Interaction energies in nematic liquid crystal suspensions

Lia Bronsard * Xavier Lamy † Dominik Stantejsky ‡
Raghavendra Venkatraman §

March 3, 2025

Abstract

We establish, as $\rho \to 0$, an asymptotic expansion for the minimal Dirichlet energy of $\mathbb{S}^2$-valued maps outside a finite number of particles of size ρ with fixed centers $x_j \in \mathbb{R}^3$, under general anchoring conditions at the particle boundaries. Up to a scaling factor, this expansion is of the form

$$E_\rho = \sum_j \mu_j - 4\pi\rho \sum_{i \neq j} \frac{\langle v_i, v_j \rangle}{|x_i - x_j|} + o(\rho)\,,$$

where μ_j is the minimal energy after zooming in at scale ρ around each particle, and $v_j \in \mathbb{R}^3$ is determined by the far-field behavior of the corresponding single-particle minimizer. The Coulomb-like interaction in this expansion agrees with the *electrostatics analogy* : a linearized approximation commonly used in the physics literature for colloid interactions in nematic liquid crystal. We obtain here for the first time a precise estimate of the energy error introduced by that linearization, by developing new tools that address the lack of convergence rate when zooming in at scale ρ.

1 Introduction

We investigate a mathematical model of interactions between colloid particles immersed in a nematic liquid crystal. Nematic liquid crystals are characterized by their orientational order: one can think of elongated molecules which tend to align along a common direction. Each immersed particle distorts this alignment

*Department of Mathematics and Statistics, McMaster University, bronsard@mcmaster.ca.
†Institut de Mathématiques de Toulouse, Université Paul Sabatier, Toulouse, France. xlamy@math.univ-toulouse.fr.
‡Institut Élie Cartan de Lorraine, University of Lorraine. dominik.stantejsky@univ-lorraine.fr.
§Department of Mathematics, The University of Utah. raghav@math.utah.edu.

at long range, inducing interactions with the other particles. When the sizes of the particles are much smaller than the distances between them, the physics literature developes an electrostatic analogy to describe their interactions, see [5, 13, 10] and the survey [12, § 2]. That analogy relies on linearizing, away from the particles, the equations which describe nematic alignment at equilibrium. Our main result gives an estimate of the error introduced by this linearization, under precise modelling assumptions which we describe next. From a purely mathematical viewpoint, this physical model corresponds to $\mathbb{S}^2$-valued harmonic maps and our study explores a new perspective on those classical geometric objects, namely the dependence of their energy on the shape of the domain.

We use the simplest order parameter to describe the nematic phase: a unit vector $n \in \mathbb{S}^2$ indicating the direction of alignment. A liquid crystal filling a domain $\Omega \subset \mathbb{R}^3$ is described by a map $n\colon \Omega \to \mathbb{S}^2$, and we assume that its energy is given by

$$E(n) = \int_\Omega |\nabla n|^2 \, dx + F(n_{\lfloor \partial\Omega}) \, ,$$

for some $F\colon H^{1/2}(\partial\Omega; \mathbb{S}^2) \to [0, +\infty]$ which accounts for the anchoring of liquid crystal molecules at the domain boundary. Note that minimizing configurations satisfy the harmonic map equation $-\Delta n = |\nabla n|^2 n$ in Ω.

Here we consider domains Ω and anchoring energies F of a specific form, to model a system with N foreign particles, all of the same small size $\rho > 0$, but not necessarily the same shape, see Figure 1. To be precise, the liquid crystal occupies the exterior domain

$$\Omega_\rho = \mathbb{R}^3 \setminus \bigcup_{j=1}^N \omega_{j,\rho}, \qquad \omega_{j,\rho} = x_j + \rho\, \hat\omega_j \, ,$$

for fixed particle centers $x_1, \ldots, x_N \in \mathbb{R}^3$ and smooth open sets

$$\hat\omega_j \subset B_1 \subset \mathbb{R}^3 \, , \qquad \text{for } j = 1, \ldots, N \, .$$

These open sets represent the particles after zooming in at scale ρ.

Rescaling by half the fixed minimal distance between these centers, we assume without loss of generality that they satisfy

$$|x_i - x_j| \geq 2 \qquad \forall i \neq j \in \{1, \ldots, N\} \, .$$

We endow each rescaled particle $\hat\omega_j$ with an anchoring energy

$$\widehat{F}_j \colon H^{1/2}(\partial\hat\omega_j; \mathbb{S}^2) \to [0, \infty], \qquad \text{weakly lower semicontinuous,}$$

with non-empty domain $\{\widehat{F}_j < \infty\} \subset H^{1/2}(\partial\hat\omega_j; \mathbb{S}^2)$, and assume that anchoring at the boundary of each small particle $\omega_{j,\rho}$ is described by the rescaled energy

$$F_{j,\rho}(n_{\lfloor \partial\omega_{j,\rho}}) = \widehat{F}_j(\hat n^\rho_{j \lfloor \partial\hat\omega_j}), \qquad \hat n^\rho_j(\hat x) = n(x_j + \rho\, \hat x) \, .$$

Examples of admissible anchoring energies $\widehat{F}_j$ are given in [2, § 1.2]. They include familiar examples of strong anchoring (Dirichlet conditions) and weak anchoring (enforced by a surface energy). With these notations, the energy of a map $n\colon \Omega_\rho \to \mathbb{S}^2$ is given by

$$E_\rho(n) = \frac{1}{\rho}\int_{\Omega_\rho} |\nabla n|^2\,dx + \sum_{j=1}^N F_{j,\rho}(n_{\lfloor \partial\omega_{j,\rho}})\,. \tag{1.1}$$

We impose far-field alignment along a fixed orientation $n_\infty \in \mathbb{S}^2$ via the condition

$$\int_{\Omega_\rho} \frac{|n(x)-n_\infty|^2}{1+|x|^2}\,dx < \infty\,. \tag{1.2}$$

Existence of a minimizer of E_ρ under this far-field alignment constraint can be proved exactly as in [2, § 1.2] for a single particle. Our main result is an asymptotic expansion, as $\rho \to 0$, of the minimal energy E_ρ. That expansion depends on minimizers of the single-particle problems

$$\mu_j = \min\left\{\widehat{E}_j(n)\colon \int_{\mathbb{R}^3\setminus\hat\omega_j} \frac{|n-n_\infty|^2}{1+|x|^2}\,dx < \infty\right\}, \tag{1.3}$$
$$\text{where } \widehat{E}_j(n) = \int_{\mathbb{R}^3\setminus\hat\omega_j} |\nabla n|^2\,dx + \widehat{F}_j(n_{\lfloor\partial\hat\omega_j})\,.$$

It is shown in [2] that any minimizer $\hat m_j$ of (1.3) has a far-field expansion

$$\hat m_j(x) = n_\infty + \frac{v_j}{|x|} + \mathcal{O}\Big(\frac{1}{|x|^2}\Big) \qquad \text{as } |x| \to +\infty\,, \tag{1.4}$$

for some $v_j \in \mathbb{R}^3$ orthogonal to n_∞. The vector v_j can be interpreted as a torque applied by the particle $\hat\omega_j$ on the nematic background [5], see also [2, Theorem 2]. The effective interaction between two particles depends on these vectors v_j.

Theorem 1.1. *There exist minimizers $\hat m_j$ of the single-particle problems* (1.3) *such that the minimum of E_ρ over maps $n\colon \Omega_\rho \to \mathbb{S}^2$ with far-field alignment* (1.2) *satisfies*

$$\min E_\rho = \sum_{j=1}^N \mu_j - 4\pi\rho\sum_{i\neq j}\frac{\langle v_i, v_j\rangle}{|x_i - x_j|} + o(\rho) \qquad \text{as } \rho \to 0\,, \tag{1.5}$$

where $\mu_j = \widehat{E}_j(\hat m_j)$ is the minimal single-particle energy (1.3)*, and $v_j \in n_\infty^\perp$ is defined by the asymptotic expansion* (1.4) *of $\hat m_j$.*

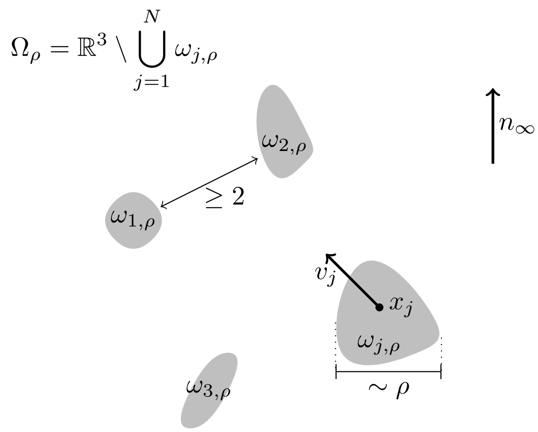

Figure 1: General setup for Theorem 1.1

The interaction potential given by the term of order ρ in the asymptotic expansion of Theorem 1.1 corresponds to solving the Poisson equation with singular source term

$$\Delta u_\rho = \sum_{j=1}^N 4\pi\rho v_j \delta_{x_j} \quad \text{in } \mathbb{R}^3 , \quad \text{that is,} \quad u_\rho(x) = \rho \sum_{j=1}^N \frac{v_j}{|x - x_j|} . \tag{1.6}$$

The infinite energy of u_ρ can indeed be renormalized to give

$$\lim_{\sigma\to 0} \left(\frac{1}{\rho} \int_{\mathbb{R}^3 \setminus \bigcup B_\sigma(x_j)} |\nabla u_\rho|^2 \, dx - \frac{4\pi}{\sigma} \rho \sum_{j=1}^N |v_j|^2 \right) = -4\pi\rho \sum_{i\neq j} \frac{\langle v_i, v_j \rangle}{|x_i - x_j|} .$$

This can be interpreted as follows:

- away from the particles, the harmonic map equation $-\Delta n = |\nabla n|^2 n$ is linearized around the uniform state n_∞, which corresponds to writing $n \approx n_\infty + u$ and $-\Delta u \approx 0$;
- the effect of the particle $\omega_{j,\rho}$ is replaced by a singular source term at x_j, and that source term is chosen to match the far-field expansion (1.4) generated by the single particle.

This linearized description is the electrostatic analogy introduced in [5] and further developed in [13, 10]. The difference in energy between the (renormalized) linearized description and the original nonlinear problem is what we estimate in Theorem 1.1. This gives the asymptotic expansion (1.5), where all the nonlinearity of the original problem is concentrated in the presence of μ_j and v_j, determined by the single-particle problem (1.3).

Ideas of proof

The proof of Theorem 1.1 consists of two parts: an upper bound, which we prove by constructing a competitor, and a lower bound which we obtain via a precise description of minimizers n_ρ.

The competitor we choose for the upper bound is equal to the single-particle minimizer $\hat{m}_j$, suitably rescaled, in small regions $B_\sigma(x_j)$. Outside these balls, we take its $\mathbb{R}^3$-valued harmonic extension (tending to n_∞ at far field) and project it back onto $\mathbb{S}^2$. For a well-chosen σ satisfying $\rho \ll \sigma \ll 1$, the energy of the competitor is controlled by the right-hand side of our expansion (1.5).

The lower bound is more challenging. Thanks to classical compactness properties of energy-minimizing maps, the blow-up at scale ρ of a minimizer n_ρ around each particle, given by $\hat{n}_j^\rho(\hat{x}) = n_\rho(x_j + \rho\,\hat{x})$, converges in H^1_{loc} to a single-particle minimizer $\hat{m}_j(\hat{x})$. This provides the first term in the asymptotic expansion (1.5) and suggests a natural route to obtain the next term: show that $\hat{n}_j^\rho - \hat{m}_j$ is small enough in $B_{\sigma/\rho}$ to produce a negligible energy error, for an adequate scale σ. If this were true, the conclusion would follow by using the energy of the harmonic extension of n_ρ as a lower bound outside the regions $B_\sigma(x_j)$. In other words, that natural route would require a quantitative rate for the convergence $\hat{n}_j^\rho \to \hat{m}_j$. However, this convergence was obtained by weak compactness arguments, and quantifying it seems out of reach. Instead, we modify our approach in order to conclude without quantitative rates. This relies on the following two ingredients.

- The first is a compensation effect between the inner and outer regions: if $\hat{n}_j^\rho$ is too different from $\hat{m}_j$ near $\partial B_{\sigma/\rho}$, the energy of the harmonic extension of n_ρ outside the regions $B_\sigma(x_j)$ is increased by an amount which partly compensates the energy error inside $B_\sigma(x_j)$. This implies an improved lower bound for the full energy. As a result, showing that the error is negligible boils down to the estimate $|\hat{n}_\rho^j(\hat{x}) - \hat{m}_j(\hat{x})| \ll 1/|\hat{x}|$ in the annulus $B_{2\sigma/\rho} \setminus B_{\sigma/\rho}$, for a choice of scale $\sigma \ll 1$ in an adequate range. In terms of scaling, such estimate is consistent with the non-quantitative L^2 convergence $\nabla\hat{n}_j^\rho \to \nabla\hat{m}_j$. In comparison, separate lower bounds in the inner and outer regions, without taking advantage of this compensation effect, would have required $|\hat{n}_\rho^j(x) - \hat{m}_j(x)| \ll \sqrt{\rho}/|\hat{x}|$ to make the error negligible.

- The second ingredient is a far-field expansion for $\hat{n}_j^\rho$ in large annuli, similar to the expansion (1.4) of $\hat{m}_j$. That far-field expansion eventually implies the estimate $|\hat{n}_\rho^j(x) - \hat{m}_j(x)| \ll 1/|\hat{x}|$, hence the conclusion thanks to the first ingredient. The proof of the expansion (1.4) in [2] uses the fact that a classical harmonic function with finite energy in the exterior domain $\mathbb{R}^3 \setminus B_\lambda$ only has radially decaying modes. Here, in order to adapt it to $\hat{n}_j^\rho$, the main difference is that we must take into account radially increasing modes which can occur in an annulus, and estimate them appropriately.

Related works

Estimating the minimal energy of harmonic maps in exterior domains, and interpreting it as an interaction energy, is a very natural mathematical problem. To the best of our knowledge, the perspective from which it has been addressed so far is different from the present one. We wish to recall here the seminal works [4] by Brezis, Coron and Lieb in three dimensions, and [3, Chapter I] by Bethuel, Brezis and Hélein in two dimensions. There, the objects of study are *smooth* $\mathbb{S}^2$ or $\mathbb{S}^1$-valued maps outside holes, and the authors investigate the minimal energy within a fixed *homotopy class*. At first sight, their holes play a role very similar to our particles. But here, on the contrary, our maps are not assumed to be smooth: near the particles they may have several singularities, about which our analysis says nothing quantitative. As a consequence, minimizing over a homotopy class would not even make sense in our setting, and instead, admissible competitors are constrained by the anchoring conditions. Finally, the results and methods in [4] and [3] are very different from each other but remain fundamentally nonlinear, while a linearization procedure is at the heart of the present work.

Note that in [3], the interaction energy is also obtained as the second term in an asymptotic expansion and is also of Coulomb type, but this comes from the fact that $\mathbb{S}^1$-valued harmonic maps can be "lifted" to $\mathbb{R}$-valued harmonic maps, rather than a linearization around a uniform state as in the present work. The analysis in [3] has initiated a rich line of research, including generalizations to maps with values into general manifolds and maps defined on higher-dimensional domains or manifolds, and we do not attempt here to give a list of these generalizations.

Finally, we mention the more recent papers [6, 8]. The paper [6] uses methods from complex analysis and analogies with potential flows in fluid dynamics to study a version of our problem in the plane. The paper [8] considers interaction energies between particles in the so-called "paranematic" regime, in which nematic order is only felt at the boundaries of the particles. Consequently, the interaction energy is much more localized to essentially overlapping boundary layers, and the analysis is largely linear.

Further directions

The physics of nematic suspensions raise many mathematical questions, and we mention here a few that are directly linked with the present work.

We considered here the simplest model for the nematic phase. Replacing the isotropic Dirichlet energy by a general anisotropic energy with three elastic constants [7, § 3.1.2] would likely be achievable at the cost of a few technical adjustments. Adapting the present analysis to a Q-tensor model (necessary to describe more symmetric single-particle minimizers, see e.g. [1]), would require new ingredients to deal with the extra length scale of phase transitions which is present in that model.

Recall that the vectors v_j in (1.5) can be interpreted as torques. As detailed in [2], it follows from that interpretation that, if the particles $\hat{\omega}_j$ are spherical, or if they are in an equilibrium orientation with respect to n_∞, then all the vectors v_j are zero. In that case, our asymptotic expansion (1.5) does not capture any interaction term. These would be described by a next-order expansion, as predicted by the electrostatics analogy [10]. An estimate of the error in that next order expansion would be very interesting.

From the physical point of view, it is also highly relevant to consider systems which are not at elastic equilibrium: either because of other physical effects (as already present in the original work of Brochard and de Gennes [5] where the particles are magnetic), or simply to describe time evolution. The present work can serve as a first step towards these more complex models.

Finally, the limit $N \to \infty$ is of course very natural to study. In that perspective, one goal could be to estimate the error in the continuum approximation proposed in [5, § II.3]. Another goal could be to establish a link between nematic suspensions and infinite point systems of Coulomb gas type, as has been done for Ginzburg-Landau vortices, see the survey [15] and references therein.

Plan of the article

In § 2 we establish preliminary estimates on the energy of harmonic functions in exterior domains. In § 3 we present the upper bound construction. In § 4 we establish the lower bound, thus proving Theorem 1.1. In the Appendices A and B we present for completeness some results about existence of decaying solutions to Poisson's equation, and estimates on the decay of harmonic functions in annuli.

Notations

We write $A \lesssim B$ to denote $A \leq CB$ for a generic constant $C > 0$, independent of ρ, but which can depend on the fixed parameters of our problem: N, $\hat{\omega}_j$, $\widehat{F}_j$, n_∞. We write $E_\rho(n; U)$ and $\widehat{E}_j(m, V)$ to denote restrictions of the integrals defining these energies to subsets $U \subset \Omega_\rho$ or $V \subset \mathbb{R}^3 \setminus \hat{\omega}_j$. If $\partial\omega_{j,\rho} \subset \partial U$ or $\partial\hat{\omega}_j \subset \partial V$, this notation is meant to include the corresponding anchoring term $F_{j,\rho}(n_{|\partial\omega_{j,\rho}})$ or $\widehat{F}_j(m_{|\partial\hat{\omega}_j})$.

Acknowledgements

LB is supported by a NSERC Discovery grant. XL is supported by the ANR project ANR-22-CE40-0006. A part of this work was carried out while DS was affiliated with McMaster University. RV is supported by the U.S. National Science Foundation under the award NSF-DMS 2407592. A part of this work was conducted during a workshop at the Mathematisches Forschungsinstitut Oberwolfach in 2024.

2 Prelude: harmonic extensions outside a union of small spheres

In this section we establish an estimate for the energy of harmonic functions u in the exterior domain

$$U_\sigma = \mathbb{R}^3 \setminus \bigcup_{j=1}^{N} B_\sigma(x_j), \qquad \text{for some } \sigma \in (\rho, 1/2)\,,$$

in terms of their boundary values on $\partial B_\sigma(x_j)$ that will be useful at several points in the proof of Theorem 1.1.

