A fully well-balanced scheme for the shallow-water model with topography and bottom friction

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Contents

The Saint-Venant equations and their source terms

$$\begin{cases} \partial_t h + \partial_x (hu) & = & 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \dfrac{1}{2} g h^2 \right) & = & -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{cases}$$

where:

- $h(x,t) > 0$ is the water height
- $u(x,t)$ is the water velocity
- $q(x,t)$ is the water discharge, equal to hu
- $Z(x)$ is the shape of the water bed
- $\eta = 7/3$ and g is the gravitational constant
- k is the so-called Manning coefficient: a higher k leads to a stronger bottom friction

Steady states

rewrite the shallow-water equations as

$$\partial_t W + \partial_x F(W) = S(W) \text{ , with:}$$

$$W = \begin{pmatrix} h \\ hu \end{pmatrix}, \; F(W) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2} g h^2 \end{pmatrix}, \; S(W) = \begin{pmatrix} 0 \\ -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{pmatrix}$$

Definition: Steady states

W is a steady state iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$

Steady states

taking $\partial_t W = 0$ in the shallow-water equations leads to

$$\left\{ \begin{array}{rcl} \partial_x q & = & 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) & = & -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{array} \right.$$

the steady states are therefore given by

$$q = \text{cst} = q_0 \quad \text{and} \quad \partial_x \left(\frac{q_0^2}{h} + \frac{1}{2} g h^2 \right) = -gh\partial_x Z - \frac{kq_0|q_0|}{h^\eta} \tag{1}$$

Objectives

$$\partial_x \left(\frac{q_0^2}{h} + \frac{1}{2} g h^2 \right) = -gh\partial_x Z - \frac{kq_0|q_0|}{h^\eta}$$

derive a fully well-balanced scheme for the shallow-water equations with friction and topography, i.e.:

- preservation of all steady states with friction and $Z = \text{cst}$
- preservation of the lake at rest steady state ($q = 0$)
- preservation of all steady states with $k = 0$ and $q \neq 0$
- preservation of some steady states with $k \neq 0$ and $Z \neq \text{cst}$ (not presented here)

Obtaining the equations

taking a flat bottom in (1), i.e. $Z = \text{cst}$, yields

$$\partial_x \left(\frac{q_0^2}{h} + \frac{1}{2} g h^2 \right) = -\frac{k q_0 |q_0|}{h^\eta},$$

which we rewrite as:

$$-\, q_0^2 \, \partial_x \frac{1}{h} + \frac{g}{2} \, \partial_x h^2 = -\frac{k q_0 |q_0|}{h^\eta} \tag{2}$$

for smooth solutions, we have

$$-\, \frac{q_0^2}{\eta - 1} \, \partial_x h^{\eta - 1} + \frac{g}{\eta + 2} \, \partial_x h^{\eta + 2} = -k q_0 |q_0| \tag{3}$$

Finding solutions

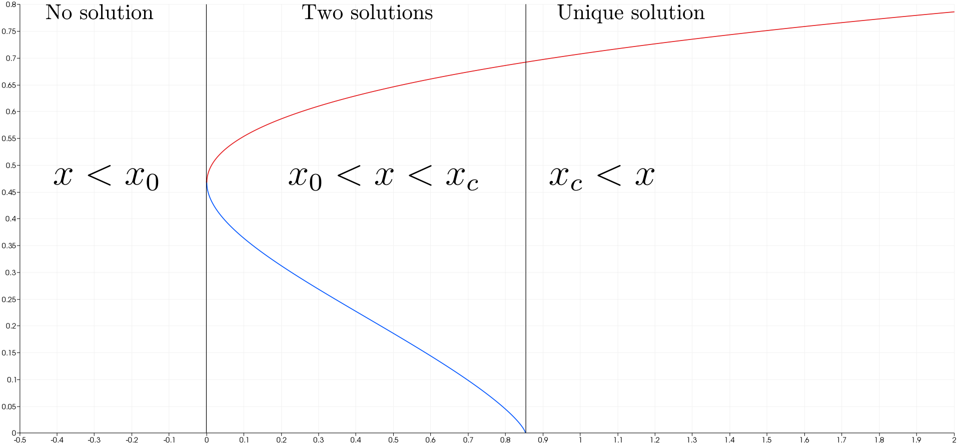
integrating (3) between some x_0 and x yields

