A conservative well-balanced hybrid SPH scheme for the shallow-water model

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Contents

History of the SPH method

- 1977: Monaghan, Gingold, Lucy - particle method for astrophysics, coined the term SPH: "smoothed particle hydrodynamics"
- 1994: Monaghan - SPH for free-surface hydrodynamics
- 1998: Vila - SPH formulation using Riemann problems
- recent & ongoing work:
  - multi-fluid SPH
  - variable mesh
  - viscous terms

The regularizing kernel

General kernel expression

$$W(r,h) = \frac{C_\theta}{h}\theta\left(\frac{|r|}{h}\right), \text{ with}$$

- θ cut-off function
- C_θ normalization constant

properties of this kernel:

1. bell-shaped even function of class C^∞
2. compact support K
3. bell parameters: r (position) and h (width)
4. $\int_{\mathbb{R}} W(r,h)\, dr = 1$
5. $\int_{\mathbb{R}} W'(r,h)\, dr = 0$

The particle approximation

- $$f(x) = (f * \delta)(x), \text{ with } f : \mathbb{R} \mapsto \mathbb{R} \text{ and } \delta \text{ the Dirac distribution}$$
  $$= \int_{\mathbb{R}} f(y)\delta(x-y)\ dy$$

- $$\Pi^h(f)(x) = (f * W)(x)$$
  $$= \int_K f(y)W(x-y,h)\ dy \simeq f(x)$$

- $$\Pi^h(f')(x) = \int_K f'(y)W(x-y,h)\ dy$$
  $$= [f(y)W(x-y,h)]_{\partial K} - \int_K f(y)(W(x-y,h))'\ dy$$
  $$= \int_K f(y)W'(x-y,h)\ dy \simeq f'(x)$$

The particle approximation

Accuracy of the continuous approximation

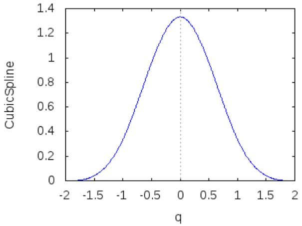
second-order accuracy requires properties 4 and 5 (Mas-Gallic - Raviart, 1987; Monaghan, 1992):

- $\displaystyle\int_{\mathbb{R}} W(r,h)\,dr = 1$, i.e. $\Pi^h(1) = 1$
- $\displaystyle\int_{\mathbb{R}} W'(r,h)\,dr = 0$, i.e. $\Pi^h(1') = 0$

Different kernels

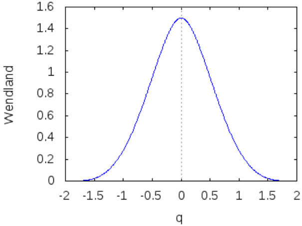
cut-off for the cubic spline kernel (Monaghan, 1998):

$$\theta(q) = \begin{cases} 4 - 6q^2 + 3q^3 & \text{if } 0 \leq q < 1 \\ (2-q)^3 & \text{if } 1 \leq q < 2 \\ 0 & \text{otherwise} \end{cases}$$

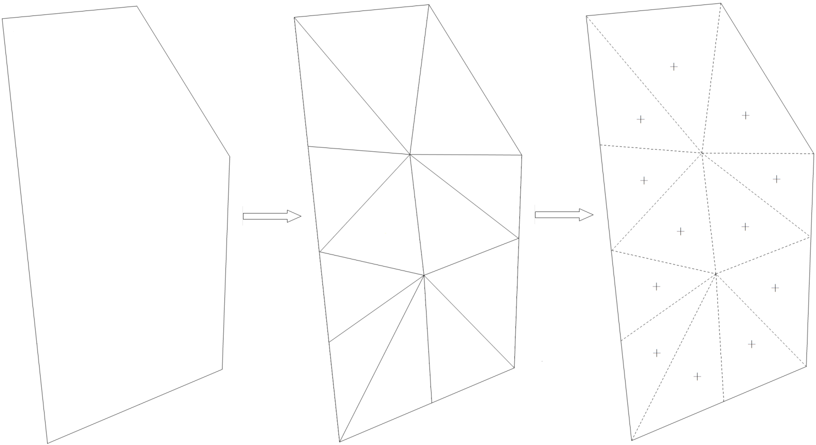


cut-off for the Wendland kernel (Wendland, 1995):

$$\theta(q) = \begin{cases} (2-q)^4(1+2q) & \text{if } 0 \leq q < 2 \\ 0 & \text{otherwise} \end{cases}$$