We first introduce some notations. We fix $\{\Phi_k\}_{k\in\mathbb{N}_0}$ an orthonormal Hilbert basis of $L^2(\mathbb{S}^2)$ which diagonalizes the Laplace-Beltrami operator,

$$-\Delta_{\mathbb{S}^2}\Phi_k = \lambda_k\Phi_k, \qquad 0 = \lambda_0 < \lambda_1 \le \lambda_2 \le \cdots .$$

The set $\{\lambda_k\}_{k\in\mathbb{N}_0}$ coincides with $\{\ell^2+\ell\}_{\ell\in\mathbb{N}_0}$, and the eigenfunctions corresponding to $\ell^2+\ell$ span the homogeneous harmonic polynomials of degree ℓ (which have dimension $2\ell+1$). The function Φ_0 is constant, equal to $1/(2\sqrt{\pi})$. Solutions $f(r)$ of $\Delta(f(r)\Phi_k(\omega)) = 0$ in $\mathbb{R}^3\setminus\{0\}$ are spanned by $f_\pm(r) = r^{\pm\gamma_k^\pm}$, with

$$\gamma_k^+ = \sqrt{\frac{1}{4}+\lambda_k} - \frac{1}{2} = \ell \quad \text{for } \lambda_k = \ell^2+\ell,$$
$$\gamma_k^- = \sqrt{\frac{1}{4}+\lambda_k} + \frac{1}{2} = \ell+1 \quad \text{for } \lambda_k = \ell^2+\ell.$$

These eigenfunctions satisfy the pointwise bound

$$|\nabla^\alpha\Phi_k| \lesssim \lambda_k^{\frac{1+\alpha}{2}} \qquad \forall\alpha\ge 0\,,\ k\ge 1\,. \tag{2.1}$$

Indeed, for $k\ge 1$ and any $\omega_0\in\mathbb{S}^2$, in local coordinates around ω_0 we can consider the rescaled function $\varphi(z) = \Phi_k(\omega_0 + z/\sqrt{\lambda_k})$ which satisfies $L\varphi=\varphi$ in a fixed ball B_1 for some elliptic operator L (with smooth coefficients depending on the local coordinates). Elliptic estimates imply $|\varphi(0)|^2 \lesssim \int_{B_1}|\varphi|^2\,dz \lesssim \lambda_k\int_{\mathbb{S}^2}|\Phi_k|^2$, hence $|\Phi_k|\lesssim\lambda_k^{1/2}$. This shows (2.1) for $\alpha=0$. The case of higher derivatives $\alpha\ge 1$ follows from elliptic estimates on $\mathbb{S}^2$ and the fact that $(-\Delta)^\beta\Phi_k = \lambda_k^\beta\Phi_k$ for any integer $\beta\ge 1$.

Proposition 2.1. *If $\Delta u = 0$ in $U_\sigma = \mathbb{R}^3\setminus\bigcup_{j=1}^N B_\sigma(x_j)$, $\int_{U_\sigma}|\nabla u|^2\,dx<\infty$ with $|u|\to 0$ as $|x|\to\infty$ and $u=g_j$ on $\partial B_\sigma(x_j)$ for $j=1,\ldots,N$, with*

$$g_j(x_j+\sigma\omega) = \sum_{k\ge 0} a_k^j\Phi_k(\omega),$$

then

$$\int_{U_\sigma} |\nabla u|^2\,dx = \sigma \sum_j \sum_{k\geq 0} \gamma_k^- |a_k^j|^2 - \sigma^2 \sum_j \sum_{i\neq j} \frac{\langle a_0^i, a_0^j\rangle}{|x_i - x_j|} + \mathcal{O}(\sigma^3)\|a\|_{\ell^2}^2 \,. \tag{2.2}$$

Proof of Proposition 2.1. Consider u_j the harmonic extension of $g_j(x_j + \cdot)$ in $\mathbb{R}^3 \setminus B_\sigma$, given by

$$u_j(r\omega) = \sum_{k\geq 0} a_k^j \left(\frac{\sigma}{r}\right)^{\gamma_k^-} \Phi_k(\omega), \tag{2.3}$$

and the function

$$\tilde{u}(x) = \sum_{j=1}^{N} u_j(x - x_j)\,,$$

which is harmonic in U_σ. The function $\tilde{u} - u$ is harmonic in U_σ and satisfies $\tilde{u} - u = h_j$ on $\partial B_\sigma(x_j)$, where h_j is smooth in $B_1(x_j)$ and given by

$$h_j(x_j + r\omega) = \sum_{i\neq j} u_i(x_j - x_i + r\omega)\,, \tag{2.4}$$

for all $r \in (0,1)$. Since u_i is decaying, this boundary error is small for small σ, and therefore, its harmonic extension $u - \tilde{u}$ is also small. Hence we expect that the energy of u should coincide, at leading order, with the energy of $\tilde{u}$. We will see that this heuristic is correct, but we will also need to include next-order contributions to capture the second term in the right-hand side of (2.1). We start from the identity

$$\int_{U_\sigma} |\nabla u|^2\,dx = \int_{U_\sigma} |\nabla \tilde{u}|^2\,dx + \int_{U_\sigma} |\nabla \tilde{u} - \nabla u|^2\,dx + 2\int_{U_\sigma} \langle \nabla u - \nabla \tilde{u}, \nabla \tilde{u}\rangle\,dx\,. \tag{2.5}$$

The rest of the proof is structured as follows. In Step 0 we gather some estimates on the boundary error h_j. In Step 1 we estimate the last integral in (2.5). The integral of $|\nabla \tilde{u} - \nabla u|^2$ is estimated in Step 2. In Step 3 we compute $\int |\nabla \tilde{u}|^2$ and finally conclude in Step 4.

Step 0. Estimates of the boundary error.

Let $i \neq j$, and $\alpha \geq 0$. Using that $|x_i - x_j| \geq 2$, $|\nabla_\omega^\alpha \Phi_k| \lesssim \lambda_k^{(1+\alpha)/2}$ and

$\gamma_k^- \lesssim 1 + \sqrt{\lambda_k}$ we obtain, for all $x \in B_\sigma(x_j)$,

$$
\begin{aligned}
|\nabla^\alpha u_i(x - x_i)| &\le C_\alpha \sum_{k\ge 0} |a_k^i| \big(1 + \lambda_k^{\frac{1+\alpha}{2}}\big) \sigma^{\gamma_k^-} \\
&\le C_\alpha \sigma \|a^i\|_{\ell^2} \Bigg(\sum_{k\ge 0} \big(1 + \lambda_k^{1+\alpha}\big) \sigma^{2\gamma_k^- - 2} \Bigg)^{1/2} \\
&\le C_\alpha \sigma \|a^i\|_{\ell^2} \,.
\end{aligned}
$$

The last inequality is valid because $\sigma \le 1/2$ and $\gamma_k^- \ge 1$. In particular we have

$$
\max_{0\le\alpha\le 4} \sup_{B_\sigma(x_j)} |\nabla^\alpha u_i(\cdot - x_i)| \lesssim \sigma \|a^j\|_{\ell^2} \,. \tag{2.6}
$$

Combining this bound with the definition (2.4) of the boundary error h_j, we infer

$$
|h_j(x_j + \sigma\omega) - h_j(x_j)| \lesssim \sigma \sum_{i\ne j} \sup_{B_\sigma(x_j)} |\nabla u_i(\cdot - x_i)| \lesssim \sigma^2 \|a\|_{\ell^2} \,, \tag{2.7}
$$

and

$$
\begin{aligned}
|\Delta_\omega^2 \left[h_j(x_j + \sigma\omega) - h_j(x_j) \right] | &\lesssim \sigma^4 \sum_{i\ne j} \max_{0\le\alpha\le 4} \sup_{B_\sigma(x_j)} |\nabla^4 u_i(\cdot - x_i)| \\
&\lesssim \sigma^5 \|a\|_{\ell^2} \,.
\end{aligned} \tag{2.8}
$$

Step 1. Estimating $\int \langle \nabla u - \nabla\tilde{u}, \nabla\tilde{u}\rangle$.

Since $\tilde{u}$ is harmonic in U_σ and $\tilde{u} - u = h_j$ on $\partial B_\sigma(x_j)$, we have

$$
\begin{aligned}
&\int_{U_\sigma} \langle \nabla u - \nabla\tilde{u}, \nabla\tilde{u}\rangle \, dx = \int_{\partial U_\sigma} \langle u - \tilde{u}, \partial_\nu \tilde{u}\rangle \, d\mathcal{H}^2 \\
&= \sigma^2 \sum_j \int_{\mathbb{S}^2} \langle h_j(x_j + \sigma\omega), (\omega\cdot\nabla)\tilde{u}(x_j + \sigma\omega)\rangle \, d\mathcal{H}^2(\omega) \\
&= \sigma^2 \sum_j \int_{\mathbb{S}^2} \langle h_j(x_j + \sigma\omega), \partial_r u_j(\sigma\omega)\rangle \, d\mathcal{H}^2(\omega) \\
&\quad + \sigma^2 \sum_j \sum_{\ell\ne j} \int_{\mathbb{S}^2} \langle h_j(x_j + \sigma\omega), (\omega\cdot\nabla) u_\ell(x_j - x_\ell + \sigma\omega)\rangle \, d\mathcal{H}^2(\omega).
\end{aligned}
$$

To control the last integral we note that the estimate (2.6) from Step 0 implies

$$
|h_j(x_j + \sigma\omega)| \lesssim \sigma N \|a\|_{\ell^2} \quad \text{and} \quad |\nabla u_\ell(x_j - x_\ell + \sigma\omega)| \lesssim \sigma \|a\|_{\ell^2} \,,
$$

for all $j \ne \ell$ and $\omega \in \mathbb{S}^2$. We deduce

$$
\begin{aligned}
&\int_{U_\sigma} \langle \nabla u - \nabla\tilde{u}, \nabla\tilde{u}\rangle \, dx \\
&= \sigma^2 \sum_j \int_{\mathbb{S}^2} \langle h_j(x_j + \sigma\omega), \partial_r u_j(\sigma\omega)\rangle \, d\mathcal{H}^2(\omega) + \mathcal{O}(\sigma^4) \|a\|_{\ell^2}^2 \,.
\end{aligned} \tag{2.9}
$$

For all $j \in \{1, \dots, N\}$, using the explicit expression (2.3) of $u_j(r\omega)$ we have

$$\begin{aligned}
&\int_{\mathbb{S}^2} \langle h_j(x_j + \sigma\omega), \partial_r u_j(\sigma\omega)\rangle \, d\mathcal{H}^2(\omega) \\
&= -\frac{1}{\sigma} \sum_{k\geq 0} \gamma_k^- \int_{\mathbb{S}^2} \langle h_j(x_j + \sigma\omega), a_k^j\rangle \, \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \\
&= -\frac{1}{2\sqrt{\pi}\sigma} \int_{\mathbb{S}^2} \langle h_j(x_j), a_0^j \rangle \, d\mathcal{H}^2 \\
&\quad - \frac{1}{\sigma} \sum_{k\geq 0} \gamma_k^- \Big\langle a_k^j, \int_{\mathbb{S}^2} \big(h_j(x_j + \sigma\omega) - h_j(x_j)\big) \, \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \Big\rangle . \qquad (2.10)
\end{aligned}$$

For the last equality we used the fact that the spherical harmonics Φ_k of order $k \geq 1$ have zero average, while Φ_0 is constant equal to $1/(2\sqrt{\pi})$. To control the last sum, we note that by the estimate (2.7) from Step 0 we have

$$\Big| \int_{\mathbb{S}^2} \big(h_j(x_j + \sigma\omega) - h_j(x_j)\big) \, \Phi_0(\omega) \, d\mathcal{H}^2(\omega) \Big| \lesssim \sigma^2 \|a\|_{\ell^2} ,$$

and, for $k \geq 1$, thanks to the fact that $\Phi_k = \lambda_k^{-2} \Delta_\omega^2 \Phi_k$ and the estimate (2.8) from Step 0,

$$\begin{aligned}
&\Big| \int_{\mathbb{S}^2} \big(h_j(x_j + \sigma\omega) - h_j(x_j)\big) \, \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \Big| \\
&= \frac{1}{\lambda_k^2} \Big| \int_{\mathbb{S}^2} \big(h_j(x_j + \sigma\omega) - h_j(x_j)\big) \, \Delta_\omega^2 \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \Big| \\
&= \frac{1}{\lambda_k^2} \Big| \int_{\mathbb{S}^2} \Delta_\omega^2 \big(h_j(x_j + \sigma\omega) - h_j(x_j)\big) \, \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \Big| \lesssim \frac{\sigma^5}{\lambda_k^2} \|a\|_{\ell^2} .
\end{aligned}$$

From this and the previous inequality for $k = 0$ we infer

$$\begin{aligned}
&\Big| \sum_{k\geq 0} \gamma_k^- \Big\langle a_k^j, \int_{\mathbb{S}^2} \big(h_j(x_j + \sigma\omega) - h_j(x_j)\big) \, \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \Big\rangle \Big| \\
&\qquad\qquad \lesssim \sigma^2 \|a\|_{\ell^2} |a_0^j| + \sigma^5 \|a\|_{\ell^2} \sum_{k\geq 1} \frac{\gamma_k^-}{\lambda_k^2} |a_k^j| \lesssim \sigma^2 \|a\|_{\ell^2}^2 .
\end{aligned}$$

The last inequality follows from the fact that $(\gamma_k^-/\lambda_k^2)^2 \lesssim 1/\lambda_k^3$ is summable. Using this to estimate the last line of (2.10) we deduce

$$\begin{aligned}
&\int_{\mathbb{S}^2} \langle h_j(x_j + \sigma\omega), \partial_r u_j(\sigma\omega)\rangle \, d\mathcal{H}^2(\omega) \\
&\qquad\qquad = -\frac{1}{2\sqrt{\pi}\sigma} \int_{\mathbb{S}^2} \langle h_j(x_j), a_0^j \rangle \; d\mathcal{H}^2(\omega) + \mathcal{O}(\sigma) \|a\|_{\ell^2}^2 \\
&\qquad\qquad = -\frac{2\sqrt{\pi}}{\sigma} \sum_{i\neq j} u_i(x_j - x_i) + \mathcal{O}(\sigma) \|a\|_{\ell^2}^2 .
\end{aligned}$$

The last equality follows from the expression (2.4) of h_j and the fact that $|\mathbb{S}^2| = 4\pi$. Plugging this back into (2.9) gives

$$\int_{U_\sigma} \langle \nabla u - \nabla \tilde{u}, \nabla \tilde{u} \rangle \, dx = -\sigma^2 \sum_j \sum_{i \neq j} \frac{2\sqrt{\pi}}{\sigma} \langle u_i(x_j - x_i), a_0^j \rangle + \mathcal{O}(\sigma^3) \|a\|_{\ell^2}^2 \,. \tag{2.11}$$

Step 2. Estimating $\int |\nabla \tilde{u} - \nabla u|^2$.

To bound the term $\int |\nabla \tilde{u} - \nabla u|^2$, we recall that $\tilde{u} - u$ is harmonic, apply Lemma 2.2 below and use (2.6) to obtain

$$\int_{U_\sigma} |\nabla \tilde{u} - \nabla u|^2 \, dx \lesssim \sigma \sum_j \|h_j\|_{C^1(\partial B_\sigma(x_j))}^2 \lesssim \sigma^3 \|a\|_{\ell^2}^2 \,. \tag{2.12}$$

Step 3. Computing $\int |\nabla \tilde{u}|^2$.

Since $\tilde{u}$ is harmonic, we have

$$\int_{U_\sigma} |\nabla \tilde{u}|^2 \, dx = \int_{\partial U_\sigma} \langle \tilde{u}, \partial_\nu \tilde{u} \rangle \, d\mathcal{H}^2 = \sigma^2 \sum_{j,\ell,\ell'} I_j[u_\ell, u_{\ell'}],$$

where, for $j, \ell, \ell' \in \{1, \ldots, N\}$,

$$\begin{aligned} I_j[u_\ell, u_{\ell'}] &= \frac{1}{\sigma^2} \int_{\partial B_\sigma(x_j)} \langle u_\ell(\cdot - x_\ell), \partial_\nu [u_{\ell'}(\cdot - x_{\ell'})] \rangle \, d\mathcal{H}^2 \\ &= -\int_{\mathbb{S}^2} \big\langle u_\ell(x_j - x_\ell + \sigma\omega), (\omega \cdot \nabla) u_{\ell'}(x_j - x'_\ell + \sigma\omega) \big\rangle \, d\mathcal{H}^2(\omega). \end{aligned}$$

Since u_ℓ is small near $x_j - x_\ell$ for $j \neq \ell$, the magnitude of this integral depends a lot on whether ℓ, ℓ' are equal to j. Main order terms will corresponding to $\ell = \ell' = j$, next-order to $\ell \neq \ell' = 1$, and all other terms will be negligible for our purposes. We present next the estimates of each type of terms.

For $\ell = \ell' = j$, using the explicit expression (2.3) of $u_j(r\omega)$ we have

$$I_j[u_j, u_j] = -\int_{\mathbb{S}^2} \langle u_j(\sigma\omega), \partial_r u_j(\sigma\omega) \rangle \, d\mathcal{H}^2(\omega) = \frac{1}{\sigma} \sum_{k \geq 0} \gamma_k^- |a_k^j|^2 \,.$$

For $\ell \neq j$ and $\ell' = j$, using again the explicit expression (2.3) of $u_j(r\omega)$, and the fact that $\Phi_0 = 1/(2\sqrt{\pi})$ while Φ_k has zero average for $k \geq 1$, we find

$$\begin{aligned} I_j[u_\ell, u_j] &= -\int_{\mathbb{S}^2} \big\langle u_\ell(x_j - x_\ell + \sigma\omega), \partial_r u_j(\sigma\omega) \big\rangle \, d\mathcal{H}^2(\omega) \\ &= \frac{1}{\sigma} \sum_{k \geq 0} \gamma_k^- \int_{\mathbb{S}^2} \big\langle u_\ell(x_j - x_\ell + \sigma\omega), a_k^j \big\rangle \, \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \\ &= \frac{2\sqrt{\pi}}{\sigma} \langle u_\ell(x_j - x_\ell), a_0^j \rangle \\ &\quad + \frac{1}{\sigma} \sum_{k \geq 1} \gamma_k^- \int_{\mathbb{S}^2} \big\langle u_\ell(x_j - x_\ell + \sigma\omega) - u_\ell(x_j - x_\ell), a_k^j \big\rangle \, \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \,. \end{aligned}$$

The last line can be estimated as in Step 1 for the last sum in (2.10), and we deduce

$$I_j[u_\ell, u_j] = \frac{2\sqrt{\pi}}{\sigma}\langle u_\ell(x_j - x_\ell), a_0^j\rangle + \mathcal{O}(\sigma)\|a\|_{\ell^2}^2 \qquad \text{for } \ell \neq j\,.$$

For $\ell = j$ and $\ell' \neq j$ we find, using the explicit expression (2.3) of $u_j(r\omega)$, and the estimate (2.6) from Step 0,

$$\begin{aligned} I_j[u_j, u_{\ell'}] &= \int_{\mathbb{S}^2} \big\langle u_j(\sigma\omega), (\omega\cdot\nabla)u_{\ell'}(x_j - x_{\ell'} + \sigma\omega)\big\rangle\, d\mathcal{H}^2(\omega) \\ &= \sum_{k\geq 0}\int_{\mathbb{S}^2} \big\langle a_k^j, (\omega\cdot\nabla)u_{\ell'}(x_j - x_{\ell'} + \sigma\omega)\big\rangle\, \Phi_k(\omega)\, d\mathcal{H}^2(\omega) \\ &= \mathcal{O}(\sigma)\|a\|_{\ell^2}^2\,. \end{aligned}$$

Finally, for $\ell, \ell' \neq j$, we can directly use (2.6) to deduce

$$I_j[u_\ell, u_\ell'] = \mathcal{O}(\sigma^2)\|a\|_{\ell^2}^2\,.$$

Gathering all these estimates on the integrals $I_j[u_\ell, u_{\ell'}]$, we obtain

$$\begin{aligned} \int_{U_\sigma} |\nabla\tilde{u}|^2\, dx = \sigma^2 &\sum_{j,\ell,\ell'} I_j[u_\ell, u_{\ell'}] \\ &= \sigma\sum_j\sum_{k\geq 0}\gamma_k^-|a_k^j|^2 + \sigma^2\sum_j\sum_{\ell\neq j}\frac{2\sqrt{\pi}}{\sigma}\langle u_\ell(x_j - x_\ell), a_0^j\rangle \\ &\quad + \mathcal{O}(\sigma^3)\|a\|_{\ell^2}^2\,. \end{aligned} \tag{2.13}$$

Step 4. Conclusion.