$$-\frac{q_0^2}{\eta-1}\left(h^{\eta-1}-h_0^{\eta-1}\right)+\frac{g}{\eta+2}\left(h^{\eta+2}-h_0^{\eta+2}\right)+kq_0|q_0|\left(x-x_0\right)=0 \tag{4}$$

with $h=h(x)$ and $h_0=h(x_0)$

(4) is a nonlinear equation with unknown h for given x; use Newton's method to find h for any x, assuming $q_0<0$

Finding solutions



- zones and variations: analytical study
- solution shape: Newton's method needed to solve (4)

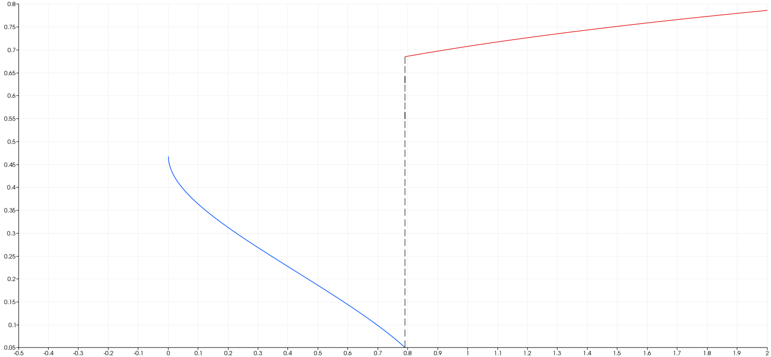
Other solutions?

1. use Rankine-Hugoniot relations to find admissible discontinuities linking two different increasing solutions, thus filling $\mathbb{R}$ by waves
   problem: we cannot fill $]-\infty, x_0[$

2. find admissible discontinuities linking any two different solutions, thus filling $\mathbb{R}$
   same problem: we cannot fill $]-\infty, x_0[$

Other solutions?

3. find admissible discontinuities linking any two solutions, without filling $\mathbb{R}$



works well on paper but appears to yield an unstable equilibrium (confirmed by numerics)

Setting

objective: approximate the solution $W(x,t)$ of the general one-dimensional system $\partial_t W + \partial_x F(W) = 0$, with suitable initial and boundary conditions

we assume:

- $\partial_t W + \partial_x F(W) = 0$ is a hyperbolic system of conservation laws, with "known" eigenvalues
- $(x,t) \mapsto W(x,t)$ is defined from $[a,b] \times [0,T]$ to $\mathbb{R}^d$

Riemann problems

a *Riemann problem* is a Cauchy problem with a piecewise constant initial condition, with one discontinuity at some $x_0 \in \mathbb{R}$:

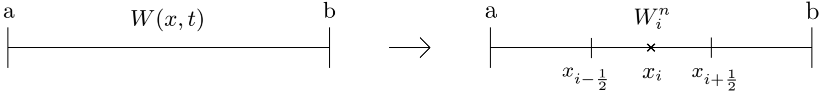
$$\begin{cases} \partial_t W + \partial_x F(W) = 0 \\ W(x,0) = \begin{cases} W_L \text{ if } x < x_0 \\ W_R \text{ if } x > x_0 \end{cases} \end{cases}$$

- we know the solution to a Riemann problem for some systems
- usually, it is not analytically known or too complicated $\rightsquigarrow$ use of an approximate Riemann solver in numerics

Space and time discretization

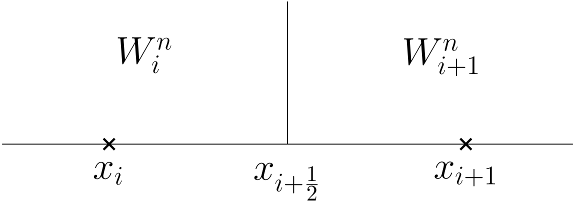
we partition $[a, b]$ in *cells*, of volume Δx and of evenly spaced centers x_i, and define:

- $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$ the boundaries of the cell i
- W_i^n an approximation of $W(x,t)$, constant in the cell i and at time t^n, and defined as $W_i^n = W\left(x_i, t^n\right)$



Using an approximate Riemann solver

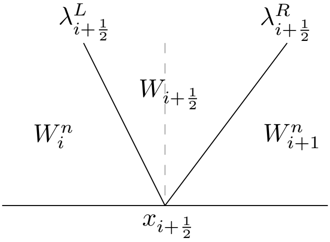
at time t^n, we have a succession of (potentially analytically unsolvable) Riemann problems at the interfaces between cells:



$$\begin{cases} \partial_t W + \partial_x F(W) = 0 \\ W(x,t^n) = \begin{cases} W_i^n \text{ if } x < x_{i+\frac{1}{2}} \\ W_{i+1}^n \text{ if } x > x_{i+\frac{1}{2}} \end{cases} \end{cases}$$

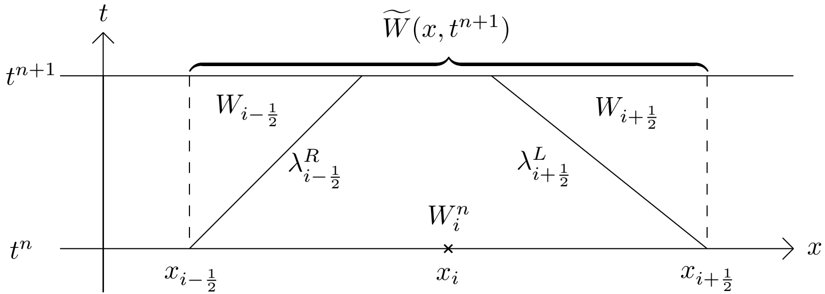
Using an approximate Riemann solver

we choose to use an approximate Riemann solver, for instance the following one-state solver:



- $W_{i+\frac{1}{2}}$ is an approximation of the interaction between W_i and W_{i+1} (i.e. of the solution to the Riemann problem)
- $\lambda^L_{i+\frac{1}{2}}$ and $\lambda^R_{i+\frac{1}{2}}$ are approximations of the wave speeds

The full Godunov-type scheme



$$\forall x \in \left[x_{i-\frac{1}{2}}, x_{i+\frac{1}{2}}\right], \widetilde{W}(x, t^{n+1}) = \begin{cases} W_{i-\frac{1}{2}} & \text{if } x < x_{i-\frac{1}{2}} + \lambda^R_{i-\frac{1}{2}} \Delta t \\ W_{i+\frac{1}{2}} & \text{if } x > x_{i+\frac{1}{2}} + \lambda^L_{i+\frac{1}{2}} \Delta t \\ W_i^n & \text{otherwise} \end{cases}$$

The full Godunov-type scheme

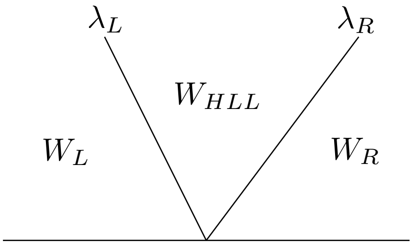
$$\text{define } W_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} \widetilde{W}(x, t^{n+1}) dx \, : \text{ then}$$

$$W_i^{n+1} = \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i-\frac{1}{2}} + \lambda^R_{i-\frac{1}{2}} \Delta t} W_{i-\frac{1}{2}} dx + \frac{1}{\Delta x} \int_{x_{i+\frac{1}{2}} + \lambda^L_{i+\frac{1}{2}} \Delta t}^{x_{i+\frac{1}{2}}} W_{i+\frac{1}{2}} dx + \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}} + \lambda^R_{i-\frac{1}{2}} \Delta t}^{x_{i+\frac{1}{2}} + \lambda^L_{i+\frac{1}{2}} \Delta t} W_i^n dx$$

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x} \left[\lambda^L_{i+\frac{1}{2}} \left(W_{i+\frac{1}{2}} - W_i^n \right) - \lambda^R_{i-\frac{1}{2}} \left(W_{i-\frac{1}{2}} - W_i^n \right) \right]$$

