Compensation effects for anisotropic energies of two-dimensional unit vector fields

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Abstract

We study the highly anisotropic energy of two-dimensional unit vector fields given by

$$E_\epsilon(u) = \int_\Omega (\operatorname{div} u)^2 + \epsilon (\operatorname{curl} u)^2 \, dx \,, \quad u\colon \Omega \subset \mathbb{R}^2 \to \mathbb{S}^1$$

in the limit $\epsilon \to 0$. This energy clearly loses control on the full gradient of u as $\epsilon \to 0$, but, adapting tools from hyperbolic conservations laws, we show that it still controls derivatives of order 1/2. In particular, any bounded energy sequence $E_\epsilon(u_\epsilon) \le C$ is compact in $W^{s,3}_{\mathrm{loc}}(\Omega)$ for $s < 1/2$. Moreover, this order 1/2 of differentiability is optimal, in the sense that any map $u \in W^{1/2,4}(\Omega; \mathbb{S}^1)$ is a limit of a bounded energy sequence. We also establish compactness of boundary traces in $L^1(\partial\Omega)$, and characterize the Γ-limit in the simpler case of maps of a single variable and in the case of a thin-film model.

1 Introduction

We consider a smooth bounded open set $\Omega \subset \mathbb{R}^2$, and, for $0 < \epsilon \ll 1$, the highly anisotropic energy functional

$$E_\epsilon(u) = \int_\Omega (\operatorname{div} u)^2 + \epsilon (\operatorname{curl} u)^2 \, dx, \qquad \text{for } u \in H^1(\Omega; \mathbb{S}^1) \,. \tag{1.1}$$

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This is a two-dimensional version of the Oseen-Frank energy for nematic liquid crystals, in a regime of very disparate elastic constants [43]. From the point of view of modelling, one of the most relevant questions is to determine the behavior of minimizers (subject to some boundary constraint) as $\epsilon \to 0$, or, more generally equicoercivity properties of E_ϵ and its Γ-limit as $\epsilon \to 0$, see [2, Definition 1.5]. Perhaps surprisingly, this seems to be a quite subtle task, and we only provide partial answers.

The mathematical study of liquid crystal models often relies on the one-constant approximation, where the elastic part of the energy is of the form $\int |\nabla u|^2 \, dx$, even though actual liquid crystals have anisotropic elastic properties [12]. It is well-known that this is not just a cosmetic simplification: anisotropic energies present significant mathematical challenges with respect to the isotropic case (see e.g. [27, 28] for anisotropic harmonic maps and [9, 20, 5, 6] for anisotropic Ginzburg-Landau energies). The effect of very disparate elastic constants has been analyzed in [23, 22, 21] for models of Ginzburg-Landau type which relax the $\mathbb{S}^1$-valued constraint, see also [7].

Here we focus on the interplay between the degenerate ellipticity $\epsilon \to 0$ and the pointwise constraint $u(x) \in \mathbb{S}^1$, and demonstrate that this interplay gives rise to interesting compensation phenomena. Similar compensations have been observed in [29], where a bound on the divergence of smooth $\mathbb{S}^1$-valued vector fields with tangential boundary conditions is enough to provide strong compactness, as well as in [39] where the divergence vanishes in the limit, and in the Aviles-Giga problem [1, 14] where the divergence is zero but the $\mathbb{S}^1$ constraint is relaxed. This is also reminiscent of the compensated integrability established in [26] where degenerate ellipticity is coupled with a cone-valued constraint.

Naively, if $u_\epsilon \colon \Omega \to \mathbb{S}^1$ minimizes the energy E_ϵ given by (1.1) with respect to some boundary conditions, as $\epsilon \to 0$ one could expect a limit map $u_* \colon \Omega \to \mathbb{S}^1$ to minimize

$$E_0(u) = \int_\Omega (\operatorname{div} u)^2 \, dx \,, \quad \text{for } u \colon \Omega \to \mathbb{S}^1 \text{ s.t. } \operatorname{div} u \in L^2(\Omega) \,. \tag{1.2}$$

Here $\operatorname{div} u$ is the distributional divergence of a vector field u. This already raises simple questions which do not have obvious answers: Is there enough compactness to obtain a limit u_* with values into $\mathbb{S}^1$? Does u_* still satisfy the same boundary data? How singular can u_* be ?

With respect to this last question, it is instructive to consider maps $u \colon \Omega \to \mathbb{S}^1$ which are divergence-free : they satisfy $E_0(u) = 0$, hence they are minimizers of (1.2). Such maps can be extremely wild: any weak solution of the eikonal equation $|\nabla\varphi| = 1$ a.e. in Ω provides a divergence-free map $u \colon \Omega \to \mathbb{S}^1$ given by $u = \nabla^\perp\varphi$, and general weak solutions of the eikonal equation are very flexible, for instance they can be uniformly close to any $\psi \colon \Omega \to \mathbb{R}$ with $|\nabla\psi| \leq 1$ [11]. Instructed by the compactness results of [1, 14, 39, 29], we may expect that such pathological behavior should be ruled out for limits of bounded energy sequences. But it is still natural to wonder about two explicit examples of elementary divergence-free singularities. The first example is that of a divergence-free jump

$$u^{\mathrm{jump}}(x) = u^- \mathbf{1}_{x_1<0} + u^+ \mathbf{1}_{x_1>0}, \qquad \begin{array}{l} u^+ \neq u^- \in \mathbb{S}^1, \\ (u^+ - u^-)\cdot \mathbf{e}_1 = 0, \end{array} \tag{1.3}$$

where $x = (x_1, x_2)$. The second example is that of a divergence-free vortex

$$u^{\mathrm{vort}}(x) = \frac{ix}{|x|} = \frac{(-x_2, x_1)}{|x|}. \tag{1.4}$$

Can these be obtained as limits of minimizers u_ϵ of E_ϵ ? Do singularities carry an energy cost ?

All these questions are related to determining the Γ-limit of E_ϵ (say, with respect to convergence in the sense of distributions), possibly under some restriction on boundary values. In that respect, a first step is to understand the class of maps with finite Γ-$\limsup$, that is, which maps can be obtained as limits of bounded energy sequences:

$$\begin{aligned} \mathcal{S} &:= \{\Gamma\text{-}\limsup\nolimits_{\epsilon\to 0} E_\epsilon < \infty\} \\ &= \Big\{ u \in L^\infty(\Omega;\mathbb{R}^2) \colon \exists \epsilon_k \searrow 0,\ u_{\epsilon_k} \in H^1(\Omega;\mathbb{S}^1) \\ &\qquad\qquad \text{s.t. } \limsup_{k\to\infty} E_{\epsilon_k}(u_{\epsilon_k}) < \infty \text{ and } u_{\epsilon_k} \to u \text{ in } \mathcal{D}'(\Omega) \Big\}. \end{aligned} \tag{1.5}$$

Our main results, to be presented in more detail below, imply that the $\mathbb{S}^1$ constraint is preserved in the limit, as well as boundary data. Concerning singularities, we find that

$$u^{\mathrm{jump}},\ u^{\mathrm{vort}} \notin \mathcal{S}.$$

In other words, these singularities cannot be approximated with finite E_ϵ energy. Further, we establish that although the functional E_ϵ in (1.1) gives up any control on $\operatorname{curl} u$ as $\epsilon \to 0$, the L^2 control on $\operatorname{div} u$ coupled with the $\mathbb{S}^1$ constraint is enough to control roughly "half a derivative", in the sense of fractional Sobolev $W^{1/2,p}$ regularity:

$$\mathcal{S} \subset W^{\frac{1}{2}-,3}_{\mathrm{loc}}(\Omega;\mathbb{S}^1).$$

This half order of differentiation turns out to be optimal, in the sense that any map with half a gradient (but higher integrability) can be approximated by bounded energy sequences:

$$W^{\frac{1}{2},4}(\Omega;\mathbb{S}^1) \subset \mathcal{S}.$$

Finally, in the simpler cases of a one-dimensional domain and a thin domain, we are able to completely determine the Γ-limit. Next we proceed to describe our results more precisely.

1.1 Compensated regularity: interior estimate

The first compensation phenomenon we observe is that energy boundedness provides ϵ-independent regularity estimates with a fractional order of derivatives. We begin with a brief review of the relevant function spaces.

1.1.1 Fractional Sobolev and Besov spaces

In a smooth bounded open set $\Omega \subset \mathbb{R}^d$, for a differentiability order $0 < s < 1$ and integrability exponent $p \geq 1$, fractional differentiability of a function $u \in L^p(\Omega)$ can be measured in terms of the difference quotient

$$\frac{\|D^h u\|_{L^p(\Omega\cap(\Omega-h))}}{|h|^s}, \qquad h \in \mathbb{R}^d \setminus \{0\},$$

where D^h denotes the finite difference operator

$$D^h u(x) = u(x+h) - u(x). \tag{1.6}$$

One can consider, for instance, the fractional Sobolev space $W^{s,p}(\Omega)$, which consists of functions $u \in L^p(\Omega)$ such that

$$|u|^p_{W^{s,p}(\Omega)} = \int_{\mathbb{R}^d} \left(\frac{\|D^h u\|_{L^p(\Omega\cap(\Omega-h))}}{|h|^s} \right)^p \frac{dh}{|h|^d} < \infty. \tag{1.7}$$

Note that this coincides with the commonly used Gagliardo seminorm [15]

$$|u|^p_{W^{s,p}(\Omega)} = \int_{\Omega\times\Omega} \frac{|u(x)-u(y)|^p}{|x-y|^{sp}} \frac{dxdy}{|x-y|^d}.$$

We will also consider the Besov space $B^s_{p,\infty}(\Omega)$ of functions $u \in L^p(\Omega)$ such that

$$|u|_{B^s_{p,\infty}(\Omega)} = \operatorname*{ess\,sup}_{h\in\mathbb{R}^d} \frac{\|D^h u\|_{L^p(\Omega\cap(\Omega-h))}}{|h|^s} < \infty. \tag{1.8}$$

Note that we have the inclusion

$$B^s_{p,\infty}(\Omega) \subset W^{s',p}(\Omega) \qquad \text{for } 0 < s' < s < 1,\ p \geq 1,$$

since

$$\begin{aligned} |u|^p_{W^{s',p}(\Omega)} &= \int_{\mathbb{R}^2} \left(\frac{\|D^h u\|_{L^p(\Omega\cap(\Omega-h))}}{|h|^{s'}} \right)^p \frac{dh}{|h|^d} \\ &\leq |u|^p_{B^s_{p,\infty}} \int_{|h|\leq 1} \frac{dh}{|h|^{d-(s-s')p}} + 2^p \|u\|^p_{L^p} \int_{|h|\geq 1} \frac{dh}{|h|^{d+s'p}}. \end{aligned}$$

Moreover, this embedding is compact, as follows e.g. from [41, Remark 1, p.233].

1.1.2 Interior estimate and compactness

We prove that the energy controls derivatives of order $1/2$ in L^3, in the sense of the Besov space $B^{1/2}_{3,\infty}$ (and therefore also in $W^{s,3}$ for all $s < 1/2$).

Theorem 1.1. *For any bounded open sets $\Omega' \subset\subset \Omega \subset \mathbb{R}^2$, there exists $C = C(\Omega', \Omega) > 0$ such that*

$$\|D^h u\|_{L^3(\Omega' \cap (\Omega' - h))} \leq C \left(1 + \|\operatorname{div} u\|_{L^2(\Omega)}\right)^{\frac{1}{2}} |h|^{\frac{1}{2}} \qquad \forall h \in \mathbb{R}^2 , \tag{1.9}$$

for all $u \in H^1(\Omega; \mathbb{S}^1)$, and more generally for all u in the finite Γ-$\limsup$ set $\mathcal{S}$ defined in (1.5)*.*

The proof of Theorem 1.1 relies on notions of entropy productions and a kinetic formulation adapted from the theory of conservation laws, see §2.

By compactness of the embedding $B^{1/2}_{3,\infty} \subset W^{s,3}$ for all $s \in (0, 1/2)$, the compensated regularity estimate (1.9) directly implies the following strong compactness properties for sequences with bounded energy.

Corollary 1.2. *If $(u_\epsilon) \subset H^1(\Omega; \mathbb{S}^1)$ satisfies $\limsup_{\epsilon \to 0} E_\epsilon(u_\epsilon) < \infty$, there exists*

$$u \in B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega; \mathbb{S}^1) \text{ with } \operatorname{div} u \in L^2(\Omega) ,$$

and a sequence $\epsilon_k \to 0$ such that

$$u_{\epsilon_k} \longrightarrow u \qquad \text{strongly in } W^{s,3}(\Omega'; \mathbb{R}^2) ,$$

for all $0 \leq s < 1/2$ and $\Omega' \subset\subset \Omega$.

Let us emphasize that the estimate (1.9) is false if we only assume that $u \in H_{\mathrm{div}}(\Omega; \mathbb{S}^1)$, that is, $u \in L^2(\Omega; \mathbb{S}^1)$ with square-integrable distributional divergence $\operatorname{div} u \in L^2(\Omega)$. For instance, the pure jump u^{jump} given in (1.3) belongs to $H_{\mathrm{div}}(B_1; \mathbb{S}^1)$ since $\operatorname{div} u^{\mathrm{jump}} = 0$, but not to $B^{1/2}_{3,\infty,\mathrm{loc}}(B_1)$ since $\|D^h u^{\mathrm{jump}}\|_{L^3(B_{1/2})} \sim c|u^+ - u^-||h_1|^{1/3}$ as $|h| = |(h_1, h_2)| \to 0$, for some $c > 0$. Hence Theorem 1.1 shows in particular that u^{jump} does not belong to the finite Γ-$\limsup$ set $\mathcal{S}$ for $\Omega = B_1$.

Remark 1.3. If $\Omega \subset \mathbb{R}^2$ is simply connected, maps $u \in H^1(\Omega; \mathbb{S}^1)$ can be lifted to $\varphi \in H^1(\Omega; \mathbb{R})$ such that $u = e^{i\varphi}$, see e.g. [3, §1]. In terms of this phase φ, we have

$$E_\epsilon(u) = \int_\Omega \langle A_\epsilon(\varphi) \nabla\varphi, \nabla\varphi \rangle \, dx , \quad A_\epsilon(\varphi) = ie^{i\varphi} \otimes ie^{i\varphi} + \epsilon \, e^{i\varphi} \otimes e^{i\varphi} \in \mathbb{R}^{2\times 2} .$$

The degenerate ellipticity is seen in the fact that the nonnegative symmetric matrix $A_\epsilon(\varphi)$ has an eigenvalue $\epsilon \to 0$, in the degenerate eigendirection $e^{i\varphi}$. If instead we had an energy of the form

$$\widetilde{E}_\epsilon(\varphi) = \int_\Omega \langle \tilde{A}_\epsilon \nabla\varphi, \nabla\varphi \rangle \, dx , \quad \tilde{A}_\epsilon = iv \otimes iv + \epsilon \, v \otimes v \in \mathbb{R}^{2\times 2} ,$$

for some constant vector $v \in \mathbb{S}^1$, then any integrable function $x \mapsto f(\langle v, x\rangle)$ could arise as a bounded energy limit, simply by taking a smooth approximation $f_j \to f$ and adjusting $\epsilon_j \to 0$. In particular there would be no regularity estimate for bounded energy limits: from that perspective, the compensation effect shown by Theorem 1.1 seems related to the specific nonlinear dependence of the degenerate direction $e^{i\varphi}$ on φ.

1.1.3 Consequence on the distributional Jacobian

Above we have seen that Theorem 1.1 implies that the pure jump u^{jump} given by (1.3) is not a bounded energy limit in $\Omega = B_1$, because it does not belong to $B^{1/2}_{3,\infty,\mathrm{loc}}(B_1)$. We cannot invoke the same reasoning to rule out the vortex u^{vort} given in (1.4) as a possible bounded energy limit, because u^{vort} belongs to $B^{2/3}_{3,\infty}(B_1) \subset B^{1/2}_{3,\infty}(B_1)$ as a consequence of the inequalities

$$\begin{aligned}\|D^h u^{\mathrm{vort}}\|^3_{L^3(B_1)} &\leq \int_{2|h|<|x|<1} |D^h u^{\mathrm{vort}}|^3\,dx + \int_{|x|<2|h|} |D^h u^{\mathrm{vort}}|^3\,dx \\ &\leq \int_{2|h|<|x|<1} \frac{|h|^3}{(|x|/2)^3}\,dx + \int_{|x|<2|h|} 8\,dx \lesssim |h|^2\,.\end{aligned}$$

However, from Theorem 1.1 it does nonetheless follow that u^{vort} cannot be a bounded energy limit in B_1, but for a subtler reason: the existence of a continuous notion of Jacobian on $W^{1/3,3}(B_1;\mathbb{S}^1)$, see [3, Corollary 8.1], which allows one to pass to the limit in the identity $Ju_\epsilon = \det(\nabla u_\epsilon) = 0$ satisfied by $u_\epsilon \in H^1(\Omega;\mathbb{S}^1)$.

Recall that a map $u \in W^{1,1} \cap L^\infty(\Omega;\mathbb{R}^2)$, has a well-defined distributional Jacobian

$$Ju = \frac{1}{2}\,\mathrm{curl}(u \wedge \nabla u)\,,$$

see [3, §1.2], which coincides with $Ju = \det(\nabla u)$ if $u \in W^{1,2}(\Omega;\mathbb{R}^2)$. In particular we have $Ju = 0$ for $u \in W^{1,2}(\Omega;\mathbb{S}^1)$ because ∇u has rank one. And the vortex satisfies $Ju^{\mathrm{vort}} = \pi\delta_0$ [3, Remark 1.9].

For $\mathbb{S}^1$-valued maps, the distributional Jacobian can be extended to the space $W^{1/p,p}(\Omega;\mathbb{S}^1)$ for any $p > 1$: there exists a continuous map $J\colon W^{1/p,p}(\Omega;\mathbb{S}^1) \to \mathcal{D}'(\Omega)$ which coincides with the usual Jacobian on $W^{1/p,p} \cap W^{1,1}(\Omega;\mathbb{S}^1)$, see [3, §8.1]. In particular, since $\mathcal{S} \subset W^{1/3,3}_{\mathrm{loc}}(\Omega;\mathbb{S}^1)$ thanks to Theorem 1.1, any map $u \in \mathcal{S}$ has a well-defined Jacobian Ju.

Recall also that maps $v \in H^1(\Omega;\mathbb{S}^1)$ have a well-defined winding number $\deg(v;\Gamma) \in \mathbb{Z}$ on any smooth curve $\Gamma \subset \Omega$ homeomorphic to a circle, which depends only on the homotopy class of Γ, see e.g. [3, §12]. This leads us to the following Corollary whose proof is detailed in §2.5.

Corollary 1.4. *If u belongs to the finite Γ-$\limsup$ set $\mathcal{S}$ defined in* (1.5)*, then we have that $Ju = 0$. Moreover, u has a well-defined winding number* $\deg(u;\Gamma) \in \mathbb{Z}$

along any smooth curve $\Gamma \subset \Omega$ *homeomorphic to a circle, such that* $\deg(u;\Gamma) = \lim \deg(u_\epsilon;\Gamma)$ *for any sequence* $u_\epsilon \to u$ *in* $\mathcal{D}'(\Omega)$ *with* $E_\epsilon(u_\epsilon) \leq C$.

In particular, the vortex u^{vort} defined in (1.4), which satisfies $Ju^{\mathrm{vort}} = \delta_0 \neq 0$, does not belong to $\mathcal{S}$ in $\Omega = B_1$.

1.2 Convergence of traces

The interior estimate of Theorem 1.1 does not provide any control on boundary data. For vector fields $u \in H^1(\Omega;\mathbb{S}^1)$ which are tangent at the boundary ($u \cdot \nu = 0$ on $\partial\Omega$, where ν is a unit normal on $\partial\Omega$) and sufficiently smooth domains Ω, an estimate up to the boundary,

$$|u|^2_{B^{1/2}_{3,\infty}(\Omega)} \lesssim 1 + \|\operatorname{div} u\|_{L^2(\Omega)}\,,$$

can actually be inferred from Theorem 1.1 via a reflection argument coupled with the use of conformal coordinates (to flatten the boundary while keeping control on the divergence). See [7] for related work. But for general maps $u \in H^1(\Omega;\mathbb{S}^1)$ it is not clear to us whether such a global estimate is valid. Nevertheless, we are able to show compactness of traces for bounded energy sequences.

Theorem 1.5. *Let* $\Omega \subset \mathbb{R}^2$ *be a bounded* C^2 *open set. For any family of maps* $u_\epsilon \colon \Omega \to \mathbb{R}^2$ *such that* $\limsup_{\epsilon\to 0} E_\epsilon(u_\epsilon) < \infty$*, the limit* u_* *provided by Corollary 1.2 has a strong* L^1 *trace on* $\partial\Omega$*, and* $\mathrm{tr}(u_{\epsilon_k}) \to \mathrm{tr}(u_*)$ *strongly in* $L^p(\partial\Omega)$ *for all* $1 \leq p < \infty$.