Inserting Equations (2.11),(2.12) and (2.13) of Steps 1-3 into (2.5), we end up with

$$\begin{aligned} \int_{U_\sigma} |\nabla\tilde{u}|^2\, dx = \sigma\sum_j\sum_{k\geq 0}\gamma_k^-|a_k^j|^2 - \sigma^2\sum_j\sum_{\ell\neq j}\frac{2\sqrt{\pi}}{\sigma}\langle u_\ell(x_j - x_\ell), a_0^j\rangle \\ + \mathcal{O}(\sigma^3)\|a\|_{\ell^2}^2\,. \end{aligned}$$

Finally, note that from (2.3) we find

$$\frac{2\sqrt{\pi}}{\sigma}u_\ell(x_j - x_\ell) = \frac{a_0^\ell}{|x_j - x_\ell|} + \mathcal{O}(\sigma)\|a\|_{\ell^2},$$

which allows us to conclude. □

In Step 2 of Proposition 2.1's proof we used the following lemma to control the energy of a harmonic function with small boundary conditions.

Lemma 2.2. *If $\Delta v = 0$ in U_σ, $\int_{U_\sigma} |\nabla v|^2 \, dx < \infty$ with $|u| \to 0$ as $|x| \to \infty$ and $v = h_j$ on $\partial B_\sigma(x_j)$ for $j = 1, \ldots, N$, with*

$$h_j(x_j + \sigma\omega) = \sum_{k \geq 0} b_k^j \Phi_k(\omega),$$

then, using the notation $\|b^j\|^2_{h^{1/2}} = \sum_{k\geq 0}(1 + \sqrt{\lambda_k})|b_k^j|^2$, we have

$$\int_{U_\sigma} |\nabla v|^2 \, dx \lesssim \sigma \sum_{j=1}^{N} \|b^j\|^2_{h^{1/2}} \, .$$

Moreover, if $h_j \in C^1(\partial B_\sigma(x_j))$ then we have $\|b^j\|_{h^{1/2}} \lesssim \|h_j\|_{L^\infty} + \sqrt{\sigma}\|h_j\|_{C^1}$.

Proof. Denote by $v_j \colon \mathbb{R}^3 \setminus B_\sigma$ the harmonic extension of $h_j(x_j + \cdot)$, that is,

$$v_j(r\omega) = \sum_{k\geq 0} b_k^j \left(\frac{\sigma}{r}\right)^{\gamma_k^-} \Phi_k(\omega),$$

so that, using the orthogonality of $\{\Phi_k\}$ and $\{\nabla_\omega \Phi_k\}$ in $L^2(\mathbb{S}^2)$, we have

$$\begin{aligned}\int_{\mathbb{R}^3\setminus B_\sigma} |\nabla v_j|^2 \, dx &= \sum_{k\geq 0}((\gamma_k^-)^2 + \lambda_k)|b_k^j|^2 \int_\sigma^\infty \left(\frac{r}{\sigma}\right)^{-2\gamma_k^-} \, dr \\ &= \sigma \sum_{k\geq 0} \frac{(\gamma_k^-)^2 + \lambda_k}{2\gamma_k^- - 1} |b_k^j|^2 \lesssim \sigma \|b^j\|^2_{h^{1/2}},\end{aligned}$$

and

$$\int_{B_2\setminus B_1} |v_j|^2 \, dx = \sum_{k\geq 0} \sigma^{2\gamma_k^-} |b_k^j|^2 \int_1^2 r^{2-2\gamma_k^-} \, dr \lesssim \sigma^2 \|b^j\|_{\ell^2}.$$

Next we fix a smooth cut-off function $\eta(r)$ such that $\mathbf{1}_{r\leq 1} \leq \eta(r) \leq \mathbf{1}_{r<2}$ and $|\eta'| \leq 2$ and set

$$\tilde{v}(x) = \sum_{j=1}^{N} \eta(|x - x_j|) v_j(x - x_j),$$

so that $v = \tilde{v}$ on ∂U_σ and by minimality of v we have

$$\begin{aligned}\int_{U_\sigma} |\nabla v|^2 \, dx \leq \int_{U_\sigma} |\nabla \tilde{v}|^2 \, dx &\leq N \sum_{j=1}^{N} \int_{B_2\setminus B_\sigma} |\nabla(\eta(r) v_j)|^2 \, dx \\ &\leq 2N \sum_{j=1}^{N} \left(4 \int_{B_2\setminus B_1} |v_j|^2 \, dx + \int_{B_2\setminus B_\sigma} |\nabla v_j|^2 \right) \, dx \, .\end{aligned}$$

Combining this with the bounds on v_j gives the estimate of $\int |\nabla v|^2\,dx$ in terms of $\|b_j\|_{h^{1/2}}$.

Assume moreover that $h_j \in C^1(\partial B_\sigma(x_j))$. For any $\alpha > 0$ we estimate

$$\begin{aligned}
2\sum_{k\geq 0}\sqrt{\lambda_k}|b_k^j|^2 &\leq \alpha\sum_{k\geq 0}|b_k^j|^2 + \frac{1}{\alpha}\sum_{k\geq 0}\lambda_k|b_k^j|^2 \\
&\lesssim \frac{\alpha}{\sigma^2}\int_{\partial B_\sigma(x_j)}|h_j|^2 d\mathcal{H}^2 + \frac{1}{\alpha}\int_{\partial B_\sigma(x_j)}|\nabla_\omega h_j|^2\,d\mathcal{H}^2 \\
&\lesssim \alpha\|h_j\|^2_{L^\infty(\partial B_\sigma(x_j))} + \frac{\sigma^2}{\alpha}\|\nabla_\omega h_j\|^2_{L^\infty(\partial B_\sigma(x_j))}.
\end{aligned}$$

We note that the claim of the lemma is trivial for constant h_j, so without loss of generality, we can assume that $\nabla_\omega h_j \not\equiv 0$, in particular $h_j \not\equiv 0$. This allows us to choose $\alpha = \sigma\|\nabla_\omega h_j\|_{L^\infty}/\|h_j\|_{L^\infty}$ gives

$$\begin{aligned}
\sum_{k\geq 0}\sqrt{\lambda_k}|b_k^j|^2 &\lesssim \sigma\|h_j\|_{L^\infty(\partial B_\sigma(x_j))}\|\nabla_\omega h_j\|_{L^\infty(\partial B_\sigma(x_j))} \\
&\lesssim \sigma\|h_j\|^2_{C^1(\partial B_\sigma(x_j))}.
\end{aligned}$$

With $\|b^j\|_{\ell^2} \lesssim \|h_j\|_{L^\infty}$, this implies $\|b^j\|_{h^{1/2}} \lesssim \|h_j\|_{L^\infty} + \sqrt{\sigma}\|h_j\|_{C^1}$. □

3 Upper bound

In this section we perform the upper bound construction.

Proposition 3.1. *The minimum of E_ρ over all $n\colon \Omega \to \mathbb{S}^2$ with far-field alignment* (1.2) *is bounded above by*

$$\min E_\rho \leq \sum_j \mu_j - \rho\sum_j\sum_{i\neq j}\frac{4\pi\langle v_i, v_j\rangle}{|x_i - x_j|} + \mathcal{O}\left(\rho^{4/3}\right), \tag{3.1}$$

for any minimizers $\hat{m}_j$ of the single-particle problem (1.3)*, where $\mu_j = \widehat{E}_j(\hat{m}_j)$ and v_j is defined by the asymptotic expansion* (1.4)*.*

The upper bound is obtained by constructing a competitor and estimating its energy. In a ball $B_\sigma(x_j)$ around each particle $\omega_{j,\rho} = x_j + \rho\widehat{\omega}_j$ we choose the competitor n to be equal to a single-particle minimizer $\hat{m}_j$, rescaled at scale ρ. In the exterior U_σ of these balls, we take n to be the $\mathbb{R}^3$-valued harmonic extension, projected to $\mathbb{S}^2$. The boundary values of this extension are determined by the maps $\hat{m}_j$, for which we have precise asymptotic estimates. If σ is large enough, the energy contribution inside each ball $B_\sigma(x_j)$ is close to $\mu_j = \widehat{E}_j(\hat{m}_j)$. If σ is not too large, the energy contribution outside the balls $B_\sigma(x_j)$ can be accurately estimated using Proposition 2.1. Choosing σ to balance error terms, we arrive at (3.1).

Proof of Proposition 3.1. We start by recalling from [2] that for each $j = 1, \dots, N$, there exists a minimizer $\hat{m}_j : \mathbb{R}^3 \setminus \hat{\omega}_j \to \mathbb{S}^2$ of $\widehat{E}_j$ under the far-field alignment constraint

$$\int_{\mathbb{R}^3 \setminus \hat{\omega}_j} \frac{|\hat{m}_j - n_\infty|^2}{1 + |x|^2} \, dx < \infty \, .$$

Furthermore, there exist $\lambda_0 > 0$ and $v_j \in n_\infty^\perp$ such that

$$\begin{aligned} &\hat{m}_j(x) = n_\infty + \frac{v_j}{r} + \hat{w}_j(x), \\ &|\hat{w}_j| + r|\nabla \hat{w}_j| + r^2 |\nabla^2 \hat{w}_j| \lesssim \frac{1}{r^2} \qquad \text{for } r = |x| > \lambda_0 \, . \end{aligned} \tag{3.2}$$

Let $\sigma \in (\rho, 1/2)$, to be fixed later on. As explained above, we define our competitor to be equal to $\hat{m}_j((\cdot - x_j)/\rho)$ in each ball $B_\sigma(x_j)$. At each boundary $\partial B_\sigma(x_j)$, it is therefore equal to $n_\infty + g_j$, where g_j is given by

$$g_j(x_j + \sigma\omega) = \hat{m}_j(\sigma\omega/\rho) - n_\infty = \frac{\rho}{\sigma} v_j + \hat{w}_j(\sigma\omega/\rho) \, . \tag{3.3}$$

We denote by u the harmonic extension to $U_\sigma = \mathbb{R}^3 \setminus \bigcup_j B_\sigma(x_j)$ satisfying $u = g_j$ on $\partial B_\sigma(x_j)$, as in Proposition 2.1. With these notations, we define our competitor $n \colon \Omega_\rho \to \mathbb{S}^2$ by setting

$$n(x) = \begin{cases} \hat{m}_j\left(\frac{x - x_j}{\rho}\right) & \text{if } |x - x_j| < \sigma, \\ \frac{n_\infty + u}{|n_\infty + u|} & \text{if } x \in U_\sigma. \end{cases}$$

We use the same notations as in Proposition 2.1 and consider the spherical harmonics coefficients

$$a_k^j = \int_{\mathbb{S}^2} g_j(x_j + \sigma\omega) \Phi_k(\omega) \, d\mathcal{H}^2(\omega) \, .$$

Taking into account the decay properties (3.2) of $\hat{w}_j$ we see that these coefficients a_k^j satisfy

$$\begin{aligned} \left| a_0^j - \frac{2\sqrt{\pi}\rho}{\sigma} v_j \right|^2 &= \left| \frac{1}{2\sqrt{\pi}} \int_{\mathbb{S}^2} \left(g_j(x_j + \sigma\omega) - \frac{\rho}{\sigma} v_j \right) d\mathcal{H}^2(\omega) \right|^2 \\ &= \left| \frac{1}{2\sqrt{\pi}} \int_{\mathbb{S}^2} \hat{w}_j(\sigma\omega/\rho) \, d\mathcal{H}^2(\omega) \right|^2 \lesssim \frac{\rho^4}{\sigma^4}, \\ \sum_{k \geq 1} \lambda_k |a_k^j|^2 &= \int_{\mathbb{S}^2} |\nabla_\omega [g_j(\sigma\omega)]|^2 \, d\mathcal{H}^2(\omega) \\ &= \int_{\mathbb{S}^2} |\nabla_\omega [\hat{w}_j(\sigma\omega/\rho)]|^2 \, d\mathcal{H}^2(\omega) \\ &\leq \frac{\sigma^2}{\rho^2} \int_{\mathbb{S}^2} |\nabla \hat{w}_j|^2 (\sigma\omega/\rho) \, d\mathcal{H}^2(\omega) \lesssim \frac{\rho^4}{\sigma^4} \, . \end{aligned}$$

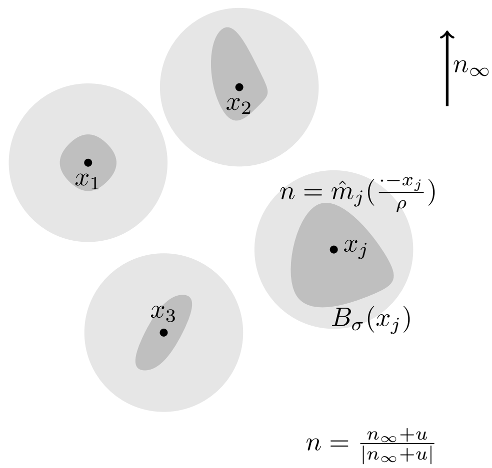

Figure 2: Structure of the competitor n constructed in Proposition 3.1.

We deduce that for each $j = 1, \ldots, N$,

$$\begin{aligned}
a_j^0 &= 2\sqrt{\pi}\frac{\rho}{\sigma}v_j + \mathcal{O}(\rho^2/\sigma^2)\,,\\
|a_j^0|^2 &= 4\pi\frac{\rho^2}{\sigma^2}|v_j|^2 + \mathcal{O}(\rho^3/\sigma^3)\,,\\
\langle a_i^0, a_j^0\rangle &= 4\pi\frac{\rho^2}{\sigma^2}\langle v_i, v_j\rangle + \mathcal{O}(\rho^3/\sigma^3)\,,\\
\sum_{k\geq 1}\gamma_k^-|a_k^j|^2 &\leq \sum_{k\geq 1}\lambda_k|a_k^j|^2 = \mathcal{O}(\rho^4/\sigma^4)\,,\\
\|a_j\|_{\ell^2}^2 &= |a_j^0|^2 + \sum_{k\geq 1}|a_j^k|^2 \leq |a_j^0|^2 + \sum_{k\geq 1}\lambda_k|a_j^k|^2 = \mathcal{O}(\rho^2/\sigma^2)\,.
\end{aligned}$$

This enables us to estimate each term appearing in the asymptotic expansion (2.1) for the energy of u provided by Proposition 2.1, namely

$$\begin{aligned}
\sigma\sum_j\sum_{k\geq 0}\gamma_k^-|a_k^j|^2 &= \frac{\rho^2}{\sigma}\sum_j 4\pi|v_j|^2 + \mathcal{O}(\rho^3/\sigma^2)\\
\sigma^2\sum_j\sum_{i\neq j}\frac{\langle a_0^i, a_0^j\rangle}{|x_i - x_j|} &= \rho^2\sum_j 4\pi\frac{\langle v_i, v_j\rangle}{|x_i - x_j|} + \mathcal{O}(\rho^3/\sigma)\\
\mathcal{O}(\sigma^3)\|a\|_{\ell^2}^2 &= \mathcal{O}(\sigma\rho^2)\,.
\end{aligned}$$

Dividing by ρ and applying Proposition 2.1, we infer

$$\begin{aligned}\frac{1}{\rho}\int_{U_\sigma}|\nabla u|^2\,dx &= \frac{\rho}{\sigma}\sum_j 4\pi|v_j|^2 - \rho\sum_j 4\pi\frac{\langle v_i, v_j\rangle}{|x_i - x_j|}\\ &\quad + \mathcal{O}(\rho^2/\sigma^2) + \mathcal{O}(\sigma\rho)\,. \end{aligned}\tag{3.4}$$

Next we need to take into account the error introduced in that energy estimate by projecting $n_\infty + u$ onto $\mathbb{S}^2$. To that end, note that the decay properties (3.2) of $\hat{w}_j$ and the fact that $\langle n_\infty, v_j\rangle = 0$ imply that the boundary condition $g_j(x_j + \sigma\omega) = (\rho/\sigma)v_j + \hat{w}_j(\sigma\omega/\rho)$ of u at $\partial B_\sigma(x_j)$, defined in (3.3), satisfies $\langle n_\infty, g_j\rangle = \mathcal{O}(\rho^2/\sigma^2)$. So by the maximum principle and since $u \to 0$ at ∞, we have $\langle n_\infty, u\rangle = \mathcal{O}(\rho^2/\sigma^2)$, and therefore

$$|n_\infty + u|^2 \geq 1 + 2\langle n_\infty, u\rangle \geq 1 - C\frac{\rho^2}{\sigma^2} \qquad \text{in } U_\sigma\,.$$

Note also that for any smooth map v with values in $\mathbb{R}^N \setminus \{0\}$, the following inequality holds

$$\begin{aligned}\Big|\partial_\alpha\Big[\frac{v}{|v|}\Big]\Big|^2 &= \frac{1}{|v|^2}\Big|\partial_\alpha v - \big\langle \partial_\alpha v, \frac{v}{|v|}\big\rangle\frac{v}{|v|}\Big|^2 = \frac{1}{|v|^2}\Big(|\partial_\alpha v|^2 - \big\langle\partial_\alpha v, \frac{v}{|v|}\big\rangle^2\Big)\\ &\leq \frac{|\partial_\alpha v|^2}{|v|^2}\,. \end{aligned}\tag{3.5}$$

Applying this to $v = n_\infty + u$, we deduce

$$\begin{aligned}\frac{1}{\rho}\int_{U_\sigma}|\nabla n|^2\,dx &\leq \frac{1}{\rho}\int_{U_\sigma}\frac{|\nabla u|^2}{|n_\infty + u|^2}\,dx\\ &\leq \frac{1}{\rho}\Big(1 + C\frac{\rho^2}{\sigma^2}\Big)\int_{U_\sigma}|\nabla u|^2\,dx\\ &\leq \frac{\rho}{\sigma}\sum_j 4\pi|v_j|^2 - \rho\sum_j\sum_{i\neq j}\frac{4\pi\langle v_i, v_j\rangle}{|x_i - x_j|} + \mathcal{O}\left(\sigma\rho + \frac{\rho^2}{\sigma^2}\right)\,. \end{aligned}$$

The last inequality comes from the estimate of $\int|\nabla u|^2$ in (3.4). In the whole domain, the energy of n is therefore bounded by

$$\begin{aligned}E_\rho(n) &\leq \sum_j \widehat{E}_j(\hat{m}_j; B_{\sigma/\rho}\setminus\hat{\omega}_j) + \frac{\rho}{\sigma}\sum_j 4\pi|v_j|^2\\ &\quad - \rho\sum_j\sum_{i\neq j}\frac{4\pi\langle v_i, v_j\rangle}{|x_i - x_j|} + \mathcal{O}\left(\sigma\rho + \frac{\rho^2}{\sigma^2}\right)\,. \end{aligned}$$

Noting that the expansion (3.2) of $\hat{m}_j$ implies, for $\sigma \geq \lambda_0\rho$,

$$\widehat{E}_j(\hat{m}_j; \mathbb{R}^3\setminus B_{\sigma/\rho}) = \int_{|\hat{x}|\geq\frac{\sigma}{\rho}}|\nabla\hat{m}_j|^2\,d\hat{x} = 4\pi\frac{\rho}{\sigma}|v_j|^2 + \mathcal{O}\left(\frac{\rho^2}{\sigma^2}\right), \tag{3.6}$$

and recalling the definition $\mu_j = \widehat{E}_j(\hat{m}_j; \mathbb{R}^3 \setminus \hat{\omega}_j)$, we deduce

$$\mu_j = \widehat{E}_j(\hat{m}_j; B_{\sigma/\rho} \setminus \hat{\omega}_j) + 4\pi \frac{\rho}{\sigma} |v_j|^2 + \mathcal{O}\left(\frac{\rho^2}{\sigma^2}\right) .$$

All together we obtain the upper bound

$$E_\rho(n) \leq \sum_j \mu_j - \rho \sum_j \sum_{i \neq j} \frac{4\pi \langle v_i, v_j \rangle}{|x_i - x_j|} + \mathcal{O}\left(\sigma\rho + \frac{\rho^2}{\sigma^2}\right) .$$

Choosing $\sigma = \sigma_\rho = \rho^{1/3}$ provides a remainder of order $\rho^{4/3}$. □

4 A matching lower bound

In this section we give the proof of a lower bound which matches the upper bound of Proposition 3.1 at order $o(\rho)$ in the following sense:

Proposition 4.1. *For any sequence $\rho \to 0$, there exist minimizers $\hat{m}_j$ of the single-particle problem* (1.3) *and a subsequence still denoted $\rho \to 0$ such that*

$$\min_{(1.2)} E_\rho \geq \sum_j \mu_j - \rho \sum_j \sum_{i \neq j} \frac{4\pi \langle v_i, v_j \rangle}{|x_i - x_j|} + o(\rho) , \tag{4.1}$$

where $\mu_j = \widehat{E}_j(\hat{m}_j)$ is the minimal value of the single particle problem (1.3) *and v_j is defined by the asymptotic expansion* (1.4)*.*

This proposition then allows us to prove Theorem 1.1.

