A high-order well-balanced scheme for the shallow-water equations with topography and Manning friction

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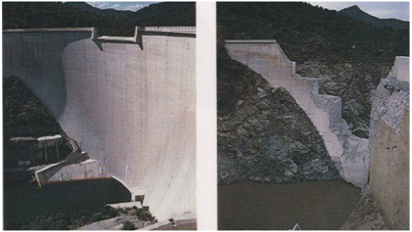
INSTITUT NATIONAL DES SCIENCES APPLIQUÉES TOULOUSE

Several kinds of destructive geophysical flows



Dam failure (Malpasset, France, 1959)

Tsunami (Tōhoku, Japan, 2011)

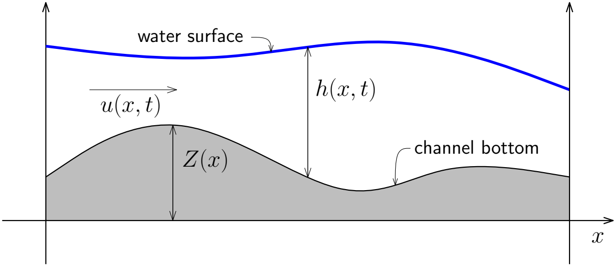
Flood (La Faute sur Mer, France, 2010)

Mudslide (Madeira, Portugal, 2010)

The shallow-water equations and their source terms

$$\begin{cases} \partial_t h + \partial_x(hu) = 0 \\ \partial_t(hu) + \partial_x\left(hu^2 + \dfrac{1}{2}gh^2\right) = -gh\partial_x Z - \dfrac{kq|q|}{h^{7/3}} \end{cases} \text{(with } q = hu\text{)}$$

We can rewrite the equations as $\partial_t W + \partial_x F(W) = S(W)$, with $W = \begin{pmatrix} h \\ q \end{pmatrix}$.



- $Z(x)$ is the known topography
- k is the Manning coefficient
- g is the gravitational constant
- we label the water discharge $q := hu$

Steady state solutions

Definition: Steady state solutions

W is a steady state solution iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$.

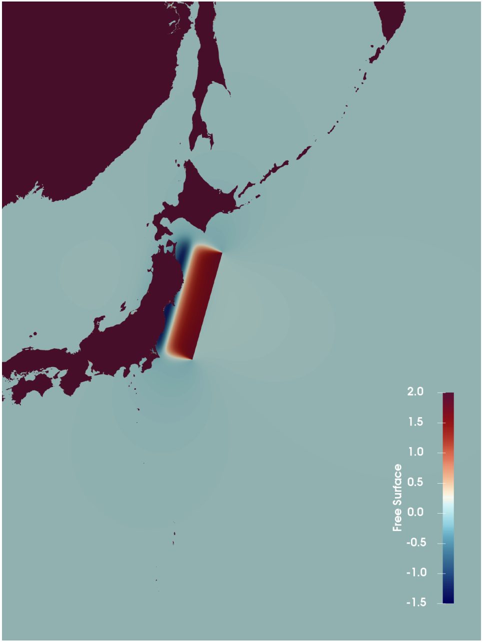
Taking $\partial_t W = 0$ in the shallow-water equations leads to

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^{7/3}}. \end{cases}$$

The steady state solutions are therefore given by

$$\begin{cases} q = \mathrm{cst} = q_0 \\ \partial_x \left(\dfrac{q_0^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq_0|q_0|}{h^{7/3}}. \end{cases}$$

A real-life simulation: the 2011 Tōhoku tsunami.
The water is close to a steady state at rest far from the tsunami.
This steady state is not preserved by a non-well-balanced scheme!



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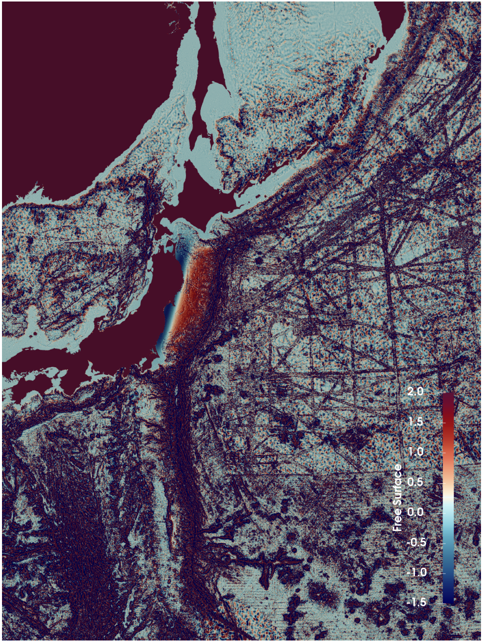


Objectives

Our goal is to derive a numerical method for the shallow-water model with topography and Manning friction that exactly preserves its stationary solutions on every mesh.

To that end, we seek a numerical scheme that:

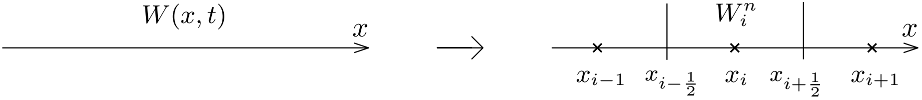
1. is well-balanced for the shallow-water equations with topography and friction, i.e. it exactly preserves and captures the steady states without having to solve the governing nonlinear differential equation;
2. preserves the non-negativity of the water height and handles wet/dry fronts;
3. ensures a discrete entropy inequality;
4. can be easily implemented in an HPC environment.

Setting: finite volume schemes

Objective: Approximate the solution $W(x,t)$ of the system $\partial_t W + \partial_x F(W) = S(W)$, with suitable initial and boundary conditions.

We partition the space domain in *cells*, of volume Δx and of evenly spaced centers x_i, and we define:

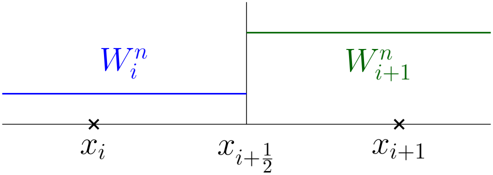
- $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$, the boundaries of the cell i;
- W_i^n, an approximation of $W(x,t)$, constant in the cell i and at time t^n, which satisfies $W_i^n \simeq \dfrac{1}{\Delta x} \displaystyle\int_{\Delta x/2}^{\Delta x/2} W(x,t^n)dx$.



Godunov-type scheme (approximate Riemann solver)

As a consequence, at time t^n, we have a succession of Riemann problems (Cauchy problems with discontinuous initial data) at the interfaces between cells:

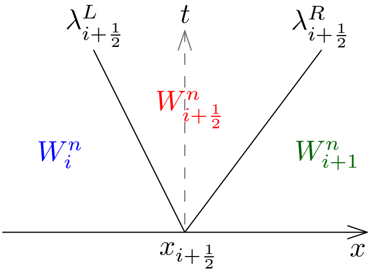
$$\begin{cases} \partial_t W + \partial_x F(W) = S(W) \\ W(x, t^n) = \begin{cases} W_i^n \text{ if } x < x_{i+\frac{1}{2}} \\ W_{i+1}^n \text{ if } x > x_{i+\frac{1}{2}} \end{cases} \end{cases}$$



For $S(W) \neq 0$, the exact solution to these Riemann problems is unknown or costly to compute $\rightsquigarrow$ we require an approximation.

