Saturn ring defect around a spherical particle immersed in a nematic liquid crystal

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Abstract

We consider a nematic liquid crystal occupying the three-dimensional domain in the exterior of a spherical colloid particle. The nematic is subject to Dirichlet boundary conditions that enforce orthogonal attachment of nematic molecules to the surface of the particle. Our main interest is to understand the behavior of energy-critical configurations of the Landau-de Gennes Q-tensor model in the limit of vanishing correlation length. We demonstrate existence of configurations with a single Saturn-ring defect approaching the equator of the particle and no other line or point defects. We show this by analyzing asymptotics of energy minimizers under two symmetry constraints: rotational equivariance around the vertical axis and reflection across the horizontal plane. Energy blow-up at the ring defect is a significant obstacle to constructing well-behaved comparison maps needed to eliminate the possibility of point defects. The boundary estimates we develop to address this issue are new and should be applicable to a wider class of problems.

1 Introduction

The study of defects in liquid crystals is well-motivated from physical considerations, and is also closely connected to many fundamental questions in analysis and geometry. The intimate connection between nematic liquid crystals and $\mathbb{S}^2$-valued harmonic maps is well-established through director-based models such as Oseen-Frank [24], and a comprehensive study of singularities in nematics will both exploit and expand the rich trove of analytical tools for studying geometrical variational problems. To better describe nematics in settings involving non-orientability, biaxiality, and the presence of line defects, physicists and mathematicians have turned to the tensorial Landau-de Gennes model, which is in some sense a relaxation of the non-convex constraints of director models. Indeed, much recent attention has concentrated on recovering the Oseen-Frank director and energy in the vanishing correlation length limit of Landau-de Gennes (see, e.g., [6, 12, 16, 22, 32, 34].)

In this paper we revisit an important model problem, that of a spherical colloid particle immersed in a nematic which fills the exterior domain, approaching a constant uniaxial state

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at infinity. We work within the Landau-de Gennes framework, with homeotropic Dirichlet boundary conditions on the colloid surface. Physicists have long expected that there are two competing candidates for minimizers in this geometry: an orientable solution with *dipolar* symmetry, consisting of a single satellite point defect lying on the axis of symmetry, and a non-orientable director pattern having a circular "Saturn ring" singularity on the equatorial plane perpendicular to the symmetry axis. The latter configuration exhibits *quadrupolar* symmetry: in addition to axial symmetry, it is invariant under reflection across the equatorial plane (see (7) for a precise definition). Both configurations are depicted in Figure 1 below. In the first mathematical treatment of this problem [1], this expectation is confirmed in the case of very small colloids (for which the quadrupolar Saturn ring solution is minimizing,) or for very large colloids (in which, assuming axial symmetry, the dipolar satellite point defect prevails.) However, Saturn ring defects have been observed both experimentally and numerically in the physics literature (see, e.g., [21, 31, 33, 38, 41]) and appear to be energetically favorable in many settings, even for larger particles. Saturn ring structures are also expected to prevail over radially symmetric point defects in the absence of colloids as well (see [27], and [17, 18, 42] for recent progress around that issue).

1.1 Main results

The goal of this paper is to produce solutions of the spherical colloid problem which exhibit Saturn-ring defects in the limit of small correlation length. To do this, we minimize the Landau-de Gennes energy in a function space enforcing the expected (quadrupolar) symmetries of such a configuration. The symmetry hypothesis will ensure the existence of at least one ring defect on the horizontal plane; a much more difficult issue is to eliminate the possibility of other defects (rings or point defects). Additional ring defects can be excluded by carefully adapting lower bound techniques developed for the Ginzburg-Landau problem [36, 26, 8, 4]. Ruling out point defects, however, presents a new and significant analytical challenge. In general, determining the precise number of point defects in a three dimensional domain is a difficult task: point defects carry only a bounded quantity of energy (see e.g. [11, 32]), which is negligible compared to that of line defects (see [13]), and thus point defects are harder to detect using energy estimates. Moreover, unlike line defects, the number of point defects cannot be deduced from topological considerations as even topologically trivial boundary conditions may give rise to an arbitrary number of point defects [25]. Only very specific examples are known where the number of point defects can be determined (see e.g. [24, 11, 1]). In the present work this task is made even harder due to presence of a line defect approaching the boundary, in that the boundary conditions are "destroyed" by energy blow-up at the ring defect. This considerably complicates the construction of well-behaved comparison maps. We overcome this obstacle by proving a very precise estimate in the blow-up region at the boundary. This estimate appears to be new, even within the context of some well-studied variational problems (such as Ginzburg-Landau with a weight).

Let us now introduce the Landau-de Gennes functional, and the variational framework which we will use in our study. In nondimensional units the colloidal particle is represented by the closed ball of radius one $B = \{|\cdot| \leq 1\} \subset \mathbb{R}^3$, so that the liquid crystal is contained in the domain $\Omega = \mathbb{R}^3 \setminus B$. In these units the Landau-de Gennes energy depends on the *nematic correlation length* $\xi > 0$, and is given by

$$E_\xi(Q) = \int_\Omega \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) dx. \tag{1}$$

The map Q takes values into the space $\mathcal{S}_0$ of 3×3 symmetric matrices with zero trace and describes nematic alignment. The nematic potential is given by

$$f(Q) = -\frac{1}{2}|Q|^2 - \operatorname{tr}(Q^3) + \frac{3}{4}|Q|^4 + C,$$

where the constant C is such that f satisfies

$$f(Q) \geq 0 \text{ with equality iff } Q \in \mathcal{U}_\star := \left\{ n \otimes n - \frac{1}{3}I \colon n \in \mathbb{S}^2 \right\}. \tag{2}$$

The correlation length ξ is typically small and therefore we are going to be interested in the limit $\xi \to 0$.

Remark 1.1. We chose specific constants for the potential $f(Q)$ in order to simplify notation. Our results remain valid for any potential of the form

$$f(Q) = -a|Q|^2 - b\operatorname{tr}(Q^3) + c|Q|^4 + C(a,b,c),$$

where a, b, c and $C(a,b,c)$ are such that (2) is verified with $\mathcal{U}_\star = \left\{ s_\star \left(n \otimes n - \frac{1}{3}I \right) : n \in \mathbb{S}^2 \right\}$ for some $s_\star = s_\star(a,b,c) > 0$.

Anchoring at the particle surface is assumed to be radial:

$$Q = Q_b := e_r \otimes e_r - \frac{1}{3}I \qquad \text{on } \partial\Omega, \qquad e_r = \frac{x}{|x|}. \tag{3}$$

At infinity, the effect of the particle is not felt and the alignment is uniform, given by

$$Q_\infty := e_3 \otimes e_3 - \frac{1}{3}I. \tag{4}$$

More precisely, this far field condition is enforced by considering configurations in the space $\mathcal{H}$ given by

$$\mathcal{H} := Q_\infty + \dot{\mathcal{H}}, \qquad \dot{\mathcal{H}} := \left\{ Q \in H^1_{loc}(\Omega; \mathcal{S}_0) \colon \int_\Omega |\nabla Q|^2 + \int_\Omega \frac{|Q|^2}{|x|^2} < \infty \right\}. \tag{5}$$

We denote by $\mathcal{H}_b$ the space of such configurations that satisfy in addition the radial anchoring condition at the particle surface:

$$\mathcal{H}_b = \{ Q \in \mathcal{H} \colon Q \text{ satisfies (3)} \}. \tag{6}$$

We seek to construct critical points of E_ξ with the quadrupolar symmetry of the Saturn ring configuration. This entails two symmetry constraints on the admissible $Q \in \mathcal{H}_b$:

- rotation symmetry around the vertical axis,
- and reflection symmetry across the equatorial plane.

See (7) for a precise definition of each, in terms of group actions. We denote the space of maps in $\mathcal{H}_b$ satisfying these two symmetry constraints by $\mathcal{H}_{sym}$, i.e.

$$\mathcal{H}_{sym} = \{Q \in \mathcal{H}_b \text{ with quadrupolar symmetry}\}.$$

A more complete discussion of the space $\mathcal{H}_{sym}$ will be given in Section 2. Minimizers of E_ξ in $\mathcal{H}_{sym}$ do exist, because Hardy's inequality ensures the coercivity of the energy. Moreover, they are critical points of E_ξ in the full space $\mathcal{H}$: this is a consequence of the principle of symmetric criticality [35] (see also [28, Appendix 1]). Our main result shows that the energy of a symmetric minimizer concentrates, as $\xi \to 0$, inside a Saturn ring shaped region around the particle. In the limit this region coincides with the equatorial circle

$$\mathcal{C} := \{(\cos\varphi, \sin\varphi, 0) \colon \varphi \in \mathbb{R}\} = \partial B \cap \{x_3 = 0\}.$$

Since the energy will blow up around $\mathcal{C}$, the resulting limit configuration will not belong to $\mathcal{H}_{sym}$. To define the limit space we cut out a small neighborhood of $\mathcal{C}$, consider the exterior domain

$$\Omega_\delta^{ext} = \{x \in \Omega \colon \operatorname{dist}(x, \mathcal{C}) > \delta\},$$

and define the limit space $\mathcal{H}^\star_{sym} = \bigcap_{\delta>0} \mathcal{H}^\star_{sym}(\Omega_\delta^{ext})$, where

$$\mathcal{H}^\star_{sym}(\Omega_\delta^{ext}) = \Big\{ Q \in H^1_{loc}(\Omega_\delta^{ext}; \mathcal{U}_\star) \text{ with quadrupolar symmetry, s.t.} \\ \int_{\Omega_\delta^{ext}} |\nabla Q|^2 + \int_{\Omega_\delta^{ext}} \frac{|Q - Q_\infty|^2}{|x|^2} < \infty, \quad \text{and} \quad Q = Q_b \text{ for } |x| = 1 \Big\}.$$

We may now state our result asserting the existence and asymptotic behavior of solutions with quadrupolar symmetry:

Theorem 1.2. *For any $\xi \in (0, 1]$ let Q_ξ minimize E_ξ in $\mathcal{H}_{sym}$. Then we have:*

(i) upper bound: *there exists a universal constant $C > 0$ such that*

$$\frac{1}{2\pi} E_\xi(Q_\xi) \le \pi \ln \frac{1}{\xi} + \pi \ln\ln \frac{1}{\xi} + C.$$

(ii) lower bound: *there exists a universal constant $C > 0$ such that for any $\delta \in (0, 1)$,*

$$\frac{1}{2\pi} E_\xi(Q_\xi; \Omega_\delta^{int}) \ge \pi \ln \frac{1}{\xi} + \pi \ln\ln \frac{1}{\xi} - 2\pi \ln \frac{1}{\delta} - C \qquad \text{as } \xi \to 0,$$

where $\Omega_\delta^{int} = \Omega \setminus \Omega_\delta^{ext} = \Omega \cap \{\operatorname{dist}(\cdot, \mathcal{C}) \le \delta\}$ is the δ-neighborhood of the ring defect $\mathcal{C}$.

(iii) limit configuration: *there is a subsequence $\xi \to 0$ such that Q_ξ converges in $C^{1,\alpha}_{loc}(\overline{\Omega}\setminus\mathcal{C})$ to a map $Q_\star \in \mathcal{H}^\star_{sym}$ which is smooth in $\overline{\Omega}\setminus\mathcal{C}$, and uniaxial,*

$$Q_\star(x) = n_\star(x)\otimes n_\star(x) - \frac{1}{3}I, \quad n_\star(x)\in\mathbb{S}^2,$$

where $n_\star$ is a smooth $\mathbb{S}^2$-valued locally minimizing harmonic map in Ω. Furthermore, $n_\star$ satisfies the additional symmetry property that $n_\star(x_1,0,x_3)\perp e_2$.

Remark 1.3. (a) The limiting map $n_\star(\rho,\varphi,z)$ is conveniently expressed in cylindrical coordinates (ρ,φ,z) and is entirely determined by its values $n(\rho,z)=n_*(\rho,0,z)$ in the upper half of the $\varphi=0$ cross-section,

$$D := \{(\rho,\varphi,z):\ \rho^2+z^2>1,\ \rho,z>0,\ \varphi=0\}.$$

Moreover, on the z-axis $n(0,z)=e_3$, $\forall z>1$, as well as on the equatorial plane: $n(\rho,0)=e_3$, $\forall\rho>1$. (See Lemma 4.2.)

(b) The additional symmetry statement in Theorem 1.2 *(iii)* amounts to $n_\star(\rho,\varphi,z)\perp e_\varphi$, i.e. $n_\star(\rho,\varphi,z)$ lies in the azimuthal plane generated by e_ρ and e_z. (See Proposition 4.4.)

(c) Aside from this symmetry property, in Section 4 we obtain rather precise information about $n_\star$. As $|x|\to\infty$, $n_\star\to e_3$, in the sense that

$$\int_\Omega \frac{|n_\star-e_3|^2}{|x|^2} \le \int_\Omega \frac{|n_\star-e_3|^2}{x_1^2+x_2^2} < \infty.$$

Near the ring defect, n resembles a point vortex of degree $-\frac{1}{2}$, with

$$n = n(1+r\cos\theta, r\sin\theta) \approx \sin\theta\, e_1 + \cos\theta\, e_3,$$

in polar coordinates (r,θ) centered at $(1,0)$ in the $\varphi=0$ cross-section. This is a consequence of Lemma 4.7.

(d) Note that we do not prove uniqueness of $n_\star$, so that in Theorem 1.2 *(iii)*, different subsequences may converge to different maps $n_\star$ with the above properties.

In proving Theorem 1.2 we rely heavily on the symmetry constraint, which reduces the problem to two dimensions. Indeed, the relevant analogy is to a two-dimensional Ginzburg-Landau energy with a weight $w=\rho$ arising from cylindrical symmetry. The asymptotic behavior of Ginzburg-Landau energies with weights have been studied in [4, 8]. The principal novelty of this work is that we must deal with Q-tensor-valued maps in an unbounded domain rather than complex-valued maps in a bounded domain. Points *(i)* and *(ii)* of Theorem 1.2 follow from careful adaptation of the techniques in [4, 8], along with classical Ginzburg-Landau methods in [40, 36, 26], and more recent arguments for Q-tensor-valued maps in [22, 12].

The most delicate part of Theorem 1.2 is the statement *(iii)* asserting that the limit is smooth everywhere away from the equatorial ring defect $\mathcal{C}$. Proving this statement amounts to eliminating the possibility of point singularities appearing on the z-axis. Indeed, from topological considerations, there is no smooth unit director field in the vertical cross-section D which can satisfy the boundary conditions on the colloid surface and at infinity, and thus the limit must exhibit one or more singularities. This topological constraint is satisfied, for instance, by a single ring defect which is negatively charged, in the sense that it resembles a degree $-\frac{1}{2}$ vortex in the cross-section. However, a configuration with a positively charged ring defect (a degree $+\frac{1}{2}$ vortex in the cross-section), combined with a pair of negatively charged point defects (hedgehogs) on the z-axis, is also topologically permissible. The energy comparison between these two configurations turns out to be nontrivial because their energies are the same to leading order in ξ, so that more precise, up to $o(1)$, estimates are required. A crucial ingredient in choosing the lowest energy configuration is to show that the energy in the core of a ring defect would be the same, up to terms of order $o(1)$ in ξ, regardless of whether the ring is negatively or positively charged; this is done in Subsection 4.3.2. Once we know the core energies of the two types of ring are essentially the same, the $O(1)$ energy cost in connecting a positive ring to a pair of point defects is shown (in Lemma 4.8) to be strictly greater than that of the single negative degree ring defect.

In calculating the energy of the defect core we encounter an additional difficulty not present in determining the $O(1)$ core energy term for classical Ginzburg-Landau vortices. Indeed, in [9] the authors crucially use the fact that the energy of Ginzburg-Landau vortices scales radially: at scale $r \gg \xi$, the energy of a vortex goes as $\ln(r/\xi) = \ln(1/\xi) + \ln r$, and the effect of phase winding is separated from the cost of core formation. Here, on the other hand, proximity to the boundary breaks this scaling invariance and influences the core shape, as seen by the presence of the $\ln \ln \xi$ term, and makes it much less clear that radial rescaling should reveal a universal $O(1)$ core energy term.

To obtain the core energy estimate, we deform the minimizers in a very narrow wedge domain emanating tangentially from the limiting equatorial defect (see Lemma 4.10). This requires a sharp lower bound on the energy in a small disc tangential to the particle surface at its equator (see Lemma 3.9). The corresponding core energy estimate is new—to the best of our knowledge. Once the core energy is determined, the added energy cost of an anti-hedgehog pair may be computed thanks to ideas in [1] (see Lemma 4.8).

1.2 Background and relevant numerical results

The mathematical study of line defects in nematics was initiated in [13], in the singular limit as the correlation length $\xi \to 0$, for domains and boundary values which induce defects along line segments. As mentioned earlier, global minimizers of the spherical colloid problem were first addressed mathematically in [1], in which the size of the colloid plays a determining role. As has long been known by physicists, equatorial ring defects can be observed even around large colloid particles, for example in the presence of external electric or magnetic fields [19, 20, 23, 30, 39] or in confinement [29, 39]. The situation with a magnetic field was studied mathematically in [2], via a Landau-de Gennes energy modified to model interaction with a constant field. The main result of [2] identifies the leading order term in an expansion of the energy, indicating the presence of an equatorial ring defect rather than a satellite point defect, provided the magnetic field is high enough $h \gg \xi\,|\ln \xi|$. In the complementary low magnetic field regime $h \ll \xi\,|\ln \xi|$, the lower bound established here in Theorem 1.2 directly implies, in view of upper bounds established in [2], that minimizers cannot have quadrupolar symmetry, thus hinting at the presence of a satellite point defect. The asymptotics of that model are further and more precisely explored in [3].

Even in the absence of external factors which appear to favor rings over satellite point defects, much physical evidence, both numerical [21] and formal [31], arguments suggest that there is a range of intermediate particle sizes for which configurations with Saturn ring defects may be stable and coexist with point defect configurations having lower energy.

We do not consider the important question of stability in this paper, but numerical simulations suggest that the solutions found here may be locally stable. To illustrate these observations, in Figs. 1a-1c we present the summary of simulations that reproduce and extend the results of [21]. We numerically solved in COMSOL [14] the equations for the gradient flow

$$\frac{\partial Q}{\partial t} = -\frac{\delta E_\xi}{\delta Q}$$

for the energy E_ξ defined in (1) in the domain in the form of a large cylinder with a spherical void of radius 1 with the same center as that of the cylinder. The admissible Q-tensor fields have values in the set of symmetric traceless matrices and are rotationally equivariant with respect to z-axis that is also the axis of the cylinder. The Q-tensors are subject to the initial condition $Q(\cdot, 0) = Q_{init}$ and the boundary conditions (4) and (3) on the surfaces of the cylinder and the sphere, respectively.

Following [21] and assuming that (ρ, φ) are polar coordinates in a plane perpendicular to the axis of the cylinder, the simulations were run starting from two initial conditions

$$Q_{init} = e_3 \otimes e_3 - \frac{1}{3} I \qquad \text{and} \qquad Q_{init} = n(\psi) \otimes n(\psi) - \frac{1}{3} I,$$

where $\mathbf{n}(\psi) = (\cos\psi\cos\varphi, \cos\psi\sin\varphi, \sin\psi)$ and

$$\psi = 2\tan^{-1}\left(\frac{\rho}{z}\right) - \tan^{-1}\left(\frac{\rho}{z + z_0}\right) - \tan^{-1}\left(\frac{\rho}{z + z_0^{-1}}\right).$$

Here the second choice of the initial condition represents an approximation of a nematic configuration with a hyperbolic point defect at distance z_0 below the sphere's center [21, 31]. We assumed that $z_0 = 1.4$ in our simulations, although equilibrium configurations attained via the gradient flow are not sensitive to the precise choice of this parameter. Note that, for $\xi < 0.005$, the simulations leading to an equatorial Saturn ring were started from the critical point obtained for $\xi = 0.005$.

Fig. 1a-1b shows the line fields of the nematic in (r, z)-coordinates when $\xi = 1/70$. For this choice of the correlation length, the critical point approached by the gradient flow simulation depends on the initial condition; the critical point in Fig. 1a has dipolar symmetry, while the critical point Fig. 1b is quadrupolar.

For larger values of ξ, once it exceeds a critical value $\xi_c \approx 0.017$, the simulations converge to the equatorial Saturn ring configuration, regardless of the initial conditions. In fact, for the initial condition with a hyperbolic point defect, this defect expands first into a small ring below the south pole of the colloid. This ring then expands and travels up the surface of the colloid, eventually stopping at its equator.

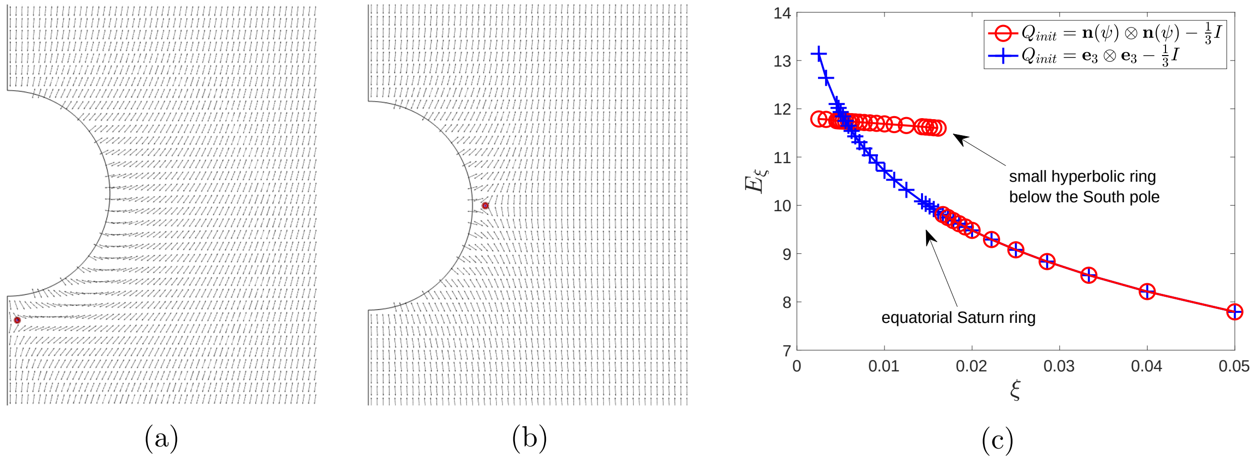

Figure 1: (a-b) Nematic configurations corresponding to critical points for E_ξ when $\xi = 1/70$. The red dot marks the location of a Saturn ring. (a) $Q_{init} = n(\psi)\otimes n(\psi) - \frac{1}{3}I$ and the critical point is a small hyperbolic ring below the south pole of the particle. The ring shrinks to a hyperbolic point defect on z-axis when $\xi \to 0$. (b) $Q_{init} = e_3 \otimes e_3 - \frac{1}{3}I$ and the critical point is the equatorial Saturn ring. The Saturn ring approaches the surface of the colloid when $\xi \to 0$. (c) Energy E_ξ vs nematic correlation length ξ. The critical point reached from the constant initial condition (blue) is always the equatorial Saturn ring. The critical points with an equatorial Saturn ring and with a small hyperbolic ring coexist and appear to be stable for all values of $\xi < \xi_c$ for which the simulations were conducted.

For $\xi < \xi_c$, the dichotomy between the initial conditions persists for all values of ξ for which the simulations were run: the hyperbolic point defect expands into a small ring below the colloid and the constant initial condition leads to the equatorial Saturn ring. Even though the energy of the equatorial ring exceeds the energy of the small ring once ξ becomes sufficiently small, the equatorial ring remains stable. These observations are summarized in Fig. 1c.

Although we do not address the question of their minimality, in the present work we establish that Saturn ring-like critical points of the Landau-de Gennes energy indeed do exist for large particles. Here, rather than varying the radius of the particle, we use an equivalent description in which the size of the particle is fixed and the nematic correlation length is assumed to converge to zero.

This paper is organized as follows. In Sections 2 and 3, we establish the upper and lower bounds given by the statements (i) and (ii) of Theorem 1.2. In Section 4 we prove the statement (iii) of Theorem 1.2; namely, a sequence of minimizers converges to a limiting map which is smooth away from the ring defect.

