus B_{1-\lambda} : \frac{x}{|x|} \in e_i^j \right\}.$$

Moreover by Fubini's theorem we may assume that

$$\int_{Q_j} |\nabla u|^2 d\mathcal{H}^j + \int_{Q_j} |\nabla v_\star|^2 d\mathcal{H}^j \lesssim \lambda^{j+1-n} \left(\int_{\partial B_1} |\nabla u|^2 d\mathcal{H}^{n-1} + \int_{\partial B_1} |\nabla v_\star|^2 d\mathcal{H}^{n-1} \right),$$

$$\int_{Q_j} |u - v_\star|^2 d\mathcal{H}^j \lesssim \lambda^{j+1-n} \int_{\partial B_1} |u - v_\star|^2 d\mathcal{H}^{n-1},$$

$$\int_{Q_j} f(u) d\mathcal{H}^j \lesssim \lambda^{j+1-n} \int_{\partial B_1} f(u) d\mathcal{H}^{n-1}.$$

In [34], the extension φ is defined on the 2-dimensional cells \widehat{e}_i^1 by interpolating linearly between u on e_i^1 and v_\star on $(1-\lambda)e_i^1$. However in our case we would like to control $\int_{\widehat{Q}_1} f(\varphi)$. A simple linear interpolation may not be sufficient: consider e.g. the situation where $u(x)$ would happen to be on \mathcal{N} , hence $f(u(x)) = 0$, but the segment between $v_\star(x)$ and $u(x)$ might contain points which are not on \mathcal{N} , and there $f(\varphi)$ would not be controlled by $f(u)$. A way around this is to first interpolate linearly between u and its projection $\pi(u) \in \mathcal{N}$, and then geodesically between $\pi(u)$ and v_\star on \mathcal{N} .

On each one dimensional cell e_i^1 , we have

$$\begin{aligned} \sup_{e_i^1} |u - v_\star|^2 &\leq \int_{e_i^1} |\nabla |u - v_\star|| + \frac{1}{\lambda} \int_{e_i^1} |u - v_\star|^2 \\ &\leq \left(\int_{e_i^1} (|\nabla u|^2 + |\nabla v_\star|^2) \right)^{\frac{1}{2}} \left(\int_{e_i^1} |u - v_\star|^2 \right)^{\frac{1}{2}} + \frac{1}{\lambda} \int_{e_i^1} |u - v_\star|^2 \\ &\lesssim \lambda^{2-n} \eta \lambda^{n-2} + \lambda^{1-n} \eta^2 \lambda^{4n-2} \lesssim \eta. \end{aligned}$$

Provided η is chosen small enough, this implies that on e_i^1 the projection $\pi_{\mathcal{N}}(u)$ is well defined, and satisfies

$$|u - \pi_{\mathcal{N}}(u)| + |\pi_{\mathcal{N}}(u) - v_\star| \lesssim |u - v_\star| \lesssim \eta^{\frac{1}{2}} \quad \text{on } e_i^1.$$

Then we define φ on \hat{e}_i^1 by setting

$$\varphi(x) = \begin{cases} u\left(\frac{x}{|x|}\right) + 2\frac{1-|x|}{\lambda} \left(\pi_{\mathcal{N}}(u)\left(\frac{x}{|x|}\right) - u\left(\frac{x}{|x|}\right) \right) & \text{if } 1 - \frac{\lambda}{2} \leq |x| \leq 1, \\ \gamma\left(2\frac{1-|x|}{\lambda} - 1, \pi_{\mathcal{N}}(u)\left(\frac{x}{|x|}\right), v_\star\left(\frac{x}{|x|}\right)\right) & \text{if } 1 - \lambda \leq |x| \leq 1 - \frac{\lambda}{2}, \end{cases}$$

Where $\gamma(\cdot, z_1, z_2): [0, 1] \rightarrow \mathcal{N}$ denotes the constant speed geodesic from z_1 to z_2 . The map γ is Lipschitz on a neighborhood of $[0, 1] \times \Delta$, where $\Delta = \{(z, z)\} \subset \mathcal{N} \times \mathcal{N}$, and its derivatives satisfy

$$|\partial_t \gamma(t, z_1, z_2)| \lesssim |z_1 - z_2|, \quad |\nabla_z \gamma| \lesssim 1.$$

Using this, we infer that

$$\begin{aligned} \int_{\hat{Q}_1} |\nabla \varphi|^2 d\mathcal{H}^2 &\lesssim \lambda \left(\int_{Q_1} |\nabla u|^2 d\mathcal{H}^1 + \int_{Q_1} |\nabla v_\star|^2 d\mathcal{H}^1 + \frac{1}{\lambda^2} \int_{Q_1} |u - v_\star|^2 d\mathcal{H}^1 \right) \\ &\lesssim \lambda \cdot \lambda^{2-n} \left(\int_{\partial B_1} |\nabla u|^2 d\mathcal{H}^{n-1} + \int_{\partial B_1} |\nabla v_\star|^2 d\mathcal{H}^{n-1} + \frac{1}{\lambda^2} \int_{\partial B_1} |u - v_\star|^2 d\mathcal{H}^{n-1} \right). \end{aligned}$$

