

On the minimality of the radial singularity in nematic liquid crystals

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Abstract

We show that the map $u_*: B_1 \subset \mathbb{R}^3 \rightarrow \mathbb{S}^2$, $x \mapsto x/|x|$, minimizes the anisotropic energy

$$\int_{B_1} \left(k_1(\operatorname{div} u)^2 + k_2(u \cdot \operatorname{curl} u)^2 + k_3|u \times \operatorname{curl} u|^2 \right) dx,$$

among $\mathbb{S}^2$ -valued maps agreeing with it on the boundary, for values of $k_1, k_2, k_3 > 0$ beyond the known regime $k_1 \leq k_2$. Our main tool is a new stability estimate for u_* as a minimizing sphere-valued harmonic map.

1 Introduction

We are interested in the Oseen-Frank energy

$$E_{\mathbf{k}}(u) = \int_{B_1} \left(k_1(\operatorname{div} u)^2 + k_2(u \cdot \operatorname{curl} u)^2 + k_3|u \times \operatorname{curl} u|^2 \right) dx, \quad (1.1)$$

of maps $u: B_1 \subset \mathbb{R}^3 \rightarrow \mathbb{S}^2 \subset \mathbb{R}^3$ which agree with the radial map

$$u_*(x) = \frac{x}{|x|}, \quad (1.2)$$

on the boundary ∂B_1 . Here $k_1, k_2, k_3 > 0$ are the elastic constants associated to splay, twist and bend deformations in a nematic liquid crystal [5], and we denote their triple by $\mathbf{k} = (k_1, k_2, k_3)$. The admissible set of maps is defined by

$$H_{u_*}^1(B_1; \mathbb{S}^2) = \left\{ u \in H^1(B_1; \mathbb{R}^3) : u(x) \in \mathbb{S}^2 \text{ for a.e. } x \in B_1 \right. \\ \left. \text{and } u = u_* \text{ on } \partial B_1 \right\}. \quad (1.3)$$

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The radial map u_* is a critical point of $E_{\mathbf{k}}$ for all values of $\mathbf{k}$, see e.g. [13, Proposition 3]. However, that u_* minimizes $E_{\mathbf{k}}$ or not, is only known for some, but not all values of $\mathbf{k}$. Our main results partially fill this gap.

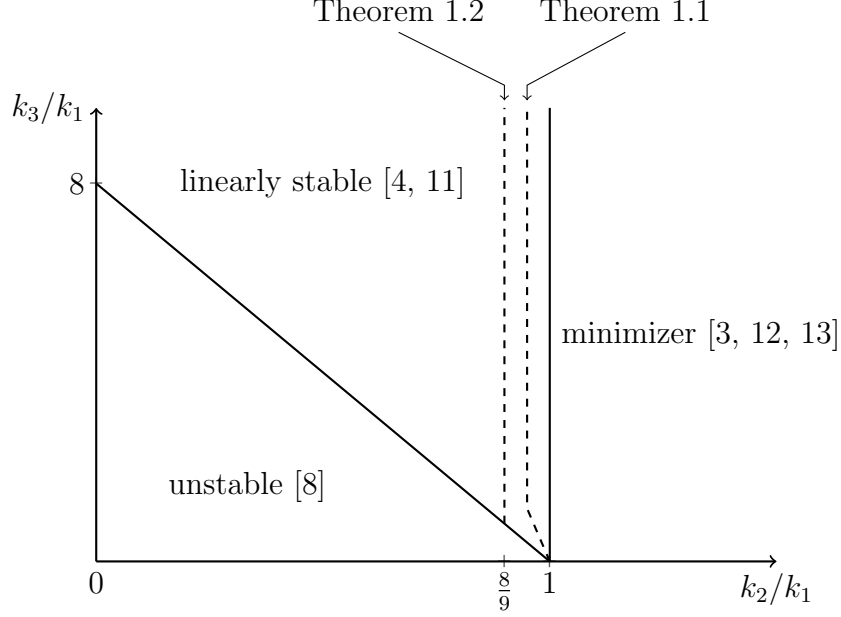


Figure 1: Known facts about the minimality of u_* , and new results proven in this article: minimality on the right of the dashed lines, among all competitors (Theorem 1.1) or those only singular at the origin (Theorem 1.2).

The first known result in that direction [3] is that u_* is the unique minimizer of the Dirichlet energy

$$D(u) = \int_{B_1} |\nabla u|^2 dx. \quad (1.4)$$

This is relevant to the case of equal elastic constants $\mathbf{k} = \mathbf{1} = (1, 1, 1)$, because $(\operatorname{div} u)^2 + |\operatorname{curl} u|^2 - |\nabla u|^2 = \operatorname{div}((\operatorname{div} u)u - (u \cdot \nabla)u)$ is a null Lagrangian: the quantity $E_1(u) - D(u)$ depends only on the fixed boundary datum $u = u_*$ on ∂B_1 and thus

$$E_1(u) - D(u) = E_1(u_*) - D(u_*) = 16\pi - 8\pi = 8\pi. \quad (1.5)$$

It follows that u_* is also the unique minimizer of the Oseen-Frank energy E_1 .

More generally, it is known that u_* is the unique minimizer of $E_{\mathbf{k}}$ provided that $k_2 \geq k_1$ [12, 13]. In the opposite direction, u_* is not a minimizer of $E_{\mathbf{k}}$ if $k_2 + k_3/8 < k_1$ [8]. For the remaining values $k_2 < k_1 \leq k_2 + k_3/8$, the radial map u_* is linearly stable (i.e. the second variation of $E_{\mathbf{k}}$ at u_* is positive definite)

[4, 11], but its minimality is open. Numerical simulations in [2] suggest that it might be non-minimizing for some of these linearly stable values.

Our main result establishes that u_* is a minimizer at least slightly beyond the previously known regime $k_2 \geq k_1$.