The “mesh”



Discrete SPH equations

A quadrature formula

$$\int_{\mathbb{R}} f(y)\, dy \simeq \sum_{j\in\mathbb{Z}} \omega(x_j) f(x_j) = \sum_{j\in\mathbb{Z}} \omega_j f_j \text{ , where:}$$

- x_j are the quadrature points, or *particles*
- $\omega_j = \omega(x_j)$ are their volumes
- f_j denotes $f(x_j)$

- $W_{ij} = W(x_i - x_j, h)$
- $\mathcal{P}$: set of interacting particles x_j close enough to particle x_i

Discrete SPH equations

Approximation of a function

$$\begin{aligned} \Pi^h(f)(x) &= \int_K f(y)W(x-y,h)\,dy \\ \text{becomes} \quad \Pi^h(f)_i &= \sum_{j\in\mathcal{P}} \omega_j f_j W_{ij} \simeq f_i \end{aligned}$$

Approximation of its derivative

$$\begin{aligned} \Pi^h(f')(x) &= \int_K f(y)W'(x-y,h)\,dy \\ \text{becomes} \quad \Pi^h(f')_i &= \sum_{j\in\mathcal{P}} \omega_j f_j W'_{ij} \simeq f'_i \end{aligned}$$

Main issue: consistency

Properties not verified in discrete form!

the discrete analogues of 4 and 5 are generally not true:

$$\sum_{j\in\mathcal{P}} \omega_j W_{ij} \neq 1 \text{ and } \sum_{j\in\mathcal{P}} \omega_j W'_{ij} \neq 0$$

$\rightsquigarrow$ loss of the consistency

aim of the SPH methods: numerical resolution of PDE's
$\longrightarrow$ we need a suitable derivation operator

Main issue: consistency

Weak formulation

reinforce the derivation operator:

$$\begin{aligned} D_h(f)_i &= \Pi^h(f')_i - f_i \Pi^h(1')_i \\ &= \sum_{j \in \mathcal{P}} \omega_j (f_j - f_i) W'_{ij} \simeq f'_i \end{aligned}$$

$\rightsquigarrow D_h(f)_i$ is exactly 0 for constant f

yet another issue: this formulation is *not conservative*!

Main issue: consistency

Conservativity

the formulation $D_h(f)_i$ will be *conservative* iff

$$\sum_{i \in \mathbb{Z}} \omega_i D_h(f)_i = 0$$

$$\begin{aligned}
\sum_{i \in \mathbb{Z}} \omega_i D_h(f)_i &= \sum_{i \in \mathbb{Z}} \sum_{j \in \mathbb{Z}} \omega_i \omega_j (f_j - f_i) W'_{ij}, \text{ with } W' \text{ odd} \\
&= -2 \sum_{i \in \mathbb{Z}} \left(\omega_i f_i \sum_{j \in \mathbb{Z}} \omega_j W'_{ij} \right) \neq 0
\end{aligned}$$

Main issue: consistency

Strong formulation

D_h^*, adjoint of D_h with respect to $\langle f, g\rangle_h = \sum_{i\in\mathbb{Z}} \omega_i f_i g_i$:

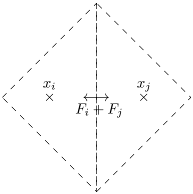
D_h^* such that $\forall (f,g), \langle D_h(f), g\rangle = -\langle f, D_h^*(g)\rangle_h$

$$\rightsquigarrow D_h^*(g)_i = \sum_{j\in\mathcal{P}} \omega_j (g_i + g_j) W'_{ij} \simeq g'_i$$

this strong formulation is conservative!