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2 Upper bound

To obtain the upper bound, i.e. part (i) of Theorem 1.2, we construct an admissible map $Q \in \mathcal{H}_{sym}$ whose energy has the expected behavior.

Let us first describe more explicitly what quadrupolar symmetry means. Symmetries are formalized in terms of the equivariant action of the orthogonal group $O(3)$ on maps $Q\colon \mathbb{R}^3 \to \mathcal{S}_0$, given by

$$(R \cdot Q)(x) = RQ(R^t x)R^t, \qquad R \in O(3).$$

The energy E_ξ is invariant under this action, consistent with the physical requirement of frame invariance. Here we consider the subgroup

$$G_{sym} = \{R \in O(3)\colon Re_3 = \pm e_3\}\,.$$

This is the largest subgroup of $O(3)$ that maps the space $\mathcal{H}$ into itself. Explicitly, G_{sym} is generated by all rotations around the vertical axis e_3 and by the reflection with respect to the horizontal plane $\langle e_1, e_2\rangle$, that is,

$$G_{sym} = \langle \{R_\varphi\}_{\varphi\in\mathbb{R}}, S\rangle\,, \quad R_\varphi = \begin{pmatrix} \cos\varphi & -\sin\varphi & 0 \\ \sin\varphi & \cos\varphi & 0 \\ 0 & 0 & 1 \end{pmatrix}, \quad S = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}. \tag{7}$$

The critical points we study in this work are minimizers of the energy among symmetric configurations belonging to the space

$$\mathcal{H}_{sym} = \{Q \in \mathcal{H}_b\colon R\cdot Q = Q,\ \forall R \in G_{sym}\}\,, \tag{8}$$

It is natural to use cylindrical coordinates to describe maps in the space $\mathcal{H}_{sym}$, i.e. coordinates (ρ, φ, z) defined by

$$x = R_\varphi \begin{pmatrix} \rho \\ 0 \\ z \end{pmatrix} = \rho R_\varphi e_1 + z e_3.$$

In these coordinates the domain Ω corresponds to $\{\rho^2 + z^2 > 1\}$, and maps $Q \in \mathcal{H}_{sym}$ can be written in the form

$$Q(\rho, \varphi, z) = R_\varphi \widetilde{Q}(\rho, z) R_\varphi^t, \tag{9}$$

where in addition $\widetilde{Q}(\rho, z) = Q(\rho, 0, z)$ satisfies the mirror symmetry constraint

$$\widetilde{Q}(\rho, -z) = S\widetilde{Q}(\rho, z)S^t. \tag{10}$$

Written in cylindrical coordinates, the energy of a map $Q \in \mathcal{H}_{sym}$ takes the form

$$\frac{1}{2\pi}E_\xi(Q) = \widetilde{E}_\xi(\widetilde{Q}) := \iint_{\widetilde{\Omega}} \left(|\nabla \widetilde{Q}|^2 + \frac{1}{\rho^2}\Xi[\widetilde{Q}] + \frac{1}{\xi^2} f(\widetilde{Q}) \right) \rho \, d\rho dz,$$

$$\text{where } \Xi[\widetilde{Q}] = |\partial_\varphi [R_\varphi \widetilde{Q} R_\varphi^t]|^2 = 8\widetilde{Q}_{12}^2 + 2\widetilde{Q}_{13}^2 + 2\widetilde{Q}_{23}^2 + 2(\widetilde{Q}_{11} - \widetilde{Q}_{22})^2,$$

$$\text{and } \widetilde{\Omega} = \{(\rho, z) \colon \rho^2 + z^2 > 1, \ \rho > 0\}.$$

Note that $\Xi[\widetilde{Q}] \leq 4\,\mathrm{dist}^2(\widetilde{Q}, \mathbb{R}Q_\infty)$, where the distance is induced by the Frobenius norm. We will construct $\widetilde{Q}$ in the region

$$D := \{\rho^2 + z^2 > 1 \colon \rho, z > 0\},$$

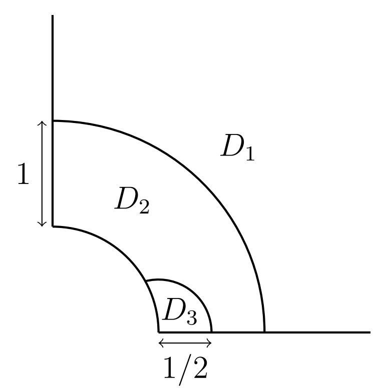
with boundary constraints

$$\begin{cases} \widetilde{Q}(\rho, z) = e_r \otimes e_r - \frac{1}{3}I, & \text{for } \rho^2 + z^2 = 1, \\ \widetilde{Q}(\rho, 0) = S\widetilde{Q}(\rho, 0)S^t, & \text{for } \rho > 1, \end{cases}$$

and then use the mirror symmetry (10) to extend $\tilde{Q}$ to the entire domain Ω. Note that $\widetilde{Q} = S\widetilde{Q}S^t$ implies that e_3 is an eigenvector of $\widetilde{Q}$ on the horizontal axis $\{z = 0\}$ in order for the mirror symmetry not to create a jump of $\tilde{Q}$.

We begin by observing that in all estimates in this and subsequent sections the letter C denotes a generic universal constant. To construct $\widetilde{Q}$ we first divide D into 3 subdomains, $\overline{D} = \overline{D}_1 \cup \overline{D}_2 \cup \overline{D}_3$, where

$$D_1 = D \cap \{\rho^2 + z^2 > 2\}, \quad D_2 = (D \setminus \overline{D}_1) \cap \{(\rho - 1)^2 + z^2 > 1/4\}, \quad D_3 = D \setminus (\overline{D}_1 \cup \overline{D}_2).$$



Let

$$\widetilde{Q} \equiv Q_\infty = e_3 \otimes e_3 - \frac{1}{3}I \qquad \text{in } D_1,$$

so that

$$\widetilde{E}_\xi(\widetilde{Q}; D_1) = 0. \tag{11}$$

Then we define a Lipschitz map $n\colon \partial D_2 \to \mathbb{S}^2$ as follows. We first set

$$n \equiv e_3 \quad \text{on } \partial D_2 \setminus \left(\partial D_3 \cup \{\rho^2+z^2=1\}\right),$$
$$n = (\rho,0,z) \quad \text{on } \partial D_2 \cap \{\rho^2+z^2=1\}.$$

To define n on $\partial D_2 \cap \partial D_3$ we use polar coordinates (r,θ) centered in $(1,0)$, that is, given by the relation $\rho + iz = 1 + re^{i\theta}$, so that

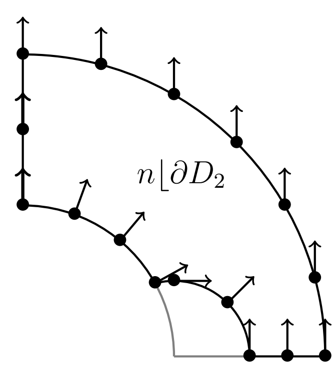
$$\partial D_2 \cap \partial D_3 = \{r = 1/2, 0 < \theta < \theta_0 := \pi/2 + \arcsin(1/4)\}.$$

We set

$$n(1/2,\theta) = (\sin\theta, 0, \cos\theta) \quad \text{for } 0<\theta<\pi/2,$$

and in the remaining part of $\partial D_2 \cap \partial D_3$ we interpolate linearly. More precisely, at the point $(1/2,\theta_0)$, continuity of n imposes $n = (\cos\theta_1, 0, \sin\theta_1)$ where $\theta_1 = 2\theta_0 - \pi = 2\arcsin(1/4)$, so we set

$$n(1/2,\theta) = (\cos(2\theta-\pi), 0, \sin(2\theta-\pi)) \quad \text{for } \pi/2<\theta<\theta_0.$$



The map $n\colon \partial D_2 \to \mathbb{S}^2$ thus defined can be written in the form

$$n = (\cos\varphi, 0, \sin\varphi), \qquad \varphi \in \mathrm{Lip}(\partial D_2, \mathbb{R}).$$

Thus, considering an H^1 extension of φ to D_2 we obtain $n\colon D_2 \to \mathbb{S}^2$ with $\int_{D_2} |\nabla n|^2 \le C$ and moreover, thanks to Hardy's inequality and since $n = e_3$ on $\partial D_2 \cap \{\rho = 0\}$, we have also $\int_{D_2} \rho^{-2}|n-e_3|^2 \le C$. Then we set

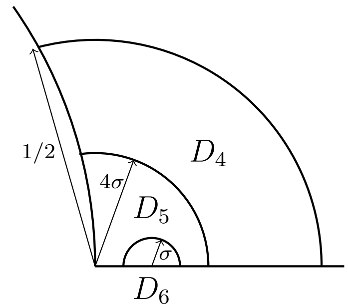
$$\widetilde{Q} = Q_n = n \otimes n - \frac{1}{3}I \qquad \text{in } D_2,$$

and, since $f(Q_n) = 0$ and $\Xi[Q_n] \le C|n-e_3|^2$, we have

$$\widetilde{E}_\xi(\widetilde{Q}; D_2) \le C. \tag{12}$$

Next we need to define $\widetilde{Q}$ in D_3. We introduce a parameter $\sigma > 0$ with $\xi < \sigma < 1/8$, and further divide D_3 into 3 subdomains:

$$D_4 = D_3 \cap \{(\rho-1)^2 + z^2 > (4\sigma)^2\},$$
$$D_5 = (D_3 \setminus \overline{D}_4) \cap \{(\rho - 1 - 2\sigma)^2 + z^2 > \sigma^2\},$$
$$D_6 = D_3 \setminus (\overline{D}_4 \cup \overline{D}_5).$$



In the polar coordinates (r,θ) centered at $(1,0)$, the domain D_4 is given by

$$\begin{aligned} &D_4 = \{1+re^{i\theta}\colon 4\sigma < r < 1/2, 0<\theta<\theta_0(r)\},\\ &\theta_0(r) := \frac{\pi}{2} + \arcsin\frac{r}{2}. \end{aligned} \tag{13}$$

In D_4 we define $\widetilde{Q}$ as

$$\widetilde{Q} = Q_n = n\otimes n - \frac{1}{3}I,$$

for some map $n\colon D_4 \to \mathbb{S}^2$. The boundary conditions on $\{\rho^2+z^2=1\}$ impose

$$\begin{aligned} &n(r,\theta_0(r)) = (\cos\theta_1(r), 0, \sin\theta_1(r)),\\ &\qquad \theta_1(r) := 2\theta_0(r) - \pi = 2\arcsin\frac{r}{2}, \end{aligned}$$

and we define

$$n(r,\theta) = \begin{cases} (\sin\theta, 0, \cos\theta) & \text{for } 0<\theta<\pi/2,\\ (\cos(2\theta-\pi), 0, \sin(2\theta-\pi)) & \text{for } \pi/2<\theta<\theta_0(r). \end{cases}$$

That way we have

$$\begin{aligned} \int_{D_4} \left|\nabla\widetilde{Q}\right|^2 \rho\, d\rho\, dz &= 2\int_{D_4} |\nabla n|^2 \rho\, d\rho\, dz\\ &= 2\int_{4\sigma}^{1/2}\int_0^{\theta_0(r)} \frac{1}{r^2}|\partial_\theta n|^2 (1+r\cos\theta)\, r\, d\theta\, dr\\ &= 2\int_{4\sigma}^{1/2}\int_0^{\pi/2} \frac{1+r\cos\theta}{r}\, d\theta\, dr\\ &\quad + 8\int_{4\sigma}^{1/2}\int_{\pi/2}^{\theta_0(r)} \frac{1+r\cos\theta}{r}\, d\theta\, dr\\ &\leq \pi\ln\frac{1}{\sigma} + C. \end{aligned}$$

Moreover, in D_4 we have $f(\widetilde{Q}) = 0$ and $\rho^{-2}\Xi[Q] \leq C$, and therefore

$$\widetilde{E}_\xi(\widetilde{Q}; D_4) \leq \pi\ln\frac{1}{\sigma} + C. \tag{14}$$

In the subdomain D_5 we are again going to define $\widetilde{Q}$ from a unit vector field $n\colon D_5 \to \mathbb{S}^2$. There we use polar coordinates (s, ϕ) centered at $(1 + 2\sigma, 0)$, that is, given by $\rho + iz = 1 + 2\sigma + se^{i\phi}$. In these coordinates the domain D_5 is of the form

$$D_5 = \left\{1 + 2\sigma + se^{i\phi} : 0 < \phi < \pi,\, \sigma < s < \bar{s}(\phi)\right\},$$

where $\bar{s}(\phi)$ is a Lipschitz function of ϕ, with $2\sigma \le \bar{s} \le 5\sigma$. On the part of ∂D_5 given by $\{s = \bar{s}(\phi)\}$, the values of n are given by the boundary conditions on $\{\rho^2 + z^2 = 1\}$ and on ∂D_4, and are of the form

$$n(\bar{s}(\phi), \phi) = (\cos\alpha(\phi), 0, \sin\alpha(\phi)) \qquad \text{for } \phi \in (0, \pi),$$

for Lipschitz function $\alpha\colon [0, \pi] \to \mathbb{R}$, which satisfies $\alpha(0) = \pi/2$ and $\alpha(\pi) = 0$. On the part of ∂D_5 given by $\{s = \sigma\}$, we set

$$n(\sigma, \phi) = (\sin(\phi/2), 0, \cos(\phi/2)) \qquad \text{for } \phi \in (0, \pi).$$

Then we define n in ∂D_5 by interpolating in the s variable, i.e. we set

$$\begin{aligned} n(s, \phi) &= (\cos\beta(s, \phi), 0, \sin\beta(s, \phi)), \\ \beta(s, \phi) &= \frac{1}{2}\frac{\bar{s}(\phi) - s}{\bar{s}(\phi) - \sigma}(\pi - \phi) + \frac{s - \sigma}{\bar{s}(\phi) - \sigma}\alpha(\phi), \qquad \text{for } \phi \in (0, \pi). \end{aligned}$$

Note that, by continuity, and since $n(\sigma, 0) = e_3 = n(\bar{s}(0), 0)$ and $n(\sigma, \pi) = e_1 = n(\bar{s}(\pi), \pi)$, this forces the trace of n on $\partial D_5 \cap \{z = 0\}$, to be given by

$$n(\rho, 0) = \begin{cases} e_1 & \text{for } 1 < \rho < 1 + \sigma, \\ e_3 & \text{for } 1 + 3\sigma < \rho < 1 + 4\sigma. \end{cases}$$

Moreover, since it is directly checked that $|\partial_s n| + |\partial_\phi n| \le C$, we deduce

$$\int_{D_5} |\nabla n|^2 \le C \int_\sigma^{5\sigma} \frac{ds}{s} \le C.$$

Therefore, setting $\widetilde{Q} = Q_n$ in D_5 we obtain

$$\widetilde{E}_\xi(\widetilde{Q}; D_5) \le C. \tag{15}$$

Finally, in D_6 we define $\widetilde{Q}$ in polar coordinates (s, ϕ) centered at $(1 + 2\sigma, 0)$ as above, by

$$\widetilde{Q} = \lambda(s) Q_n, \qquad n = (\sin(\phi/2), 0, \cos(\phi/2)), \qquad \lambda(s) = \min(1, s/\xi).$$

Then we have

$$\begin{aligned} \int_{D_6} |\nabla \widetilde{Q}|^2 \rho\, d\rho\, dz &\le C + 2\int_\xi^\sigma \int_0^\pi \frac{|\partial_\phi n|^2}{s^2}(1 + 2\sigma + s\cos\phi) s\, d\phi\, ds \\ &\le C + \frac{\pi}{2}(1 + 2\sigma)\ln\frac{\sigma}{\xi} \\ &\le \frac{\pi}{2}\ln\frac{\sigma}{\xi} + \pi\sigma\ln\frac{1}{\xi} + C, \end{aligned}$$

and therefore, since $f(\widetilde{Q}_n) = 0$ for $s > \xi$ and $\leq C$ for $s < \xi$, we deduce that

$$\widetilde{E}_\xi(\widetilde{Q}; D_6) \leq \frac{\pi}{2} \ln \frac{\sigma}{\xi} + \pi\sigma \ln \frac{1}{\xi} + C. \tag{16}$$

Gathering (11)-(16), we obtain

$$\widetilde{E}_\xi(\widetilde{Q}; D) \leq \frac{\pi}{2} \ln \frac{1}{\xi} + \frac{\pi}{2} \ln \frac{1}{\sigma} + \pi\sigma \ln \frac{1}{\xi} + C.$$

Optimizing in σ, we are led to choosing

$$\sigma = \frac{1}{2 \ln \frac{1}{\xi}},$$

and conclude that

$$\widetilde{E}_\xi(\widetilde{Q}; D) \leq \frac{\pi}{2} \ln \frac{1}{\xi} + \frac{\pi}{2} \ln \ln \frac{1}{\xi} + C,$$

which, upon applying the mirror symmetry, proves part (i) of Theorem 1.2.

3 Lower bound

In this section we prove the lower bound, part (ii) of Theorem 1.2. The strategy of proof is inspired by vortex ball constructions [9, 26, 36, 40] in the Ginzburg-Landau context, an idea which has been successfully exploited in various previous works on Q-tensors. (See e.g., [12, 22].) Because we expect the ring defect to approach the boundary as $\xi \to 0$, we employ methods introduced for Ginzburg-Landau energies with a weight which attains its minimum on the boundary [4, 7, 8]. A considerable sharpening of these results is necessary in order to apply the ideas to the cross-sections of the exterior domain, and this is the aim of the first part of this section. Some of the technical lemmas (such as Lemma 3.9 below) will also be needed in our analysis of singularities of the limit problem in section 4.

We use the notation $\lesssim$ to denote inequality up to a universal multiplicative constant. We denote by $\widetilde{Q}_\xi$ the map defined by

$$Q_\xi(\rho, \varphi, z) = R_\varphi \widetilde{Q}_\xi(\rho, z) R_\varphi^t,$$

which satisfies in addition the mirror symmetry (10). The map $\widetilde{Q}_\xi$ is thus defined in

$$\widetilde{\Omega} = \{(\rho, z) \colon \rho^2 + z^2 > 1,\ \rho > 0\},$$

uniquely determined by its values in the region

$$D = \{\rho^2 + z^2 > 1 \colon \rho, z > 0\},$$

and minimizes

$$\widetilde{E}_\xi(\widetilde{Q}) = \int_D \left(|\nabla \widetilde{Q}|^2 + \frac{1}{\rho^2} \Xi[\widetilde{Q}] + \frac{1}{\xi^2} f(\widetilde{Q}) \right) \rho \, d\rho dz$$

under the boundary constraints

$$\widetilde{Q}(\rho,z) = e_r \otimes e_r - \frac{1}{3}I \qquad \text{for } \rho^2+z^2=1, \qquad \text{where } e_r=(\rho,0,z),$$
$$\text{and } \widetilde{Q}(\rho,0) = S\widetilde{Q}(\rho,0)S^t \qquad \text{for } \rho>1.$$

For any $X \in \widetilde{\Omega}$ we will denote by $B(X,r)$ the disc of radius $r>0$ centered at X.

Note that our potential f vanishes quadratically near the manifold $\mathcal{U}_\star$: more specifically, given $C>0$, there exist constants c_1, c_2 with

$$c_1 f(Q) \le \operatorname{dist}^2(Q,\mathcal{U}_\star) \le c_2 f(Q) \qquad \text{for } |Q| \le C. \tag{17}$$

The upper and lower estimates on $f(Q)$ are proven in [12, Section 2.2] for Q in a neighborhood of $\mathcal{U}_\star$; by adjusting the constants, the same bounds hold in any compact set of tensors Q. As a consequence, we may fix $\eta>0$ such that in the region $\{Q \in \mathcal{S}_0 \colon f(Q) < 2\eta\}$, the nearest neighbor projection π onto the smooth submanifold $\mathcal{U}_\star$ is well defined and smooth.

Lemma 3.1. *The minimizers Q_ξ are smooth in $\widetilde{\Omega}$, and satisfy the uniform bounds,*

$$\|Q_\xi\|_{L^\infty(\Omega)} \lesssim 1 \quad \textit{and} \quad \|\nabla Q_\xi\|_{L^\infty} \lesssim \frac{1}{\xi}.$$

Proof. The L^∞ bound is proved in [1, Lemma 5], and the regularity and gradient bound follow from elliptic estimates. □

Away from the vertical axis of symmetry the energy is almost two-dimensional, and we exploit this observation to use η-compactness methods developed for the Ginzburg-Landau functional by Struwe [40].

Lemma 3.2. *For any $\alpha \in (0,1]$ there exists $C_\alpha > 0$ such that for any $X=(\rho_0,z_0) \in \widetilde{\Omega} \cap \{\rho \ge \frac{1}{2}\}$,*

$$\int_{\widetilde{\Omega}\cap B(X,\xi^\alpha)} \left[\frac{1}{\rho^2}\Xi(\widetilde{Q}_\xi) + \frac{1}{\xi^2} f(\tilde{Q})\right] \rho\, d\rho dz \le C_\alpha.$$

Proof. Fix any $X=(\rho_0,z_0) \in \widetilde{\Omega} \cap \{\rho \ge \frac{1}{2}\}$. Define

$$F(r) = F(r;X) := r \int_{\partial B(X,r)\cap\widetilde{\Omega}} \left[|\nabla \widetilde{Q}_\xi|^2 + \frac{1}{\rho^2}\Xi(\widetilde{Q}_\xi) + \frac{1}{\xi^2} f(\widetilde{Q}_\xi)\right] \rho\, ds_r,$$

where ds_r denotes arclength measure on $\partial B(X,r)$.