Theorem 1.1. *There exists $\delta \in (0, 1/8]$ such that, for any $k_1, k_2, k_3 > 0$ with $k_1 \leq k_2 + \delta \min(k_2, k_3)$, the radial hedgehog u_* is the unique minimizer of the Oseen-Frank energy $E_{\mathbf{k}}$ in $H_{u_*}^1(B_1; \mathbb{S}^2)$.*

The proof of Theorem 1.1 relies on a compactness argument and we are not able to provide an optimal value of δ . However, if we restrict ourselves to competitors which are continuous away from the origin, we can actually show minimality under the condition $k_1 \leq k_2 + \min(k_2, k_3)/8$, which, for $k_2 \geq k_3$, corresponds exactly to the regime of linear stability $k_1 \leq k_2 + k_3/8$.

Theorem 1.2. *For any $k_1, k_2, k_3 > 0$ with $k_1 \leq k_2 + \min(k_2, k_3)/8$, the radial hedgehog u_* is the unique minimizer of the Oseen-Frank energy $E_{\mathbf{k}}$ among maps in $H_{u_*}^1(B_1; \mathbb{S}^2)$ which are continuous in $\overline{B}_1 \setminus \{0\}$.*

Remark 1.3. As noted in [4], even though the second variation of $E_{\mathbf{k}}$ at u_* is positive definite for $k_1 < k_2 + k_3/8$, and this implies local minimality of u_* with respect to small Lipschitz perturbations, it is not clear whether this implies local minimality of u_* with respect to the natural H^1 distance. This is because the functional setting is not smooth enough, as can be seen e.g. by moving the singularity from the origin [4, Proposition 5]. In fact, the map u_* is a minimizer if and only if it is a local minimizer (in the H^1 distance). Assume indeed that u_* is not a minimizer. Then, there exists $\tilde{u} \in H_{u_*}^1(B_1; \mathbb{S}^2)$ such that $E_{\mathbf{k}}(\tilde{u}) < E_{\mathbf{k}}(u_*)$. Extend $\tilde{u}$ to $\mathbb{R}^3$ by u_* , and define $\tilde{u}_\epsilon(x) = \tilde{u}(x/\epsilon)$. Then, $\tilde{u}_\epsilon \rightarrow u_*$ in H^1 as $\epsilon \rightarrow 0$, and $E_{\mathbf{k}}(\tilde{u}_\epsilon) - E_{\mathbf{k}}(u_*) = \epsilon(E_{\mathbf{k}}(\tilde{u}) - E_{\mathbf{k}}(u_*)) < 0$, so u_* is not a local minimizer.

Both Theorem 1.1 and Theorem 1.2 rely on stability estimates for u_* as a minimizer of the Dirichlet energy $D(u) = \int_{B_1} |\nabla u|^2 dx$ among competitors with one singularity, which are of independent interest.

Theorem 1.4. *There exists $c_* > 0$ such that, if $u \in H_{u_*}^1(B_1; \mathbb{S}^2) \cap C^0(\overline{B}_1 \setminus \{x_0\})$ for some $x_0 \in B_1$, we have*

$$c_* \int_{B_1} |\nabla u_*(\cdot - x_0) - \nabla u|^2 dx \leq D(u) - D(u_*). \quad (1.6)$$

When $x_0 = 0$, one can take $c_* = 1/9$.

Even if the above statement is restricted to maps u having just one singularity, it is sufficient for the proof of Theorem 1.1, due to the fact that for $\mathbf{k} \approx \mathbf{1}$, minimizers of $E_{\mathbf{k}}$ have only one singularity [1]. However, we expect that a suitable formulation of (1.6) is still valid on the whole set $H_{u_*}^1(B_1; \mathbb{S}^2)$:

Conjecture 1.5. *There exists $c_* > 0$ such that*

$$c_* \inf_{x_0 \in B_1} \int_{B_1} |\nabla u_*(\cdot - x_0) - \nabla u|^2 dx \leq D(u) - D(u_*), \quad (1.7)$$

for all $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$.

Since the map u_* is the unique minimizer of the Dirichlet energy D in $H_{u_*}^1(B_1; \mathbb{S}^2)$, and the second variation of D at u_* is a positive definite quadratic form [4], one might be tempted to expect the nondegenerate, quadratic stability estimate:

$$D(u) - D(u_*) \geq c_* \int_{B_1} |\nabla u - \nabla u_*|^2 dx. \quad (1.8)$$

But the example

$$u^{x_0}(x) = u_*(x - \eta(|x|)x_0),$$

for a smooth function η with $\mathbf{1}_{r \leq 1/2} \leq \eta(r) \leq \mathbf{1}_{r \leq 3/4}$ and $|x_0| \leq 1/4$, actually shows that (1.8) is not possible: one checks by direct calculation that

$$\int_{B_1} |\nabla u^{x_0} - \nabla u_*|^2 dx \gtrsim \sqrt{D(u^{x_0}) - D(u_*)},$$

see Lemma 2.4 below. The main feature of this example is to translate the singularity, and Conjecture 1.5 asserts that this should be the only source of degeneracy.

Remark 1.6. In higher dimensions $n \geq 4$, the radial map $u_*(x) = x/|x|$ is also a minimizer of the Dirichlet energy among $\mathbb{S}^{n-1}$ -valued maps agreeing with u_* on the boundary of $B_1 \subset \mathbb{R}^n$ [12]. For $n \geq 7$ the nondegenerate stability estimate (1.8) is known : see the proof of [10, Theorem A (ii)] and remark a in that paper. For $n = 4$, translations u^{x_0} also provide a counterexample, see Remark 2.5. The validity of (1.8) for $n \in \{5, 6\}$ seems open. Note that, thanks to [9, Theorem 2.1] it would be enough to check it on maps u which are continuous in $\overline{B_1} \setminus \{0\}$.

From (1.6), using that $|\operatorname{curl} v|^2 \leq 2|\nabla v|^2$ and that u_* as well as its translations are curl-free, one gets

$$c_* \int_{B_1} |\operatorname{curl} u|^2 dx \leq D(u) - D(u_*). \quad (1.9)$$

As a matter of fact, this weaker form of (1.6) is sufficient to establish Theorems 1.1 and 1.2.