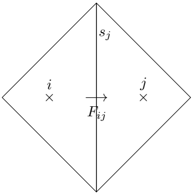
Hybridization SPH - Finite Volumes

SPH



$$\partial_x F_i \simeq \sum_{j \in \mathcal{P}} \omega_j (F_i + F_j) W'_{ij}$$

Finite Volumes



$$\partial_x F_i \simeq \sum_{j \in \gamma(i)} s_j F_{ij}$$

(F_{ij}: any conservative FV flux)

Hybrid formulation FV-SPH

$$\partial_x F_i \simeq \sum_{j \in \mathcal{P}} \omega_j 2 F_{ij} W'_{ij}$$

Summary

SPH-FV approximation of a PDE

consider a general PDE of the form $\partial_t \Phi + \partial_x F(\Phi) = S(\Phi)$

$$\text{SPH approximation of } \partial_x F(\Phi)\text{:} \ \sum_{j \in \mathcal{P}} 2\omega_j F_{ij} W'_{ij}$$

- F_{ij}: any conservative FV flux from particle i to particle j
- choice to make to discretize $\partial_t \Phi$ and $S(\Phi)$
- conservative flux discretization

The shallow-water equations with topography

$$\begin{cases} \partial_t h + \partial_x (hu) & = & 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \frac{1}{2} g h^2 \right) & = & -gh\partial_x Z \end{cases}$$

where

- $h \geq 0$: water height
- $u \in \mathbb{R}$: water velocity in the x direction
- $g > 0$: gravity constant
- Z: smooth topography

they can be rewritten as $\partial_t \Phi + \partial_x F(\Phi) = S(\Phi)$, with

$$\Phi = \begin{pmatrix} h \\ hu \end{pmatrix}, F(\Phi) = \begin{pmatrix} hu \\ hu^2 + \frac{1}{2} g h^2 \end{pmatrix}, S(\Phi) = \begin{pmatrix} 0 \\ -gh\partial_x Z \end{pmatrix}$$

Discretization

goal: discretize $\partial_t \Phi + \partial_x F(\Phi) = S(\Phi)$

$$\begin{aligned}
(x,t) \in \mathbb{R} \times \mathbb{R}_+ \quad &\longrightarrow \quad (x_i, t^n), \text{ with } (i,n) \in \mathbb{Z} \times \mathbb{N} \text{ and steps } \Delta x \text{ and } \Delta t \\
\Phi(x,t) \quad &\longrightarrow \quad \Phi_i^n \\
\partial_t \Phi \quad &\longrightarrow \quad \frac{\Phi_i^{n+1} - \Phi_i^n}{\Delta t} \text{ (explicit Euler)} \\
\partial_x F(\Phi) \quad &\longrightarrow \quad \sum_{j \in \mathcal{P}} 2\omega_j F_{ij} W'_{ij} \text{ (SPH discretization)} \\
S(\Phi) \quad &\longrightarrow \quad S_i \text{ (any discretization)}
\end{aligned}$$

Discretized shallow-water equations

Hybrid scheme applied to the shallow-water equations

$$\begin{cases} \dfrac{h_i^{n+1} - h_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j (hu)_{ij} W'_{ij} & = & 0 \\ \dfrac{h_i^{n+1} u_i^{n+1} - h_i^n u_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j \left(hu^2 + \frac{1}{2} g h^2 \right)_{ij} W'_{ij} & = & s_i \end{cases}$$

- s_i: discretization of $-gh\partial_x Z$
- $F^{\Delta x}(\Phi_i^n, \Phi_j^n)$: numerical flux, such that $F^{\Delta x}(\Phi_i^n, \Phi_j^n) = \begin{pmatrix} (hu)_{ij} \\ \left(hu^2 + \frac{1}{2} g h^2\right)_{ij} \end{pmatrix}$

The lake at rest steady state

Steady states

a solution Φ of a PDE will be a *steady state* iff

$$\partial_t \Phi = 0$$

for the shallow-water equations:

$$\begin{cases} \partial_t h & = & 0 \\ \partial_t (hu) & = & 0 \end{cases} \Rightarrow \begin{cases} \partial_x (hu) & = & 0 \\ \partial_x \left(hu^2 + \frac{1}{2} g h^2 \right) & = & -gh\partial_x Z \end{cases}$$