Proof of Theorem 1.1. Combining the upper bound of Proposition 3.1 with the lower bound of Proposition 4.1, we obtain the energy asymptotics

$$\min E_\rho = \sum_j \mu_j - 4\pi\rho\, G + o(\rho), \quad \text{where } G = \sup \left\{ \sum_{i \neq j} \frac{\langle v_i, v_j \rangle}{|x_i - x_j|} \right\},$$

and the supremum is over all collections of admissible vectors v_j, $j = 1, \ldots, N$, which satisfy (1.4) for some minimizer $\hat{m}_j$. For generic n_∞, there is a unique admissible v_j for each j, see [2], so the supremum is not needed and this proves Theorem 1.1. If some v_j's are not unique, we need to check that the supremum in G is attained to conclude the proof of Theorem 1.1. To that end, it suffices to show that, for each particle $\hat{\omega}_j$, the set of admissible vectors v_j's is compact. This follows from two basic facts. First, the set of minimizers $\hat{m}_j$ of the single-particle problem (1.3) is compact in H^1_{loc} [9, 11]. Second, the vector v_j defined by (1.4) depends continuously on the minimizer $\hat{m}_j$ in that topology. Assume indeed that $\hat{m}_j$ and $\tilde{m}_j$ are two minimizers with corresponding v_j and $\tilde{v}_j$. Then, using the asymptotic expansion (3.2) for both minimizers we infer

$$|v_j - \tilde{v}_j|^2 = R \int_{B_{2R} \setminus B_R} |\nabla \hat{m}_j - \nabla \tilde{m}_j|^2 \, dx + \mathcal{O}\Big(\frac{1}{R}\Big) \quad \text{as } R \to \infty.$$

(It can be checked from the proof [2, Theorem 1] of the expansion (3.2) that the constant in the estimate of the remainder can be taken independent from the minimizer.) For any $\varepsilon > 0$ we may choose R large enough that the last term is smaller than $\varepsilon/2$, and we deduce that $|v_j - \tilde{v}_j|^2 \leq \varepsilon$ provided $\hat{m}_j - \tilde{m}_j$ is small enough in $H^1(B_{2R} \setminus B_R)$, for this fixed radius R. $\square$

In this whole section, we consider $n_\rho \colon \Omega_\rho \to \mathbb{S}^2$ a minimizer of E_ρ. We extend n_ρ to $\mathbb{R}^3$ by filling the holes $\omega_{j,\rho} = x_j + \rho\hat{\omega}_j$ with $\mathbb{S}^2$-valued maps minimizing the Dirichlet energy, and define the rescaled map

$$\hat{n}_j^\rho(\hat{x}) = n_\rho(x_j + \rho\hat{x}) , \tag{4.2}$$

around each particle $j \in \{1, \ldots, N\}$. We will freely extract subsequences and never make this explicit in the notations.

We divide the proof of Proposition 4.1 into four subsections. In the first two we apply classical properties of energy-minimizing maps: small energy estimates and local H^1 compactness. These provide, in § 4.1, pointwise bounds that will be used throughout the following sections; and in § 4.2, strong H^1_{loc} convergence of $\hat{n}_j^\rho$ to a minimizer $\hat{m}_j$ of the single-particle problem (1.3). Then, in § 4.3 we take advantage of a compensation effect to obtain, for any $1 \ll \lambda \ll 1/\rho$, a lower bound which depends on the smallness of $|\hat{n}_j^\rho - \hat{m}_j|$ on the annulus $B_{2\lambda} \setminus B_\lambda$. Finally, in § 4.4 we establish a far-field expansion for $\hat{n}_j^\rho$ which we then combine with the far-field expansion (1.4) of $\hat{m}_j$ and the lower bound from the previous step in order to conclude the proof of Proposition 4.1.

4.1 Pointwise estimates

In this section we gather pointwise estimates on $|\nabla\hat{n}_j^\rho|^2$ and $|\hat{n}_j^\rho - n_\infty|^2$ that follow from classical small energy estimates [14] for the harmonic map $\hat{n}_j^\rho$ in the exterior of a large enough ball.

Lemma 4.2. *There exists $\lambda_0 > 2$ such that, for all $\rho \in (0, 1/(2\lambda_0))$ and $\lambda \in [\lambda_0, 1/(2\rho)]$, we have*

$$\sup_{\partial B_\lambda} |\nabla\hat{n}_j^\rho|^2 \lesssim \fint_{B_{5\lambda/4}\setminus B_{3\lambda/4}} |\nabla\hat{n}_j^\rho|^2 dx , \tag{4.3}$$

$$\lambda^2 \sup_{\partial B_\lambda} |\nabla\hat{n}_j^\rho|^2 \lesssim \fint_{B_{5\lambda/4}\setminus B_{3\lambda/4}} |\hat{n}_j^\rho - n_\infty|^2 \, dx , \tag{4.4}$$

$$\sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2 \lesssim \fint_{B_{5\lambda/4}\setminus B_{3\lambda/4}} |\hat{n}_j^\rho - n_\infty|^2 \, dx . \tag{4.5}$$

Proof of Lemma 4.2. Let $\eta > 0$ be such that the small energy estimate

$$\begin{aligned} &\int_{B_1} |u - u_*|^2 \, dx + \int_{B_1} |\nabla u|^2 dx \leq \eta \\ \Rightarrow \quad &\sup_{B_{1/2}} |\nabla u|^2 \lesssim \int_{B_1} |u - u_*|^2 dx , \end{aligned} \tag{4.6}$$

is valid for any map $u\colon B_1 \to \mathbb{S}^2$ minimizing the Dirichlet energy with respect to its own boundary conditions, and any $u_* \in \mathbb{R}^3$ (this is proved in [14], see [16, § 2.3] for this precise statement). Choosing u_* in (4.6) to be the average of u on B_1, applying Poincaré's inequality, and decreasing η if necessary, we also have the implication

$$\begin{aligned} &\int_{B_1} |\nabla u|^2 dx \leq \eta \\ &\Rightarrow \quad \sup_{B_{1/2}} |\nabla u|^2 \lesssim \int_{B_1} \Big| u - \fint_{B_1} u \Big|^2 dx \lesssim \int_{B_1} |\nabla u|^2 \, dx \,, \end{aligned} \tag{4.7}$$

for any map $u\colon B_1 \to \mathbb{S}^2$ minimizing the Dirichlet energy.

We introduce the notation $A_{\theta,\lambda}$ for the annulus of width $2\theta\lambda$ around the sphere ∂B_λ, that is,

$$A_{\theta,\lambda} = B_{(1+\theta)\lambda} \setminus B_{(1-\theta)\lambda}, \qquad \text{for } 0 < \theta < 1 \,.$$

Let $\lambda_0 > 16$, to be chosen large enough later on. We fix $\lambda \in [\lambda_0, 1/(2\rho)]$ and $\theta \in [\lambda^{-1/4}, 1/2]$. For any $x_0 \in \partial B_\lambda$, we consider the harmonic map

$$u(y) = \hat{n}_j^\rho(x_0 + \theta\lambda y) \,,$$

and check that it satisfies the smallness assumptions in (4.6)-(4.7). We have

$$\begin{aligned} \int_{B_1} |\nabla u|^2 \, dy &= \frac{1}{\theta\lambda} \int_{B_{\theta\lambda}(x_0)} |\nabla \hat{n}_j^\rho|^2 \, dx \leq \frac{1}{\theta\lambda} \int_{A_{\theta,\lambda}} |\nabla \hat{n}_j^\rho|^2 \, dx \\ &\leq \frac{1}{\theta\lambda} E_\rho(n_\rho) \lesssim \frac{1}{\lambda^{3/4}} \,, \end{aligned}$$

since $\lambda^{1/4}\theta \geq 1$ and $E_\rho(n_\rho) \lesssim 1$, so u satisfies the smallness assumption in (4.7) provided λ_0 is large enough. In addition, we have

$$\begin{aligned} \int_{B_1} |u - n_\infty|^2 \, dy &= \frac{1}{(\theta\lambda)^3} \int_{B_{\theta\lambda}(x_0)} |\hat{n}_j^\rho - n_\infty|^2 \, dx \\ &\leq \frac{1}{\theta^3\lambda^3} \int_{A_{\theta,\lambda}} |\hat{n}_j^\rho - n_\infty|^2 \, dx \\ &\leq \frac{1}{\theta^3\lambda} \int_{\mathbb{R}^3 \setminus B_1} \frac{|\hat{n}_j^\rho - n_\infty|^2}{|x|^2} \, dx \lesssim \frac{1}{\lambda^{1/4}}, \end{aligned}$$

since $\lambda^{3/4}\theta^3 \geq 1$ and by Hardy's inequality combined with the bound $E_\rho(n_\rho) \lesssim 1$ we have $\int |\hat{n}_j^\rho - n_\infty|^2/(1 + |x|^2) \, dx \lesssim 1$, see also (4.10) later. So u satisfies the smallness assumption in (4.6) with $u_* = n_\infty$, provided λ_0 is large enough.

The estimate (4.7) thus gives

$$(\theta\lambda)^2 |\nabla \hat{n}_j^\rho(x_0)|^2 = |\nabla u(0)|^2 \lesssim \frac{1}{\theta\lambda} \int_{A_{\theta,\lambda}} |\nabla \hat{n}_j^\rho|^2 \, dx \,,$$

for any $x_0 \in \partial B_\lambda$. For $\theta = 1/4$, this proves (4.3). The estimate (4.6) with $u_* = n_\infty$ gives

$$(\theta\lambda)^2 |\nabla \hat{n}_j^\rho(x_0)|^2 = |\nabla u(0)|^2 \lesssim \frac{1}{\theta^3\lambda^3}\int_{A_{\theta,\lambda}} |\hat{n}_j^\rho - n_\infty|^2 \, dx \,,$$

which, for $\theta = 1/4$, proves (4.4).

It remains to prove the pointwise estimate (4.5) on $|\hat{n}_\rho^j - n_\infty|$. To that end we use the pointwise estimate (4.4) and the fundamental theorem of calculus to bound the oscillation of $\hat{n}_j^\rho$ on the annulus $A_{\theta/2,\lambda}$, whose diameter is $\leq 4\lambda$. Namely, for any $x, y \in A_{\theta/2,\lambda}$ we have

$$|\hat{n}_j^\rho(x) - \hat{n}_j^\rho(y)|^2 \lesssim \lambda^2 \sup_{A_{\theta/2,\lambda}} |\nabla \hat{n}_j^\rho|^2 \lesssim \frac{1}{\theta^5\lambda^3}\int_{A_{\theta,\lambda}} |\hat{n}_j^\rho - n_\infty|^2 \, dz \,.$$

Taking $x \in \partial B_\lambda$ and integrating with respect to y, this implies

$$\sup_{x\in\partial B_\lambda} \left| \hat{n}_j^\rho(x) - \fint_{A_{\theta/2,\lambda}} \hat{n}_j^\rho(y)\, dy \right|^2 \lesssim \frac{1}{\theta^5\lambda^3}\int_{A_{\theta,\lambda}} |\hat{n}_j^\rho - n_\infty|^2 \, dz \,.$$

Inserting n_∞ gives

$$\begin{aligned} \sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2 &\lesssim \left| n_\infty - \fint_{A_{\theta/2,\lambda}} \hat{n}_j^\rho(y)\, dy \right|^2 + \frac{1}{\theta^5\lambda^3}\int_{A_{\theta,\lambda}} |\hat{n}_j^\rho - n_\infty|^2 \, dy \,, \\ &\lesssim \fint_{A_{\theta/2,\lambda}} |\hat{n}_j^\rho - n_\infty|^2 \, dy + \frac{1}{\theta^5\lambda^3}\int_{A_{\theta,\lambda}} |\hat{n}_j^\rho - n_\infty|^2 \, dy \,, \end{aligned}$$

which, for $\theta = 1/4$, implies (4.5). □

4.2 Compactness of rescaled minimizers

In this section we exploit another classical property: minimizing harmonic maps are compact in H^1_{loc} [9, 11]. We apply the proof of this property to obtain that the limit of $\hat{n}_j^\rho$ along a subsequence $\rho \to 0$ is a global minimizer of $\widehat{E}_j$. From there, little extra effort is required to deduce the strong L^2 convergence of ∇n_j^ρ in $B_{1/\rho} \setminus \hat{\omega}_j$, see (4.8), so we include a proof of that fact, even though Proposition 4.1 is only going to require the H^1_{loc} compactness.

Lemma 4.3. *For any sequence $\rho \to 0$ and $j \in \{1, \ldots, N\}$, there exists a minimizer $\hat{m}_j$ of $\widehat{E}_j$ and a subsequence, still denoted $\rho \to 0$, such that*

$$\int_{B_{1/\rho}\setminus\hat{\omega}_j} |\nabla \hat{n}_j^\rho - \nabla \hat{m}_j|^2 \, dx \longrightarrow 0, \tag{4.8}$$

as $\rho \to 0$.

Proof of Lemma 4.3. Since $|x_i - x_j| \geq 2$ for $i \neq j$ we have

$$E_\rho(n_\rho) \;\geq\; \sum_{j=1}^{N} E_\rho(n_\rho; B_1 \setminus \omega_{j,\rho}) \;=\; \sum_{j=1}^{N} \hat{E}_j(\hat{n}_j^\rho; B_{1/\rho} \setminus \hat{\omega}_j)\,.$$

Combining this with the upper bound of Proposition 3.1 we deduce

$$\sum_{j=1}^{N} \hat{E}_j(\hat{n}_j^\rho; B_{1/\rho} \setminus \hat{\omega}_j) \leq \sum_{j=1}^{N} \mu_j + \mathcal{O}(\rho). \tag{4.9}$$

We use this bound in order to extract more information on the convergence of $\hat{n}_j^\rho$.

First recall that n_ρ has been extended to $\mathbb{R}^3$ so as to minimize the Dirichlet energy inside each hole $\omega_{\ell,\rho}$, and so the rescaled map $\hat{n}_\ell^\rho$ minimizes the Dirichlet energy inside $\hat{\omega}_\ell$. Moreover, we can construct an energy competitor $w \in H^1(\hat{\omega}_\ell; \mathbb{S}^2)$ such that $w = \hat{n}_\ell^\rho$ on $\partial\hat{\omega}_\ell$ and

$$\int_{\hat{\omega}_\ell} |\nabla w|^2\, dx \lesssim \int_{B_2 \setminus \hat{\omega}_\ell} |\nabla \hat{n}_\ell^\rho|^2\, dx\,.$$

This follows by applying [9, Lemma A.1] (the proof of which is valid in any domain) to an $\mathbb{R}^3$-valued extension with the same estimate. The existence of this $\mathbb{R}^3$-valued extension follows e.g. from composing a bounded extension operator $H^{1/2}(\partial\hat{\omega}_\ell) \to H^1(\hat{\omega}_\ell)$ with the trace operator $H^1(B_2 \backslash \hat{\omega}_\ell) \to H^{1/2}(\partial\hat{\omega}_\ell)$. Thanks to that energy competitor w, the minimality of $\hat{n}_\ell^\rho$ in $\hat{\omega}_\ell$ implies

$$\int_{\hat{\omega}_\ell} |\nabla \hat{n}_\ell^\rho|^2\, dx \lesssim \int_{B_2 \setminus \hat{\omega}_\ell} |\nabla \hat{n}_\ell^\rho|^2\, dx\,, \qquad \text{for } \ell = 1\, \ldots, N\,.$$

As a result, the Dirichlet energy of $\hat{n}_j^\rho$ in the whole space $\mathbb{R}^3$ is controlled by

$$\int_{\mathbb{R}^3} |\nabla \hat{n}_j^\rho|^2\, dx \leq E_\rho(n_\rho) + \sum_{\ell=1}^{N} \int_{\hat{\omega}_\ell} |\nabla \hat{n}_\ell^\rho|^2 dx \lesssim E_\rho(n_\rho)\,.$$

Using also Hardy's inequality, we deduce

$$\int_{\mathbb{R}^3} \frac{|\hat{n}_j^\rho - n_\infty|^2}{1+|x|^2}\, dx \lesssim \int_{\mathbb{R}^3} |\nabla \hat{n}_j^\rho|^2\, dx \lesssim E_\rho(n_\rho)\,. \tag{4.10}$$

Thanks to this bound, for every $j \in \{1, \ldots, N\}$, there exists a map $\hat{m}_j \in H^1_{\mathrm{loc}}(\mathbb{R}^3; \mathbb{S}^2)$ and a subsequence $\rho \to 0$ such that

$$\begin{aligned}
&\nabla \hat{n}_j^\rho \rightharpoonup \nabla \hat{m}_j \text{ weakly in } L^2(\mathbb{R}^3),\\
&\hat{n}_j^\rho \rightharpoonup \hat{m}_j \text{ weakly in } L^2\left(\mathbb{R}^3; \frac{dx}{1+|x|^2}\right),\\
\text{and } &\int_{\mathbb{R}^3} \frac{|\hat{m}_j - n_\infty|^2}{1+|x|^2}\, dx \lesssim \int_{\mathbb{R}^3} |\nabla \hat{m}_j|^2\, dx < \infty.
\end{aligned}$$

Since $\hat{n}_j^\rho$ minimizes the Dirichlet energy locally in $\mathbb{R}^3 \setminus \overline{\hat{\omega}_j}$, the compactness results of [9, 11] imply that $\hat{n}_j^\rho \to \hat{m}_j$ strongly in $H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \overline{\hat{\omega}_j})$, and $\hat{m}_j$ is a local minimizer of the Dirichlet energy. Next we adapt these arguments to show that $\hat{m}_j$ is a global minimizer of the energy $\widehat{E}_j$.