Remark 1.7. Another indication supporting Conjecture 1.5 is that the nondegenerate estimate (1.9) on $\operatorname{curl} u$ is actually valid among all competitors $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$ and not just for those which are continuous on $\overline{B_1}$ except at one

point. To check this, note that, in view of Theorem 1.1, there exists $\delta > 0$ such that the radial map u_* minimizes $E_{\mathbf{k}}$, with $\mathbf{k} = (1 + \delta, 1, 1)$. Using this minimality and the fact that $\operatorname{curl} u_* = 0$, we obtain that for every $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$,

$$\begin{aligned} \frac{\delta}{1 + \delta} \int_{B_1} |\operatorname{curl} u|^2 dx &= E_1(u) - \frac{1}{1 + \delta} E_{\mathbf{k}}(u) \\ &\leq E_1(u) - \frac{1}{1 + \delta} E_{\mathbf{k}}(u_*) = E_1(u) - E_1(u_*), \end{aligned}$$

and we have already seen that the last expression equals $D(u) - D(u_*)$, see (1.5).

Organization of the article

We prove the stability estimate of Theorem 1.4 in §2 and the minimality statements of Theorem 1.1 and Theorem 1.2 in §3.

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2 Stability among maps with one singularity

In this section we prove the quadratic stability of u_* modulo translations for the Dirichlet energy among sphere-valued maps which are discontinuous at just one point, as stated in Theorem 1.4.

2.1 Singularity at the origin

We start by presenting the short proof of Theorem 1.4 for maps $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$ which are continuous away from the origin $x_0 = 0$. This assumption ensures that the restrictions of u to concentric spheres have topological degree equal to one, hence enables us to use the minimality of $u_*: \mathbb{S}^2 \rightarrow \mathbb{S}^2$ in its homotopy class.

Lemma 2.1. *If $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$ is continuous in $\overline{B_1} \setminus \{0\}$, then*

$$\frac{1}{9} \int_{B_1} |\nabla u - \nabla u_*|^2 dx \leq \int_{B_1} |\nabla u|^2 dx - 8\pi. \quad (2.1)$$

Equality occurs if and only if $u = u_$.*

Proof. For a.e. $r \in (0, 1)$, the map $u^{(r)}: \omega \mapsto u(r\omega)$ belongs to $H^1(\mathbb{S}^2; \mathbb{S}^2)$, and it has degree one thanks to the continuity assumption. Classically, the inequality

$|\nabla_\omega u^{(r)}|^2 \geq 2|\det(\nabla_\omega u^{(r)})|$ (where $\nabla_\omega u^{(r)}$ is seen as a linear map from $T_\omega \mathbb{S}^2$ to $T_{u^{(r)}(\omega)} \mathbb{S}^2$) therefore implies

$$\int_{\mathbb{S}^2} |\nabla_\omega u^{(r)}|^2 d\mathcal{H}^2(\omega) \geq 8\pi |\deg(u^{(r)})| = 8\pi.$$

Thus we have, using also classically that $-\Delta_\omega u_* = 2u_*$ and $2u_* \cdot (u_* - u^{(r)}) = |u^{(r)} - u_*|^2$,

$$\begin{aligned} 0 &\leq \int_{\mathbb{S}^2} |\nabla_\omega u^{(r)}|^2 d\mathcal{H}^2(\omega) - 8\pi \\ &= \int_{\mathbb{S}^2} |\nabla_\omega u^{(r)}|^2 d\mathcal{H}^2(\omega) - \int_{\mathbb{S}^2} |\nabla_\omega u_*|^2 d\mathcal{H}^2(\omega) \\ &= \int_{\mathbb{S}^2} |\nabla_\omega u^{(r)} - \nabla_\omega u_*|^2 d\mathcal{H}^2(\omega) - 2 \int_{\mathbb{S}^2} \nabla_\omega u_* \cdot \nabla_\omega (u_* - u^{(r)}) d\mathcal{H}^2(\omega) \\ &= \int_{\mathbb{S}^2} |\nabla_\omega u^{(r)} - \nabla_\omega u_*|^2 d\mathcal{H}^2(\omega) - 4 \int_{\mathbb{S}^2} u_* \cdot (u_* - u^{(r)}) d\mathcal{H}^2(\omega) \\ &= \int_{\mathbb{S}^2} |\nabla_\omega u^{(r)} - \nabla_\omega u_*|^2 d\mathcal{H}^2(\omega) - 2 \int_{\mathbb{S}^2} |u^{(r)} - u_*|^2 d\mathcal{H}^2(\omega). \end{aligned}$$

For any $\theta \in [0, 1]$, we deduce

$$\begin{aligned} &\int_{\mathbb{S}^2} |\nabla_\omega u^{(r)}|^2 d\mathcal{H}^2(\omega) - 8\pi \\ &\geq \theta \left(\int_{\mathbb{S}^2} |\nabla_\omega u^{(r)} - \nabla_\omega u_*|^2 d\mathcal{H}^2(\omega) - 2 \int_{\mathbb{S}^2} |u^{(r)} - u_*|^2 d\mathcal{H}^2(\omega) \right). \end{aligned}$$

Integrating this with respect to $r \in (0, 1)$ and adding the integral of $|\partial_r u|^2$, we find

$$\begin{aligned} &\int_{B_1} |\nabla u|^2 dx - 8\pi \\ &\geq \theta \int_{B_1} |\nabla u - \nabla u_*|^2 dx + (1 - \theta) \int_{B_1} |\partial_r u|^2 dx - 2\theta \int_{B_1} \frac{|u - u_*|^2}{|x|^2} dx. \end{aligned}$$

Further, applying Hardy's inequality

$$\frac{1}{4} \int_0^1 |f(r)|^2 dr \leq \int_0^1 |f'(r)|^2 r^2 dr \quad \text{if } f(1) = 0,$$

to $f(r) = f_\omega(r) = u(r\omega) - u_*(r\omega)$ for a.e. $\omega \in \mathbb{S}^2$, and integrating with respect to ω , we get

$$\int_{B_1} |\partial_r u|^2 dx \geq \frac{1}{4} \int_{B_1} \frac{|u - u_*|^2}{|x|^2} dx.$$