Lake at rest steady state

$$\begin{cases} u & = & 0 \qquad \text{(lake at } \textit{rest}\text{)} \\ h + Z & = & cst \end{cases}$$

Well-balanced schemes

Well-balancedness

the scheme will be *well-balanced* for a steady state iff

$$\text{for } \Phi^n \text{ verifying the steady state, } \forall i \in \mathbb{Z},\ \Phi_i^{n+1} = \Phi_i^n$$

Properties to be verified by the desired hybrid scheme

- well-balancedness (preservation of the lake at rest steady state)
- conservativity
- based on SPH approximations

Variables H and X

consider the following variables (Berthon-Foucher, 2012):

- $H = h + Z$: the free surface
- $X = \dfrac{h}{H}$: the water volume fraction

from now on, $V = \begin{pmatrix} H \\ Hu \end{pmatrix}$ is a new set of variables

Case of the lake at rest

for the lake at rest steady state, V is *constant throughout the domain*: indeed,

$$\left.\begin{aligned} u = 0 \\ H = h + Z = cst \end{aligned}\right\} \Rightarrow V = \begin{pmatrix} H \\ 0 \end{pmatrix} = cst$$

Variables H and X

rewrite the shallow-water equations for weak solutions:

$$\begin{cases} \partial_t h + \partial_x (X(Hu)) = 0, \\ \partial_t (hu) + \partial_x \left(X(Hu^2 + \frac{1}{2} g H^2) \right) = \frac{g}{2} \partial_x (hZ) - gh\partial_x Z \end{cases}$$

this system can be written as

$$\partial_t \Phi + \partial_x (XF(V)) = \tilde{S}(\Phi)$$

with *the same flux function* F as the original shallow-water equations

Numerical scheme used on for this reformulation

Hybrid scheme applied to the reformulation

$$\begin{cases} \dfrac{h_i^{n+1} - h_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j X_{ij} (Hu)_{ij} W'_{ij} = 0 \\ \dfrac{h_i^{n+1} u_i^{n+1} - h_i^n u_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j X_{ij} \left(Hu^2 + \frac{1}{2} g H^2 \right)_{ij} W'_{ij} = \\ \qquad \left(\dfrac{g}{2} \partial_x (hZ) - gh\partial_x Z \right)_i \end{cases}$$

still to be defined:

- X_{ij}, an average of X_i and X_j
- $\left(\dfrac{g}{2} \partial_x (hZ) - gh\partial_x Z \right)_i$, the source term discretization

Source term discretization

What is left to do?

ensuring the well-balancedness by finding a suitable discretization of the source term $\frac{g}{2}\partial_x(hZ) - gh\partial_x Z$

noting that $h = HX$ and $Z = H(1-X)$ yields

$$\frac{g}{2}\partial_x(hZ) - gh\partial_x Z = \frac{g}{2}\partial_x(XH^2(1-X)) - gXH\partial_x(H(1-X))$$

First naïve approach: brutal SPH discretization

$$\left(\frac{g}{2}\partial_x(hZ) - gh\partial_x Z\right)_i = \frac{g}{2}\sum_{j\in\mathcal{P}} 2\omega_j \left(X_{ij}H_{ij} - 2\bar{X}_i\bar{H}_i\right)\left(H_{ij}(1-X_{ij})\right)W'_{ij}$$

still to determine: the averages X_{ij}, H_{ij}, $\bar{X}_i$, $\bar{H}_i$

however, this discretization leads to a *non-conservative* scheme!

$$\left(\text{i.e. } \sum_{i\in\mathbb{Z}}\sum_{j\in\mathcal{P}} X_{ij}F^{\Delta x}(V_i^n, V_j^n) \neq \sum_{i\in\mathbb{Z}}\left(\frac{g}{2}\partial_x(hZ) - gh\partial_x Z\right)_i\right)$$

Idea behind the fix

main idea: add a term to make the previous naïve discretization conservative

$$\text{we have } \sum_{j \in \mathcal{P}} \omega_j W'_{ij} \simeq 0$$

$\rightsquigarrow$ to add a factor of $\sum_{j \in \mathcal{P}} \omega_j W'_{ij}$ is to add a representation of zero

