FAR-FIELD EXPANSIONS FOR HARMONIC MAPS AND THE ELECTROSTATICS ANALOGY IN NEMATIC SUSPENSIONS

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Abstract. For a smooth bounded domain $G \subset \mathbb{R}^3$ we consider maps $n\colon \mathbb{R}^3 \setminus G \to \mathbb{S}^2$ minimizing the energy $E(n) = \int_{\mathbb{R}^3\setminus G} |\nabla n|^2 + F_s(n_{\lfloor \partial G})$ among $\mathbb{S}^2$-valued map such that $n(x) \approx n_0$ as $|x| \to \infty$. This is a model for a particle G immersed in nematic liquid crystal. The surface energy F_s describes the anchoring properties of the particle, and can be quite general. We prove that such minimizing map n has an asymptotic expansion in powers of $1/r$. Further, we show that the leading order $1/r$ term is uniquely determined by the far-field condition n_0 for almost all $n_0 \in \mathbb{S}^2$, by relating it to the gradient of the minimal energy with respect to n_0. We derive various consequences of this relation in physically motivated situations: when the orientation of the particle G is stable relative to a prescribed far-field alignment n_0; and when the particle G has some rotational symmetries. In particular, these corollaries justify some approximations that can be found in the physics literature to describe nematic suspensions via a so-called electrostatics analogy.

1. Introduction

The goal of this work is to investigate the so-called electrostatics analogy in the analysis of nematic suspensions or colloids: these consist of small particles immersed in a nematic liquid crystal matrix. The presence of these particles and their alignment induces elastic strains in nematic medium; what results is a complex strain-alignment coupling yielding novel high-functional composite materials. Examples include dilute ferronematics, where the suspended particles are ferromagnetic inclusions; organizing carbon nanotubes using liquid crystals; ferroelectrics; and living liquid crystals, where the suspended particles are swimming bodies (e.g. flagellated bacteria). Further details on the numerous applications of such systems may be found in the review articles [21, 24].

Mathematical studies of colloid inclusions in nematics have tended to follow two different directions. Several papers have addressed homogenization of nematics with a dense array of colloids (see, e.g., [8, 9, 12, 14, 13]), while others consider the presence of point or ring singularities induced by a single colloid particle (see, e.g., [3, 4, 6, 2, 5]). In this paper we adopt the setting of the second set of papers,

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but concentrate on the effect of the colloid geometry on the far-field behavior of the nematic rather than the local structure of singularities near the colloid surface.

The electrostatics analogy is commonly used to describe colloidal suspensions in the case of a dilute concentration of particles. It originates in the work [11] by Brochard and de Gennes, and has been developed further by several authors in the physics literature [20, 26, 22]. It relies on considering each single particle separately and postulating that:

- far away from the particle the distortion in nematic alignment can be viewed as a perturbation of uniform alignment and taken to solve the corresponding linearized equation – the representation formula for solutions of that linearized equation then provides a specific asymptotic expansion,
- the first few coefficients of that asymptotic expansion are characterized by the properties (size, symmetries, etc.) of the particle.

Then one formally replaces the nonlinear effect of each colloid particle by some singular source terms (derivatives of Dirac masses) in the linearized equation, according to the terms in the asymptotic expansion, which are derivatives of the fundamental solution (see Remark 1.7). In the one-constant approximation for the elastic energy of the nematic, this amounts to the equation satisfied by an electric potential in the presence of charged multipoles, hence the name "electrostatics analogy". This simplification, intuitively valid for dilute enough suspensions, allows for an explicit calculation of the energy of a given configuration in terms of the respective positions and properties of each particle, leading to the ultimate goal: computation of interparticle interactions.

In this article we provide a few elements towards mathematically quantifying the electrostatics analogy, rigorously obtaining an asymptotic expansion for solutions of the original non-linear and non-convex minimization problem, and comparing it with a multipole expansion of a harmonic function. What seems to us the most challenging part is the second bullet-point above: relating the coefficients of the asymptotic expansion to the particle's properties. Indeed, various mathematical obstacles defy a straightforward calculation of an expansion of minimizers: for instance, minimizers may not be unique, and it is unknown whether the symmetry of the particle system imposes a corresponding symmetry on the minimizing nematic configuration. Nevertheless we do obtain some results in that direction for the leading-order term of the expansion.

Specifically, we consider a single particle $G \subset \mathbb{R}^3$ (smooth and bounded) surrounded by nematic liquid crystal. A configuration of nematic alignment is represented by a director field $n\colon \mathbb{R}^3 \setminus G \to \mathbb{S}^2$, and its energy (within the one-constant approximation) is given by

$$E(n) = \int_{\mathbb{R}^3\setminus G} |\nabla n|^2 + F_s(n_{\lfloor \partial G}),$$

where $F_s\colon H^{1/2}(\partial G;\mathbb{S}^2) \to [0,\infty]$ can be a very general surface energy reflecting the particle's anchoring properties. Uniform alignment at far field, loosely expressed as $n(x) \approx n_0 \in \mathbb{S}^2$ for $r = |x| \to \infty$, is imposed through the condition

$$\int_{\mathbb{R}^3\setminus G} \frac{|n-n_0|^2}{1+r^2} \lesssim \int_{\mathbb{R}^3\setminus G} |\nabla n|^2 < \infty.$$

In other words, we are imposing that $n - n_0$ belongs to the completion of smooth maps with bounded support, with respect to the distance induced by the H^1 semi-norm; the weight $1/(1+r^2)$ is given by Hardy's inequality. Here and in the rest of the article, $A \lesssim B$ means $A \leq CB$ for some absolute constant $C > 0$. Equilibrium configurations satisfy the harmonic map equation

$$-\Delta n = |\nabla n|^2 n \qquad \text{in } \mathbb{R}^3 \setminus G.$$

Loosely speaking, we prove that:

- minimizing configurations have an asymptotic expansion determined by the linearized equation $\Delta n = 0$, however one cannot discard non-harmonic corrections – see Theorem 1.1;
- generically, the leading-order $\mathcal{O}(1/r)$ term in that expansion is uniquely determined by the particle G and the far-field uniform alignment n_0 – see Theorem 1.4.

The first point is a result about minimizing harmonic maps in an exterior domain, independent of the presence of a particle (since we do not explicitly relate the expansion's coefficients to the particle). The second point is obtained by connecting the leading-order term to the variation of minimal energy induced by keeping the particle G fixed and rotating the far-field alignment n_0. This is related to formal calculations in [11] for the torque exerted by the particle on the nematic (see Remark 1.5). We have not been able yet to obtain similar characterizations for the next-order terms in the expansion.

In terms of the electrostatics analogy developed in the physics literature, the main input of our results is to clarify the first postulate (that the far field distortions generated by a particle are purely harmonic to large order) by sheding new light on the second postulate (that these distortions are uniquely characterized by the particle). More precisely, in [11, 20, 26, 22], the possible presence of nonharmonic corrections is either not considered, or implicitly deduced from a hypothetical uniqueness principle which would ensure that symmetry properties of the particle directly translate into symmetry properties of the full configuration (such uniqueness/symmetry principle seems however difficult to prove). Here instead we deduce that nonharmonic corrections are negligible from our characterization of the leading-order term, bypassing any uniqueness or symmetry properties of the full configuration. This is valid for instance in the case of a spherical particle (see Corollary 1.8), but also when the orientation of the particle is at equilibrium (locally minimizing relative to variations in the prescribed far-field alignment, see Remark 1.7), independently of its symmetry properties. Moreover we stress that, for an axisymmetric particle, it is not evident that the equilibrium orientation should be the most symmetric one (see Remark 1.9).

Below we state our results in more detail.

1.1. **Far-field expansion for harmonic maps.** Our first main result is a far-field expansion for harmonic maps in an exterior domain, which (by rescaling) we may without loss of generality assume to contain $\mathbb{R}^3 \setminus \overline{B}_1$. Our first main result is a far-field expansion for such minimizing maps.

Theorem 1.1. *Let $n_0 \in \mathbb{S}^2$. Assume that $n \in H^1_{loc}(\mathbb{R}^3 \setminus \overline{B}_1; \mathbb{S}^2)$ satisfies*

$$\int_{\mathbb{R}^3\setminus\overline{B}_1} \frac{|n-n_0|^2}{r^2} \lesssim \int_{\mathbb{R}^3\setminus\overline{B}_1} |\nabla n|^2 < \infty, \tag{1.1}$$

and n is locally energy-minimizing, that is,

$$\int_{\mathbb{R}^3\setminus\overline{B}_1} |\nabla n|^2 \leq \int_{\mathbb{R}^3\setminus\overline{B}_1} |\nabla \tilde{n}|^2,$$

for any $\mathbb{S}^2$-valued map $\tilde{n}$ which agrees with n outside of a compact subset of $\mathbb{R}^3 \setminus \overline{B}_1$. Then there exist $v_0, p_j, c_{k\ell} \in \mathbb{R}^3$ $(1 \leq j,k,\ell \leq 3)$ such that, as $r = |x| \to \infty$,

$$\begin{aligned} n &= n_0 + n_{harm} + n_{corr} + \mathcal{O}\left(\frac{1}{r^4}\right), \qquad (1.2)\\ n_{harm} &= \frac{1}{r}v_0 + \sum_{j=1}^{3} p_j \partial_j\left(\frac{1}{r}\right) + \sum_{k,\ell=1}^{3} c_{k\ell}\partial_k\partial_\ell\left(\frac{1}{r}\right), \quad v_0, p_j, c_{k\ell} \in \mathbb{R}^3,\\ n_{corr} &= -\frac{|v_0|^2}{2r^2}n_0 - \frac{|v_0|^2}{6r^3}v_0 - \frac{1}{3r}\sum_{j=1}^{3} v_0\cdot p_j\,\partial_j\left(\frac{1}{r}\right) n_0. \end{aligned}$$

Moreover the vectors v_0, p_j $(j = 1,2,3)$ are orthogonal to n_0.