To that end we fix a competitor $m \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \hat{\omega}_j; \mathbb{S}^2)$ such that

$$\int_{\mathbb{R}^3 \setminus \hat{\omega}_j} \frac{|m - n_\infty|^2}{1 + |x|^2} \, dx < \infty ,$$

and a radius $R > 0$. The argument in [9, Proposition 5.1] provides a sequence $\delta_\rho \searrow 0$ and a map $u_\rho \in H^1_{\mathrm{loc}}(\mathbb{R}^3 \setminus \hat{\omega}_j; S^2)$ such that

$$u_\rho = \begin{cases} m & \text{in } B_R, \\ \hat{n}_j^\rho & \text{in } \mathbb{R}^3 \setminus B_{R+\delta_\rho}, \end{cases} \qquad \text{and} \qquad \int_{B_{R+\delta_\rho} \setminus B_R} |\nabla u_\rho|^2 \, dx \to 0 . \tag{4.11}$$

The minimality of n_ρ for E_ρ implies $E_\rho(n_\rho) \le E_\rho(u_\rho((\cdot - x_j)/\rho))$. Denoting by Ω_j^ρ the rescaled domain

$$\Omega_j^\rho = \frac{1}{\rho}(\Omega_\rho - x_j) ,$$

and taking into account the properties (4.11) of u_ρ, this turns into

$$\begin{aligned} E_\rho(n_\rho) &= \int_{\Omega_j^\rho} |\nabla \hat{n}_j^\rho|^2 \, dx + F_j(\hat{n}_j^\rho \lfloor \partial\hat{\omega}_j) + \sum_{i \neq j} F_i(\hat{n}_j^\rho \lfloor \partial\hat{\omega}_i) \\ &\le \int_{\Omega_j^\rho} |\nabla u_\rho|^2 \, dx + F_j(u_\rho \lfloor \partial\hat{\omega}_j) + \sum_{i \neq j} F_i(u_\rho((x_i - x_j)/\rho + \cdot\,) \lfloor \partial\hat{\omega}_i) \\ &= \int_{B_R \setminus \hat{\omega}_j} |\nabla m|^2 \, dx + F_j(m \lfloor \partial\hat{\omega}_j) + \int_{\Omega_j^\rho \setminus B_{R+\delta_\rho}} |\nabla \hat{n}_j^\rho|^2 \, dx \\ &\quad + \sum_{i \neq j} F_i(\hat{n}_i^\rho \lfloor \partial\hat{\omega}_i) + o(1) , \end{aligned}$$

and therefore

$$\begin{aligned} &\int_{B_R \setminus \hat{\omega}_j} |\nabla \hat{n}_j^\rho|^2 \, dx + F_j(\hat{n}_j^\rho \lfloor \partial\hat{\omega}_j) \\ &\le \int_{B_{R+\delta_\rho} \setminus \hat{\omega}_j} |\nabla \hat{n}_j^\rho|^2 \, dx + F_j(\hat{n}_j^\rho \lfloor \partial\hat{\omega}_j) \\ &= E_\rho(n_\rho) - \int_{\Omega_j^\rho \setminus B_{R+\delta_\rho}} |\nabla \hat{n}_j^\rho|^2 \, dx - \sum_{i \neq j} F_i(\hat{n}_i^\rho \lfloor \partial\hat{\omega}_i) \\ &\le \int_{B_R \setminus \hat{\omega}_j} |\nabla m|^2 \, dx + F_j(m \lfloor \partial\hat{\omega}_j) + o(1). \end{aligned} \tag{4.12}$$

Both terms in the first line of (4.12) are lower semicontinuous with respect to the weak convergence $\hat{n}_j^\rho \rightharpoonup \hat{m}_j$ in $H^1(B_R)$ and weak convergence of the traces in $H^{1/2}(\partial\hat{\omega}_j)$. So we deduce

$$\int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{m}_j|^2\,dx + F_j(\hat{m}_j\lfloor\partial\hat{\omega}_j) \leq \int_{B_R\setminus\hat{\omega}_j} |\nabla m|^2\,dx + F_j(m\lfloor\partial\hat{\omega}_j)\,,$$

and sending $R \to +\infty$ we conclude that $\hat{m}_j$ is a minimizer of $\widehat{E}_j$.

Moreover, applying the inequality (4.12) to $m = \hat{m}_j$, and using again the lower semicontinuity of both terms in its left-hand side, we deduce the chain of inequalities

$$\begin{aligned}
&\int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{m}_j|^2\,dx + F_j(\hat{m}_j\lfloor\partial\hat{\omega}_j)\\
&\leq \liminf_{\rho\to 0}\int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{n}_j^\rho|^2\,dx + \liminf_{\rho\to 0} F_j(\hat{n}_j^\rho\lfloor\partial\hat{\omega}_j)\\
&\leq \liminf_{\rho\to 0}\Big(\int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{n}_j^\rho|^2\,dx + F_j(\hat{n}_j^\rho\lfloor\partial\hat{\omega}_j)\Big)\\
&\leq \limsup_{\rho\to 0}\Big(\int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{n}_j^\rho|^2\,dx + F_j(\hat{n}_j^\rho\lfloor\partial\hat{\omega}_j)\Big)\\
&\leq \int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{m}_j|^2\,dx + F_j(\hat{m}_j\lfloor\partial\hat{\omega}_j)\,.
\end{aligned}$$

All these inequalities must therefore be equalities, which implies

$$\begin{aligned}
\int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{m}_j|^2\,dx &= \lim_{\rho\to 0}\int_{B_R\setminus\hat{\omega}_j} |\nabla \hat{n}_j^\rho|^2\,dx\,,\\
F_j(\hat{m}_j\lfloor\partial\hat{\omega}_j) &= \lim_{\rho\to 0} F_j(\hat{n}_j^\rho\lfloor\partial\hat{\omega}_j)\,.
\end{aligned}\tag{4.13}$$

By definition of $\mu_j = \widehat{E}_j(\hat{m}_j)$, for any $\varepsilon > 0$ we can choose $R > 1$ such that

$$\mu_j - \varepsilon \leq \widehat{E}_j(\hat{m}_j; B_R\setminus\hat{\omega}_j) = \lim_{\rho\to 0}\widehat{E}_j(\hat{n}_j^\rho; B_R\setminus\hat{\omega}_j)\,.$$

The last equality follows from (4.13). Since this is valid for any $\varepsilon > 0$, we infer

$$\liminf_{\rho\to 0}\hat{E}_j(\hat{n}_j^\rho; B_{1/\rho}\setminus\hat{\omega}_j) \geq \mu_j,$$

for all $j \in \{1,\ldots,N\}$. Combining this with (4.9) implies

$$\hat{E}_j(\hat{n}_j^\rho; B_{1/\rho}\setminus\hat{\omega}_j) \to \mu_j, \quad \text{as } \rho \to 0\,,$$

and, since by (4.13) we also have $F_j(\hat{n}_j^\rho\lfloor\partial\hat{\omega}_j) \to F_j(\hat{m}_j\lfloor\partial\hat{\omega}_j)$,

$$\int_{B_{1/\rho}\setminus\hat{\omega}_j} |\nabla \hat{n}_j^\rho|^2\,dx - \int_{B_{1/\rho}\setminus\hat{\omega}_j} |\nabla \hat{m}_j|^2\,dx \to 0.$$

We deduce

$$\begin{aligned}\int_{B_{1/\rho}\setminus\hat\omega_j} |\nabla \hat n_j^\rho - \nabla\hat m_j|^2\,dx &= \int_{B_{1/\rho}\setminus\hat\omega_j} |\nabla\hat m_j|^2\,dx - \int_{B_{1/\rho}\setminus\hat\omega_j} |\nabla\hat n_j^\rho|^2\,dx \\ &\quad + 2\int_{\mathbb{R}^3\setminus\hat\omega_j} \langle \mathbf{1}_{B_{1/\rho}}\nabla\hat m_j, \nabla\hat n_j^\rho - \nabla\hat m_j\rangle\,dx \\ &\to 0,\end{aligned}$$

thanks to the weak convergence $\nabla\hat n_j^\rho - \nabla\hat m_j \rightharpoonup 0$ and the strong convergence $\mathbf{1}_{B_{1/\rho}}\nabla\hat m_j \to \nabla\hat m_j$ in $L^2(\mathbb{R}^3\setminus\hat\omega_j)$. □

4.3 Lower bound in terms of $\hat n_\rho^j - \hat m_j$

Recall that our goal is to obtain the asymptotic expansion

$$E_\rho(n_\rho) = \sum_j \mu_i - 4\pi\rho\sum_{i\neq j}\frac{\langle v_i, v_j\rangle}{|x_i - x_j|} + o(\rho) \qquad \text{as } \rho\to 0.$$

The upper bound was obtained in Proposition 3.1, via a competitor equal to the rescaled single-particle minimizers $\hat m_j$ inside small balls $B_\sigma(x_j)$, harmonically extended (and projected onto $\mathbb{S}^2$) outside these balls. In particular, at the gluing scale σ around each x_j, that competitor was equal to the rescaled $\hat m_j$. In this section we establish a converse estimate: if there is a scale $\sigma = \lambda\rho$ such that n_ρ is close enough to the rescaled $\hat m_j$ near $\partial B_\sigma(x_j)$, then the lower bound is satisfied with a small error. Such an estimate is natural: here the important point is that we manage to obtain one that is sharp enough to conclude using only the convergence (4.8).

Proposition 4.4. *There exist $C > 0$ and $\lambda_0 \geq 2$ such that*

$$E_\rho(n_\rho) \geq \sum_j \mu_i - 4\pi\rho\sum_{i\neq j}\frac{\langle v_i, v_j\rangle}{|x_i - x_j|} - C\rho\,\Xi_\lambda(\rho)\,,$$

$$\text{where}\quad \Xi_\lambda(\rho) := \lambda\rho + \frac{1}{\lambda} + \Theta_\lambda(\rho)^{1/2} + \Theta_\lambda(\rho) + \frac{1+\Theta_\lambda(\rho)^2}{\lambda^3\rho}\,, \tag{4.14}$$

$$\text{and}\quad \Theta_\lambda(\rho) := \lambda^2\sum_j \fint_{B_{2\lambda}\setminus B_{\lambda/2}} |\hat n_j^\rho - \hat m_j|^2\,dx\,,$$

for all $\rho\in(0, 1/(2\lambda_0))$ and $\lambda\in[\lambda_0, 1/(2\rho)]$.

Remark 4.5. Recall that $|\hat m_j - n_\infty|^2 \lesssim 1/|x|^2$ for $|x|\gg 1$, due to the asymptotic expansion (1.4). It seems reasonable to hope that $|\hat n_j^\rho - n_\infty|^2$ could satisfy the same bound, which would imply $\Theta_\lambda(\rho)\lesssim 1$. If we manage to take this to the next order and find a scale $\lambda = \lambda_\rho$ such that

$$\Theta_{\lambda_\rho}(\rho) \ll 1\,,\quad \lambda_\rho \ll \frac{1}{\rho}\,,\quad \text{and } \lambda_\rho \gg \frac{1}{\rho^{1/3}}\,,$$

then we deduce that the error in (4.14) satisfies $\Xi_{\lambda_\rho}(\rho) \ll 1$. This is precisely how, in the next section, we are going to prove the lower bound of Proposition 4.1.

The proof of Proposition 4.4 relies on two separate lower bounds: in the domain $U_{\lambda\rho}$ outside the balls $B_{\lambda\rho}(x_j)$, and in each ball $B_{\lambda\rho}(x_j)$. Specifically, we establish:

- In Lemma 4.6, a lower bound for $E_\rho(n_\rho; U_{\lambda\rho})$ in terms of the boundary values of $\hat{n}_j^\rho$ at ∂B_λ. It is simply obtained as the energy of the harmonic extension of n_ρ from $\partial U_{\lambda\rho}$, for which Proposition 2.1 provides a precise expression in terms of the boundary values.
- In Lemma 4.7, a lower bound for $\widehat{E}_j(\hat{n}_j^\rho; B_\lambda)$ in terms of $\mu_j = \widehat{E}_j(\hat{m}_j)$ and the boundary values of $\hat{n}_j^\rho$ at ∂B_λ. It follows from an upper bound on μ_j obtained by constructing a competitor equal to $\hat{n}_j^\rho$ inside B_λ, and equal to its $\mathbb{S}^2$-projected harmonic extension outside B_λ.

When summing these two lower bounds, it turns out that the main contributions from the boundary values of $\hat{n}_j^\rho$ at ∂B_λ cancel each other, leaving us with the rather precise lower bound of Proposition 4.4.

Both lower bounds are expressed in terms of the spherical harmonics coefficients

$$\hat{a}_k^j(\lambda, \rho) = \int_{\mathbb{S}^2} (\hat{n}_j^\rho(\lambda\omega) - n_\infty)\Phi_k(\omega)\, d\mathcal{H}^2(\omega)\,. \tag{4.15}$$

We start with the lower bound in the exterior domain $U_{\lambda\rho}$.

Lemma 4.6. *There exist $C > 0$ and $\lambda_0 \geq 2$ such that*

$$\begin{aligned} E_\rho(n_\rho; U_{\lambda\rho}) \geq \lambda \sum_j \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda, \rho)|^2 - 4\pi\rho \sum_{i\neq j} \frac{\langle v_j, v_j \rangle}{|x_i - x_j|} \\ - C\,\rho\left(\theta_\lambda(\rho) + \theta_\lambda(\rho)^{1/2} + \frac{1}{\lambda} + \lambda\rho\right), \end{aligned} \tag{4.16}$$

where

$$\theta_\lambda(\rho) := \lambda^2 \sum_j \fint_{\partial B_\lambda} |\hat{n}_j^\rho - \hat{m}_j|^2\, d\mathcal{H}^2\,, \tag{4.17}$$

for all $\rho \in (0, 1/\lambda_0)$ and $\lambda \in [\lambda_0, 1/\rho]$.

Then, complementary to the exterior lower bound of Lemma 4.6, we have the following interior lower bound for each ball $B_{\lambda\rho}(x_j)$.

Lemma 4.7. *There exist $C > 0$ and $\lambda_0 \geq 2$ such that*

$$\widehat{E}_j(\hat{n}_j^\rho; B_\lambda \setminus \hat{\omega}_j) \geq \mu_j - \lambda \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda, \rho)|^2 - C\frac{1 + \widetilde{\Theta}_\lambda(\rho)^2}{\lambda^3}\,, \tag{4.18}$$

$$\textit{where} \quad \widetilde{\Theta}_\lambda(\rho) := \lambda^2 \fint_{B_{5\lambda/4}\setminus B_{3\lambda/4}} |\hat{n}_j^\rho - \hat{m}_j|^2\, dx\,,$$

for all $\rho \in (0, 1/(2\lambda_0))$ and $\lambda \in [\lambda_0, 1/(2\rho)]$.

Before proving the lower bounds of Lemma 4.6 and Lemma 4.7, we give the quick proof of how they imply Proposition 4.4.

Proof of Proposition 4.4. For any $\rho \in (0, 1/(2\lambda_0))$ and $\lambda \in [\lambda_0, 1/(2\rho)]$, we can find $\lambda' \in [3\lambda/4, 5\lambda/4]$ such that $\theta_{\lambda'}(\rho)$ is less than its average over that interval, and then we have

$$\theta_{\lambda'}(\rho) \lesssim \Theta_\lambda(\rho), \qquad \widetilde{\Theta}_{\lambda'}(\rho) \lesssim \Theta_\lambda(\rho)\,.$$

Summing the lower bounds (4.16) and (4.18) taken at $\lambda = \lambda'$, we deduce the lower bound of Proposition 4.4. □

Now we prove the lower bound (4.16) in the exterior domain $U_{\lambda\rho}$.

Proof of Lemma 4.6. The map $n_\rho - n_\infty$ has higher Dirichlet energy in the domain $U_{\lambda\rho}$ than the harmonic extension of its boundary values. The harmonic extension is given by

$$n_\rho(x_j + \lambda\rho\omega) - n_\infty = \hat{n}_j^\rho(\lambda\omega) - n_\infty = \sum_{k\geq 0} \hat{a}_k^j(\lambda,\rho)\Phi_k(\omega)\,,$$

Proposition 2.1 provides the lower bound

$$\begin{aligned} E_\rho(n_\rho; U_{\lambda\rho}) &= \frac{1}{\rho}\int_{U_{\lambda\rho}} |\nabla n_\rho|^2\,dx \\ &\geq \lambda \sum_j \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda,\rho)|^2 \\ &\quad - \lambda^2\rho \sum_j \sum_{i\neq j} \frac{\langle \hat{a}_0^i(\lambda,\rho), \hat{a}_0^j(\lambda,\rho)\rangle}{|x_i - x_j|} \\ &\quad + \mathcal{O}\left(\lambda^3\rho^2\right) \|\hat{a}(\lambda,\rho)\|_{\ell^2}^2\,. \end{aligned}$$

We define

$$\hat{b}_0^j(\lambda,\rho) = \hat{a}_0^j(\lambda,\rho) - \frac{2\sqrt{\pi}}{\lambda} v_j\,. \tag{4.19}$$

With this notation, we can rewrite the scalar product $\langle \hat{a}_0^i(\lambda,\rho), \hat{a}_0^j(\lambda,\rho)\rangle$ as

$$\begin{aligned} \langle \hat{a}_0^i(\lambda,\rho), \hat{a}_0^j(\lambda,\rho)\rangle &= \Big\langle \frac{2\sqrt{\pi}}{\lambda} v_i + \hat{b}_0^i(\lambda,\rho), \frac{2\sqrt{\pi}}{\lambda} v_j + \hat{b}_0^j(\lambda,\rho) \Big\rangle \\ &= \frac{4\pi}{\lambda^2}\langle v_i, v_j\rangle + \mathcal{O}\Big(|\hat{b}_0(\lambda,\rho)|^2 + \frac{|\hat{b}_0(\lambda,\rho)|}{\lambda}\Big)\,. \end{aligned}$$

Hence the above lower bound becomes

$$\begin{aligned} E_\rho(n_\rho; U_{\lambda\rho}) &\geq \lambda \sum_j \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda,\rho)|^2 - 4\pi\rho \sum_{i\neq j} \frac{\langle v_j, v_j\rangle}{|x_i - x_j|} \\ &\quad - C\,\rho \Big(\lambda^2 |\hat{b}_0(\lambda,\rho)|^2 + \lambda|\hat{b}_0(\lambda,\rho)| + \lambda^3\rho\|\hat{a}(\lambda,\rho)\|_{\ell^2}^2\Big)\,. \end{aligned} \tag{4.20}$$

Next we estimate the error terms in the last line. Recalling the definitions (4.19) of $\hat{b}_0^j$ and (4.15) of $\hat{a}_0^j$, and the fact that $\Phi_0 = 1/(2\sqrt{\pi})$, we have

$$\hat{b}_0^j(\lambda,\rho) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{S}^2} \Big(\hat{n}_\rho^j(\lambda\omega) - n_\infty - \frac{1}{\lambda} v_j \Big)\, d\mathcal{H}^2(\omega)\,.$$

Recalling also the asymptotic expansion (3.2) of $\hat{m}_j$, we can further rewrite this as

$$\hat{b}_0^j(\lambda,\rho) = \frac{1}{2\sqrt{\pi}} \int_{\mathbb{S}^2} \Big(\hat{n}_\rho^j(\lambda\omega) - \hat{m}_j(\lambda\omega) \Big)\, d\mathcal{H}^2(\omega) + \mathcal{O}\Big(\frac{1}{\lambda^2}\Big)\,.$$

This implies

$$\begin{aligned} |\hat{b}_0^j(\lambda,\rho)|^2 &\lesssim \int_{\mathbb{S}^2} |\hat{n}_j^\rho - \hat{m}_j|^2(\lambda\omega)\, d\mathcal{H}^2(\omega) + \mathcal{O}\Big(\frac{1}{\lambda^4}\Big) \\ &\lesssim \fint_{\partial B_\lambda} |\hat{n}_j^\rho - \hat{m}_j|^2\, d\mathcal{H}^2 + \mathcal{O}\Big(\frac{1}{\lambda^4}\Big)\,. \end{aligned}$$

Hence, recalling the definition (4.17) of θ_λ,

$$\lambda^2 |\hat{b}_0(\lambda,\rho)|^2 \lesssim \theta_\lambda(\rho) + \frac{1}{\lambda^2}\,. \tag{4.21}$$

Moreover, by definition (4.15) of the coefficients $\hat{a}_k^j$ and by orthonormality of (Φ_k) in $L^2(\mathbb{S}^2)$ we have

$$\begin{aligned} \|\hat{a}^j(\lambda,\rho)\|_{\ell^2}^2 = \sum_{k\geq 0} |\hat{a}_k^j(\lambda,\rho)|^2 &= \int_{\mathbb{S}^2} |\hat{n}_j^\rho(\lambda\omega) - n_\infty|^2\, d\mathcal{H}^2(\omega) \\ &\lesssim \fint_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2\, d\mathcal{H}^2 \\ &\lesssim \fint_{\partial B_\lambda} |\hat{n}_j^\rho - \hat{m}_j|^2\, d\mathcal{H}^2 + \sup_{\partial B_\lambda} |\hat{m}_j - n_\infty|^2\,. \end{aligned}$$

Recalling the asymptotic expansion (3.2) of $\hat{m}_j$, we are left with

$$\lambda^2 \|a^j(\lambda,\rho)\|_{\ell^2}^2 \lesssim 1 + \lambda^2 \fint_{\partial B_\lambda} |\hat{n}_j^\rho - \hat{m}_j|^2\, d\mathcal{H}^2 \lesssim 1 + \theta_\lambda(\rho)\,,$$

where the last inequality follows from the definition (4.17) of θ_λ. Combining this with the bound (4.21) on $\hat{b}_0$ we deduce

$$\lambda^2 |\hat{b}_0(\lambda,\rho)|^2 + \lambda |\hat{b}_0(\lambda,\rho)| + \lambda^3 \rho \|\hat{a}(\lambda,\rho)\|_{\ell^2}^2 \lesssim \theta_\lambda(\rho) + \theta_\lambda(\rho)^{1/2} + \frac{1}{\lambda} + \lambda\rho\,.$$

Plugging this into (4.20) we obtain (4.16). □

Finally we prove the lower bound (4.18) in each ball $B_{\lambda\rho}(x_j)$.