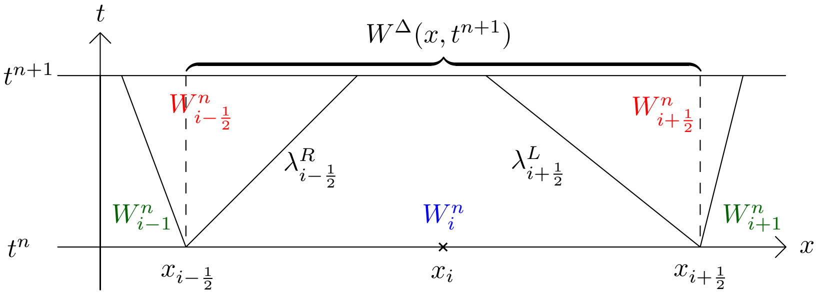
Godunov-type scheme (approximate Riemann solver)

We choose to use an approximate Riemann solver, as follows.



- $W^n_{i+\frac{1}{2}}$ is an approximation of the interaction between W^n_i and W^n_{i+1} (i.e. of the solution to the Riemann problem), possibly made of several constant states separated by discontinuities.
- $\lambda^L_{i+\frac{1}{2}}$ and $\lambda^R_{i+\frac{1}{2}}$ are approximations of the largest wave speeds of the system.

Godunov-type scheme (approximate Riemann solver)



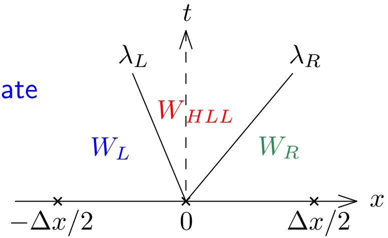
We define the time update as follows:

$$W_i^{n+1} := \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^{\Delta}(x, t^{n+1}) dx.$$

Since $W^n_{i-\frac{1}{2}}$ and $W^n_{i+\frac{1}{2}}$ are made of constant states, the above integral is easy to compute.

The HLL approximate Riemann solver

To approximate solutions of $\partial_t W + \partial_x F(W) = 0$, the HLL approximate Riemann solver (Harten, Lax, van Leer (1983)) may be chosen; it is denoted by W^Δ and displayed on the right.



The consistency condition (as per Harten and Lax) holds if:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W^\Delta(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x; W_L, W_R)dx,$$

which gives $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \begin{pmatrix} h_{HLL} \\ q_{HLL} \end{pmatrix}$.

Note that, if $h_L > 0$ and $h_R > 0$, then $h_{HLL} > 0$ for $|\lambda_L|$ and $|\lambda_R|$ large enough.

Modification of the HLL approximate Riemann solver

The shallow-water equations with the topography and friction source terms read as follows:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) + g h \partial_x Z + \dfrac{k q |q|}{h^{7/3}} = 0. \end{cases}$$

Modification of the HLL approximate Riemann solver

With $Y(t,x) := x$, we can add the equations $\partial_t Z = 0$ and $\partial_t Y = 0$, which correspond to the fixed geometry of the problem:

$$
\begin{cases}
\partial_t h + \partial_x q = 0, \\
\partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) + g h \partial_x Z + \dfrac{k q |q|}{h^{7/3}} \partial_x Y = 0, \\
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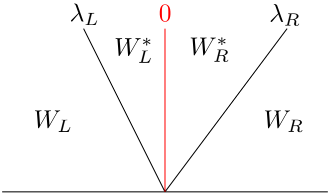
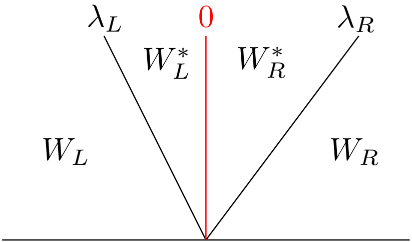
The equations $\partial_t Y = 0$ and $\partial_t Z = 0$ induce stationary waves associated to the source term (of which q is a Riemann invariant).

To approximate solutions of $\partial_t W + \partial_x F(W) = S(W)$, we thus use the approximate Riemann solver displayed on the right (assuming $\lambda_L < 0 < \lambda_R$).



Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q_L^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q_R^* \end{pmatrix}$.



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  - $$q^* = q_{HLL} + \frac{1}{\lambda_R - \lambda_L} \frac{1}{\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} S(W_{\mathcal{R}}(x,t))\, dt\, dx$$

    then $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$ (relation 3),

    where $\overline{S} \simeq \dfrac{1}{\Delta x} \dfrac{1}{\Delta t} \displaystyle\int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} S(W_{\mathcal{R}}(x,t))\, dt\, dx$

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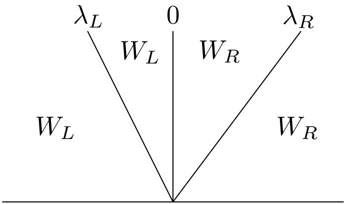
- **next step**: obtain a fourth relation

Obtaining an additional relation

Assume that W_L and W_R define a steady state, i.e. that they satisfy the following discrete version of the steady relation $\partial_x F(W) = S(W)$ (where $[X] = X_R - X_L$):

$$\frac{1}{\Delta x}\left(q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right]\right) = \overline{S}.$$

For the steady state to be preserved, it is sufficient to have $h_L^* = h_L$, $h_R^* = h_R$ and $q^* = q_0$.



Assuming a steady state, we show that $q^* = q_0$, as follows:

$$q^* = q_{HLL} + \frac{\overline{S}\Delta x}{\lambda_R - \lambda_L} = q_0 - \frac{1}{\lambda_R - \lambda_L}\left(q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] - \overline{S}\Delta x\right) = q_0.$$

Obtaining an additional relation

In order to determine an additional relation, we consider the discrete steady relation, satisfied when W_L and W_R define a steady state:

$$q_0^2 \left(\frac{1}{h_R} - \frac{1}{h_L} \right) + \frac{g}{2}\left((h_R)^2 - (h_L)^2 \right) = \overline{S} \Delta x.$$

To ensure that $h_L^* = h_L$ and $h_R^* = h_R$, we impose that h_L^* and h_R^* satisfy the above relation, as follows:

$$q_0^2 \left(\frac{1}{h_R^*} - \frac{1}{h_L^*} \right) + \frac{g}{2}\left((h_R^*)^2 - (h_L^*)^2 \right) = \overline{S} \Delta x.$$

Determination of h_L^* and h_R^*

The intermediate water heights satisfy the following relation:

$$-q_0^2 \left(\frac{h_R^* - h_L^*}{h_L^* h_R^*} \right) + \frac{g}{2}(h_L^* + h_R^*)(h_R^* - h_L^*) = \overline{S}\Delta x.$$

Recall that q^* is known and is equal to q_0 for a steady state.
Instead of the above relation, we choose the following linearization:

$$\frac{-(q^*)^2}{h_L h_R}(h_R^* - h_L^*) + \frac{g}{2}(h_L + h_R)(h_R^* - h_L^*) = \overline{S}\Delta x,$$

which can be rewritten as follows:

$$\underbrace{\left(\frac{-(q^*)^2}{h_L h_R} + \frac{g}{2}(h_L + h_R) \right)}_{\alpha}(h_R^* - h_L^*) = \overline{S}\Delta x.$$

Determination of h_L^* and h_R^*

With the consistency relation between h_L^* and h_R^*, the intermediate water heights satisfy the following linear system:

$$\begin{cases} \alpha(h_R^* - h_L^*) = \overline{S}\Delta x, \\ \lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}. \end{cases}$$

Using both relations linking h_L^* and h_R^*, we obtain

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \end{cases}$$

where $\alpha = \left(\dfrac{-(q^*)^2}{h_L h_R} + \dfrac{g}{2}(h_L + h_R) \right)$ with $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$.