Let $\beta = \alpha/2 > 0$. As in the proof of [40, Lemma 2.3 (i)], by Fubini's Theorem there exists $r_\xi \in (\xi^\alpha, \xi^\beta)$ for which

$$\widetilde{E}_\xi(\widetilde{Q}_\xi) \ge \widetilde{E}_\xi(\widetilde{Q}_\xi; B(X,\xi^\beta)) \ge \int_{\xi^\alpha}^{\xi^\beta} F(r)\frac{dr}{r} \ge \frac{\alpha}{2} F(r_\xi) \ln\frac{1}{\xi}.$$

In particular,

$$F(r_\xi) \le \frac{2\widetilde{E}_\xi(\widetilde{Q}_\xi; B(X,\xi^\beta))}{\alpha|\ln\xi|} \le \frac{2\widetilde{E}_\xi(\widetilde{Q}_\xi)}{\alpha|\ln\xi|} \le C_\alpha. \tag{18}$$

To obtain the desired bound we require a version of the Pohozaev identity. We treat our problem as if it were two dimensional, with domain $\widetilde{\Omega}$ and a nonconstant weight function $w=\rho$. Consider a solution u of the equation

$$-\frac{1}{w}\nabla\cdot(w\nabla u) + \partial_u g(x,u) = 0 \qquad x\in\mathbb{R}^2.$$

For a smooth vector field Y, we multiply by $Y\cdot\nabla u$ and integrate by parts on $D\subset\widetilde{\Omega}$, to obtain:

$$\begin{aligned}\int_D &\left[\frac{1}{2}\nabla\cdot(wY)|\nabla u|^2 - w\,(\partial_j Y_k)\partial_j u\partial_k u\right] dx \\ &+ \int_D [g(x,u)\nabla\cdot(wY) + \partial_x g(x,u)\cdot wY]\, dx \\ &\qquad = \int_{\partial D}\left[g(x,y)Y\cdot\nu + \frac{1}{2}Y\cdot\nu|\nabla u|^2 - (\nabla u\cdot\nu)(Y\cdot\nabla u)\right] w\, ds.\end{aligned}$$

In our case, we choose the domain $D=D_r=B(X,r)\cap\widetilde{\Omega}$, and the vector field $Y=(\rho-\rho_0, z-z_0)$. The Euler-Lagrange equations have the form

$$-\Delta\widetilde{Q}_\xi + \frac{1}{2}\partial_Q\left(\frac{1}{\rho^2}\Xi(\widetilde{Q}_\xi) + \frac{1}{\xi^2}f(\widetilde{Q}_\xi)\right) = L(\widetilde{Q}_\xi), \tag{19}$$

where $L=\frac{1}{6}|\widetilde{Q}|^2\mathbf{I}$ is a Lagrange multiplier due to the vanishing trace condition. Since the test functions $Y\cdot\nabla\widetilde{Q}_\xi$ are trace-free, (as observed in [12, Remark 4.3],) $L(\widetilde{Q}_\xi)$ plays no role in the resulting identity, and so we may take $g(x,u)=g(\rho,\widetilde{Q}_\xi)=\frac{1}{2\rho^2}\Xi(\widetilde{Q}_\xi)+\frac{1}{2\xi^2}f(\widetilde{Q}_\xi)$. Substituting in the above identity, with $u=\widetilde{Q}_{\xi,ij}$ and summing over i,j, we obtain:

$$\begin{aligned}I_1 &:= \int_{D_r}(3\rho-\rho_0)g(\rho,\widetilde{Q}_\xi)d\rho\, dz + \int_{D_r}\left[\frac{1}{2}|\nabla\widetilde{Q}_\xi|^2 - \frac{1}{\rho^2}\Xi(\widetilde{Q}_\xi)\right](\rho-\rho_0)\, d\rho\, dz \\ &= \int_{\partial D_r}\left[g(\rho,\widetilde{Q}_\xi)(Y\cdot\nu) + \frac{1}{2}(Y\cdot\nu)|\nabla\widetilde{Q}_\xi|^2 - (\nabla\widetilde{Q}_\xi\cdot\nu)(Y\cdot\nabla\widetilde{Q}_\xi)\right]\rho\, ds_r =: I_2.\end{aligned} \tag{20}$$

Recalling $X=(\rho_0,z_0)$ with $\rho_0\ge\frac{1}{2}$, we will apply the above identity (here and in the next lemma) for $r\in(\xi,\xi^\beta)$, and so for ξ sufficiently small the domain D_r will be strongly starshaped, in the sense that $Y\cdot\nu\ge\frac{r}{4}$ on ∂D_r. Following [40, Lemma 2.3 (ii)], the right-hand side of (20) may be estimated as:

$$I_2 \le C\,F(r) + C\,r\int_{\partial\widetilde{\Omega}\cap B(X,r)}|\nabla Q_b|^2\, ds \le C\,F(r) + O(r^2), \tag{21}$$

with constant C independent of X, and recalling $\widetilde{Q}_\xi = Q_b$ on $\partial B(0,1)$. For the left-hand side, we note that in D_r, $|\rho-\rho_0| < r \le 3\rho\xi^\beta$, and hence

$$I_1 \ge 2\int_{D_r} g(\rho,\widetilde{Q}_\xi)\,\rho\,d\rho\,dz - O(\xi^\beta|\ln\xi|).$$

Thus, we have for any $r \in (\xi, \xi^\beta)$,

$$\int_{D_r} g(\rho,\widetilde{Q}_\xi)\,\rho\,d\rho\,dz \le C\,F(r) + O(\xi^\beta|\ln\xi|). \tag{22}$$

Choosing $r = r_\xi$ as in (18), and noting $B(X,\xi^\alpha) \subset B(X,r_\xi)$, the desired inequality is established. □

The following is an adaptation of the η-compactness (η-ellipticity) condition to our setting.

Lemma 3.3. *There exists $\gamma > 0$ such that, for any $\alpha \in (0,1]$ there is $\xi_0(\alpha) > 0$ with the following property. If $\xi \in (0,\xi_0)$ and $r \in [\xi,\xi^\alpha]$ are such that*

$$F(r;X) \le \gamma\rho(X) \qquad \text{for some } X \in \widetilde{\Omega} \cap \{\rho \ge \frac{1}{2}\},$$

then $f(\widetilde{Q}_\xi) \le \eta$ in $B_r(X) \cap \widetilde{\Omega}$.

Proof. The proof is as in [40, Lemma 2.3(ii)]. Suppose the contrary: there exists $X' = (\rho', z') \in B(X,r)$ for which $f(\widetilde{Q}_\xi(X')) > \eta$. By Lemma 3.1, there exists $c > 0$ for which $f(\widetilde{Q}_\xi(x)) > \eta/2$ for all $x \in B(X', c\xi)$. Thus, there is a constant $C_0 > 0$, independent of X, ξ, for which

$$\int_{B(X',c\xi)\cap\widetilde{\Omega}} \frac{1}{\xi^2} f(\widetilde{Q}_\xi)\,\rho\,d\rho\,dz \ge C_0\rho(X'). \tag{23}$$

On the other hand, by (22) we then have

$$C_0\rho(X') \le \int_{B(X',c\xi)\cap\widetilde{\Omega}} \frac{1}{\xi^2} f(\widetilde{Q}_\xi)\,\rho\,d\rho\,dz \le \int_{D_r} g(\rho,\widetilde{Q}_\xi)\,\rho\,d\rho\,dz \le C\,F(r) \le \gamma\rho(X).$$

For any $\gamma < C_0$ this is impossible, as $|\rho(X') - \rho(X)| < r \ll 1$, and hence the conclusion must hold. □

As in the Ginzburg-Landau case, we may now define the "bad balls" which contain the eventual defects:

Lemma 3.4. *There exist $M_0 \in \mathbb{N}$ and $A_0 \ge 1$ such that*

$$\{f(\widetilde{Q}_\xi) > \eta\} \subset \{\rho \le A_0\},$$

and for any disjoint collection of balls $\{B(X_j, \frac{\xi}{5})\}_{j\in J}$ with centers

$$X_j \in S_\xi := \{f(\widetilde{Q}_\xi) > \eta\} \cap \{\rho \ge \frac{1}{2}\},$$

the cardinality of J must be $\le M_0$.

Proof. For the first assertion, let $X' \in \{x \in \widetilde{\Omega} : \ f(\widetilde{Q}_\xi(x)) > \eta)\}$. As in the proof of Lemma 3.3, there exists $c > 0$ for which $f(\widetilde{Q}_\xi(x)) > \eta/2$ in $B(X', c\xi)$, with the same lower bound (23). Taking $r = r_\xi$ as in (18), we obtain $C_0\rho(X') \le CF(r_\xi) \le C'$, and hence $\rho(X')$ is uniformly bounded in ξ.

The second assertion now follows as in the proof of [40, Lemma 3.2], relying on Lemmas 3.2 and 3.3. Indeed, thanks to the first assertion, we have $S_\xi \subset \{1/2 \le \rho \le A_0\}$ and for $X \in \widetilde{\Omega} \cap \{1/2 \le \rho \le A_0\}$, the factor ρ can be replaced by a constant both in the lower bound (23) and in the upper bound appearing in the statement of Lemma 3.3. Hence, the identical arguments as in [40], based on Vitali covering of S_ξ by balls of radius $\xi^{\frac{1}{4}}$, assure the bounded cardinality of the collection of "bad balls" $\{B(X_j, \frac{\xi}{5})\}_{j\in J}$. □

Recall that given any Lipschitz simply connected bounded domain $R \subset \mathbb{R}^2$ and any continuous map $U : \partial R \to \mathcal{U}_\star$ we can consider its homotopy class in $\pi_1(\mathcal{U}_\star) \approx \mathbb{Z}/2\mathbb{Z}$. A loop is trivial in $\pi_1(\mathcal{U}_\star)$ if and only if it is orientable, i.e. it is of the form $\gamma \otimes \gamma - \frac{1}{3}I$, for some continuous loop $\gamma : \mathbb{S}^1 \to \mathbb{S}^2$.

Lemma 3.5. *Consider for some $z_0 \in (0, 1/2]$ and $\rho_0 \ge A_0$ the domain*

$$R_0 = \{|z| < z_0, \ \rho < \rho_0\} \cap \widetilde{\Omega}.$$

and assume that $f(\widetilde{Q}_\xi) \le \eta$ on ∂R_0 and that $\widetilde{Q}_\xi$ restricted to ∂R_0 is continuous. Then the projected map

$$U_\xi = \pi(\widetilde{Q}_\xi) : \partial R_0 \to \mathcal{U}_\star,$$

has non trivial homotopy class in $\pi_1(\mathcal{U}_\star)$, that is, U_ξ is non-orientable.

Proof of Lemma 3.5. Recall that $Q_\xi \in \mathcal{H}_{sym}$, hence

$$\int_{\widetilde{\Omega}} \left(|\nabla \widetilde{Q}_\xi|^2 + \frac{|\widetilde{Q}_\xi - Q_\infty|^2}{\rho^2 + z^2} \right) \rho d\rho dz < \infty.$$

In particular, for any $\xi \in (0, \xi_0)$ and any $\delta > 0$ we may choose $\rho_1 \ge \rho_0$ such that

$$\int_{-1}^{1} |\partial_z \widetilde{Q}_\xi(\rho_1, z)|^2 dz + \int_{-1}^{1} |\widetilde{Q}_\xi(\rho_1, z) - Q_\infty|^2 dz < \delta,$$

which implies

$$|\widetilde{Q}_\xi(\rho_1, z) - Q_\infty|^2 \lesssim \delta \quad \forall z \in [-1, 1]. \tag{24}$$

We denote by R_1 the domain

$$R_1 = \widetilde{\Omega} \cap \{|z| < z_0, \, \rho < \rho_1\},$$

and define $V_\xi = \pi(\widetilde{Q}_\xi) : \partial R_1 \to \mathcal{U}_\star$. Since $\pi(\widetilde{Q}_\xi)$ is well defined and has finite energy in $R_1 \setminus R_0$, the maps U_ξ and V_ξ are homotopically equivalent. To prove Lemma 3.5 it thus suffices to prove that V_ξ is non orientable. Assume that V_ξ is orientable: there exists a continuous $n\colon \partial R_1 \to \mathbb{S}^2$ such that

$$V_\xi = n\otimes n - \frac{1}{3}I.$$

Since n is uniquely defined up to a sign and $V_\xi = e_1\otimes e_1 - \frac{1}{3}I$ at $(\rho,z)=(1,0)$, we may assume that $n(1,0)=e_1$. The symmetry assumption (10) implies that

$$n(\rho,-z) = \tau S n(\rho,z) \qquad \text{for some } \tau\in\{\pm 1\}.$$

Evaluating this at $(\rho,z)=(1,0)$ gives $e_1=\tau S e_1 = \tau e_1$, hence $\tau=1$. This implies that at $(\rho,z)=(\rho_1,0)$ one must have $n(\rho_1,0)=Sn(\rho_1,0)$, i.e. $n(\rho_1,0)\perp e_3$, and therefore $|V_\xi(\rho_1,0)-Q_\infty|^2=2$. For small enough δ this contradicts (24). □

The next lemmas deal with universal lower bounds in annular regions.

Lemma 3.6. *There exists $C>0$ such that for any $\xi>0$, for any annulus $\omega\subset\mathbb{R}^2$ of the form*

$$\omega = B(0,R)\setminus \overline{B(0,r)}, \qquad \xi\le r<R\le \frac{1}{2},$$

and any H^1 map $Q\colon\omega\to\mathcal{S}_0$ satisfying $f(Q)\le\eta$ in ω and such that the trace on $\partial B(0,R)$ of $U=\pi(Q)\colon\omega\to\mathcal{U}_\star$ is continuous and nonorientable, we have

$$\int_\omega \left(|\nabla Q|^2 + \frac{1}{\xi^2}f(Q)\right) \ge \pi\ln\frac{R}{r} - C\xi\left(\frac{1}{r}-\frac{1}{R}\right).$$

Proof of Lemma 3.6. Very similar results are proved in [12, Proposition 3.12] and [22, Lemma 7], but for completeness we provide a proof, following the method in [26].

Let $D=|Q-U|=\operatorname{dist}(Q,\mathcal{U}_*)$, and denote by $P\colon\mathcal{U}_\star\to\mathcal{L}(\mathcal{S}_0)$ the smooth map given by $P(u)=P_{(T_u\mathcal{U}_\star)^\perp}$ the orthogonal projection onto the normal space to $\mathcal{U}_\star$ at u. Note that by definition $Q-U=P(U)(Q-U)$. Then for any direction k we compute

$$\begin{aligned}
|\partial_k Q|^2 &= |\partial_k U|^2 + |\partial_k(Q-U)|^2 + 2\partial_k U\cdot\partial_k(Q-U)\\
&\ge |\partial_k U|^2 + |\partial_k D|^2 + 2\partial_k U\cdot \partial_k\left[P(U)(Q-U)\right]\\
&= |\partial_k U|^2 + |\partial_k D|^2 + 2\partial_k U\cdot\partial_k\left[P(U)\right](Q-U) + 2\partial_k U\cdot P(U)\partial_k(Q_U).
\end{aligned}$$

The last term is zero because $\partial_k U\in T_U\mathcal{U}_\star$ and therefore $P(U)\partial_k U=0$. So we have

$$\begin{aligned}
|\partial_k Q|^2 &\ge |\partial_k U|^2 + |\partial_k D|^2 - 2\|DP(U)\| D|\partial_k U|^2\\
&\ge (1-cD)|\partial_k U|^2 + |\partial_k D|^2,
\end{aligned}$$

for $c=2\sup_{\mathcal{U}_\star}\|DP(U)\|$. This computation is very similar to [12, Lemma 2.6] and [22, Lemma 4]. Recalling now that $f(Q)\ge\alpha^2D^2$ for some $\alpha>0$ we find

$$|\nabla Q|^2 + \frac{1}{\xi^2}f(Q) \ge (1-cD)|\nabla U|^2 + |\nabla D|^2 + \frac{\alpha^2}{\xi^2}D^2.$$

Hence for any $s \in (r,R)$ we have, letting $d(s) = \max_{\partial B(0,s)} D$,

$$\int_{\partial B(0,s)} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) \geq (1 - c d(s)) \int_{\partial B(0,s)} |\partial_\tau U|^2 \\ + \int_{\partial B(0,s)} \left(|\partial_\tau D|^2 + \frac{\alpha^2}{\xi^2} D^2 \right). \tag{25}$$

The latter term is bounded from below by

$$\int_{\partial B(0,s)} \left(|\partial_\tau D|^2 + \frac{\alpha^2}{\xi^2} D^2 \right) \gtrsim \frac{1}{\xi} d(s)^2. \tag{26}$$

The proof of (26) follows the argument of [26, Lemma 2.3] which we reproduce here for the reader's convenience. Denoting

$$\gamma := \int_{\partial B(0,s)} |\partial_\tau D|^2,$$

Morrey's inequality gives

$$|D(x) - D(y)| \lesssim \gamma^{\frac{1}{2}} |x-y|^{\frac{1}{2}} \qquad \forall x, y \in \partial B(0,s).$$

Letting $x_{max} \in \partial B(0,s)$ be such that $D(x_{max}) = d(s)$, we deduce that

$$D(x) \geq \frac{d(s)}{2} \qquad \forall x \in \partial B(0,s) \text{ with } |x - x_{max}| \lesssim \frac{d(s)^2}{\gamma},$$

and therefore

$$\mathcal{H}^1 \left(\left\{ x \in \partial B(0,s) \colon D(x) \geq \frac{d(s)}{2} \right\} \right) \gtrsim \min \left(s, \frac{d(s)^2}{\gamma} \right), \qquad .$$

This implies

$$\int_{\partial B(0,s)} \left(|\partial_\tau D|^2 + \frac{\alpha^2}{\xi^2} D^2 \right) \gtrsim \gamma + \frac{1}{\xi^2} \min \left(s, \frac{d(s)^2}{\gamma} \right) d(s)^2 \\ \gtrsim \min \left(\frac{s}{\xi^2} d(s)^2, \inf_{\gamma \geq 0} \left\{ \gamma + \frac{d(s)^4}{\gamma \xi^2} \right\} \right),$$

from which (26) follows because $s \geq r \geq \xi$. Since U is H^1 in ω and non orientable on $\partial B(0,R)$, it is continuous and nonorientable on $\partial B(0,s)$ for a.e. $s \in [r,R]$, and for such s we have (see e.g. [12, Corollary 3.8])

$$\int_{\partial B(0,s)} |\partial_\tau U|^2 \geq \frac{\pi}{s},$$

and therefore, recalling (25)-(26), there is a constant $\hat{\alpha} > 0$ such that

$$\int_{\partial B(0,s)} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) \geq \frac{\pi}{s} (1 - c d(s)) + \frac{\hat{\alpha}}{\xi} d(s)^2 \\ \geq \inf_{d \geq 0} \left(\frac{\pi}{s} (1 - cd) + \frac{\hat{\alpha}}{\xi} d^2 \right) \\ \geq \frac{\pi}{s} - \frac{c^2 \pi^2}{4 \hat{\alpha}} \frac{\xi}{s^2}.$$

Integrating we deduce

$$\int_\omega \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) \geq \pi \ln \frac{R}{r} - \frac{c^2\pi^2}{4}\hat{\alpha}\xi \left(\frac{1}{r} - \frac{1}{R} \right).$$

□

Using Lemma 3.6 and the growing ball construction of Jerrard or Sandier [26, 36] (which is adapted to our setting in [22, Lemma 7]), one obtains the following lower bound on perforated domains:

Lemma 3.7. *There exists $C > 0$ such that for any $\xi \in (0, 1]$, for any perforated domain $\omega \subset \mathbb{R}^2$ of the form*

$$\omega = B(0, R) \setminus \bigcup_{j=1}^{N} \overline{B(x_j, r)}, \qquad B(x_j, r) \subset B(0, \frac{R}{2}) \text{ disjoint, } r \geq \xi,$$

and any H^1 map $Q : \omega \to \mathcal{S}_0$ satisfying $f(Q) \leq \eta$ in ω and such that the trace on $\partial B(0, R)$ of $U = \pi(Q) : \overline{\omega} \to \mathcal{U}_\star$ is continuous and nonorientable, we have

$$\int_\omega \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) \geq \pi \ln \frac{R}{Nr} - C.$$

We will also need a boundary version of Lemma 3.6.

Lemma 3.8. *There exists $C > 0$ such that:*

- *for any $\xi > 0$, for any annulus $\omega \subset \mathbb{R}^2$ of the form*

$$\omega = B(0, R) \setminus B(0, r), \qquad \xi \leq r < R \leq \frac{1}{2},$$

- *for any H^1 map $Q : \omega \to \mathcal{S}_0$ satisfying $f(Q) \leq \eta$ in ω and such that the trace on $\partial B(0, R)$ of $U = \pi(Q) : \overline{\omega} \to \mathcal{U}_\star$ is continuous and nonorientable, and such that in addition $Q = U_0$ on the left half $\omega_- = \omega \cap \{(x_1, x_2) : x_1 < 0\}$ of the annulus, for some Lipschitz $U_0 : \mathbb{R}^2 \to \mathcal{U}_\star$ with $|\nabla U_0| \leq 1$,*
- *and for any function w such that $w(x) \geq 1 - |x|^2$,*

then:

$$\int_{\omega_+} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) w(x)\, dx \geq 2\pi \ln \frac{R}{r} - C \left(1 + \frac{\xi}{r} \right), \tag{27}$$

where $\omega_+ = \omega \cap \{(x_1, x_2) : x_1 > 0\}$ is the right half of the annulus.

Proof of Lemma 3.8. We claim that

$$\int_{\partial B(0,s)\cap\omega_+} |\partial_\tau U|^2 \geq \frac{2\pi}{s} - 16 \qquad \text{for a.e. } s \in [r,R]. \tag{28}$$

Then, going through the computations in the proof of Lemma 3.6 gives

$$\begin{aligned}\int_{\omega_+} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q)\right) w(x)\, dx &\geq \int_r^R \left(\frac{2\pi}{s} - \frac{c^2\pi^2}{\hat{\alpha}}\frac{\xi}{s^2} - 16\right)(1-s^2)\, s\, ds \\ &\geq 2\pi \ln\frac{R}{r} - C\left(\frac{c^2\pi^2}{\hat{\alpha}}\frac{\xi}{r} + 1\right),\end{aligned}$$

which proves Lemma 3.8, provided we show (28). We fix $s \in [r,R]$ such that $U_{\lfloor \partial B(0,s)}$ is in H^1 and non-orientable. Since $\mathbb{R}^2$ and $\partial B(0,s) \cap \overline{\omega}_+$ are simply connected we may orient U_0 and U on these domains, i.e. write

$$\begin{aligned} U_0 &= n_0 \otimes n_0 - \frac{1}{3} I, && n_0 \colon \mathbb{R}^2 \to \mathbb{S}^2, \\ U(se^{i\theta}) &= n_s(\theta) \otimes n_s(\theta) - \frac{1}{3} I && n_s \colon [-\frac{\pi}{2}, \frac{\pi}{2}] \to \mathbb{S}^2. \end{aligned}$$

Since $U(se^{-i\frac{\pi}{2}}) = U_0(se^{-i\frac{\pi}{2}})$ we may, up to switching the orientation of n_0, assume that

$$n_s(-\frac{\pi}{2}) = n_0(se^{-i\frac{\pi}{2}}).$$

Since $U(se^{i\frac{\pi}{2}}) = U_0(se^{i\frac{\pi}{2}})$ we have $n_s(\frac{\pi}{2}) = \pm n_0(se^{i\frac{\pi}{2}})$, and because U is non-orientable it must be

$$n_s(\frac{\pi}{2}) = -n_0(se^{i\frac{\pi}{2}}).$$

Note also that, as $|\nabla n_0| = \frac{1}{\sqrt{2}}|\nabla U_0| \leq 1$, we have

$$|n_0(se^{-i\frac{\pi}{2}}) - n_0(se^{i\frac{\pi}{2}})| \leq 2s \leq 1,$$

which implies that

$$\mathrm{dist}_{\mathbb{S}^2}(n_0(se^{-i\frac{\pi}{2}}), -n_0(se^{i\frac{\pi}{2}})) \geq \pi - 4s$$

Hence we find

$$\begin{aligned}\int_{\partial B(0,s)\cap\omega_+} |\partial_\tau U|^2 &= \frac{1}{s}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} 2|n_s'(\theta)|^2\, d\theta \\ &\geq \frac{1}{s}\frac{2}{\pi}\left(\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |n_s'(\theta)| d\theta\right)^2 \\ &\geq \frac{1}{s}\frac{2}{\pi}\, \mathrm{dist}_{\mathbb{S}^2}(n_s(-\frac{\pi}{2}), n_s(\frac{\pi}{2}))^2 \\ &\geq \frac{1}{s}\frac{2\pi^2 - 16\pi s + 32 s^2}{\pi} \\ &\geq \frac{2\pi}{s} - 16,\end{aligned}$$

thus proving (28). ∎

In Section 4 we will also need the following refinement of Lemma 3.8. In particular, we show that (to $O(1)$) we obtain the same lower bound in a smaller domain which pulls away from the boundary of the half-disk along a curve $x_1 = \lambda x_2^\beta$ with $\lambda > 0$ and $\beta > 1$. While the statement is quite technical, the result is essential for the very precise control of the energy of the core of the defect demanded in the proof of Lemma 4.10.