Proof of Lemma 4.7. First note that, if λ_0 is large enough, then thanks to the pointwise estimate (4.5) and Hardy's inequality (4.10), we have

$$\sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2 \leq \frac{1}{2}, \qquad \forall \lambda \in [\lambda_0, 1/\rho].$$

Now, in order to bound the minimal energy $\mu_j = \widehat{E}_j(\hat{m}_j)$ from above, we consider a competitor $\tilde{n}_\rho^j \colon \mathbb{R}^3 \to \mathbb{S}^2$ defined by

$$\tilde{n}_\rho^j = \begin{cases} \hat{n}_j^\rho & \text{in } B_\lambda, \\ \frac{n_\infty + \tilde{u}}{|n_\infty + \tilde{u}|} & \text{outside } B_\lambda, \end{cases}$$

where $\tilde{u} \colon \mathbb{R}^3 \setminus B_\lambda \to \mathbb{R}^3$ is the harmonic extension agreeing with $\hat{n}_\rho^j - n_\infty$ on ∂B_λ. By definition (4.15) of the coefficients $\hat{a}_k^j$, the extension $\tilde{u}$ is given by

$$\tilde{u}(r\omega) = \sum_{k\geq 0} a_k^j(\lambda,\rho)\Big(\frac{r}{\lambda}\Big)^{\gamma_k^-} \Phi_k(\omega),$$

and its energy by

$$\int_{\mathbb{R}^3\setminus B_\lambda} |\nabla \tilde{u}|^2 \, dx = -\int_{\partial B_\lambda} \langle \tilde{u}, \partial_r \tilde{u}\rangle \, d\mathcal{H}^2 = \lambda \sum_{k\geq 0} \gamma_k^- |a_k^j(\lambda,\rho)|^2. \tag{4.22}$$

By minimality of $\hat{m}_j$ we have

$$\begin{aligned} \widehat{E}_j(\hat{m}_j; \mathbb{R}^3 \setminus \hat{\omega}_j) &\leq \widehat{E}_j(\tilde{n}_\rho^j; \mathbb{R}^3 \setminus \hat{\omega}_j) \\ &= \widehat{E}_j(\hat{n}_j^\rho; B_\lambda \setminus \hat{\omega}_j) + \int_{\mathbb{R}^3\setminus B_\lambda} |\nabla \tilde{n}_\rho^j|^2 \, dx \\ &\leq \widehat{E}_j(\hat{n}_j^\rho; B_\lambda \setminus \hat{\omega}_j) + \int_{\mathbb{R}^3\setminus B_\lambda} \frac{|\nabla \tilde{u}|^2}{|n_\infty + \tilde{u}|^2} \, dx. \end{aligned} \tag{4.23}$$

The last inequality follows from the inequality $|\nabla(v/|v|)|^2 \leq |\nabla v|^2/|v|^2$ applied to $v = n_\infty + \tilde{u}$, see (3.5). Since the harmonic function $\langle n_\infty, \tilde{u}\rangle$ is either positive or attains its minimum at the boundary ∂B_λ, we have

$$|n_\infty + \tilde{u}|^2 = 1 + 2\langle n_\infty, \tilde{u}\rangle + |\tilde{u}|^2 \geq 1 - 2\sup_{\partial B_\lambda} |\langle n_\infty, \hat{n}_j^\rho - n_\infty\rangle|.$$

Using also that

$$|\langle n_\infty, \hat{n}_j^\rho - n_\infty\rangle| = 1 - \langle n_\infty, \hat{n}_j^\rho\rangle = \frac{1}{2}|n_\infty - \hat{n}_j^\rho|^2,$$

we deduce

$$|n_\infty + \tilde{u}|^2 \geq 1 - \sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2 \qquad \text{in } \mathbb{R}^3 \setminus B_\lambda.$$

Since we have $|\hat{n}_j^\rho - n_\infty|^2 \leq 1/2$ on ∂B_λ, this implies

$$\frac{1}{|n_\infty + \tilde{u}|^2} \leq 1 + 2\sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2 \qquad \text{in } \mathbb{R}^3 \setminus B_\lambda\,.$$

Using this to bound the last term in the energy estimate (4.23) we obtain

$$\begin{aligned}
\mu_j &= \widehat{E}_j(\hat{m}_j; \mathbb{R}^3 \setminus \hat{\omega}_j) \\
&\leq \widehat{E}_j(\hat{n}_j^\rho; B_\lambda \setminus \hat{\omega}_j) + \Big(1 + 2\sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2\Big) \int_{\mathbb{R}^3 \setminus B_\lambda} |\nabla \tilde{u}|^2 \, dx. \\
&= \widehat{E}_j(\hat{n}_j^\rho; B_\lambda \setminus \hat{\omega}_j) + \Big(1 + 2\sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2\Big) \lambda \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda,\rho)|^2\,.
\end{aligned}$$

The last equality follows from the explicit expression (4.22) of the energy of $\tilde{u}$. Rearranging, we deduce

$$\begin{aligned}
\widehat{E}_j(\hat{n}_j^\rho; B_\lambda \setminus \hat{\omega}_j) &\geq \mu_j - \lambda \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda,\rho)|^2 \\
&\qquad - 2\lambda \sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2 \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda,\rho)|^2\,. \qquad (4.24)
\end{aligned}$$

Using that $\gamma_k^- \leq 1 + (\gamma_k^-)^2 \lesssim 1 + \lambda_k$ and the definition (4.15) of the coefficients $\hat{a}_j^k$ we obtain

$$\begin{aligned}
\sum_{k\geq 0} \gamma_k^- |a_k^j(\lambda,\rho)|^2 &\lesssim \sum_{k\geq 0} |a_k^j(\lambda,\rho)|^2 + \sum_{k\geq 0} \lambda_k |a_k^j(\lambda,\rho)|^2 \\
&= \int_{\mathbb{S}^2} |\hat{n}_j^\rho(\lambda\omega) - n_\infty|^2 \, d\mathcal{H}^2(\omega) + \int_{\mathbb{S}^2} |\nabla_\omega \hat{n}_j^\rho(\lambda\omega)|^2 \, d\mathcal{H}^2(\omega) \\
&\lesssim \fint_{B_{5\lambda/4} \setminus B_{3\lambda/4}} |\hat{n}_j^\rho - n_\infty|^2 \, dx\,.
\end{aligned}$$

The last inequality follows from the pointwise estimates of Lemma 4.2, which also imply

$$\sup_{\partial B_\lambda} |\hat{n}_j^\rho - n_\infty|^2 \lesssim \fint_{B_{5\lambda/4} \setminus B_{3\lambda/4}} |\hat{n}_j^\rho - n_\infty|^2 \, dx\,.$$

Using the last two estimates to bound the last line in the lower bound (4.24), we deduce

$$\begin{aligned}
\widehat{E}_j(\hat{n}_j^\rho; B_\lambda \setminus \hat{\omega}_j) &\geq \mu_j - \lambda \sum_{k\geq 0} \gamma_k^- |\hat{a}_k^j(\lambda,\rho)|^2 \\
&\qquad - \frac{C}{\lambda^3} \Big(\lambda^2 \fint_{B_{5\lambda/4} \setminus B_{3\lambda/4}} |\hat{n}_j^\rho - n_\infty|^2 \, dx \Big)^2\,.
\end{aligned}$$

To conclude we recall that the asymptotics (3.2) of $\hat{m}_j$ ensure

$$\lambda^2 \fint_{B_{5\lambda/4} \setminus B_{3\lambda/4}} |\hat{n}_j^\rho - n_\infty|^2 \, dx \lesssim 1 + \widetilde{\Theta}_\lambda(\rho),$$

with $\widetilde{\Theta}_\lambda$ as in (4.18). □

4.4 Far field asymptotics of $\hat{n}_j^\rho$

As noted in Remark 4.5, the proof of the sharp lower bound of Proposition 4.1 relies on proving that $|\hat{n}_j^\rho - \hat{m}_j|$ on some large annulus $B_{2\lambda} \setminus B_{\lambda/2}$ is much smaller than the leading asymptotics of $\hat{m}_j - n_\infty$ which is of order $1/\lambda$. In this section we show that $\hat{n}_j^\rho$ has an asymptotic expansion similar to that of $\hat{m}_j$ in (3.2). This will allow us to control the error terms $\Theta_\lambda(\rho)$ and $\Xi(\rho)$ in (4.14), leading to the proof of Proposition 4.1.

Proposition 4.8. *There exist $\lambda_0 > 2$, $\rho_0 \in (0,1)$ and, for every $\rho \in (0, \rho_0)$, vectors $n_\infty^\rho, v_j^\rho \in \mathbb{R}^3$ such that*

$$\hat{n}_j^\rho(x) = n_\infty^\rho + \frac{v_j^\rho}{|x|} + w_j^\rho(x)\,, \qquad |n_\infty^\rho - n_\infty|^2 \lesssim \rho \ln^2 \rho\,, \tag{4.25}$$

$$|w_j^\rho|^2 \lesssim \rho + \frac{\ln^6 |x|}{|x|^4} \qquad \textit{for } \lambda_0 \le |x| \le \frac{|\ln \rho|}{\sqrt{\rho}}\,,$$

and $v_j^\rho \to v_j$ along the sequence $\rho \to 0$ provided by Lemma 4.3.

Before proving Proposition 4.8 we give the short argument of how to combine it with Proposition 4.4 to deduce the sharp lower bound on $E_\rho(n_\rho)$.

Proof of Proposition 4.1. Using the asymptotic expansions (4.25) of $\hat{n}_j^\rho$ and (3.2) of $\hat{m}_j$, we have, for $\lambda_0 \le |x| \le |\ln \rho|/\sqrt{\rho}$,

$$|\hat{n}_j^\rho - \hat{m}_j|^2 \lesssim \rho \ln^2 \rho + \frac{\ln^6 \rho}{|x|^4} + \frac{|v_j - v_j^\rho|^2}{|x|^2}\,,$$

and recalling the definition of Θ_λ in (4.14) we infer

$$\Theta_\lambda(\rho) \lesssim \lambda^2 \rho \ln^2 \rho + \frac{\ln^6 \rho}{\lambda^2} + |v_j - v_j^\rho|^2\,, \qquad \text{for } 2\lambda_0 \le \lambda \le \frac{|\ln \rho|}{2\sqrt{\rho}}\,.$$

Choosing, for small ρ, the admissible value

$$\lambda = \lambda_\rho = \frac{|\ln \rho|}{\rho^{1/4}}\,,$$

we deduce

$$\Theta_{\lambda_\rho}(\rho) \lesssim \sqrt{\rho} \ln^4 \rho + |v_j^\rho - v_j|^2 \longrightarrow 0,$$

along the sequence $\rho \to 0$ provided by Lemma 4.3, since $v_j^\rho \to v_j$ by Proposition 4.8. Since we also have $\rho^{-1/3} \ll \lambda_\rho \ll \rho^{-1}$ we conclude, see Remark 4.5, that the lower bound error Ξ_λ in (4.14) satisfies $\Xi_{\lambda_\rho}(\rho) \to 0$, thus proving Proposition 4.1. □

Now we turn to the proof of Proposition 4.8. It is based on the strategy in [2] for the asymptotic expansion (1.4) of $\hat{m}_j$. That strategy relies on repeated application of two basic principles:

- if the Laplacian Δn is small in some region, then n is close to a classical harmonic function u, that is, $\Delta u = 0$;
- harmonic functions u in $\mathbb{R}^3 \setminus B_\lambda$ have an asymptotic expansion determined by their spherical harmonics decomposition.

Here our main issue is the last point: we can only hope to control $\hat{n}_j^\rho$ in an annulus $B_{1/\rho} \setminus B_\lambda$, where the spherical harmonics decomposition can have radially increasing modes. We have to take into account additional error terms coming from these increasing modes, and this is reflected here in the fact that we are only able to control the error $\hat{w}_j$ in a smaller annulus, of amplitude slightly larger than $1/\sqrt{\rho}$. As we will see, we also need to deal with with borderline cases in our application of the first principle, i.e. that a function with small Laplacian is close to a harmonic function.

Proof of Proposition 4.8. We follow essentially the first two steps of [2, Theorem 1.1], with adaptations for estimates in an annulus $B_{1/\rho} \setminus B_\lambda$ instead of the whole exterior domain $\mathbb{R}^3 \setminus B_\lambda$.

The initial decay of $|\nabla \hat{n}_j^\rho|$ that we start with is provided by the small energy estimate (4.3) and the fact that the energy $\widehat{E}_j(\hat{n}_j^\rho)$ is bounded: we have

$$|\nabla \hat{n}_j^\rho|^2 \lesssim \frac{1}{|x|^3} \qquad \text{for } \lambda_0 \leq |x| \leq \frac{1}{2\rho}\,.$$

Together with the harmonic map equation

$$-\Delta \hat{n}_j^\rho = |\nabla \hat{n}_j^\rho|^2 \hat{n}_j^\rho\,,$$

this implies $|\Delta \hat{n}_j^\rho| \lesssim 1/|x|^3$, which is not precise enough to capture the first decaying harmonic term of order $1/|x|$ in the expansion of $\hat{n}_j^\rho$.

In order to obtain a stronger estimate on $|\nabla \hat{n}_j^\rho|$, we proceed as in the alternative proof of Step 1 in [2, Theorem 1.1] and consider the map $g = \partial_\alpha n_j^\rho$, which is pointwise orthogonal to $\hat{n}_j^\rho$ and solves the linearized equation

$$-\Delta g = 2\langle \nabla \hat{n}_j^\rho, \nabla g\rangle + |\nabla \hat{n}_j^\rho|^2 g. \tag{4.26}$$

For $R \in [2\lambda_0, 1/(6\rho)]$ we multiply this with $\chi^2 g$ for a cut off function χ satisfying

$$\mathbf{1}_{R \leq |\hat{x}| \leq 2R} \leq \chi(\hat{x}) \leq \mathbf{1}_{R/2 \leq |\hat{x}| \leq 3R} \quad \text{and} \quad |\nabla \chi| \lesssim 1/R\,.$$

Since g is orthogonal to $\hat{n}_j^\rho$, the first term in the right-hand side drops out and we are left with

$$-\chi^2 \langle g, \Delta g\rangle = \chi^2 |\nabla \hat{n}_j^\rho|^2 |g|^2\,.$$

Integrating by parts in the left-hand side, we deduce

$$\begin{aligned}
&\int_{B_{3R}\setminus B_{R/2}} |\nabla g|^2 \chi^2 \, dx \\
&= \int_{B_{3R}\setminus B_{R/2}} |\nabla \hat{n}_j^\rho|^2 |g|^2 \chi^2 \, dx - 2\int_{B_{3R}\setminus B_{R/2}} \langle g, (\nabla\chi\cdot\nabla) g\rangle \chi \, dx \\
&\leq \int_{B_{3R}\setminus B_{R/2}} |\nabla \hat{n}_j^\rho|^2 \chi^2 \, dx + 2\int_{B_{3R}\setminus B_{R/2}} |g|^2 |\nabla\chi|^2 \, dx \\
&\quad + \frac{1}{2}\int_{B_{3R}\setminus B_{R/2}} |\nabla g|^2 \chi^2 \, dx \,.
\end{aligned}$$

Absorbing the last term into the left-hand side and using that $|\nabla\chi| \lesssim 1/R$ and $|g|^2 \leq |\nabla \hat{n}_j^\rho|^2 \lesssim 1/R^3$, we infer

$$\int_{B_{3R}\setminus B_{R/2}} |\nabla g|^2 \chi^2 \, dx \lesssim \frac{1}{R^2} \,,$$

and therefore

$$⨍_{B_{2R}\setminus B_R} |\nabla g|^2 \, dx \lesssim \frac{1}{R^5} \,.$$

Using this, and once more $|g|^2 \leq |\nabla \hat{n}_j^\rho|^2 \lesssim 1/R^3$, to estimate the right-hand side of (4.26), we find

$$\left(⨍_{B_{2R}\setminus B_R} |\Delta g|^2 \, dx\right)^{1/2} \lesssim \frac{1}{R^4} \qquad \text{for } \lambda_0 \leq R \leq \frac{1}{6\rho} \,.$$

Applying Lemma A.1 with $d = 3$, $\gamma = 2$ and $f = \mathbf{1}_{B_{1/(6\rho)}} \Delta g$, we obtain the existence of a map $u \colon B_{1/(6\rho)} \setminus B_{\lambda_0} \to \mathbb{R}^3$ such that $\Delta(u - g) = 0$ and

$$⨍_{B_{2R}\setminus B_R} |u|^2 \, dx \lesssim \frac{\ln^2 R}{R^4} \qquad \text{for } \lambda_0 \leq R \leq \frac{1}{6\rho} \,. \tag{4.27}$$