Correction to ensure non-negative h_L^* and h_R^*

However, these expressions of h_L^* and h_R^* do not guarantee that the intermediate heights are non-negative: instead, we use the following cutoff (see Audusse, Chalons, Ung (2015)):

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right). \end{cases}$$

Note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$;
- the well-balance property, since it is not activated when W_L and W_R define a steady state.

Summary

The two-state approximate Riemann solver with intermediate states $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$ given by

$$\begin{cases} q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}, \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL} \right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL} \right), \end{cases}$$

is consistent, non-negativity-preserving, entropy preserving and well-balanced.

next step: determination of $\overline{S}$ according to the source term definition (topography or friction).

The topography source term

We now consider $S(W) = S^t(W) = -gh\partial_x Z$:
the smooth steady states are governed by

$$\left.\begin{aligned} \partial_x\left(\frac{q_0^2}{h}\right) + \frac{g}{2}\partial_x(h^2) &= -gh\partial_x Z, \\ \frac{q_0^2}{2}\partial_x\left(\frac{1}{h^2}\right) + g\partial_x(h+Z) &= 0, \end{aligned}\right\} \xrightarrow[\text{discretization}]{} \left\{\begin{aligned} q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] &= \overline{S}^t\Delta x, \\ \frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h+Z] &= 0. \end{aligned}\right.$$

We can exhibit an expression of q_0^2 and thus obtain

$$\overline{S}^t = -g\frac{2h_L h_R}{h_L + h_R}\frac{[Z]}{\Delta x} + \frac{g}{2\Delta x}\frac{[h]^3}{h_L + h_R}.$$

However, when $Z_L = Z_R$, we have $\overline{S}^t \neq \mathcal{O}(\Delta x)$, i.e. a loss of consistency with S^t (see for instance Berthon, Chalons (2016)).

The topography source term

Instead, we set, for some constant $C > 0$,

$$\begin{cases} \overline{S}^t = -g \dfrac{2 h_L h_R}{h_L + h_R} \dfrac{[Z]}{\Delta x} + \dfrac{g}{2\Delta x} \dfrac{[h]_c^3}{h_L + h_R}, \\ [h]_c = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C\Delta x, \\ \operatorname{sgn}(h_R - h_L)\, C\Delta x & \text{otherwise.} \end{cases} \end{cases}$$

Theorem: Well-balance for the topography source term

If W_L and W_R define a smooth steady state, i.e. if they satisfy

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h + Z] = 0,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced. By construction, the Godunov-type scheme using this approximate Riemann solver is consistent, well-balanced, non-negativity-preserving and entropy preserving.

The friction source term

We consider, in this case, $S(W) = S^f(W) = -kq|q|h^{-\eta}$, where we have set $\eta = 7/3$.

The average of S^f we choose is $\overline{S}^f = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$, with

- $\bar{q}$ the harmonic mean of q_L and q_R (note that $\bar{q} = q_0$ at the equilibrium);
- $\overline{h^{-\eta}}$ a well-chosen discretization of $h^{-\eta}$, depending on h_L and h_R, and ensuring the well-balance property.

We determine $\overline{h^{-\eta}}$ using the same technique (with $\mu_0 = \operatorname{sgn}(q_0)$):

$$\left.\begin{aligned} &\partial_x\left(\frac{q_0^2}{h}\right) + \frac{g}{2}\partial_x\left(h^2\right) = -kq_0|q_0|h^{-\eta}, \\ &q_0^2\frac{\partial_x h^{\eta-1}}{\eta-1} - g\frac{\partial_x h^{\eta+2}}{\eta+2} = kq_0|q_0|, \end{aligned}\right\} \xrightarrow[\text{discretization}]{} \left\{\begin{aligned} &q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = -k\mu_0 q_0^2\overline{h^{-\eta}}\Delta x, \\ &q_0^2\frac{\left[h^{\eta-1}\right]}{\eta-1} - g\frac{\left[h^{\eta+2}\right]}{\eta+2} = k\mu_0 q_0^2\Delta x. \end{aligned}\right.$$

The friction source term

The expression for q_0^2 we obtained is now used to get:

$$\overline{h^{-\eta}} = \frac{[h^2]}{2}\frac{\eta+2}{[h^{\eta+2}]} - \frac{\mu_0}{k\Delta x}\left(\left[\frac{1}{h}\right] + \frac{[h^2]}{2}\frac{[h^{\eta-1}]}{\eta-1}\frac{\eta+2}{[h^{\eta+2}]}\right),$$

which gives $\overline{S}^f = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$ ($\overline{h^{-\eta}}$ is consistent with $h^{-\eta}$ if a cutoff is applied to the second term of $\overline{h^{-\eta}}$).

Theorem: Well-balance for the friction source term

If W_L and W_R define a smooth steady state, i.e. verify

$$q_0^2\frac{[h^{\eta-1}]}{\eta-1} + g\frac{[h^{\eta+2}]}{\eta+2} = -kq_0|q_0|\Delta x,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced. By construction, the Godunov-type scheme using this approximate Riemann solver is consistent, well-balanced, non-negativity-preserving and entropy preserving.

Friction and topography source terms

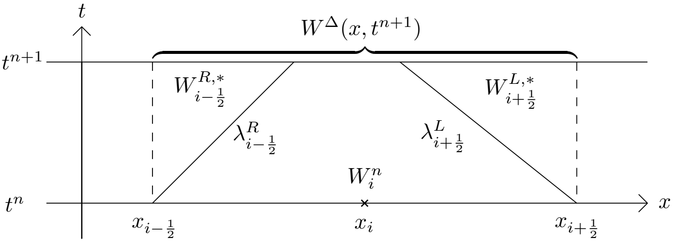
With both source terms, the scheme preserves the following discretization of the steady relation $\partial_x F(W) = S(W)$:

$$q_0^2 \left[\frac{1}{h} \right] + \frac{g}{2} \left[h^2 \right] = \overline{S}^t \Delta x + \overline{S}^f \Delta x.$$

The intermediate states are therefore given by:

$$\begin{cases} q^* = q_{HLL} + \dfrac{(\overline{S}^t + \overline{S}^f)\Delta x}{\lambda_R - \lambda_L}; \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L} \right) h_{HLL} \right); \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R} \right) h_{HLL} \right). \end{cases}$$

The full Godunov-type scheme



We recall $W_i^{n+1} = \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^{\Delta}(x, t^{n+1}) dx$: then

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left[\lambda^L_{i+\frac{1}{2}} \left(W^{L,*}_{i+\frac{1}{2}} - W_i^n \right) - \lambda^R_{i-\frac{1}{2}} \left(W^{R,*}_{i-\frac{1}{2}} - W_i^n \right) \right],$$

which can be rewritten, after straightforward computations,

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}} \right) + \Delta t \left(\begin{pmatrix} 0 \\ \dfrac{(\mathcal{S}^t)^n_{i-\frac{1}{2}} + (\mathcal{S}^t)^n_{i+\frac{1}{2}}}{2} \end{pmatrix} + \begin{pmatrix} 0 \\ \dfrac{(\mathcal{S}^f)^n_{i-\frac{1}{2}} + (\mathcal{S}^f)^n_{i+\frac{1}{2}}}{2} \end{pmatrix} \right).$$

Summary

We have presented a scheme that:

- is consistent with the shallow-water equations with friction and topography;
- is well-balanced for friction and topography steady states;
- preserves the non-negativity of the water height;
- ensures a discrete entropy inequality;
- is easily implemented in a HPC solution;
- is not able to correctly approximate wet/dry interfaces due to the stiffness of the friction $kq|q|h^{-7/3}$: the friction term should be treated implicitly.

next step: introduction of this semi-implicit scheme

Semi-implicit finite volume scheme

We use a splitting method with an explicit treatment of the flux and the topography and an implicit treatment of the friction.