Note that vector fields with L^2 divergence always have a weak normal trace which ensures the validity of the Gauss-Green formula, see e.g. [8], but this weak normal trace may not be attained in any pointwise sense [31, Example 2.7], and tangential components may not even have a weak trace [8, Remark 2.4]. Here we really take advantage of the fact that $u_* \in \mathcal{S}$ to define its strong trace, see Proposition 3.1. As for Theorem 1.1, the proof of Theorem 1.5 relies on ideas from the theory of conservation laws, in particular from [8] and [42].

1.3 Recovery sequences

Thanks to Theorem 1.1, if a map $u \colon \Omega \to \mathbb{S}^1$ is a limit of maps u_ϵ with bounded energy, then it must have a certain fractional regularity of order 1/2, namely $B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega)$. Conversely, it is natural to ask which sufficient conditions on u ensure that it can be approximated with bounded energy. In other words, we would like to identify large subsets of the finite Γ-$\limsup$ set $\mathcal{S}$ defined in (1.5). We are only able to obtain an incomplete converse to Theorem 1.1, which does however confirm the optimality of the order 1/2 of differentiability: any map in $W^{1/2,4}(\Omega)$, see (1.7), satisfies the condition (1.10) and, according to our next main result, belongs therefore to $\mathcal{S}$.

Theorem 1.6. *Let $\Omega \subset \mathbb{R}^2$ be a C^3 open set, and $u\colon \Omega \to \mathbb{S}^1$ possess square-integrable distributional divergence* $\operatorname{div} u \in L^2(\Omega)$*, and such that*

$$\fint_{B_\delta} \left(\frac{\|D^h u\|_{L^4(\Omega\cap(\Omega-h))}}{\delta^{1/2}} \right)^4 dh \longrightarrow 0 \qquad \text{as } \delta \to 0\,, \tag{1.10}$$

where D^h is the finite difference operator defined in (1.6) *and $\fint$ denotes integral average. Then there exists $(u_\epsilon) \subset H^1(\Omega;\mathbb{S}^1)$ such that*

$$u_\epsilon \to u \text{ in } L^2(\Omega;\mathbb{R}^2) \quad \text{and } E_\epsilon(u_\epsilon) \to \int_\Omega (\operatorname{div} u)^2\, dx,$$

as $\epsilon \to 0$.

The recovery sequence u_ϵ will be constructed by convolving u with a smoothing kernel and projecting it onto $\mathbb{S}^1$. Convolution commutes with taking the divergence, but projection onto $\mathbb{S}^1$ does not: the main step in the proof of Theorem 1.6 is to estimate the extra error thus introduced, by adapting commutator estimates from [10, 13].

From Theorems 1.1 and 1.6, we infer that the finite Γ-$\limsup$ set $\mathcal{S}$ given by (1.5) satisfies the inclusions

$$W^{\frac{1}{2},4}(\Omega;\mathbb{S}^1) \subset \mathcal{S} \subset B^{\frac{1}{2}}_{3,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1) \subset W^{\frac{1}{2}-,3}_{\mathrm{loc}}(\Omega;\mathbb{S}^1)\,.$$

In fact, for $u \in W^{\frac{1}{2},4}(\Omega;\mathbb{S}^1)$ or more generally u satisfying (1.10), the Γ-limit of E_ϵ at u (see e.g. [2, Definition 1.5]) is completely determined,

$$\Big(\Gamma\text{-}\lim_{\epsilon\to 0} E_\epsilon\Big)(u) = \int_\Omega (\operatorname{div} u)^2\, dx\,.$$

Indeed, the upper bound is covered by Theorem 1.6, and the lower bound follows from the lower semicontinuity of $\int (\operatorname{div} u_\epsilon)^2\, dx$ with respect to weak L^2 convergence of $\operatorname{div} u_\epsilon$. This leaves the determination of the Γ-limit open in the set of maps $u \in B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1)$ which, recalling Corollary 1.4, satisfy $Ju = 0$, but not (1.10).

1.4 The one-dimensional case and a thin-film dimension reduction limit

For $\mathbb{S}^1$-valued maps $u = (u_1, u_2)$ depending only on one variable, say $x_1 \in I$ for a finite interval $I \subset \mathbb{R}$, we have $\operatorname{div} u = \frac{du_1}{dx_1}$. Since $|u_2| = \sqrt{1-u_1^2}$, and $\sqrt{\cdot}$ is $C^{1/2}$, it is not so surprising that L^2 control on $\operatorname{div} u$ implies fractional regularity of order $1/2$ for u, as in Theorem 1.1. In fact, using this simple idea, one can completely determine the Γ-limit [2, Definition 1.5] in this one-dimensional case.

Theorem 1.7. *The energy functionals $E_\epsilon\colon \mathcal{D}'(I;\mathbb{R}^2)\to\mathbb{R}$ given for $\epsilon>0$ by*

$$E_\epsilon^{1\mathrm{D}}(u)=\begin{cases}\int_I \left[(u_1')^2+\epsilon(u_2')^2\right]dx & \text{if } u\in H^1(I;\mathbb{S}^1)\,,\\ +\infty & \text{otherwise}\,,\end{cases}$$

Γ-converge, as $\epsilon\to 0$, to the limit functional $E_0\colon \mathcal{D}'(I;\mathbb{R}^2)\to\mathbb{R}$ given by

$$E_0^{1\mathrm{D}}(u)=\begin{cases}\int_I (u_1')^2\,dx & \text{if } u\in B^{1/2}_{4,\infty}(I;\mathbb{S}^1) \text{ and } u_1\in H^1(I)\,,\\ +\infty & \text{otherwise}\,.\end{cases}$$

Moreover, any sequence $(u_\epsilon)\subset H^1(I;\mathbb{S}^1)$ with $\sup_\epsilon E_\epsilon^{1\mathrm{D}}(u_\epsilon)<\infty$ for some sequence $\epsilon\to 0$ is bounded in $B^{1/2}_{4,\infty}(I;\mathbb{S}^1)$ and therefore compact in $W^{s,4}(I;\mathbb{S}^1)$ for any $s\in(0,1/2)$.

The main subtlety in the proof of Theorem 1.7 lies in the construction of the recovery sequence. In contrast with the two-dimensional case, where we had to impose (1.10), here we combine the $B^{1/2}_{4,\infty}$−regularity with the assumption that $\operatorname{div} u = du_1/dx_1\in L^2$ in order to deduce the validity of (1.10).

We would like to use this one-dimensional observation in order to gain more insight into the two-dimensional case. Therefore we consider a somewhat simplified model which resembles the one-dimensional model, while introducing some mild two-dimensionality.

Specifically, for $\epsilon,\delta>0$, we study the energy E_ϵ given by (1.1) in a thin film $\Omega_\delta=(-1,1)\times(-\delta,\delta)$, subject to Dirichlet boundary conditions in the first variable and periodic conditions in the thin variable x_2. After rescaling the x_2 variable to unit length, we are led to the energy functional

$$\begin{aligned}E_{\epsilon,\delta}(u)&=\delta E_\epsilon(u(\cdot,\cdot/\delta);\Omega_\delta)\\ &=\frac{1}{2}\int_{[-1,1]^2}\left(\partial_1u_1+\frac{1}{\delta}\partial_2u_2\right)^2+\epsilon\left(\partial_1u_2-\frac{1}{\delta}\partial_2u_1\right)^2dx\,,\end{aligned}$$

for maps $u\in H^1((-1,1)^2;\mathbb{S}^1)$ with boundary conditions

$$\begin{aligned}&u(\pm1,x_2)=e_\pm\in\mathbb{S}^1 && \forall x_2\in(-1,1)\,,\\ &u(x_1,+1)=u(x_1,-1) && \forall x_1\in(-1,1).\end{aligned}\tag{1.11}$$

The latter imposes periodicity with respect to the second variable. We can consider the energy $E_{\epsilon,\delta}\colon\mathcal{D}'((-1,1)^2;\mathbb{R}^2)\to\mathbb{R}$ by setting it to be $+\infty$ unless $u\in H^1((-1,1)^2;\mathbb{S}^1)$ with the boundary conditions (1.11).

Theorem 1.8. *The Γ-limit in $\mathcal{D}'((-1,1)^2;\mathbb{R}^2)$ as $(\epsilon,\delta)\to(0,0)$ of the energy functionals $E_{\epsilon,\delta}$ is the functional $\widetilde{E}_0^{1\mathrm{D}}\colon\mathcal{D}'((-1,1)^2;\mathbb{R}^2)\to\mathbb{R}$ given by*

$$\widetilde{E}_0^{1\mathrm{D}}(u)=\begin{cases}\int_{-1}^1(\partial_1u_1)^2\,dx_1 & \text{if } \partial_2u=0,\ u\in B^{\frac12}_{4,\infty}((-1,1);\mathbb{S}^1),\\ & \qquad \text{and } u(\pm1)=e_\pm\,,\\ +\infty & \text{otherwise.}\end{cases}$$

Moreover any sequence $(u^{(j)}) \subset H^1((-1,1)^2;\mathbb{S}^1)$ *such that* $\sup E_{\epsilon_j,\delta_j}(u_j) < \infty$ *for some* $(\epsilon_j,\delta_j) \to (0,0)$ *is bounded in* $B^{1/2}_{3,\infty}((-1,1)^2;\mathbb{S}^1)$*, hence compact in* $W^{s,3}((-1,1)^2;\mathbb{S}^1)$ *for any* $s \in (0,1/2)$*.*

Most of Theorem 1.8 follows directly from Theorem 1.1 and Theorem 1.7, except for one slightly subtle part: the fact that bounded energy limits have the $B^{1/2}_{4,\infty}$−regularity, see Lemma 6.2.

Remark 1.9. Theorem 1.8 implies, in particular, that minimizers of $E_{\epsilon,\delta}$ converge to a one-dimensional configuration. But one can actually also check directly that minimizers of $E_{\epsilon,\delta}$ are one-dimensional for any value of $\epsilon,\delta > 0$, see Appendix B.

1.5 Plan of the article

In §2 we prove the compensated regularity estimate of Theorem 1.1, and Corollary 1.4 about the Jacobian. In §3 we prove Theorem 1.5 about the compactness of traces. In §4 we construct the recovery sequence of Theorem 1.6. In §5 we treat the one-dimensional case and prove Theorem 1.7. In §6 we prove the Theorem 1.8 about the thin-domain limit. In Appendix A we adapt to our context the proof, due to [42], of the fact that limit maps admit strong traces. In Appendix B we prove the one-dimensional symmetry of minimizers with periodic boundary conditions. Throughout the article, we will use the symbol $\lesssim$ to denote inequality up to an absolute multiplicative constant.

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2 Compensated regularity

In this section we prove Theorem 1.1. The proof relies on the strategy used in [17] to show regularity of a class of divergence-free unit vector fields, based on ideas from [24, 25] where the authors introduced quantitative improvements on the div-curl lemma from the theory of compensated compactness [35, 40]. Before going into the details, we describe briefly the structure and main ingredients of the proof:

- The assumptions on u imply $\operatorname{div}\Phi(u) \in L^2(\Omega)$ for a large family of vector fields $\Phi \in C^1(\mathbb{S}^1;\mathbb{R}^2)$, called *entropies* by analogy with scalar conservations laws, see (2.2) and Lemma 2.4.

- These L^2 entropy productions provide a *kinetic equation* satisfied by the characteristic function $\chi(s,x) = \mathbf{1}_{e^{is}\cdot u(x)>0}$, where $s \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$ is the kinetic variable, see Lemma 2.1.

- This kinetic equation implies a *compensation identity* relating the divergnce $\operatorname{div} u \in L^2$ to a family of quantities $\Delta^{\varphi,\chi}(x,h)$ depending quadratically on the finite difference $D_x^h\chi$, see (2.8) and Lemma 2.6.

- For a particular choice of the parameter φ, the quantity $\Delta^{\varphi,\chi}$ controls variations of u via a coercivity estimate $\Delta^{\varphi,\chi}(x,h) \gtrsim |D^h u(x)|^3$, see (2.9).

- Estimating the terms in the compensation identity provides control on the integral $\int_{B_1} \Delta^{\varphi,\chi}(x,h)\,dx$ in terms of $\|\operatorname{div} u\|_{L^2(B_2)}$, which, combined with the previous ingredient, proves Theorem 1.1.

In the next subsections we detail all these ingredients: entropies and kinetic formulation are presented in §2.1, the compensation identity in §2.2, estimates of the terms involved in that identity in §2.3, to conclude the proof of the regularity estimate (1.9) in §2.4.

2.1 Entropies and kinetic formulation

We rely on the notion of entropies for the eikonal equation, introduced in [14] in analogy with scalar conservation laws. These are the vector fields in the class

$$\mathrm{ENT} = \Big\{ \Phi \in C^1(\mathbb{S}^1;\mathbb{R}^2) \colon \exists \lambda_\Phi \in C^0(\mathbb{S}^1;\mathbb{R}), \\ \frac{d}{d\theta}\Phi(e^{i\theta}) = \lambda_\Phi(e^{i\theta}) i e^{i\theta} \quad \forall \theta \in \mathbb{R} \Big\}. \tag{2.1}$$

Their relevance comes from a direct application of the chain rule, which shows that, for all $u \in H^1(\Omega;\mathbb{S}^1)$, we have

$$\operatorname{div} \Phi(u) = \lambda_\Phi(u) \operatorname{div} u \qquad \forall \Phi \in \mathrm{ENT}. \tag{2.2}$$

In the context of hyperbolic conservation laws, it was first observed in [33] that information about a large family of entropy productions, as in (2.2), can be succinctly expressed as a kinetic formulation, see also [37]. In the context of divergence-free unit vector fields, a kinetic formulation was introduced in [30] and reformulated in [17], for the indicator function

$$\chi(s,x) = \mathbf{1}_{u(x)\cdot e^{is}>0}, \quad s \in \mathbb{T} = \mathbb{R}/2\pi\mathbb{Z},\ x \in \Omega, \tag{2.3}$$

where $z_1 \cdot z_2 = \Re\mathrm{e}(\bar{z}_1 z_2)$ is the scalar product of $z_1, z_2 \in \mathbb{C}$. We adapt it here to our setting.

Lemma 2.1. *For any $u \in H_{\mathrm{div}}(\Omega;\mathbb{S}^1)$ satisfying* (2.2)*, the indicator function $\chi \in L^\infty(\mathbb{T}\times\Omega)$ defined in* (2.3) *satisfies*

$$e^{is}\cdot\nabla_x\chi = \Theta \quad \text{in } \mathcal{D}'(\mathbb{T}\times\Omega)\,, \tag{2.4}$$

for $\Theta \in L^2(\Omega;\mathcal{M}(\mathbb{T}))$ given explicitly by

$$\Theta(s,x) = \Big(\delta_{\theta(x)+\frac{\pi}{2}}(s) + \delta_{\theta(x)-\frac{\pi}{2}}(s)\Big)\operatorname{div} u(x)\,,$$

where $\theta(x)\in\mathbb{T}$ is such that $u(x) = e^{i\theta(x)}$.

Proof of Lemma 2.1. Consider the family of entropies given by

$$\begin{aligned}\Phi_g(e^{i\theta}) &= \int_{\mathbb{T}} g(s)\mathbf{1}_{e^{is}\cdot e^{i\theta}>0}\, e^{is}\,ds\\ &= \int_{\theta-\frac{\pi}{2}}^{\theta+\frac{\pi}{2}} g(s)e^{is}\,ds\,, \qquad g\in C^0(\mathbb{T};\mathbb{R})\,,\end{aligned} \tag{2.5}$$

which satisfies (2.1) with $\lambda_{\Phi_g}(e^{i\theta}) = g(\theta+\pi/2)+g(\theta-\pi/2)$. The family (2.5) is naturally obtained by taking a 2π-periodic antiderivative of the identity $\partial_\theta\Phi = \lambda_\Phi ie^{i\theta}$ in the case where λ_Φ is π-periodic, and it has the advantage of being defined in terms of the singular entropies

$$\Phi^{\{s\}}(z) = \mathbf{1}_{z\cdot e^{is}>0}e^{is}\,,$$

which play an important role in [14, 36]. This enables us to obtain the kinetic formulation (2.4). Namely, testing (2.2) against a test function $\zeta(x)$ for all entropies in the family (2.5), we have

$$\int_\Omega \langle \Phi_g(u(x)),\nabla\zeta(x)\rangle\,dx = \int_\Omega \lambda_{\Phi_g}(u(x))\operatorname{div} u(x)\,\zeta(x)\,dx\,.$$

Using (2.5) and the expression of

$$\lambda_{\Phi_g}(e^{i\theta}) = g(\theta+\frac{\pi}{2}) + g(\theta-\frac{\pi}{2}) = \int_{\mathbb{T}} g(s)(\delta_{\theta+\frac{\pi}{2}}+\delta_{\theta-\frac{\pi}{2}})(ds)\,,$$

this can be rewritten as

$$\int_{\mathbb{T}\times\Omega}\mathbf{1}_{u(x)\cdot e^{is}>0}\,e^{is}\cdot\nabla_x[g(s)\zeta(x)]\,dsdx = \langle\Theta, g(s)\zeta(x)\rangle\,,$$

for all $g\in C^0(\mathbb{T})$ and $\zeta\in C^1_c(\Omega)$, where $\langle\Theta,g(s)\zeta(x)\rangle$ denotes the value of the distribution $\Theta\in L^2(\Omega;\mathcal{M}(\mathbb{T}))\subset\mathcal{D}'(\mathbb{T}\times\Omega)$ applied to the function $(s,x)\mapsto g(s)\zeta(x)$. In other words, the indicator function $\chi(s,x) = \mathbf{1}_{u(x)\cdot e^{is}>0}$ satisfies the kinetic formulation (2.4). $\square$

Remark 2.2. For another interpretation of the kinetic formulation (2.4), note that

$$e^{is}\mathbf{1}_{e^{is}\cdot z>0} = \partial_s \Psi(s,z) + \frac{1}{\pi} z,$$

where $\Psi(s,\cdot)$ is the Lipschitz entropy given by

$$\Psi(s,z) = \int_{\mathbb{T}} f_0(t-s)\mathbf{1}_{e^{it}\cdot z>0}\, e^{it}\, dt, \qquad f_0(s) = \frac{s}{2\pi} \text{ for } 0 \le s < 2\pi\,.$$

Therefore we have

$$\begin{aligned} e^{is}\cdot\nabla_x\chi = \operatorname{div}[e^{is}\mathbf{1}_{e^{is}\cdot u>0}] &= \partial_s \operatorname{div}\Psi(s,u) + \frac{1}{\pi}\operatorname{div} u(x) \\ &= \left(\frac{1}{\pi} + \partial_s[\lambda_0(e^{-is}u)]\right)\operatorname{div} u, \end{aligned}$$

where $\lambda_0(e^{i\theta}) = f_0(\theta+\pi/2) + f_0(\theta-\pi/2)$. Since $f_0'(s) = 1/(2\pi) - \delta_0(s)$, this is exactly (2.4).

The kinetic formulation (2.4) is all we require to prove the compensated regularity estimate (1.9). In other words, we will obtain Theorem 1.1 as a consequence of the following slightly more general statement.

Theorem 2.3. *Let $u \in H_{\mathrm{div}}(\Omega;\mathbb{S}^1)$ be such that $\chi(s,x) = \mathbf{1}_{u(x)\cdot e^{is}>0}$ satisfies the kinetic formulation* (2.4)*. Then u satisfies the $B^{1/2}_{3,\infty,\mathrm{loc}}$ regularity estimate* (1.9).

Proof of Theorem 1.1 from Theorem 2.3. We have already seen that maps $u \in H^1(\Omega;\mathbb{S}^1)$ satisfy (2.2) and therefore (2.4), so Theorem 2.3 implies Theorem 1.1 in the case $u \in H^1(\Omega;\mathbb{S}^1)$. In order to treat the case $u \in \mathcal{S}$ we just need to check that Theorem 2.3 also implies that limits of bounded energy sequences satisfy (2.2) and therefore (2.4). By definition (1.5) of $\mathcal{S}$, there exists a sequence $(u_j) \subset H^1(\Omega;\mathbb{S}^1)$ such that $u_j \to u$ in $\mathcal{D}'(\Omega)$ and $E_{\epsilon_j}(u_j) \le C$ for some $\epsilon_j \to 0$. This energy bound implies $\|\operatorname{div} u_j\|^2_{L^2(\Omega)} \le C$, so, according to the next Lemma, u does indeed satisfy (2.2). □

Lemma 2.4. *If a sequence $(u_j) \subset H^1(\Omega;\mathbb{S}^1)$ satisfies $\sup_j \|\operatorname{div} u_j\|_{L^2(\Omega)} < \infty$, there exists $u \in B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1)$ with $\operatorname{div} u \in L^2(\Omega)$ such that*

$$\begin{aligned} &u_j \to u \qquad \text{in } L^1(\Omega) \text{ and a.e.}, \\ &\operatorname{div}\Phi(u_j) \rightharpoonup \operatorname{div}\Phi(u) = \lambda_\Phi(u)\operatorname{div} u \quad \text{weakly in } L^q(\Omega) \text{ for } 1<q<2\,, \end{aligned}$$

for all $\Phi \in \mathrm{ENT}$, along a non-relabeled subsequence.