The HLL scheme

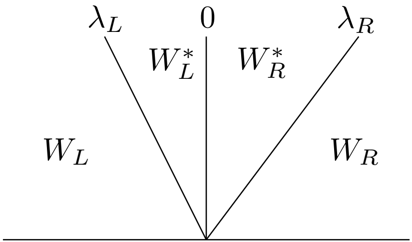
our scheme is based on the HLL scheme (Harten, Lax, van Leer (1983)), which uses the following approximate Riemann solver:



where $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L}$

Modification of the HLL scheme

to include the source term contribution, we use the following approximate Riemann solver (assuming $\lambda_L < 0 < \lambda_R$) :



$\rightsquigarrow$ 3 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$

Properties to be verified

we want our scheme to be:

1. consistent with the shallow-water equations, and
2. well-balanced for the friction steady states

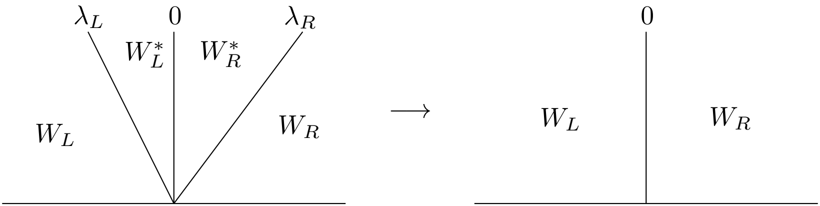
consistency is done thanks to a theorem from Harten and Lax (1983), which ensures that consistency holds if

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}(x,\Delta t) dx,$$

i.e. the mean of the approximation by the Riemann solver is equal to the mean of the exact solution to the Riemann problem

Properties to be verified

ensuring the well-balancedness means enforcing that, if W^n is a steady state, then $\forall i,\ W_i^{n+1} = W_i^n$



we need $W_L^* = W_L$ and $W_R^* = W_R$ at the equilibrium

Properties to be verified

we therefore have the following relations between the unknowns:

- consistency gives us
  - $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$
  - $q^* = q_{HLL} - \dfrac{\bar{S}\Delta t}{\lambda_R - \lambda_L}$

  with h_{HLL} and q_{HLL} defined earlier
- if equilibrium is reached, i.e. we have $q_L = q_R = q_0$ and (4), then for well-balancedness, we need:
  - $q^* = q_0$
  - $h_L^* = h_L$ and $h_R^* = h_R$

What is left to do?

we have 3 unknowns $\rightsquigarrow$ we need 3 equations

- we have 2 from consistency, provided we find a suitable expression for $\bar{S}$
- we need a third relation, between h_L^* and h_R^*, that will ensure the well-balancedness

Determination of q^*

recall $q^* = q_{HLL} - \dfrac{\bar{S}\Delta t}{\lambda_R - \lambda_L}$

we have to find a suitable expression of $\bar{S}$ that:

- is consistent with the source term
- ensures that $q^* = q_0$ when the equilibrium is reached

we use $\bar{S} = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$, with

- $\bar{q}$ the harmonic mean of q_L and q_R (note that $\bar{q} = q_0$ at equilibrium), and
- $\overline{h^{-\eta}}$ a well-chosen representation of $h^{-\eta}$, depending on h_L and h_R, and ensuring the well-balancedness

Determination of q^*

to determine $\overline{h^{-\eta}}$, we assume equilibrium and use the discrete equivalents of (2) and (3):

$$-q_0^2\left[\frac{1}{h}\right]+g\frac{[h^2]}{2}=-k\Delta x q_0|q_0|\overline{h^{-\eta}} \tag{5}$$

$$-q_0^2\frac{[h^{\eta-1}]}{\eta-1}+g\frac{[h^{\eta+2}]}{\eta+2}=-k\Delta x q_0|q_0| \tag{6}$$

where $[X]$ denotes the jump of a quantity X, i.e.