1. explicitly solve $\partial_t W + \partial_x F(W) = S^t(W)$ as follows:

$$W_i^{n+\frac{1}{2}} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}}\right) + \Delta t \begin{pmatrix} 0 \\ \frac{1}{2}\left((\mathcal{S}^t)^n_{i-\frac{1}{2}} + (\mathcal{S}^t)^n_{i+\frac{1}{2}}\right) \end{pmatrix}$$

2. implicitly solve $\partial_t W = S^f(W)$ as follows:

$$\begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\ \text{IVP: } \begin{cases} \partial_t q = -kq|q|(h_i^{n+1})^{-\eta} \\ q(x_i, t^n) = q_i^{n+\frac{1}{2}} \end{cases} \rightsquigarrow q_i^{n+1} \end{cases}$$

Semi-implicit finite volume scheme

Solving the IVP yields:

$$q_i^{n+1} = \frac{(h_i^{n+1})^\eta q_i^{n+\frac{1}{2}}}{(h_i^{n+1})^\eta + k\,\Delta t \left| q_i^{n+\frac{1}{2}} \right|}.$$

We use the following approximation of $(h_i^{n+1})^\eta$, which provides us with an expression of q_i^{n+1} that is equal to q_0 at the equilibrium:

$$(\overline{h^\eta})_i^{n+1} = \frac{2\mu_i^{n+\frac{1}{2}}\mu_i^n}{\left(\overline{h^{-\eta}}\right)_{i-\frac{1}{2}}^{n+1} + \left(\overline{h^{-\eta}}\right)_{i+\frac{1}{2}}^{n+1}} + k\,\Delta t\,\mu_i^{n+\frac{1}{2}} q_i^n.$$

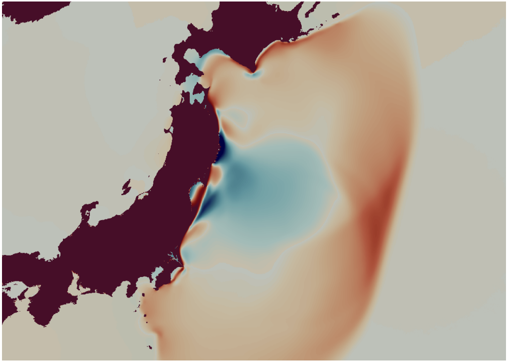
- semi-implicit treatment of the friction source term
  $\rightsquigarrow$ scheme able to model wet/dry transitions
- scheme still well-balanced and non-negativity-preserving

Two-dimensional extension

2D shallow-water model: $\partial_t W + \boldsymbol{\nabla} \cdot \boldsymbol{F}(W) = \boldsymbol{S}^t(W) + \boldsymbol{S}^f(W)$

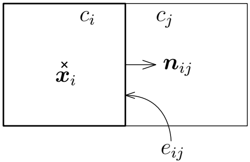
$$\begin{cases} \partial_t h + \boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \\ \partial_t \boldsymbol{q} + \boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{q} \otimes \boldsymbol{q}}{h} + \dfrac{1}{2} g h^2 \mathbb{I}_2 \right) = -gh \boldsymbol{\nabla} Z - \dfrac{k \boldsymbol{q} \|\boldsymbol{q}\|}{h^\eta} \end{cases}$$

to the right: simulation of the 2011 Japan tsunami



Two-dimensional extension

space discretization: Cartesian mesh



With $\mathcal{F}_{ij}^n = \mathcal{F}(W_i^n, W_j^n; \boldsymbol{n}_{ij})$ and ν_i the neighbors of c_i, the scheme reads:

$$W_i^{n+\frac{1}{2}} = W_i^n - \Delta t \sum_{j \in \nu_i} \frac{|e_{ij}|}{|c_i|} \mathcal{F}_{ij}^n + \frac{\Delta t}{2} \sum_{j \in \nu_i} (\boldsymbol{\mathcal{S}}^t)_{ij}^n.$$

W_i^{n+1} is obtained from $W_i^{n+\frac{1}{2}}$ with a splitting strategy:

$$\begin{cases} \partial_t h = 0 \\ \partial_t \boldsymbol{q} = -k\, \boldsymbol{q} \|\boldsymbol{q}\| h^{-\eta} \end{cases} \rightsquigarrow \begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\ \boldsymbol{q}_i^{n+1} = \dfrac{(\overline{h^\eta})_i^{n+1} \boldsymbol{q}_i^{n+\frac{1}{2}}}{(\overline{h^\eta})_i^{n+1} + k\, \Delta t \left\| \boldsymbol{q}_i^{n+\frac{1}{2}} \right\|} \end{cases}$$

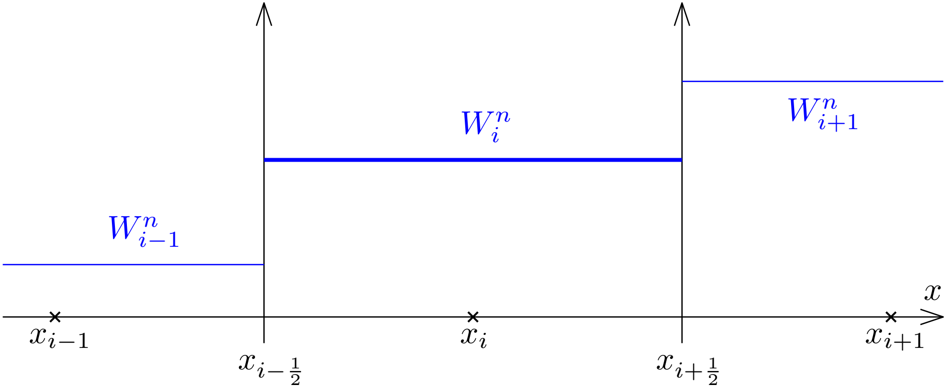
Two-dimensional extension

The 2D scheme is:

- non-negativity-preserving for the water height:
  $\forall i \in \mathbb{Z},\ h_i^n \geq 0 \implies \forall i \in \mathbb{Z},\ h_i^{n+1} \geq 0$;
- able to deal with wet/dry transitions thanks to the semi-implicitation with the splitting method;
- well-balanced by direction for the shallow-water equations with friction and/or topography, i.e.:
  - it preserves all steady states at rest,
  - it preserves friction and/or topography steady states in the x-direction and the y-direction,
  - it does not preserve the general 2D steady states such that $\boldsymbol{\nabla} \cdot \boldsymbol{q} = 0$.