The far-field expansion (1.2) consists of a harmonic part n_{harm} solving the linearized equation $\Delta n_{harm} = 0$, and of a non-harmonic correction n_{corr}. Interestingly, if the coefficient v_0 of the leading-order term in n_{harm} vanishes, then the non-harmonic correction vanishes and n admits a harmonic expansion up to $\mathcal{O}(1/r^4)$. Higher-order non-harmonic corrections would not have that property. This is why we stop the expansion at this order, even though it will be clear from the proof that one can obtain an expansion at any arbitrary order. The relations $v_0 \cdot n_0 = p_j \cdot n_0 = 0$ simply come from the constraint $|n|^2 = 1$, which also imposes similar relations about the higher order coefficients $c_{k\ell}$, but we do not write them explicitly because they do not have such a precise geometric interpretation.

Remark 1.2. The proof of Theorem 1.1 can be generalized to obtain far-field expansions for any manifold-valued map $u\colon \mathbb{R}^d \setminus \overline{B}_1 \to \mathcal{N} \subset \mathbb{R}^k$ ($d \geq 3$) with given far-field value $u_0 \in \mathcal{N}$ in the sense $\int r^{-2}|u-u_0|^2 < \infty$, minimizing the Dirichlet energy. In the context of nematic liquid crystals with unequal elastic constants, it is interesting to consider more general energies of the form $\int A(u)[\nabla u, \nabla u]$, where $A(u)$ is a positive definite bilinear form on $\mathbb{R}^{k\times n}$ depending smoothly on u. Far field asymptotics should then be dictated by the linearized system $\nabla \cdot A(u_0)\nabla v = 0$, for which multipole expansions in terms of derivatives of the fundamental solution are described e.g. in [7]. We expect that the tools developed in the present work will apply to that generalized setting, but do not provide the technical details here.

We will obtain below various sufficient conditions ensuring that $v_0 = 0$, and so $n_{corr} = 0$. For now, it is worth noting that v_0 vanishes for axisymmetric configurations. The map $n : \mathbb{R}^3 \setminus \overline{B}_1 \to \mathbb{S}^2$ is axisymmetric about n_0 if for any rotation R of axis n_0 one has

$$n(Rx) = Rn(x) \qquad \forall x \in \Omega.$$

Using the far-field expansion (1.2) in this identity implies $Rv_0 = v_0$ for all rotations R of axis n_0, and therefore $v_0 = 0$ since $v_0 \cdot n_0 = 0$.

Corollary 1.3. *If the minimizing map n is axisymmetric about n_0, then $n = n_0 + n_{harm} + \mathcal{O}(1/r^4)$ as $r = |x| \to \infty$, with $\Delta n_{harm} = 0$.*

Corollary 1.3 is stated here for minimizing maps that are axisymmetric, but it is hard in general to prove that a minimizing map is symmetric. However, the proof of Theorem 1.1 can be reproduced for an axisymmetric map which is minimizing merely among axisymmetric configurations (see Remark 2.3), and Corollary 1.3 is valid also in that case.

1.2. **Characterization of the leading-order term.** Next we take into account the presence of the particle, a smooth bounded open subset $G \subset \mathbb{R}^3$, and consider the energy

$$E(n) = \int_{\mathbb{R}^3\setminus G} |\nabla n|^2 + F_s(n_{\lfloor \partial G}),$$

where

$$F_s\colon H^{1/2}(\partial G; \mathbb{S}^2) \to [0,\infty] \text{ is weakly lower semicontinuous and } \{F_s < \infty\} \neq \emptyset. \tag{1.3}$$

This ensures that, for any $n_0 \in \mathbb{S}^2$, the energy E admits a minimizer among maps $n\colon \mathbb{R}^3 \setminus G \to \mathbb{S}^2$ such that

$$\int_{\mathbb{R}^3\setminus G} \frac{|n-n_0|^2}{1+r^2} + \int_{\mathbb{R}^3\setminus G} |\nabla n|^2 < \infty.$$

To check this, note first that a boundary map $n_b \in H^{1/2}(\partial G; \mathbb{S}^2)$ with finite surface energy $F_s(n_b) < \infty$ can be extended to a map $n \in H^1_{loc}(\mathbb{R}^3 \setminus G; \mathbb{S}^2)$ such that $n \equiv n_0$ outside of a compact set using e.g. [18, Lemma A.1], so the infimum is finite. Moreover the energy is coercive thanks to Hardy's inequality, and weakly lower semicontinuous as a sum of two weakly lower semicontinuous functions. Therefore we may define

$$\begin{aligned} \hat{E}(n_0) = \min\Big\{ & E(n)\colon n \in H^1_{loc}(\mathbb{R}^3 \setminus G; \mathbb{S}^2), \\ & \int_{\mathbb{R}^3\setminus G} \frac{|n-n_0|^2}{1+r^2} + \int_{\mathbb{R}^3\setminus G} |\nabla n|^2 < \infty \Big\}. \end{aligned} \tag{1.4}$$

Examples of admissible surface energies F_s include

$$F_s(n) = \begin{cases} 0 & \text{if } n = n_D, \\ +\infty & \text{otherwise,} \end{cases}$$

for some fixed map $n_D \in H^{1/2}(\partial G;\mathbb{S}^2)$, which corresponds to imposing Dirichlet boundary conditions $n = n_D$ on ∂G; or

$$F_s(n) = \int_{\partial G} g(n,x)\, d\mathcal{H}^2(x),$$

for some measurable function $g\colon \mathbb{S}^2 \times \partial G \to [0,\infty)$ which is continuous with respect to n; for instance $g(n,x) = |n - n_D(x)|^2$ which relaxes Dirichlet boundary conditions (strong anchoring) to weak anchoring.

Our second main result relates the vector v_0 appearing in the leading-order term of the expansion (1.2) to the gradient of the function $\hat{E}$ at n_0.

Theorem 1.4. *Let $F_s\colon H^{1/2}(\partial G;\mathbb{S}^2) \to [0,\infty]$ satisfy* (1.3)*. Then the function $\hat{E}$ defined by* (1.4) *is Lipschitz, and for a.e. $n_0 \in \mathbb{S}^2$ we have*

$$\nabla \hat{E}(n_0) = -8\pi v_0, \tag{1.5}$$

where $v_0 = \lim_{r\to\infty} r(n - n_0)$ for any minimizing n such that $\hat{E}(n_0) = E(n)$. Moreover $\hat{E}$ is semiconcave: for all $n_0, m_0 \in \mathbb{S}^2$ and $v_0 = \lim_{r\to\infty} r(n - n_0)$ for any minimizer n achieving $\hat{E}(n_0)$, we have the one-sided inequality

$$\hat{E}(m_0) \le \hat{E}(n_0) - 8\pi v_0 \cdot (m_0 - n_0) + C|m_0 - n_0|^2,$$

for some constant $C = C(G, F_s) \ge 0$.

Remark 1.5. Formula (1.5) relates v_0 to the torque applied by the particle G on the nematic, in agreement with formal calculations in [11] for an axisymmetric particle. These formal calculations can be made rigorous (and then they show that $\hat{E}$ is differentiable everywhere) if one knows that the minimization problem (1.4) admits a unique minimizer n which moreover depends smoothly on n_0. Such uniqueness and smoothness results seem very hard to obtain in general, and we use a somewhat different method to prove (1.5) and Theorem 1.4.

Different minimizers n in (1.4) may a priori have different asymptotic expansions (1.2). However, a crucial nontrivial consequence of Theorem 1.4 is that at any differentiability point n_0 of $\hat{E}$, the coefficient v_0 of the leading-order term is uniquely determined by n_0, even though (1.4) may have several minimizers. We do not know whether $\hat{E}$ can have non-differentiable points, and whether v_0 can be multivalued at such points. The semiconcavity inequality in Theorem 1.4 implies that all possible values of v_0 are included in the subdifferential of $-\frac{1}{8\pi}\hat{E}$. It would be interesting to characterize values of v_0 in terms of this subdifferential.

One may pose an analogous question for $\mathbb{S}^1$-valued minimizers in exterior domains $\mathbb{R}^2 \setminus G$ in the plane which approach a constant $n_0 = e^{i\phi_0}$ at infinity. However the situation is completely different, because finite-energy configurations don't exist in general. One way around that issue is to relax the $\mathbb{S}^1$-valued constraint via a Ginzburg-Landau approximation. This approach is implemented in [1], with the asymptotic value $n_0 = e^{i\phi_0}$ left free.

An interesting consequence of the semiconcavity of $\hat{E}$ is that it must be differentiable, of zero gradient, at any local minimum point.

Corollary 1.6. *If $n_0 \in \mathbb{S}^2$ is locally minimizing for $\hat{E}$, then $v_0 = 0$ and $n = n_{harm} + \mathcal{O}(1/r^4)$ as $r = |x| \to \infty$ with $\Delta n_{harm} = 0$, for any minimizing n such that $E(n) = \hat{E}(n_0)$.*

Remark 1.7. In the physical system it is formally equivalent to rotate the far-field alignment n_0 or the particle G. Hence Corollary 1.6 tells us that, when the particle is in a stable equilibrium position, all minimizing configurations n have a far-field expansion which is harmonic up to $\mathcal{O}(1/r^4)$, and whose leading order is given by the harmonic term $\sum_j p_j \partial_j (1/r)$ for some vectors $p_j \in n_0^\perp$. Such leading-order term corresponds to solutions of the equation

$$\Delta n = \frac{1}{4\pi} \sum_{j=1}^{3} p_j \partial_j \delta \qquad \text{in } \mathbb{R}^3,$$

where the singular source term can be interpreted as a dipole-moment, as described e.g. in [22].