Proof of Lemma 2.4 from Theorem 2.3. Since $u_j \in H^1(\Omega;\mathbb{S}^1)$, it satisfies (2.2) and therefore the kinetic formulation (2.4) by Lemma 2.1. We may therefore apply Theorem 2.3, ensuring that the sequence (u_j) is bounded in $B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1)$. By compactness of the embedding $B^{1/2}_{3,\infty,\mathrm{loc}} \subset L^1_{\mathrm{loc}}$, we can extract a subsequence converging a.e. in Ω, hence in $L^1(\Omega)$ by dominated convergence. Invoking also the weak compactness of the sequence $(\operatorname{div} u_j) \subset L^2(\Omega)$, we can therefore find $u \in B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1)$ such that $\operatorname{div} u \in L^2(\Omega)$ and

$$
\begin{aligned}
&u_j \to u \qquad \text{in } L^1(\Omega) \text{ and } a.e.,\\
&\operatorname{div} u_j \rightharpoonup \operatorname{div} u \qquad \text{weakly in } L^2(\Omega)\,,
\end{aligned}
$$

along a non-relabeled subsequence. Applying (2.2) to u_j, we have

$$
\operatorname{div}\Phi(u_j) = \lambda_\Phi(u_j)\operatorname{div} u_j \qquad \forall \Phi \in \mathrm{ENT}\,.
$$

Since $\operatorname{div} u_j \rightharpoonup \operatorname{div} u$ weakly in $L^2(\Omega)$ and $\lambda_\Phi(u_j) \to \lambda_\Phi(u)$ strongly in $L^p(\Omega)$ for any $p \in [1,\infty)$, we have

$$
\lambda_\Phi(u_j)\operatorname{div} u_j \rightharpoonup \lambda_\Phi(u)\operatorname{div} u \quad \text{weakly in } L^q(\Omega) \text{ for } 1 < q < 2\,.
$$

We also have $\operatorname{div}\Phi(u_j) \to \operatorname{div}\Phi(u)$ in $\mathcal{D}'(\Omega)$ since $\Phi(u_j) \to \Phi(u)$ in $L^1(\Omega)$, hence

$$
\operatorname{div}\Phi(u_j) \rightharpoonup \operatorname{div}\Phi(u) = \lambda_\Phi(u)\operatorname{div} u \quad \text{weakly in } L^q(\Omega) \text{ for } 1 < q < 2\,,
$$

for all $\Phi \in \mathrm{ENT}$. □

In the next subsections we provide the proof of Theorem 2.3, starting with the compensation identity relating finite differences $D^h_x\chi$ with $\operatorname{div} u$.

2.2 A compensation identity

We denote by T^h and D^h the translation and finite difference operators in the x variable:

$$
T^h f(x) = f(x+h), \quad D^h f = T^h f - f\,, \tag{2.6}
$$

for any function $f(x)$, and we will frequently use the identity

$$
D^h(fg) = f D^h g + T^h g D^h f\,. \tag{2.7}
$$

As noticed first in [25] in a slightly different context, the kinetic formulation (2.4) provides an identity for some bilinear functions of $D^h\chi$, see (2.8). This identity can be thought of as a generalization of the integration by parts at the origin of compensated compactness phenomena [35, 40], see Remark 2.5. Note also that this compensation identity does not use the specific form of the characteristic function χ in Lemma 2.1, but is valid for any function χ satisfying the kinetic formulation (2.4).

For any $\varphi \in L^1(\mathbb{T})$ odd and π-periodic, and for the indicator function $\chi(s,x) = \mathbf{1}_{u(x)\cdot e^{is}>0}$, we define the function

$$\Delta^{\varphi,\chi}(x,h) = \frac{1}{2}\int_{\mathbb{T}^2} \varphi(t-s) D^h\chi(t,x) D^h\chi(s,x) \sin(t-s)\, dtds, \tag{2.8}$$

on $\Omega^* = \{(x,h) \in \Omega \times \mathbb{R}^2 \colon x+h \in \Omega\}$. (Note that this quantity is zero if $\varphi \in L^1(\mathbb{T})$ is even or anti-π-periodic.) The same quantity (without the 1/2 factor) is considered in [17, § 3.2], for the specific choice of φ given by

$$\varphi_0(s) = \mathbf{1}_{\cos(s)\sin(s)>0} - \mathbf{1}_{\cos(s)\sin(s)<0}\,,$$

as can be seen by taking $(\xi,\eta) = (e^{is}, e^{it})$ in the expression of Δ in [17, § 3.2]. It is shown in [17, Lemma 3.8] that we have

$$\Delta^{\varphi_0,\chi}(x,h) \gtrsim |D^h u|^3(x)\,. \tag{2.9}$$

Given this coercivity property, it is clear that the regularity estimate (1.9) will follow from an estimate on the integral of $\Delta^{\varphi_0,\chi}$ in terms of $\operatorname{div} u$. This is made possible by Lemma 2.1 below, where we establish a compensation identity relating $\Delta^{\varphi,\chi}$ to $\operatorname{div} u$ via the distribution Θ in the kinetic formulation (2.4).

Remark 2.5. It was realized in [25] (and adapted to $\mathbb{S}^1$-valued maps in [17]) that the kinetic formulation (2.4) provides a compensation identity satisfied by the quantity $\Delta^{\varphi,\chi}$, and which allows one to estimate its integral in an efficient way. In the context of Burgers' equation, this compensation identity has been interpreted in [18] as a modification of the so-called Kármán-Howarth-Monin formula in fluid dynamics. Another way to understand why some compensations can be expected in the quantity $\Delta^{\varphi,\chi}$ is to rewrite it as

$$\Delta^{\varphi,\chi} = \int_{\mathbb{T}^2} \varphi(t-s) D^h[\Phi^{\{t\}}(u)] \wedge D^h[\Phi^{\{s\}}(u)]\, dtds\,,$$

where $\Phi^{\{s\}}(z) = e^{is}\mathbf{1}_{z\cdot e^{is}>0}$ is the singular entropy which already appeared in the proof of Lemma 2.1, and $X \wedge Y = \det(X,Y)$ for any $X, Y \in \mathbb{R}^2$. Classical div-curl estimates (see. e.g. [24, § 3.4]) provide improved control on the integral of $A \wedge B$ for vector fields A, B with controlled divergence, and would therefore provide control on the integral of $\Delta^{\varphi,\chi}$ if the entropies $\Phi^{\{s\}}$ were regular, as in [17, § 4.3] or [19, Proposition 6.1]. In that sense, the compensation identity (2.11) satisfied by $\Delta^{\varphi,\chi}$ can also be interpreted as a formula of div-curl type, adapted to these singular entropies.

Lemma 2.6. *Let $\Omega \subset \mathbb{R}^2$, $\mathbb{T} = \mathbb{R}/2\pi\mathbb{Z}$, $\chi \in L^\infty(\mathbb{T}\times\Omega)$ such that $x \mapsto \chi(s,x)$ is C^1, and $\Theta \in L^1(\mathbb{T}\times\Omega)$ such that*

$$e^{is}\cdot\nabla_x\chi = \Theta \quad \text{a.e. in } \mathbb{T}\times\Omega\,. \tag{2.10}$$

Then, for any $\varphi \in L^1(\mathbb{T})$ odd and π-periodic, the function $\Delta^{\varphi,\chi}$ defined in (2.8) *satisfies*

$$\frac{d}{d\tau}\Delta^{\varphi,\chi}(x,\tau\mathbf{e}_1) = I^\tau(x) + \operatorname{div}_x A^\tau(x)\,, \tag{2.11}$$

where I^τ and $A^\tau = (A_1^\tau, A_2^\tau)$ are given by

$$\begin{aligned} I^\tau(x) = &\int_{\mathbb{T}^2}(\Theta + T^{\tau\mathbf{e}_1}\Theta)(s,x)\varphi(t-s)D^{\tau\mathbf{e}_1}\chi(t,x)\sin(t)\,dsdt \\ &- D^{\tau\mathbf{e}_1}\left[\int_{\mathbb{T}^2}\Theta(s,x)\varphi(t-s)\chi(t,x)\sin(t)\,dsdt\right] \end{aligned} \tag{2.12}$$

$$A_1^\tau(x) = \int_{\mathbb{T}^2}\varphi(t-s)\sin(t)\cos(s)T^{\tau\mathbf{e}_1}\chi(t,x)D^{\tau\mathbf{e}_1}\chi(s,x)\,dsdt \tag{2.13}$$

$$A_2^\tau(x) = \int_{\mathbb{T}^2}\varphi(t-s)\sin(t)\ \sin(s)\,\chi(t,x)D^{\tau\mathbf{e}_1}\chi(s,x)\,dsdt\,. \tag{2.14}$$

Moreover, the identity (2.11) *is valid in the sense of distributions if $\chi \in L^\infty(\mathbb{T}\times\Omega)$ and $\Theta \in L^2(\Omega;\mathcal{M}(\mathbb{T}))$ satisfy* (2.10), *and $\varphi \in C^0(\mathbb{T})$ is odd and π-periodic.*

The compensation identity in Lemma 2.6 is the same as in [17, Lemma 3.8], but the quantity I^τ in (2.11) is rewritten in a way that makes further compensations appear. These further compensations were not needed in [17], but were already used in [34, Lemma 4.8].

Proof of Lemma 2.6. The last statement follows from applying the first statement to χ and Θ mollified with respect to x, which preserves (2.10), and passing to the limit in (2.11). Thus we assume that $x \mapsto \chi(s,x)$ is C^1 and that $\Theta \in L^1(\mathbb{T}\times\Omega)$.

The identity (2.11) follows from the calculations in [17, Lemma 3.9], which we reproduce here for the readers' convenience. To simplify notations we simply write $\Delta = \Delta^{\varphi,\chi}(x,\tau\mathbf{e}_1)$. First we have

$$\begin{aligned} \frac{d}{d\tau}\Delta = &\frac{1}{2}\int_{\mathbb{T}^2}\varphi(t-s)T^{\tau\mathbf{e}_1}\partial_1\chi(t,x)D^{\tau\mathbf{e}_1}\chi(s,x)\sin(t-s)\,dtds \\ &+ \frac{1}{2}\int_{\mathbb{T}^2}\varphi(t-s)D^{\tau\mathbf{e}_1}\chi(t,x)T^{\tau\mathbf{e}_1}\partial_1\chi(s,x)\sin(t-s)\,dtds\,. \end{aligned}$$

Changing variable $(s,t)\mapsto(t,s)$ in the second integral and using the fact that φ

is odd, this becomes

$$\begin{aligned}\frac{d}{d\tau}\Delta &= \int_{\mathbb{T}^2} \varphi(t-s) T^{\tau \mathbf{e}_1} \partial_1 \chi(t,x) D^{\tau \mathbf{e}_1} \chi(s,x) \sin(t-s)\, dtds \\ &= \int_{\mathbb{T}^2} \varphi(t-s) T^{\tau \mathbf{e}_1} \partial_1 \chi(t,x) D^{\tau \mathbf{e}_1} \chi(s,x) \sin(t)\cos(s)\, dtds \\ &\quad - \int_{\mathbb{T}^2} \varphi(t-s) T^{\tau \mathbf{e}_1} \partial_1 \chi(t,x) D^{\tau \mathbf{e}_1} \chi(s,x) \cos(t)\sin(s)\, dtds \\ &= \partial_1 A_1^\tau - \int_{\mathbb{T}^2} \varphi(t-s) T^{\tau \mathbf{e}_1} \chi(t,x) D^{\tau \mathbf{e}_1} \partial_1 \chi(s,x) \sin(t)\cos(s)\, dtds \\ &\quad - \int_{\mathbb{T}^2} \varphi(t-s) T^{\tau \mathbf{e}_1} \partial_1 \chi(t,x) D^{\tau \mathbf{e}_1} \chi(s,x) \cos(t)\sin(s)\, dtds\,,\end{aligned}$$

where A_1^τ the first component of the vector field A^τ in (2.11). Using again the oddness of φ and the change of variable $(s,t) \mapsto (t,s)$ in the last integral, we find

$$\begin{aligned}\frac{d}{d\tau}\Delta &= \partial_1 A_1^\tau + \int_{\mathbb{T}^2} \varphi(t-s) T^{\tau \mathbf{e}_1} \chi(t,x) \partial_1 \chi(s,x) \sin(t)\cos(s)\, dtds \\ &\quad - \int_{\mathbb{T}^2} \varphi(t-s) \chi(t,x) T^{\tau \mathbf{e}_1} \partial_1 \chi(s,x) \cos(s)\sin(t)\, dtds\,.\end{aligned}$$

Now we use the kinetic equation (2.10), which gives

$$\cos(s)\partial_1 \chi(s,x) = -\sin(s)\partial_2 \chi(s,x) + \Theta(s,x)\,,$$

to rewrite the above as

$$\begin{aligned}\frac{d}{d\tau}\Delta &= \partial_1 A_1^\tau + \int_{\mathbb{T}^2} \Theta(s,x) \varphi(t-s) T^{\tau \mathbf{e}_1} \chi(t,x) \sin(t)\, dtds \\ &\quad - \int_{\mathbb{T}^2} T^{\tau \mathbf{e}_1} \Theta(s,x) \varphi(t-s) \chi(t,x) \sin(t)\, dtds \\ &\quad - \int_{\mathbb{T}^2} \varphi(t-s) T^{\tau \mathbf{e}_1} \chi(t,x) \partial_2 \chi(s,x) \sin(t)\sin(s)\, dtds \\ &\quad + \int_{\mathbb{T}^2} \varphi(t-s) \chi(t,x) T^{\tau \mathbf{e}_1} \partial_2 \chi(s,x) \sin(s)\sin(t)\, dtds\,.\end{aligned}$$

Using again the oddness of φ and the change of variable $(s,t) \mapsto (t,s)$, we see that the last two lines correspond to $\partial_2 A_2^\tau$, where A_2^τ is the second component of the vector field A^τ in (2.11). Rewriting slightly the first two integrals, we are left with

$$\begin{aligned}\frac{d}{d\tau}\Delta &= \operatorname{div} A^\tau + \int_{\mathbb{T}^2} \Theta(s,x) \varphi(t-s) D^{\tau \mathbf{e}_1} \chi(t,x) \sin(t)\, dtds \\ &\quad - \int_{\mathbb{T}^2} D^{\tau \mathbf{e}_1} \Theta(s,x) \varphi(t-s) \chi(t,x) \sin(t)\, dtds\,.\end{aligned}$$

Using the identity (2.7) to rewrite the last line, we obtain exactly (2.11). $\square$

2.3 Estimating the right-hand side of the compensation identity

For any odd π-periodic $\varphi \in C^0(\mathbb{R})$ we can apply the compensation identity of Lemma 2.6 to the kinetic formulation (2.4) obtained in Lemma 2.1. In this subsection we estimate the quantities I^τ and A^τ appearing in (2.11).

Lemma 2.7. *Let $u\colon \Omega \to \mathbb{S}^1$ measurable and $\chi(s,x) = \mathbf{1}_{e^{is}\cdot u(x)>0}$. Then the vector field A^τ in* (2.11) *satisfies*

$$|A^\tau| \lesssim \|\varphi\|_{L^1(\mathbb{T})}\, |D^{\tau e_1} u| \qquad \text{a.e. in } \Omega \cap (\Omega - \tau \mathbf{e}_1)\,.$$

Proof of Lemma 2.7. Recalling the definitions of A_1^τ and A_2^τ from (2.13) and (2.14), as well as the finite difference operator D^h and the translation operator T^h defined in (2.6), one finds that A_1^τ and A_2^τ can be rewritten as

$$\begin{aligned} A_1^\tau &= F_1(T^{\tau\mathbf{e}_1}u, T^{\tau\mathbf{e}_1}u) - F_1(T^{\tau\mathbf{e}_1}u, u)\,,\\ A_2^\tau &= F_2(u, T^{\tau\mathbf{e}_1}u) - F_2(u,u)\,, \end{aligned}$$

where $F_1, F_2\colon \mathbb{S}^1 \times \mathbb{S}^1 \to \mathbb{R}$ are given by

$$\begin{aligned} F_1(e^{i\theta}, e^{i\alpha}) &= \int_{\alpha-\frac{\pi}{2}}^{\alpha+\frac{\pi}{2}} \int_{\theta-\frac{\pi}{2}}^{\theta+\frac{\pi}{2}} \varphi(t-s)\sin(t)\,dt\,\cos(s)\,ds\,,\\ F_2(e^{i\theta}, e^{i\alpha}) &= \int_{\alpha-\frac{\pi}{2}}^{\alpha+\frac{\pi}{2}} \int_{\theta-\frac{\pi}{2}}^{\theta+\frac{\pi}{2}} \varphi(t-s)\sin(t)\,dt\,\sin(s)\,ds\,. \end{aligned}$$

These satisfy

$$\left|\frac{d}{d\alpha} F_j(e^{i\theta}, e^{i\alpha})\right| \lesssim \int_{\mathbb{T}} |\varphi|\,ds,$$

and this implies the claimed inequality on A^τ. $\square$

Lemma 2.8. *Let $u \in H_{\mathrm{div}}(\Omega;\mathbb{S}^1)$ and $\chi(s,x) = \mathbf{1}_{e^{is}\cdot u(x)>0}$. Then, for any $\eta \in C_c^1(\Omega;[0,1])$, the function I^τ in* (2.11) *satisfies*

$$\begin{aligned} \int_\Omega I^\tau\, \eta^2\,dx &\lesssim \|\varphi\|_{L^\infty(\mathbb{T})} \big\|\eta^2 |D^{\tau\mathbf{e}_1}u|\big\|_{L^2(\Omega)} \|\operatorname{div} u\|_{L^2(\Omega)} \\ &\quad + |\tau| \|\varphi\|_{L^1(\mathbb{T})} \|\nabla \eta\|_\infty \|\operatorname{div} u\|_{L^1(\Omega)}\,, \end{aligned}$$

for $|\tau| \le \operatorname{dist}(\operatorname{supp}(\eta), \partial\Omega)$.