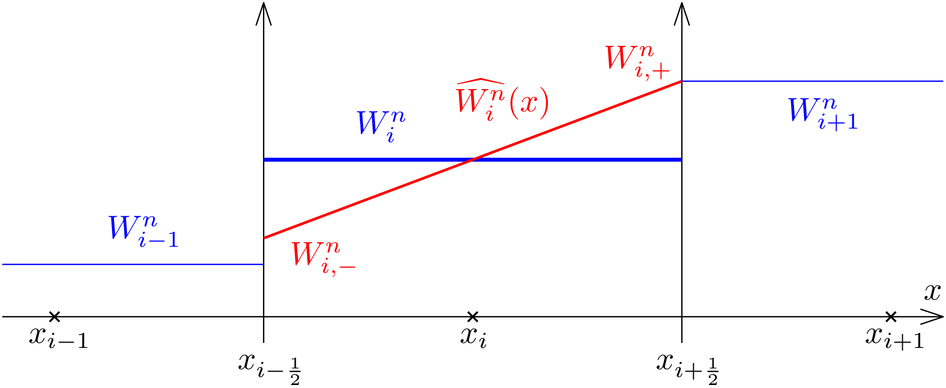
next step: high-order extension of this 2D scheme

High-order extension: the basics, in 1D



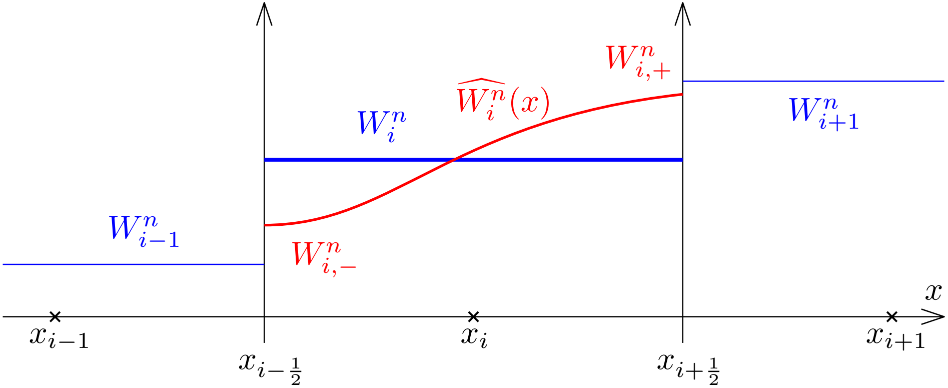
$W_i^n \in \mathbb{P}_0$: constant (order 1 scheme)

High-order extension: the basics, in 1D



$\widehat{W_i^n} \in \mathbb{P}_1$: linear (order 2 scheme)

High-order extension: the basics, in 1D



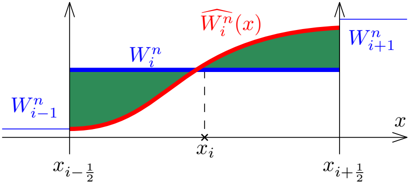
$\widehat{W_i^n} \in \mathbb{P}_d$: polynomial (order $d+1$ scheme)

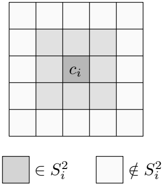
High-order extension: the polynomial reconstruction

polynomial reconstruction (see Diot, Clain, Loubère (2012)):

$$\widehat{W}_i^n(x) = W_i^n + \sum_{|k|=1}^{d} \alpha_i^k \left[(x - x_i)^k - M_i^k \right]$$

- We have $M_i^k = \dfrac{1}{|c_i|} \displaystyle\int_{c_i} (x - x_i)^k dx$ such that the conservation property is verified: $\dfrac{1}{|c_i|} \displaystyle\int_{c_i} \widehat{W}_i^n(x) dx = W_i^n$.





- The polynomial coefficients α_i^k are chosen to minimize the least squares error between the reconstruction and W_j^n, for all j in the stencil S_i^d.

High-order extension: the scheme

High-order space accuracy

$$W_i^{n+1} = W_i^n - \Delta t \sum_{j \in \nu_i} \frac{|e_{ij}|}{|c_i|} \sum_{r=0}^{R} \xi_r \mathcal{F}_{ij,r}^n + \Delta t \sum_{q=0}^{Q} \eta_q \Big((\mathcal{S}^t)_{i,q}^n + (\mathcal{S}^f)_{i,q}^n \Big)$$

- $\mathcal{F}_{ij,r}^n = \mathcal{F}(\widehat{W}_i^n(\sigma_r), \widehat{W}_j^n(\sigma_r); \boldsymbol{n}_{ij})$
- $(\mathcal{S}^t)_{i,q}^n = S^t(\widehat{W}_i^n(x_q))$ and $(\mathcal{S}^f)_{i,q}^n = S^f(\widehat{W}_i^n(x_q))$

We have set:

- $(\xi_r, \sigma_r)_r$, a quadrature rule on the edge e_{ij};
- $(\eta_q, x_q)_q$, a quadrature rule on the cell c_i.

The high-order time accuracy is achieved by the use of SSPRK methods (see Gottlieb, Shu (1998)).

Well-balance recovery (1D): a convex combination

reconstruction procedure $\rightsquigarrow$ the scheme no longer preserves steady states

Well-balance recovery

We suggest a convex combination between the high-order scheme W_{HO} and the well-balanced scheme W_{WB}:

$$W_i^{n+1} = \theta_i^n (W_{HO})_i^{n+1} + (1 - \theta_i^n)(W_{WB})_i^{n+1},$$

with θ_i^n the parameter of the convex combination, such that:

- if $\theta_i^n = 0$, then the well-balanced scheme is used;
- if $\theta_i^n = 1$, then the high-order scheme is used.

next step: derive a suitable expression for θ_i^n

Well-balance recovery (1D): a steady state detector

Steady state detector

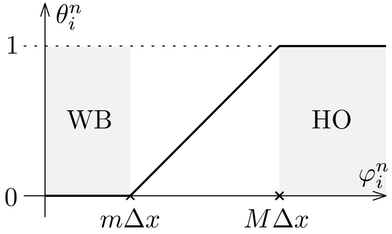
steady state solution: $$\begin{cases} q_L = q_R = q_0, \\ \mathcal{E} := \dfrac{q_0^2}{h_R} - \dfrac{q_0^2}{h_L} + \dfrac{g}{2}\left(h_R^2 - h_L^2\right) - \left(\overline{S}^t + \overline{S}^f\right)\Delta x = 0 \end{cases}$$

steady state detector: $$\varphi_i^n = \left\| \begin{pmatrix} q_i^n - q_{i-1}^n \\ [\mathcal{E}]_{i-\frac{1}{2}}^n \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} q_{i+1}^n - q_i^n \\ [\mathcal{E}]_{i+\frac{1}{2}}^n \end{pmatrix} \right\|_2$$

$\varphi_i^n = 0$ if there is a steady state between W_{i-1}^n, W_i^n and W_{i+1}^n

$\rightsquigarrow$ in this case, we take $\theta_i^n = 0$

$\rightsquigarrow$ otherwise, we take $0 < \theta_i^n \leq 1$



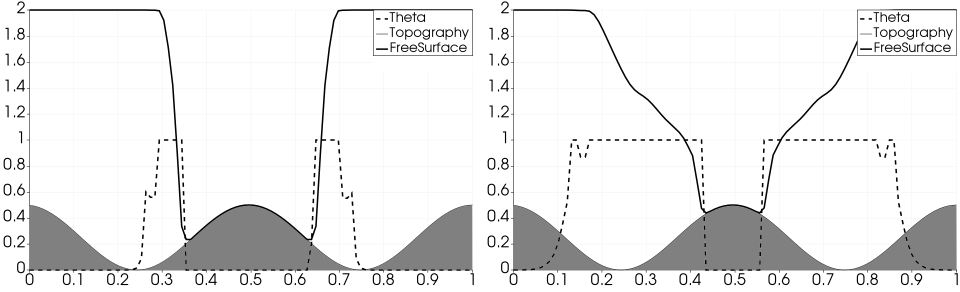
MOOD method

High-order schemes induce oscillations: we adapt the MOOD framework (Multidimensional Optimal Order Detection) to get rid of the oscillations and to restore the non-negativity preservation (see Clain, Diot, Loubère (2011)).