Another remarkable consequence of Theorem 1.4 concerns the important case where the particle G, together with its anchoring properties described by the surface energy F_s, possess some rotational symmetry. As mentioned earlier, we may not necessarily infer the same symmetry for all minimizers, but we can make some strong geometrical conclusions concerning the vector v_0 in the expansion (1.2) of minimizers. To make this precise, we define the symmetry group of the particle (and its anchoring properties) (G, F_s) as a subgroup of the orthogonal transformations $O(3)$ given by

$$\mathrm{Sym}(G,F_s) = \Big\{ R \in O(3) \colon RG = G, \text{ and } \\ F_s(Rn \circ R^{-1}) = F_s(n)\ \forall n \in H^{1/2}(\partial G; \mathbb{S}^2) \Big\}.$$

For any symmetry-preserving transformation $R \in \mathrm{Sym}(G, F_s)$, the energy E is conserved under the transformation $n \mapsto Rn \circ R^{-1}$, and therefore $\hat{E}(n_0) = \hat{E}(Rn_0)$.

Corollary 1.8. *If the particle has an axis of symmetry $\mathbf{u} \in \mathbb{S}^2$, i.e. $\mathrm{Sym}(G, F_s)$ contains all rotations $R \in SO(3)^{\mathbf{u}}$ about axis $\mathbf{u}$, then for almost all $n_0 \in \mathbb{S}^2$ we have*

$$v_0(n_0) \cdot (\mathbf{u} \times n_0) = 0, \tag{1.6}$$

where $v_0(n_0) = \lim_{r\to\infty} r(n - n_0)$ for any minimizing map n achieving $\hat{E}(n_0)$. If $\hat{E}$ is differentiable at $\mathbf{u}$ then $v_0(\mathbf{u}) = 0$.

If the particle is spherically symmetric, i.e. $\mathrm{Sym}(G, F_s)$ contains all rotations $SO(3)$, then $v_0(n_0) = 0$ for all $n_0 \in \mathbb{S}^2$.

Note that since v_0 is orthogonal to n_0, if $\mathbf{u}$ and n_0 are not parallel, then the identity $v_0 \cdot (\mathbf{u} \times n_0) = 0$ forces v_0 to belong to a fixed line determined by n_0 and $\mathbf{u}$. This link between symmetry properties of G and of v_0 gives a rigorous justification to assertions in [11, § II.1.a] where this is deduced from the assumption, false in general, that minimizers n in (1.4) are unique.

Remark 1.9. In the axisymmetric setting, Corollary 1.8 leaves open the case when $\hat{E}$ is not differentiable at $n_0 = \mathbf{u}$, the axis of symmetry: the $1/r$ asymptotic might be nonzero. If that situation occurs, that is, there is a minimizer n with far-field alignment $\mathbf{u}$ but with $v_0 \neq 0$, then all its axial rotations $Rn \circ R^{-1}$ are minimizers for $\hat{E}(\mathbf{u})$ too, with $1/r$ asymptotic term equal to Rv_0. The semiconcavity inequality

$$\hat{E}(n_0) \leq \hat{E}(\mathbf{u}) - 8\pi R v_0 \cdot (n_0 - \mathbf{u}) + C|n_0 - \mathbf{u}|^2,$$

is then valid for all rotations R of axis $\mathbf{u}$, and we deduce

$$\hat{E}(n_0) \leq \hat{E}(\mathbf{u}) - 8\pi|n_0 - \mathbf{u}| + C|n_0 - \mathbf{u}|^2.$$

Hence $\hat{E}$ has a local maximum at $\mathbf{u}$, and its graph near $\mathbf{u}$ looks locally like a cone. While none of the results above preclude this scenario in the axisymmetric setting, it is natural to ask the open question: can this situation really occur?

1.3. **Plan of the article.** In section 2 we prove Theorem 1.1 and in section 3 we prove Theorem 1.4. In Appendix A we provide proofs of some familiar (but not easily found) decay estimates for Poisson's equation for the reader's convenience.

2. Far-field expansion

In this section we prove Theorem 1.1. The minimizing map n solves, in the weak sense, the harmonic map equation

$$-\Delta n = |\nabla n|^2 n \qquad \text{in } \mathbb{R}^3 \setminus \overline{B}_1. \tag{2.1}$$

If the right-hand side decays like $\mathcal{O}(1/|x|^\gamma)$ for some $\gamma > 3$, decay estimates for the Poisson equation (see Lemma A.1) enable one to start a harmonic expansion for n, and this process can then be iterated including relevant non-harmonic corrections. Hence the main new ingredient in the proof of Theorem 1.1 is to obtain an initial strong enough decay estimate on $|\nabla n|$.

Note that, since $\int_{|x|\geq R}|\nabla n|^2 \to 0$ as $R\to\infty$, small energy estimates for harmonic maps [27, 28] ensure that n is smooth outside of a finite ball of large enough radius. Specifically, given $x_0\in\mathbb{R}^3$, $|x_0|=R$, the small energy regularity estimate for harmonic maps [27, Theorem 2.2] applied to $\hat n(\hat x) = n(x_0+(R/2)\hat x)$ implies the existence of $R_0\geq 1$ (depending on n) such that

$$|x_0| = R\geq R_0 \quad \Longrightarrow \quad |\nabla n|^2(x_0) \lesssim R^{-3}\int_{\frac R2\leq |x|\leq \frac{3R}{2}} |\nabla n|^2. \tag{2.2}$$

In particular we have the decay estimate $|\nabla n(x)|^2 = o(1/|x|^3)$. At this point we would like to use decay estimates of Poisson's equation from Lemma A.1 in an iterative process to generate the far-field expansion, but the decay given in (2.2) is just not enough to start applying the Lemma. Consequently, we require an algebraic decay $\mathcal{O}(1/R^\delta)$, for some $\delta>0$, of the integral $\int_{|x|\geq R}|\nabla n|^2$. This we obtain in Lemma 2.2 and Step 1 of Theorem 1.1's proof, using the minimizing property of n in order to compare the decay of that integral with the decay of the same integral for minimizers of the Dirichlet energy with values into the plane $T_{n_0}\mathbb{S}^2$, that is, solutions of the linearized equation $\Delta n = 0$.

First recall that for harmonic functions we have the following decay estimates.

Lemma 2.1. *Let* $u\colon \mathbb{R}^3\setminus \overline{B}_1\to\mathbb{R}$ *satisfy* $\int_{\mathbb{R}^3\setminus\overline{B}_1}|\nabla u|^2<\infty$ *and* $\Delta u = 0$ *in* $\mathbb{R}^3\setminus\overline{B}_1$. *Then for all* $R\geq 1$, $\hat u(\hat x) = u(R\hat x)$ *satisfies*

$$\int_{|\hat x|\geq 1}|\nabla\hat u|^2 = \frac 1R\int_{|x|\geq R}|\nabla u|^2 \leq \frac{1}{R^2}\int_{|x|\geq 1}|\nabla u|^2.$$

Proof. Since u is harmonic and $\int_{\mathbb{R}^3\setminus\overline{B}_1}|\nabla u|^2<\infty$, its spherical harmonics expansion is of the form

$$u(x) = u(r\omega) = u_0 + \sum_k \frac{a_k}{r^{\gamma_k}}\phi_k(\omega),$$

where we decompose $x\neq 0$ in polar coordinates as $x=r\omega, r=|x|$, and $\omega=\frac{x}{|x|}\in\mathbb{S}^2$, and $\{\phi_k\}_k$ is an $L^2(\mathbb{S}^2)$-orthonormal system of spherical harmonics and $\gamma_k>0$. Then we compute

$$\begin{aligned}\int_{|x|\geq R}|\nabla u|^2 &= \int_{|x|\geq R}\nabla\cdot(u\nabla u) = -\int_{|x|=R}u\partial_r u\\ &= \sum_k\frac{\gamma_k a_k^2}{R^{2\gamma_k+1}}\leq \frac 1R\sum_k\gamma_k a_k^2 = \frac 1R\int_{|x|\geq 1}|\nabla u|^2.\end{aligned}$$

□

We obtain almost the same decay for our minimizing map n, via the following decay improvement result. The estimate obtained in Lemma 2.2 will be needed in Step 1 of the proof of Theorem 1.1. After the proof of the theorem we present a second proof of that step, replacing the estimate of Lemma 2.2 by a different approach inspired by asymptotic expansions of minimal surfaces in [29]. Note that, as pointed out by the anonymous referee, this second proof makes use of minimality of n only for the small energy estimate, and therefore it applies also to nonminimizing stationary harmonic maps [10]. We find it worth including both proofs here, as their ranges of applicability to the anisotropic energies mentioned in Remark 1.2 may differ.