Proof. Integrating the expression (2.12) for I^τ against η^2 and performing a discrete integration by parts using (2.7), we obtain

$$\begin{aligned} \int_\Omega I^\tau\,\eta^2\,dx &= \int_\Omega \int_{\mathbb{T}} (\Theta + T^{\tau\mathbf{e}_1}\Theta)(s,x) D^{\tau\mathbf{e}_1}\Lambda(s,x)\,ds\,\eta^2\,dx \\ &\quad + \int_\Omega \int_{\mathbb{T}} \Theta(s,x) T^{\tau\mathbf{e}_1}\Lambda(s,x)\,ds\, D^{\tau\mathbf{e}_1}[\eta^2]\,dx\,, \end{aligned}$$

$$\text{where } \Lambda(s,x) = \int_{\mathbb{T}} \varphi(t-s)\chi(t,x)\sin(t)\,dt\,.$$

Recalling that $\chi(t,x) = \mathbf{1}_{e^{it}\cdot u(x)>0}$, we see that

$$\Lambda(s,x) = F(e^{is}, u(x)), \quad F(e^{is}, e^{i\theta}) = \int_{\theta-\frac{\pi}{2}}^{\theta+\frac{\pi}{2}} \varphi(t-s)\sin(t)\,dt\,.$$

(Note that F is well-defined: this expression does not depend on the choice of s modulo 2π since φ is 2π-periodic, nor on the choice of θ modulo 2π since $\int_0^{2\pi}\varphi(t-s)\sin(t)\,dt = 0$ as a consequence of the π-periodicity of φ.) From the inequalities

$$|F| \le \|\varphi\|_{L^1(\mathbb{T})}, \quad \left|\frac{d}{d\theta}F(e^{is}, e^{i\theta})\right| \le 2\|\varphi\|_{L^\infty(\mathbb{T})}\,,$$

we deduce therefore

$$\sup_{s\in\mathbb{T}} |\Lambda(s,x)| \lesssim \|\varphi\|_{L^1(\mathbb{T})}\,, \quad \sup_{s\in\mathbb{T}} |D^{\tau\mathbf{e}_1}\Lambda(s,x)| \lesssim \|\varphi\|_{L^\infty(\mathbb{T})}|D^{\tau\mathbf{e}_1}u|(x)\,.$$

Since the explicit expression

$$\Theta(s,x) = \big(\delta_{\theta(x)+\frac{\pi}{2}}(s) + \delta_{\theta(x)-\frac{\pi}{2}}(s)\big)\operatorname{div} u(x)\,,$$

obtained in Lemma 2.1 implies

$$\int_{\mathbb{T}} \Theta(s,x)G(s,y)\,ds \le 2|\operatorname{div} u|(x)\sup_{s\in\mathbb{T}}|G(s,y)|\,,$$

for any bounded function G, we deduce that

$$\begin{aligned}\int_\Omega I\,\eta^2\,dx \lesssim \|\varphi\|_{L^\infty(\mathbb{T})} &\int_\Omega \big(|\operatorname{div} u| + T^{\tau\mathbf{e}_1}|\operatorname{div} u|\big)|D^{\tau\mathbf{e}_1}u|\,\eta^2\,dx \\ &+ \|\varphi\|_{L^1(\mathbb{T})}\int_\Omega |\operatorname{div} u|\,|D^{\tau\mathbf{e}_1}[\eta^2]|\,dx.\end{aligned}$$

Using Cauchy-Schwarz' inequality and the fact that $|D^{\tau\mathbf{e}_1}[\eta^2]| \le 2|\tau|\|\nabla\eta\|_\infty$ since $|\eta| \le 1$, we infer

$$\begin{aligned}\int_\Omega I\,\eta^2\,dx \lesssim\; &\|\varphi\|_{L^\infty(\mathbb{T})}\|\operatorname{div} u\|_{L^2(\operatorname{supp}(\eta))}\|\eta^2|D^{\tau\mathbf{e}_1}u|\|_{L^2(\Omega)} \\ &+ \|\varphi\|_{L^\infty(\mathbb{T})}\|T^{\tau\mathbf{e}_1}|\operatorname{div} u|\|_{L^2(\operatorname{supp}(\eta))}\|\eta^2|D^{\tau\mathbf{e}_1}u|\|_{L^2(\Omega)} \\ &+ |\tau|\|\varphi\|_{L^1(\mathbb{T})}\|\nabla\eta\|_\infty\|\operatorname{div} u\|_{L^1(\Omega)},\end{aligned}$$

which implies the conclusion since $\|T^{\tau\mathbf{e}_1}|\operatorname{div} u|\|_{L^2(\operatorname{supp}(\eta))} \le \|\operatorname{div} u\|_{L^2(\Omega)}$. $\square$

2.4 Proof of the regularity estimate

Proof of Theorem 2.3. Thanks to Lemma 2.6, for the functions χ and Θ given by Lemma 2.1, and any odd, π-periodic $\varphi \in C^0(\mathbb{T})$, we have the compensation identity (2.11). Integrating this compensation identity against $\eta^2\,dx$ for some $\eta \in C_c^1(\Omega;[0,1])$ and then with respect to τ, we deduce

$$\int_\Omega \Delta^{\varphi,\chi}(x,\tau\mathbf{e}_1)\eta^2\,dx = \int_0^\tau \left(\int_\Omega I^{\tau'}\eta^2\,dx\,d\tau' - \int_\Omega \langle A^{\tau'},\nabla\eta\rangle\eta\,dx\right)d\tau'\,.$$

Thanks to Lemma 2.8 we have

$$\begin{aligned}\int_\Omega I^{\tau'}\eta^2\,dx &\lesssim |\tau'|\|\varphi\|_{L^1(\mathbb{T})}\|\nabla\eta\|_\infty\|\operatorname{div} u\|_{L^1(\Omega)} \\ &\quad + \|\varphi\|_{L^\infty(\mathbb{T})}\|\operatorname{div} u\|_{L^2(\Omega)}\left\|\eta^2|D^{\tau'\mathbf{e}_1}u|\right\|_{L^2(\Omega)}\,,\end{aligned}$$

and thanks to Lemma 2.7 we have

$$-\int_\Omega \langle A^{\tau'},\nabla\eta\rangle\eta\,dx \lesssim \|\varphi\|_{L^1(\mathbb{T})}\|\nabla\eta\|_\infty\|\eta|D^{\tau'\mathbf{e}_1}u|\|_{L^1}\,.$$

Plugging these two inequalities into the previous identity and using also that $\|\varphi\|_{L^1} \lesssim \|\varphi\|_{L^\infty}$, we obtain

$$\begin{aligned}\int_\Omega \Delta^{\varphi,\chi}(x,\tau\mathbf{e}_1)\eta^2\,dx &\lesssim \tau^2\|\varphi\|_{L^\infty}\|\nabla\eta\|_\infty\|\operatorname{div} u\|_{L^1} \\ &\quad + |\tau|\|\varphi\|_{L^\infty}\|\operatorname{div} u\|_{L^2}\sup_{|\tau'|\le|\tau|}\|\eta^2|D^{\tau'\mathbf{e}_1}u|\|_{L^2} \\ &\quad + |\tau|\|\varphi\|_{L^\infty}\|\nabla\eta\|_\infty\sup_{|\tau'|\le|\tau|}\|\eta|D^{\tau'\mathbf{e}_1}u|\|_{L^1}\,, \end{aligned}\tag{2.15}$$

for $|\tau| \le \operatorname{dist}(\operatorname{supp}(\eta),\partial\Omega)$. We have established this inequality for any continuous, odd and π-periodic function φ, but claim that it is also valid for any odd, π-periodic function $\varphi \in L^\infty(\mathbb{T})$. Consider indeed a sequence of smooth, odd, π-periodic functions $\varphi_j \to \varphi$ a.e. on $\mathbb{T}$ and such that $\|\varphi_j\|_{L^\infty} \le \|\varphi\|_{L^\infty}$ and apply (2.15) to φ_j. The right-hand side depends linearly on $\|\varphi_j\|_{L^\infty} \le \|\varphi\|_{L^\infty}$, and the left-hand side converges by dominated convergence, so we do conclude that (2.15) is satisfied by the limit map $\varphi \in L^\infty(\mathbb{T})$.

Next we make a specific choice of such a function, given by

$$\varphi(t) = 1 \quad \text{for } 0 < t \le \frac{\pi}{2}\,,$$

and extended as an odd π-periodic function. As recalled above (2.9), for this choice of function φ, the quantity $\Delta^{\varphi,\chi}$ satisfies

$$\Delta^{\varphi,\chi}(x,h) \gtrsim |D^h u(x)|^3\,,$$

hence (2.15) implies

$$\int_\Omega |D^{\tau \mathbf{e}_1} u|^3 \eta^2 \, dx \lesssim \tau^2 \|\nabla \eta\|_\infty \| \operatorname{div} u\|_{L^1} \\ + |\tau| \| \operatorname{div} u\|_{L^2} \sup_{|\tau'| \le |\tau|} \|\eta^2 |D^{\tau' \mathbf{e}_1} u|\|_{L^2} \\ + |\tau| \|\nabla \eta\|_\infty \sup_{|\tau'| \le |\tau|} \|\eta |D^{\tau' \mathbf{e}_1} u|\|_{L^1} \, .$$

Setting

$$X := \sup_{|\tau'| \le |\tau|} \|\eta |D^{\tau' \mathbf{e}_1} u|\|_{L^3} \, ,$$

and using Hölder's inequality and the fact that $|\eta| \le 1$, we deduce

$$X^3 \lesssim \tau^2 \|\nabla \eta\|_\infty |\Omega|^{\frac{1}{2}} \| \operatorname{div} u\|_{L^2} + |\tau| \| \operatorname{div} u\|_{L^2} |\Omega|^{\frac{1}{6}} X + |\tau| \|\nabla \eta\|_\infty |\Omega|^{\frac{2}{3}} X \, .$$

To estimate the last two terms we use Young's inequality

$$ab \le \frac{2}{3} \lambda^{-\frac{1}{2}} a^{\frac{3}{2}} + \frac{\lambda}{3} b^3 \qquad \forall a, b \ge 0 \, , \lambda > 0 \, .$$

We obtain, for any $\lambda > 0$,

$$X^3 \lesssim \tau^2 \|\nabla \eta\|_\infty |\Omega|^{\frac{1}{2}} \| \operatorname{div} u\|_{L^2} + \lambda^{-\frac{1}{2}} |\tau|^{\frac{3}{2}} \left(\| \operatorname{div} u\|_{L^2}^{\frac{3}{2}} |\Omega|^{\frac{1}{4}} + \|\nabla \eta\|_\infty^{\frac{3}{2}} |\Omega| \right) \\ + 2\lambda X^3 \, .$$

Choosing λ small enough, we infer

$$X^3 \lesssim C \, (1 + \| \operatorname{div} u\|_{L^2})^{\frac{3}{2}} |\tau|^{\frac{3}{2}} \, ,$$

for some $C = C(\Omega, \eta) > 0$. Recall $\Omega' \subset\subset \Omega$. Fixing $\eta \in C_c^1(\Omega; [0,1])$ such that $\eta = 1$ on Ω', this gives exactly the estimate (1.9), that is,

$$\|D^h u\|_{L^3(\Omega' \cap (\Omega' - h))} \le C (1 + \| \operatorname{div} u\|_{L^2(\Omega)})^{\frac{1}{2}} |h|^{\frac{1}{2}} \, ,$$

for $h = \tau \mathbf{e}_1$ such that $|h| \le \delta_0$ for some $\delta_0 = \delta_0(\Omega, \Omega') > 0$. That last restriction can be automatically removed, using simply for $|h| \ge \delta_0$ that $|D^h u| \le 2$. So this is valid for all $h = \tau \mathbf{e}_1$, $\tau \in \mathbb{R}$. Finally, the direction $\mathbf{e}_1$ does not play a particular role, we can apply this same estimate to $\tilde{u}(x) = R^{-1} u(Rx)$ for any rotation R (noting that this preserves the L^2 norm of the divergence) and deduce (1.9) for $h = \tau R \mathbf{e}_1$, hence for all $h \in \mathbb{R}^2$. $\square$

2.5 Proof of Corollary 1.4

In this section we give the proof of Corollary 1.4 about the Jacobian.

Proof of Corollary 1.4. Recall that the Jacobian distribution Ju is well-defined and continuous on $W^{1/3,3}_{\mathrm{loc}}(\Omega;\mathbb{S}^1)$ thanks to [3, Corollary 8.1], coincides with $Ju = (1/2)\,\mathrm{curl}(u\wedge\nabla u)$ for $u\in W^{1,1}_{\mathrm{loc}}\cap L^\infty$, and with $Ju=\det(\nabla u)=0$ for $u\in H^1_{\mathrm{loc}}(\Omega;\mathbb{S}^1)$.

Let $u\in\mathcal{S}$, then there exists $u_\epsilon\to u$ in $\mathcal{D}'(\Omega)$ such that $\sup E_\epsilon(u_\epsilon)<\infty$ along a sequence $\epsilon\to 0$. Thanks to Corollary 1.2 we have $u_\epsilon\to u$ in $W^{1/3,3}(\Omega')$ and therefore $Ju=\lim Ju_\epsilon = 0$ in $\mathcal{D}'(\Omega')$ for every $\Omega'\subset\subset\Omega$, hence $Ju=0$ in $\mathcal{D}'(\Omega)$.

We also note that since $B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega)\subset W^{3/7,3}_{\mathrm{loc}}(\Omega)$, we have $u\in W^{3/7,3}_{\mathrm{loc}}(\Omega;\mathbb{S}^1)$ thanks to Theorem 1.1, so [3, Corollary 14.3] implies the existence of $v\in C^\infty(\Omega;\mathbb{S}^1)$ and $\varphi\in W^{2/5,3}_{\mathrm{loc}}(\Omega;\mathbb{R})$ such that $u=e^{i\varphi}v$. (Here we used that the regularity $W^{3/7,3}+W^{1,9/7}$ given by [3, Corollary 14.3] for the function φ implies $W^{2/5,3}$ regularity since $3/7>2/5$.)

Let $\Gamma\subset\Omega$ be a smooth curve homeomorphic to a circle. Choose any $\delta>0$ such that the δ-neighborhood $\omega_\delta=\{x\in\mathbb{R}^2\colon\ \mathrm{dist}(x,\Gamma)\le\delta\}$ of Γ is contained inside Ω and there is a C^1-diffeomorphism $\Theta\colon\Gamma\times[-\delta,\delta]\to\omega_\delta$ such that $\Theta(\Gamma\times\{0\})=\Gamma$. Then, letting $\Gamma_t=\Theta(\Gamma\times\{t\})$, by Fubini's theorem we have $\varphi\in W^{2/5,3}(\Gamma_t;\mathbb{R})\subset C^0(\Gamma_t;\mathbb{R})$ for a.e. $t\in(-\delta,\delta)$. Hence there is a well-defined winding number $\deg(u;\Gamma_t)=\deg(v;\Gamma_t)=\deg(v;\Gamma)$. This winding number being independent of t and of the choice of the diffeomorphism Θ, this gives a well-defined meaning to $\deg(u;\Gamma)$. Moreover, for any sequence $u_\epsilon\to u$ with $E_\epsilon(u_\epsilon)\le C$, thanks to the $W^{2/5,3}_{\mathrm{loc}}$ convergence provided by Corollary 1.2 and invoking again Fubini and the embedding $W^{2/5,3}(\Gamma_t)\subset C^0(\Gamma_t)$, for a.e. $t\in(-\delta,\delta)$ we have uniform convergence of u_ϵ on Γ_t along a subsequence, hence $\deg(u_\epsilon;\Gamma)=\deg(u_\epsilon;\Gamma_t)\to\deg(u;\Gamma_t)=\deg(u;\Gamma)$. $\square$

3 Compactness of traces

In this section we prove Theorem 1.5.

Let $(u_j)\subset H^1(\Omega;\mathbb{S}^1)$ be such that $\sup_j\|\operatorname{div}u_j\|_{L^2(\Omega)}<\infty$. Then from Lemma 2.4, we have, along a non-relabeled subsequence,

$$\begin{aligned}
&u_j\to u \qquad \text{in } L^1_{\mathrm{loc}}(\Omega) \text{ and } a.e.,\\
&\operatorname{div}\Phi(u_j)\rightharpoonup\operatorname{div}\Phi(u)=\lambda_\Phi(u)\operatorname{div}u \quad \text{weakly in } L^q(\Omega) \text{ for } 1<q<2\,,
\end{aligned}$$

for all $\Phi\in\mathrm{ENT}$ (2.1). In particular the limit map $u\in H_{\mathrm{div}}(\Omega;\mathbb{S}^1)$ satisfies the chain rule property (2.2) and therefore the kinetic formulation (2.4). The arguments in [42] thus ensure that u has a trace on $\partial\Omega$, attained in the strong L^1 sense.

Proposition 3.1. *If* $u \in B^{1/2}_{3,\infty,\mathrm{loc}}(\Omega;\mathbb{S}^1)$ *satisfies* $\operatorname{div} u \in L^2(\Omega)$ *and* $\operatorname{div}\Phi(u) = \lambda_\Phi(u)\operatorname{div} u$ *for all* $\Phi \in \mathrm{ENT}$*, then* u *admits a strong trace on* $\partial\Omega$*, that is, there exists a measurable map* $\mathrm{tr}(u)\colon \partial\Omega \to \mathbb{S}^1$ *such that*

$$\operatorname*{ess\,lim}_{\delta\to 0} \int_{\partial\Omega} |u(x-\delta\nu(x)) - \mathrm{tr}(u)(x)|\, d\mathcal{H}^1(x) = 0\,, \tag{3.1}$$

where $\nu(x)$ *denote the exterior unit normal to* $x \in \partial\Omega$*, and* $\operatorname{ess\,lim}$ *denotes the limit as* $\delta \to 0$ *with* δ *outside a negligible subset of* $(0,\delta_0)$*, for a small enough* $\delta_0 > 0$*.*

The proof of Proposition 3.1 is a direct adaptation of the arguments in [42]. For the readers' convenience, we sketch this adaptation in Appendix A. Next we establish that traces of bounded energy sequences converge strongly to the trace of their limit.

Lemma 3.2. *If* (u_j) *and* u *are as in Lemma 2.4, then* $\mathrm{tr}(u_j) \to \mathrm{tr}(u)$ *in* $L^1(\partial\Omega)$*.*

Proof of Lemma 3.2. We consider any Young measure $\{\mu_x\} \in \mathcal{P}(\mathbb{R}^2)^{\partial\Omega}$ generated by the bounded sequence $\{\mathrm{tr}(u_j)\} \subset L^\infty(\partial\Omega)$ and prove that μ_x is, for $\mathcal{H}^1$-almost every $x \in \partial\Omega$, a Dirac mass at $\mathrm{tr}(u)(x)$, which implies the strong convergence $\mathrm{tr}(u_j) \to \mathrm{tr}(u)$ in $L^2(\partial\Omega)$, see e.g. [16, § 1.5.3]. Here, recall that μ_x is characterized by the convergence, along a (nonrelabeled) subsequence $j \to \infty$,

$$\int_{\partial\Omega} \zeta\, F(\mathrm{tr}(u_j))\, d\mathcal{H}^1 \to \int_{\partial\Omega} \zeta(x) \int_{\mathbb{R}^2} F(y)\, d\mu_x(y)\, d\mathcal{H}^1(x)\,,$$

for all $F \in C^0(\mathbb{R}^2)$ and $\zeta \in L^1(\partial\Omega)$.

Thanks to Proposition 3.1, u has a strong trace $\mathrm{tr}(u)$ on $\partial\Omega$, see (3.1), and it satisfies the Gauss-Green identity

$$\int_\Omega \eta \operatorname{div}\Phi(u)\, dx + \int_\Omega \langle \nabla\eta, \Phi(u)\rangle\, dx = \int_{\partial\Omega} \eta\, \langle \nu, \Phi(\mathrm{tr}(u))\rangle\, d\mathcal{H}^1\,,$$

for all $\eta \in C^1_c(\mathbb{R}^2)$, see [8]. We can also apply this formula to $u_j \in H^1(\Omega;\mathbb{S}^1)$. Using that $\operatorname{div}\Phi(u_j) \rightharpoonup \operatorname{div}\Phi(u)$ weakly in $L^{3/2}(\Omega)$ and $\Phi(u_j) \to \Phi(u)$ strongly in $L^1(\Omega)$, we deduce

$$\int_{\partial\Omega} \eta\langle \nu, \Phi(\mathrm{tr}(u_j))\rangle\, d\mathcal{H}^1 \to \int_{\partial\Omega} \eta\langle \nu, \Phi(\mathrm{tr}(u))\rangle\, d\mathcal{H}^1\,,$$

for all $\eta \in C^1_c(\mathbb{R}^2)$. By definition of the Young measure μ_x, this implies

$$\int_{\partial\Omega} \eta(x)\Big\langle \nu(x), \int_{\mathbb{R}^2} \widetilde{\Phi}(y) d\mu_x(y) - \Phi(\mathrm{tr}(u)(x))\Big\rangle\, d\mathcal{H}^1(x) = 0,$$

for all $\eta \in C_c^1(\mathbb{R}^2)$ and $\Phi \in \mathrm{ENT}$, and any extension $\widetilde{\Phi} \in C_c^1(\mathbb{R}^2;\mathbb{R}^2)$ such that $\widetilde{\Phi}_{|\mathbb{S}^1} = \Phi$. This being valid for all η, we infer

$$\Big\langle \nu(x), \int_{\mathbb{R}^2} \widetilde{\Phi}(y) d\mu_x(y) - \Phi(\mathrm{tr}(u)(x)) \Big\rangle = 0\,, \qquad \text{for a.e. } x \in \partial\Omega\,. \tag{3.2}$$

We fix $x \in \partial\Omega$ such that this identity is satisfied. First note that we may add to the extension $\widetilde{\Phi}$ any map $\Psi \in C_c^1(\mathbb{R}^2)$ such that $\Psi_{|\mathbb{S}^1} = 0$. This implies that

$$\Big\langle \nu(x), \int_{\mathbb{R}^2} \Psi(y) d\mu_x(y) \Big\rangle = 0\,, \qquad \forall \Psi \in C_c^1(\mathbb{R}^2) \text{ with } \Psi_{|\mathbb{S}^1} = 0\,.$$

As a consequence, μ_x is supported in $\mathbb{S}^1$. Then, applying (3.2) to the entropies Φ_g given in (2.5), we obtain

$$\int_{\mathbb{T}} g(s)\, \langle e^{is}, \nu(x) \rangle \bigg(\int_{\mathbb{S}^1} \mathbf{1}_{e^{is}\cdot y>0}\, d\mu_x(y) - \mathbf{1}_{e^{is}\cdot \mathrm{tr}(u)(x)>0} \bigg)\, ds = 0\,,$$

for all $g \in C^0(\mathbb{T})$, and therefore, letting $\xi = \mathrm{tr}(u)(x) \in \mathbb{S}^1$,

$$\int_{\mathbb{S}^1} \mathbf{1}_{e^{is}\cdot y>0}\, d\mu_x(y) = \mathbf{1}_{e^{is}\cdot\xi>0} \qquad \text{for a.e. } s \in \mathbb{T}\,,$$

so the semi-circle $C_s^+ = \{y \in \mathbb{S}^1 \colon e^{is}\cdot y > 0\}$ satisfies

$$\mu_x(C_s^+) = 0 \qquad \text{for a.e. } s \in \mathbb{T}_\xi^- = \{s \in \mathbb{T} \colon e^{is}\cdot\xi < 0\}\,.$$

Applying this to a dense sequence $\{s_k\} \subset \mathbb{T}_\xi^-$ we deduce that

$$\mu_x(\mathbb{S}^1 \setminus \{\xi\}) = \mu_x\Big(\bigcup_k C_{s_k}^+ \Big) = 0\,.$$

We conclude that the Young measure μ_x is a Dirac mass at $\xi = \mathrm{tr}(u)(x)$ for a.e. $x \in \partial\Omega$, and therefore $\mathrm{tr}(u_j) \to \mathrm{tr}(u)$ strongly in $L^1(\partial\Omega)$. □

Proof of Theorem 1.5. Theorem 1.5 follows directly from Lemma 2.4, Proposition 3.1 and Lemma 3.2. □

4 Recovery sequences

In this section we prove Theorem 1.6. It relies on the properties of the convolution

$$u_\delta = u * \rho_\delta \qquad \text{in } \Omega_\delta = \{x \in \Omega \colon\ \mathrm{dist}(x, \partial\Omega) > \delta\}\,, \tag{4.1}$$

where $\rho_\delta(x) = \delta^{-2}\rho(x/\delta)$ and $\rho \in C_c^1(B_1;[0,1])$ satisfies $|\nabla\rho| \le 2$ and $\int_{B_1} \rho\, dx = 1$. We will obtain the recovery sequence by projecting u_δ onto $\mathbb{S}^1$, namely setting

$v_\delta = u_\delta/|u_\delta|$, and then modifying this map near the boundary in order to extend it to the whole Ω instead of the smaller domain Ω_δ. The main step is to establish convergence of v_δ and $\operatorname{div} v_\delta$ in this smaller domain. This is done in the following lemma, by relying on commutator estimates which provide an improved control on the oscillations of $|u_\delta|^2$ thanks to the commutator structure $1-|u_\delta|^2 = |u|^2_\delta - |u_\delta|^2$.