This implies in particular

$$\int_{B_{1/(6\rho)}\setminus B_{\lambda_0}} |u|^2 \, dx \lesssim 1 \,.$$

This, together with the inequality $|g|^2 \leq |\nabla \hat{n}_j^\rho|^2$ and the fact that $\widehat{E}_j(\hat{n}_j^\rho) \lesssim 1$, implies

$$\int_{B_{1/(6\rho)}\setminus B_{\lambda_0}} |u - g|^2 \, dx \lesssim 1 \,.$$

We may therefore apply Lemma B.1 to the harmonic function $u - g$. This gives

$$⨍_{B_{2R}\setminus B_R} |u - g|^2 \, dx \lesssim \frac{\lambda_0}{R^4} + \frac{\rho}{R^2} \qquad \text{for } 2\lambda_0 \leq R \leq \frac{1}{24\rho}.$$

Combining this with the decay (4.27) of u and raising the value of λ_0, we deduce

$$\fint_{B_{2R}\setminus B_R} |g|^2\,dx \lesssim \frac{\ln^2 R}{R^4} + \frac{\rho}{R^2} \qquad \text{for } \lambda_0 \le R \le \frac{1}{24\rho}.$$

Recalling the definition $g = \partial_\alpha \hat{n}_j^\rho$, applying this for all $\alpha = 1, 2, 3$ and using the small energy estimate (4.3), we obtain

$$|\nabla \hat{n}_j^\rho|^2 \lesssim \frac{\ln^2 |x|}{|x|^4} + \frac{\rho}{|x|^2} \qquad \text{for } \lambda_0 \le |x| \le \frac{1}{24\rho}. \tag{4.28}$$

Since $r \mapsto r^{-2}\ln^2 r$ is decreasing for $r \ge e$, we have $\rho \lesssim r^{-2}\ln^2 r$ for all $r \in [e, 8|\ln\rho|/\sqrt{\rho}]$, and we deduce

$$|\Delta \hat{n}_j^\rho| = |\nabla \hat{n}_j^\rho|^2 \lesssim \frac{\ln^2 |x|}{|x|^4} \qquad \text{for } \lambda_0 \le |x| \le R_\rho := \frac{8|\ln\rho|}{\sqrt{\rho}}\,.$$

Applying Lemma A.1 with $\gamma = 2 = \theta$ (and elliptic estimates to turn its conclusion into a pointwise bound), we find $\tilde{u}\colon B_{R_\rho}\setminus B_{\lambda_0} \to \mathbb{R}^3$ such that $\Delta(\hat{n}_\rho^j - \tilde{u}) = 0$ and

$$\frac{|\tilde{u}|}{|x|} + |\nabla \tilde{u}| \lesssim \frac{\ln^3 |x|}{|x|^3} \qquad \text{for } \lambda_0 \le |x| \le R_\rho\,.$$

Since $\hat{n}_\rho^j - \tilde{u}$ is a harmonic function which satisfies

$$|\nabla(\hat{n}_\rho^j - \tilde{u})| \lesssim \frac{\ln|x|}{|x|^2} \qquad \text{for } \lambda_0 \le |x| \le R_\rho\,,$$

Lemma B.2 allows us to decompose it as

$$\hat{n}_\rho^j - \tilde{u} = n_\infty^\rho + \frac{v_j^\rho}{|x|} + v + \tilde{w},$$

where $n_\infty^\rho, v_j^\rho \in \mathbb{R}^3$, v is harmonic in $\mathbb{R}^3 \setminus B_{\lambda_0}$, and (possibly raising the value of λ_0),

$$|v_j^\rho| \lesssim 1, \qquad |v| + |x|\,|\nabla v| \lesssim \frac{1}{|x|^2} \qquad \text{for } |x| \ge \lambda_0\,,$$

$$|\tilde{w}| + |x|\,|\nabla \tilde{w}| \lesssim \frac{\ln R_\rho}{R_\rho} \lesssim \sqrt{\rho} \qquad \text{for } \lambda_0 \le |x| \le \frac{R_\rho}{8} = \frac{|\ln\rho|}{\sqrt{\rho}}\,.$$

Setting $w_j^\rho = \tilde{u} + v + \tilde{w}$, we obtain

$$\hat{n}_j^\rho = n_\infty^\rho + \frac{v_j^\rho}{|x|} + w_j^\rho\,, \tag{4.29}$$

$$|w_j^\rho| + |x|\,|\nabla w_j^\rho| \lesssim \frac{\ln^3 |x|}{|x|^2} + \sqrt{\rho} \qquad \text{for } \lambda_0 \le |x| \le \frac{|\ln\rho|}{\sqrt{\rho}}\,.$$

To complete the proof of Proposition 4.8, it remains to obtain the estimate (4.25) on $|n_\infty^\rho - n_\infty|$ and that $|v_j^\rho - v_j| \to 0$ as $\rho \to 0$.

From the expansion (4.29) and the fact that $|v_j^\rho| \lesssim |\ln \rho|$, we infer

$$\begin{aligned}
|n_\infty^\rho - n_\infty|^2 &= \Big|\hat{n}_j^\rho - n_\infty - \frac{v_j^\rho}{|x|} - w_j^\rho\Big|^2 \\
&\lesssim |\hat{n}_j^\rho - n_\infty|^2 + \frac{\ln^2\rho}{|x|^2} + \frac{\ln^6\rho}{|x|^4} + \rho \qquad \text{for } \lambda_0 \le |x| \le \frac{|\ln\rho|}{\sqrt{\rho}}\,. \qquad (4.30)
\end{aligned}$$

Moreover, the pointwise bound (4.28) on $|\nabla \hat{n}_j^\rho|$ and the fundamental theorem of calculus ensure, for $|\ln\rho|/\sqrt{\rho} \le |x| \le 1/(24\rho)$,

$$\begin{aligned}
\sup_{\partial B_{|\ln\rho|/\sqrt{\rho}}} |n_j^\rho - n_\infty|^2 &\lesssim |\hat{n}_j^\rho - n_\infty|^2(x) + \Big(\int_{|\ln\rho|/\sqrt{\rho}}^{|x|} \frac{\ln r + \sqrt{\rho}\, r}{r^2} dr\Big)^2 \\
&\lesssim |\hat{n}_j^\rho - n_\infty|^2(x) + \rho \ln^2|x|\,.
\end{aligned}$$

So the inequality (4.30) at $|x| = |\ln\rho|/\sqrt{\rho}$ implies

$$|n_\infty^\rho - n_\infty|^2 \lesssim |\hat{n}_j^\rho - n_\infty|^2 + \rho\ln^2|x| \qquad \text{for } \frac{|\ln\rho|}{\sqrt{\rho}} \le |x| \le \frac{1}{24\rho}\,.$$

Dividing by $|x|^2$ and integrating on the annulus $1/(48\rho) \le |x| \le 1/(24\rho)$, we deduce

$$\begin{aligned}
\frac{|n_\infty^\rho - n_\infty|^2}{\rho} &\lesssim \int_{B_{\frac{1}{48\rho}} \setminus B_{\frac{1}{24\rho}}} \frac{|n_\infty^\rho - n_\infty|^2}{|x|^2}\, dx \\
&\lesssim \int_{B_{1/\rho}\setminus B_1} \frac{|\hat{n}_j^\rho - n_\infty|^2}{|x|^2}\, dx + \ln^2\rho\,.
\end{aligned}$$

Recalling Hardy's inequality (4.10), we obtain the claimed estimate

$$|n_\infty^\rho - n_\infty|^2 \lesssim \rho\ln^2\rho\,.$$

Finally we turn to the estimate on $|v_j^\rho - v_j|$. Using the expansions (4.29) and (3.2) of $\hat{n}_j^\rho$ and $\hat{m}_j$ we express

$$\begin{aligned}
\frac{|v_j^\rho - v_j|^2}{|x|^4} &= \Big|\nabla\Big(\frac{v_j^\rho}{|x|} - \frac{v_j}{|x|}\Big)\Big|^2 = \big|\nabla(\hat{n}_j^\rho - \hat{m}_j + \hat{w}_j - w_j^\rho)\big|^2 \\
&\lesssim |\nabla\hat{n}_j^\rho - \nabla\hat{m}_j|^2 + \frac{\ln^6|x|}{|x|^6} + \frac{\rho}{|x|^2} \qquad \text{for } \lambda_0 \le |x| \le \frac{|\ln\rho|}{\sqrt{\rho}}\,.
\end{aligned}$$

Integrating this inequality over an annulus $\lambda \le |x| \le 2\lambda$ for any $\lambda \in [\lambda_0, 1/\sqrt{\rho}]$, we find

$$\frac{|v_j^\rho - v_j|^2}{\lambda} \lesssim \int_{B_{2\lambda}\setminus B_\lambda} |\nabla\hat{n}_j^\rho - \nabla\hat{m}_j|^2\, dx + \frac{\ln^6\lambda}{\lambda^3} + \lambda\,\rho\,.$$

Along the sequence $\rho \to 0$ provided by Lemma 4.3, the first integral in the right-hand side converges to zero. Hence we deduce, along that sequence,

$$\limsup_{\rho\to 0} |v_j^\rho - v_j|^2 \lesssim \frac{\ln^6 \lambda}{\lambda^2} \qquad \forall \lambda \geq \lambda_0.$$

Sending $\lambda \to \infty$ concludes the proof that $v_j^\rho \to v_j$. □

Appendix A Decaying solutions of Poisson's equation

We include here, for the readers' convenience, a proof of a folklore result about existence of decaying solutions to Poisson's equations. We follow and adapt the proof in [2, Lemma A.2] in the case $\theta = 0$ and γ non-integer.

Lemma A.1. *Let $d \geq 3$, $\gamma \geq d-2$ and $\theta \geq 0$, $\lambda \geq 1$ and f a function in $\mathbb{R}^d \setminus B_\lambda$ satisfying*

$$\left(\fint_{R<|x|<2R} f^2 \, dx \right)^{\frac{1}{2}} \leq \frac{\ln^\theta (2R/\lambda)}{R^{\gamma+2}} \qquad \forall R \geq \lambda.$$

Then there exists a function u such that $\Delta u = f$ in $\mathbb{R}^d \setminus B_\lambda$ and

$$\left(\fint_{R<|x|<2R} u^2 \, dx \right)^{\frac{1}{2}} \leq C \frac{\ln^{1+\theta} (2R/\lambda)}{R^{\gamma}} \qquad \forall R \geq \lambda,$$

where $C > 0$ depends only on d, γ and θ.

Proof of Lemma A.1. By scaling, we assume without loss of generality that $\lambda = 1$. We fix, as in § 2, an orthonormal Hilbert basis $\{\Phi_j\}$ of $L^2(\mathbb{S}^{d-1})$ which diagonalizes the Laplace-Beltrami operator,

$$-\Delta_{\mathbb{S}^{d-1}} \Phi_j = \lambda_j \Phi_j, \qquad 0 = \lambda_0 \leq \lambda_1 \leq \cdots$$

The set $\{\lambda_j\}_{j\in\mathbb{N}}$ coincides with $\{k^2 + k(d-2)\}_{k\in\mathbb{N}}$. The eigenfunctions corresponding to $k^2+k(d-2)$ span the homogeneous harmonic polynomials of degree k. For a $W^{2,2}_{loc}$ function $w\colon (0,\infty) \to \mathbb{R}$ we have

$$\Delta(w(r)\Phi_j(\omega)) = (\mathcal{L}_j w)(r)\Phi_j(\omega), \qquad \mathcal{L}_j = \partial_{rr} + \frac{d-1}{r}\partial_r - \frac{\lambda_j}{r^2}. \tag{A.1}$$

The solutions of $\mathcal{L}_j w = 0$ are linear combinations of $r^{\gamma_j^+}$ and $r^{-\gamma_j^-}$, where $\gamma_j^\pm \geq 0$ are given by

$$\gamma_j^\pm = \sqrt{\left(\frac{d-2}{2}\right)^2 + \lambda_j} \pm \frac{2-d}{2}\,, \tag{A.2}$$

that is,

$$\gamma_j^+ = k \qquad \text{for } \lambda_j = k^2 + k(d-2)\,,$$
$$\gamma_j^- = k + d - 2 \qquad \text{for } \lambda_j = k^2 + k(d-2)\,.$$

The decay rate $\gamma \geq d-2$ is fixed and we denote by $j_0 = j_0(\gamma)$ the integer $j_0 \geq 0$ such that

$$\big\{j \in \mathbb{N} \colon \gamma_j^- < \gamma\big\} = \{0, \ldots, j_0\},$$
$$\big\{j \in \mathbb{N} \colon \gamma_j^- \geq \gamma\big\} = \{j_0+1, j_0+2, \ldots\}.$$

The function $f \in L^2(\mathbb{R}^d \setminus B_1)$ admits a spherical harmonics expansion

$$f = \sum_{j \geq 0} f_j(r) \Phi_j(\omega),$$

and the decay assumption on f implies

$$\begin{aligned}\sum_{j \geq 0} \int_R^\infty f_j(r)^2 r^{d+1}\, dr &\lesssim \sum_{k \geq 0} (2^k R)^2 \int_{2^k R \leq |x| \leq 2^{k+1} R} f^2\, dx \\ &\lesssim \sum_{k \geq 0} (2^k R)^{d-2\gamma-2} \ln^{2\theta}(2^{k+1} R) \\ &\lesssim R^{d-2\gamma-2} \ln^{2\theta}(2R)\,, \end{aligned} \tag{A.3}$$

for all $R \geq 1$. We define u as

$$u := \sum_{j \geq 0} u_j(r) \Phi_j(\omega),$$

where $u_j \in W^{2,2}_{loc}(0,\infty)$ satisfy

$$\mathcal{L}_j u_j = f_j.$$

To write down an explicit formula for u_j we rewrite $\mathcal{L}_j$, defined in (A.1), as

$$\mathcal{L}_j u = r^{-d+1+\gamma_j^-} \partial_r [r^{d-1-2\gamma_j^-} \partial_r (r^{\gamma_j^-} u)],$$

and define

$$u_j(r) = \begin{cases} r^{-\gamma_j^-} \displaystyle\int_r^\infty t^{2\gamma_j^- + 1 - d} \int_t^\infty s^{d-1-\gamma_j^-} f_j(s)\, ds\, dt & \text{if } j \in \{0, \ldots, j_0\}, \\ r^{-\gamma_j^-} \displaystyle\int_1^r t^{2\gamma_j^- + 1 - d} \int_t^\infty s^{d-1-\gamma_j^-} f_j(s)\, ds\, dt & \text{if } j \geq j_0 + 1. \end{cases} \tag{A.4}$$

This is well defined because, for any $t \geq 1$, using Cauchy-Schwarz, (A.3) with the choice $R = t$, and the fact that $\gamma_j^- \geq d-2 > 0$, we can estimate the inner integral by

$$\int_t^\infty s^{d-1-\gamma_j^-}|f_j(s)|\,ds \le \left(\int_t^\infty s^{-2-2\gamma_j^-}s^{d-1}ds\right)^{\frac{1}{2}}\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)^{\frac{1}{2}}$$
$$\lesssim \frac{1}{\sqrt{2\gamma_j^-+2-d}}t^{\frac{d}{2}-\gamma_j^--1}t^{\frac{d}{2}-\gamma-1}\ln^\theta(2t)$$
$$= \frac{1}{\sqrt{2\gamma_j^-+2-d}}t^{d-\gamma-\gamma_j^--2}\ln^\theta(2t). \qquad \text{(A.5)}$$

Furthermore, as $t \mapsto t^{2\gamma_j^-+1-d}t^{d-2-\gamma-\gamma_j^-}\ln^\theta t = t^{\gamma_j^--\gamma-1}\ln^\theta t$ is integrable near ∞ if $\gamma_j^- < \gamma$, i.e., if $j \le j_0$, the functions u_j in (A.4) are well-defined.

Let $j \le j_0$ and set

$$\alpha := \gamma + \gamma_{j_0}^- + 1 - d,$$

so that $2\gamma + 1 - d > \alpha > 2\gamma_j^- + 1 - d$. By (A.5) and Cauchy-Schwarz we have

$$|u_j(r)|^2 \le \frac{r^{-2\gamma_j^-}}{2+2\gamma_j^--d}\left(\int_r^\infty t^{\gamma_j^--\frac{d}{2}}\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)^{\frac{1}{2}}dt\right)^2$$
$$= \frac{r^{-2\gamma_j^-}}{2+2\gamma_j^--d}\left(\int_r^\infty t^{\gamma_j^--\frac{d}{2}-\frac{\alpha}{2}}t^{\frac{\alpha}{2}}\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)^{\frac{1}{2}}dt\right)^2$$
$$\le \frac{r^{-2\gamma_j^-}}{2+2\gamma_j^--d}\int_r^\infty t^{2\gamma_j^--d-\alpha}\,dt\int_r^\infty t^\alpha\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt$$
$$= \frac{r^{-d+1-\alpha}}{(2+2\gamma_j^--d)(\alpha-2\gamma_j^-+d-1)}\int_r^\infty t^\alpha\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt$$
$$\le \frac{r^{-d+1-\alpha}}{(d-2)(\gamma-\gamma_{j_0}^-)}\int_r^\infty t^\alpha\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt,$$

where in the last line, we used that $\gamma_j^- \geqslant d-2$ so that $2+2\gamma_j^- - d \geqslant d-2$, and that $\gamma + \gamma_{j_0}^- - 2\gamma_j^- \geqslant \gamma - \gamma_{j_0}^-$, when $j \leqslant j_0$. Summing and using (A.3), we deduce

$$\sum_{j=0}^{j_0}\frac{|u_j(r)|^2}{r^2} \le \frac{r^{-d-1-\alpha}}{(d-2)(\gamma-\gamma_{j_0}^-)}\int_r^\infty t^\alpha\left(\sum_{j=0}^{j_0}\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt$$
$$\le \frac{r^{-d-1-\alpha}}{(d-2)(\gamma-\gamma_{j_0}^-)}\int_r^\infty t^{\alpha+d-2\gamma-2}\ln^{2\theta}(2t)\,dt$$
$$\lesssim \frac{r^{-2\gamma-2}\ln^{2\theta}(2r)}{(d-2)(\gamma-\gamma_{j_0}^-)(2\gamma+1-d-\alpha)}$$
$$\le \frac{r^{-2\gamma-2}\ln^{2\theta}(2r)}{(d-2)(\gamma-\gamma_{j_0}^-)^2}.$$

For $j \geq j_0 + 1$ we need to distinguish cases if $\gamma = \gamma^-_{j_0+1}$. We introduce $j_1 \geq j_0$ such that

$$\begin{aligned} &\gamma = \gamma_j^- \quad \text{for } j \in \{j_0+1, \dots, j_1\}, \\ &\gamma < \gamma_j^- \quad \text{for } j \geq j_1 + 1. \end{aligned}$$

For $j \geq j_1 + 1$ we set

$$\beta = \gamma + \gamma^-_{j_0+1} + 1 - d,$$

which satisfies $2\gamma + 1 - d < \beta < 2\gamma_j^- + 1 - d$. Using (A.5) and Cauchy-Schwarz we find

$$\begin{aligned} |u_j(r)|^2 &\leq \frac{r^{-2\gamma_j^-}}{2 + 2\gamma_j^- - d} \int_1^r t^{2\gamma_j^- - d - \beta}\, dt \int_1^r t^\beta \left(\int_t^\infty s^2 f_j(s)^2 s^{d-1} ds \right) dt \\ &\lesssim \frac{r^{-d+1-\beta}}{(d-2)(\gamma - \gamma^-_{j_1+1})} \int_1^r t^\beta \left(\int_t^\infty s^2 f_j(s)^2 s^{d-1} ds \right) dt \end{aligned}$$

so that from (A.3) we obtain that

$$\sum_{j=j_1+1}^{\infty} \frac{|u_j(r)|^2}{r^2} \lesssim \frac{r^{-2\gamma-2} \ln^{2\theta}(2r)}{(d-2)(\gamma^-_{j_1+1} - \gamma)^2}.$$

It remains to treat $j_0 + 1 \leq j \leq j_1$, where $\gamma = \gamma_j^-$. In that case, the same manipulations, with $\beta = 2\gamma + 1 - d$, lead to

$$|u_j(r)|^2 \leq \frac{r^{-2\gamma}}{d-2} \ln r \int_1^r t^{2\gamma+1-d} \left(\int_t^\infty s^2 f_j(s)^2 s^{d-1} ds \right) dt\,,$$

and, using (A.3),

$$\sum_{j=j_0+1}^{j_1} \frac{|u_j(r)|^2}{r^2} \lesssim \frac{r^{-2\gamma-2} \ln^{2\theta+2}(2r)}{d-2}.$$

We conclude that

$$\sum_{j=0}^{\infty} \frac{|u_j(r)|^2}{r^2} \lesssim \frac{1}{d-2} \left(\frac{1}{(\gamma - \gamma^-_{j_0})^2} + \frac{1}{(\gamma^-_{j_1+1} - \gamma)^2} + \ln^2(2r) \right) r^{-2\gamma-2} \ln^{2\theta}(2r)$$

Therefore, since $\gamma \geq d-2$,

$$\begin{aligned} &\frac{1}{R^d} \int_{|x| \geq R} \frac{|u|^2}{|x|^2}\, dx = \frac{1}{R^d} \int_R^\infty \left(\sum_{j=0}^{\infty} \frac{|u_j(r)|^2}{r^2} \right) r^{d-1}\, dr \\ &\lesssim \left(\frac{1}{(\gamma - \gamma^-_{j_0})^2} + \frac{1}{(\gamma^-_{j_0+1} - \gamma)^2} + \ln^2(2R) \right) \frac{R^{-2\gamma-2} \ln^{2\theta}(2R)}{(d-2)^2}\,, \end{aligned}$$

which implies the conclusion. □

Appendix B Decay of harmonic functions in annuli

In this appendix we gather two results about pointwise control of harmonic functions in annuli. We use the same notations as in § 2 and § A, denoting by $\{\Phi_j\}$ an orthonormal system of eigenfunctions of the Laplacian on $\mathbb{S}^{d-1}$, with eigenvalues λ_j and associated powers $\gamma_j^\pm$ as in (A.2).