Lemma 2.2. *For any* $\alpha<2$, *there exist* $\delta>0$ *and* $R_1>1$ *such that for any* $n_0\in\mathbb{S}^2$ *and any map* $n\colon\mathbb{R}^3\setminus\overline{B}_1\to\mathbb{S}^2$ *with*

$$\int_{\mathbb{R}^3\setminus\overline{B}_1}\frac{|n-n_0|^2}{r^2}\lesssim\int_{\mathbb{R}^3\setminus\overline{B}_1}|\nabla n|^2<\infty,$$

which is energy minimizing, i.e.

$$\int_{\mathbb{R}^3\setminus\overline{B}_1} |\nabla n|^2 \leq \int_{\mathbb{R}^3\setminus\overline{B}_1} |\nabla \tilde{n}|^2,$$

for all $\mathbb{S}^2$*-valued maps* $\tilde{n}$ *that agree with* n *outside of a compact subset of* $\mathbb{R}^3\setminus\overline{B}_1$*, we have*

$$\int_{|x|\geq 1} |\nabla n|^2 \leq \delta^2 \quad\Rightarrow\quad \frac{1}{R_1}\int_{|x|\geq R_1} |\nabla n|^2 \leq \frac{1}{R_1^\alpha}\int_{|x|\geq 1}|\nabla n|^2.$$

Proof of Lemma 2.2. The proof follows quite closely the strategy in [23, Proposition 1] (see also [17, Theorem 2.4]). By rotational symmetry we may assume $n_0 = (0,0,1)$. We fix $\alpha < 2$. Since $T_{n_0}\mathbb{S}^2 = n_0^\perp = \mathbb{R}^2\times\{0\}$, thanks to Lemma 2.1 we may choose any $R_\star > 1$ such that for any $T_{n_0}\mathbb{S}^2$-valued energy minimizing map v in $\mathbb{R}^3\setminus\overline{B}_1$ with $\int |v|^2|x|^{-2} \lesssim \int |\nabla v|^2 < \infty$,

$$\frac{1}{R_\star}\int_{|x|\geq R_\star} |\nabla v|^2 \leq \frac{1}{4}\frac{1}{R_\star^\alpha}\int_{|x|\geq 1}|\nabla v|^2. \tag{2.3}$$

Then we fix $R_1 = 2R_\star$ and argue by contradiction, assuming Lemma 2.2 to be false for this value of R_1. Hence there exist $\delta_j\to 0$ and minimizing $\mathbb{S}^2$-valued maps n_j such that

$$\int_{|x|\geq 1}\frac{|n_j-n_0|^2}{|x|^2} \lesssim \int_{|x|\geq 1}|\nabla n_j|^2 = \delta_j^2$$
$$\text{and}\quad \frac{1}{R_1}\int_{|x|\geq R_1}|\nabla n_j|^2 > \frac{1}{R_1^\alpha}\int_{|x|\geq 1}|\nabla n_j|^2.$$

We set

$$v_j := \frac{n_j - n_0}{\delta_j},$$

so that

$$\int_{|x|\geq 1}\frac{|v_j|^2}{|x|^2} \lesssim \int_{|x|\geq 1}|\nabla v_j|^2 = 1 \quad\text{and}\quad \frac{1}{R_1}\int_{|x|\geq R_1}|\nabla v_j|^2 > \frac{1}{R_1^\alpha}\int_{|x|\geq 1}|\nabla v_j|^2. \tag{2.4}$$

Up to a subsequence (that we do not relabel), there exists $v_\star \in H^1_{loc}(\mathbb{R}^3\setminus\overline{B}_1;\mathbb{R}^3)$ such that $v_j \rightharpoonup v_\star$ weakly in H^1_{loc}, strongly in L^2_{loc}, and almost everywhere. Note that $v_\star(x)\in T_{n_0}\mathbb{S}^2$ for a.e. $x\in\mathbb{R}^3\setminus\overline{B}_1$. Indeed, considering a subsequence of v_j converging a.e., we see that $v_\star(x)$ is the limit of vectors of the form $(z_j - n_0)/\delta_j$ for some $z_j\in\mathbb{S}^2$ and $\delta_j\to 0$, which implies first that $z_j\to n_0$, and then that $v_\star(x)\in T_{n_0}\mathbb{S}^2$. Furthermore, by lower semi-continuity,

$$\int_{|x|\geq 1}\frac{|v_\star|^2}{|x|^2}\lesssim \int_{|x|\geq 1}|\nabla v_\star|^2 \leq 1.$$

By Fubini's theorem we may moreover pick $r\in[1,2]$ such that

$$\int_{|x|=r}|\nabla v_\star|^2 \lesssim 1 \qquad\text{and}\qquad \int_{|x|=r}|\nabla v_j|^2\lesssim 1.$$

By continuity of the trace operator and compactness of the embedding $H^{\frac12}(\partial B_r)\subset L^2(\partial B_r)$ we have $\int_{|x|=r}|v_j - v_\star|^2\to 0$. We claim that $v_\star$ is a $T_{n_0}\mathbb{S}^2$-valued minimizing map in $\Omega_r = \{|x|>r\}$. Let $v\in H^1_{loc}(\Omega_r;T_{n_0}\mathbb{S}^2)$ agree with $v_\star$ outside of a compact subset of Ω_r. We will show that $\int|\nabla v_\star|^2\leq \int|\nabla v|^2$, thus proving the claim. Let

$$\tilde{v}_j = \frac{\delta_j^{-\frac12}v}{\max(\delta_j^{-\frac12},|v|)},\qquad \tilde{n}_j = \pi_{\mathbb{S}^2}(n_0+\delta_j\tilde{v}_j),$$

where $\pi_{\mathbb{S}^2}$ is the orthogonal projection onto $\mathbb{S}^2$ (well-defined in a neighborhood of it), so that

$$|\nabla\tilde{n}_j|^2 \leq \delta_j^2\left(1+O(\delta_j^{\frac12})\right)|\nabla v|^2,\qquad \tilde{v}_j\to v \text{ in } H^1_{loc}(\overline{\Omega}_r;T_{n_0}\mathbb{S}^2).$$

Since $v = v_\star$ on ∂B_r and $\int_{|x|=r} |v_j - v_\star|^2 \to 0$, we also have

$$\gamma_j^2 := \int_{\partial B_r} |v_j - \tilde{v}_j|^2 \to 0.$$

Moreover, using that $\pi_{\mathbb{S}^2}$ is smooth in a small neighborhood of n_0 and $\tilde{v}_j \cdot n_0 = 0$, we obtain

$$\tilde{n}_j - n_j = \pi_{\mathbb{S}^2}(n_0 + \delta_j \tilde{v}_j) - n_0 - \delta_j v_j = \delta_j(\tilde{v}_j - v_j) + \mathcal{O}(\delta_j^2 |v|^2),$$

so

$$\int_{\partial B_r} |n_j - \tilde{n}_j|^2 \leq \delta_j^2(\gamma_j^2 + c^2\delta_j^2),$$

where $c > 0$ is a constant depending on v. Luckhaus' extension lemma [23, Lemma 1] ensures, for any $\lambda \in (0,1)$, the existence of $\varphi_j \colon B_{(1+\lambda)r} \setminus B_r \to \mathbb{R}^3$ such that

$$\begin{aligned}
\varphi_j = &n_j \text{ on } \partial B_r, \quad \varphi_j = \tilde{n}_j((1+\lambda)\cdot) \text{ on } \partial B_{(1+\lambda)r},\\
\int_{B_{(1+\lambda)r}\setminus B_r} |\nabla \varphi_j|^2 &\lesssim \lambda \int_{\partial B_r} \left(|\nabla n_j|^2 + |\nabla \tilde{n}_j|^2 \right) + \lambda^{-1} \int_{\partial B_r} |n_j - \tilde{n}_j|^2 \\
&\lesssim \delta_j^2 \left(\lambda + \lambda^{-1}(\gamma_j^2 + c^2\delta_j^2) \right),\\
\sup_{B_{(1+\lambda)r}\setminus B_r} \operatorname{dist}^2(\varphi_j, \mathbb{S}^2) &\lesssim \lambda^{-1} \left(\int_{\partial B_r} \left(|\nabla n_j|^2 + |\nabla \tilde{n}_j|^2 \right) \right)^{\frac{1}{2}} \left(\int_{\partial B_r} |n_j - \tilde{n}_j|^2 \right)^{\frac{1}{2}} \\
&\quad + \lambda^{-2} \int_{\partial B_r} |n_j - \tilde{n}_j|^2 \\
&\lesssim \delta_j^2 \left(\lambda^{-1}\gamma_j + \lambda^{-2}\gamma_j^2 \right)
\end{aligned}$$

Choosing $\lambda = \lambda_j = \gamma_j + c\delta_j \to 0$, we may thus define $\psi_j = \pi_{\mathbb{S}^2}(\varphi_j) \colon B_{(1+\lambda_j)r} \setminus B_r \to \mathbb{S}^2$ satisfying

$$\begin{aligned}
&\psi_j = n_j \text{ on } \partial B_r, \quad \psi_j = \tilde{n}_j((1+\lambda_j)\cdot) \text{ on } \partial B_{(1+\lambda_j)r},\\
&\text{and } \delta_j^{-2} \int_{B_{(1+\lambda_j)r}\setminus B_r} |\nabla \psi_j|^2 \to 0.
\end{aligned}$$

Then we set

$$\hat{n}_j(x) = \begin{cases} \psi_j(x) & \text{for } r \leq |x| \leq (1+\lambda_j)r, \\ \tilde{n}_j((1+\lambda_j)x) & \text{for } |x| \geq (1+\lambda_j)r. \end{cases}$$

Note that $\hat{n}_j$ agrees with n_j on ∂B_r and satisfies

$$\int_{|x|\geq 2} \frac{|\hat{n}_j - n_0|^2}{|x|^2} \lesssim \int_{|x|\geq r} \frac{|\tilde{n}_j - n_0|^2}{|x|^2} \lesssim \delta_j^2 \int_{|x|\geq r} \frac{|\tilde{v}_j|^2}{|x|^2} \lesssim \delta_j^2 \int_{|x|\geq r} \frac{|v|^2}{|x|^2} < \infty,$$

since $v = v_\star$ outside of a compact set and $\int_{|x|\geq 1} |v_\star|^2 |x|^{-2} < \infty$. Therefore the $\hat{n}_j$ must have greater energy than n_j, hence

$$\begin{aligned}
\int_{|x|\geq r} |\nabla v_j|^2 = \delta_j^{-2} \int_{|x|\geq r} |\nabla n_j|^2 &\leq \delta_j^{-2} \int_{|x|\geq r} |\nabla \hat{n}_j|^2 \\
&\leq (1+o(1))\delta_j^{-2} \int_{|x|\geq r} |\nabla \tilde{n}_j|^2 + o(1) \\
&\leq (1+o(1)) \int_{|x|\geq r} |\nabla v|^2 + o(1)
\end{aligned}$$