Plugging this into the above estimate yields

$$\begin{aligned} & \int_{B_1} |\nabla u|^2 dx - 8\pi \\ & \geq \theta \int_{B_1} |\nabla u - \nabla u_*|^2 dx + \left(\frac{1-\theta}{4} - 2\theta \right) \int_{B_1} \frac{|u - u_*|^2}{|x|^2} dx. \end{aligned}$$

Choosing $\theta = 1/9$ gives (2.1). Moreover, if equality occurs, then there must be equality in Hardy's inequality, which implies $f_\omega = 0$ for a.e. ω , and thus $u = u_*$. $\square$

2.2 Singularity away from the origin

This section is devoted to the proof of Theorem 1.4. It relies on a generalization of Lemma 2.1 where u_* is replaced by $u_*(\cdot - x_0)$ for any $x_0 \in B_1$, see Lemma 2.2. We also need an estimate of x_0 in terms of the energy deficit $D(u) - D(u_*)$, whenever u is continuous except at x_0 . This inequality, which is encapsulated in Lemma 2.3, is a particular case of more general estimates established in [3].

Lemma 2.2. *For any $x_0 \in B_1$ and $u \in H_{u_*}^1(B_1; \mathbb{S}^2) \cap C^0(\overline{B_1} \setminus \{x_0\})$, we have*

$$\frac{1}{9} \int_{B_1} |\nabla u - \nabla u_*(\cdot - x_0)|^2 dx \leq \int_{B_1} |\nabla u|^2 dx - 8\pi + \frac{32\pi}{9} |x_0|^2. \quad (2.2)$$

Proof of Lemma 2.2. Consider u as defined on $\mathbb{R}^2$ by setting $u = u_*$ outside B_1 , and define $\tilde{u}(\tilde{x}) = u(x_0 + \tilde{x})$, which has degree one on ∂B_r for a.e. $r > 0$. Arguing exactly as in Lemma 2.1, we obtain therefore

$$\begin{aligned} & \int_{B_2} |\nabla \tilde{u}|^2 dx - 16\pi \\ & \geq \theta \int_{B_2} |\nabla \tilde{u} - \nabla u_*|^2 dx + (1-\theta) \int_{B_2} |\partial_r \tilde{u}|^2 dx - 2\theta \int_{B_2} \frac{|\tilde{u} - u_*|^2}{|x|^2} dx, \end{aligned}$$

for any $\theta \in (0, 1)$. Then we apply the Hardy inequality

$$\frac{1}{4} \int_0^2 |f(r)|^2 dr \leq \int_0^2 |f'(r)|^2 r^2 dr + |f(2)|^2,$$

which follows from the identity

$$(f')^2 r^2 + \frac{1}{2} (r f^2)' - \frac{1}{4} f^2 = r ((\sqrt{r} f)')^2 \geq 0,$$

to $f(r) = f_\omega(r) = \tilde{u}(r\omega) - u_*(r\omega)$ for a.e. $\omega \in \mathbb{S}^2$, and integrate with respect to ω , to find

$$\int_{B_2} |\partial_r \tilde{u}|^2 dx \geq \frac{1}{4} \int_{B_2} \frac{|\tilde{u} - u_*|^2}{|x|^2} dx - \frac{1}{4} \int_{\partial B_2} |\tilde{u} - u_*|^2 d\mathcal{H}^2.$$

Plugging this into the previous estimate and using that $\tilde{u} = u_*(\cdot + x_0)$ on ∂B_2 gives

$$\begin{aligned} & \int_{B_2} |\nabla \tilde{u}|^2 dx - 16\pi + \frac{1-\theta}{4} \int_{\partial B_2} |u_*(\cdot + x_0) - u_*|^2 d\mathcal{H}^2 \\ & \geq \theta \int_{B_2} |\nabla \tilde{u} - \nabla u_*|^2 dx + \frac{1-9\theta}{4} \int_{B_2} \frac{|\tilde{u} - u_*|^2}{|x|^2} dx. \end{aligned}$$

Choosing $\theta = 1/9$ and replacing $\tilde{u} = u(\cdot + x_0)$, we infer

$$\begin{aligned} & \frac{1}{9} \int_{B_2(x_0)} |\nabla u - \nabla u_*(\cdot - x_0)|^2 dx \\ & \leq \int_{B_2(x_0)} |\nabla u|^2 - 16\pi + \frac{2}{9} \int_{\partial B_2} |u_*(\cdot + x_0) - u_*|^2 d\mathcal{H}^2. \end{aligned} \quad (2.3)$$

To estimate the right-hand side, first note that

$$\begin{aligned} \frac{1}{4} \int_{\partial B_2} |u_*(\cdot + x_0) - u_*|^2 d\mathcal{H}^2 &= \int_{\mathbb{S}^2} \left| \frac{\omega + x_0/2}{|\omega + x_0/2|} - \omega \right|^2 d\mathcal{H}^2(\omega) \\ &\leq 4\pi|x_0|^2, \end{aligned} \quad (2.4)$$

where we used the fact that, for any $a, b \in \mathbb{R}^3 \setminus \{0\}$,

$$\left| \frac{a}{|a|} - \frac{b}{|b|} \right| = \frac{|(|b| - |a|)a + |a|(a - b)|}{|a||b|} \leq 2 \frac{|a - b|}{|b|}, \quad (2.5)$$

applied to $a = \omega + x_0/2$ and $b = \omega$. Moreover, since $u = u_*$ outside B_1 and $B_1 \subset B_2(x_0)$, we have

$$\begin{aligned} & \int_{B_2(x_0)} |\nabla u|^2 dx - 16\pi - \left(\int_{B_1} |\nabla u|^2 dx - 8\pi \right) \\ &= \int_{B_2(x_0) \setminus B_1} |\nabla u_*|^2 dx - 8\pi = \int_{B_2(x_0)} |\nabla u_*|^2 dx - \int_{B_2} |\nabla u_*|^2 dx. \end{aligned}$$