Proposition 4.1. *Assume $u\colon \Omega \to \mathbb{S}^1$ is such that $\operatorname{div} u \in L^2(\Omega)$, and it satisfies* (1.10)*, that is,*

$$\fint_{B_\delta} \left(\frac{\|D^h u\|_{L^4(\Omega\cap(\Omega-h))}}{\delta^{1/2}} \right)^4 dh \longrightarrow 0 \qquad \text{as } \delta \to 0\,.$$

Then, for small enough $\delta > 0$, the convolution u_δ defined in (4.1) *does not vanish in Ω_δ, and the map $v_\delta = u_\delta/|u_\delta| \in C^1(\Omega_\delta;\mathbb{S}^1)$ satisfies*

$$\int_{\Omega_\delta} |v_\delta - u|^2\, dx \to 0 \quad \text{and} \quad \int_{\Omega_\delta} (\operatorname{div} v_\delta - \operatorname{div} u)^2\, dx \to 0\,,$$

as $\delta \to 0$.

Proposition 4.1 relies on the chain rule and on the properties of $|u_\delta|$, namely:

Lemma 4.2. *Assume $u\colon \Omega \to \mathbb{S}^1$ satisfies* (1.10)*, then the convolution u_δ defined in* (4.1) *satisfies*

$$\sup_{\Omega_\delta} \left|1 - |u_\delta|^2\right| \to 0 \quad \text{and} \quad \int_{\Omega_\delta} |\nabla[|u_\delta|^2]|^2\, dx \to 0\,,$$

as $\delta \to 0$.

Proof of Lemma 4.2. One approach to establish the first property is to notice that u is VMO thanks to the assumption (1.10), and this is well-known to imply the uniform convergence of $\operatorname{dist}(u_\delta, \mathbb{S}^1)$ to zero, see e.g. [4, eqn.(7)]). The second property will follow from commutator estimates (as used e.g. in [10, 13]) for $1 - |u_\delta|^2 = |u|^2 * \rho_\delta - |u * \rho_\delta|^2$. We present here a self-contained proof of both properties relying on these commutator estimates.

We start from the identity

$$\begin{aligned} 1 - |u_\delta(x)|^2 &= |u(x-y)|^2 - |u_\delta(x)|^2 \\ &= |u(x-y) - u_\delta(x)|^2 + 2\langle u_\delta(x), u(x-y) - u_\delta(x)\rangle\,, \end{aligned}$$

valid for all $x \in \Omega_\delta$ and $y \in B_\delta$. Integrating with respect to $\rho_\delta(y)\, dy$, the last term integrates to zero and we are left with

$$\begin{aligned} 1 - |u_\delta(x)|^2 &= \int_{B_\delta} |u(x-y) - u_\delta(x)|^2 \rho_\delta(y)\, dy \\ &= \int_\Omega \left| \int_\Omega \big(u(y) - u(z)\big)\rho_\delta(x-z)\, dz \right|^2 \rho_\delta(x-y)\, dy \end{aligned} \tag{4.2}$$

As a first consequence of (4.2), we deduce, using Jensen's inequality and the fact that $\rho_\delta \leq \delta^{-2}$,

$$
\begin{aligned}
\big|1 - |u_\delta(x)|^2\big|^2 &\leq \int_\Omega \int_\Omega \big|u(y) - u(z)\big|^4 \rho_\delta(x-y)\, dy\, \rho_\delta(x-z)\, dz \\
&= \int_{B_\delta(x)} \int_{B_\delta(x-z)} \big|u(z+h) - u(z)\big|^4 \rho_\delta(x-z-h)\, dh\, \rho_\delta(x-z)\, dz \\
&\leq \frac{1}{\delta^4} \int_{B_\delta(x)} \int_{B_{2\delta}} \mathbf{1}_{z+h\in\Omega} \big|D^h u(z)\big|^4\, dh\, dz \\
&\leq \int_{B_{2\delta}} \frac{1}{\delta^4} \|D^h u\|^4_{L^4(\Omega\cap(\Omega-h))}\, dh \\
&= \pi \fint_{B_{2\delta}} \left(\frac{\|D^h u\|_{L^4(\Omega\cap(\Omega-h))}}{\delta^{1/2}} \right)^4 dh\,.
\end{aligned}
$$

This last quantity is independent of $x \in \Omega_\delta$, and tends to zero by assumption (1.10), hence the first convergence property.

To establish the convergence of $\nabla[|u_\delta|^2]$ in L^2, we differentiate (4.2) and obtain

$$
\begin{aligned}
-\nabla[|u_\delta|^2](x) = \frac{1}{\delta} &\int_\Omega \left| \int_\Omega \big(u(y) - u(z)\big) \rho_\delta(x-z)\, dz \right|^2 (\nabla\rho)_\delta(x-y)\, dy \\
+ \frac{2}{\delta} &\int_\Omega \Big\langle \int_\Omega \big(u(y) - u(z')\big) \rho_\delta(x-z')\, dz', \\
&\qquad \int_\Omega \big(u(y) - u(z)\big)(\nabla\rho)_\delta(x-z)\, dz \Big\rangle\, \rho_\delta(x-y)\, dy\,.
\end{aligned}
$$

Recalling that $|\nabla\rho| \leq 2$ and using again Jensen's inequality this implies

$$
\begin{aligned}
\big|\nabla[|u_\delta|^2]\big|^2(x) &\lesssim \frac{1}{\delta^2} \fint_{B_\delta(x)} \fint_{B_\delta(x)} |u(y) - u(z)|^4\, dy\, dz \\
&= \frac{1}{\delta^2} \fint_{B_\delta} \fint_{B_\delta} |u(x+h) - u(x+k)|^4\, dh\, dk \\
&\lesssim \frac{1}{\delta^2} \fint_{B_\delta} |u(x+h) - u(x)|^4\, dh = \fint_{B_\delta} \frac{|D^h u(x)|^4}{\delta^2}\, dh
\end{aligned}
$$

Integrating this inequality we infer

$$
\int_{\Omega_\delta} \big|\nabla[|u_\delta|^2]\big|^2\, dx \lesssim \fint_{B_\delta} \left(\frac{\|D^h u\|_{L^4(\Omega\cap(\Omega-h))}}{\delta^{1/2}} \right)^4 dh\,,
$$

and this tends to zero by assumption (1.10). □

Proof of Proposition 4.1. Thanks to the first convergence property in Lemma 4.2, for small enough $\delta > 0$ we have $|u_\delta| \geq 1/2$ in Ω_δ. Hence the map $v_\delta = u_\delta/|u_\delta|$ is well defined and C^1 in Ω_δ. The map $\xi \mapsto \xi/|\xi|$ is Lipschitz on $\{|\xi| \geq 1/2\}$, so we have

$$\int_{\Omega_\delta} |v_\delta - u|^2 \, dx \lesssim \int_{\Omega_\delta} |u_\delta - u|^2 \, dx \, ,$$

and this last integral tends to zero thanks to classical properties of convolutions. Moreover, using the chain rule we compute

$$\operatorname{div} v_\delta = \frac{\operatorname{div} u_\delta}{|u_\delta|} - \frac{1}{2|u_\delta|^3} \langle u_\delta, \nabla[|u_\delta|^2] \rangle \, .$$

Since $\operatorname{div} u_\delta = (\operatorname{div} u)_\delta$, this implies

$$\operatorname{div} v_\delta - \operatorname{div} u = (\operatorname{div} u)_\delta - \operatorname{div} u + (\operatorname{div} u)_\delta \frac{|u_\delta| - 1}{|u_\delta|} - \frac{1}{2|u_\delta|^3} \langle u_\delta, \nabla[|u_\delta|^2] \rangle \, ,$$

and therefore

$$\begin{aligned} \| \operatorname{div} v_\delta - \operatorname{div} u \|_{L^2(\Omega_\delta)} \leq \; & \| (\operatorname{div} u)_\delta - \operatorname{div} u \|_{L^2(\Omega_\delta)} \\ & + 2 \| \operatorname{div} u \|_{L^2(\Omega)} \sup_{\Omega_\delta} |1 - |u_\delta|| \\ & + 2 \| \nabla [|u_\delta|^2] \|_{L^2(\Omega_\delta)} \, . \end{aligned}$$

The first term in the right-hand side tends to 0 by classical properties of convolutions, and the last two terms tend to zero thanks to Lemma 4.2. □

In order to prove Theorem 1.6 from Proposition 4.1, we will need to modify the map v_δ defined on the slightly smaller domain Ω_δ, into a map $\tilde{v}_\delta$ defined on the full domain Ω. This will require some control on v_δ in H^1, which we establish before showing the proof of Theorem 1.6.

Lemma 4.3. *Let $\Omega \subset \mathbb{R}^2$ a bounded open set and assume $u \colon \Omega \to \mathbb{S}^1$ satisfies* (1.10). *Then the map $v_\delta \in C^1(\Omega_\delta; \mathbb{S}^1)$, defined in Proposition 4.1 for small $\delta > 0$, satisfies*

$$\delta \int_{\Omega_\delta} |\nabla v_\delta|^2 \, dx \to 0 \qquad \text{as } \delta \to 0 \, ,$$

where Ω_δ is defined in (4.1).

Proof of Lemma 4.3. Recall that $v_\delta = u_\delta / |u_\delta|$ and $|u_\delta| \geq 1/2$ for small $\delta > 0$. The map $\xi \mapsto \xi/|\xi|$ is Lipschitz on $\{|\xi| \geq 1/2\}$, so it suffices to establish the estimate for $u_\delta = u * \rho_\delta$, that is,

$$\delta \int_{\Omega_\delta} |\nabla u_\delta|^2 \, dx \to 0 \qquad \text{as } \delta \to 0 \, ,$$

For all $x \in \Omega_\delta$ we have

$$\begin{aligned}\nabla u_\delta(x) &= \int_\Omega u(y)\nabla\rho_\delta(x-y)\,dy = \frac{1}{\delta}\int_{B_1} u(x-\delta y)\nabla\rho(y)\,dy \\ &= \frac{1}{\delta}\int_{B_1} D^{-\delta y}u(x)\,\nabla\rho(y)\,dy\,.\end{aligned}$$

The last equality is obtained by subtracting $u(x)$ in the integral with respect to the zero-average measure $\nabla\rho(y)\,dy$. Squaring, integrating, and applying Jensen's inequality, we deduce

$$\begin{aligned}\left(\delta\int_{\Omega_\delta}|\nabla u_\delta|^2\,dx\right)^2 &\lesssim |\Omega_\delta|\int_{B_1}\int_{\Omega_\delta}\frac{|D^{-\delta y}u|^4}{\delta^2}\,dx\,dy \\ &\lesssim |\Omega_\delta|\fint_{B_\delta}\left(\frac{\|D^h u\|_{L^4(\Omega\cap(\Omega-h))}}{\delta^{1/2}}\right)^4 dh\,,\end{aligned}$$

and this last expression tends to zero by assumption (1.10). $\square$

Finally we use Proposition 4.1 to prove Theorem 1.6.

Proof of Theorem 1.6. We first need to adjust the maps v_δ provided by Proposition 4.1 in the slightly smaller domain Ω_δ in order to obain approximating maps in the whole domain Ω. This is done by performing a small dilation in a neighborhood of $\partial\Omega$.

Denote by $\nu\colon\partial\Omega\to\mathbb{S}^1$ the outer unit normal to $\partial\Omega$, and fix $\delta_0>0$ such that

$$\begin{aligned}\Xi\colon \partial\Omega\times(0,\delta_0) &\to \Omega\setminus\overline{\Omega}_{\delta_0} = \{x\in\Omega\colon\ \mathrm{dist}(x,\partial\Omega)<\delta_0\} \\ (x,t) &\mapsto x - t\nu(x)\,,\end{aligned}$$

is a C^2 diffeomorphism. We use it to define, for $0<\delta<\delta_0$, a C^2 diffeomorphism

$$\Psi_\delta\colon\Omega\setminus\Omega_{\delta_0}\to\Omega_\delta\setminus\Omega_{\delta_0}\,,\quad \Psi_\delta = \Xi\circ\widetilde{\Psi}_\delta\circ\Xi^{-1}\,,$$

where $\widetilde{\Psi}_\delta = \Xi^{-1}\circ\Psi_\delta\circ\Xi\colon\partial\Omega\times(0,\delta_0)\to\partial\Omega\times(\delta,\delta_0)$ is given by

$$\widetilde{\Psi}_\delta(x,t) = \big(x,\delta+(1-\delta/\delta_0)t\big) = (x,t)+\frac{\delta}{\delta_0}(0,\delta_0-t)\,.$$

Using that $|\widetilde{\Psi}_\delta - \mathrm{id}_{\partial\Omega\times\mathbb{R}}|\le\delta$ and $|D\widetilde{\Psi}_\delta - \mathrm{id}_{T(\partial\Omega)\times\mathbb{R}}|\le\delta/\delta_0$ on $\partial\Omega\times(0,\delta_0)$, we find that

$$|D\Psi_\delta - \mathrm{id}_{\mathbb{R}^2}|\lesssim\delta\left(\frac{\|D\Xi\|_\infty}{\delta_0}+\mathrm{Lip}(D\Xi)\right)\|D\Xi^{-1}\|_\infty\quad\text{on }\Omega\setminus\Omega_{\delta_0}\,.$$

Then we define $\tilde{v}_\delta\in H^1(\Omega;\mathbb{S}^1)$ by setting

$$\tilde{v}_\delta = \begin{cases} v_\delta & \text{in }\Omega_{\delta_0}\,,\\ v_\delta\circ\Psi_\delta & \text{in }\Omega\setminus\Omega_{\delta_0}\,.\end{cases}$$

Using the above estimate on $D\Psi_\delta$ we see that

$$\begin{aligned} |\operatorname{div}\tilde{v}_\delta - (\operatorname{div} v_\delta)\circ\Psi_\delta| &= \left|\operatorname{tr}\left(Dv_\delta(\Psi_\delta)(D\Psi_\delta - \mathrm{id}_{\mathbb{R}^2})\right)\right| \\ &\le c\,\delta|\nabla v_\delta|\circ\Psi_\delta \qquad \text{on } \Omega\setminus\Omega_{\delta_0}\,, \\ \text{and } |\det(D\Psi_\delta^{-1}) - 1| \le c\,\delta \qquad &\text{on } \Omega_\delta\setminus\Omega_{\delta_0}\,, \end{aligned}$$

for some $c > 0$ depending on Ω, and therefore

$$\left|\int_{\Omega\setminus\Omega_{\delta_0}} (\operatorname{div}\tilde{v}_\delta)^2\,dx - \int_{\Omega_\delta\setminus\Omega_{\delta_0}} (\operatorname{div} v_\delta)^2\,dx\right| \lesssim c\delta \int_{\Omega_\delta\setminus\Omega_{\delta_0}} |\nabla v_\delta|^2\,dx$$

Thanks to Lemma 4.3, this last quantity tends to zero as $\delta \to 0$. Invoking Proposition 4.1 and using that $|\Omega\setminus\Omega_\delta| \to 0$, we are therefore able to conclude that

$$\int_\Omega (\operatorname{div}\tilde{v}_\delta)^2\,dx \to \int_\Omega (\operatorname{div} u)^2\,dx\,.$$

Moreover, the bi-Lipschitzness of Ψ_δ and Lemma 4.3 ensure that

$$\int_\Omega |\nabla\tilde{v}_\delta|^2\,dx \lesssim \int_{\Omega_\delta} |\nabla v_\delta|^2\,dx = o(1/\delta)\,,$$

as $\delta \to 0$, hence we see that

$$E_\epsilon(\tilde{v}_\delta) - \int_\Omega (\operatorname{div}\tilde{v}_\delta)^2\,dx = \epsilon\int_\Omega (\operatorname{curl}\tilde{v}_\delta)^2\,dx = o(\epsilon/\delta)\,.$$

Therefore, choosing $\delta = \delta_\epsilon = \epsilon$ we conclude that

$$E_\epsilon(\tilde{v}_\delta) \to \int_\Omega (\operatorname{div} u)^2\,dx\,.$$

Finally we also check that $\tilde{v}_\delta \to u$ in $L^2(\Omega)$. This follows from the facts that $v_\delta \to u$ in $L^2(\Omega_{\delta_0})$, $u\circ\Psi_\delta \to u$ in $L^2(\Omega\setminus\Omega_{\delta_0})$ by dominated convergence, and

$$\int_{\Omega\setminus\Omega_{\delta_0}} |\tilde{v}_\delta - u\circ\Psi_\delta|^2\,dx \le (1 + c\,\delta)\int_{\Omega_\delta\setminus\Omega_{\delta_0}} |v_\delta - u|^2\,dy \to 0\,,$$

where we used $|\det(D\psi_\delta^{-1})| \le 1 + c\,\delta$. □

5 The one-dimensional case

In this section we prove Theorem 1.7. We start with a sharp version of the compensated regularity estimates, which can be obtained rather easily in this one-dimensional case.

Proposition 5.1. *We have*

$$|u|^4_{B^{1/2}_{4,\infty}(I)} \lesssim \int_I (u_1')^2\,dx\,, \tag{5.1}$$

for all $u \in H^1(I;\mathbb{S}^1)$.