By weak lower semi-continuity of the Dirichlet energy with respect to H^1_{loc} convergence we infer

$$\int_{|x|\geq r} |\nabla v_\star|^2 \leq \liminf \int_{|x|\geq r} |\nabla v_j|^2 \leq \int_{|x|\geq r} |\nabla v|^2,$$

so that $v_\star$ is a $T_{n_0}\mathbb{S}^2$-valued energy minimizing map in Ω_r, and moreover applying the above to $v = v_\star$ we deduce that

$$\int_{|x|\geq r} |\nabla v_j - \nabla v_\star|^2 \to 0.$$

In particular, since $\int_{|x|\geq 1} |\nabla v_j|^2 = 1$, (2.4) implies that $\int_{|x|\geq R_1} |\nabla v_\star|^2 > 0$. Moreover, recalling that $r \in [1,2]$ and taking $j \to \infty$ in (2.4) we obtain

$$\frac{1}{R_1}\int_{|x|\geq R_1} |\nabla v_\star|^2 \geq \frac{1}{R_1^\alpha}\int_{|x|\geq 2} |\nabla v_\star|^2,$$

hence, for $\hat{v}_\star(\hat{x}) = v_\star(2\hat{x})$, recalling that $R_1 = 2R_\star$ and $\alpha < 2$, we have

$$\frac{1}{R_\star}\int_{|x|\geq R_\star} |\nabla \hat{v}_\star|^2 \geq \frac{2^{1-\alpha}}{R_\star^\alpha}\int_{|x|\geq 1} |\nabla \hat{v}_\star|^2 \geq \frac{1}{2}\frac{1}{R_\star^\alpha}\int_{|x|\geq 1} |\nabla \hat{v}_\star|^2.$$

Since $\hat{v}_\star$ is a $T_{n_0}\mathbb{S}^2$-valued energy minimizing map in $\mathbb{R}^3 \setminus \overline{B}_1$ and $\int_{|x|\geq 1} |\nabla \hat{v}_\star|^2 > 0$, this contradicts (2.3). □

We will plug the initial decay provided by Lemma 2.2 into the equilibrium equation (2.1) in order to deduce the expansion (1.2) (implying in particular *a posteriori* that Lemma 2.2 is also valid for $\alpha = 2$). The main tool to obtain the expansion will be decay estimates for Poisson equation. These estimates are familiar but not easily found in the form we require here, and so we have provided a proof in the Appendix A. With these preliminary lemmas, we are now ready to present the proof of Theorem 1.1.

Proof of Theorem 1.1. Let R_1 and δ be as in Lemma 2.2.
Step 1. Picking $R_0 > 1$ (depending on n) such that $\frac{1}{R_0}\int_{|x|\geq R_0} |\nabla n|^2 \leq \delta^2$ we may apply Lemma 2.2 iteratively to $x \mapsto n(R_1^k R_0 x)$ for $k \geq 0$ and obtain

$$\frac{1}{R_1^k R_0}\int_{|x|\geq R_1^k R_0} |\nabla n|^2 \leq \frac{\delta^2}{(R_1^k)^\alpha},$$

and therefore

$$\frac{1}{R}\int_{|x|\geq R} |\nabla n|^2 \leq \frac{C(n,\alpha)}{R^\alpha} \quad \forall R \geq R_0(n), \alpha < 2.$$

Thanks to (2.2) this implies

$$|\nabla n| \leq \frac{C(n,\sigma)}{r^{2-\sigma}} \qquad \text{for } r \geq R_0(n), \sigma > 0.$$

Here we are interested in small values of $\sigma > 0$, and $C(n,\sigma) > 0$ denotes a generic constant depending on n and σ, whose precise value may change from line to line in the rest of the proof. Integrating this along radial rays yields $|n - n_0| \leq C(n,\sigma)/r^{1-\sigma}$. Moreover, since $-\Delta n = |\nabla n|^2 n$ we have (redefining σ appropriately)

$$|\Delta n| \leq \frac{C(n,\sigma)}{r^{4-\sigma}} \qquad \text{for } r \geq R_0(n), \sigma > 0.$$

Step 2. Applying Lemma A.1 to $f_1 = \Delta n = \Delta(n - n_0)$ we obtain the existence of $u_1 \colon \mathbb{R}^3 \setminus B_{R_0} \to \mathbb{R}^3$ such that $\Delta u_1 = \Delta(n - n_0)$ and

$$\frac{|u_1|}{r} + |\nabla u_1| \leq \frac{C(n,\sigma)}{r^{3-\sigma}} \qquad \text{for } r \geq R_0(n). \tag{2.5}$$

The map $n - n_0 - u_1$ is harmonic in $\mathbb{R}^3 \setminus B_{R_0}$. Writing down its spherical harmonics expansion, we modify u_1 to include the part of the expansion that decays faster than $1/r$. Specifically, we have

$$n - n_0 - u_1 = \frac{1}{r}v_0 + \tilde{u}_1,$$

for some $v_0 \in \mathbb{R}^3$ and a remainder $\tilde{u}_1$ satisfying $|\tilde{u}_1|/r + |\nabla \tilde{u}_1| = \mathcal{O}(1/r^3)$. Therefore, replacing u_1 by $u_1 + \tilde{u}_1$ (without renaming it), we obtain

$$n = n_0 + \frac{1}{r} v_0 + u_1, \tag{2.6}$$

with u_1 still satisfying (2.5). The vector v_0 is, a posteriori, uniquely determined by the map n, since $v_0 = \lim_{r\to\infty} r(n - n_0)$. Moreover, this implies

$$1 = |n|^2 = 1 + \frac{2}{r} v_0 \cdot n_0 + \mathcal{O}\left(\frac{1}{r^{2-\sigma}}\right),$$

so we must have

$$v_0 \cdot n_0 = 0.$$

Step 3. With an eye toward obtaining the next term in the far-field expansion, we plug in (2.6) into the harmonic maps PDE (2.1), and isolate terms that are higher order than $\mathcal{O}(\frac{1}{r^5})$ on the right hand side. Specifically, we have

$$\begin{aligned} 0 = \Delta n + |\nabla n|^2 n &= \Delta u_1 + \frac{1}{r^4} |v_0|^2 n_0 + \mathcal{O}\left(\frac{1}{r^{5-\sigma}}\right) \\ &= \Delta\left(u_1 + \frac{1}{r^2}\frac{|v_0|^2}{2} n_0\right) + \mathcal{O}\left(\frac{1}{r^{5-\sigma}}\right), \end{aligned}$$

that is,

$$\Delta\left(u_1 + \frac{1}{r^2}\frac{|v_0|^2}{2} n_0\right) = f_2,$$

where f_2 has decay rate given by $|f_2| \leq C(n,\sigma)/r^{5-\sigma}$ for $r \geq R_0(n)$. By Lemma A.1, we obtain $u_2 \colon \mathbb{R}^3 \setminus B_{R_0} \to \mathbb{R}^3$ such that $\Delta u_2 = f_2$ and

$$\frac{|u_2|}{r} + |\nabla u_2| \leq \frac{C(n,\sigma)}{r^{4-\sigma}} \qquad \text{for } r \geq R_0(n).$$

The map $u_1 + r^{-2}|v_0|^2 n_0/2 - u_2$ is harmonic in $\mathbb{R}^3 \setminus B_{R_0}$, hence including the higher decay part of its spherical harmonics expansion into u_2 we deduce the existence of $P_1 \in \mathbb{R}[X]^3$, a vector of homogeneous harmonic polynomials of degree 1 (i.e. linear forms) such that

$$u_1 = -\frac{1}{r^2}\frac{|v_0|^2}{2} n_0 + \frac{1}{r^3} P_1(x) + u_2,$$

i.e.

$$n = \left(1 - \frac{|v_0|^2}{2r^2}\right) n_0 + \frac{1}{r} v_0 + \frac{1}{r^3} P_1(x) + u_2. \tag{2.7}$$

Note that the unit norm constraint on n implies $n_0 \cdot P_1(x) = 0$ for all x. Indeed, taking the norm square of (2.7), we find

$$1 = |n|^2 = 1 + \frac{2\, n_0 \cdot P_1(x/r)}{r^2} + \mathcal{O}\left(\frac{1}{r^{3-\sigma}}\right),$$

which implies $n_0 \cdot P_1(x) \equiv 0$. Writing $P_1(x)/r^3 = \sum p_j \partial_j(1/r)$, we must have $p_j \cdot n_0 = 0$ for $j = 1, 2, 3$.