$$[X]=X_R-X_L$$

Determination of q^*

replacing $q_0|q_0|$ with $\delta_0 q_0^2$ in (6), where $\delta_0 = \operatorname{sgn}(q_0)$, yields

$$q_0^2 \frac{\left[h^{\eta-1}\right]}{\eta-1} + g\frac{\left[h^{\eta+2}\right]}{\eta+2} + k\Delta x \delta_0 q_0^2 = 0$$

therefore, when equilibrium is reached, we have an expression of q_0 depending on h_L and h_R:

$$q_0^2 = \frac{g\dfrac{\left[h^{\eta+2}\right]}{\eta+2}}{\dfrac{\left[h^{\eta-1}\right]}{\eta-1} - k\delta_0\Delta x}$$

Determination of q^*

injecting the new expression of q_0 in (5) ultimately yields:

$$\overline{h^{-\eta}} = \frac{-\delta_0}{k\Delta x}\left(\left[\frac{1}{h}\right] + \frac{[h^2]}{2}\frac{\eta+2}{[h^{\eta+2}]}\left(\frac{[h^{\eta-1}]}{\eta-1} - k\delta_0\Delta x\right)\right),$$

which we can inject into $\bar{S} = -k\bar{q}|\bar{q}|\overline{h^{-\eta}}$, and then into

$$q^* = q_{HLL} - \frac{\bar{S}\Delta t}{\lambda_R - \lambda_L}$$

Determination of h_L^* and h_R^*

two unknowns left $\rightsquigarrow$ we need two equations

- we have $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$ from consistency
- we choose $\alpha_R h_R^* + \alpha_L h_L^* = \bar{S}\Delta x$;
  α_L and α_R to be determined to ensure well-balancedness

we determine α_L and α_R such that:

- if $\bar{S} = 0$, we have $h_L^* = h_{HLL} = h_R^*$ (for stability)
- if equilibrium is reached, i.e. $\bar{S}\Delta x = -q_0^2 \left[\frac{1}{h}\right] + g\frac{[h^2]}{2}$ (5),
  we have $h_L^* = h_L$ and $h_R^* = h_R$ (to preserve steady states)

Determination of h_L^* and h_R^*

computing such α_L and α_R easily leads us to

$$\xi\left(h_R^* - h_L^*\right) = \bar{S}\Delta x, \quad \text{with } \xi = \frac{-\bar{q}^2}{h_L h_R} + \frac{g}{2}\left(h_L + h_R\right)$$

using the consistency relation, we eventually obtain

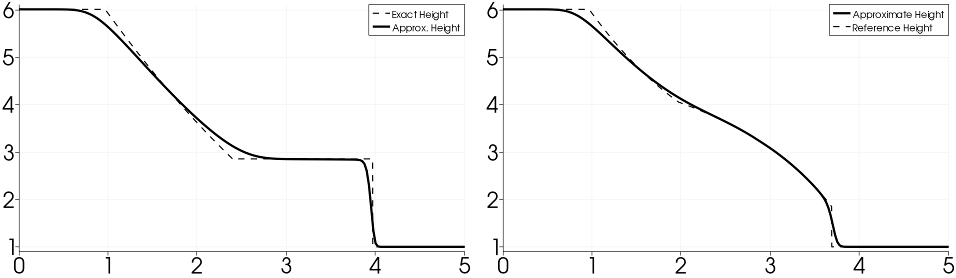
$$\begin{cases} h_L^* &= h_{HLL} - \dfrac{\lambda_R \bar{S}\Delta x}{\xi\left(\lambda_R - \lambda_L\right)} \\ h_R^* &= h_{HLL} - \dfrac{\lambda_L \bar{S}\Delta x}{\xi\left(\lambda_R - \lambda_L\right)} \end{cases}$$

Summary

we have presented an approximate Riemann solver leading to a scheme that:

- is consistent with the shallow-water equations with friction
- is fully well-balanced for friction steady states