The first result is an annulus version of the fact that if a harmonic function in $\mathbb{R}^3 \setminus B_\lambda$ is square-integrable, then it decays like $1/|x|^2$.

Lemma B.1. *Let $R_*/8 > \lambda \geq 1$. Assume $\Delta u = 0$ in $B_{R_*} \setminus B_\lambda \subset \mathbb{R}^3$ and*

$$\int_{B_{R_*}\setminus B_\lambda} |u|^2 \, dx \leq 1.$$

Then we have

$$|u| + |x||\nabla u| \lesssim \frac{\sqrt{\lambda}}{|x|^2} + \frac{R_*^{-1/2}}{|x|}, \qquad \text{for } 2\lambda \leq |x| \leq \frac{R_*}{4}.$$

The second result is an annulus version of the fact that a harmonic function with finite energy in $\mathbb{R}^d \setminus B_\lambda$ only has decaying modes.

Lemma B.2. *Let $R_*/8 > \lambda \geq 1$ and assume $\Delta u = 0$ in $B_{R_*} \setminus B_\lambda \subset \mathbb{R}^d$. Then we can decompose u as a sum of two harmonic functions $u = v + w$, with*

$$v(r\omega) = a_0\Phi_0 + \sum_{j\geq 0} b_j r^{-\gamma_j^-} \Phi_j(\omega)\,,$$

$$\sum_{j\geq 0} \frac{|b_j|^2}{(4\lambda)^{2\gamma_j^-}} \lesssim \frac{1}{\lambda^{d-2}} \int_{B_{2\lambda}\setminus B_\lambda} |\nabla u|^2 \, dx\,,$$

$$\text{and } |w|^2 + |x|^2|\nabla w|^2 \lesssim \frac{1}{R_*^{d-2}} \int_{B_{R_*}\setminus B_{R_*/2}} |\nabla u|^2 \, dx \quad \text{for } 2\lambda \leq |x| \leq \frac{R_*}{4}\,.$$

The multiplicative constants depend on d.

In the proofs of both lemmas, the main tool is an elementary estimate on the coefficients of a harmonic function generated by one single spherical harmonic, that is, $u(r\omega) = \big(ar^{\gamma_k^+} + br^{-\gamma_k^-}\big)\Phi_k(\omega)$, in terms of integrals of $|u|^2$.

Lemma B.3. *For any $d \geq 2$, any $a, b \in \mathbb{R}$ and $\gamma^\pm \geq 0$, such that $\gamma^+ + \gamma^- \geq 1$ and $\gamma^- - \gamma^+ = d - 2$, the function*

$$u(r) = a\, r^{\gamma^+} + br^{-\gamma^-} \qquad \text{for } r > 0\,,$$

satisfies, for any $\mu > 1$, the estimates

$$|a|^2 R^{2\gamma^+} \lesssim \frac{\mu^{-2\gamma^+}}{R^d} \int_R^{\mu^3 R} |u|^2 \, r^{d-1} \, dr\,,$$

$$|b|^2 R^{-2\gamma^-} \lesssim \frac{\mu^{4\gamma^-}}{R^d} \int_R^{\mu^3 R} |u|^2 \, r^{d-1} \, dr\,, \qquad \forall R > 0\,,$$

where the multiplicative constant depends on μ and d, but not on $\gamma^{\pm}$.

Proof of Lemma B.3. We denote by A the average of $|u|^2$ on $[R, \mu^3 R]$ with respect to $r^{d-1}\,dr$, that is,

$$A = \frac{d}{(\mu^{3d}-1)R^d} \int_R^{\lambda R} |u|^2\, r^{d-1}\, dr, \,.$$

By the mean value theorem, we can find $R_1 \in [R, \mu R]$ and $R_2 \in [\mu^2 R, \mu^3 R]$ such that

$$\begin{aligned}
|u(R_1)|^2 &\leq \frac{d}{(\mu^d - 1)R^d} \int_R^{\mu R} |u|^2\, r^{d-1}\, dr = \frac{\mu^{3d}-1}{\mu^d - 1} A\,, \\
|u(R_2)|^2 &\leq \frac{d}{\mu^{2d}(\mu^d - 1)R^d} \int_{\mu^2 R}^{\mu^3 R} |u|^2\, r^{d-1}\, dr = \frac{\mu^{3d}-1}{\mu^{2d}(\mu^d - 1)} A\,.
\end{aligned}$$

Inverting the system

$$\begin{aligned}
a\, R_1^{\gamma^+} + b\, R_1^{-\gamma^-} &= u(R_1)\,, \\
a\, R_2^{\gamma^+} + b\, R_2^{-\gamma^-} &= u(R_2)\,,
\end{aligned}$$

we obtain

$$\begin{aligned}
a &= \frac{-R_2^{-\gamma^-} u(R_1) + R_1^{-\gamma^-} u(R_2)}{R_2^{\gamma^+} R_1^{-\gamma^-} - R_1^{\gamma^+} R_2^{-\gamma^-}}\,, \\
b &= \frac{R_2^{\gamma^+} u(R_1) - R_1^{\gamma^+} u(R_2)}{R_2^{\gamma^+} R_1^{-\gamma^-} - R_1^{\gamma^+} R_2^{-\gamma^-}}\,.
\end{aligned}$$

Using that $R_1 \leq \mu R \leq \mu^2 R \leq R_2$ we find

$$R_2^{\gamma^+} R_1^{-\gamma^-} - R_1^{\gamma^+} R_2^{-\gamma^-} \geq \frac{\mu^{\gamma^+} - \mu^{-\gamma^-}}{(\mu R)^{\gamma^- - \gamma^+}}\,,$$

and, using also that $R_1 \in [R, \mu R]$, $R_2 \in [\mu^2 R, \mu^3 R]$, we deduce

$$\begin{aligned}
|a| &\leq \frac{1+\mu^{-2\gamma^-}}{1-\mu^{-(\gamma^+ + \gamma^-)}} \mu^{\gamma^- - \gamma^+} (\mu R)^{-\gamma^+} \max\big(|u(R_1)|, |u(R_2)|\big) \\
|b| &\leq \frac{1+\mu^{-2\gamma^+}}{1-\mu^{-(\gamma^+ + \gamma^-)}} \mu^{\gamma^+ - \gamma^-} (\mu^2 R)^{\gamma^-} \max\big(|u(R_1)|, |u(R_2)|\big)\,.
\end{aligned}$$

Squaring these inequalities and using that $|u(R_j)|^2 \lesssim A$, $\gamma^+ + \gamma^- \geq 1$ and $\gamma^- - \gamma^+ = d - 2$, we conclude. □

With Lemma B.3 now proven, we establish the estimates of Lemma B.1 and Lemma B.2.

Proof of Lemma B.1. We write the spherical harmonics decomposition

$$u(r\omega) = \sum_{j\geq 0} u_j(r)\Phi_j(\omega), \qquad u_j(r) = a_j r^{\gamma_j^+} + b_j r^{-\gamma_j^-},$$

and denote

$$A_j(R) = \frac{1}{R^3}\int_R^{2R} |u_j|^2\, r^2\, dr, \qquad \tilde{A}_j(R) = R^3 A_j(R),$$

so that

$$\sum_{j\geq 0} \tilde{A}_j(R) \lesssim \int_{B_{2R}\setminus B_R} |u|^2\, dx \lesssim 1 \qquad \forall R \in [\lambda, R_*/2]. \tag{B.1}$$

Applying Lemma B.3 to each u_j with $\mu = 2^{1/3}$ we have the inequalities

$$\begin{aligned} |a_j|^2 &\lesssim (2^{1/3}R)^{-2\gamma_j^+ - 3}\tilde{A}_j(R), \\ |b_j|^2 &\lesssim (2^{2/3}R)^{2\gamma_j^- - 3}\tilde{A}_j(R), \qquad \forall R \in [\lambda, R_*/2]. \end{aligned}$$

Recall $\gamma_0^- = 1$ and $\gamma_j^- \geq 2$ for $j \geq 1$, so the sign of the exponent of R in the inequality for b_j is different for $j = 0$ and $j \geq 1$. Choosing $R = R_*/2$ in the estimate on a_j and b_0, and $R = \lambda$ in the estimate on b_j for $j \geq 1$, we obtain

$$\begin{aligned} |a_j|^2 &\lesssim \frac{2^{\frac{4}{3}\gamma_j^+}}{R_*^{2\gamma_j^+ + 3}}\tilde{A}_j(R_*/2) \qquad \text{for } j \geq 0, \\ |b_0|^2 &\lesssim \frac{1}{R_*}, \qquad |b_j|^2 \lesssim 2^{\frac{4}{3}\gamma_j^-}\lambda^{2\gamma_j^- - 3}\tilde{A}_j(\lambda) \qquad \text{for } j \geq 1. \end{aligned}$$

We use this to obtain

$$\begin{aligned} \fint_{B_{2R}\setminus B_R} |u|^2\, dx &\lesssim \sum_{j\geq 0} \frac{1}{R^3}\int_R^{2R} |a_j r^{\gamma_j^+} + b_j r^{-\gamma_j^-}|^2\, r^2\, dr \\ &\lesssim \sum_{j\geq 0} |a_j|^2 (2R)^{2\gamma_j^+} + \sum_{j\geq 0} |b_j|^2 R^{-2\gamma_j^-} \\ &\lesssim \frac{1}{R_*^3}\sum_{j\geq 0}\Big(\frac{2^{5/3}R}{R_*}\Big)^{2\gamma_j^+}\tilde{A}_j(R_*/2) + \frac{1/R_*}{R^2} \\ &\quad + \frac{\lambda}{R^4}\sum_{j\geq 1}\Big(\frac{2^{2/3}\lambda}{R}\Big)^{2\gamma_j^- - 4}\tilde{A}_j(\lambda), \end{aligned}$$

and therefore, using the summability property (B.1) of the $\tilde{A}_j$'s,

$$\fint_{B_{2R}\setminus B_R} |u|^2\, dx \lesssim \frac{1/R_*}{R^2} + \frac{\lambda}{R^4} \qquad \text{for } 2^{2/3}\lambda \leq R \leq \frac{R_*}{2^{5/3}}.$$

Using elliptic estimates for the harmonic function u, this implies the conclusion of Lemma B.1. ☐

Proof of Lemma B.2. We write the spherical harmonics decomposition

$$u(r\omega) = \sum_{j\geq 0} u_j(r)\Phi_j(\omega), \qquad u_j(r) = a_j r^{\gamma_j^+} + b_j r^{-\gamma_j^-}\,,$$

and denote

$$A_j(R) = \frac{1}{R^d}\int_R^{2R} |u_j|^2\, r^{d-1}\, dr\,, \qquad \hat{A}_j(R) = R^{d-2}A_j(R)\,,$$

so that

$$\begin{aligned}\sum_{j\geq 1}\hat{A}_j(R) &\lesssim \int_{B_{2R}\setminus B_R} \frac{|\nabla_\omega u|^2}{|x|^2}\, dx \\ &\lesssim \int_{B_{2R}\setminus B_R} |\nabla u|^2\, dx \qquad \forall R\in[\lambda, R_*/2]\,. \end{aligned} \tag{B.2}$$

Applying Lemma B.3 to each u_j with $\mu = 2^{1/3}$ we have the inequalities

$$\begin{aligned}|a_j|^2 &\lesssim (2^{1/3}R)^{-2\gamma_j^+ - d+2}\hat{A}_j(R)\,,\\ |b_j|^2 &\lesssim (2^{2/3}R)^{2\gamma_j^- - d+2}\hat{A}_j(R)\,, \qquad \forall R\in[\lambda, R_*/2]\,.\end{aligned}$$

Choosing $R = R_*/2$ in the estimate on a_j, and $R=\lambda$ in the estimate on b_j, we obtain, for all $j\geq 1$,

$$\begin{aligned}|a_j|^2 &\lesssim \Big(\frac{2^{\frac{2}{3}}}{R_*}\Big)^{2\gamma_j^+ + d-2}\hat{A}_j(R_*/2)\,,\\ |b_j|^2 &\lesssim \big(2^{\frac{2}{3}}\lambda\big)^{2\gamma_j^- - d+2}\hat{A}_j(\lambda)\,.\end{aligned}$$

Using (B.2) we deduce

$$\sum_{j\geq 1}\frac{|b_j|^2}{(4\lambda)^{2\gamma_j^-}} \lesssim \frac{1}{\lambda^{d-2}}\sum_{j\geq 1}\hat{A}_j(\lambda) \lesssim \frac{1}{\lambda^{d-2}}\int_{B_{2\lambda}\setminus B_\lambda} |\nabla u|^2\, dx\,.$$

For $j=0$ we can use that $\partial_r u_0 = -\gamma_0^- b_0 r^{-\gamma_0^- -1}$, to obtain

$$\frac{|b_0|^2}{(4\lambda)^{2\gamma_0^-}} \lesssim \frac{1}{\lambda^{d-2}}\int_{B_{2\lambda}\setminus B_\lambda} |\partial_r u_0|^2\, dx \lesssim \frac{1}{\lambda^{d-2}}\int_{B_{2\lambda}\setminus B_\lambda} |\nabla u|^2\, dx\,.$$

This shows that the function v given by

$$v(r\omega) = a_0\Phi_0 + \sum_{j\geq 0} b_j r^{-\gamma_j^-}\,\Phi_j(\omega)\,,$$

does satisfy the claimed estimate. It remains to prove the estimate on the function $w = u - v$ given by

$$w(r\omega) = \sum_{j\geq 1} a_j r^{\gamma_j^+}\,\Phi_j(\omega)\,.$$

For any $R \in [\lambda, R_*/2^{2/3}]$, we use the above estimate on a_j and the control (B.2) on the sum of the $\hat{A}_j$'s to calculate

$$\begin{aligned}
\fint_{B_{2R}\setminus B_R} |w|^2\,dx &\lesssim \sum_{j\geq 1} |a_j|^2 R^{2\gamma_j^+} \\
&\lesssim \frac{1}{R_*^{d-2}} \sum_{j\geq 1} \Big(\frac{2^{2/3}R}{R_*}\Big)^{2\gamma_j^+} \hat{A}_j(R_*/2) \\
&\lesssim \frac{1}{R_*^{d-2}} \sum_{j\geq 1} \hat{A}_j(R_*/2) \\
&\lesssim \frac{1}{R_*^{d-2}} \int_{B_{R_*}\setminus B_{R_*/2}} |\nabla u|^2\,dx\,.
\end{aligned}$$

The conclusion follows from elliptic estimates for the harmonic function w. □

References

[1] Alama, S., Bronsard, L., and Lamy, X. Minimizers of the Landau-de Gennes energy around a spherical colloid particle. *Arch. Ration. Mech. Anal. 222*, 1 (2016), 427–450.

[2] Alama, S., Bronsard, L., Lamy, X., and Venkatraman, R. Far-field expansions for harmonic maps and the electrostatics analogy in nematic suspensions. *J. Nonlinear Sci. 33*, 3 (2023), 27. Id/No 39.

[3] Bethuel, F., Brezis, H., and Hélein, F. *Ginzburg-Landau vortices*, vol. 13 of *Progress in Nonlinear Differential Equations and their Applications*. Birkhäuser Boston, Inc., Boston, MA, 1994.

[4] Brezis, H., Coron, J.-M., and Lieb, E. H. Harmonic maps with defects. *Comm. Math. Phys. 107*, 4 (1986), 649–705.

[5] Brochard, F., and de Gennes, P.G. Theory of magnetic suspensions in liquid crystals. *J. Phys. France 31*, 7 (1970), 691–708.

[6] Chandler, T. G. J., and Spagnolie, S. E. Exact and approximate solutions for elastic interactions in a nematic liquid crystal. *SIAM Journal on Applied Mathematics 84*, 6 (2024), 2476–2501.

[7] De Gennes, P.-G., and Prost, J. *The Physics of Liquid Crystals*. Oxford university press, 1993.

[8] Golovaty, D., Taylor, J., Venkatraman, R., and Zarnescu, A. Interaction energies in paranematic colloids. *arXiv:2308.06331* (2023).

[9] Hardt, R., Kinderlehrer, D., and Lin, F.-H. Stable defects of minimizers of constrained variational principles. *Ann. Inst. H. Poincaré Anal. Non Linéaire 5*, 4 (1988), 297–322.

[10] Lubensky, T. C., Pettey, D., Currier, N., and Stark, H. Topological defects and interactions in nematic emulsions. *Phys. Rev. E 57* (Jan 1998), 610–625.

[11] Luckhaus, S. Partial Hölder continuity for minima of certain energies among maps into a Riemannian manifold. *Indiana Univ. Math. J. 37*, 2 (1988), 349–367.

[12] Muševič, I. Interactions, topology and photonic properties of liquid crystal colloids and dispersions. *Eur. Phys. J. Special Topics 227*, 17 (2019), 2455–2485.

[13] Ramaswamy, S., Nityananda, R., Raghunathan, V. A., and Prost, J. Power-law forces between particles in a nematic. *Mol. Cryst. Liq. Cryst. 288*, 1 (1996), 175–180.

[14] Schoen, R., and Uhlenbeck, K. A regularity theory for harmonic maps. *J. Differ. Geom. 17* (1982), 307–335.

[15] Serfaty, S. Ginzburg-Landau vortices, Coulomb gases, and renormalized energies. *J. Stat. Phys. 154*, 3 (2014), 660–680.

[16] Simon, L. *Theorems on regularity and singularity of energy minimizing maps. Based on lecture notes by Norbert Hungerbühler*. Basel: Birkhäuser, 1996.