Lemma 3.9. *Let $C, \xi, \omega, \omega_+, r, R, Q, \eta$, and U_0 be as in Lemma 3.8. If Q also satisfies a matching upper bound, in the sense that*

$$\int_{\omega_+} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) w(x)\, dx \le 2\pi \ln \frac{R}{r} + K,$$

for some $K > 0$, then the lower bound (27) is also valid in the slightly smaller domain

$$\tilde{\omega}_\beta = \omega \cap \{(x_1, x_2) \colon x_1 > \lambda x_2^\beta\}, \qquad \lambda > 0, \beta > 1,$$

in the sense that

$$\int_{\tilde{\omega}_\beta} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) w(x)\, dx \ge 2\pi \ln \frac{R}{r} - C\left(1 + \frac{\xi}{r}\right) - CK - \frac{C\lambda}{\beta - 1}.$$

Proof of Lemma 3.9. We use the same notations as in the proof of Lemma 3.8, and refine the lower bound obtained on each slice $\partial B(0, s) \cap \omega_+$. The crucial computation is

$$\begin{aligned}
\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| |n_s'(\theta)| - 1 \right|^2 d\theta &= \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |n_s'(\theta)|^2\, d\theta - 2 \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |n_s'(\theta)|\, d\theta + \pi \\
&\le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |n_s'(\theta)|^2\, d\theta - 2\, \mathrm{dist}_{\mathbb{S}^2}(n_s(-\frac{\pi}{2}), n_s(\frac{\pi}{2})) + \pi \\
&\le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |n_s'(\theta)|^2\, d\theta - 2(\pi - 4s) + \pi \\
&\le \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} |n_s'(\theta)|^2\, d\theta - \pi + 8s.
\end{aligned}$$

This can be interpreted as a stability estimate for geodesics in $\mathbb{S}^2$: if the lower bound for the energy of a curve (minimized by constant speed geodesics) is almost saturated, then this curve's speed must be close to constant. Plugging this back into the estimates performed in Lemma 3.6, we deduce

$$\begin{aligned}
&\int_{\partial B(0,s) \cap \omega_+} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q) \right) \\
&\ge (1 - cd(s)) \left(\frac{2\pi}{s} - 16 + \frac{2}{s} \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left| |n_s'(\theta)| - 1 \right|^2 d\theta \right) + \frac{\hat{\alpha}}{\xi} d(s)^2,
\end{aligned}$$

where recall that $c > 0$ depends only on $\mathcal{U}_\star$ and $0 \leq d(s) \lesssim \eta$. Hence, possibly lowering η we can assume $cd(s) \leq \frac{1}{2}$, and arguing as in Lemma 3.6 we obtain

$$\begin{aligned}
&\int_{\partial B(0,s)\cap\omega_+} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q)\right) \\
&\geq (1 - cd(s))\frac{2\pi}{s} + \frac{\hat{\alpha}}{\xi} d(s)^2 - 16 + \frac{1}{s}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left||n_s'(\theta)| - 1\right|^2 \, d\theta \\
&\geq \frac{2\pi}{s} - \frac{c^2\pi^2}{\hat{\alpha}} \frac{\xi}{s^2} - 16 + \frac{1}{s}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left||n_s'(\theta)| - 1\right|^2 \, d\theta.
\end{aligned}$$

Integrating we deduce

$$\begin{aligned}
\int_{\omega_+} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q)\right) w(x)\, dx \geq 2\pi \ln \frac{R}{r} - C\left(\frac{\xi}{r} + 1\right) \\
+ \int_r^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left||n_s'(\theta)| - 1\right|^2 d\theta \frac{ds}{s}.
\end{aligned}$$

Combining this with the assumption that we have a matching upper bound gives

$$\int_r^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left||n_s'(\theta)| - 1\right|^2 d\theta \frac{ds}{s} \leq K + C\left(\frac{\xi}{r} + 1\right). \tag{29}$$

We will use this to show that the part that we "forget" by integrating over $\tilde{\omega}_\beta$ instead of ω_+ is bounded. Indeed we have

$$\begin{aligned}
\int_{\partial B(0,s)\cap\tilde{\omega}_\beta} |\partial_\tau U|^2 &= \int_{\partial B(0,s)\cap\omega_+} |\partial_\tau U|^2 - \int_{\partial B(0,s)\cap\omega_+\setminus\tilde{\omega}_\beta} |\partial_\tau U|^2 \\
&\geq \frac{2\pi}{s} - 16 \\
&\quad - \frac{2}{s}\int_{-\frac{\pi}{2}}^{-\frac{\pi}{2}+2\lambda s^{\beta-1}} |n_s'(\theta)|^2 \, d\theta - \frac{2}{s}\int_{\frac{\pi}{2}-2\lambda s^{\beta-1}}^{\frac{\pi}{2}} |n_s'(\theta)|^2 \, d\theta \\
&\geq \frac{2\pi}{s} - 16 - \frac{2}{s}\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left||n_s'(\theta)| - 1\right|^2 d\theta - 4\lambda s^{\beta-2}.
\end{aligned}$$

Notice that since $\beta > 1$ the last term is summable with respect to s small . Hence upon arguing as above we find

$$\begin{aligned}
\int_{\tilde{\omega}_\beta} \left(|\nabla Q|^2 + \frac{1}{\xi^2} f(Q)\right) w(x)\, dx \geq 2\pi \ln \frac{R}{r} - C\left(\frac{\xi}{r} + 1\right) - \frac{4\lambda}{\beta - 1} \\
- 2\int_r^R \int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} \left||n_s'(\theta)| - 1\right|^2 d\theta \frac{ds}{s},
\end{aligned}$$

and combining this with (29) enables us to conclude. □

We are finally ready to prove the lower bound on the energy stated in Theorem 1.2.

Proof of Part (ii) of Theorem 1.2. First, as in the Ginzburg-Landau problem [9] it will be convenient to extend the minimizers $\widetilde{Q}_\xi$ into the colloid. That is we fix a common (ie, ξ-independent) Lipschitz $\mathcal{U}_\star$-valued map U_0 which is defined in a thin neighborhood $\{(\rho, z) : \frac{1}{4} < \rho^2 + z^2 < 1\}$ inside the colloid, and such that the extension

$$\overline{\widetilde{Q}}_\xi = \begin{cases} \widetilde{Q}_\xi, & \text{if } x \in \widetilde{\Omega}, \\ U_0, & \text{if } \frac{1}{4} < \rho < 1, \end{cases}$$

is Lipschitz across the colloid surface $\{\rho^2 + z^2 = 1\}$. For instance, once could simply choose $U_0 = e_r \otimes e_r - \frac{1}{3}I$. By abuse of notation, we will denote this extension by $\widetilde{Q}_\xi$ in the proof. With this understanding, for any circle $\partial B(a, r)$ with $a \in \overline{\widetilde{\Omega}}$ and $0 < r < 3/4$ such that $f(\widetilde{Q}_\xi) < \eta$ on $\partial B(a, r) \cap \widetilde{\Omega}$, it makes sense to consider $U_\xi = \pi(\widetilde{Q}_\xi)$ on all of $\partial B(a, r)$ (even if part of this circle lies outside of $\widetilde{\Omega}$). Moreover U_ξ is orientable on $\partial B(a, r)$ if and only if it is orientable on $\partial(B(a, r) \cap \widetilde{\Omega})$.

Step 1: we construct "bad balls" as in [9, 40], and show that at least one must carry topological charge and converge to the equator $a = (1, 0)$ of the colloid.

Thanks to Lemma 3.4 and arguing as in [9, Theorem IV.1], there exist $\lambda > 0$ and a family of disjoint balls $B(x_j^\xi, \lambda\xi)$ such that

$$\begin{aligned}
&\{f(\widetilde{Q}_\xi) > \eta\} \cap \{\rho \geq \frac{1}{2}\} \subset \bigcup_{j=1}^{J_\xi} B(x_j^\xi, \lambda\xi), \qquad J_\xi \leq M_0, \\
&|x_i^\xi - x_j^\xi| > 8\lambda\xi \qquad \forall i \neq j, \\
&x_i^\xi \in \{z = 0\} \quad \text{or} \quad \operatorname{dist}(x_i^\xi, \{z = 0\}) > 8\lambda\xi \qquad \forall i, \\
&x_i^\xi \in \partial\widetilde{\Omega} \quad \text{or} \quad \operatorname{dist}(x_i^\xi, \partial\widetilde{\Omega}) > 8\lambda\xi \qquad \forall i
\end{aligned}$$

Let us now consider a sequence $\xi = \xi_n \to 0$. To keep notation simple we will not write explicitly the dependence on n, and will not relabel subsequences. We may extract converging subsequences of the x_j^ξ that lie inside $\{|z| \leq \frac{1}{2}, \rho \leq A_0\}$, and we denote by $a_1, \ldots, a_K$ those of the limit points that lie in the axis $\{z = 0\}$, and $z_1 = \min(\frac{1}{2}, \frac{d}{2})$, where d is the minimal distance of any other limit point to the axis $\{z = 0\}$ or of any of the a_j's to the other a_j's. Then for any $\delta \in (0, z_1)$ the balls $B(a_j, \delta)$ are disjoint and we have

$$\{f(\widetilde{Q}_\xi) > \eta\} \cap \{|z| \leq z_1\} \subset \bigcup_{j=1}^{K} B(a_j, \delta) \qquad \text{for small enough } \xi.$$

By Fubini's theorem we may find $z_0 \in [z_1/2, z_1]$ and $\rho_0 \geq A_0$ such that $\widetilde{Q}_\xi$ restricted to ∂R is continuous, where $R = \{|z| < z_0, \rho < \rho_0\} \cap \widetilde{\Omega}$. By the above $f(\widetilde{Q}_\xi) \leq \eta$ on ∂R for small enough ξ, so we may apply Lemma 3.5 to deduce that the $U_\xi = \pi(\widetilde{Q}_\xi)$ is non-orientable on ∂R. As a consequence, U_ξ must be non-orientable on $\partial B(a_j, z_1/2)$ for at least one index j.

Relabeling we assume $j = 1$. We claim that a_1 must be the leftmost possible point $a = (1, 0)$. Assume indeed that $\rho(a_1) > 1$ and fix $\delta \in (0, z_1)$ such that $\rho(a_1) - \delta > 1$. Then applying Lemma 3.7 on $B(a_1,\delta)\setminus\bigcup_j B(x_j^\xi,\lambda\xi)$ for small enough ξ we deduce that

$$\widetilde{E}_\xi(\widetilde{Q}_\xi)\geq(\rho(a_1)-\delta)\pi\ln\frac{\delta}{\xi}-C,$$

but since $\rho(a_1)-\delta>1$ this implies that $\widetilde{E}_\xi(\widetilde{Q}_\xi)-\pi\ln\frac{1}{\xi}\to\infty$ as $\xi\to 0$, thus contradicting the upper bound obtained in Section 2.

Gathering the above, we have that $a=(1,0)$ is a limit of some x_j^ξ and for any $\delta>0$, if ξ is small enough then $U_\xi=\pi(\widetilde{Q}_\xi)$ is well defined and non-orientable on $\partial B(a,\delta)$. Of the above defined x_j^ξ, we now consider only those which converge to a.

Step 2: We identify a critical annulus $B(a,\mu\hat{\sigma}_\xi)\setminus B(a,\hat{\sigma}_\xi)$ on which energy concentrates.

Since there is a bounded number of x_j^ξ, arguing as in [7, Lemma 2.1] one may find a constant $\mu>1$ and radii σ_i^ξ such that

$$\xi=\sigma_0^\xi\ll\cdots\ll\sigma_L^\xi=\delta,\qquad L\leq M_0,$$

(and so $\mu\sigma_{\ell-1}^\xi<\sigma_\ell^\xi$,) and such that each ball $B(x_i^\xi,\lambda\xi)$ is contained either in $B(a,\mu\sigma_0^\xi)$, or for some $\ell\geq 1$, in the annulus $B(a,\mu\sigma_\ell^\xi)\setminus B(a,\sigma_\ell^\xi)$ for $\ell\geq 1$. Inside each annulus $A_\ell=B(a,\sigma_\ell^\xi)\setminus B(a,\mu\sigma_{\ell-1}^\xi)$ the map U_ξ is well defined and its homotopy class on $C_s=\partial B(a,s)$ is constant for $s\in[\mu\sigma_{\ell-1}^\xi,\sigma_\ell^\xi]$. We will refer to this constant homotopy class as the homotopy class of U_ξ on the annulus A_ℓ. For $\ell=L$ we know that U_ξ is non-orientable on A_L.

We claim that there exists $\ell_0\in\{1,\ldots,L\}$ such that U_ξ is orientable on A_ℓ. Otherwise, U_ξ is non-orientable on all annuli A_ℓ which lie outside $B(a,\mu\xi)$. Thus, flattening the boundary $\partial\widetilde{\Omega}$ we may for small enough δ apply Lemma 3.8 on each annulus. We deduce the lower bound

$$\begin{aligned}\widetilde{E}_\xi(\widetilde{Q}_\xi;\widetilde{\Omega}\cap B(a,\delta)\setminus B(a,\mu\xi))&\geq\sum_{\ell=1}^{L}2\pi\ln\frac{\sigma_\ell}{\mu\sigma_{\ell-1}}-C\\&\geq 2\pi\ln\frac{\delta}{\xi}-C,\end{aligned}$$

and this contradicts the upper bound obtained in Section 2.

Let us then fix the largest $\ell_0\in\{1,\ldots,L-1\}$ such that U_ξ is orientable on A_ℓ, and set $\hat{\sigma}_\xi=\sigma_{\ell_0}^\xi$. In particular U_ξ is non-orientable on A_ℓ for all $\ell\geq\ell_0+1$, hence arguing as above we find the lower bound

$$\begin{aligned}\widetilde{E}_\xi(\widetilde{Q}_\xi;\widetilde{\Omega}\cap B(a,\delta)\setminus B(a,2\mu\hat{\sigma}_\xi))&\geq\sum_{\ell=\ell_0+1}^{L}2\pi\ln\frac{\sigma_\ell}{\mu\sigma_{\ell-1}}-C\\&\geq 2\pi\ln\frac{\delta}{\hat{\sigma}_\xi}-C.\end{aligned}\tag{30}$$

Moreover, since U_ξ is orientable on $\partial B(a,\hat{\sigma}_\xi)$ and non-orientable on $\partial B(a,\mu\hat{\sigma}_\xi)$, there must be at least one x_j^ξ such that $B(x_j^\xi,\lambda\xi)\subset B(a,\mu\hat{\sigma}_\xi)\setminus B(a,\hat{\sigma}_\xi)$ and U_ξ is non-orientable on $\partial B(x_j^\xi,\lambda\xi)$.

Step 3: We show that a bad ball must be centered on the plane $\{z = 0\}$, and estimate its contribution to the energy.

Now we consider only those x_j^ξ which are in $B(a, \mu\hat{\sigma}_\xi) \setminus B(a, \hat{\sigma}_\xi)$. Considering the balls $B(x_j^\xi, c_0\hat{\sigma}_\xi)$ for some small enough $c_0 > 0$ (depending only on the number of x_j^ξ's) and applying the procedure in [9, Theorem IV.1], we obtain a bounded number $c_0 \le \gamma \le \frac{1}{16}$ and a collection of balls $B(y_j^\xi, \gamma\hat{\sigma}_\xi)$ satisfying the following:

$$
\begin{aligned}
&\{f(\widetilde{Q}_\xi) > \eta\} \cap B(a, \mu\hat{\sigma}_\xi) \setminus B(a, \hat{\sigma}_\xi) \subset \bigcup_{j=1}^{\hat{J}} B(y_j^\xi, \gamma\hat{\sigma}_\xi), \qquad \hat{J} \le M_0,\\
&|y_i^\xi - y_j^\xi| > 8\gamma\hat{\sigma}_\xi \qquad \forall i \neq j,\\
&y_j^\xi \in \{z = 0\} \quad \text{or} \quad \mathrm{dist}(y_j^\xi, \{z = 0\}) > 8\gamma\hat{\sigma}_\xi.
\end{aligned}
$$

Since all balls $B(y_j^\xi, 2\gamma\hat{\sigma}_\xi)$ are contained in the annulus $B(a, 2\mu\hat{\sigma}_\xi) \setminus B(a, \hat{\sigma}_\xi/2)$ and for small enough ξ by the above U_ξ is non-orientable on $\partial B(a, 2\mu\hat{\sigma}_\xi)$ and orientable on $\partial B(a, \hat{\sigma}_\xi/2)$, we deduce that U_ξ must be non-orientable on $\partial B(y_j^\xi, 2\gamma\hat{\sigma}_\xi)$ for at least one y_j^ξ. Relabeling we assume this is y_1^ξ.

We may apply Lemma 3.7 to obtain a lower bound on the energy in $B(y_1^\xi, 2\gamma\hat{\sigma}_\xi)$. This ball may happen to intersect $\partial\widetilde{\Omega}$, but considering the fixed Lipschitz extension U_0 of $\widetilde{Q}_\xi$ outside $\widetilde{\Omega}$ we still have the same lower bound.

Let us first assume that this y_1^ξ does not lie on the axis $\{z = 0\}$. Then, taking into account that $\rho \ge 1 - \hat{\sigma}_\xi^2$ we obtain

$$
\widetilde{E}_\xi(\widetilde{Q}_\xi; B(y_1^\xi, 2\gamma\hat{\sigma}_\xi) \ge (1 - \hat{\sigma}_\xi^2)\pi \ln \frac{\hat{\sigma}_\xi}{\xi} - C.
$$

By symmetry, if y_1^ξ lies above the axis $\{z = 0\}$ then the same lower bound will be obtained below the axis, so we deduce

$$
\widetilde{E}_\xi(\widetilde{Q}_\xi; \widetilde{\Omega} \cap B(a, 2\mu\hat{\sigma}_\xi) \setminus B(a, \hat{\sigma}_\xi/2)) \ge 2(1 - \hat{\sigma}_\xi^2)\pi \ln \frac{\hat{\sigma}_\xi}{\xi} - O(1).
$$

Adding this to the lower bound (30) on $B(a, \delta) \setminus B(a, 2\mu\hat{\sigma}_\xi)$, we obtain

$$
\begin{aligned}
\widetilde{E}_\xi(\widetilde{Q}_\xi) &\ge 2\pi \ln \frac{\delta}{\hat{\sigma}_\xi} + 2\pi(1 - \hat{\sigma}_\xi^2) \ln \frac{\hat{\sigma}_\xi}{\xi} - O(1)\\
&\ge 2\pi(1 - \hat{\sigma}_\xi^2) \ln \frac{1}{\xi} - O(1).
\end{aligned}
$$

Since $\hat{\sigma}_\xi \to 0$ this contradicts the upper bound from Section 2.

Thus the point y_1^ξ must lie on the axis $\{z = 0\}$, hence $\rho \ge 1 + \hat{\sigma}_\xi/2$ on $B(y_1^\xi, 2\gamma\hat{\sigma}_\xi)$, and applying Lemma 3.7 we have

$$
\widetilde{E}_\xi(\widetilde{Q}_\xi; B(y_1^\xi, 2\gamma\hat{\sigma}_\xi)) \ge (1 + \frac{1}{2}\hat{\sigma}_\xi)\pi \ln \frac{\hat{\sigma}_\xi}{\xi} - C.
$$

Step 4: Completing the lower bound.

Together with the lower bound (30) on $B(a,\delta)\setminus B(a,2\mu\hat{\sigma}_\xi)$, this implies

$$\begin{aligned}\widetilde{E}_\xi(\widetilde{Q}_\xi;\widetilde{\Omega}\cap B(a,\delta)) &\geq 2\pi\ln\frac{\delta}{\hat{\sigma}_\xi}+\pi(1+\frac{1}{2}\hat{\sigma}_\xi)\ln\frac{\hat{\sigma}_\xi}{\xi}-C\\ &\geq \pi\ln\frac{1}{\xi}+\pi\ln\frac{1}{\hat{\sigma}_\xi}+\frac{\pi}{2}\hat{\sigma}_\xi\ln\frac{1}{\xi}-2\pi\ln\frac{1}{\delta}-C.\end{aligned}$$

Now notice that

$$\inf_{0<\sigma<1}\left\{\pi\ln\frac{1}{\sigma}+\frac{\pi}{2}\sigma\ln\frac{1}{\xi}\right\}$$

is attained at $\sigma=2/\ln(\frac{1}{\xi})$, so the above lower bound implies

$$\widetilde{E}_\xi(\widetilde{Q}_\xi;\widetilde{\Omega}\cap B(a,\delta))\geq\pi\ln\frac{1}{\xi}+\pi\ln\ln\frac{1}{\xi}-2\pi\ln\frac{1}{\delta}-C.$$

This is the lower bound we have been after. Now recall we have been arguing on an arbitrary sequence $\xi=\xi_n\to 0$ and taking subsequences, so what this proves is that for any $\delta\in(0,\delta_0)$,

$$\liminf_{\xi\to 0}\left(\widetilde{E}(\widetilde{Q}_\xi;\widetilde{\Omega}\cap B(a,\delta))-\pi\ln\frac{1}{\xi}-\pi\ln\ln\frac{1}{\xi}\right)\geq -2\pi\ln\frac{1}{\delta}-C,$$

and this is part (ii) of Theorem 1.2. □

Remark 3.10. In the above proof, we obtain lower bounds on $B(a,\delta)\setminus B(a,2\mu\hat{\sigma}_\xi)$ and on $B(y_1^\xi,2\gamma\hat{\sigma}_\xi)$ and realize that their sum corresponds to the upper bound obtained in Section 2. Therefore in those sets the lower bounds must be sharp, in the sense that matching upper bounds are valid. In particular we have

$$\widetilde{E}_\xi(\widetilde{Q}_\xi;\widetilde{\Omega}\cap B(a,\delta)\setminus B(a,2\mu\hat{\sigma}_\xi))\leq 2\pi\ln\frac{\delta}{\hat{\sigma}_\xi}+C,$$

and find ourselves in the situation of Lemma 3.9. As a consequence, a lower bound similar to the one in part (ii) of Theorem 1.2 is valid on the slightly smaller set

$$D_\delta^{int}\cap\{\rho\geq 1+\lambda z^\beta\},$$

for any $\lambda>0$ and $\beta>1$. Combining this with the upper bound obtained in Section 2 then yields

$$\widetilde{E}_\xi(\widetilde{Q}_\xi;D_\delta^{ext}\cup\{\rho\leq 1+\lambda z^\beta\})\leq 2\pi\ln\frac{1}{\delta}+C(\beta,\lambda)\qquad\forall\delta\in(0,1),\ \lambda>0,\ \beta>1.$$

4 Limit configuration

In this section we prove part (iii) of Theorem 1.2. Although in the previous sections we have already established tight upper and lower bounds on the energy away from the equator defect, extracting precise information about the structure of the minimizers and the absence of point singularities will be a multi-step process:

- In § 4.1 we use standard arguments to establish strong H^1 convergence away from the ring defect.
- In § 4.2 we then establish the additional symmetry property that the director points inside the azimuthal plane (in the sense of Remark 1.3). The argument of [37] implies that as long as this symmetry is satisfied at the boundary, it should also be satisfied inside the domain. Applying this result in our context is however not straightforward: the limiting map is only minimizing away from the ring defect and it is necessary to cut out a small disc around the defect. The boundary conditions are not fixed there and they need to be carefully estimated using the rigidity imposed by the energy asymptotics (in the spirit of Lemma 3.9).
- Finally, in § 4.3, we use an argument of [1] to rule out point defects which are not topologically necessary. That argument requires even more precise estimates on the boundary conditions. We can only rule out point defects provided that the ring defect is negatively charged—otherwise a positively charged ring defect would have to be compensated by a pair of point defects. The possibility of a positively charged ring defect is eventually eliminated by establishing that in a small region around the ring, it could not—compared to a negatively charged ring—improve the energy by more than $o(1)$ as $\xi \to 0$. Effectively, this amounts to showing that the core energy of a positively or negatively charged ring defect are the same.

4.1 Strong convergence

We start by establishing strong H^1 convergence along a subsequence. The limiting maps $Q \in \mathcal{H}^\star_{sym}$ will be minimizers of the harmonic map energy,

$$E_\star(Q; A) = \int_A |\nabla Q|^2, \quad A \subset \Omega^{ext}_\delta = \{x \in \Omega : \ \mathrm{dist}(x, \mathcal{C}) > \delta\},$$

away from the singularity at the defect.

Lemma 4.1. *There is a subsequence $\xi \to 0$ and $Q_\star$ such that for all $\delta > 0$, the sequence Q_ξ converges strongly in $H^1_{loc}(\Omega^{ext}_\delta)$ to $Q_\star \in \mathcal{H}^\star_{sym}$ and*

$$E_\xi(Q_\xi; \Omega^{ext}_\delta) \longrightarrow E_\star(Q_\star; \Omega^{ext}_\delta) \quad \text{and} \quad \frac{1}{\xi^2}\int_{\Omega^{ext}_\delta} f(Q_\xi) \longrightarrow 0,$$

as $\xi \to 0$. Moreover, in any relatively open subset $U \subset \overline{\Omega}$ where $Q_\star$ is smooth, we have $Q_\xi \to Q_\star$ in $C^{1,\alpha}_{loc}(U)$ for all $\alpha \in (0, 1)$.