MOOD loop

1. compute a candidate solution W^c with the high-order scheme
2. determine whether W^c is admissible, i.e.
   - if h^c is non-negative (PAD criterion)
   - if W^c does not present spurious oscillations (DMP and u2 criteria)
3. where necessary, decrease the degree of the reconstruction or set $\theta = 0$
4. compute a new candidate solution

Pseudo-1D double dry dam-break on a sinusoidal bottom



The $\mathbb{P}_5^{\mathrm{WB}}$ scheme is used in the whole domain:

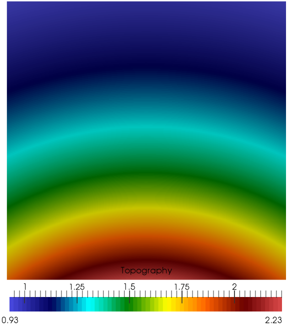
- near the boundaries, steady state at rest $\rightsquigarrow$ well-balanced scheme;
- away from the boundaries, far from steady state $\rightsquigarrow$ high-order scheme;
- center, dry area $\rightsquigarrow$ well-balanced scheme.

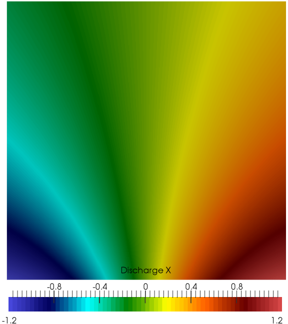
Order of accuracy assessment

To assess the order of accuracy, we take the following exact steady solution of the 2D shallow-water system, where $\boldsymbol{r} = {}^t(x,y)$:

$$h = 1 \; ; \; \boldsymbol{q} = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|} \; ; \; Z = \frac{2k\|\boldsymbol{r}\| - 1}{2g\|\boldsymbol{r}\|^2}.$$

With $k = 10$, this solution is depicted below on the space domain $[-0.3, 0.3] \times [0.4, 1]$.



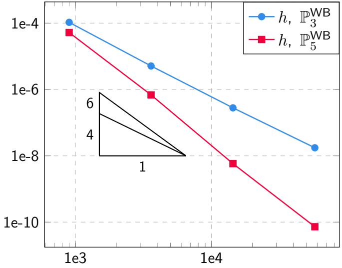


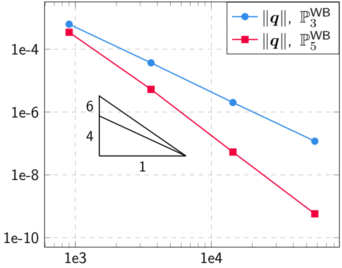
Order of accuracy assessment

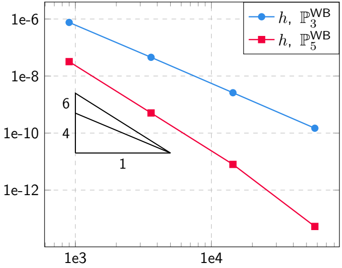
L^2 errors with respect to the number of cells

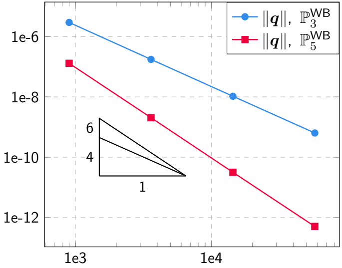
top graphs: 2D steady solution with topography

bottom graphs: 2D steady solution with friction and topography

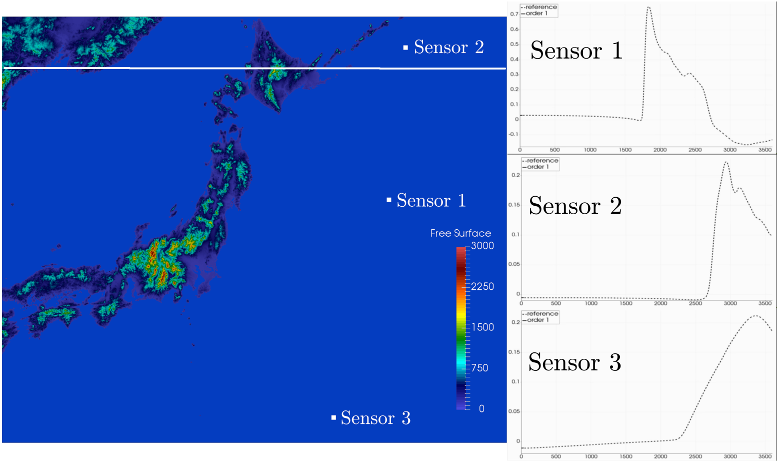






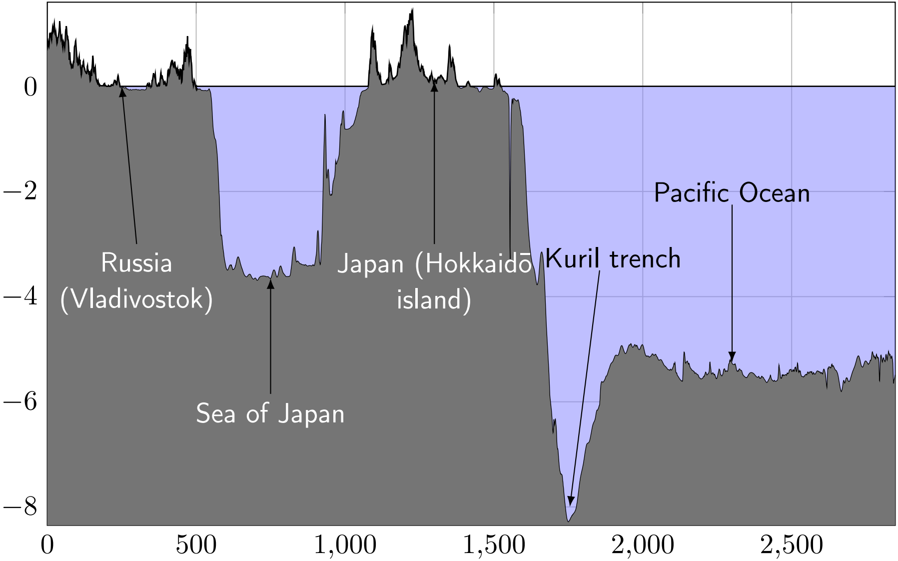


2011 Tōhoku tsunami



Tsunami simulation on a Cartesian mesh: 13 million cells, Fortran code parallelized with OpenMP, run on 48 cores.

2011 Tōhoku tsunami

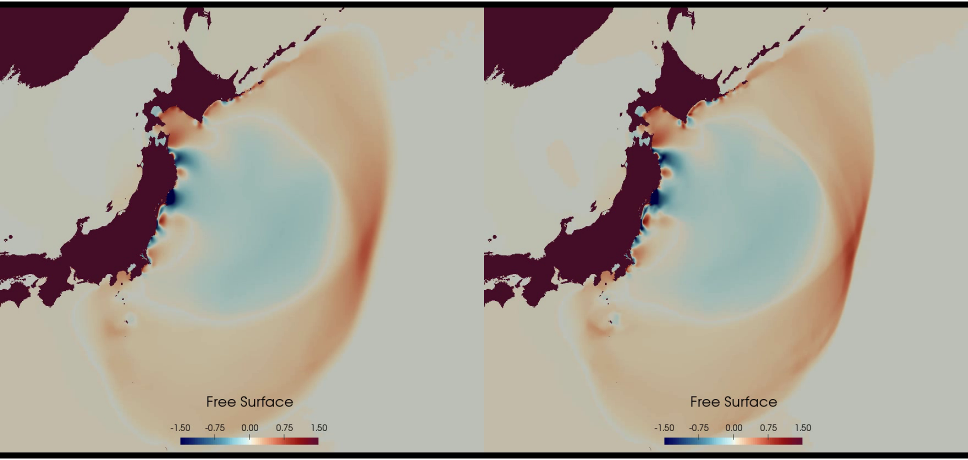


1D slice of the topography (unit: kilometers).