Moreover for $1 - \lambda \leq |x| \leq 1 - \frac{\lambda}{2}$ it holds $f(\varphi(x)) = 0$, and for $1 - \frac{\lambda}{2} \leq |x| \leq 1$ it holds

$$\begin{aligned} f(\varphi(x)) &\lesssim \text{dist}^2(\varphi(x), \mathcal{N}) \lesssim |u - \pi_{\mathcal{N}}(u)|^2 \left(\frac{x}{|x|}\right) \\ &\lesssim \text{dist}^2(u, \mathcal{N}) \left(\frac{x}{|x|}\right) \lesssim f(u) \left(\frac{x}{|x|}\right), \end{aligned}$$

and this implies

$$\int_{\hat{Q}_1} f(\varphi) d\mathcal{H}^2 \lesssim \lambda \int_{Q_1} f(u) d\mathcal{H}^1 \lesssim \lambda \cdot \lambda^{2-n} \int_{\partial B_1} f(u) d\mathcal{H}^{n-1}.$$

On the higher dimensional skeletons \hat{Q}_j ($j \geq 2$) we extend φ by induction, via 0-homogenous extensions: identifying \hat{e}_i^j with B_λ^{j+1} (through a bilipschitz homeomorphism that we omit to write here), one may set

$$\varphi(x) = \varphi\left(\lambda \frac{x}{|x|}\right),$$

thus defining φ on \hat{e}_i^j through its boundary values on $\partial\hat{e}_i^j$, which either correspond to values of u and v_\star on Q_j or to previously defined values of φ on \hat{Q}_{j-1} . Since $j \geq 2$, such 0-homogenous extension has finite energy, and we obtain by induction the estimates

$$\int_{\hat{Q}_j} |\nabla\varphi|^2 d\mathcal{H}^{j+1} \lesssim \lambda \cdot \lambda^{j+1-n} \left(\int_{\partial B_1} |\nabla u|^2 d\mathcal{H}^{n-1} + \int_{\partial B_1} |\nabla v_\star|^2 d\mathcal{H}^{n-1} + \frac{1}{\lambda^2} \int_{\partial B_1} |u - v_\star|^2 d\mathcal{H}^{n-1} \right),$$

$$\int_{\hat{Q}_j} f(\varphi) d\mathcal{H}^{j+1} \lesssim \lambda \cdot \lambda^{j+1-n} \int_{\partial B_1} f(u) d\mathcal{H}^{n-1}.$$

For $j = n - 1$ this concludes the proof. \square

Now we turn to the proof of the compactness result.

Proof of Proposition B.1. Along a subsequence $\varepsilon_\ell \rightarrow 0$ we have $E_{\varepsilon_\ell}(u_\ell; B_1) \lesssim 1$, where we denote $u_\ell = u_{\varepsilon_\ell}$. Therefore, up to taking a subsequence there is $u_\star \in H^1(B_1; \mathcal{N})$ such that $u_{\varepsilon_\ell} \rightharpoonup u_\star$ weakly in H^1 and strongly in L^2 . Let $\rho \in (0, 1)$ be fixed. By Fubini's theorem we may find $r \in [\rho, 1]$ such that $E_{\varepsilon_\ell}(u_\ell; \partial B_r) \leq 1$ and we infer that u_ℓ converges strongly towards u_\star in $L^2(\partial B_r)$, and moreover $E_\star(u_\star; \partial B_r) \leq 1$.

Let v_\star minimize $E_\star(\cdot; B_r)$ with $v_\star = u_\star$ on ∂B_r . Using Lemma B.2 we construct $v_\ell \in H^1(B_r; \mathbb{R}^k)$ such that $v_\ell = u_\ell$ on ∂B_r , and $E_{\varepsilon_\ell}(v_\ell; B_r) \rightarrow E_\star(v_\star; B_r)$. Explicitly, since $\mu_\ell := \int_{\partial B_r} |u_\ell - v_\star|^2 \rightarrow 0$, we may for large enough ℓ apply Lemma B.2 to find $\lambda_\ell \rightarrow 0$ and $\varphi_\ell \in H^1(B_r \setminus B_{(1-\lambda_\ell)r})$ satisfying $\varphi_\ell = u_\ell$ on ∂B_r , $\varphi_\ell = v_\star(\cdot/(1-\lambda_\ell))$ on $\partial B_{(1-\lambda_\ell)r}$ and $E_{\varepsilon_\ell}(\varphi_\ell; B_r \setminus B_{(1-\lambda_\ell)r}) \lesssim \lambda_\ell \rightarrow 0$. Then we set $v_\ell = \varphi_\ell$ in $B_r \setminus B_{(1-\lambda_\ell)r}$ and $v_\ell = v_\star(\cdot/(1-\lambda_\ell))$ in $B_{(1-\lambda_\ell)r}$ and obtain indeed $E_{\varepsilon_\ell}(v_\ell; B_r) \rightarrow E_\star(v_\star; B_r)$.

By the minimizing property of u_ℓ we infer that $\limsup E_{\varepsilon_\ell}(u_\ell; B_r) \leq \limsup E_{\varepsilon_\ell}(v_\ell; B_r) \leq E_\star(v_\star; B_r)$, and since by weak lower semicontinuity $\liminf E_{\varepsilon_\ell}(u_\ell; B_r) \geq E_\star(u_\star; B_r)$ we conclude that u_\star minimizes $E_\star(\cdot; B_r)$ and u_ℓ converges in fact strongly to u_\star in $H^1(B_r)$. \square

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