Proof of Lemma 4.1. For any $\delta > 0$, the upper and lower bound imply

$$\frac{1}{2\pi} E_\xi(Q_\xi; \Omega^{ext}_\delta) \leq 2\pi \ln \frac{1}{\delta} + C.$$

By a diagonal argument, this bound implies the existence a limit map $Q_\star \in \mathcal{H}^\star_{sym}$ such that, up to a subsequence, Q_ξ converges weakly to $Q_\star$ in $H^1_{loc}(\Omega^{ext}_\delta)$ for every $\delta > 0$. We denote by T_δ the part of $\partial\Omega^{ext}_\delta$ that lies inside Ω, namely in cylindrical coordinates

$$T_\delta = \Omega \cap \{(\rho - 1)^2 + z^2 = \delta^2\}.$$

By Fubini's theorem we may (upon extracting a further subsequence) fix δ arbitrarily small such that

$$E_\xi(Q_\xi;T_\delta) + E_\star(Q_\star;T_\delta) \lesssim \frac{1}{\delta}\left(E_\xi(Q_\xi;\Omega^{ext}_{\delta/2}) + E_\star(Q_\star;\Omega^{ext}_{\delta/2})\right) \lesssim C(\delta). \tag{31}$$

In particular the trace of Q_ξ on T_δ is bounded in $H^1(T_\delta)$, and converges therefore also weakly in $H^1(T_\delta)$. The map Q_ξ is of the form

$$Q_\xi(\rho,\varphi,z) = R_\varphi \widetilde{Q}_\xi(\rho,z) R_\varphi^t,$$

and $\widetilde{Q}_\xi$ converges weakly in $H^1_{loc}(\widetilde{\Omega}^{ext}_\delta)$ and in $H^1(\widetilde{T}_\delta)$ to $\widetilde{Q}_\star$ such that

$$Q_\star(\rho,\varphi,z) = R_\varphi \widetilde{Q}_\star(\rho,z) R_\varphi^t,$$

where as before, $\widetilde{\Omega}^{ext}_\delta$ and $\widetilde{T}_\delta$ denote the two-dimensional cross-sections of Ω^{ext}_δ and T_δ corresponding to the (ρ,z) coordinates. The limiting harmonic map energy is then expressed as

$$\widetilde{E}_\star(\widetilde{Q}_\star;\widetilde{A}) = \iint_{\widetilde{A}} \left(\left|\nabla \widetilde{Q}_\star\right|^2 + \frac{1}{\rho^2}\Xi[\widetilde{Q}_\star]\right) \rho \, d\rho dz = \frac{1}{2\pi} E_\star(Q_\star;A), \quad \widetilde{A} \subset \widetilde{\Omega}^{ext}_\delta. \tag{32}$$

Consider a map $Q_0 \in \mathcal{H}^\star_{sym}(\Omega^{ext}_\delta)$ satisfying

$$E_\star(Q_0;\Omega^{ext}_\delta) = \min\left\{ E_\star(Q;\Omega^{ext}_\delta)\colon Q \in \mathcal{H}^\star_{sym}(\Omega^{ext}_\delta),\, Q = Q_\star \text{ on } T_\delta \right\}.$$

Since $\widetilde{Q}_\xi$ converges weakly in $H^1(\widetilde{T}_\delta)$ to $(\widetilde{Q}_\star)_{\lfloor \widetilde{T}_\delta} = (\widetilde{Q}_0)_{\lfloor \widetilde{T}_\delta}$ and $\widetilde{T}_\delta$ is a one-dimensional curve, on this curve $\widetilde{Q}_\xi$ converges to $\widetilde{Q}_0$ in $C^{0,\alpha}(T_\delta)$ for any $\alpha \in (0,1/2)$. We claim that this allows to construct a map $\overline{Q}_\xi \in \mathcal{H}_{sym}$ such that

$$\overline{Q}_\xi = Q_\xi \quad \text{in } \Omega^{int}_\delta = \Omega \setminus \Omega^{ext}_\delta \qquad \text{and } E_\xi(\overline{Q}_\xi;\Omega^{ext}_\delta) - E_\star(Q_0;\Omega^{ext}_\delta) \to 0 \quad \text{as } \xi \to 0. \tag{33}$$

In order to define $\overline{Q}_\xi$ in Ω^{ext}_δ we introduce the cross-sectional domains

$$D = \left\{(\rho,z)\colon \rho, z > 0,\, \rho^2 + z^2 > 1\right\},$$

and

$$\begin{aligned} D^{ext}_\delta &= \left\{(\rho,z) \in D\colon \ (\rho-1)^2 + z^2 > \delta^2, \ z > 0\right\}, \\ D^{int}_\delta &= \left\{(\rho,z) \in D\colon \ (\rho-1)^2 + z^2 \le \delta^2, \ z > 0\right\} = D \setminus D^{ext}_\delta. \end{aligned} \tag{34}$$

We use polar coordinates (r,θ) centered at $(\rho,z) = (1,0)$. In these coordinates the domains D and D^{ext}_δ are given by

$$D = \{(r,\theta) : \ 0 < \theta < \theta_0(r), \ r > 0\}, \quad D^{ext}_\delta = \{(r,\theta) : \ 0 < \theta < \theta_0(r), \ r > \delta\}$$
$$\text{where } \theta_0(r) = \begin{cases} \frac{\pi}{2} + \arcsin \frac{r}{2}, & \text{if } r \le \sqrt{2}, \\ \frac{\pi}{2} + \arcsin \frac{1}{r}, & \text{if } r > \sqrt{2}. \end{cases} \tag{35}$$

Then, the desired map $\overline{Q}_\xi$ will have the form

$$\overline{Q}_\xi(\rho,\varphi,z) = R_\varphi \widehat{Q}_\xi(\rho,z) R_\varphi^t,$$

and it suffices to define $\widehat{Q}_\xi$ in the domain D_δ^{ext}.

The standard idea is to introduce a thin slice $\{\delta < r < (1+\lambda)\delta\}$ where we interpolate from $\widehat{Q}_\xi = \widetilde{Q}_\xi$ at $r=\delta$ to $\widehat{Q}_\xi = \widetilde{Q}_0$ at $r=(1+\lambda)\delta$, and choose λ in order that the energy of $\widehat{Q}_\xi$ in that thin slice be negligible. Then we extend by $\widetilde{Q}_0$ (slightly rescaled) in $D_{(1+\lambda)\delta}$. Here, however, we need to be more careful because we have to preserve the boundary condition at $\rho^2+z^2=1$, i.e., in polar coordinates at $\theta=\theta_0(r)=\frac{\pi}{2}+\arcsin\frac{r}{2}$. Hence we interpolate instead in a deformed slice S_λ of the form

$$S_\lambda = \{\delta < r < (1+\tilde\lambda(\theta))\delta,\ 0<\theta<\theta_0(\delta)\}, \qquad \tilde\lambda(\theta) = (\theta_0(\delta)-\theta)\lambda,$$

for some $\lambda>0$ to be chosen later. Next we define $\widehat{Q}_\xi$ in the deformed slice S_λ. As noted in [15, Lemma B.2], with a simple linear interpolation we might fail at controlling the potential part of the energy $\int f(\widehat{Q}_\xi)$, and we need to proceed in two steps as in [15], namely first interpolate linearly between $\widetilde{Q}_\xi$ and its projection $\pi(\widetilde{Q}_\xi)$ onto $\mathcal{U}_\star$, and then interpolate geodesically between $\pi(\widetilde{Q}_\xi)$ and $\widetilde{Q}_0$ inside $\mathcal{U}_\star$. To this end we denote by

$$\gamma\colon [0,1]\times V \to \mathcal{U}_\star, \qquad V \text{ is a neighborhood of } \{(U,U)\colon U\in\mathcal{U}_\star\} \text{ in } \mathcal{U}_\star\times\mathcal{U}_\star,$$

the smooth map such that $t\mapsto\gamma(t;U_1,U_2)$ is the constant speed geodesic from U_1 to U_2 in $\mathcal{U}_\star$, which is indeed unique, well-defined and depends smoothly on U_1,U_2 provided U_1,U_2 are close enough to each other. This map satisfies the bounds

$$|\partial_t\gamma(t;U_1,U_2)| \lesssim |U_1-U_2|, \qquad |\partial_U\gamma|\lesssim 1.$$

In S_λ we define

$$\widehat{Q}(r,\theta) = \widetilde{Q}_\xi(\delta,\theta) + \mu_1(r,\theta)\left(\pi(\widetilde{Q}_\xi(\delta,\theta)) - \widetilde{Q}_\xi(\delta,\theta)\right) \quad \text{for } \delta < r < \delta + \frac{\tilde\lambda(\theta)}{2}\delta,$$
$$\text{where } \mu_1(r,\theta) = \frac{2}{\tilde\lambda(\theta)\delta}(r-\delta),$$
$$\widehat{Q}(r,\theta) = \gamma\left(\mu_2(r,\theta); \pi(\widetilde{Q}_\xi(\delta,\theta), \widetilde{Q}_0(\delta,\theta)\right) \quad \text{for } \delta+\frac{\tilde\lambda(\theta)}{2}\delta < r < \delta+\tilde\lambda(\theta)\delta,$$
$$\text{where } \mu_2(r,\theta) = \frac{2}{\tilde\lambda(\theta)\delta}(r-\delta-\frac{\tilde\lambda(\theta)}{2}\delta).$$

Note that $\pi(\widetilde{Q}_\xi(\delta,\theta))$ is well-defined for small ξ because $\widetilde{Q}_\xi$ converges uniformly to $\widetilde{Q}_0$ on

$\widetilde{T}_\delta$. Direct computations then show that

$$
\begin{aligned}
|\partial_r \widehat{Q}| &\lesssim \frac{1}{\delta\lambda}\frac{1}{\theta_0(\delta)-\theta}\left(|\pi(\widetilde{Q}_\xi(\delta,\theta)) - \widetilde{Q}_\xi(\delta,\theta)| + |\pi(\widetilde{Q}_\xi(\delta,\theta)) - \widetilde{Q}_0(\delta,\theta)|\right)\\
&\lesssim \frac{1}{\delta\lambda}\frac{1}{\theta_0(\delta)-\theta}|\widetilde{Q}_\xi(\delta,\theta) - \widetilde{Q}_0(\delta,\theta)|\\
|\partial_\theta \widehat{Q}| &\lesssim |\partial_\theta \widetilde{Q}(\delta,\theta)| + |\partial_\theta \widetilde{Q}_0(\delta,\theta)|\\
&\quad + \frac{1}{\theta_0(\delta)-\theta}\left(|\pi(\widetilde{Q}_\xi(\delta,\theta)) - \widetilde{Q}_\xi(\delta,\theta)| + |\pi(\widetilde{Q}_\xi(\delta,\theta)) - \widetilde{Q}_0(\delta,\theta)|\right),\\
&\lesssim |\partial_\theta \widetilde{Q}(\delta,\theta)| + |\partial_\theta \widetilde{Q}_0(\delta,\theta)| + \frac{1}{\theta_0(\delta)-\theta}|\widetilde{Q}_\xi(\delta,\theta) - \widetilde{Q}_0(\delta,\theta)|
\end{aligned}
$$

Next, we denote by $\sigma = \sigma(\xi)$ the quantity

$$
\sigma = \|\widetilde{Q}_\xi - \widetilde{Q}_0\|_{C^{1/4}(\widetilde{T}_\delta)} \longrightarrow 0 \qquad \text{as } \xi \to 0,
$$

and notice that since the boundary conditions on $\rho^2 + z^2 = 1$ ensure that $\widetilde{Q}_\xi(\delta, \theta_0(r)) = \widetilde{Q}_0(\delta, \theta_0(r))$, we have

$$
|\widetilde{Q}_\xi(\delta,\theta) - \widetilde{Q}_0(\delta,\theta)| \le \sigma \cdot (\theta_0(\delta) - \theta)^{\frac{1}{4}}.
$$

Therefore, the above estimates on the derivatives of $\widehat{Q}$ imply

$$
\begin{aligned}
|\nabla \widehat{Q}|^2 &= |\partial_r \widehat{Q}|^2 + \frac{1}{r^2}|\partial_\theta \widehat{Q}|^2\\
&\lesssim \frac{1}{\delta^2\lambda^2}\frac{1}{(\theta_0(\delta)-\theta)^{\frac{3}{2}}}\sigma^2 + \frac{1}{\delta^2}\left(|\partial_\theta \widetilde{Q}(\delta,\theta)|^2 + |\partial_\theta \widetilde{Q}_0(\delta,\theta)|^2\right),
\end{aligned}
$$

and

$$
\begin{aligned}
\int_{S_\lambda} |\nabla \widehat{Q}|^2 &= \int_0^{\theta_0(\delta)} \int_\delta^{\delta+\delta\lambda(\theta_0(\delta)-\theta)} r\,dr |\nabla \widehat{Q}|^2 d\theta\\
&\lesssim \delta^2\lambda \int_0^{\theta_0(\delta)} (\theta_0(\delta)-\theta)\\
&\qquad \cdot \left[\frac{1}{\delta^2\lambda^2}\frac{1}{(\theta_0(\delta)-\theta)^{\frac{3}{2}}}\sigma^2 + \frac{1}{\delta^2}\left(|\partial_\theta \widetilde{Q}(\delta,\theta)|^2 + |\partial_\theta \widetilde{Q}_0(\delta,\theta)|^2\right)\right] d\theta\\
&\lesssim \frac{\sigma^2}{\lambda} + \lambda C(\delta).
\end{aligned}
$$

For the last inequality we used the fact that, thanks to (31), the L^2 norms of $\partial_\theta \widetilde{Q}_\xi(\delta,\theta)$ and $\partial_\theta \widetilde{Q}_0(\delta,\theta)$ are bounded for δ fixed. Note moreover that the definition of $\widehat{Q}$ ensures that

$$
f(\widehat{Q}) \lesssim \mathrm{dist}^2(\widehat{Q}, \mathcal{U}_\star) \le \mathrm{dist}^2(\widetilde{Q}_\xi(\delta,\theta), \mathcal{U}_\star) \lesssim f(\widetilde{Q}_\xi(\delta,\theta)),
$$

thanks to the nonegeneracy property (17) of the potential f. Using this pointwise inequality and (31) we infer that

$$\frac{1}{\xi^2}\int_{S_\lambda} f(\widehat{Q}_\xi) \lesssim \lambda C(\delta).$$

Finally, since $\rho \approx 1$ and $\Xi[\widehat{Q}] \lesssim 1$ in S_λ we deduce from the above that

$$\widetilde{E}_\xi(\widehat{Q}_\xi; S_\lambda) \lesssim \frac{\sigma^2}{\lambda} + \lambda C(\delta).$$

Choosing $\lambda = \sigma$ then implies that

$$\widetilde{E}_\xi(\widehat{Q}_\xi; S_\lambda) \lesssim C(\delta)\sigma \longrightarrow 0 \quad \text{as } \xi \to 0.$$

Next we define $\widehat{Q}_\xi$ in $D_\delta^{ext} \setminus S_\lambda$ by setting

$$\begin{aligned}\widehat{Q}_\xi(r,\theta) &= \widetilde{Q}_0\left(\delta + \frac{1}{1-\tilde{\lambda}(\theta)}(r - \delta - \tilde{\lambda}(\theta)\delta), \theta\right) \qquad \text{for } \delta + \tilde{\lambda}(\theta)\delta < r < 2\delta,\\ \widehat{Q}_\xi &= \widetilde{Q}_0 \qquad \text{in } D_{2\delta}.\end{aligned}$$

Then we have

$$\begin{aligned}|\widetilde{E}_\xi(\widehat{Q}_\xi; D_\delta^{ext} \setminus S_\lambda) - \widetilde{E}_\star(\widetilde{Q}_0; D_\delta^{ext})| &= |\widetilde{E}_\star(\widehat{Q}_\xi; D_\delta^{ext} \setminus S_\lambda) - \widetilde{E}_\star(\widetilde{Q}_0; D_\delta^{ext})|\\ &\lesssim \lambda \widetilde{E}_\star(\widetilde{Q}_0; D_\delta^{ext}) \longrightarrow 0,\end{aligned}$$

as $\xi \to 0$, since $\lambda = \sigma \to 0$. Combining this and the fact that $\widetilde{E}_\xi(\widehat{Q}_\xi; S_\lambda) \to 0$, we deduce that

$$\widetilde{E}_\xi(\widehat{Q}_\xi; D_\delta^{ext}) - \widetilde{E}_\star(\widetilde{Q}_0; D_\delta^{ext}) \longrightarrow 0,$$

which implies the desired estimate (33). By minimality of Q_ξ we then have

$$E_\xi(Q_\xi; \Omega_\delta^{ext}) \le E_\xi(\overline{Q}_\xi; \Omega_\delta^{ext}) \le E_\star(Q_0; \Omega_\delta^{ext}) + o(1),$$

and by lower semicontinuity we deduce

$$E_\star(Q_\star; \Omega_\delta^{ext}) \le \liminf E_\xi(Q_\xi; \Omega_\delta^{ext}) \le \limsup E_\xi(\overline{Q}_\xi; \Omega_\delta^{ext}) \le E_\star(Q_0; \Omega_\delta^{ext}).$$

It follows that $Q_\star$ minimizes $E_\star(\cdot; \Omega_\delta^{ext})$ among all maps $Q \in \mathcal{H}_{sym}^\star(\Omega_\delta^{ext})$ such that $Q = Q_\star$ on T_δ, and that all above inequalities are in fact equalities. Thus we have

$$\int_{\Omega_\delta^{ext}} |\nabla Q_\xi|^2 \longrightarrow \int_{\Omega_\delta^{ext}} |\nabla Q_\star|^2 \quad \text{and} \quad \frac{1}{\xi^2}\int_{\Omega_\delta^{ext}} f(Q_\xi) \longrightarrow 0.$$

In particular, together with the weak convergence, this implies that ∇Q_ξ converges strongly in $L^2(\Omega_\delta^{ext})$ towards $\nabla Q_\star$, that is, the convergence is in fact strong.

The local $C^{1,\alpha}$ convergence away from singularities follows from the analysis in [32, 34]. There the authors consider minimizers which are not subject to the constraint of axial symmetry, but they only use the minimizing property to obtain the strong H^1 convergence, which we have just obtained. A close reading of their results reveals that the steps leading to $C^{1,\alpha}$ convergence apply equally well for *smooth* critical points, as are Q_ξ. (See Lemma 3.1.) □

4.2 Reduction to a director in a cross-section

The limiting Q-tensor Q_* inherits the symmetries (8) of the space $\mathcal{H}_{sym}$, but it also exhibits further symmetry by virtue of energy minimization. Here we show that the map $Q_\star$ may be represented by a uniaxial tensor with a unit director field $n = (n_1(\rho, z), 0, n_3(\rho, z))$, expressed in cylindrical coordinates in a cross-section of Ω.

In the sequel, whenever we refer to $\xi \to 0$, we will always mean convergence along the subsequence obtained in Lemma 4.1.

Since the limit $Q_\star$ is symmetric, it is characterized by a map defined in the two-dimensional domain D. To describe $Q_\star$ further, it will also be convenient to introduce the following notations:

$$\hat{E}(n; D_\delta^{ext}) = \int_{D_\delta^{ext}} \left(|\nabla n|^2 + \frac{n_1^2 + n_2^2}{\rho^2} \right) \rho \, d\rho \, dz \quad \text{for } n \in H^1_{loc}(D_\delta^{ext}; \mathbb{S}^2), \tag{36}$$

$$\hat{\mathcal{H}}(D_\delta^{ext}) = \Big\{ n \in H^1_{loc}(D_\delta^{ext}; \mathbb{S}^2) \colon \hat{E}(n; D_\delta^{ext}) < \infty, \\ n \otimes n = e_r \otimes e_r \text{ on } \partial D_\delta^{ext} \cap \{\rho^2 + z^2 = 1\}, \\ n \otimes n = e_3 \otimes e_3 \text{ on } \partial D_\delta^{ext} \cap \{z = 0\} \Big\}.$$

$$\hat{\mathcal{H}} = \bigcap_{\delta > 0} \hat{\mathcal{H}}(D_\delta^{ext}),$$

where D_δ^{ext} is defined in (34). The symmetry and minimizing property of $Q_\star$ allow us to express it in terms of a map n which minimizes $\hat{E}$ in $\hat{\mathcal{H}}$. Specifically, we have:

Lemma 4.2. *The map $Q_\star$ is given in cylindrical coordinates by*

$$Q_\star(\rho, \varphi, z) = R_\varphi n(\rho, z) \otimes R_\varphi n(\rho, z) - \frac{1}{3} I,$$

where $n \in H^1_{loc}(\widetilde{\Omega})$ satisfies $n(\rho, -z) = -Sn(\rho, z)$ and, when restricted to $\{z > 0\}$, $n \in \hat{\mathcal{H}}$ minimizes $\hat{E}(\cdot; D_\delta^{ext})$ for all $\delta > 0$, among all maps $m \in \hat{\mathcal{H}}(D_\delta^{ext})$ such that $m \otimes m = n \otimes n$ on $\partial D_\delta^{ext} \cap D$.

Moreover, up to replacing n by $-n$, we have

$$n = e_r \text{ on } \partial D_\delta^{ext} \cap \{\rho^2 + z^2 = 1\} \quad \text{and} \quad n = \tau e_3 \text{ on } \partial D_\delta^{ext} \cap \{z = 0\}, \tag{37}$$

for some $\tau \in \{\pm 1\}$.

Proof of Lemma 4.2. Since Ω_δ^{ext} is simply connected, $Q_\star$ can be lifted [10, 5]: there exists a map $n_\star \in H^1_{loc}(\Omega; \mathbb{S}^2)$ such that

$$Q_\star = n_\star \otimes n_\star - \frac{1}{3} I.$$

As $|\nabla Q_\star|^2 = 2|\nabla n_\star|^2$, the symmetry of $Q_\star$ implies that $n(\rho, z) = n_\star(\rho, 0, z)$ belongs to $H^1_{loc}(\widetilde{\Omega}_\delta^{ext})$, and we have

$$\frac{1}{8\pi} E_\star(Q_\star; \Omega_\delta^{ext}) = \hat{E}(n; D_\delta^{ext}).$$

Since $Q_\star$ is minimizing in $\mathcal{H}^\star_{sym}(\Omega^{ext}_\delta)$, we deduce that n minimizes $\hat{E}(\cdot; D^{ext}_\delta)$ among maps $m\colon D^{ext}_\delta \to \mathbb{S}^2$ satisfying the boundary conditions

$$
\begin{aligned}
&m\otimes m = e_r\otimes e_r \quad \text{ on } \partial D^{ext}_\delta \cap \{\rho^2+z^2=1\},\\
&m\otimes m = n\otimes n \quad \text{ on } \partial D^{ext}_\delta \cap D,\\
&m\otimes m = (Sm)\otimes(Sm) \quad \text{ on } \partial D^{ext}_\delta \cap \{z=0\}.
\end{aligned}
$$

Note that $|m\otimes m - e_3\otimes e_3|^2 = 2(m_1^2+m_2^2)$, so that the far-field condition which requires $\int |Q-Q_\infty|^2|x|^{-2}<\infty$ is obsolete here: any map m with finite energy satisfies

$$
\int_{D^{ext}_\delta} \frac{m_1^2+m_2^2}{\rho^2+z^2}\rho\, d\rho\, dz \leq \hat{E}(m; D^{ext}_\delta) < \infty.
$$

The boundary condition on $\{z=0\}$ comes from the requirement that $m\otimes m$ can be extended to an H^1 map in $\widetilde{\Omega}^{ext}_\delta$ via the mirror symmetry. It is equivalent to $m_3(\rho,0)\in\{0,\pm 1\}$ for almost all $\rho>1+\delta$. Since the trace $\rho\mapsto m_3(\rho,0)$ has $H^{1/2}_{loc}$ regularity, being integer valued it has to be constant: there exists $\tau\in\{0,\pm1\}$ such that $m_3(\rho,0)=\tau$ for almost all $\rho>1+\delta$. One can rule out $\tau=0$. Indeed, assume by contradiction that $\tau=0$, i.e. $m_3(\rho,0)=0$. Then we have

$$
\begin{aligned}
m_1^2+m_2^2 = 1-m_3^2 &= 1-\left(\int_0^z \partial_3 m_3\right)^2\\
&\geq 1-|z|\int_0^z |\partial_3 m|^2,
\end{aligned}
$$

and therefore, for almost all $z_0>0$,

$$
\begin{aligned}
+\infty = \int_2^\infty \frac{d\rho}{\rho} &\leq \int_2^\infty \frac{m_1^2+m_2^2}{\rho}\, d\rho + |z_0|\int_2^\infty \frac{1}{\rho^2}\int_0^\infty |\partial_3 m|^2\, dz\, \rho\, d\rho\\
&\leq \int_2^\infty \frac{m_1^2+m_2^2}{\rho} d\rho + |z_0|\int_{D_1^{ext}} |\nabla m|^2 \rho\, d\rho\, dz.
\end{aligned}
$$

This clearly contradicts the finiteness of $\int (m_1^2+m_2^2)/\rho\, d\rho dz$. We deduce that $\tau=\pm 1$, that is,

$$
m\otimes m = e_3\otimes e_3 \quad \text{for } z=0,
$$

so that $m\in\hat{\mathcal{H}}(D^{ext}_\delta)$ and, as a consequence, n minimizes $\hat{E}(\cdot;D^{ext}_\delta)$ in $\hat{\mathcal{H}}(D^{ext}_\delta)$ for all $\delta>0$ with respect to its own boundary conditions on $\partial D^{ext}_\delta\cap D$.