2011 Tōhoku tsunami

2011 Tōhoku tsunami

physical time of the simulation: 1 hour



first-order scheme
wall time: ~ 1.1 hour

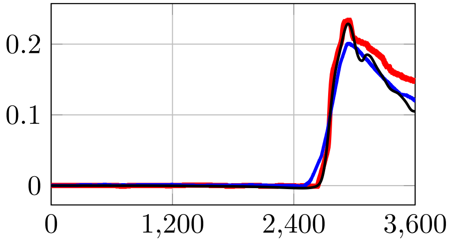
second-order scheme
wall time: ~ 2.7 hours

2011 Tōhoku tsunami

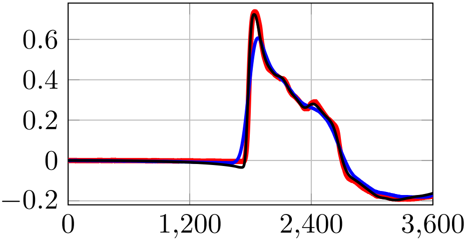
Water depth at the sensors:

- #1: 5700 m;
- #2: 6100 m;
- #3: 4400 m.

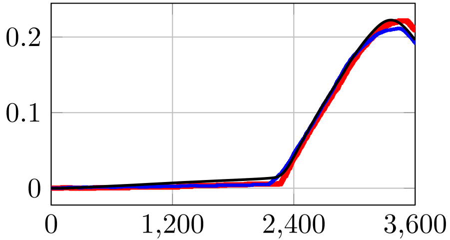
Graphs of the time variation of the water height (in meters).
data in black, order 1 in blue, order 2 in red



Sensor #2



Sensor #1



Sensor #3

Conclusion

- We have presented a well-balanced, non-negativity-preserving and entropy preserving numerical scheme for the shallow-water equations with topography and Manning friction, able to be applied to other source terms or combinations of source terms.
- We have also displayed results from the 2D high-order extension of this numerical method, coded in Fortran and parallelized with OpenMP.

This work has been published in international journals:

V. M.-D., C. Berthon, S. Clain and F. Foucher.
"A well-balanced scheme for the shallow-water equations with topography".
Comput. Math. Appl. 72(3):568–593, 2016.

V. M.-D., C. Berthon, S. Clain and F. Foucher.
"A well-balanced scheme for the shallow-water equations with topography or Manning friction". *J. Comput. Phys.* 335:115–154, 2017.

C. Berthon, R. Loubère, and V. M.-D.
"A second-order well-balanced scheme for the shallow-water equations with topography". *Springer Proc. Math. Stat.*, 2018.

C. Berthon and V. M.-D.
"A simple fully well-balanced and entropy preserving scheme for the shallow-water equations". *Appl. Math. Lett.* 86:284–290, 2018.

Work in progress and perspectives

Work in progress

- high-order simulation of the 2011 Tōhoku tsunami
- application to other source terms:
    - Coriolis force source term
    - breadth variation source term

Long-term perspectives

- ensure the entropy preservation for the high-order scheme (use of an e-MOOD method)
- simulation of rogue waves

Thank you for your attention!

The discrete entropy inequality

The following non-conservative entropy inequality is satisfied by the shallow-water system:

$$\partial_t \eta(W) + \partial_x G(W) \leq \frac{q}{h} S(W); \ \eta(W) = \frac{q^2}{2h} + \frac{gh^2}{2}; \ G(W) = \frac{q}{h}\left(\frac{q^2}{2h} + gh^2\right).$$

At the discrete level, we show that:

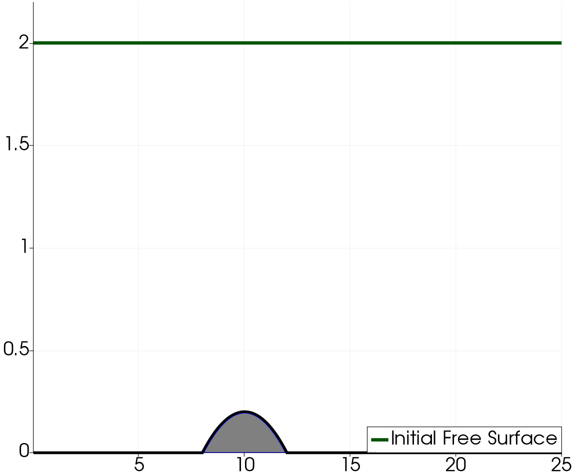
$$\lambda_R(\eta_R^* - \eta_R) - \lambda_L(\eta_L^* - \eta_L) + (G_R - G_L) \leq \frac{q_{HLL}}{h_{HLL}} \overline{S} \Delta x + \mathcal{O}(\Delta x^2).$$

main ingredients:

- $h_L^* = h_{HLL} - \overline{S} \Delta x \dfrac{\lambda_R}{\alpha(\lambda_R - \lambda_L)}$

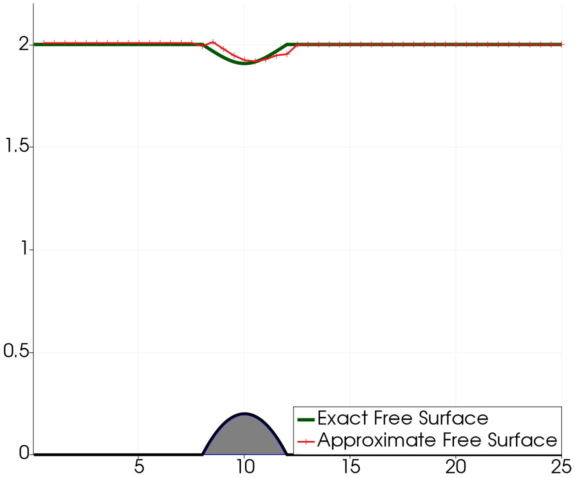
  (and similar expressions for h_R^* and q^*)
- $(\lambda_R - \lambda_L)\eta_{HLL} \leq \lambda_R \eta_R - \lambda_L \eta_L - (G_R - G_L)$ from Harten, Lax, van Leer (1983)

Verification of the well-balance: topography



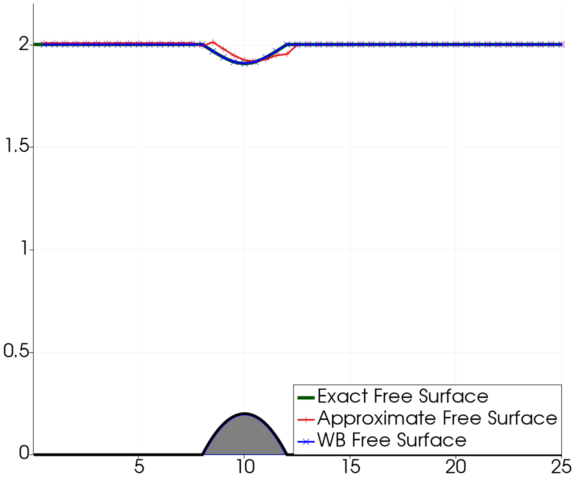
The initial condition is at rest; water is injected through the left boundary.

Verification of the well-balance: topography



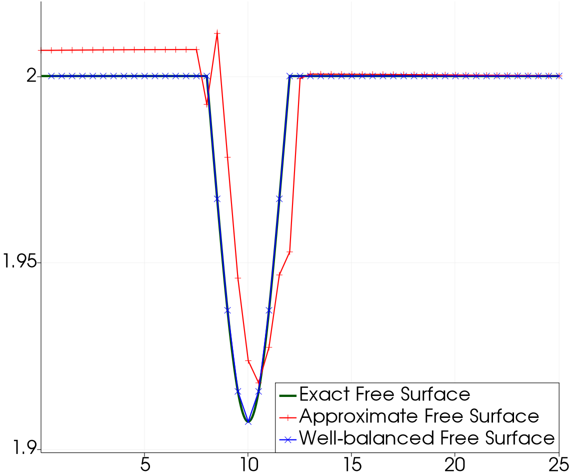
The non-well-balanced HLL scheme converges towards a numerical steady state which does not correspond to the physical one.