Using the fact that the function $x_0 \mapsto \int_{B_2(x_0)} |\nabla u_*|^2 dx$ is radial and decreasing because $|\nabla u_*|^2$ is radial and decreasing, we thus get

$$\int_{B_2(x_0)} |\nabla u|^2 dx - 16\pi \leq \int_{B_1} |\nabla u|^2 dx - 8\pi.$$

Plugging this inequality and (2.4) into (2.3) gives (2.2). $\square$

Lemma 2.3 ([3]). *For any $x_0 \in B_1$ and $u \in H_{u_*}^1(B_1; \mathbb{S}^2) \cap C^0(\overline{B_1} \setminus \{x_0\})$, we have*

$$\int_{B_1} |\nabla u|^2 dx - 8\pi \geq \frac{8\pi}{3} |x_0|^2. \quad (2.6)$$

Proof of Lemma 2.3. This is a special case of estimates proved in [3, §7]. We recall the proof for the reader's convenience. We have

$$\int_{B_1} |\nabla u|^2 dx \geq 2 \int_{B_1} |D| dx, \quad (2.7)$$

where the vector field D corresponds to the 2-form obtained by pulling back the volume form on $\mathbb{S}^2$ under the map u , and is explicitly given by

$$D = (u \cdot u_y \wedge u_z, u \cdot u_z \wedge u_x, u \cdot u_x \wedge u_y).$$

If x, y are orthonormal local coordinates on $\mathbb{S}^2$, then $D \cdot n = u \cdot (u_x \wedge u_y)$ on $\mathbb{S}^2$ (here, n is the outer unit normal to $\mathbb{S}^2$). Since $n = u_* = u$ on $\mathbb{S}^2$, it follows that the normal component of D is equal to 1. Moreover, since u has only one singularity of degree 1 at x_0 , one has $\operatorname{div} D = 4\pi\delta_{x_0}$ in B_1 , see [3, Appendix B].

Let us introduce the 1-Lipschitz function $\zeta(x) = |x - x_0|$. Since $|D| \geq |D \cdot \nabla \zeta|$, it follows from (2.7) that

$$\int_{B_1} |\nabla u|^2 dx \geq 2 \left| \int_{B_1} D \cdot \nabla \zeta dx \right|.$$

Integrating by parts in the right-hand side and using the fact that $\operatorname{div} D = 4\pi\delta_{x_0}$ while $\zeta(x_0) = 0$, we deduce

$$\int_{B_1} |\nabla u|^2 dx \geq 2 \left| \int_{\partial B_1} (D \cdot n) \zeta d\mathcal{H}^2 \right| = 2 \int_{\mathbb{S}^2} |\omega - x_0| d\mathcal{H}^2(\omega).$$

The right-hand side only depends on the norm of x_0 . Let us write

$$g : t \in (-1, 1) \mapsto 2 \int_{\mathbb{S}^2} |\omega - te| d\mathcal{H}^2(\omega)$$

for some fixed vector $e \in \mathbb{S}^2$. Then, using spherical coordinates, one gets

$$g(t) = 4\pi \int_0^\pi \sqrt{1 + t^2 - 2t \cos \theta} \sin \theta d\theta = 8\pi \left(1 + \frac{t^2}{3} \right),$$

and (2.6) follows. □

We now have all the ingredients to prove Theorem 1.4.

Proof of Theorem 1.4. Plugging (2.6) into (2.2) gives

$$\frac{1}{9} \int_{B_1} |\nabla u - \nabla u_*(\cdot - x_0)|^2 dx \leq \left(1 + \frac{4}{3} \right) \left(\int_{B_1} |\nabla u|^2 - 8\pi \right),$$

thus proving (1.6) with $c_* = 1/21$. For $x_0 = 0$ one can take $c_* = 1/9$ thanks to Lemma 2.1. □

2.3 Necessity of the translation

In this section, we prove that the quadratic stability estimate (1.8) fails to be true in dimension 3. Our counterexample shows that taking translations in Conjecture 1.5 seems unavoidable.

Lemma 2.4. *Fix an even cut-off function $\eta \in C_c^\infty(B_1)$ such that*

$$\mathbf{1}_{|x| \leq 1/2} \leq \eta(x) \leq \mathbf{1}_{|x| \leq 3/4},$$

and define, for $|x_0| < 1/4$,

$$u^{x_0}(x) = u_*(x - \eta(x)x_0).$$

There exists a constant $C > 0$ such that, for all $x_0 \in B_{1/4}$,

$$C \int_{B_1} |\nabla u^{x_0} - \nabla u_*|^2 dx \geq |x_0| \quad \text{and} \quad \int_{B_1} |\nabla u^{x_0}|^2 dx - 8\pi \leq C|x_0|^2. \quad (2.8)$$

Proof. Let $x_0 \in B_{1/4} \setminus \{0\}$. Using that $\eta = 1$ on $B_{1/2}$ and ∇u_* is homogeneous of degree -1 , and letting $\omega_0 = x_0/|x_0| \in \mathbb{S}^2$, we find

$$\begin{aligned} \int_{B_1} |\nabla u_* - \nabla u^{x_0}|^2 dx &\geq \int_{B_{1/2}} |\nabla u_*(x) - \nabla u_*(x - x_0)|^2 dx \\ &= \frac{1}{|x_0|^2} \int_{B_{1/2}} \left| \nabla u_* \left(\frac{x}{|x_0|} \right) - \nabla u_* \left(\frac{x - x_0}{|x_0|} \right) \right|^2 dx \\ &= |x_0| \int_{B_{1/(2|x_0|)}} |\nabla u_*(y) - \nabla u_*(y - \omega_0)|^2 dy \\ &\geq |x_0| \int_{B_1} |\nabla u_*(y) - \nabla u_*(y - \omega_0)|^2 dy. \end{aligned} \quad (2.9)$$

The last integral is independent of x_0 by rotational invariance, and positive since $\nabla u_*(y) = \nabla u_*(z)$ if and only if $y = \pm z$. This completes the proof of the first inequality in (2.8).