Moreover, the boundary conditions on $\partial D^{ext}_\delta \cap\{\rho^2+z^2=1\}$ require that the $H^{1/2}$ trace $n\cdot e_r$ take values into $\{\pm1\}$ and thus be constant. Then, up to changing n to $-n$ and τ to $-\tau$, one obtains the boundary conditions (37).

It remains to show that $n(\rho,-z) = -Sn(\rho,z)$. The mirror symmetry implies that $n(\rho,-z)=\pm Sn(\rho,z)$, and therefore the H^1 function $n(\rho,-z)\cdot Sn(\rho,z)$ takes values into $\{\pm1\}$ and must be constant. By the above, its trace on $\{z=0\}$ is equal to $\tau^2 e_3\cdot(-e_3)=-1$. We conclude that $n(\rho,-z)=-Sn(\rho,z)$. $\square$

Next we turn to proving the additional symmetry property $n_2 \equiv 0$. The idea is that the use of an appropriate comparison map as in [37] implies that symmetry provided it is satisfied at the boundary. But we can only use energy comparison in D_δ^{ext}, and then we actually do not know that $n_2 = 0$ on the whole boundary, due to the undetermined part $\partial D_\delta^{ext} \cap D$. This will make the proof quite technical.

To gather more information about the behavior of n on $\partial D_\delta^{ext} \cap D$, it is natural to use polar coordinates (r, θ) around $(\rho = 1, z = 0)$, so that $\partial D_\delta^{ext} \cap D$ corresponds to fixing $r = \delta$. In those coordinates, the domain D is given by $0 < \theta < \theta_0(r)$, where $\theta_0(r)$ is defined in (35). The upper and lower bound, together with strong H^1 convergence, provide the estimate

$$\frac{1}{2\pi} E(Q_\star; \Omega_\delta^{ext}) \leq 2\pi \ln \frac{1}{\delta} + C,$$

which for $n(\rho, z)$ translates into

$$\hat{E}(n; D_\delta^{ext}) \leq \frac{\pi}{2} \ln \frac{1}{\delta} + C.$$

In coordinates (r, θ) this implies

$$\int_\delta^1 \left[\int_0^{\theta_0(r)} |\partial_\theta n|^2 d\theta - \frac{\pi}{2} \right] \frac{dr}{r} + \int_\delta^1 \int_0^{\theta_0(r)} |\partial_r n|^2 d\theta \, r \, dr \leq C. \tag{38}$$

For $r < \sqrt{2}$, the boundary conditions (37) become

$$n(r, \theta_0(r)) = \cos\theta_1 e_1 + \sin\theta_1 e_3, \quad \text{where} \quad \theta_1 = 2\theta_0 - \pi, \tag{39}$$

and $n(r, 0) = \tau e_3$. Here, recall (13) that $\theta_0(r) = \pi/2 + O(r)$ and so $\theta_1(r) = O(r)$. Remarking (as in Lemma 3.9) that we also have the lower bound

$$\begin{aligned}
\int_0^{\theta_0} |\partial_\theta n|^2 d\theta &\geq \frac{1}{\theta_0} \left(\int_0^{\theta_0} |\partial_\theta n| d\theta \right)^2 \\
&\geq \frac{1}{\theta_0} [\mathrm{dist}_{\mathbb{S}^2}(n(r,0), n(r,\theta_0))]^2 \\
&= \frac{(\tau\pi/2 - \theta_1)^2}{\theta_0} = \frac{\pi}{2} + O(r),
\end{aligned}$$

from (38) we deduce

$$\int_0^1 \left| \int_0^{\theta_0(r)} |\partial_\theta n|^2 d\theta - \frac{\pi}{2} \right| \frac{dr}{r} + \int_0^1 \int_0^{\theta_0(r)} |\partial_r n|^2 d\theta \, r \, dr < \infty. \tag{40}$$

Finiteness of the first integral in (40) tells us that $\int_0^{\theta_0} |\partial_\theta n|^2 d\theta$ is close to the length of the geodesic from $n(r, 0)$ to $n(r, \theta_0)$ in $\mathbb{S}^2$, as this length is $|\tau\pi/2 - \theta_1| = \pi/2 + O(r)$. This implies that n must be close to the actual geodesic, thanks to the following stability estimate for geodesics of length $\ell < \pi$ on $\mathbb{S}^2$.

Lemma 4.3. *Let $\ell \in (0,\pi)$ and $m_0\colon [0,\ell] \to \mathbb{S}^2$ be a geodesic on $\mathbb{S}^2$ parametrized with unit speed $|m_0'| = 1$. Then for any $m \in H^1((0,\ell);\mathbb{S}^2)$ such that $m(0) = m_0(0)$ and $m(\ell) = m_0(\ell)$, we have*

$$\int_0^\ell |m' - m_0'|^2 \leq \frac{\pi^2}{\pi^2 - \ell^2}\left(\int_0^\ell |m'|^2 - \ell\right)$$

Proof of Lemma 4.3. We have the identity

$$\int_0^\ell |m' - m_0'|^2 = \int_0^\ell |m'|^2 - \ell + 2\int_0^\ell m_0' \cdot (m_0' - m').$$

Note that the geodesic m_0 is smooth and satisfies $m_0'' = -m_0$. Moreover we know that $m_0 - m$ vanishes at 0 and ℓ, so integrating by parts in the last term we obtain

$$\int_0^\ell |m' - m_0'|^2 = \int_0^\ell |m'|^2 - \ell + 2\int_0^\ell m_0 \cdot (m_0 - m). \tag{41}$$

Remarking that $|m - m_0|^2 = 2 - 2m_0 \cdot m$ we have

$$2\int_0^\ell m_0 \cdot (m_0 - m) = \int_0^\ell |m - m_0|^2 \leq \frac{\ell^2}{\pi^2}\int_0^\ell |m' - m_0'|^2,$$

by Poincaré inequality, since the lowest positive eigenvalue of the Dirichlet Laplacian on $(0,\ell)$ corresponds to the eigenfunction $\theta \mapsto \sin(\pi\theta/\ell)$ and is therefore equal to π^2/ℓ^2. Hence we can absorb the last term of (41) into the left-hand side to conclude the proof. □

We now define $m_0\colon D_1^{int} \to \mathbb{S}^2$ such that for all $r \in (0,1)$, the map $(0,\theta_0(r)) \ni \theta \mapsto m_0(r,\theta)$ is a constant speed geodesic from $n(r,0)$ to $n(r,\theta_0(r))$, that is,

$$m_0(r,\theta) = \cos(\tau\frac{\pi}{2} - \lambda^\tau(r)\theta)\, e_1 + \sin(\tau\frac{\pi}{2} - \lambda^\tau(r)\theta)\, e_3, \qquad \lambda^\tau(r) = \frac{\tau\pi/2 - \theta_1(r)}{\theta_0(r)}.$$

Note that $m_0(r,\cdot)$ is a geodesic of speed $1 + O(r)$ and of length $\pi/2 + O(r)$, so applying Lemma 4.3 to appropriately rescaled versions of $n(r,\cdot)$ and $m_0(r,\cdot)$ and invoking (40) we deduce

$$\int_0^1 \int_0^{\theta_0(r)} |\partial_\theta n - \partial_\theta m_0|^2 d\theta\, \frac{dr}{r} < \infty. \tag{42}$$

We are now ready to prove:

Proposition 4.4. *We have $n_2 \equiv 0$ in D.*

Proof of Proposition 4.4. The starting idea is to use as a comparison map

$$\tilde{n} = \left(\sqrt{n_1^2 + n_2^2}, 0, n_3\right),$$

which has lower energy than n since $n_1^2 + n_2^2 = \tilde{n}_1^2 + \tilde{n}_2^2$ and

$$|\partial_j \tilde{n}|^2 - |\partial_j n|^2 = -\frac{(n_1 \partial_j n_2 - n_2 \partial_j n_1)^2}{n_1^2 + n_2^2},$$

and to conclude that $n_2 = 0$. One just needs to take care of technical difficulties that arise due to the undetermined part of the boundary $\partial D_\delta^{ext} \cap D$: there, one does not know that $n_2 = 0$ and $n_1 \geq 0$, hence $\tilde{n}$ cannot be used directly as a comparison map. One does know however thanks to (42) that n is very close to a map satisfying these conditions: (42) implies that the quantity

$$\sigma = \sigma(\delta) := \left(\int_0^{\theta_0(\delta)} |\partial_\theta n(\delta, \theta) - \partial_\theta m_0(\delta, \theta)|^2 \, d\theta \right)^{\frac{1}{2}}, \tag{43}$$

tends to 0 along a subsequence $\delta = \delta_k \to 0$. This readily provides some uniform control on $n(\delta, \cdot) - \tilde{n}(\delta, \cdot)$, since by Morrey's inequality we have

$$|n(\delta, \theta) - m_0(\delta, \theta)| \lesssim \sigma \, (\theta_0(\delta) - \theta)^{\frac{1}{2}} \qquad \forall \theta \in (0, \theta_0(\delta)).$$

Using that $m_0 = \tilde{m}_0$ and that the transformation $n \mapsto \tilde{n}$ is Lipschitz, this implies

$$\begin{aligned} |n(\delta, \theta) - \tilde{n}(\delta, \theta)| &\leq |n(\delta, \theta) - m_0(\delta, \theta)| + |\tilde{m}_0(\delta, \theta) - \tilde{n}(\delta, \theta)| \\ &\lesssim |n(\delta, \theta) - m_0(\delta, \theta)| \\ &\lesssim \sigma \, (\theta_0(\delta) - \theta)^{\frac{1}{2}} \qquad \forall \theta \in (0, \theta_0(\delta)). \end{aligned} \tag{44}$$

Next we define a good comparison map $\bar{n}$ in D_δ^{ext}. In $D_{2\delta}^{ext}$ we simply set

$$\bar{n} = \tilde{n} \qquad \text{in } D_{2\delta}^{ext},$$

and it remains to construct a well-behaved transition layer in $D_\delta^{ext} \setminus D_{2\delta}^{ext}$. In polar coordinates (r, θ), the domain $D_\delta^{ext} \setminus D_{2\delta}^{ext}$ is given by $\delta \leq r < 2\delta$, $0 < \theta < \theta_0(r)$, and we want to define $\bar{n}$ such that $\bar{n} = n$ for $r = \delta$, $\bar{n} = \tilde{n}$ for $r = 2\delta$, and $\bar{n} = n = \tilde{n}$ for $\theta \in \{0, \theta_0(r)\}$. To this end we introduce a small parameter $\hat{\lambda} \in (0, 1/2)$, to be fixed later, and proceed similarly to the construction in the proof of Lemma 4.1. Namely, we interpolate between $n(\delta, \cdot)$ and $\tilde{n}(\delta, \cdot)$ in a thin slice $\delta \leq r < (1 + \hat{\lambda})\delta$, and then slightly rescale $\tilde{n}$ in the remaining part $(1+\hat{\lambda})\delta \leq r < 2\delta$. As in Lemma 4.1, this simple picture actually requires a small modification to accomodate the non-constant boundary condition at $\theta = \theta_0(r)$. To do this, we first define

$$\tilde{\lambda}(\theta) = \hat{\lambda} \frac{(\theta_0(\delta) - \theta)_+}{\theta_0(\delta)}, \tag{45}$$

and partition $D_\delta^{ext} \setminus D_{2\delta}^{ext}$ as

$$\begin{aligned} D_\delta^{ext} \setminus D_{2\delta}^{ext} &= R_1 \cup R_2, \\ R_1 &= \{\delta \leq r < (1 + \tilde{\lambda}(\theta))\delta, \ 0 < \theta < \theta_0(r)\}, \\ R_2 &= \{(1 + \tilde{\lambda}(\theta))\delta \leq r < 2\delta, \ 0 < \theta < \theta_0(r)\}. \end{aligned}$$

Then we set

$$\bar{n}(r,\theta) = \begin{cases} \pi_{\mathbb{S}^2}\left(n(\delta,\theta) + \dfrac{\ln(r/\delta)}{\ln(1+\tilde{\lambda}(\theta))}(\tilde{n}(\delta,\theta) - n(\delta,\theta))\right) & \text{in } R_1, \\ \tilde{n}\left(\delta + \dfrac{r-(1+\tilde{\lambda}(\theta))\delta}{1-\tilde{\lambda}(\theta)}, \theta\right) & \text{in } R_2, \end{cases}$$

where $\pi_{\mathbb{S}^2}$ denotes the nearest point projection onto $\mathbb{S}^2$, which is well-defined and Lipschitz thanks to the uniform estimate (44) on $n(\delta,\cdot) - \tilde{n}(\delta,\cdot)$. With that definition, the map $\bar{n}\colon D_\delta^{ext} \to \mathbb{S}^2$ agrees with n on ∂D_δ^{ext} and from the minimizing property of n we have

$$\hat{E}(n; D_\delta^{ext}) \leq \hat{E}(\bar{n}; D_\delta^{ext}). \tag{46}$$

In order to make use of that inequality, we need to estimate the energy of $\bar{n}$ in the transition layer $D_\delta^{ext} \setminus D_{2\delta}^{ext}$. We start by estimating the energy in R_1. Recalling the definition of $\tilde{\lambda}$ in (45), by direct calculation we have

$$|\nabla \bar{n}|^2 \lesssim \frac{1}{r^2}|\partial_\theta n(\delta,\theta)|^2 + \frac{1}{r^2}|\partial_\theta \tilde{n}(\delta,\theta)|^2 + \frac{1}{r^2}\frac{|n(\delta,\theta)-\tilde{n}(\delta,\theta)|^2}{\hat{\lambda}^2(\theta_0(\delta)-\theta)^2} \qquad \text{in } R_1,$$

so, taking into account that $\int_\delta^{(1+\tilde{\lambda})\delta} r^{-2}\, rdr = \ln(1+\tilde{\lambda}) \lesssim \tilde{\lambda}$ and $|\partial_\theta \tilde{n}| \leq |\partial_\theta n|$, we deduce

$$\int_{R_1} |\nabla \bar{n}|^2 \lesssim \hat{\lambda}\int_0^{\theta_0(\delta)} |\partial_\theta n(\delta,\theta)|^2\, d\theta + \frac{1}{\hat{\lambda}}\int_0^{\theta_0(\delta)} \frac{|n(\delta,\theta)-\tilde{n}(\delta,\theta)|^2}{(\theta_0(\delta)-\theta)}\, d\theta.$$

Using the uniform estimate (44) on $n(\delta,\cdot) - \tilde{n}(\delta,\cdot)$ to bound the last term, this implies

$$\int_{R_1} |\nabla \bar{n}|^2 \lesssim \hat{\lambda}\int_0^{\theta_0(\delta)} |\partial_\theta n(\delta,\theta)|^2\, d\theta + \frac{1}{\hat{\lambda}}\sigma^2.$$

Appealing to the facts that $\sigma^2 = \int_0^{\theta_0} |\partial_\theta n - \partial_\theta m_0|^2\, d\theta \ll 1$ and $|\partial_\theta m_0| \lesssim 1$ to bound the first term in the right-hand side, we obtain

$$\int_{R_1} |\nabla \bar{n}|^2 \lesssim \hat{\lambda} + \frac{1}{\hat{\lambda}}\sigma^2.$$

From the definition (36) of the energy

$$\hat{E}(\bar{n}; R_1) = \int_{R_1} |\nabla \bar{n}|^2 \rho + \int_{R_1} \frac{\bar{n}_1^2 + \bar{n}_2^2}{\rho},$$

and taking into account that $\rho = 1 + r\cos\theta = 1 + O(\delta)$ in R_1, we deduce

$$\hat{E}(\bar{n}; R_1) \lesssim \hat{\lambda} + \frac{1}{\hat{\lambda}}\sigma^2. \tag{47}$$

Next we turn to estimating the energy in R_2. Direct calculation gives

$$
\begin{aligned}
|\partial_r \bar{n}|^2 &= \frac{1}{(1-\tilde{\lambda}(\theta))^2}|\partial_r \tilde{n}|^2 \\
|\partial_\theta \bar{n}|^2 &\leq |\partial_\theta \tilde{n}|^2 + C\hat{\lambda}^2\delta^2|\partial_r \tilde{n}|^2 + C\hat{\lambda}\delta|\partial_\theta \tilde{n}|\,|\partial_r \tilde{n}| \\
&\leq (1+\hat{\lambda})|\partial_\theta \tilde{n}|^2 + C\hat{\lambda}\delta^2|\partial_r \tilde{n}|^2,
\end{aligned}
$$

where it is implicitly understood that all derivatives of $\tilde{n}$ appearing in the right-hand sides are evaluated at $\left(\delta + \frac{r-(1+\tilde{\lambda}(\theta))\delta}{1-\tilde{\lambda}(\theta)}, \theta\right)$. Integrating and changing variables we deduce

$$
\begin{aligned}
\int_{R_2} |\nabla \bar{n}|^2 \rho - \int_{D_\delta^{ext}\setminus D_{2\delta}^{ext}} |\nabla \tilde{n}|^2 \rho &\lesssim \hat{\lambda} \int_{D_\delta^{ext}\setminus D_{2\delta}^{ext}} |\nabla \tilde{n}|^2 \\
&\lesssim \hat{\lambda} \int_{D_\delta^{ext}\setminus D_{2\delta}^{ext}} |\nabla n|^2.
\end{aligned}
$$

Note that

$$
\begin{aligned}
\int_{D_\delta^{ext}\setminus D_{2\delta}^{ext}} |\nabla n|^2 &\lesssim \int_\delta^{2\delta}\int_0^{\theta_0(r)} |\partial_\theta m_0|^2 d\theta\, \frac{dr}{r} + \int_\delta^{2\delta}\int_0^{\theta_0(r)} |\partial_\theta n - \partial_\theta m_0|^2 d\theta\, \frac{dr}{r} \\
&\quad + \int_\delta^{2\delta}\int_0^{\theta_0(r)} |\partial_r n|^2 d\theta\, r\, dr \\
&\lesssim C + \int_0^{1}\int_0^{\theta_0(r)} |\partial_\theta n - \partial_\theta m_0|^2 d\theta\, \frac{dr}{r} + \int_0^{1}\int_0^{\theta_0(r)} |\partial_r n|^2 d\theta\, r\, dr,
\end{aligned}
$$

where we used the fact that $|\partial_\theta m_0| \lesssim 1$. Therefore, recalling that the two last integrals are finite thanks to (40) and (42), we obtain

$$
\int_{R_2} |\nabla \bar{n}|^2 \rho - \int_{D_\delta^{ext}\setminus D_{2\delta}^{ext}} |\nabla \tilde{n}|^2 \rho \lesssim \hat{\lambda}.
$$

The second term in the energy (36) can be estimated in a similar way, so that

$$
\hat{E}(\bar{n}; R_2) - \hat{E}(\tilde{n}; D_\delta^{ext} \setminus D_{2\delta}^{ext}) \lesssim \hat{\lambda}.
$$

Combining this with (47), the estimate on the energy of the transition layer $D_\delta^{ext} \setminus D_{2\delta}^{ext}$ becomes

$$
\hat{E}(\bar{n}; D_\delta^{ext} \setminus D_{2\delta}^{ext}) - \hat{E}(\tilde{n}; D_\delta^{ext} \setminus D_{2\delta}^{ext}) \lesssim \hat{\lambda} + \frac{1}{\hat{\lambda}}\sigma^2.
$$

Choosing $\hat{\lambda} = \sigma$, the right-hand side is $\lesssim \sigma \ll 1$. From the minimizing property (46) of n we thus obtain

$$
\begin{aligned}
0 &\leq \hat{E}(\bar{n}; D_\delta^{ext}) - \hat{E}(n; D_\delta^{ext}) \\
&\leq \int_{D_\delta^{ext}} \left(|\nabla \tilde{n}|^2 - |\nabla n|^2\right) \rho + C\,\sigma.
\end{aligned}
$$

Since $\sigma = \sigma(\delta_k) \to 0$ as $\delta = \delta_k \to 0$, and $|\nabla \tilde{n}| \leq |\nabla n|$ in D, we deduce that we have in fact $|\nabla \tilde{n}| = |\nabla n|$ in D, which implies

$$n_1 \nabla n_2 - n_2 \nabla n_1 = 0 \quad \text{in } D.$$

In particular, for any fixed $\rho, z > 0$ with $\rho^2 + z^2 = 1$, the map $m\colon t \mapsto (n_1, n_2)(t\rho, tz)$ satisfies $m \wedge \dot{m} = 0$, and thus $\dot{m} = \alpha m$, where $\alpha = (m \cdot \dot{m})/|m|^2$ is continuous on $[1, \infty)$ because $|m|^2$ does not vanish there. Since $m_2(1) = 0$ this implies that $m_2(t) = 0$ for all $t \geq 1$, hence $n_2 \equiv 0$ in D. □

Thanks to Proposition 4.4, we can apply the regularity results on symmetric harmonic maps in [24] to deduce that $n_\star$ is analytic in $\overline{\Omega} \setminus (\mathcal{C} \cup Z)$, where $Z \subset \Omega \cap \{\rho = 0\}$ is a discrete set of singular points on the vertical axis. By Lemma 4.1 we therefore have $Q_\xi \to Q_\star$ in $C^{1,\alpha}_{loc}(\overline{\Omega} \setminus (\mathcal{C} \cup Z))$.

4.3 The absence of point defects

To conclude the proof of Theorem 1.2 (*iii*) it remains to show that Z is empty. The starting idea is to try and apply the argument of [1, Theorem 13] (using reflections of the image in $\mathbb{S}^2$ and analyticity) to eliminate all point defects that are not required by topology. If the defect ring is negatively charged, that is $\tau = +1$ in Lemma 4.2, this argument may be applied to eliminate all point defects, and we carry out this analysis in § 4.3.1. However, if the ring happens to be positively charged—corresponding to $\tau = -1$—then an additional pair of point defects would be required. To complete the proof we therefore need to show that the case $\tau = -1$ can not occur, and this is demonstrated in § 4.3.2.

4.3.1 The case of a negatively charged ring $\tau = +1$

The argument of [1, Theorem 13] used to eliminate extraneous point defects relies on the construction of comparison maps, and thus we again face the sticky issue of controlling the boundary conditions on $\partial D^{ext}_\delta \cap D$. Specifically, we need to know that n_3 does not change sign on that boundary part, and our first step is therefore to gather stronger information about the trace of n there.