Second approach

$$\left(\frac{g}{2}\partial_x(hZ) - gh\partial_x Z\right)_i = \frac{g}{2}\sum_{j \in \mathcal{P}} 2\omega_j \left(X_{ij}H_{ij} - 2\bar{X}_i\bar{H}_i\right)\left(H_{ij}(1 - X_{ij})\right)W'_{ij} + Y_i \sum_{j \in \mathcal{P}} \omega_j W'_{ij}$$

The correction term

How to choose Y_i?

such that the additional term *corrects* the non-conservativity
$\rightsquigarrow$ add what was missing for the naïve approach to be conservative

Final form of the discretization

$$\left(\frac{g}{2}\partial_x(hZ) - gh\partial_x Z\right)_i = \frac{g}{2}\sum_{j\in\mathcal{P}} 2\omega_j \left(X_{ij}H_{ij} - 2\bar{X}_i\bar{H}_i\right)\left(H_{ij}(1-X_{ij})\right)W'_{ij}$$
$$+ g\sum_{j\in\mathcal{P}} 2\omega_j \bar{H}_i^2\bar{X}_i(1-\tilde{X}_i)W'_{ij}$$

averages $\bar{X}_i$, $\bar{H}_i$, $\tilde{X}_i$, X_{ij} and H_{ij} still to be determined
$\rightsquigarrow$ whatever the averages, the scheme *will be conservative*!

The theorem

Well-balancedness of the scheme

Assume both free surface averages to satisfy:

$$H_{ij} = \bar{H}_i = H, \qquad \text{as soon as } H_i = H_j = H.$$

Assume $\bar{X}_i$ is defined by

$$\bar{X}_i = \frac{1}{2} \frac{\sum_j \omega_j X_{ij}^2 W'_{ij}}{\sum_j \omega_j \left(X_{ij} - 1\right) W'_{ij} + (\tilde{X}_i - 1) \sum_j \omega_j W'_{ij}}.$$

Then the scheme defined by the SPH hybridization and source term discretization preserves the lake at rest.

Outline of the proof

What to prove? *Preservation of the lake at rest*

assume Φ^n at rest, i.e. $\forall i \in \mathbb{Z}, \begin{cases} h_i^n + Z_i = H \geq 0 \\ u_i^n = 0 \end{cases}$

$$\rightsquigarrow \text{ prove that } \forall i \in \mathbb{Z},\ \Phi_i^{n+1} = \Phi_i^n$$

An important property of numerical fluxes

recall $F^{\Delta x}(\Phi_i^n, \Phi_j^n) = \begin{pmatrix} (hu)_{ij} \\ \left(hu^2 + \frac{1}{2}gh^2\right)_{ij} \end{pmatrix}$

$$\forall \Phi, F^{\Delta x}(\Phi, \Phi) = F(\Phi) \ (\mathit{consistency})$$

Outline of the proof

use the consistency: we have, $\forall i \in \mathbb{Z}$, $V_i^n = \begin{pmatrix} H \\ 0 \end{pmatrix}$

therefore, $\forall (i,j) \in \mathbb{Z}^2$,

$$F^{\Delta x}(V_i^n, V_j^n) = F^{\Delta x}\left(\begin{pmatrix} H \\ 0 \end{pmatrix}, \begin{pmatrix} H \\ 0 \end{pmatrix}\right) = F\begin{pmatrix} H \\ 0 \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}gH^2 \end{pmatrix}$$

$$\begin{pmatrix} (Hu)_{ij} \\ \left(Hu^2 + \frac{1}{2}gH^2\right)_{ij} \end{pmatrix} = \begin{pmatrix} 0 \\ \frac{1}{2}gH^2 \end{pmatrix}$$

Outline of the proof

recall the scheme:

$$\begin{cases} \dfrac{h_i^{n+1} - h_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j X_{ij} (Hu)_{ij} W'_{ij} & = 0 \\ \dfrac{h_i^{n+1} u_i^{n+1} - h_i^n u_i^n}{\Delta t} + \displaystyle\sum_{j \in \mathcal{P}} 2\omega_j X_{ij} \left(Hu^2 + \dfrac{1}{2} g H^2 \right)_{ij} W'_{ij} & = \\ \qquad \left(\dfrac{g}{2} \partial_x (hZ) - gh \partial_x Z \right)_i & \end{cases}$$