The second inequality follows from the fact that the even function

$$h : x_0 \in B_{1/4} \mapsto \int_{B_1} |\nabla u^{x_0}|^2 dx$$

is smooth. This can be checked by fixing $\chi \in C_c^\infty(B_{1/2})$ such that $\chi \equiv 1$ on $B_{1/3}$, so that we have

$$h(x_0) = \int_{\mathbb{R}^3} \chi(x) |\nabla u_*(x - x_0)|^2 dx + \int_{B_1 \setminus B_{1/3}} (1 - \chi(x)) |\nabla u^{x_0}|^2 dx.$$

The first integral depends smoothly on x_0 as the convolution of $\chi \in C_c^\infty(B_{1/2})$ with $|\nabla u_*|^2 \in L_{\text{loc}}^1(\mathbb{R}^3)$. The second integral depends smoothly on $x_0 \in B_{1/4}$ because u_* is smooth on $\mathbb{R}^3 \setminus 0$ and $|x - \eta(x)x_0| \geq 1/12$ for all $x \in B_1 \setminus B_{1/3}$ and $x_0 \in B_{1/4}$. $\square$

Remark 2.5. In dimension $n \geq 4$, consider u^{x_0} defined as in Lemma 2.4 in $B_1 \subset \mathbb{R}^n$. For every $x_0 \in B_{1/4} \setminus \{0\}$, letting $\omega_0 = x_0/|x_0|$ we find, as in (2.9),

$$\int_{B_{\frac{1}{2}}} |\nabla u_* - \nabla u^{x_0}|^2 dx = |x_0|^{n-2} \int_{B_{\frac{1}{2|x_0|}}} |\nabla u_*(y) - \nabla u_*(y - \omega_0)|^2 dy. \quad (2.10)$$

We denote by $C = C(n) > 0$ a generic dimensional constant whose precise value may change from line to line in what follows. Direct calculation shows that

$$\frac{1}{C} \leq |\nabla^2 u_*(z) \cdot \omega_0| \leq C, \quad \forall z \in \mathbb{S}^{n-1}.$$

The homogeneity of ∇u_* and a contradiction argument therefore imply

$$\frac{1}{C} \leq |y|^2 |\nabla u_*(y) - \nabla u_*(y - \omega_0)| \leq C, \quad \forall y \in \mathbb{R}^3 \setminus B_2.$$

Using also that

$$\int_{B_2} |\nabla u_*(y) - \nabla u_*(y - \omega_0)|^2 dy \leq 4 \int_{B_3} |\nabla u_*|^2 dy \leq C,$$

we deduce

$$\begin{aligned} \frac{1}{C} \int_2^{\frac{1}{2|x_0|}} r^{n-5} dr &\leq \int_{B_{\frac{1}{2|x_0|}} \setminus B_2} |\nabla u_*(y) - \nabla u_*(y - \omega_0)|^2 dy \\ &\leq \int_{B_{\frac{1}{2|x_0|}}} |\nabla u_*(y) - \nabla u_*(y - \omega_0)|^2 dy \\ &\leq C + C \int_2^{\frac{1}{2|x_0|}} r^{n-5} dr. \end{aligned}$$

Combining these inequalities with (2.10) and the fact that the function

$$x_0 \mapsto \int_{B_1 \setminus B_{1/2}} |\nabla u_* - \nabla u^{x_0}|^2 dx$$

is smooth on $B_{1/4}$ and minimal at $x_0 = 0$, we deduce the existence of $\Theta_n(x_0) \in [C^{-1}, C]$ such that

$$\int_{B_1} |\nabla u_* - \nabla u^{x_0}|^2 dx = \Theta_n(x_0) \cdot \begin{cases} |x_0|^2 \ln \frac{1}{|x_0|} & \text{if } n = 4, \\ |x_0|^2 & \text{if } n \geq 5, \end{cases}$$

for all $x_0 \in B_{1/4}$. For the energy deficit, the same argument as in Lemma 2.4 provides the upper bound $D(u^{x_0}) - D(u_*) \leq C|x_0|^2$. For $n = 4$, this shows that $D(u^{x_0} - u_*) \gg D(u^{x_0}) - D(u_*)$ as $|x_0| \rightarrow 0$.

3 Minimality for the Oseen-Frank energy

In this section, we prove Theorem 1.1 and Theorem 1.2; that is, the minimality of u_* for some values of $\mathbf{k}$ beyond the known regime $k_1 \leq k_2$.

3.1 Link with curl stability estimates

The proofs of Theorem 1.1 and Theorem 1.2 crucially rely on a quantitative control of $\operatorname{curl} u$ in terms of the Dirichlet energy deficit $D(u) - D(u_*)$:

Lemma 3.1. *If $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$ satisfies*

$$c \int_{B_1} |\operatorname{curl} u|^2 dx \leq \int_{B_1} |\nabla u|^2 - \int_{B_1} |\nabla u_*|^2, \quad (3.1)$$

for some $c \in (0, 1)$, then, for any $\mathbf{k} = (k_1, k_2, k_3)$ such that

$$0 < k_1, k_2, k_3, \quad k_1 \leq k_2 + \frac{c}{1-c} \min(k_2, k_3), \quad (3.2)$$

we have

$$E_{\mathbf{k}}(u) \geq E_{\mathbf{k}}(u_*).$$

If equality occurs, then equality holds in (3.1), and either $u = u_$, or $k_1 = k_2 + ck_3/(1-c)$.*

Proof. We write

$$E_{\mathbf{k}}(u) = k_1 I_1 + k_2 I_2 + k_3 I_3,$$

where

$$I_1 = \int_{B_1} (\operatorname{div} u)^2 dx, \quad I_2 = \int_{B_1} (u \cdot \operatorname{curl} u)^2 dx, \quad I_3 = \int_{B_1} |u \times \operatorname{curl} u|^2 dx.$$

By [13, Theorem 1], if $k_2 \geq k_1$, then $E_{\mathbf{k}}(u) \geq E_{\mathbf{k}}(u_*)$ with equality if and only if $u = u_*$. In particular, equality holds in (3.1).