Lemma 4.5. *There exists $r_n \to 0$ such that $\theta \mapsto n_3(r_n, \theta)$ is strictly monotone on $[0, \theta_0(r_n)]$.*

Proof of Lemma 4.5. Since D is simply connected, there exists a lifting $\varphi \in H^1_{loc}(D; \mathbb{R})$ such that

$$n = (\cos\varphi, 0, \sin\varphi).$$

This lifting φ is in fact smooth up to the boundary of D, except at points of the singular set Z and at $(\rho, z) = (1, 0)$. It is defined up to a constant multiple of 2π, that one may fix by imposing $\varphi(r, \theta_0(r)) = \theta_1(r)$, where we recall the definition (39) of $\theta_1(r)$. For $\theta = 0$ we then have $\varphi \equiv \tau\pi/2 + 2N\pi$ for some $N \in \mathbb{Z}$. This implies

$$\begin{aligned}\int_0^{\theta_0} |\partial_\theta n|^2 d\theta = \int_0^{\theta_0} |\partial_\theta \varphi|^2 d\theta &\geq \frac{1}{\theta_0} \left(\int_0^{\theta_0} \partial_\theta \varphi \, d\theta \right)^2 \\ &= \frac{(\theta_1 - \tau\pi/2 - 2N\pi)^2}{\theta_0} = \frac{\pi}{2}(1 + 4\tau N)^2 + O(r).\end{aligned}$$

Recalling (40), we deduce that necessarily $N = 0$.

Moreover, the estimate (42) suggests that for small $r > 0$, n is close to the map

$$n_0 = (\cos\varphi_0, 0, \tau\sin\varphi_0), \qquad \varphi_0 = \pi/2 - \theta, \tag{48}$$

and hence we expect the same behavior for its lifting, φ. More precisely, we have

$$\begin{aligned}\int_0^{\theta_0} |\partial_\theta\varphi - \tau\partial_\theta\varphi_0|^2\, d\theta &= \int_0^{\theta_0} |\partial_\theta\varphi|^2\, d\theta - \frac{\pi}{2} + (1+4\tau)\arcsin\frac{r}{2} \\ &= \int_0^{\theta_0} |\partial_\theta n|^2 d\theta - \frac{\pi}{2} + O(r),\end{aligned}$$

and since $\varphi = \tau\varphi_0 + O(r)$ at $\theta = 0$ and $\theta = \theta_0(r)$, together with (40) this shows that

$$\int_0^1 \|\varphi(r,\cdot) - \tau\varphi_0(r,\cdot)\|^2_{H^1(0,\theta_0(r))} \frac{dr}{r} < \infty. \tag{49}$$

In particular, there are arbitrarily small δ's such that n is very close to n_0 on $\partial D_\delta^{ext} \cap D$, in H^1 and thus also in L^∞. Using the equation satisfied by φ one can obtain a stronger estimate. Since φ minimizes

$$F(\varphi; D_\delta^{ext}) = \hat{E}(n; D_\delta^{ext}) = \int_{D_\delta^{ext}} \left[|\nabla\varphi|^2 + \frac{\cos^2\varphi}{\rho^2}\right]\rho\, d\rho\, dz, \tag{50}$$

it solves the Euler Lagrange equation

$$\Delta\varphi + \frac{1}{\rho}\partial_\rho\varphi = -\frac{1}{\rho^2}\sin(2\varphi).$$

Using also $\Delta\varphi_0 = 0$, rescaled elliptic estimates enable us to obtain a stronger control on $\varphi - \tau\varphi_0$ in the annular domain

$$A_\lambda = \left\{\frac{\lambda}{2} \le r \le 2\lambda\right\} \cap D.$$

Specifically, elliptic estimates in the rescaled domain $\lambda^{-1}A_\lambda$ (which is Lipschitz independently of λ) give control on the following scale invariant quantity:

$$\begin{aligned}g(\lambda) &:= \|\varphi - \tau\varphi_0\|^2_{L^\infty(A_\lambda)} + \|\nabla\varphi - \tau\nabla\varphi_0\|^2_{L^2(A_\lambda)} + \|\nabla^2\varphi - \tau\nabla^2\varphi_0\|^2_{L^1(A_\lambda)} \\ &\le C\left(\lambda^2 + \|\varphi - \tau\varphi_0\|^2_{L^\infty(\{r=\lambda/2\})} + \|\varphi - \tau\varphi_0\|^2_{L^\infty(\{r=2\lambda\})}\right) \\ &\le C\Big(\lambda^2 + \|\varphi(\lambda/2,\cdot) - \tau\varphi_0(\lambda/2,\cdot)\|^2_{H^1(0,\theta_0(\lambda/2))} \\ &\qquad\qquad + \|\varphi(2\lambda,\cdot) - \tau\varphi_0(2\lambda,\cdot)\|^2_{H^1(0,\theta_0(2\lambda))}\Big).\end{aligned}$$

Hence from (49) we deduce

$$\int_0^1 g(\lambda)\frac{d\lambda}{\lambda} < \infty,$$

and there exists $\lambda_n \to 0$ such that $g(\lambda_n) \to 0$. Moreover, by Sobolev embedding, we have

$$\begin{aligned}
&\int_{\lambda/2}^{2\lambda} \|\partial_\theta \varphi(r,\cdot) - \tau \partial_\theta \varphi_0(r,\cdot)\|_{L^\infty(0,\theta_0(r))} \frac{dr}{r} \\
&\leq C \int_{\lambda/2}^{2\lambda} \left(\|\partial_\theta \varphi(r,\cdot) - \tau \partial_\theta \varphi_0(r,\cdot)\|_{L^2(0,\theta_0(r))} + \|\partial_\theta^2 \varphi(r,\cdot) - \tau \partial_\theta^2 \varphi_0\|_{L^1(0,\theta_0(r))} \right) \frac{dr}{r} \\
&\leq C\sqrt{g(\lambda)},
\end{aligned}$$

thus we may find $r_n \in [\lambda_n/2, 2\lambda_n]$ such that

$$\|\partial_\theta \varphi(r_n,\cdot) - \tau \partial_\theta \varphi_0(r_n,\cdot)\|_{L^\infty(0,\theta_0(r_n))} \leq C\sqrt{g(\lambda_n)} \longrightarrow 0.$$

Since $\partial_\theta \varphi_0 = -1$, this implies in particular that $\varphi(r_n,\cdot)$ is strictly monotone for n large enough. ☐

Equipped with Lemma 4.5, we are now in a position to apply an argument similar to the one in [1, Theorem 13]. It enables us to conclude that Z is empty if $\tau = +1$:

Corollary 4.6. *The set of singular points $Z \cap \{z > 1\}$ is empty if $\tau = 1$.*

Proof of Corollary 4.6. Assume that $\tau = +1$. Take $\delta = r_n$ provided by Lemma 4.5. Then $n_3 > 0$ on $\partial D_\delta^{ext} \cap \{\rho > 0\}$. Therefore, the map $\tilde{n} = (n_1, n_2, |n_3|)$ is an admissible comparison map in D_δ^{ext} and has the same energy as n, hence is a minimizer and must be analytic inside D_δ^{ext}. This implies that $n_3 \geq 0$ inside D_δ^{ext}. Since Z corresponds to changes of sign of n_3, we deduce that $Z = \emptyset$. ☐

4.3.2 Ruling out the case of a positively charged ring $\tau = -1$

The end of the proof will consist in ruling out the case $\tau = -1$. This is the most delicate part of the argument, and it has two main steps:

- In the first step we consider the complement of a small region around the ring defect, and show that the energy cost of a point defect away from this neighborhood is a strictly positive quantity of order $O(1)$. This is done in Lemma 4.8, using a variation of the argument already used in Corollary 4.6, but with more precise boundary estimates.

- In the second step we derive a more precise estimate of the energy concentrated in a small region around the ring defect, in order to conclude that the $O(1)$ increase obtained in the first step (away from the ring defect) leads to a strict increase of the total energy. Specifically we show that the core energy of the ring defect is independent of the ring's charge, up to an error of smaller order. To this end, we construct (in Lemma 4.10) a comparison map which modifies boundary values in a singular region between the particle and a curve tangent to it; the error thus introduced can be controlled thanks to Lemma 3.9.

For later use, we derive the following useful estimates, based on those established in the proof of Lemma 4.5; recall $n = (\cos\varphi, 0, \sin\varphi)$, and $\varphi_0 = \frac{\pi}{2} - \theta$ (the lifting of the comparison map n_0). Then:

Lemma 4.7. *As $\delta \to 0$, we have*

$$\|\varphi - \tau\varphi_0\|_{L^\infty(D_\delta^{int})} + \|\nabla\varphi - \tau\nabla\varphi_0\|_{L^2(D_\delta^{int})} \longrightarrow 0,$$

where $D_\delta^{int} = D \setminus D_\delta^{ext} = \{(\rho, z) \in D : \ (\rho-1)^2 + z^2 \leq \delta^2\}$.

That is, as we approach the ring defect, the director n resembles $(\tau \sin\theta, 0, \cos\theta)$, a symmetric vortex of degree $-\tau/2$.

Proof of Lemma 4.7. It is shown in Lemma 4.5 that

$$f(\lambda) = \|\varphi - \tau\varphi_0\|^2_{L^\infty(A_\lambda)} + \|\nabla\varphi - \tau\nabla\varphi_0\|^2_{L^2(A_\lambda)},$$

satisfies

$$\int_0^1 f(\lambda)\frac{d\lambda}{\lambda} = \sum_{n=0}^\infty \int_{1/2}^1 f(2^{-n}\lambda)\frac{d\lambda}{\lambda} < \infty.$$

We may thus pick $\lambda_n \in [1/2, 1]$ such that

$$\sum_{n=0}^\infty f(2^{-n}\lambda_n) < \infty.$$

Since $A_{(2^{-n-1}\lambda_{n+1})}$ overlaps with $A_{(2^{-n}\lambda_n)}$, we deduce that

$$\begin{aligned}&\|\varphi - \tau\varphi_0\|^2_{L^\infty(D\setminus D^{ext}_{2^{-N}})} + \|\nabla\varphi - \tau\nabla\varphi_0\|^2_{L^2(D\setminus D^{ext}_{2^{-N}})}\\ &\leq \sum_{n=N-1}^\infty f(2^{-n}\lambda_n) \longrightarrow 0,\end{aligned}$$

as $N \to \infty$. □

In order to rule out the case $\tau = -1$, we wish to estimate the difference of total energy between the two cases $\tau = \pm 1$. To this end, we recall the definition (39) of $\theta_0(r)$ and $\theta_1(r)$ in representing the boundary condition on the colloid, and introduce the notation:

$$\begin{aligned}\mathbb{E}^\tau[\delta] = \min\Big\{F(\varphi; D_\delta^{ext})\colon\ & \varphi = \tau\pi/2 \text{ for } \theta = 0,\\ & \varphi(r, \theta_0(r)) = \theta_1(r) \text{ for } \delta < r < \sqrt{2}\\ & \varphi = \phi_0^\tau \text{ for } r = \delta\Big\},\end{aligned}$$

$$\phi_0^\tau(r,\theta) = \tau\frac{\pi}{2} - \lambda^\tau(r)\theta, \quad \lambda^\tau(r) = \frac{\tau\pi/2 - \theta_1(r)}{\theta_0(r)}. \tag{51}$$

Note that ϕ_0^τ is the phase corresponding to the constant speed geodesic m_0 used in the proof of Lemma 4.4. Then we compare $\mathbb{E}^+[\delta]$ and $\mathbb{E}^-[\delta]$ for small δ:

Lemma 4.8. *we have*

$$\limsup_{\delta\to 0} \left(\mathbb{E}^+[\delta] - \mathbb{E}^-[\delta]\right) < 0.$$

Proof of Lemma 4.8. We denote by φ_δ^τ the minimizer in $\mathbb{E}^\tau[\delta]$. We claim the following upper bound holds:

$$F(\varphi_\delta^\tau; D_\eta^{ext}) \le \frac{\pi}{2}\ln\frac{1}{\eta} + C, \qquad \text{for } 0 < \delta \le \eta \le \sqrt{2}. \tag{52}$$

Indeed, using a comparison function $\tilde{\varphi}$ satisfying

$$\tilde{\varphi} = \phi_0^\tau \quad \text{for } \delta < r < \sqrt{2}, \qquad \tilde{\varphi} = \tau\frac{\pi}{2} \quad \text{for } r > 2\sqrt{2},$$

and any admissible Lipschitz extension in the region $\sqrt{2} \le r \le 2\sqrt{2}$, we find on the one hand the upper bound

$$F(\varphi_\delta^\tau; D_\delta^{ext}) \le \frac{\pi}{2}\ln\frac{1}{\delta} + C.$$

On the other hand, for $0 < \delta \le \eta \le \sqrt{2}$, the boundary conditions enforce the lower bound

$$F(\varphi_\delta^\tau; D_\delta^{ext} \setminus D_\eta^{ext}) \ge \int_\delta^\eta \int_0^{\theta_0(r)} (\partial_\theta \varphi_\delta^\tau)^2 \, d\theta \, \frac{dr}{r} \ge \frac{\pi}{2}\ln\frac{\eta}{\delta} - C,$$

which, together with the previous upper bound, implies (52).

Arguing as in Lemma 4.1 we thus have, up to extracting a further subsequence, a limit $\varphi_\delta^\tau \to \varphi^\tau$ in $H^1_{loc}(D)$ as $\delta \to 0$, and $F(\varphi_\delta^\tau; D_\eta^{ext}) \to F(\varphi^\tau; D_\eta^{ext})$ for all $\eta > 0$. The function φ^τ satisfies the boundary conditions

$$\varphi = \tau\pi/2 \text{ for } \theta = 0,$$
$$\varphi(r, \theta_0(r)) = \theta_1(r) = 2\theta_0(r) - \pi \text{ for } \delta < r < \sqrt{2},$$

and minimizes $F(\cdot; D_\eta^{ext})$ among functions that agree with φ^τ on $\partial D_\eta^{ext} \cap \{\rho > 0\}$, for all $\eta > 0$. The arguments in Lemma 4.5 and in Lemma 4.7 carry over, and we find that φ^τ is analytic in $\overline{D} \setminus (Z \cup \{(1,0)\})$, where $Z = \emptyset$ if $\tau = +1$, and $Z = \{(0, z_0)\}$ for some $z_0 > 0$ if $\tau = -1$. Moreover we have

$$\|\varphi^\tau - \tau\varphi_0\|_{L^\infty(D_\eta^{int})} + \|\nabla\varphi^\tau - \tau\nabla\varphi_0\|_{L^2(D_\eta^{int})} \to 0 \quad \text{as } \eta \to 0.$$

We set $\psi = |\varphi^-|$, so that

$$F(\psi; D_\delta^{ext}) = F(\varphi^-; D_\delta^{ext}) \quad \text{and } \psi = \varphi^+ \text{ on } \partial D_\delta^{ext} \cap \{\rho > 0\}.$$

for all $\delta > 0$. Next, from the estimates for φ^- and easy estimates on $(|\varphi_0| - \varphi_0)$ and $(\varphi_0 - \phi_0^+)$ we deduce that

$$\|\psi - \phi_0^+\|_{L^\infty(D_\eta^{int})} + \|\nabla\psi - \nabla\phi_0^+\|_{L^2(D_\eta^{int})} \to 0 \quad \text{as } \eta \to 0. \tag{53}$$

Also note that, since φ^- changes sign at one point of $\partial D_\delta^{ext} \cap D$ for small enough δ, ψ can not be locally analytic inside D_δ^{ext}. In particular, it is certainly not a minimizer of $F(\cdot; D_\delta^{ext})$

for any δ small enough. Therefore, for some small fixed δ_0 there exists a function ξ such that $\xi = \psi$ on $\partial D^{ext}_{\delta_0} \cap \{\rho > 0\}$, and

$$\varepsilon := F(\psi; D^{ext}_{\delta_0}) - F(\xi; D^{ext}_{\delta_0}) > 0.$$

Let $\eta < \delta_0/2$. Consistently with its boundary conditions, we may extend ξ to D by setting

$$\xi = \psi \text{ in } D \setminus D^{ext}_{\delta_0}.$$

Next we introduce a modified function ξ_η given by

$$\xi_\eta = \mu\xi + (1-\mu)\phi_0^+,$$
$$\mu = \mu(r) = \begin{cases} 1 & \text{for } r > 2\eta, \\ 0 & \text{for } r < \eta, \\ \frac{1}{\ln 2} \ln \frac{r}{\eta} & \text{for } \eta < r < 2\eta, \end{cases}$$

so that $\xi_\eta = \xi$ in $D^{ext}_{2\eta}$, $\xi_\eta = \varphi^+_\eta$ on $\partial D^{ext}_\eta \cap \{\rho > 0\}$, and

$$\begin{aligned} &\|\xi - \xi_\eta\|^2_{L^\infty(D^{int}_{2\eta})} + \|\nabla\xi - \nabla\xi_\eta\|^2_{L^2(D^{int}_{2\eta})} \\ &\leq \left(1 + 2\int_\eta^{2\eta} (\mu')^2 r\, dr\right) \|\psi - \phi_0^+\|^2_{L^\infty(D^{int}_{2\eta})} + 2\|\nabla\psi - \nabla\phi_0^+\|^2_{L^2(D^{int}_{2\eta})} \\ &= \left(1 + \frac{2}{\ln 2}\right) \|\psi - \phi_0^+\|^2_{L^\infty(D^{int}_{2\eta})} + 2\|\nabla\psi - \nabla\phi_0^+\|^2_{L^2(D^{int}_{2\eta})}. \end{aligned}$$

Thanks to (53), we therefore have a function $R(\eta)$ which tends to zero as $\eta \to 0$, such that

$$\begin{aligned} \varepsilon &= F(\psi; D^{ext}_\eta) - F(\xi; D^{ext}_\eta) = F(\varphi^-; D^{ext}_\eta) - F(\xi; D^{ext}_\eta) \\ &\leq F(\varphi^-; D^{ext}_\eta) - F(\xi_\eta; D^{ext}_\eta) + R(\eta) \\ &\leq F(\varphi^-; D^{ext}_\eta) - \mathbb{E}^+[\eta] + R(\eta). \end{aligned}$$

The last inequality holds by definition of $\mathbb{E}^+$ because $\xi_\eta = \varphi^+_\eta$ on $\partial D^{ext}_\eta \cap \{\rho > 0\}$. Recalling the definition of $\mathbb{E}^-$ and taking the limit as $\eta \to 0$, we find

$$\limsup_{\eta\to 0} \left(\mathbb{E}^+[\eta] - \mathbb{E}^-[\eta]\right) \leq -\varepsilon + \liminf_{\eta\to 0} \left(F(\varphi^-; D^{ext}_\eta) - F(\varphi^-_\eta; D^{ext}_\eta)\right).$$

The lemma will be proven once we show that

$$\limsup_{\eta\to 0} \left(F(\varphi^-; D^{ext}_\eta) - F(\varphi^-_\eta; D^{ext}_\eta)\right) \leq 0. \tag{54}$$

To this end we may, consistently with its boundary conditions, extend φ^-_η to D by setting

$$\varphi^-_\eta = \phi^-_0 \quad \text{in } D^{int}_\eta.$$

We also introduce a parameter $\nu < \eta/2$. We have

$$F(\varphi^-; D^{ext}_\eta) - F(\varphi^-_\eta; D^{ext}_\eta) \leq F(\varphi^-; D^{ext}_\nu) - F(\varphi^-_\eta; D^{ext}_\nu) + CU(\eta),$$

where

$$U(\eta) = \|\varphi^- - \phi_0^-\|_{L^\infty(D_\eta^{int})} + \|\nabla\varphi^- - \nabla\phi_0^-\|_{L^2(D_\eta^{int})} \to 0 \quad \text{as } \eta \to 0.$$

Next we modify φ_η^- in order to use the minimizing property of φ^- in D_ν^{ext}. Similarly to the above definition of ξ_η, we set

$$\tilde{\varphi}_\nu = \mu\varphi_\eta^- + (1-\mu)\varphi^-,$$
$$\mu = \mu(r) = \begin{cases} 1 & \text{for } r > 2\nu, \\ 0 & \text{for } r < \nu, \\ \frac{1}{\ln 2}\ln\frac{r}{\nu} & \text{for } \nu < r < 2\nu, \end{cases}$$

so that $\tilde{\varphi}_\nu = \varphi_\eta^-$ in $D_{2\nu}^{ext}$, and $\tilde{\varphi}_\nu = \varphi^-$ on $\partial D_\nu^{ext} \cap \{\rho > 0\}$. Moreover, since $\varphi_\eta^- = \phi_0^-$ in $D_{2\nu}^{int}$, we have

$$\begin{aligned} &F(\varphi_\eta^-; D_\nu^{ext}) - F(\tilde{\varphi}_\nu; D_\nu^{ext}) \\ &\le C\left(\|\varphi^- - \phi_0^-\|_{L^\infty(D_{2\nu}^{int})} + \|\nabla\varphi^- - \nabla\phi_0^-\|_{L^2(D_{2\nu}^{int})}\right) \\ &\le CU(\eta). \end{aligned}$$

We deduce

$$\begin{aligned} F(\varphi^-; D_\eta^{ext}) - F(\varphi_\eta^-; D_\eta^{ext}) &\le F(\varphi^-; D_\nu^{ext}) - F(\tilde{\varphi}_\nu; D_\nu^{ext}) + CU(\eta) \\ &\le CU(\eta). \end{aligned}$$

The last inequality holds because $\tilde{\varphi}_\nu = \varphi^-$ on $\partial D_\nu^{ext} \cap \{\rho > 0\}$ and φ^- is minimizing in D_ν^{ext}. This obviously implies (54). □

We would like to use Lemma 4.8 to show that τ must be $+1$. From now on we assume that $\tau = -1$. We will then construct a map $P_\xi \in \mathcal{H}_{sym}$ with lower energy than Q_ξ, hence contradicting the minimality of Q_ξ and proving that $\tau = +1$ and $Z = \emptyset$. We introduce the notations

$$n_\delta^\pm = (\cos\varphi_\delta^\pm, 0, \sin\varphi_\delta^\pm),$$
$$n_0^\pm = (\cos\phi_0^\pm, 0, \sin\phi_0^\pm),$$

where $\varphi_\delta^\pm$ are the minimizers corresponding to the minimization problems $\mathbb{E}^\pm[\delta]$. Note in particular that $n_\delta^\pm = n_0^\pm$ for $r = \delta$.

First we show that we may, without messing too much with the energy of $\widetilde{Q}_\xi$ inside D_δ^{int}, replace it with a map that equals $(n_0^- \otimes n_0^- - \frac{1}{3}I)$ on $D \cap \partial D_\delta^{ext}$.

Lemma 4.9. *Consider $\widetilde{R}_\xi \colon D_\delta^{int} \to \mathcal{S}_0$ minimizing $\widetilde{E}_\xi(\cdot; D_\delta^{int})$ among all maps R with the boundary constraints*

$$R = e_r \otimes e_r - \frac{1}{3}I \quad \textit{for } \rho^2 + z^2 = 1,$$
$$R = n_0^- \otimes n_0^- - \frac{1}{3}I \quad \textit{for } r = \delta,$$
$$R = SRS^t \quad \textit{for } z = 0.$$

Then we have that

$$\widetilde{E}_\xi(\widetilde{R}_\xi; D_\delta^{int}) \le \widetilde{E}_\xi(\widetilde{Q}_\xi; D_\delta^{int}) + \sigma_1(\delta,\xi) + \zeta_1(\delta),$$

where $\zeta_1(\delta) \to 0$ *as* $\delta \to 0$, *and* $\sigma_1(\delta,\xi) \to 0$ *as* $\xi \to 0$ *for all fixed* $\delta > 0$.

Proof of Lemma 4.9. We construct a test configuration R_ξ in D_δ^{int} and evaluate its energy $\widetilde{E}_\xi$. We systematically denote by $\sigma(\delta,\xi)$ (resp. $\zeta(\delta)$) functions that tend to 0 as $\xi \to 0$ for any fixed δ (resp. $\delta \to 0$), although they may change from one line to another.