Step 4. As before, we plug (2.7) back again into the equation (2.1) and isolate terms that are $\mathcal{O}(\frac{1}{r^6})$ on the right hand side. We find

$$\begin{aligned} 0 = \Delta n + |\nabla n|^2 n &= \Delta u_2 + \frac{1}{r^5}|v_0|^2 v_0 + \frac{4}{r^6}(v_0 \cdot P_1(x))\, n_0 + \mathcal{O}\left(\frac{1}{r^{6-\sigma}}\right) \\ &= \Delta\left(u_2 + \frac{1}{6r^3}|v_0|^2 v_0 + \frac{1}{3r^4}(v_0 \cdot P_1(x))\, n_0\right) + \mathcal{O}\left(\frac{1}{r^{6-\sigma}}\right). \end{aligned}$$

Applying Lemma A.1 and arguing as in Steps 2 and 3, we deduce the existence of $P_2 \in \mathbb{R}[X]^3$ a vector of homogeneous harmonic polynomials of degree 2 (i.e. harmonic quadratic forms) such that we have the expansion

$$n = \left(1 - \frac{|v_0|^2}{2r^2}\right) n_0 + \frac{1}{r} v_0 + \frac{1}{r^3} P_1(x) - \frac{|v_0|^2}{6r^3} v_0 - \frac{1}{3r^4} (v_0 \cdot P_1)\, n_0 + \frac{1}{r^5} P_2(x) + u_3,$$
$$\frac{|u_3|}{r} + |\nabla u_3| \le \frac{C(n,\sigma)}{r^{5-\sigma}} \qquad \text{for } r \ge R_0.$$

With one more iteration we realize that the decay $u_3 = \mathcal{O}(1/r^{4-\sigma})$ improves to $u_3 = \mathcal{O}(1/r^4)$. Writing $P_1(x)/r^3 = \sum p_j \partial_j(1/r)$ and $P_2(x)/r^5 = \sum c_{k\ell} \partial_k \partial_\ell (1/r)$, the proof of Theorem 1.1 is complete. □

Alternative proof of Step 1. We present here another proof of Step 1, inspired by [29, Proposition 3]. The map $w = \partial_k n$ solves, for $r = |x| \ge R_0$, the system

$$-\Delta w = 2\nabla n : \nabla w\, n + |\nabla n|^2 w, \tag{2.8}$$

where for matrices A, B we use the notation $A : B := \mathrm{tr}(A^T B)$, for their Frobenius inner product. Testing (2.8) with $\eta^2 w$ for some smooth cut-off function η we obtain

$$\int \eta^2 |\nabla w|^2 \lesssim \int |\eta|\, |\nabla \eta|\, |w|\, |\nabla w| + \int \eta^2 |w| |\nabla n| |\nabla w| + \int \eta^2 |\nabla n|^2 |w|^2$$
$$\le \frac{1}{2} \int \eta^2 |\nabla w|^2 + C \int |\nabla \eta|^2 |w|^2 + \eta^2 |\nabla n|^2 |w|^2.$$

Absorbing the first term of the last line in the left-hand side, choosing $\mathbf{1}_{R \le |x| \le 2R} \le \eta \le \mathbf{1}_{R/2 \le |x| \le 3R}$ with $|\nabla \eta| \lesssim 1/R$, and using $|w|^2 \le |\nabla n|^2 \lesssim 1/r^3$ thanks to (2.2), we deduce

$$\int_{R \le |x| \le 2R} |\nabla w|^2 \lesssim \frac{1}{R^2},$$

hence

$$\int_{|x| \ge R} |\nabla w|^2 \le \sum_{k \ge 0} \int_{2^k R \le |x| \le 2^{k+1} R} |\nabla w|^2 \lesssim \sum_{k \ge 0} \frac{1}{2^{2k} R^2} \lesssim \frac{1}{R^2} \tag{2.9}$$

Therefore, plugging in (2.2) and (2.9) in (2.8), we find that the right-hand side of (2.8) has $\mathcal{O}(R^{-4})$ decay in an appropriate L^2 sense. To be precise,

$$-\Delta w = f, \qquad \left(\frac{1}{R^3} \int_{|x| \ge R} |x|^2 |f|^2\right)^{\frac{1}{2}} \lesssim \frac{1}{R^3}.$$

Applying Lemma A.2 with the choice $\gamma = 3 - \sigma/2$ for any small $\sigma > 0$, we deduce the existence of a map u such that $-\Delta u = f$ and

$$\left(\frac{1}{R^3} \int_{|x| \ge R} \frac{|u|^2}{|x|^2}\right)^{\frac{1}{2}} \lesssim \frac{1}{R^{3-\sigma/2}},$$

which implies

$$\int_{|x| \ge R} |u|^2 \le \sum_{k \ge 0} 2^{2k+2} R^2 \int_{2^k R \le |x| \le 2^{k+1} R} \frac{|u|^2}{|x|^2} \lesssim \sum_{k \ge 0} \frac{1}{2^{(1-\sigma)k}} \frac{1}{R^{1-\sigma}} \lesssim \frac{1}{R^{1-\sigma}},$$

for any $\sigma > 0$.

Since $w - u$ is harmonic and square integrable at ∞, we have $w - u = \mathcal{O}(1/r^2)$ as $r \to \infty$, and deduce from this and the above that

$$\int_{|x| \ge R} |w|^2 \lesssim \frac{1}{R^{1-\sigma}}.$$

Recalling $w = \partial_k n$ this implies, together with (2.2), $|\nabla n|^2 \lesssim 1/r^{4-\sigma}$ and the iteration starting in Step 2 of Theorem 1.1's proof can now be applied. □

Remark 2.3. We sketch here how to modify the proof of Theorem 1.1 for maps n which are minimizing only among axisymmetric configurations, so Corollary 1.3 applies also in that case. First of all, n is smooth outside of a large finite ball thanks to small energy estimates which are valid also in that setting: see e.g. [19, Lemma 4.1] where the symmetry condition is slightly more restrictive but the proof can be adapted, or note that n is stationary harmonic thanks to the methods in [15, § 2.1] and apply [10, Theorem I.4]. Then the alternative proof of Step 1 applies without modification, as do the rest of the steps. The first proof of Step 1 can also be applied, with the constraint that the constructed comparison map needs to be axisymmetric.

3. The leading-order term

In this section we prove Theorem 1.4 and Corollary 1.8.

Proof of Theorem 1.4. Without loss of generality assume $G \subset B_1$ and fix a C^1 function $\chi\colon \mathbb{R}^3 \to [0,1]$ such that $\chi \equiv 0$ on B_1 and $\int_{|x|\geq 1} |x|^{-2}(\chi-1)^2\,dx \lesssim \int_{|x|\geq 1} |\nabla\chi|^2\,dx < \infty$. Here, as stated in the introduction, $\lesssim$ denotes inequality up to an absolute constant, the cut-off function χ being fixed. In what follows, for any $m_0 \in \mathbb{S}^2$, we denote by

$$H(m_0) = \left\{ m \in H^1_{loc}(\mathbb{R}^3 \setminus G; \mathbb{S}^2)\colon \int_{\mathbb{R}^3\setminus G} \frac{|m-m_0|^2}{1+r^2} + \int_{\mathbb{R}^3\setminus G} |\nabla m|^2 + F_s(m_{\lfloor \partial G}) < \infty \right\},$$

the class of admissible competitors in the minimization problem (1.4) defining $\hat{E}(m_0)$. This class depends also on ∂G and F_s, which remain fixed throughout the proof.

Step 1: The map $\hat{E}$ is Lipschitz.

Let n_1, n_2 be minimizers with far-field alignments n_1^∞, n_2^∞. For any angle $\vartheta \in \mathbb{R}$, we denote by $\mathcal{R}(\vartheta) \in SO(3)$ the rotation of axis e_1 and angle ϑ. We choose the frame such that $n_1^\infty = e_3$ and $n_2^\infty = \mathcal{R}(\theta)e_3$, where θ is an angle satisfying $|n_1^\infty - n_2^\infty| \leq \theta \leq 2|n_1^\infty - n_2^\infty|$. Consider now the map $\tilde{n}_1 \in H(n_1^\infty)$ given by

$$\tilde{n}_1(x) = \mathcal{R}(-\chi(x)\theta)\, n_2(x).$$

We have

$$\begin{aligned} |\nabla \tilde{n}_1|^2 &\leq \theta^2 |\nabla\chi|^2 + |\nabla n_2|^2 + 2\theta\, |\nabla\chi|\, |\nabla n_2| \\ &\leq (1+\lambda^{-1})\theta^2 |\nabla\chi|^2 + (1+\lambda)|\nabla n_2|^2, \end{aligned}$$

for any $\lambda > 0$, hence

$$\hat{E}(n_1^\infty) \leq C(1+\lambda^{-1})|n_1^\infty - n_2^\infty|^2 + (1+\lambda)\hat{E}(n_2^\infty).$$

Applying this to $\lambda = 1$ and a fixed n_2^∞ we deduce in particular that $\hat{E}$ is bounded on $\mathbb{S}^2$. Moreover, choosing $\lambda = |n_1^\infty - n_2^\infty|$ we obtain

$$\hat{E}(n_1^\infty) - \hat{E}(n_2^\infty) \leq |n_1^\infty - n_2^\infty| \left(\hat{E}(n_2^\infty) + C + C|n_1^\infty - n_2^\infty| \right).$$

Reversing the roles of n_1, n_2 and recalling that $\hat{E}(n^\infty)$ is bounded on $\mathbb{S}^2$, we conclude that $\hat{E}$ is Lipschitz.

Step 2: At every differentiability point $n_0 \in \mathbb{S}^2$ of $\hat{E}$ we have $\nabla\hat{E}(n_0) = -8\pi v_0$, where $v_0 = \lim_{r\to\infty} r(n-n_0) \in T_{n_0}\mathbb{S}^2$ for any minimizer n such that $E(n) = \hat{E}(n_0)$. Here recall that $r = |x|$ and the limit v_0 is well-defined for any such map n, thanks to Theorem 1.1.