Verification of the well-balance: topography



The non-well-balanced HLL scheme yields a numerical steady state which does not correspond to the physical one. The well-balanced scheme exactly yields the physical steady state.

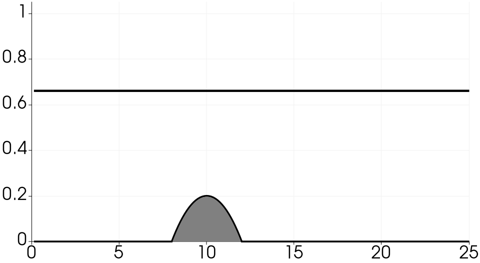
Verification of the well-balance: topography

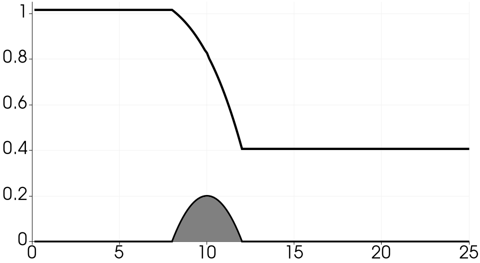


The non-well-balanced HLL scheme yields a numerical steady state which does not correspond to the physical one. The well-balanced scheme exactly yields the physical steady state.

Verification of the well-balance: topography

transcritical flow test case (see Goutal, Maurel (1997))





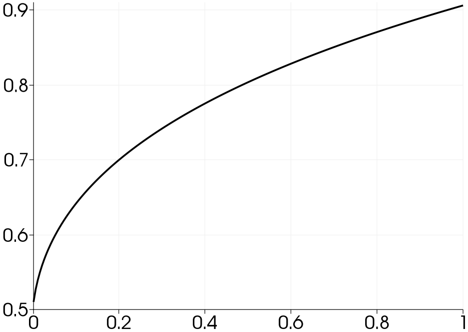
left panel: initial free surface at rest; water is injected from the left boundary

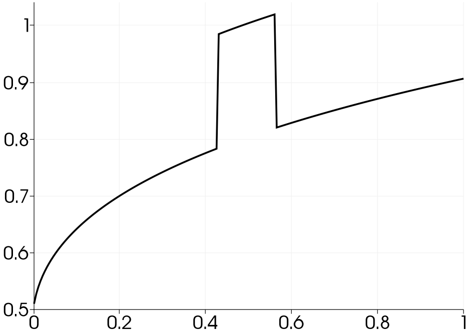
right panel: free surface for the steady state solution, after a transient state

$$\Phi = \frac{u^2}{2} + g(h+Z)$$

	L^1	L^2	L^∞
errors on q	1.47e-14	1.58e-14	2.04e-14
errors on Φ	1.67e-14	2.13e-14	4.26e-14

Verification of the well-balance: friction

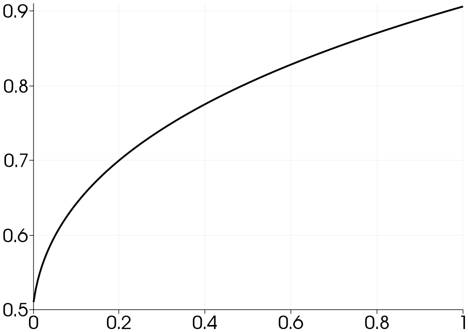


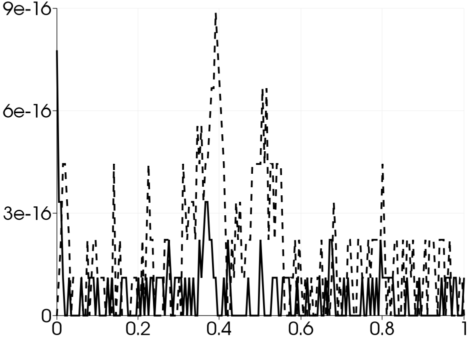


left panel: water height for the subcritical steady state solution

right panel: water height for the perturbed steady state solution

Verification of the well-balance: friction

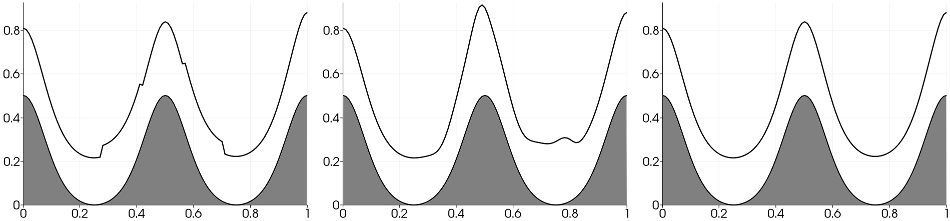




left panel: convergence to the unperturbed steady state

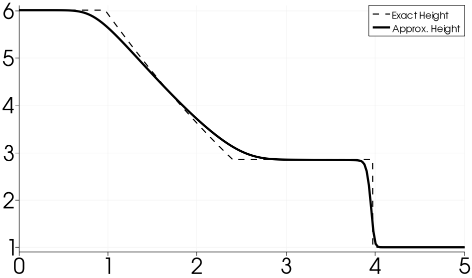
right panel: errors to the steady state (solid: h, dashed: q)

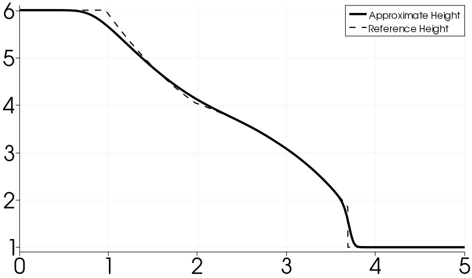
Perturbed pseudo-1D friction and topography steady state



	h			$\|\boldsymbol{q}\|$		
	L^1	L^2	L^∞	L^1	L^2	L^∞
$\mathbb{P}_0$	1.22e-15	1.71e-15	6.27e-15	2.34e-15	3.02e-15	9.10e-15
$\mathbb{P}_5$	5.01e-05	1.47e-04	1.16e-03	2.32e-04	2.63e-04	1.18e-03
$\mathbb{P}_5^{\mathsf{WB}}$	8.50e-14	1.05e-13	3.35e-13	2.82e-13	3.37e-13	6.76e-13

Riemann problems between two wet areas

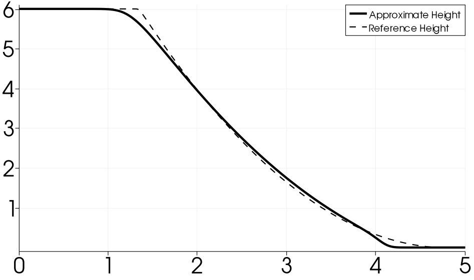


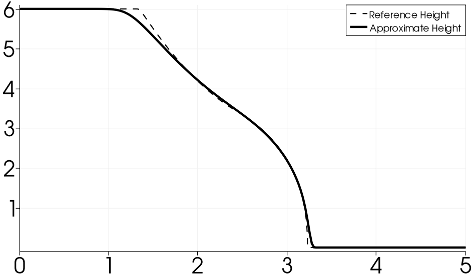


left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

Riemann problems with a wet/dry transition





left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.15s

Double dry dam-break on a sinusoidal bottom