Proof of Proposition 5.1. Let $u \in H^1(I;\mathbb{S}^1)$ and $\varphi \in H^1(I;\mathbb{R})$ such that $u = e^{i\varphi}$, and note that $|u_1'| = |(\cos\varphi)'| = |\sin\varphi|\,|\varphi'| = |F(\varphi)'|$, where $F \in C^1(\mathbb{R};\mathbb{R})$ is given by

$$F(t) = \int_0^t |\sin s|\,ds \qquad \forall t \in \mathbb{R}\,. \tag{5.2}$$

Note also that F is bijective and its inverse F^{-1} is $C^{1/2}$. Explicitly, F and F^{-1} are characterized by

$$\begin{aligned}
&F(t) = 1 - \cos t \text{ for } t \in [0,\pi],\\
&F(t+\ell\pi) = 2\ell + F(t) \text{ for } t \in \mathbb{R}, \ell \in \mathbb{Z},\\
&F^{-1}(s) = \arccos(1-s) \text{ for } s \in [0,2],\\
&F^{-1}(s+2\ell) = \ell\pi + F^{-1}(s) \text{ for } s \in \mathbb{R}, \ell \in \mathbb{Z},
\end{aligned}$$

where $\arccos\colon [-1,1] \to [0,\pi]$ is the left inverse of cos on $[0,\pi]$. Since the function F^{-1} is $C^{1/2}$, we have, letting $w = F(\varphi)$ and $I_h = I \cap (I-h)$,

$$\begin{aligned}
\int_{I_h} |\varphi(x+h) - \varphi(x)|^4\,dx &\lesssim \int_{I_h} |w(x+h) - w(x)|^2\,dx\\
&\lesssim |h|^2 \int_I (w')^2\,dx = |h|^2 \int_I (u_1')^2\,dx,
\end{aligned}$$

which implies (5.1) since u is a Lipschitz function of φ. $\square$

Proposition 5.1 implies the boundedness and compactness statement in Theorem 1.7. The lower bound of the Γ-convergence statement also follows rather directly: any sequence $(u_\epsilon) \subset H^1(I;\mathbb{S}^1)$ such that $\sup E_\epsilon^{1D}(u_\epsilon) < \infty$ and $u_\epsilon \to u$ in $\mathcal{D}'(\Omega)$ is bounded in $B^{1/2}_{4,\infty}(I;\mathbb{S}^1)$, so u must belong to that space, and the simple inequality $E_\epsilon^{1D}(u_\epsilon) \geq E_0^{1D}(u_\epsilon)$ implies that $(u_{\epsilon 1})$ is bounded, hence weakly convergent, in $H^1(I)$ and therefore

$$\liminf_{\epsilon\to 0} E_\epsilon^{1D}(u_\epsilon) \geq E_0^{1D}(u)\,,$$

thanks to the weak lower semicontinuity of $v \mapsto \int_I (v')^2\,dx$ on $H^1(I)$. Hence it only remains to prove the upper bound part of the Γ-convergence.

Proposition 5.2. *Let $u \in B^{1/2}_{4,\infty}(I;\mathbb{S}^1)$ such that $u_1 \in H^1(I)$. Then there exists $(u_\epsilon) \subset H^1(I;\mathbb{S}^1)$ such that $u_\epsilon \to u$ in $\mathcal{D}'(I)$ and $E_\epsilon^{1D}(u_\epsilon) \to E_0(u)$ as $\epsilon \to 0$.*

Proof. Since $B^{1/2}_{4,\infty}(I) \subset C^0(\overline{I})$, the continuous $\mathbb{S}^1$-valued map u admits a lifting $\varphi \in C^0(\overline{I};\mathbb{R})$ such that $u = e^{i\varphi}$. We extend φ to a continuous function $\varphi\colon \mathbb{R} \to \mathbb{R}$ which is smooth with compact support outside I and constant on $[a-1,a]$ and $[b,b+1]$, where $I = (a,b)$. This also provides an extension of $u = e^{i\varphi}$ to $\mathbb{R}$.

Locally (that is, on any small interval where the image of u is contained in an arc of length less than π), the lifting φ is given locally by $\varphi = \Theta(u)$ where Θ is a smooth left inverse of $t \mapsto e^{it}$ on an open interval of size π (that is, $\Theta(e^{it}) = t$ for t in that interval). This implies that φ inherits the regularity of u, hence we have $\varphi \in B^{1/2}_{4,\infty}(\mathbb{R})$. Moreover we have $u_1 = \cos\varphi \in H^1_{\mathrm{loc}}(\mathbb{R})$, which implies $F(\varphi) \in H^1_{\mathrm{loc}}(\mathbb{R})$, where F is defined in (5.2).

In what follows, we denote convolution with a smooth compactly supported kernel $\rho_\delta(x) = \delta^{-1}\rho(x/\delta)$, $\rho \in C^\infty_c(-1,1)$, $\int\rho = 1$ by a subscript δ.

According to this notation, we consider the mollified phase function $\varphi_\delta = \varphi * \rho_\delta$, and define $u^{[\delta]} = e^{i\varphi_\delta} \in C^1_c(\mathbb{R};\mathbb{S}^1)$. Then we have $u^{[\delta]} \to e^{i\varphi} = u$ in $\mathcal{D}'(\mathbb{R})$ as $\delta \to 0$, and we claim that

$$\int_I |(u_1^{[\delta]})'|^2\,dx \to \int_I (u_1')^2\,dx = E_0(u). \tag{5.3}$$

Granted this, noting that

$$\varphi_\delta'(x) = \frac{1}{\delta}\int (\varphi(x-\delta y) - \varphi(x))\rho'(y)\,dy,$$

implies

$$\begin{aligned}\int_I (\varphi_\delta')^2\,dx &\leq \frac{\|\rho'\|_{L^1}}{\delta^2}\int \|\varphi(\cdot-\delta y) - \varphi\|_{L^2}^2 |\rho'(y)|\,dy \\ &\lesssim \frac{\|\rho'\|_{L^1}}{\delta^2}\int \|\varphi(\cdot-\delta y) - \varphi\|_{L^4}^2 |\rho'(y)|\,dy,\end{aligned}$$

and therefore

$$\int_I |(u_2^{[\delta]})'|^2\,dx \leq \int_I (\varphi_\delta')^2\,dx \lesssim \frac{1}{\delta}\|\varphi\|^2_{B^{1/2}_{4,\infty}},$$

we see that choosing $\delta = \delta_\epsilon = \sqrt{\epsilon}$ ensures the recovery sequence property

$$E_\epsilon(u^{[\delta]}) \to \int_I (u_1')^2\,dx,$$

which proves Proposition 5.2.

Thus it all boils down to showing (5.3), which is equivalent to $F(\varphi_\delta)' \to F(\varphi)'$ in $L^2(I)$ since $|(u_1^{[\delta]})'| = |F(\varphi_\delta)'|$, and we devote the remainder of the proof to establishing this convergence.

To that end we compute

$$\begin{aligned}
&\frac{d}{dx}\left[F(\varphi)_\delta(x) - F(\varphi_\delta(x))\right] \\
&= \int F(\varphi(y))\rho_\delta'(x-y)\,dy - F'(\varphi_\delta(x))\varphi_\delta'(x) \\
&= \int \Big(F(\varphi(y)) - F'(\varphi_\delta(x))\varphi(y)\Big)\rho_\delta'(x-y)\,dy \\
&= \int \Big(F(\varphi(y)) - F(\varphi_\delta(x)) - F'(\varphi_\delta(x))(\varphi(y)-\varphi_\delta(x))\Big)\rho_\delta'(x-y)\,dy \\
&= \int R(\delta,x,y)(\varphi(y)-\varphi_\delta(x))^2\rho_\delta'(x-y)\,dy,
\end{aligned}$$

where

$$R(\delta,x,y) = \int_0^1 (1-t)F''(\varphi_\delta(x) + t(\varphi(y)-\varphi_\delta(x)))\,dt.$$

Since $|R| \le \sup|F''| \le 1$, we deduce

$$|F(\varphi_\delta)' - [F(\varphi)_\delta]'| \le \frac{1}{\delta}\int (\varphi(x-\delta y) - \varphi(x-\delta z))^2\,\rho(z)\,dz\,|\rho'(y)|\,dy,$$

and by Jensen's (or Cauchy-Schwarz') inequality,

$$\begin{aligned}
&(F(\varphi_\delta)' - [F(\varphi)_\delta]')^2 \\
&\le \frac{\|\rho'\|_{L^1}}{\delta^2}\int (\varphi(x-\delta y) - \varphi(x-\delta z))^4\,\rho(z)\,dz\,|\rho'(y)|\,dy.
\end{aligned} \tag{5.4}$$

Integrating this on I gives a bounded right-hand side thanks to the fact that $\varphi \in B^{1/2}_{4,\infty}(\mathbb{R})$, but we would really like to obtain a bound that tends to 0 as $\delta \to 0$, in order to deduce that $F(\varphi_\delta)' \to F(\varphi)'$ in L^2.

In order to obtain that convergence, we observe that the above crude estimate can be refined by using the additional information that $u_1 \in H^1$. Since $|u_1'| = |F(\varphi)'| = |F'(\varphi)|\,|\varphi'|$, that additional information implies indeed that, away from the points where $|\sin\varphi| = |F'(\varphi)| = 0$, we actually have H^1 control on φ.

Observe that $B^{1/2}_{4,\infty} \subset C^{1/4}$ [41, (2.7.1/12)], that is $[\varphi]_{1/4} \le C\|\varphi\|_{B^{1/2}_{4,\infty}}$, where we denote by $[\varphi]_{1/4}$ the $C^{1/4}$ seminorm of φ. We fix a small parameter $\lambda \ge 2[\varphi]_{1/4}\delta^{1/4}$. Then, from the definition of this seminorm, we have that

$$|\varphi(x) - \varphi(y)| \le \frac{\lambda}{2\delta^{\frac14}}|x-y|^{\frac14} \quad \forall x,y \in \mathbb{R}\,, \tag{5.5}$$

and we deduce in particular the property

$$|x-y| \ge 16\,\delta \quad \text{ whenever } x,y \text{ satisfy } |\sin\varphi(y) - \sin\varphi(x)| \ge \lambda\,. \tag{5.6}$$

This property allows us to write I as a union of mutually disjoint and not-too-small 'elliptic' intervals $\{I_j^{e,\lambda}\}$ and 'nonelliptic' intervals $\{I_k^{ne,\lambda}\}$. Specifically, we claim

$$
\begin{aligned}
&I = \bigsqcup_j I_j^{e,\lambda} \sqcup \bigsqcup_k I_k^{ne,\lambda}, \qquad |I_j^{e,\lambda}|, |I_k^{ne,\lambda}| \geq 8\delta\,, \\
&|\sin\varphi| \geq \lambda \text{ in each } I_j^{e,\lambda}, \quad |\sin\varphi| \leq 2\lambda \text{ in each } I_k^{ne,\lambda}\,.
\end{aligned}
\tag{5.7}
$$

On the intervals $I_j^{e,\lambda}$ the integrand $|u_1'|^2 = |\sin(\varphi)|^2|\varphi'|^2$ is 'elliptic' because $|\sin\varphi|$ stays away from 0, while on the intervals $I_k^{ne,\lambda}$ it is 'nonelliptic' because $|\sin\varphi|$ stays close to zero. Note that two disjoint elliptic intervals are separated by a non-elliptic interval of length at least 8δ, so that the slightly enlarged elliptic intervals $I_j^{e,\lambda} + (-2\delta, 2\delta)$ (whose left/right endpoint are that of $I_j^{e,\lambda}$ shifted left/right by 2δ) are still disjoint. And the same holds for the slightly enlarged non-elliptic intervals $I_k^{ne,\lambda} + (-2\delta, \delta)$.

In order to obtain this decomposition of I, one can for instance monitor the variations of the continuous function $f(x) = |\sin\varphi(x)|$ along the interval $\overline{I} = [a,b]$. We set $x_0 = a$ and define by induction a strictly increasing sequence (x_ℓ) satisfying

$$
\begin{aligned}
x_{\ell+1} &= \max\left(\sup J_\ell^{ne}, \sup J_\ell^e\right)\,, \\
\text{where } J_\ell^{ne} &= \{x > x_\ell \colon f(x) \leq 2\lambda \text{ on } [x_\ell, x)\}, \\
J_\ell^e &= \{x > x_\ell \colon f(x) \geq \lambda \text{ on } [x_\ell, x)\}\,.
\end{aligned}
$$

Here we adopt the convention that the supremum of an empty interval is $-\infty$. If one of these two intervals is empty, the other interval must have length at least 16δ due to (5.6). And if both intervals are non-empty (which actually happens only in the first iteration), then the largest of them must contain a transition of f between λ and 2λ, which again has length at least 16δ due to (5.6). Therefore the sequence (x_ℓ) defined that way satisfies $x_{\ell+1} - x_\ell \geq 16\delta$. Moreover, the definition of $x_{\ell+1}$ clearly implies that the interval $I_\ell = (x_\ell, x_{\ell+1}]$ is either elliptic (i.e. $f \geq \lambda$ on I_ℓ) or nonelliptic (i.e. $f \leq 2\lambda$ on I_ℓ). There exists an integer $N \geq 0$ such that $x_N < b \leq x_{N+1}$. If $b - x_N \geq 8\delta$ we set $I_N = (x_n, b)$, and the disjoint intervals $I_0, \ldots, I_N$ are all of length at least 8δ and either elliptic or nonelliptic. If $b - x_N < 8\delta$, then thanks to (5.6) and since $f(x_N) \in \{\lambda, 2\lambda\}$, the interval $I_N = (b - 8\delta, b)$ must be either elliptic or nonelliptic, and we modify I_{N-1} into $I_{N-1} = (x_{N-1}, b - 8\delta]$, which is still either nonelliptic or elliptic, and has length at least 8δ. In that case again, the disjoint intervals $I_0, \ldots, I_N$ are all of length at least 8δ and either elliptic or nonelliptic. In all cases, we have obtained (5.7).

Now we come back to estimating $F(\varphi_\delta)' - [F(\varphi)_\delta]'$, taking advantage of that decomposition. On any elliptic interval $I_j^{e,\lambda}$ we have $|\sin\varphi| \geq \lambda$, recall (5.7). Thanks to the Hölder regularity (5.5), we have in fact the similar lower bound $|\sin\varphi| \geq c\lambda$ for $c = 1 - 2^{-3/4} > 0$, on the enlarged interval $I_j^{e,\lambda} + (-2\delta, 2\delta)$. As a consequence, we obtain

$$\int_{I_j^{e,\lambda}+(-2\delta,2\delta)} (\varphi')^2\,dx \lesssim \frac{1}{\lambda^2}\int_{I_j^{e,\lambda}+(-2\delta,2\delta)} (F(\varphi)')^2\,dx, \tag{5.8}$$

and for any $y, z \in [-1, 1]$,

$$\begin{aligned}
&\int_{I_j^{e,\lambda}} (\varphi(x-\delta y)-\varphi(x-\delta z))^4\,dx \\
&\le \sqrt{2}[\varphi]_{1/4}^2\delta^{1/2}\int_{I_j^{e,\lambda}} (\varphi(x-\delta y)-\varphi(x-\delta z))^2\,dx \\
&= \sqrt{2}[\varphi]_{1/4}^2\delta^{1/2}\int_{I_j^{e,\lambda}} \left(\int_0^1 \varphi'(x-\delta y+t\delta(y-z))\,dt\delta(y-z)\right)^2\,dx \\
&\le 4\sqrt{2}[\varphi]_{1/4}^2\delta^{1/2+2}\int_{I_j^{e,\lambda}}\int_0^1 \left(\varphi'(x-\delta y+t\delta(y-z))\right)^2\,dt\,dx \\
&\le 4\sqrt{2}[\varphi]_{1/4}^2\delta^{1/2+2}\int_{I_j^{e,\lambda}+(-2\delta,2\delta)} (\varphi')^2\,dx.
\end{aligned}$$

Then, invoking (5.8), it follows that

$$\int_{I_j^{e,\lambda}} (\varphi(x-\delta y)-\varphi(x-\delta z))^4\,dx \lesssim [\varphi]_{1/4}^2\frac{\delta^{1/2+2}}{\lambda^2}\int_{I_j^{e,\lambda}+(-2\delta,2\delta)} (F(\varphi)')^2\,dx.$$

Since the intervals $I_j^{e,\lambda}+(-2\delta,2\delta)$ are disjoint, on the union $I^{e,\lambda}=\bigcup_j I_j^{e,\lambda}$ we deduce, recalling also (5.4),

$$\int_{I^{e,\lambda}} \left(F(\varphi_\delta)'-[F(\varphi)_\delta]'\right)^2\,dx \lesssim [\varphi]_{1/4}^2\frac{\delta^{1/2}}{\lambda^2}\int_I (F(\varphi)')^2\,dx.$$

On the nonelliptic intervals we have simply, recalling that F^{-1} is $C^{1/2}$,

$$\begin{aligned}
&\int_{I_k^{ne,\lambda}} (\varphi(x-\delta y)-\varphi(x-\delta z))^4\,dx \\
&\lesssim \int_{I_k^{ne,\lambda}} (F(\varphi(x-\delta y))-F(\varphi(x-\delta z)))^2\,dx \\
&\lesssim \delta^2\int_{I_k^{ne,\lambda}+(-2\delta,2\delta)} (F(\varphi)')^2\,dx,
\end{aligned}$$

and on their union $I^{ne,\lambda}=\bigcup_k I_k^{ne,\lambda}$ we obtain, using again (5.4),

$$\begin{aligned}
\int_{I^{ne,\lambda}} \left(F(\varphi_\delta)'-[F(\varphi)_\delta]'\right)^2\,dx &\lesssim \int_{I^{ne,\lambda}+(-2\delta,2\delta)} (F(\varphi)')^2\,dx \\
&\le \int_{\{|\sin\varphi|\le 3\lambda\}} (F(\varphi)')^2\,dx.
\end{aligned}$$

For the last inequality we used the fact that $|\sin(\varphi)| \le 2\lambda$ on $I^{ne,\lambda}$ and the Hölder continuity (5.5) of φ. Summing the estimates on $I^{e,\lambda}$ and $I^{ne,\lambda}$ we infer

$$\int_I \left(F(\varphi_\delta)' - [F(\varphi)_\delta]'\right)^2 \, dx \lesssim [\varphi]_{1/4}^2 \frac{\delta^{1/2}}{\lambda^2} \int_I (F(\varphi)')^2 \, dx + \int_{\{|\sin\varphi| \le 3\lambda\}} (F(\varphi)')^2 \, dx,$$

hence, for any $\lambda > 0$,

$$\limsup_{\delta\to 0} \int_I \left(F(\varphi_\delta)' - [F(\varphi)_\delta]'\right)^2 \, dx \lesssim \int_{\{|\sin\varphi| \le 3\lambda\}} (F(\varphi)')^2 \, dx.$$

Letting $\lambda \to 0$, and recalling $u = e^{i\varphi}$, we deduce

$$\begin{aligned}\limsup_{\delta\to 0} \int_I \left(F(\varphi_\delta)' - [F(\varphi)_\delta]'\right)^2 \, dx &\lesssim \int_{\{\sin\varphi = 0\}} (F(\varphi)')^2 \, dx \\ &= \int_{\{|u_1| = 1\}} (|u_1|')^2 \, dx = 0.\end{aligned}$$

The last equality follows e.g. from [32, Lemma 3.60]. Since $|u_1'| = |F(\varphi)'| \in L^2$, we have $[F(\varphi)_\delta]' \to F(\varphi)'$ strongly in L^2, and we deduce that $F(\varphi_\delta)'$ converges to $F(\varphi)'$ strongly in L^2, which implies (5.3). $\square$

6 A thin film limit

In this section we prove Theorem 1.8 about the Γ-convergence of the energy

$$E_{\epsilon,\delta}(u) = \frac{1}{2} \int_{[-1,1]^2} \left(\partial_1 u_1 + \frac{1}{\delta}\partial_2 u_2\right)^2 + \epsilon \left(\partial_1 u_2 - \frac{1}{\delta}\partial_2 u_1\right)^2 \, dx,$$

for $u \in H^1([-1,1]^2; \mathbb{S}^1)$ subject to the boundary conditions $u(\pm 1, x_2) = e_\pm$ for $x_2 \in (-1,1)$, and $u(x_1,-1) = u(x_1,+1)$.

We start by adapting the compensated regularity estimate of Theorem 1.1 to this thin periodic case, thereby proving the equiboundedness and compactness statement in Theorem 1.8.

Lemma 6.1. *For any $u \in H^1([-1,1]^2; \mathbb{S}^1)$ satisfying $u(\pm 1, x_2) = e_\pm$ for $x_2 \in (-1,1)$, and $u(x_1,-1) = u(x_1,+1)$ for $x_1 \in (-1,1)$, we have*

$$\int_{[-1,1]^2} |D^{te_1}u|^3 \, dx \lesssim |t|^{\frac{3}{2}} \left(1 + E_{\epsilon,\delta}(u)^{\frac{3}{2}}\right), \tag{6.1}$$

$$\int_{[-1,1]^2} |D^{te_2}u|^3 \, dx \lesssim \delta^{\frac{3}{2}} |t|^{\frac{3}{2}} \left(1 + E_{\epsilon,\delta}(u)^{\frac{3}{2}}\right), \tag{6.2}$$

for all $t \in \mathbb{R}$. Here, u is extended to $\mathbb{R}^2$ by setting

$$u(x_1, x_2) = \begin{cases} e_+, & \text{if } x_1 \geq 1, \\ e_-, & \text{if } x_1 \leq -1\,, \end{cases}$$

and $u(x_1, x_2 + 2k) = u(x_1, x_2)$ for all $k \in \mathbb{Z}$.