Let $n_0 \in \mathbb{S}^2$ be a differentiability point of $\hat{E}$. For any axis $e \in \mathbb{S}^2$ let $\mathcal{R}(\theta)$ be the rotation of axis e and angle θ, and set $n_\theta^\infty = \mathcal{R}(\theta)n_0$, so that

$$\hat{E}(n_\theta^\infty) - \hat{E}(n_0) = \nabla\hat{E}(n_0) \cdot (\mathcal{R}'(0)n_0) + o(\theta) \qquad \text{as } \theta \to 0.$$

Define $\tilde{n} \in H(n_\theta^\infty)$ by $\tilde{n} = \mathcal{R}(\chi\theta)n$, where n is a minimizer such that $E(n) = \hat{E}(n_0)$. Using the equation satisfied by n and the fact that $\tilde{n} = n$ in ∂G, for all $R > 1$ we have

$$\begin{aligned}
&\int_{B_R\setminus G} |\nabla\tilde{n}|^2 - \int_{B_R\setminus G} |\nabla n|^2 \\
&= \int_{B_R\setminus G} \left(2\nabla n\cdot\nabla(\tilde{n}-n) + |\nabla(\tilde{n}-n)|^2\right) \\
&= 2\int_{\partial B_R} \partial_r n\cdot(\tilde{n}-n) + \int_{B_R\setminus G} \left(-2\Delta n\cdot(\tilde{n}-n) + |\nabla(\tilde{n}-n)|^2\right) \\
&= 2\int_{\partial B_R} \partial_r n\cdot(\tilde{n}-n) + \int_{B_R\setminus G} 2|\nabla n|^2 n\cdot(\tilde{n}-n) + \int_{B_R\setminus G} |\nabla(\tilde{n}-n)|^2 \\
&= 2\int_{\partial B_R} \partial_r n\cdot(\tilde{n}-n) - \int_{B_R\setminus G} |\nabla n|^2|\tilde{n}-n|^2 + \int_{B_R\setminus G} |\nabla(\tilde{n}-n)|^2
\end{aligned}$$

Using the asymptotic expansion of the minimizing map n ($n = n_0 + v_0/r + u_1$, see (2.6), with $|u_1|/r + |\nabla u_1| = \mathcal{O}(1/r^3)$ thanks to (2.7)) we have

$$\int_{B_R} \partial_r n\cdot(\tilde{n}-n) = -8\pi v_0\cdot(n_\theta^\infty - n_0) + O(1/R) \quad \text{as } R\to\infty,$$

where $v_0 = \lim_{r\to\infty} r(n-n_0) \in T_{n_0}\mathbb{S}^2$. We deduce that

$$\begin{aligned}
&\hat{E}(n_\theta^\infty) - \hat{E}(n_0) \\
&\le E(\tilde{n}) - E(n) = \lim_{R\to\infty}\left(\int_{B_R\setminus G} |\nabla\tilde{n}|^2 - \int_{B_R\setminus G} |\nabla n|^2\right) \\
&= -8\pi v_0\cdot(n_\theta^\infty - n_0) - \int_{\mathbb{R}^3\setminus G} |\nabla n|^2|\tilde{n}-n|^2 + \int_{\mathbb{R}^3\setminus G} |\nabla(\tilde{n}-n)|^2 \\
&\le -8\pi v_0\cdot(n_\theta^\infty - n_0) + C\left(1 + \int_{\mathbb{R}^3\setminus G} |\nabla n|^2\right)\theta^2. \qquad (3.1)
\end{aligned}$$

The last estimate follows from the explicit form of $\tilde{n} = \mathcal{R}(\chi\theta)n$, and the constant C depends only on the fixed cut-off function χ. In particular we have

$$\hat{E}(n_\theta^\infty) - \hat{E}(n_0) \le -8\pi v_0\cdot(\mathcal{R}'(0)n_0) + O(\theta^2),$$

which implies

$$(\nabla\hat{E}(n_0) + 8\pi v_0)\cdot(\mathcal{R}'(0)n_0) \le 0.$$

Since $\mathcal{R}'(0)n_0$ can be any tangent vector in $T_{n_0}\mathbb{S}^2$ we infer that $\nabla\hat{E}(n_0) + 8\pi v_0 = 0$.

Step 3. It remains to prove that $\hat{E}$ is semiconcave. This follows directly from the inequality (3.1) obtained in Step 2, as any $m_0 \in \mathbb{S}^2$ can be written as $m_0 = n_\theta^\infty$ for some $0 \le \theta \le 2|m_0 - n_0|$. This completes the proof of Theorem 1.4. □

Proof of Corollary 1.8. Consider first the axisymmetric case $\mathrm{Sym}(G) \supset SO(3)^{\mathbf{u}}$. Then we have $\hat{E}(Rn_0) = \hat{E}(n_0)$ for any rotation R of axis $\mathbf{u}$ and $n_0 \in \mathbb{S}^2$. At a differentiable point n_0, differentiating this identity with respect to R implies $\nabla\hat{E}(n_0)\cdot An_0 = 0$ for any antisymmetric matrix A with $A\mathbf{u} = 0$, i.e. $\nabla\hat{E}(n_0)\cdot(\mathbf{u}\times n_0) = 0$. Recalling from Theorem 1.4 that $\nabla\hat{E}(n_0) = -8\pi v_0$, we deduce $v_0\cdot(\mathbf{u}\times n_0) = 0$.

Moreover, if $\mathbf{u}$ is a differentiability point, then differentiating that same identity with respect to n_0 at $n_0 = \mathbf{u}$ gives $R^{-1}\nabla\hat{E}(\mathbf{u}) = \nabla\hat{E}(\mathbf{u})$ for any rotation R of axis $\mathbf{u}$, hence $\nabla\hat{E}(\mathbf{u}) = 0$ since $\nabla\hat{E}(u) \in T_{\mathbf{u}}\mathbb{S}^2 = \mathbf{u}^\perp$. So $v_0(\mathbf{u}) = 0$.

In the spherically symmetric case $\mathrm{Sym}(G) \supset SO(3)$ we have $\hat{E}(Rn_0) = \hat{E}(n_0)$ for all $R \in SO(3)$, hence $\hat{E}$ is constant, and $\nabla\hat{E} = 0$ on $\mathbb{S}^2$. So $v_0(n_0) = 0$ for all $n_0 \in \mathbb{S}^2$. □

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Appendix A. Decay estimates for Poisson's equation

We collect here some folklore decay estimates for Poisson's equation. For the reader's convenience we include a self-contained proof (similar arguments can be found e.g. in [25, § 2.2.3] for Hölder decay at the origin). The elementary arguments we present here don't seem to apply directly for general systems as in Remark 1.2, in that case one should refer to [7, § 5-6].

Lemma A.1. *Let $d \geq 3$, $\gamma > d-2$, $\gamma \notin \mathbb{N}$, and f a function in $\mathbb{R}^d \setminus B_1$ satisfying*

$$|f(x)| \leq \frac{1}{r^{\gamma+2}} \qquad \text{for } r = |x| \geq 1.$$

Then there exists a function u such that $\Delta u = f$ in $\mathbb{R}^d \setminus B_1$ and

$$\frac{|u(x)|}{r} + |\nabla u(x)| \lesssim \frac{1}{r^{\gamma+1}}, \tag{A.1}$$

where the constant depends only on d and γ.

Note that (A.1) doesn't determine u uniquely, as we may add any faster-decaying harmonic terms to u without changing the equation $\Delta u = f$, but the proof does determine an explicit right inverse $f \mapsto u$ to the Laplacian in that decay range.

We will obtain Lemma A.1 as a consequence of an L^2 version of it, that we state now.

Lemma A.2. *Let $d \geq 3$, $\gamma > d-2$, $\gamma \notin \mathbb{N}$, and f a function in $\mathbb{R}^d \setminus B_1$ satisfying*

$$\left(\frac{1}{R^d} \int_{|x|>R} |x|^2 f^2 \, dx \right)^{\frac{1}{2}} \leq \frac{1}{R^{\gamma+1}} \qquad \forall R \geq 1, \tag{A.2}$$

Then there exists a function u such that $\Delta u = f$ in $\mathbb{R}^d \setminus B_1$ and

$$\left(\frac{1}{R^d} \int_{|x|\geq R} \frac{|u|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \lesssim \frac{1}{R^{\gamma+1}} \qquad \forall R \geq 1, \tag{A.3}$$

where the implicit constant depends only on d and γ.

Before proving Lemma A.2, we explain why, together with rescaled elliptic estimates, it implies Lemma A.1.

Proof of Lemma A.1. The assumption on f implies that it satisfies the L^2 decay in the assumption of Lemma A.2, so we obtain u such that $\Delta u = f$ in $\mathbb{R}^d \setminus B_1$ and

$$\left(\frac{1}{R^d} \int_{|x|\geq R} \frac{|u|^2}{|x|^2} \, dx \right)^{\frac{1}{2}} \lesssim \frac{1}{R^{\gamma+1}} \qquad \forall R \geq 1,$$

and the pointwise bound (A.1) in the conclusion of Lemma A.1 follows from rescaled elliptic estimates. Explicitly, consider $\hat{u}(\hat{x}) = u(R\hat{x})$ which solves $\Delta \hat{u} = \hat{f}$, where $\hat{f}(\hat{x}) := R^2 f(R\hat{x})$, then from interior elliptic estimates (see e.g. [16]) we have

$$\begin{aligned}\sup_{B_3\setminus B_2}(|\hat{u}|+|\nabla\hat{u}|) &\lesssim \left(\int_{B_4\setminus B_1}|\hat{u}|^2\right)^{\frac{1}{2}}+\sup_{B_4\setminus B_1}|\hat{f}|\\ &\lesssim R\left(\frac{1}{R^d}\int_{|x|\geq R}\frac{|u|^2}{|x|^2}\right)^{\frac{1}{2}}+\frac{1}{R^\gamma},\end{aligned}$$

from which, scaling back, we infer (A.1). □

Next we prove Lemma A.2. Before doing so, we recall some facts concerning spherical harmonics (that is, homogeneous harmonic polynomials), referring the reader to [30] for details. The Laplace-Beltrami operator on $\mathbb{S}^{d-1}$ diagonalizes as

$$-\Delta_{\mathbb{S}^{d-1}}\Phi_j=\lambda_j\Phi_j,\qquad 0=\lambda_0\leq\lambda_1\leq\cdots$$