By Fubini's theorem, we may choose $\hat{\delta} \in (\frac{\delta}{4}, \frac{\delta}{3})$ such that (possibly along a subsequence)

$$\widetilde{E}_\xi(\widetilde{Q}_\xi; D \cap \partial D_{\hat{\delta}}^{ext}) + \widetilde{E}_\star(\widetilde{Q}_\star; D \cap \partial D_{\hat{\delta}}^{ext}) \lesssim \frac{1}{\delta}\left(\widetilde{E}_\xi(\widetilde{Q}_\xi; D_{\frac{\delta}{4}}^{ext}) + \widetilde{E}_\star(\widetilde{Q}_\star; D_{\frac{\delta}{4}}^{ext})\right).$$

Arguing exactly as in the proof of (33) in Lemma 4.1, this enables us to construct R_ξ in $D_{\hat{\delta}}^{ext} \setminus D_{\delta/2}^{ext}$ such that R_ξ satisfies the boundary conditions of $\widetilde{R}_\xi$ for $z = 0$ and $\rho^2 + z^2 = 1$, and

$$R_\xi = \widetilde{Q}_\xi \quad \text{for } r = \hat{\delta}, \qquad R_\xi = \widetilde{Q}_\star \quad \text{for } r = \frac{\delta}{2},$$

$$\text{and } \sigma(\delta,\xi) = \widetilde{E}_\xi(R_\xi; D_{\hat{\delta}}^{ext} \setminus D_{\delta/2}^{ext}) - \widetilde{E}_\star(\widetilde{Q}_\star; D_{\hat{\delta}}^{ext} \setminus D_{\delta/2}^{ext}) \longrightarrow 0 \quad \text{as } \xi \to 0,$$

for any fixed δ. In $D_{\hat{\delta}}^{int}$ we set $R_\xi = \widetilde{Q}_\xi$, so that the above estimate combined with Lemma 4.1 implies

$$\widetilde{E}_\xi(R_\xi; D_{\frac{\delta}{2}}^{int}) \le \widetilde{E}_\xi(\widetilde{Q}_\xi; D_{\frac{\delta}{2}}^{int}) + \sigma(\delta,\xi).$$

Finally, we interpolate in $\mathcal{U}_\star$ between $\widetilde{Q}_\star = n \otimes n - \frac{1}{3}I$ and $Q_0^- := n_0^- \otimes n_0^- - \frac{1}{3}I$ to define R_ξ in the remaining region $D_{\delta/2}^{ext} \setminus D_\delta^{ext}$. This we do via the phase variables φ and ϕ_0^- which satisfy $n = (\cos\varphi, 0, \sin\varphi)$ and $n_0^- = (\cos\phi_0^-, 0, \sin\phi_0^-)$: we set

$$\hat{\phi}(r,\theta) = \frac{2}{\delta}\left(r - \frac{\delta}{2}\right)\varphi + \frac{2}{\delta}(\delta - r)\,\phi_0^-, \qquad \frac{\delta}{2} < r < \delta,\ 0 < \theta < \theta_0(r).$$

This defines a director $\hat{n} := (\cos\hat{\phi}, 0, \sin\hat{\phi})$ and an associated uniaxial Q-tensor $R_\xi = \hat{n} \otimes \hat{n} - \frac{1}{3}I$. In this way, R_ξ will be continuous in D_δ^{int} and satisfy each of the desired conditions on $\partial(D_\delta^{int})$. As $R_\xi \in \mathcal{U}_\star$ in this region, $f(R_\xi) = 0$ and moreover,

$$\widetilde{E}_\xi(R_\xi; D_{\delta/2}^{ext} \setminus D_\delta^{ext}) = 2\hat{E}(\hat{n}; D_{\delta/2}^{ext} \setminus D_\delta^{ext}) = 2F(\hat{\phi}; D_{\delta/2}^{ext} \setminus D_\delta^{ext}).$$

Thanks to Lemma 4.7 and the explicit form of φ_0 (48) and ϕ_0^- (51), we have

$$\|\varphi - \phi_0^-\|_{H^1 \cap L^\infty(D_\delta^{int})} \longrightarrow 0, \qquad \text{as } \delta \to 0, \tag{55}$$

and use this fact to estimate the energy of R_ξ in $D_{\delta/2}^{ext} \setminus D_\delta^{ext}$. For instance, we have

$$\begin{aligned}\partial_r\hat{\phi}(r,\theta) &= \frac{2}{\delta}\left(r - \frac{\delta}{2}\right)\partial_r\varphi + \frac{2}{\delta}(\delta - r)\,\partial_r\phi_0^- + \frac{2}{\delta}(\varphi - \phi_0^-)\\ &= \partial_r\phi_0^- + \left\{\frac{2}{\delta}\left(r - \frac{\delta}{2}\right)[\partial_r\phi_0^- - \partial_r\varphi] + \frac{2}{\delta}(\varphi - \phi_0^-)\right\},\end{aligned}$$

and similarly,

$$\partial_\theta \hat{\phi}(r,\theta) = \partial_\theta \phi_0^- + \left\{ \frac{2}{\delta}\left(r - \frac{\delta}{2}\right)[\partial_\theta \phi_0^- - \partial_\theta \varphi] \right\}.$$

The estimate (55) ensures that the bracketed terms on the right-hand side of each of the above equations tend to zero in $L^2(D^{ext}_{\delta/2} \setminus D^{ext}_\delta)$ as $\delta \to 0$. As a consequence we have

$$\begin{aligned}\int_{D^{ext}_{\delta/2}\setminus D^{ext}_\delta} |\nabla \hat{\phi}|^2 \,\rho\, d\rho\, dz &\leq \int_{D^{ext}_{\delta/2}\setminus D^{ext}_\delta} |\nabla \phi_0^-|^2 \,\rho\, d\rho\, dz + \zeta(\delta) \\ &\leq \int_{D^{ext}_{\delta/2}\setminus D^{ext}_\delta} |\nabla \varphi|^2 \,\rho\, d\rho\, dz + \zeta(\delta),\end{aligned}$$

where $\zeta(\delta) \to 0$ as $\delta \to 0$, and we used again (55) for the last inequality. As

$$\int_{D^{ext}_{\delta/2}\setminus D^{ext}_\delta} \frac{\hat{n}_1^2}{\rho} d\rho\, dz = O(\delta^2),$$

we deduce that

$$F(\hat{\phi}; D^{ext}_{\delta/2} \setminus D^{ext}_\delta) \leq F(\varphi; D^{ext}_{\delta/2} \setminus D^{ext}_\delta) + \zeta(\delta).$$

Finally,

$$\begin{aligned}\widetilde{E}_\xi(R_\xi; D^{ext}_{\delta/2} \setminus D^{ext}_\delta) = 2F(\hat{\phi}; D^{ext}_{\delta/2} \setminus D^{ext}_\delta) &\leq 2[F(\varphi_*) + \zeta(\delta)] \\ &\leq \widetilde{E}_\xi(\widetilde{Q}_\xi; D^{ext}_{\delta/2} \setminus D^{ext}_\delta) + \zeta(\delta) + \sigma(\delta,\xi),\end{aligned}$$

by Lemma 4.1. Combined with the above estimate in $D^{int}_{\delta/2}$ this yields

$$\widetilde{E}_\xi(R_\xi; D^{int}_\delta) \leq \widetilde{E}_\xi(\widetilde{Q}_\xi; D^{int}_\delta) + \zeta(\delta) + \sigma(\delta,\xi),$$

and since $\widetilde{R}_\xi$ minimizes $\widetilde{E}_\xi$ with the same boundary conditions as R_ξ this completes the proof of Lemma 4.9. □

The final step consists in proving that we may "transform" the boundary conditions on $D \cap \partial D^{ext}_\delta$ from n_0^- to n_0^+ without increasing the energy too much. This establishes the crucial core energy estimate mentioned in the Introduction: that the energy of a positively or negatively charged line defect is the same up to $o(1)$.

Lemma 4.10. *Consider $\widetilde{P}_\xi \colon D^{int}_\delta \to \mathcal{S}_0$ minimizing $\widetilde{E}_\xi(\cdot; D^{int}_\delta)$ among all maps P with the boundary constraints*

$$P = e_r \otimes e_r - \frac{1}{3}I \quad \text{for } \rho^2 + z^2 = 1,$$
$$P = n_0^+ \otimes n_0^+ - \frac{1}{3}I \quad \text{for } r = \delta,$$
$$P = SPS^t \quad \text{for } z = 0.$$

Then

$$\widetilde{E}_\xi(\widetilde{P}_\xi; D^{int}_\delta) \leq \widetilde{E}_\xi(\widetilde{R}_\xi; D^{int}_\delta) + \zeta_3(\delta),$$

where $\zeta_3(\delta) \to 0$ as $\delta \to 0$.

Proof of Lemma 4.10. We define a map $\widehat{P}_\xi$ satisfying the boundary condition of $\widetilde{P}_\xi$ and an adequate upper bound on its energy. We do this in two steps: first we define $\widehat{P}_\xi$ in the domain

$$X_\delta := \{0 < \theta < \frac{\pi}{2} - r^{\frac{1}{2}}\} \cap \{0 < r < \delta - \delta^{\frac{3}{2}}\},$$

(see Figure 2) as the reflected map $S\widetilde{R}_\xi S$, appropriately rescaled to "fit" into this smaller domain, and then define $\widehat{P}_\xi$ on the remaining part by interpolating in $\mathcal{U}_\star$ between the boundary values of $S\widetilde{R}_\xi S$ and the boundary values of $\widetilde{P}_\xi$.

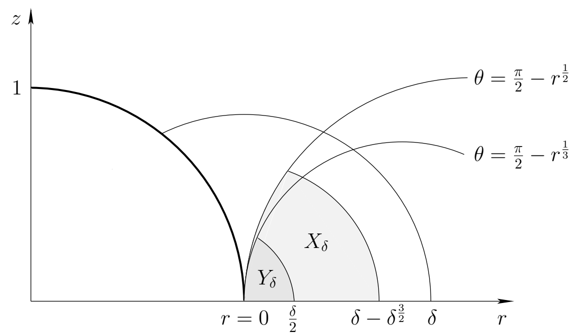


Figure 2: The domain X_δ and its subdomain Y_δ.

For the first step we start by defining a bi-Lipschitz change of variables which transforms X_δ into

$$D_\delta^{int} = \{0 < \theta < \frac{\pi}{2} + \arcsin\frac{r}{2}\} \cap \{0 < r < \delta\},$$

and keeps the subdomain

$$Y_\delta := \{0 < \theta < \frac{\pi}{2} - r^{\frac{1}{3}}\} \cap \{0 < r < \frac{\delta}{2}\}$$

fixed. Explicitly, we set

$$\begin{aligned}
\Phi(r,\delta) &= (r + g_1(r), \theta + g_2(r,\theta)),\\
g_1, g_2 &\equiv 0 \qquad \text{in } Y_\delta,\\
g_1(r) &= 2\delta^{\frac{1}{2}}\frac{1}{1-2\delta^{\frac{1}{2}}}(r - \frac{\delta}{2}) \qquad \text{for } \frac{\delta}{2} < r < \delta - \delta^{\frac{2}{3}},\\
g_2(r,\theta) &= r^{\frac{1}{6}}\frac{1 + r^{-\frac{1}{2}}\arcsin\frac{r}{2}}{1 - r^{\frac{1}{6}}}(\theta - \frac{\pi}{2} + r^{\frac{1}{3}}) \qquad \text{for } \frac{\pi}{2} - r^{\frac{1}{3}} < \theta < \frac{\pi}{2} + \arcsin\frac{r}{2}.
\end{aligned}$$

Direct computations show that Φ is one-to-one from X_δ into D_δ^{int}, that $\Phi = id$ in Y_δ, and that

$$|\det(D\Phi) - 1| + |g_1'| + |\partial_\theta g_2| + r|\partial_r g_2| \lesssim \delta^{\frac{1}{6}}.$$

Hence for any function $u(r,\theta)$, the function $\tilde{u} = u \circ \Phi$ satisfies

$$
\begin{aligned}
|\nabla \tilde{u}|^2 &= |\partial_r \tilde{u}|^2 + \frac{1}{r^2}|\partial_\theta \tilde{u}|^2 \\
&\le [(1+|g_1'|)|\partial_r u| \circ \Phi + |\partial_r g_2||\partial_\theta u| \circ \Phi]^2 + (1+|\partial_\theta g_2|)^2 |\partial_\theta u|^2 \circ \Phi \\
&\le (1 + C\delta^{\frac{1}{6}})|\nabla u|^2 \circ \Phi,
\end{aligned}
$$

for some constant $C > 0$. Therefore, setting

$$
\widehat{P}_\xi = S\widetilde{R}_\xi S \circ \Phi \qquad \text{in } X_\delta,
$$

and recalling that Φ is the identity in Y_δ, we have

$$
\begin{aligned}
\widetilde{E}_\xi(\widehat{P}_\xi; X_\delta) &= \widetilde{E}_\xi(\widehat{P}_\xi; Y_\delta) + \widetilde{E}_\xi(\widehat{P}_\xi; X_\delta \setminus Y_\delta) \\
&\le \widetilde{E}_\xi(\widetilde{R}_\xi; Y_\delta) + (1 + C\delta^{\frac{1}{6}})\widetilde{E}_\xi(\widetilde{R}_\xi; D_\delta^{int} \setminus Y_\delta) \\
&\le \widetilde{E}_\xi(\widetilde{R}_\xi; D_\delta^{int}) + C\delta^{\frac{1}{6}}\widetilde{E}_\xi(\widetilde{R}_\xi; D_\delta^{int} \setminus Y_\delta).
\end{aligned}
$$

Next, note that the proof of upper and lower bound in Sections 2 and 3 can be adapted to prove similar bounds on $\widetilde{R}_\xi$. Moreover, thanks to Remark 3.10 and the inclusion

$$
Y_\delta \supset \{r \le \frac{\delta}{2}\} \cap \{\rho \ge 1 + z^{\frac{4}{3}}\},
$$

the lower bound can actually be obtained in Y_δ. As a consequence we have the upper bound

$$
\widetilde{E}_\xi(\widetilde{R}_\xi; D_\delta^{int} \setminus Y_\delta) \le C \ln \frac{1}{\delta},
$$

and deduce that

$$
\widetilde{E}_\xi(\widehat{P}_\xi; X_\delta) \le \widetilde{E}_\xi(\widetilde{R}_\xi; D_\delta^{int}) + C\delta^{\frac{1}{6}} \ln \frac{1}{\delta} \tag{56}
$$

On $\partial X_\delta \cap \{z = 0\}$, $\widehat{P}_\xi$ satisfies the boundary condition $S\widehat{P}_\xi S = \widehat{P}_\xi$, since $\widetilde{R}_\xi$ satisfies it as well.

Finally, we define $\widehat{P}_\xi$ in the region $D_\delta^{int} \setminus X_\delta$ to satisfy the desired boundary conditions, via interpolation in this thin region of width $O(\delta^{3/2})$. We decompose (up to sets of measure zero),

$$
D_\delta^{int} \setminus X_\delta = Z_\delta^1 \cup Z_\delta^2 \cup Z_\delta^3,
$$

using the arcs $\{r = \delta - \delta^{3/2},\ \theta \in (\theta_0(r), \underline{\theta}(r))\}$ and $\{\theta = \underline{\theta}(r),\ r \in (\delta - \delta^{3/2}, \delta)\}$, where we denote $\underline{\theta}(r) := \frac{\pi}{2} - r^{1/2}$. Explicitly, we set

$$
\begin{aligned}
Z_\delta^1 &= \{\theta \in (\underline{\theta}(r), \theta_0(r)),\ 0 < r \le \delta - \delta^{3/2}\}, \\
Z_\delta^2 &= \{0 < \theta < \underline{\theta}(r),\ r \in (\delta - \delta^{3/2}, \delta)\}, \\
Z_\delta^3 &= \{r \in (\delta - \delta^{3/2}, \delta),\ \theta \in (\underline{\theta}(r), \theta_0(r))\}
\end{aligned}
$$

As the boundary data are all taken with values in $\mathcal{U}_*$, we may define $\widehat{P}_\xi = \hat{n}\otimes\hat{n} - \frac{1}{3}I$, $\hat{n} = (\cos\hat{\phi}, 0, \sin\hat{\phi})$, by specifying its phase $\hat{\phi}$. Similarly, we define the boundary data for R_ξ in terms of a director characterized by its phase, $R_\xi|_{\partial X_\delta} = n_R\otimes n_R - \frac{1}{3}I$, $n_R = (\cos\varphi, 0, \sin\varphi)$, with $\varphi = \underline{\varphi}(r)$ on $\underline{\Gamma} = \{\theta = \underline{\theta}(r),\ r\in(0,\delta-\delta^{3/2})\}$, (corresponding to $n = Se_r\circ\Phi$,) and $\varphi = \overline{\varphi}(\theta) = -\phi_0^-\circ\Phi$ on $\overline{\Gamma} = \{r = \delta-\delta^{3/2},\ \theta\in(0,\underline{\theta}(r))\}$.

In $Z_\delta^1 = \{\theta\in(\underline{\theta}(r),\theta_0(r)),\ 0<r\le\delta-\delta^{3/2}\}$, we interpolate in $\theta\in(\underline{\theta}(r),\theta_0(r))$ for each fixed r:

$$\hat{\phi}(r,\theta) = \frac{\theta_0(r)-\theta}{\theta_0(r)-\underline{\theta}(r)}\underline{\varphi}(r) + \frac{\theta-\underline{\theta}(r)}{\theta_0(r)-\underline{\theta}(r)}\theta_1(r),$$

where we recall that $\theta_1(r) = 2\theta_0(r)-\pi$ gives the Dirichlet condition along the circle $\rho^2+z^2 = 1$. As $\underline{\varphi}(r)-\theta_1(r) = O(r)$, we calculate

$$(\partial_r\hat{\phi})^2 + \frac{1}{r^2}(\partial_\theta\hat{\phi})^2 = O(r^{-2}), \tag{57}$$

and hence

$$\widetilde{E}_\xi(\widehat{P}_\xi; Z_\delta^1) = 2F(\hat{\phi}; Z_\delta^1) \lesssim \delta^{1/2}.$$

In $Z_\delta^2 = \{0<\theta<\underline{\theta}(r),\ r\in(\delta-\delta^{3/2},\delta)\}$ we set

$$\hat{\phi}(r,\theta) = \frac{\delta-r}{\delta^{3/2}}\overline{\varphi}(\theta) + \frac{r-\delta-\delta^{3/2}}{\delta^{3/2}}\phi_0^+(\theta,\delta).$$

As the phase difference $|\overline{\varphi}(\theta)-\phi_0^+(\theta,\delta)| = O(\delta)$, we again may estimate the gradient as in (57) to obtain

$$\widetilde{E}_\xi(\widehat{P}_\xi; Z_\delta^2) = 2F(\hat{\phi}; Z_\delta^2) \lesssim \delta^{1/2}.$$

Lastly, we consider the domain $Z_\delta^3 = \{r\in(\delta-\delta^{3/2},\delta),\ \theta\in(\underline{\theta}(r),\theta_0(r))\}$, for which $\hat{\phi}$ has already been defined on ∂Z_δ^3 via the previous two steps. Indeed, $\hat{\phi}|_{\partial Z_\delta^3} = \frac{\pi}{2} + h_\delta$, for h_δ Lipschitz continuous, with sup norm of order δ, and $|\partial_\tau h_\delta| = O(\delta^{-1/2})$ on each edge. Define v_δ as the minimizer of the Dirichlet energy $\int_{Z_\delta^3}|\nabla v|^2$ with $v|_{\partial Z_\delta^3} = h_\delta$. The domain Z_δ^3 is nearly square, with side length $\delta^{3/2}$; indeed, after rescaling lengths by $\delta^{3/2}$, $\delta^{-3/2}Z_\delta^3$ approaches the unit square as $\delta\to 0$. In particular elliptic estimates give

$$\int_{Z_\delta^3}|\nabla v_\delta|^2\,dx \lesssim \|h_\delta\|_{H^{1/2}(\partial Z_\delta^3)}^2 \lesssim \|\partial_\tau h_\delta\|_{L^2(\partial Z_\delta^3)}\|h_\delta\|_{L^2(\partial Z_\delta^3)} \lesssim \delta^2.$$

Setting $\hat{\phi} = \frac{\pi}{2} + v_\delta$ in Z_δ^3, we then have that

$$\widetilde{E}_\xi(P_\xi; Z_\delta^3) = 2F(\hat{\phi}; Z_\delta^3) = 2\int_{Z_\delta^3}\left[|\nabla v_\delta|^2 + \frac{\sin^2 v_\delta}{\rho^2}\right] = O(\delta^2).$$

Together with the previous two constructions, we have defined $\widehat{P}_\xi$ in all D_δ^{int}, satisfying the desired boundary conditions, with

$$\widetilde{E}_\xi(\widehat{P}_\xi; D_\delta^{int}) = \widetilde{E}_\xi(\widehat{P}_\xi; X_\delta) + O(\delta^{1/2}) \le \widetilde{E}_\xi(\widetilde{R}_\xi; D_\delta^{int}) + O(\delta^{\frac{1}{6}}\ln\frac{1}{\delta}),$$

by (56). Using $\widehat{P}_\xi$ as a comparison map thus proves Lemma 4.10. □

To conclude, we extend $\widetilde{P}_\xi$ to D_δ^{ext} by setting

$$P_\xi = n_\delta \otimes n_\delta - \frac{1}{3}I \qquad \text{in } D_\delta^{ext}.$$

From the estimates in Lemma 4.9 and 4.10 we have that

$$\widetilde{E}_\xi(\widetilde{P}_\xi) \le \widetilde{E}_\xi(\widetilde{Q}_\xi) + \widetilde{E}_\xi(\widetilde{P}_\xi; D_\delta^{ext}) - E_\xi(\widetilde{Q}_\xi; D_\delta^{ext}) + \sigma_4(\delta,\xi) + \zeta_4(\delta),$$

where $\zeta_4(\delta) \to 0$ as $\delta \to 0$, and $\sigma_4(\delta,\xi) \to 0$ as $\xi \to 0$ for all fixed $\delta > 0$. Recalling from the definition of n_δ that $\widetilde{E}_\xi(\widetilde{P}_\xi; D_\delta^{ext}) = 2\mathbb{E}^+[\delta]$, we also have that

$$\begin{aligned}\frac{1}{2}\widetilde{E}_\xi(\widetilde{P}_\xi; D_\delta^{ext}) - \frac{1}{2}\widetilde{E}_\xi(\widetilde{Q}_\xi; D_\delta^{ext}) &= \mathbb{E}_+[\delta] - \mathbb{E}_-[\delta] + \sigma_5(\delta,\xi) + \zeta_5(\delta), \\ \sigma_5(\delta,\xi) &= \frac{1}{2}\widetilde{E}_\star(\widetilde{Q}_\star; D_\delta^{ext}) - \frac{1}{2}\widetilde{E}_\xi(\widetilde{Q}_\xi; D_\delta^{ext}), \\ \zeta_5(\delta) &= \mathbb{E}^-[\delta] - \frac{1}{2}\widetilde{E}_\star(\widetilde{Q}_\star; D_\delta^{ext}) = F(\varphi_\delta^-; D_\delta^{ext}) - F(\varphi; D_\delta^{ext}).\end{aligned}$$

Note that $\sigma_5(\delta,\xi) \to 0$ as $\xi \to 0$, thanks to Lemma 4.1. Since φ minimizes $F(\cdot; D_\eta^{ext})$ for every $\eta > 0$ and satisfies the estimates of Lemma 4.7, one can argue exactly as for (54) to prove that $\max(\zeta_5(\delta), 0) \to 0$ as δ to 0. (In fact similar arguments will show that $\zeta_5(\delta) \to 0$, but here we only need the upper bound.)

Gathering the above estimates and recalling Lemma 4.8, we deduce that

$$\limsup_{\delta\to 0}\limsup_{\xi\to 0}\left[\widetilde{E}_\xi(\widetilde{P}_\xi) - \widetilde{E}_\xi(\widetilde{Q}_\xi)\right] \le 2\limsup_{\delta\to 0}\left(\mathbb{E}^+[\delta] - \mathbb{E}^-[\delta]\right) < 0,$$

and so we can find $\delta, \xi > 0$ such that the map $\widetilde{P}_\xi$ has strictly lower energy than $\widetilde{Q}_\xi$. This contradicts minimality of $\widetilde{Q}_\xi$ and concludes the proof of Theorem 1.2.

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