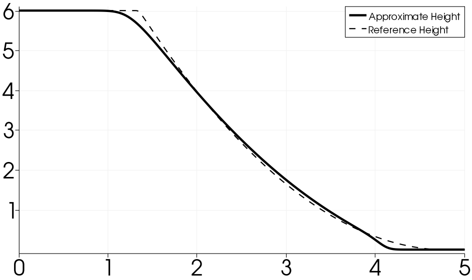
Well-defined Riemann problems

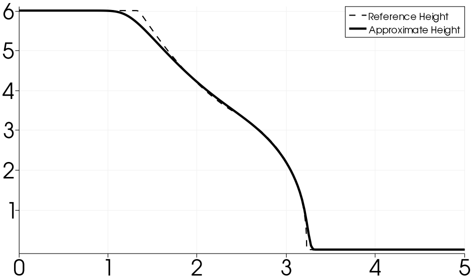


left: $k = 0$ right: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

Riemann problems with a dry/wet transition

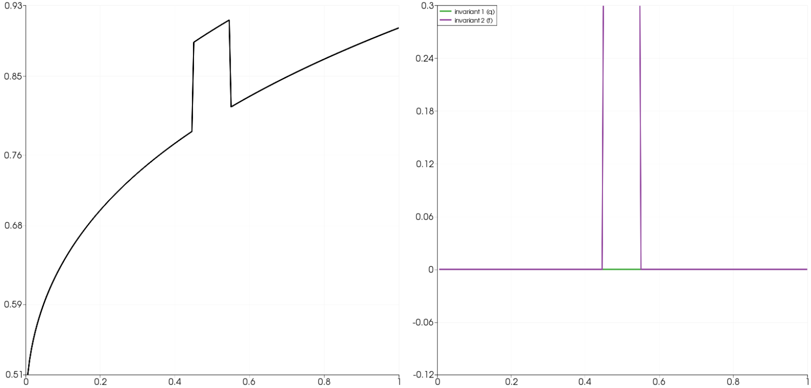




left: $k = 0$ right: $k = 10$

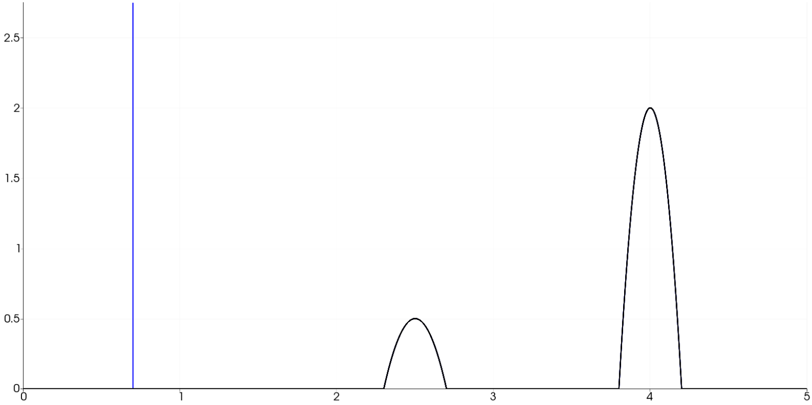
both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.15s

Verification of the well-balancedness



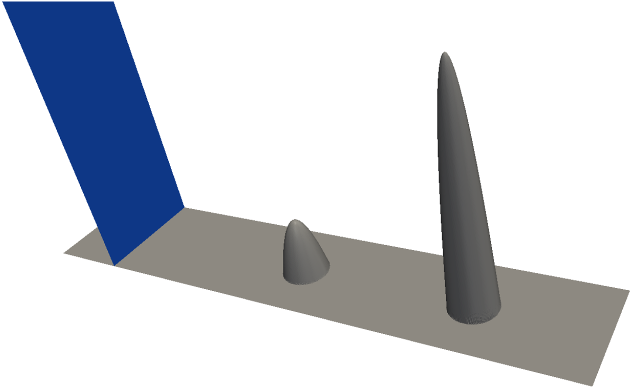
small perturbation of a steady state solution
left: height; right: errors to the equilibrium

A more complex test case, with topography



$$k = 0.1,\ W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \text{ and } W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}, \text{ final time } 3.5\text{s}$$

Two-dimensional extension



$$k = 0.1,\ W_L = \begin{pmatrix} 6 \\ 0 \\ 0 \end{pmatrix} \text{ and } W_R = \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}, \text{ final time 3.5s}$$

Conclusion

- well-balanced scheme for the shallow-water equations with friction
- well-balancedness for friction and topography as well (not presented here)
- Cartesian two-dimensional extension
- MUSCL extension with a MOOD technique

Perspectives

- higher order extension, using MOOD-like techniques to stay well-balanced
- extend to general 2D meshes

Thank you for your attention!