- $(Hu)_{ij} = 0 \Longrightarrow h_i^{n+1} = h_i^n$ directly
- $\left(Hu^2 + \dfrac{1}{2} g H^2 \right)_{ij} = \dfrac{1}{2} g H^2 \Longrightarrow h_i^{n+1} u_i^{n+1} = h_i^n u_i^n$ (after a few easy steps)

$\rightsquigarrow$ the scheme is well-balanced! $\square$

An important remark: consistency of the average

recall that $\bar{X}_i = \dfrac{1}{2} \dfrac{\sum_j \omega_j X_{ij}^2 W'_{ij}}{\sum_j \omega_j \left(X_{ij} - 1\right) W'_{ij} + (\tilde{X}_i - 1) \sum_j \omega_j W'_{ij}}$

is $\bar{X}_i$ consistent with X?

- $\sum_j \omega_j X_{ij}^2 W'_{ij}$ is consistent with $\partial_x X^2$
- $\sum_j \omega_j \left(X_{ij} - 1\right) W'_{ij}$ is consistent with $\partial_x (X - 1)$
- $\sum_j \omega_j W'_{ij}$ is consistent with 0

$\rightsquigarrow$ therefore $\bar{X}_i$ is consistent with $\dfrac{1}{2} \dfrac{\partial_x X^2}{\partial_x (X - 1)} = X$

Definitions of the other averages

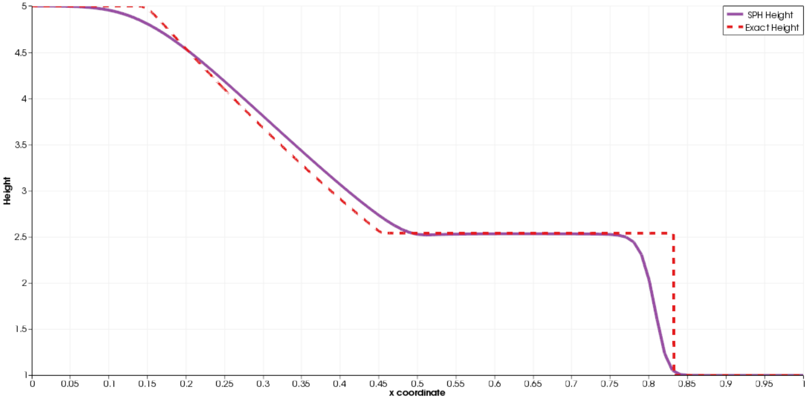
the well-balancedness is *independent from the definitions of the 4 other averages*: some usual expressions can be used

- $\bar{H}_i = H_i^n = h_i^n + Z_i$
- $\tilde{X}_i = X_i^n = \dfrac{h_i^n}{h_i^n + Z_i}$
- $H_{ij} = \begin{cases} H_i^n \text{ if } (Hu)_{ij} > 0 \\ H_j^n \text{ otherwise} \end{cases}$ (*upwind* expression)
- $X_{ij} = \begin{cases} X_i^n \text{ if } (Hu)_{ij} > 0 \\ X_j^n \text{ otherwise} \end{cases}$

Inconsistency of the SPH method

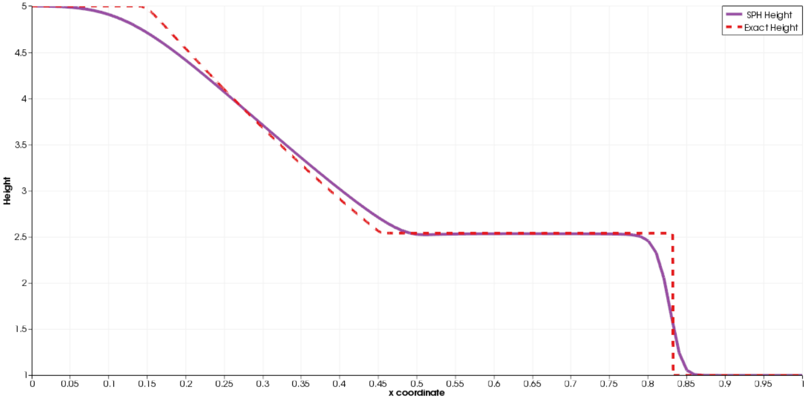
consider the dam-break defined by $\Phi = \Phi_L$ if $x < 0.5$, Φ_R otherwise