We may therefore assume from now on that $k_1 > k_2$. In view of (1.5),

$$I_1 + I_2 + I_3 = E_{(1,1,1)}(u) = \int_{B_1} |\nabla u|^2 dx + 8\pi,$$

and thus, our assumption (3.1) can be rewritten as

$$c(I_2 + I_3) \leq I_1 + I_2 + I_3 - 16\pi,$$

or equivalently

$$I_1 + (1 - c)I_2 + (1 - c)I_3 \geq 16\pi. \quad (3.3)$$

By [13, Proposition 4], we have

$$I_1 + I_2 = E_{(1,1,0)}(u) \geq E_{(1,1,0)}(u_*) = \int_{B_1} (\operatorname{div} u_*)^2 dx = 16\pi. \quad (3.4)$$

For any $t \in [0, 1]$, multiplying (3.3) by tk_1 and (3.4) by $(1 - t)k_1$, we find

$$k_1 I_1 + (1 - tc)k_1 I_2 + t(1 - c)k_1 I_3 \geq 16\pi k_1 = E_{\mathbf{k}}(u_*). \quad (3.5)$$

By (3.2), one has

$$(1 - c)k_1 \leq (1 - c)k_2 + ck_2 = k_2,$$

and thus $k_1 - k_2 \leq ck_1$. Together with the fact that $k_2 < k_1$, this entitles one to choose

$$t = \frac{k_1 - k_2}{ck_1} \in (0, 1].$$

Observe that $(1 - tc)k_1 = k_2$ and also that

$$t(1 - c)k_1 = \frac{k_1 - k_2}{\delta},$$

where we have set $\delta = c/(1 - c)$. It thus follows from (3.5) that

$$k_1 I_1 + k_2 I_2 + \frac{k_1 - k_2}{\delta} I_3 \geq E_{\mathbf{k}}(u_*),$$

and therefore

$$k_1 I_1 + k_2 I_2 + k_3 I_3 = E_{\mathbf{k}}(u) \geq E_{\mathbf{k}}(u_*) + \frac{\delta k_3 + k_2 - k_1}{\delta} I_3.$$

By (3.2) again, $k_1 \leq k_2 + \delta k_3$, so that $E_{\mathbf{k}}(u) \geq E_{\mathbf{k}}(u_*)$.

Assume that equality occurs; that is, u minimizes $E_{\mathbf{k}}$. Since $t > 0$ we must have equality in (3.3); that is, equality in the assumption (3.1). We must also have either $k_1 = k_2 + \delta k_3$ or $I_3 = 0$. By the regularity properties of the minimizers of $E_{\mathbf{k}}$ developed in [6, 7], the singular set of u is closed and has zero Lebesgue measure. We can thus repeat the proof of [13, Theorem 1] showing that if $I_3 = 0$, then $u = u_*$. $\square$

3.2 Proof of Theorem 1.2

From Lemma 3.1 and the stability estimate of Lemma 2.1 for maps which are continuous away from the origin, we can readily prove Theorem 1.2.

Proof of Theorem 1.2. Let $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$. We introduce the map $v = u - u_*$. Since $\operatorname{curl} u_* = 0$, one has

$$\int_{B_1} |\operatorname{curl} u|^2 dx = \int_{B_1} |\operatorname{curl} v|^2 dx \leq \int_{B_1} \left((\operatorname{div} v)^2 + |\operatorname{curl} v|^2 \right) dx.$$

Using that $v = u - u_*$ vanishes on ∂B_1 and that $|\nabla v|^2 - |\operatorname{curl} v|^2 - (\operatorname{div} v)^2$ is a null Lagrangian, we thus get

$$\int_{B_1} |\operatorname{curl} u|^2 dx \leq \int_{B_1} |\nabla v|^2 dx = \int_{B_1} |\nabla u - \nabla u_*|^2 dx.$$

Assuming in addition that u is continuous in $\overline{B_1} \setminus \{0\}$, we can apply Lemma 2.1 to deduce that

$$\frac{1}{9} \int_{B_1} |\operatorname{curl} u|^2 dx \leq \int_{B_1} |\nabla u|^2 dx - 8\pi. \quad (3.6)$$

Lemma 3.1 therefore implies that $E_{\mathbf{k}}(u) \geq E_{\mathbf{k}}(u_*)$ if $k_1 \leq k_2 + \min(k_2, k_3)/8$. Moreover, equality in this inequality implies equality in (2.1), hence $u = u_*$. $\square$

3.3 Proof of Theorem 1.1

The proof of Theorem 1.1 follows the same strategy, but we first need to invoke an argument in [1] showing that for $\mathbf{k} \approx \mathbf{1}$, the minimizers of $E_{\mathbf{k}}$ have a single singularity. For the readers' convenience, we provide a sketch of the proof.