The set $\{\lambda_j\}_{j\in\mathbb{N}}$ coincides with $\{k^2+k(d-2)\}_{k\in\mathbb{N}}$. The eigenfunctions corresponding to $k^2+k(d-2)$ span the homogeneous harmonic polynomials of degree k. We choose them normalized in $L^2(\mathbb{S}^{d-1})$ so they form an orthonormal Hilbert basis of this space. For a $W^{2,2}_{loc}$ function $w\colon(0,\infty)\to\mathbb{R}$ we have

$$\Delta(w(r)\Phi_j(\omega))=(\mathcal{L}_jw)(r)\Phi_j(\omega),\qquad \mathcal{L}_j=\partial_{rr}+\frac{d-1}{r}\partial_r-\frac{\lambda_j}{r^2}. \tag{A.4}$$

The solutions of $\mathcal{L}_jw=0$ are linear combinations of $r^{\gamma_j^+}$ and $r^{-\gamma_j^-}$, where $\gamma_j^\pm\geq 0$ are given by

$$\begin{aligned}\gamma_j^+&=\sqrt{\left(\frac{d-2}{2}\right)^2+\lambda_j}-\frac{d-2}{2}=k &&\text{for }\lambda_j=k^2+k(d-2),\\ \gamma_j^-&=\sqrt{\left(\frac{d-2}{2}\right)^2+\lambda_j}+\frac{d-2}{2}=k+d-2 &&\text{for }\lambda_j=k^2+k(d-2).\end{aligned}$$

The decay rate $\gamma>d-2$, $\gamma\notin\mathbb{N}$, is fixed and we denote by $j_0=j_0(\gamma)$ the integer $j_0\geq 0$ such that

$$\begin{aligned}\left\{j\in\mathbb{N}\colon\gamma_j^-<\gamma\right\}&=\{0,\ldots,j_0\},\\ \left\{j\in\mathbb{N}\colon\gamma_j^->\gamma\right\}&=\{j_0+1,j_0+2,\ldots\}.\end{aligned}$$

Proof of Lemma A.2. We extend f to be defined in $\mathbb{R}^d$, with the property that

$$\left(\int_{|x|\leq 1}|x|^2f^2\,dx\right)^{\frac{1}{2}}\leq 1,$$

and will construct a function u such that $\Delta u=f$ in $\mathbb{R}^d\setminus\{0\}$. The function $f\in L^2(\mathbb{R}^d)$ admits a spherical harmonics expansion

$$f=\sum_{j\geq 0}f_j(r)\Phi_j(\omega),$$

and the decay assumption (A.2) on f amounts to

$$\sum_{j\geq 0}\int_R^\infty f_j(r)^2r^{d+1}\,dr\leq R^{d-2\gamma-2}. \tag{A.5}$$

We define u as

$$u:=\sum_{j\geq 0}u_j(r)\Phi_j(\omega),$$

where $u_j\in W^{2,2}_{loc}(0,\infty)$ satisfy

$$\mathcal{L}_ju_j=f_j.$$

To write down an explicit formula for u_j we rewrite $\mathcal{L}_j$, defined in (A.4), as

$$\mathcal{L}_j u = r^{-d+1+\gamma_j^-}\partial_r[r^{d-1-2\gamma_j^-}\partial_r(r^{\gamma_j^-}u)],$$

and define

$$u_j(r) = \begin{cases} r^{-\gamma_j^-}\displaystyle\int_r^\infty t^{2\gamma_j^-+1-d}\int_t^\infty s^{d-1-\gamma_j^-} f_j(s)\,ds\,dt & \text{if } j\in\{0,\dots,j_0\},\\ r^{-\gamma_j^-}\displaystyle\int_0^r t^{2\gamma_j^-+1-d}\int_t^\infty s^{d-1-\gamma_j^-} f_j(s)\,ds\,dt & \text{if } j\geq j_0+1.\end{cases} \tag{A.6}$$

This is well defined because for any $t>0$ using Cauchy-Schwarz, (A.5) with the choice $R=t$, and the fact that $\gamma_j^-\geq d-2>0$, we can estimate the inner integral by

$$\begin{aligned}\int_t^\infty s^{d-1-\gamma_j^-}|f_j(s)|\,ds &\leq \left(\int_t^\infty s^{-2-2\gamma_j^-}s^{d-1}ds\right)^{\frac12}\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)^{\frac12}\\ &\leqslant \frac{1}{\sqrt{2\gamma_j^-+2-d}}t^{\frac d2-\gamma_j^--1}t^{\frac d2-\gamma-1} = \frac{1}{\sqrt{2\gamma_j^-+2-d}}t^{d-\gamma-\gamma_j^--2}.\end{aligned} \tag{A.7}$$

Furthermore, as $t\mapsto t^{2\gamma_j^-+1-d}t^{d-2-\gamma-\gamma_j^-}=t^{\gamma_j^--\gamma-1}$ is integrable near ∞ if $\gamma_j^-<\gamma$, i.e., if $j\leq j_0$; and is integrable near 0 if $\gamma_j^->\gamma$, i.e., if $j\geq j_0+1$, the functions u_j in (A.6) are well-defined.

Let $j\leq j_0$ and set

$$\alpha := \gamma+\gamma_{j_0}^-+1-d,$$

so that $2\gamma+1-d>\alpha>2\gamma_j^-+1-d$. By (A.7) and Cauchy-Schwarz we have

$$\begin{aligned}|u_j(r)|^2 &\leq \frac{r^{-2\gamma_j^-}}{2+2\gamma_j^--d}\left(\int_r^\infty t^{\gamma_j^--\frac d2}\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)^{\frac12}dt\right)^2\\ &= \frac{r^{-2\gamma_j^-}}{2+2\gamma_j^--d}\left(\int_r^\infty t^{\gamma_j^--\frac d2-\frac\alpha2}t^{\frac\alpha2}\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)^{\frac12}dt\right)^2\\ &\leq \frac{r^{-2\gamma_j^-}}{2+2\gamma_j^--d}\int_r^\infty t^{2\gamma_j^--d-\alpha}\,dt\int_r^\infty t^\alpha\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt\\ &= \frac{r^{-d+1-\alpha}}{(2+2\gamma_j^--d)(\alpha-2\gamma_j^-+d-1)}\int_r^\infty t^\alpha\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt\\ &\leq \frac{r^{-d+1-\alpha}}{(d-2)(\gamma-\gamma_{j_0}^-)}\int_r^\infty t^\alpha\left(\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt,\end{aligned}$$

where in the last line, we used that $\gamma_j^-\geqslant d-2$ so that $2+2\gamma_j^--d\geqslant d-2$, and that $\gamma+\gamma_{j_0}^--2\gamma_j^-\geqslant\gamma-\gamma_{j_0}^-$, when $j\leqslant j_0$. Summing and using (A.5), we deduce

$$\begin{aligned}\sum_{j=0}^{j_0}\frac{|u_j(r)|^2}{r^2} &\leq \frac{r^{-d-1-\alpha}}{(d-2)(\gamma-\gamma_{j_0}^-)}\int_r^\infty t^\alpha\left(\sum_{j=0}^{j_0}\int_t^\infty s^2 f_j(s)^2 s^{d-1}ds\right)dt\\ &\leq \frac{r^{-d-1-\alpha}}{(d-2)(\gamma-\gamma_{j_0}^-)}\int_r^\infty t^{\alpha+d-2\gamma-2}\,dt\\ &= \frac{r^{-2\gamma-2}}{(d-2)(\gamma-\gamma_{j_0}^-)(2\gamma+1-d-\alpha)}\\ &\leq \frac{r^{-2\gamma-2}}{(d-2)(\gamma-\gamma_{j_0}^-)^2}.\end{aligned}$$

Similarly, for $j \geq j_0 + 1$ we set

$$\beta = \gamma + \gamma^-_{j_0+1} + 1 - d,$$

which satisfies $2\gamma + 1 - d < \beta < 2\gamma^-_j + 1 - d$. Using (A.7) and Cauchy-Schwarz we find

$$|u_j(r)|^2 \leq \frac{r^{-d+1-\beta}}{(d-2)(\gamma^-_{j_0+1} - \gamma)} \int_0^r t^{\alpha_j} \left(\int_t^\infty s^2 f_j(s)^2 s^{d-1} ds \right) dt,$$

so that, we similarly obtain from (A.5) that

$$\sum_{j=j_0+1}^{\infty} \frac{|u_j(r)|^2}{r^2} \leq \frac{r^{-2\gamma-2}}{(d-2)(\gamma^-_{j_0+1} - \gamma)^2}.$$

We conclude that

$$\sum_{j=0}^{\infty} \frac{|u_j(r)|^2}{r^2} \leq \frac{1}{d-2} \left(\frac{1}{(\gamma - \gamma^-_{j_0})^2} + \frac{1}{(\gamma^-_{j_0+1} - \gamma)^2} \right) r^{-2\gamma-2}.$$

Therefore, since $\gamma > d - 2$,

$$\begin{aligned} \frac{1}{R^d} \int_{|x|\geq R} \frac{|u|^2}{|x|^2}\, dx &= \frac{1}{R^d} \int_R^\infty \left(\sum_{j=0}^{\infty} \frac{|u_j(r)|^2}{r^2} \right) r^{d-1}\, dr \\ &\leq \frac{1}{d-2} \left(\frac{1}{(\gamma - \gamma^-_{j_0})^2} + \frac{1}{(\gamma^-_{j_0+1} - \gamma)^2} \right) \frac{R^{-2\gamma-2}}{2\gamma + 2 - d} \\ &\leq \frac{1}{(d-2)^2} \left(\frac{1}{(\gamma - \gamma^-_{j_0})^2} + \frac{1}{(\gamma^-_{j_0+1} - \gamma)^2} \right) R^{-2\gamma-2}, \end{aligned}$$

which proves (A.3). □

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