Proof of Lemma 6.1. Define $\tilde{u}(x_1, x_2) = u(x_1, x_2/\delta)$ for $(x_1, x_2) \in \mathbb{R}^2$, so that $\tilde{u} \in H^1_{\mathrm{loc}}$ and

$$\int_{[-1,1]\times[-\delta,\delta]} (\nabla \cdot \tilde{u})^2 \, dx \leq 2\delta \, E_{\epsilon,\delta}(u).$$

Note that, from the definition of the extension of u to $\mathbb{R}^2$, we have $\tilde{u}(x_1, x_2+2\delta) = \tilde{u}(x_1, x_2)$ and $\tilde{u}(x_1, x_2) = \tilde{u}(\pm 1, x_2)$ for $x_1 > 1$ or $x_1 < -1$. Thus we find

$$\int_{[-2,2]^2} (\nabla \cdot \tilde{u})^2 \, dx \lesssim \frac{1}{\delta} \int_{[-1,1]\times[-\delta,\delta]} (\nabla \cdot \tilde{u})^2 \, dx \leq 2E_{\epsilon,\delta}(u),$$

and, for any $p \in [1, \infty)$,

$$\int_{[-1,1]^2} |D^{te_1} u|^p \, dx = \frac{1}{\delta} \int_{[-1,1]\times[-\delta,\delta]} |D^{te_1} \tilde{u}|^p \, dx \lesssim \int_{[-1,1]^2} |D^{te_1} \tilde{u}|^p \, dx\,.$$

Theorem 1.1 implies, for $h \in \mathbb{R}^2$,

$$\begin{aligned} \int_{[-1,1]^2} |D^h \tilde{u}|^3 \, dx &\lesssim |h|^{\frac{3}{2}} \left(\int_{[-2,2]^2} (\nabla \cdot \tilde{u})^2 \, dx + 1 \right)^{\frac{3}{2}} \\ &\lesssim |h|^{\frac{3}{2}} \left(1 + E_{\epsilon,\delta}(u)^{\frac{3}{2}} \right). \end{aligned} \tag{6.3}$$

Applying this to increments $h = te_1$ in the x_1-direction, we obtain (6.1). For increments in the x_2-direction, we have

$$\begin{aligned} D^{te_2} u(x) &= u(x_1, x_2 + t) - u(x_1, x_2) = \tilde{u}(x_1, \delta x_2 + \delta t) - \tilde{u}(x_1, \delta x_2) \\ &= D^{\delta te_2} \tilde{u}(x_1, \delta x_2), \end{aligned}$$

hence

$$\int_{[-1,1]^2} |D^{te_2} u|^p \, dx = \frac{1}{\delta} \int_{[-1,1]\times[-\delta,\delta]} |D^{\delta te_2} \tilde{u}|^p \, dx \lesssim \int_{[-1,1]^2} |D^{\delta te_2} \tilde{u}|^p \, dx\,,$$

so (6.3) applied to $h = \delta te_2$ implies (6.2). ◻

Lemma 6.1 will enable us to show that limits of bounded energy sequences for $E_{\epsilon,\delta}$ are one-dimensional and belong to $B^{1/2}_{3,\infty}([-1,1]^2; \mathbb{S}^1)$. Next we prove a lemma which will allow us to conclude that such limits actually belong to the smaller space $B^{1/2}_{4,\infty}([-1,1]; \mathbb{S}^1)$.

Lemma 6.2. *Assume* $u\colon [-1,1] \to \mathbb{S}^1$ *is continuous and* $u_1 \in H^1([-1,1])$*. Then* $u \in B^{1/2}_{4,\infty}([-1,1];\mathbb{S}^1)$*.*

Proof of Lemma 6.2. Note that $|u_2| = \sqrt{1-u_1^2}$ is a composition of the H^1 function u_1 and the $C^{1/2}$ function $x \mapsto \sqrt{1-x^2}$. Hence, by the same argument as in Proposition 5.1, we have $|u_2| \in B^{1/2}_{4,\infty}([-1,1])$. But this does not directly imply that $u_2 \in B^{1/2}_{4,\infty}([-1,1])$, and we use a slightly different argument to prove Lemma 6.2.

Since u is continuous, one can define a continuous phase φ such that $u = e^{i\varphi}$ on $[-1,1]$. Recall the function $F(t) = \int_0^t |\sin(s)|\,ds$ defined in (5.2). If φ were differentiable, we would have $|F(\varphi)'| = |\sin\varphi||\varphi'| = |u_1'| \in L^2([-1,1])$, and conclude that $\varphi \in B^{1/2}_{4,\infty}([-1,1])$ since F^{-1} is $C^{1/2}$, as in Proposition 5.1.

This calculation using the chain rule is only formal, because we do not know that φ is (even weakly) differentiable. But we do show in what follows that $F(\varphi) \in H^1([-1,1])$. First we establish that

$$\begin{aligned} &\forall x \in [-1,1],\ \exists r_x > 0 \text{ such that, for all } y_1 < y_2 \in (x-r_x, x+r_x)\,, \\ &\text{we have} \quad |F(\varphi(y_1)) - F(\varphi(y_2))| \leq \int_{y_1}^{y_2} |u_1'(y)|\,dy\,. \end{aligned} \tag{6.4}$$

To prove (6.4) we distinguish two cases depending on the value of $\sin\varphi(x)$.

Let us first assume $\sin\varphi(x) \neq 0$. There exists $r_x > 0$ such that, on the interval $(x-r_x, x+r_x)$, the continuous function φ takes values in an interval where sin does not vanish. In such an interval, we have $F(\varphi(y)) = C + \alpha\cos\varphi(y)$ for some constants $C \in \mathbb{Z}$, $\alpha \in \{\pm 1\}$, so $F(\varphi) = C + \alpha u_1$. We deduce that $F(\varphi) \in H^1(x-r_x, x+r_x)$ and $|F(\varphi)'| = |u_1'|$ in $(x-r_x, x+r_x)$, which implies (6.4).

Let us now consider the remaining case $\sin\varphi(x) = 0$. There exists $r_x > 0$ such that, on the interval $(x-r_x, x+r_x)$, the continuous function φ takes values in an interval of length less than π. In that interval we have

$$F(\varphi(y)) = \begin{cases} C - \alpha\cos\varphi(y) & \text{if } \varphi(y) > \varphi(x), \\ C - 2 + \alpha\cos\varphi(y) & \text{if } \varphi(y) < \varphi(x), \end{cases}$$

for some constant $C \in \mathbb{Z}$, $\alpha = \cos\varphi(x) \in \{\pm 1\}$. Let $y_1 < y_2 \in (x-r_x, x+r_x)$. If $\varphi(y_1) < \varphi(x) < \varphi(y_2)$, then there exists $\bar{x} \in (y_1, y_2)$ such that $\varphi(\bar{x}) = \varphi(x)$, and we deduce

$$\begin{aligned} F(\varphi(y_2)) - F(\varphi(y_1)) &= -\alpha\cos\varphi(y_2) + 2 - \alpha\cos\varphi(y_1) \\ &= -\alpha\cos\varphi(y_2) + 2\alpha\cos\varphi(\bar{x}) - \alpha\cos\varphi(y_1) \\ &= \alpha\left(u_1(\bar{x}) - u_1(y_2) + u_1(\bar{x}) - u_1(y_1)\right). \end{aligned}$$

Therefore,

$$\begin{aligned}|F(\varphi(y_1)) - F(\varphi(y_2))| &\le |u_1(y_1) - u_1(\bar{x})| + |u_1(y_2) - u_1(\bar{x})| \\ &\le \int_{y_1}^{\bar{x}} |u_1'|\, dy + \int_{\bar{x}}^{y_2} |u_1'|\, dy \le \int_{y_1}^{y_2} |u_1'|\, dy\,.\end{aligned}$$

If $\varphi(y_1)$ and $\varphi(y_2)$ both lie on the same side of $\varphi(x)$, the inequality (6.4) also holds because in that case we have $|F(\varphi(y_2)) - F(\varphi(y_1))| = |\cos\varphi(y_2) - \cos\varphi(y_1)| = |u_1(y_2) - u_1(y_1)|$. In all cases, we have proved (6.4).

Applying (6.4) to any $y_1 < \cdots < y_M \in (x - r_x, x + r_x)$, we see that

$$\sum_{i=1}^{M-1} |F(\varphi(y_{i+1})) - F(\varphi(y_i))| \le \int_{y_1}^{y_M} |u_1'|\, dy\,,$$

hence the function $F(\varphi)$ is absolutely continuous on $(x - r_x, x + r_x)$, so it is differentiable almost everywhere. Moreover, the inequality (6.4) implies that its derivative satisfies $|F(\varphi)'| \le |u_1'|$ almost everywhere on $(x - r_x, x + r_x)$.

This is true for any $x \in [-1, 1]$, so we deduce that $F(\varphi)$ is absolutely continuous on $[-1, 1]$ and its derivative satisfies $|F(\varphi)'| \le |u_1'|$. Recalling that $u_1 \in H^1([-1, 1])$, this implies $F(\varphi) \in H^1([-1, 1])$. We deduce as in Proposition 5.1 that $\varphi \in B^{1/2}_{4,\infty}(-1, 1)$, and therefore $u \in B^{1/2}_{4,\infty}([-1, 1]; \mathbb{S}^1)$ since $u = e^{i\varphi}$ is a Lipschitz function of φ. □

Proof of Theorem 1.8. Let $(u^{(j)}) \subset H^1([-1, 1]^2; \mathbb{S}^1)$ such that $E_{\epsilon_j,\delta_j}(u^{(j)}) \le C$ for some $(\epsilon_j, \delta_j) \to (0, 0)$. In light of Jensen's inequality and the x_2-periodicity one has

$$\begin{aligned}E_{\epsilon_j,\delta_j}(u^{(j)}) &\ge \int_{-1}^{1} \left[\left(\frac{d\bar{u}_1^{(j)}}{dx_1} \right)^2 + \epsilon \left(\frac{d\bar{u}_2^{(j)}}{dx_1} \right)^2 \right] dx_1, \\ \bar{u}^{(j)}(x_1) &:= \frac{1}{2} \int_{-1}^{1} u^{(j)}(x_1, x_2)\, dx_2.\end{aligned}$$

Thanks to (6.1), the averaged maps $\bar{u}^{(j)}$ are equibounded in $B^{1/2}_{3,\infty}([-1, 1]; \mathbb{R}^2)$. We can extract a subsequence such that $\bar{u}^{(j)} \to \bar{u}$ strongly in L^2, and $d\bar{u}_1^{(j)}/dx_1$ converges weakly in L^2. Then we have $\bar{u}_1' \in L^2$ and

$$\liminf_{\epsilon \to 0} E_{\epsilon,\delta}(u_\epsilon) \ge \int_{-1}^{1} (\bar{u}_1')^2\, dx.$$

Moreover, Lemma 6.1 implies that $(u^{(j)})$ is bounded in $B^{1/2}_{3,\infty}([-1, 1]^2; \mathbb{S}^1)$, and any weak limit u in that space satisfies $D^{te_2} u = 0$ for all $t \in \mathbb{R}$. Hence we have $u = u(x_1) = \bar{u}(x_1) \in B^{1/2}_{3,\infty}([-1, 1]; \mathbb{S}^1) \subset C([-1, 1]; \mathbb{S}^1)$. Further, applying Lemma 6.2 we deduce that $u = \bar{u} \in B^{1/2}_{4,\infty}([-1, 1]; \mathbb{S}^1)$. This proves the lower bound part of the Γ-convergence statement of Theorem 1.8, as well as the equiboundedness and compactness statement.

For the upper bound, since bounded energy limits are one-dimensional, we can directly use the construction from §5, concluding the proof of Theorem 1.8. □

Appendix A Traces

In this Appendix we sketch the proof of Proposition 3.1, adapting directly the arguments in [42] to our slightly different context.

Proof of Proposition 3.1. There are two main steps: first, the existence of a weak trace for the indicator function χ, and, second, the proof that this weak trace is an indicator function. This last fact then implies that the trace is strong.

Step 1: Existence of a weak trace. There exists $\mathrm{tr}(\chi) \in L^\infty(\mathbb{T} \times \partial\Omega)$ a weak-$*$ trace of the indicator function $\chi(s,x) = \mathbf{1}_{e^{is}\cdot u(x)>0}$, in the following sense. For any $x_0 \in \partial\Omega$ and any C^1-diffeomorphism

$$\begin{aligned}
&\psi\colon V \to U, \quad 0 \in V,\ x_0 \in U\,, \\
&\psi^{-1}(U\cap\Omega) = V \cap H_+, \quad H_+ = \{x \in \mathbb{R}^2 \colon x_2 > 0\}\,, \\
&\overline{U} \cap \partial\Omega = \overline{U \cap \partial\Omega}\,, \\
&\psi^{-1}(U \cap \partial\Omega) = V \cap \partial H_+ = V \cap \{x_2 = 0\}\,,
\end{aligned} \tag{A.1}$$

we have

$$\chi \circ \big(\mathrm{id}_{\mathbb{T}} \otimes \psi(\cdot,\delta)\big) \overset{*}{\rightharpoonup} \mathrm{tr}(\chi) \circ \big(\mathrm{id}_{\mathbb{T}} \otimes \psi(\cdot,0)\big)\,,$$

weakly-$*$ in $L^\infty(\mathbb{T} \times (V \cap \partial H_+))$ as $\delta \to 0$.

Proof of Step 1. Let $\omega_\delta = \psi(V \cap \{x_2 > \delta\})$, $\gamma_\delta = \psi(V \cap \{x_2 = \delta\})$. First note that $u \in B^{1/2}_{3,\infty}(\omega_\delta)$, hence classical trace properties (see e.g. [41, § 3.3.3]) imply that u has a strong trace on γ_δ, and therefore $\tilde\chi_\delta := \chi \circ (\mathrm{id}_{\mathbb{T}} \otimes \psi(\cdot,\delta))$ can be interpreted as the strong trace

$$\tilde\chi_\delta(s,x_1) = \mathbf{1}_{e^{is}\cdot u(\psi(x_1,\delta))>0}\,.$$

We also define $\chi_\delta \colon \mathbb{T} \times (U \cap \partial\Omega) \to \{0,1\}$ by setting

$$\chi_\delta(s,\psi(x_1,0)) = \tilde\chi_\delta(s,x_1) \qquad \forall (s,x_1) \in \mathbb{T} \times (V \cap \partial H_+)\,.$$

This bounded sequence in $L^\infty(\mathbb{T}\times U\cap\partial\Omega)$ admits a weak-$*$ converging subsequence

$$\chi_{\delta_j} \overset{*}{\rightharpoonup} \chi_0 \in L^\infty(\mathbb{T} \times (U \cap \partial\Omega))\,.$$

Next we show that this weak-$*$ limit χ_0 does not depend on the subsequence $\delta_j \to 0$ nor the choice of the diffeomorphism ψ. That is, different choices of ψ

give the same limit χ_0 almost everywhere on the intersection of their domains. As a consequence, we can set $\mathrm{tr}(\chi) = \chi_0$ and it is a well-defined weak-$*$ trace, thus proving Step 1.

To check this, note that thanks to the strong trace property on γ_δ, we have the Gauss-Green formula

$$\int_{\omega_\delta} \eta \operatorname{div} \Phi(u)\, dx + \int_{\omega_\delta} \langle \nabla \eta, \Phi(u) \rangle\, dx = \int_{\gamma_\delta} \eta \,\langle \nu_\delta, \Phi(\mathrm{tr}_{\gamma_\delta}(u)) \rangle\, d\mathcal{H}^1 ,$$

for all $\eta \in C^1_c(U)$ and $\Phi \in \mathrm{ENT}$, where ν_δ is the exterior unit normal along γ_δ, see e.g. [8]. Since $\Phi(u) \in L^\infty(\Omega)$, $\operatorname{div} \Phi(u) \in L^2(\Omega)$, and $|U \cap \Omega \setminus \omega_\delta| \to 0$, we deduce that the function

$$f^{\psi}_{\eta,\Phi}(\delta) = \int_{\gamma_\delta} \eta \,\langle \nu_\delta, \Phi(\mathrm{tr}_{\gamma_\delta}(u)) \rangle\, d\mathcal{H}^1 ,$$

is Cauchy as $\delta \to 0$. Moreover, if ψ, ψ' are two different diffeomorphisms whose image contains the support of η, then the Gauss-Green formula applied in the corresponding sets ω_δ, ω'_δ also implies that

$$f^{\psi}_{\eta,\Phi}(\delta) - f^{\psi'}_{\eta,\Psi}(\delta) \to 0 \quad \text{as } \delta \to 0 .$$

We deduce that there exists $\ell_{\eta,\Phi} \in \mathbb{R}$ such that

$$\int_{\gamma_\delta} \eta \,\langle \nu_\delta, \Phi(\mathrm{tr}_{\gamma_\delta}(u)) \rangle\, d\mathcal{H}^1 \to \ell_{\eta,\Phi} \quad \text{as } \delta \to 0 ,$$

for any choice of diffeomorphism ψ whose image contains the support of η. Applying this to the entropies Φ_g given in (2.5), and using that $\psi(\cdot,\delta) \circ \psi(\cdot,0)^{-1} \to \mathrm{id}_{U\cap\partial\Omega}$ in $C^1_{\mathrm{loc}}(U \cap \partial\Omega)$, we infer

$$\int_{\mathbb{T}\times(U\cap\partial\Omega)} g(s)\eta(x)\chi_\delta(s,x)\langle e^{is}, \nu(x)\rangle\, ds d\mathcal{H}^1(x) \to \ell_{\eta,\Phi_g} \quad \text{as } \delta \to 0 ,$$

for all $\eta \in C^1_c(\mathbb{R}^2)$ and $g \in C^0(\mathbb{T})$, Recalling the weak star convergence $\chi_{\delta_j} \overset{*}{\rightharpoonup} \chi_0$, this implies that the quantity

$$\int_{\mathbb{T}\times(U\cap\partial\Omega)} g(s)\eta(x)\chi_0(s,x)\langle e^{is}, \nu(x)\rangle\, ds d\mathcal{H}^1(x) = \ell_{\eta,\Phi_g} ,$$

is independent of the sequence $\delta_j \to 0$ and of the choice of ψ. This being true for all η and g, and since $\langle e^{is}, \nu(x)\rangle \neq 0$ a.e. in $\mathbb{T}\times\partial\Omega$, we deduce that $\chi_0(s,x)$ is independent of the sequence $\delta_j \to 0$ and that two different choices of diffeomorphisms ψ give the same limit χ_0 on the intersection of their domains.

Step 2. The weak trace is an indicator function. We have $\chi_0 = \mathrm{tr}(\chi) \in \{0,1\}$ a.e. on $\mathbb{T} \times \partial\Omega$.

Proof that Step 2 implies the conclusion of Proposition 3.1. With the notations of Step 1, if χ_0 takes values in $\{0,1\}$, then we have $(\chi_0)^2 = \chi_0$ and therefore

$$\begin{aligned}\int_{\mathbb{T}\times(U\cap\partial\Omega)} (\chi_\delta)^2\, dsd\mathcal{H}^1 &= \int_{\mathbb{T}\times(U\cap\partial\Omega)} \chi_\delta\, dsd\mathcal{H}^1 \\ &\to \int_{\mathbb{T}\times(U\cap\partial\Omega)} \chi_0\, dsd\mathcal{H}^1 = \int_{\mathbb{T}\times(U\cap\partial\Omega)} (\chi_0)^2\, dsd\mathcal{H}^1 ,\end{aligned}$$

so that $\chi_\delta \to \chi_0$ strongly in $L^2(\mathbb{T}\times(U\cap\partial\Omega))$, and this in turn implies that

$$\mathrm{tr}_{\gamma_\delta}(u)(\psi(x_1,\delta)) = \frac{1}{2}\int_{\mathbb{T}} \chi_\delta(s,\psi(x_1,0))e^{is}\, ds ,$$

converges strongly in $L^2(V\cap\partial H_+)$. Covering $\partial\Omega$ with coordinate patches and choosing adequate diffeomorphisms ψ concludes the proof of Proposition 3.1. $\square$

Proof of Step 2. This is obtained via a blow-up argument in the kinetic equation (2.4). The main idea is that, when blowing up at a generic boundary point $x_0 \in \partial\Omega$, the blow-up limit is an indicator function which satisfies the free transport equation $e^{is}\cdot\nabla\chi = 0$ and has therefore a strong boundary trace. That strong trace of the blow-up limit can then be shown to coincide with the blow-up limit of the weak trace χ_0, which must therefore take values in $\{0,1\}$.