Lemma 3.2 ([1, §6.1]). *There exists $\eta > 0$ such that, if $k_1, k_2, k_3 > 0$ satisfy $|k_2/k_1 - 1| + |k_3/k_1 - 1| < \eta$, then any minimizer $E_{\mathbf{k}}$ in $H_{u_*}^1(B_1; \mathbb{S}^2)$ is smooth away from a single point $x_0 \in B_1$.*

Proof. Dividing $E_{\mathbf{k}}$ by k_1 , we assume without loss of generality that $k_1 = 1$. Then we argue by contradiction and assume the existence of a sequence of triples $\mathbf{k}^{(j)} = (1, k_2^{(j)}, k_3^{(j)}) \rightarrow \mathbf{1}$ and maps $u^{(j)}$ minimizing $E_{\mathbf{k}^{(j)}}$ in $H_{u_*}^1(B_1; \mathbb{S}^2)$ such that $u^{(j)}$ has at least two singular points $x^{(j)} \neq y^{(j)} \in B_1$. We extract a weak limit $u^{(j)} \rightharpoonup \bar{u}$ in $H^1(B_1; \mathbb{R}^3)$. Then $\bar{u} \in H_{u_*}^1(B_1; \mathbb{S}^2)$. By weak lower semicontinuity of E_1 and minimality of each $u^{(j)}$, this implies that

$$E_1(\bar{u}) \leq \liminf E_{\mathbf{k}^{(j)}}(u^{(j)}) \leq \limsup E_{\mathbf{k}^{(j)}}(u^{(j)}) \leq \limsup E_{\mathbf{k}^{(j)}}(u_*) = E_1(u_*).$$

Hence, $\bar{u} = u_*$ by uniqueness of the minimizer of $D = E_1 - 8\pi$, see [3, Theorem 7.1]. Moreover, all the above inequalities are equalities. This implies that $u^{(j)} \rightarrow u_*$

strongly in H^1 , see e.g. [7, Lemma 5.2]. We next rely on the small energy regularity for minimizers of $E_{\mathbf{k}}$ developed in [6, Section 2]: given a compact subset $K \subset (0, \infty)^3$, there exists $\varepsilon > 0$ such that for every $a \in B_1$, for every $R > 0$ such that $B_{2R}(a) \subset B_1$, for every $\mathbf{k} \in K$, if w is a minimizer of $E_{\mathbf{k}}$ on B_1 such that

$$\frac{1}{R} \int_{B_{2R}(a)} |\nabla w|^2 \leq \varepsilon,$$

then w is smooth in $B_R(a)$. We proceed to apply this statement for $K = \{\mathbf{k}^{(j)}\}_{j \geq 1} \cup \{\mathbf{1}\}$. Using the fact that u_* is smooth away from 0 and the strong convergence of $u^{(j)}$ to u_* , the above small energy regularity result implies that for every $\delta \in (0, 1)$, there exists $j_\delta \geq 1$ such that $u^{(j)}$ is smooth away from B_δ for every $j \geq j_\delta$. As a consequence, the singularities $x^{(j)} \neq y^{(j)}$ must both converge to 0. Now consider the rescaled maps

$$\hat{u}^{(j)}(\hat{x}) = u^{(j)}(x^{(j)} + r^{(j)}\hat{x}), \quad r^{(j)} = |x^{(j)} - y^{(j)}| \rightarrow 0.$$

For any $R > 0$, we have $x^{(j)} + r^{(j)}B_R \subset B_1$ if j is large enough, and $\hat{u}^{(j)}$ is a minimizer of $E_{\mathbf{k}^{(j)}}$ among maps in $H^1(B_R; \mathbb{S}^2)$ with the same boundary values. The compactness result [7, Theorem 5.3]¹ and a diagonal argument imply therefore that, after extracting a subsequence, $\hat{u}^{(j)} \rightarrow \hat{u}$ strongly in $H_{loc}^1(\mathbb{R}^3; \mathbb{R}^3)$, and $\hat{u} \in H_{loc}^1(\mathbb{R}^3; \mathbb{S}^2)$ is a local minimizer of the Dirichlet energy D . According to [1, Theorem 2.2], the map $\hat{u}$ is smooth away from at most one singular point $\hat{z}$. By strong convergence and using again the small energy regularity for the minimizers of $E_{\mathbf{k}}$, this implies that $\hat{u}^{(j)}$ is smooth in $B_2(0) \setminus B_\delta(\hat{z})$ for any small $\delta > 0$ and large enough j , in contradiction with the fact that $\hat{u}^{(j)}$ has a singularity at 0 and another singularity on ∂B_1 . $\square$

Proof of Theorem 1.1. By Lemma 3.2, there exists $\eta > 0$ such that, if $|k_2/k_1 - 1| + |k_3/k_1 - 1| \leq \eta$, then any minimizer u of $E_{\mathbf{k}}$ in $H_{u_*}^1(B_1; \mathbb{S}^2)$ has a single singularity $x_0 \in B_1$. Setting $v = u - u_*(\cdot - x_0)$, and using that $\text{curl } u_* = 0$ and $|\text{curl } v|^2 \leq 2|\nabla v|^2$, we find

$$\int_{B_1} |\text{curl } u|^2 dx = \int_{B_1} |\text{curl } v|^2 dx \leq 2 \int_{B_1} |\nabla v|^2 dx = 2 \int_{B_1} |\nabla u - \nabla u_*(\cdot - x_0)|^2 dx.$$

Combining this with the stability estimate of Theorem 1.4 we deduce

$$\frac{c_*}{2} \int_{B_1} |\text{curl } u|^2 dx \leq \int_{B_1} |\nabla u|^2 dx - 8\pi.$$

According to Lemma 3.1 this implies that u_* is the unique minimizer of $E_{\mathbf{k}}$ if $k_1 < k_2 + \delta \min(k_2, k_3)$, with $\delta = c_*/(2 - c_*)$. Here, the parameters k_1, k_2 and

¹This result in [7] is stated for a single functional $E_{\mathbf{k}}$ but the proof generalizes at once for a family of functionals $E_{\mathbf{k}^{(j)}}$ provided that the parameters $\mathbf{k}^{(j)}$ remain in a compact subset of $(0, \infty)^3$, as this is the case in our situation.

k_3 are also subject to the additional restriction $|k_2/k_1 - 1| + |k_3/k_1 - 1| \leq \eta$. We deduce in particular that for any $\eta' > 0$ sufficiently small, the map u_* is the unique minimizer of $E_{\mathbf{k}}$ for $\mathbf{k} = (1 + \eta', 1, 1)$. We can thus repeat the calculation presented in Remark 1.7 to infer the validity of (3.1) for $c = \eta'/(1 + \eta') > 0$ and all $u \in H_{u_*}^1(B_1; \mathbb{S}^2)$. The conclusion thus follows from Lemma 3.1. $\square$

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