To perform the blow-up argument, we fix $x_0 \in \partial\Omega$, and a diffeomorphism $\psi\colon V\to U$ as in (A.1). Then we define $\tilde{\chi}\colon \mathbb{T}\times(V\cap H_+)\to\{0,1\}$ by setting

$$\tilde{\chi}(s,x) = \chi(s,\psi(x)) = \mathbf{1}_{e^{is}\cdot\tilde{u}(x)>0}, \quad \tilde{u}(x) = u(\psi(x)) ,$$

and from the kinetic equation (2.4) satisfied by χ we deduce

$$[D\psi(x)^{-1}e^{is}]\cdot\nabla_x\tilde{\chi} = \widetilde{\Theta} \in L^2(V\cap H_+;\mathcal{M}(\mathbb{T})) .$$

For any $x_* \in V\cap\partial H_+$ and $\epsilon>0$ we define

$$\begin{aligned}&\hat{\chi}_\epsilon(s,\hat{x}) = \tilde{\chi}(s,x_*+\epsilon\hat{x}) = \mathbf{1}_{e^{is}\cdot\hat{u}_\epsilon(\hat{x})>0}, \quad \hat{u}_\epsilon(\hat{x}) = \tilde{u}(x_*+\epsilon\hat{x}) ,\\ &\widehat{\Theta}(s,\hat{x}) = \frac{1}{\epsilon}\widetilde{\Theta}(s,x_*+\epsilon\hat{x}) ,\end{aligned}$$

which satisfy

$$[D\psi(x_*+\epsilon\hat{x})^{-1}e^{is}]\cdot\nabla\hat{\chi}_\epsilon = \widehat{\Theta}_\epsilon ,$$

in $\mathcal{D}'(\mathbb{T}\times(B_2\cap H_+))$ for small $\epsilon>0$. In order to simplify the expression of $D\psi(x)^{-1}$ we choose coordinates on U in which $\nu(x_0) = -\mathbf{e}_2$, and fix a diffeomorphism ψ in graphical form

$$\psi(x_1,x_2) = (x_1, h(x_1)+x_2) ,$$

for some $h \in C^1(V \cap \{x_2 = 0\})$. That way, we have

$$D\psi(x)^{-1}v = (v_1, v_2 - h'(x_1)v_1) \quad \forall v \in \mathbb{R}^2 ,$$

and can therefore rewrite the above kinetic equation for $\hat{\chi}_\epsilon$ as

$$\begin{aligned}
[D\psi(x_*)^{-1}e^{is}] \cdot \nabla\hat{\chi}_\epsilon &= (\cos(s)\partial_{\hat{x}_1} + (\sin(s) - h'(x_{*1})\cos(s))\partial_{\hat{x}_2})\hat{\chi}_\epsilon \\
&= \widehat{\Theta}_\epsilon + \partial_{\hat{x}_2}\hat{g}_\epsilon , \\
\hat{g}_\epsilon &= [(h'(x_{*1}) - h'(x_{*1} + \epsilon\hat{x}_1))\cos(s)\hat{\chi}_\epsilon] .
\end{aligned} \tag{A.2}$$

The terms in the right-hand side of that kinetic equation satisfy $\hat{g}_\epsilon \to 0$ uniformly on $\mathbb{T} \times (B_2 \cap H_+)$, and

$$\begin{aligned}
\int_{\mathbb{T}\times(B_2\cap H_+)} |\widehat{\Theta}_\epsilon(s,\hat{x})|\, ds d\hat{x} &= \frac{1}{\epsilon}\int_{\mathbb{T}\times(B_{2\epsilon}(x_*)\cap H_+)} |\widetilde{\Theta}(s,x)|\, dsdx \\
&\lesssim \|\widetilde{\Theta}\|_{L^2(B_{2\epsilon}(x_*)\cap H_+;\mathcal{M}(\mathbb{T}))} \to 0 ,
\end{aligned}$$

as $\epsilon \to 0$. Arguing as in [42, Proposition 3] relying e.g. on the velocity averaging results of [38], we deduce that we can extract a (non-relabeled) sequence $\epsilon \to 0$ such that

$$\hat{u}_\epsilon(\hat{x}) = \frac{1}{2}\int_{\mathbb{T}} \hat{\chi}_\epsilon(s,\hat{x})e^{is}\, ds \to \hat{u}_*(\hat{x}) \quad \text{in } L^1_{\mathrm{loc}}(B_2 \cap H_+) ,$$

for some $\mathbb{S}^1$-valued map $\hat{u}_*$. Letting $\hat{\chi}_*(s,\hat{x}) = \mathbf{1}_{e^{is}\cdot\hat{u}_*(\hat{x})>0}$, and since

$$\int_{\mathbb{T}} |\hat{\chi}_\epsilon(s,\hat{x}) - \hat{\chi}_*(s,\hat{x})|\, ds \lesssim |\hat{u}_\epsilon(\hat{x}) - \hat{u}_*(\hat{x})| ,$$

we deduce also that

$$\hat{\chi}_\epsilon \to \hat{\chi}_* \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{T} \times (B_2 \cap H_+)) .$$

Moreover, if we consider $\hat{\chi}_\epsilon$ and $\hat{\chi}_*$ as extended by 0 in $B_1 \setminus H_+$, then we have $\hat{\chi}_\epsilon \to \hat{\chi}_*$ in $L^1_{\mathrm{loc}}(\mathbb{T} \times K)$ for any compact set $K \subset B_2 \setminus \{x_2 = 0\}$. Since $(\hat{\chi}_\epsilon)$ is weakly compact in $L^2_{\mathrm{loc}}(B_2)$, we also have $\hat{\chi}_\epsilon \to \hat{\chi}_*$ weakly in $L^2_{\mathrm{loc}}(B_2)$. But we have already seen that this, combined with the fact that $\hat{\chi}_\epsilon$ and $\hat{\chi}_*$ take values into $\{0,1\}$, implies strong convergence. Thus we have

$$\mathbf{1}_{H_+}\hat{\chi}_\epsilon \to \mathbf{1}_{H_+}\hat{\chi}_* \quad \text{in } L^1_{\mathrm{loc}}(\mathbb{T} \times B_2) .$$

Further, recall that the right-hand side of the kinetic equation (A.2) satisfied by $\hat{\chi}_\epsilon$ converges to 0 in distributions, so that its limit $\hat{\chi}_*$ satisfies the free transport equation

$$[D\psi(x_*)^{-1}e^{is}] \cdot \nabla_{\hat{x}}\hat{\chi}_* = 0 \quad \text{in } \mathcal{D}'(B_2 \cap H_+) .$$

This implies that, for a.e. $s \in \mathbb{T}$, the function $\hat{\chi}_*(s,\cdot)$ depends only on the variable $\hat{x} \cdot (iD\psi(x_*)^{-1}e^{is})$ orthogonal to the direction $D\psi(x_*)^{-1}e^{is}$. For a.e. $s \in \mathbb{T}$, that variable is transverse to $\{x_2 = 0\}$, so that $\hat{\chi}_*(s,\cdot)$ has a strong L^1_{loc} trace onto $B_2 \cap \{x_2 = 0\}$. By dominated convergence, we deduce that $\hat{\chi}_*$ has a local strong $\{0,1\}$-valued trace $\mathrm{tr}(\hat{\chi}_*)$ on $\mathbb{T} \times (B_2 \cap \{\hat{x}_2 = 0\})$, and in particular

$$\operatorname*{ess\,lim}_{\delta\to 0} \int_{\mathbb{T}\times[-1,1]} |\hat{\chi}_*(s,(\hat{x}_1,\delta)) - \mathrm{tr}(\hat{\chi}_*)(s,(\hat{x}_1,0))| \, ds d\hat{x}_1 = 0$$

Finally we make the link between this trace of $\hat{\chi}_*$ and the weak trace χ_0, by considering, for any $\zeta \in C^1_c(\mathbb{T}\times(-1,1))$, the function

$$\begin{aligned}
\rho_\epsilon(\hat{x}_2) &= \int_{\mathbb{T}\times[-1,1]} (\hat{a}_\epsilon(s,\hat{x})\hat{\chi}_\epsilon(s,\hat{x}) - \hat{a}_*(s)\hat{\chi}_*(s,\hat{x}))\zeta(s,\hat{x}_1) \, ds d\hat{x}_1 \, , \\
\hat{a}_\epsilon(s,\hat{x}) &= \sin(s) - h(x_{*1} + \epsilon\hat{x}_1)\cos(s), \quad \hat{a}_*(s) = \sin(s) - h(x_{*1})\cos(s) \, ,
\end{aligned}$$

defined for $0 < \hat{x}_2 \leq 1$. The weak trace property established in Step 1 ensures

$$\begin{aligned}
\rho_\epsilon(0^+) &= \lim_{\hat{x}_2\to 0^+} \rho_\epsilon(\hat{x}_2) \\
&= \int_{\mathbb{T}\times[-1,1]} (\hat{a}_\epsilon(s,\hat{x})\tilde{\chi}_0(s,x_* + \epsilon\hat{x}_1) - \hat{a}_*(s)\hat{\chi}_*(s,\hat{x}_1))\zeta(s,\hat{x}_1) \, ds d\hat{x}_1 \, ,
\end{aligned}$$

where $\tilde{\chi}_0(s,x) = \chi_0(s,\psi(x))$. Moreover, the kinetic equation (A.2) ensures that ρ_ϵ is weakly differentiable, with

$$\begin{aligned}
\partial_{\hat{x}_2}\rho_\epsilon(\hat{x}_2) &= \int_{\mathbb{T}\times[-1,1]} \zeta(s,\hat{x}_1) \, \widehat{\Theta}_\epsilon(s,\hat{x}) \, ds \, d\hat{x}_1 \\
&\quad + \int_{\mathbb{T}\times[-1,1]} \cos(s)(\hat{\chi}_\epsilon(s,\hat{x}) - \mathrm{tr}(\hat{\chi}_*)(s,\hat{x}))\partial_{\hat{x}_1}\zeta(s,\hat{x}_1) \, ds d\hat{x}_1 \, ,
\end{aligned}$$

hence

$$\begin{aligned}
\|\rho_\epsilon\|_{L^\infty(0,1)} &\lesssim \int_0^1 |\rho_\epsilon| \, d\hat{x}_2 + \int_0^1 |\partial_{\hat{x}_2}\rho_\epsilon| \, d\hat{x}_2 \\
&\lesssim \|\zeta\|_\infty \|\hat{a}_\epsilon - \hat{a}_*\|_{L^1(\mathbb{T}\times(B_2\cap H_+))} \\
&\quad + \|\zeta\|_\infty \|\hat{a}_*\|_\infty \|\hat{\chi}_\epsilon - \hat{\chi}_*\|_{L^1(\mathbb{T}\times(B_2\cap H_+))} \\
&\quad + \|\zeta\|_\infty \|\widehat{\Theta}_\epsilon\|_{L^1(\mathbb{T}\times(B_2\cap H_+))} \\
&\quad + \|\partial_{\hat{x}_1}\zeta\|_\infty \|\hat{\chi}_\epsilon - \hat{\chi}_*\|_{L^1(\mathbb{T}\times(B_2\cap H_+))} \, .
\end{aligned}$$

We infer that $\rho_\epsilon \to 0$ uniformly on $(0,1)$ as $\epsilon \to 0$. Given the above expression of $\rho_\epsilon(0^+)$ and the uniform convergence $\hat{a}_\epsilon \to \hat{a}_*$, this implies

$$\lim_{\epsilon\to 0} \int_{\mathbb{T}\times[-1,1]} \hat{a}_*(s)(\tilde{\chi}_0(s,x_* + \epsilon\hat{x}_1) - \mathrm{tr}(\hat{\chi}_*)(s,\hat{x}_1))\zeta(s,\hat{x}_1) \, ds d\hat{x}_1 = 0 \, ,$$

for all $\zeta \in C_c^1(\mathbb{T}\times(-1,1))$. Recall that this is valid for any $x_* \in V \cap H_+$. The blow-up function $\hat{\chi}_*$ depends on x_* and on the sequence $\epsilon \to 0$, and its strong trace $\mathrm{tr}(\hat{\chi}_*)$ takes values in $\{0,1\}$. Further, for almost every choice of $x_* \in V \cap H_+$ we have

$$\int_{\mathbb{T}\times[-1,1]} |\tilde{\chi}_0(s, x_* + \epsilon\hat{x}_1) - \tilde{\chi}_0(s, x_*)|\, ds d\hat{x}_1 \to 0\,,$$

as $\epsilon \to 0$. (This can be interpreted as almost every x_* being a Lebesgue point of $x \mapsto \chi_0(\cdot, x) \in L^1(\mathbb{T})$, and proved in the same way as the usual Lebesgue differentiation theorem. See also [42, Lemma 3] for a proof of this fact up to extracting a subsequence, which is sufficient for our purpose here.) Combining this with the above and the fact that $\hat{a}_*(s) = \sin(s) - h(x_{*1})\cos(s) \neq 0$ for a.e. $s \in \mathbb{T}$, we conclude that $\tilde{\chi}_0(s, x_*) \in \{0,1\}$ for a.e. $(s, x_*) \in \mathbb{T}\times(V\cap\partial H_+)$, hence $\chi_0 = \tilde{\chi}_0 \circ (\mathrm{id}_{\mathbb{T}} \otimes \psi(\cdot,0)^{-1})$ is an indicator function on $\mathbb{T}\times(\partial\Omega\cap U)$. $\square$

Appendix B A symmetry question

In the non-thin periodic case, is it true that any minimizer u of E_ϵ in $[-1,1]^2$ subject to the boundary conditions $u(\pm 1, x_2) = e_\pm$ and $u(x_1,-1) = u(x_1,+1)$, depends only on x_1 ? In this Appendix we answer this question affirmatively.

Denote by $u_\epsilon^{1\mathrm{D}}(x_1) = e^{i\varphi_\epsilon^{1\mathrm{D}}(x_1)}$ the one-dimensional minimizer of the energy $E_\epsilon^{1\mathrm{D}}$ restricted to maps which depend only on x_1, and $\mathfrak{e}_\epsilon^{1\mathrm{D}}$ the value of its energy. Denote also by $\varphi_\pm$ the boundary values of the phase $\varphi_\epsilon^{1\mathrm{D}}$, which are such that $e_\pm = e^{i\varphi_\pm}$. Since

$$E_\epsilon^{1\mathrm{D}}(e^{i\varphi}) = \int_{-1}^1 (\sin^2\varphi + \epsilon\cos^2\varphi)(\varphi')^2\, dx = \int_{-1}^1 \left([F_\epsilon(\varphi)]'\right)^2\, dx,$$
$$\text{where } F_\epsilon(t) = \int_0^t \sqrt{\sin^2 s + \epsilon\cos^2 s}\, ds,$$

we have that $\varphi_\epsilon^{1\mathrm{D}}$ solves

$$\frac{d^2}{dx_1^2}\big[F_\epsilon(\varphi_\epsilon^{1\mathrm{D}})\big] = 0\,.$$

It follows that the derivative of $F_\epsilon(\varphi_\epsilon^{1\mathrm{D}})$ with respect to x_1 is constant, and integrating on $(-1,1)$ we find the value of this constant:

$$\frac{d}{dx_1}\big[F_\epsilon(\varphi_\epsilon^{1\mathrm{D}})\big] = \frac{F_\epsilon(\varphi_+) - F_\epsilon(\varphi_-)}{2}\,.$$

Hence the value of the minimal 1D energy is

$$\mathfrak{e}_\epsilon^{1\mathrm{D}} = \int_{-1}^1 \left(\frac{d}{dx_1}\big[F_\epsilon(\varphi_\epsilon^{1D})\big]\right)^2 dx_1 = \frac{1}{2}\big(F_\epsilon(\varphi_+) - F_\epsilon(\varphi_-)\big)^2\,,$$

and $\varphi_\epsilon^{\rm 1D}$ solves

$$\frac{d\varphi_\epsilon^{\rm 1D}}{dx_1} = \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi_\epsilon^{\rm 1D})}\,, \qquad \tau = \operatorname{sign}(\varphi_+ - \varphi_-)\,.$$

Guided by this equation, we use the fact that $F_\epsilon'(\varphi)^2 = \sin^2\varphi + \epsilon\cos^2\varphi$ to rewrite the energy of any two-dimensional configuration $u = e^{i\varphi}$ with $\varphi(\pm 1, x_2) = \varphi_\pm$, as

$$\begin{aligned}
2E_\epsilon(e^{i\varphi}) &= \iint \Big(-\sin\varphi\,\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)} + \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)}\Big) + \cos\varphi\,\partial_2\varphi\Big)^2 \\
&\qquad + \epsilon\Big(\cos\varphi\,\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)} + \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)}\Big) + \sin\varphi\,\partial_2\varphi\Big)^2\,dx \\
&= \iint F_\epsilon'(\varphi)^2\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)} + \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)}\Big)^2 \\
&\qquad + (\cos^2\varphi + \epsilon\sin^2\varphi)(\partial_2\varphi)^2 \\
&\qquad + 2(\epsilon - 1)\cos\varphi\,\sin\varphi\,\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)} + \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)}\Big)\partial_2\varphi\;dx\,.
\end{aligned}$$

Expanding the integrand gives

$$\begin{aligned}
&2\big(E_\epsilon(e^{i\varphi}) - \mathfrak{e}_\epsilon^{\rm 1D}\big) \\
&= \iint F_\epsilon'(\varphi)^2\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)}\Big)^2 + (\cos^2\varphi + \epsilon\sin^2\varphi)(\partial_2\varphi)^2 \\
&\qquad + 2(\epsilon - 1)\cos\varphi\,\sin\varphi\,\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}}{F_\epsilon'(\varphi)}\Big)\partial_2\varphi \\
&\qquad + 2\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}\Big(F_\epsilon'(\varphi)\partial_1\varphi - \tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}\Big) \\
&\qquad + 2(\epsilon - 1)\tau\sqrt{\mathfrak{e}_\epsilon^{\rm 1D}/2}\frac{\cos\varphi\,\sin\varphi}{F_\epsilon'(\varphi)}\partial_2\varphi\;dx\,.
\end{aligned}$$

The last line integrates to zero by x_2-periodicity, and so does the next-to-last line, because

$$\int_{-1}^{1} F_\epsilon'(\varphi)\partial_1\varphi\,dx_1 = \int_{-1}^{1}\partial_1[F_\epsilon(\varphi)]\,dx_1 = F_\epsilon(\varphi_+) - F_\epsilon(\varphi_-) = \tau\sqrt{2\mathfrak{e}_\epsilon^{\rm 1D}}\,.$$

Thus we obtain, recalling also that $F_\epsilon'(t)^2 = \sin^2 t + \epsilon\cos^2 t$,

$$\begin{aligned}
&2\big(E_\epsilon(e^{i\varphi}) - \mathfrak{e}_\epsilon^{1\mathrm{D}}\big)\\
&= \iint (\sin^2\varphi + \epsilon\cos^2\varphi)\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{1\mathrm{D}}/2}}{F_\epsilon'(\varphi)}\Big)^2 + (\cos^2\varphi + \epsilon\sin^2\varphi)(\partial_2\varphi)^2\\
&\qquad + 2(\epsilon-1)\cos\varphi\,\sin\varphi\,\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{1\mathrm{D}}/2}}{F_\epsilon'(\varphi)}\Big)\partial_2\varphi\,dx\\
&= \iint \Big(\frac{1+\epsilon}{2} - \frac{1-\epsilon}{2}\cos(2\varphi)\Big)\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{1\mathrm{D}}/2}}{F_\epsilon'(\varphi)}\Big)^2\\
&\qquad + \Big(\frac{1+\epsilon}{2} + \frac{1-\epsilon}{2}\cos(2\varphi)\Big)(\partial_2\varphi)^2\\
&\qquad + (\epsilon-1)\sin(2\varphi)\,\Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{1\mathrm{D}}/2}}{F_\epsilon'(\varphi)}\Big)\partial_2\varphi\,dx
\end{aligned}$$

For any $C, S \in \mathbb{R}$ such that $C^2 + S^2 = 1$, the quadratic form on $\mathbb{R}^2$ given by

$$\begin{aligned}
q(X,Y) = {} & ((1+\epsilon) - (1-\epsilon)C)X^2\\
& + ((1+\epsilon) + (1-\epsilon)C)Y^2\\
& + 2(\epsilon-1)SXY\,,
\end{aligned}$$

has determinant $\det(q) = 4\epsilon$ and trace $\mathrm{tr}(q) = 2(1+\epsilon)$, hence it satisfies

$$q(X,Y) \geq 2\min(1,\epsilon)(X^2+Y^2)\,.$$

Applying this to the integrand in the above expression of the energy difference, we deduce that

$$E_\epsilon(e^{i\varphi}) - \mathfrak{e}_\epsilon^{1\mathrm{D}} \geq \frac{\min(1,\epsilon)}{2}\iint (\partial_2\varphi)^2 + \Big(\partial_1\varphi - \frac{\tau\sqrt{\mathfrak{e}_\epsilon^{1\mathrm{D}}/2}}{F_\epsilon'(\varphi)}\Big)^2\,dx\,,$$

for any H^1 function φ satisfying the boundary conditions $\varphi(\pm1, x_2) = \varphi_\pm$ and $\varphi(x_1,-1) = \varphi(x_1,+1)$. The assertion made in this appendix is thus proved.

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