$$\Phi_L = \begin{pmatrix} h_L \\ h_L u_L \end{pmatrix} = \begin{pmatrix} 5 \\ 0 \end{pmatrix} \text{ and } \Phi_R = \begin{pmatrix} h_R \\ h_R u_R \end{pmatrix} = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$$

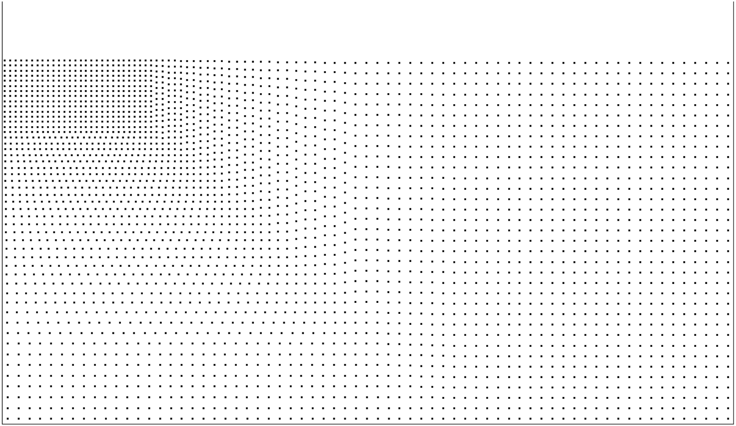


Inconsistency of the SPH method

renormalization: $D_{h,r}(f)_i = B_i^{-1} \sum_{j \in \mathcal{P}} \omega_j (f_j - f_i) W'_{ij}$, with $B_i = \sum_{j \in \mathcal{P}} \omega_j (x_j - x_i) W'_{ij}$



Non-well-balancedness in 2D



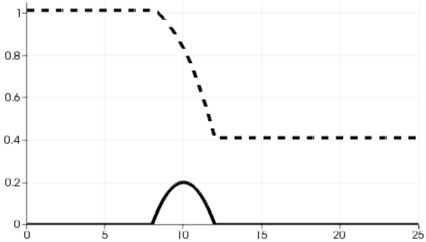
Steady test cases

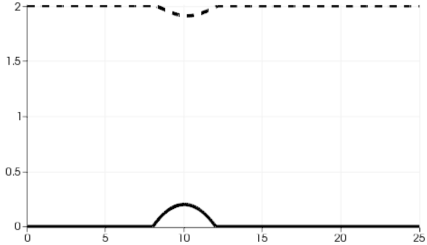
Three steady test cases (Goutal - Maurel, 1997)

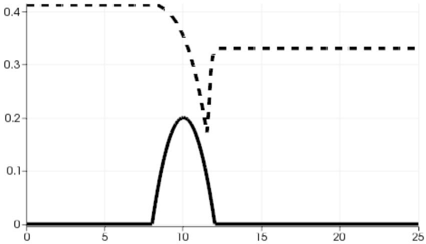
- computational domain: $x \in [0, 25]$
- topography: $Z(x) = \begin{cases} 0.2 - 0.05(x-10)^2 & \text{if } 8 < x < 12, \\ 0 & \text{otherwise} \end{cases}$
- boundary conditions: $h(0,t)u(0,t) = q_L$ and $h(25,t) = h_R$
- initial conditions: $h(x,0)u(x,0) = 0$ and $h(x,0) + Z(x) = h_R$
- individual values:
  - transcritical flow without shock (TF): $q_L = 1.53$; $h_R = 0.66$
  - transcritical flow with shock: (TFS) $q_L = 0.18$; $h_R = 0.33$
  - subcritical flow (SF): $q_L = 4.42$; $h_R = 2$

steady test cases: a *steady state* (i.e. constant discharge) is obtained after a transition period

Steady test cases







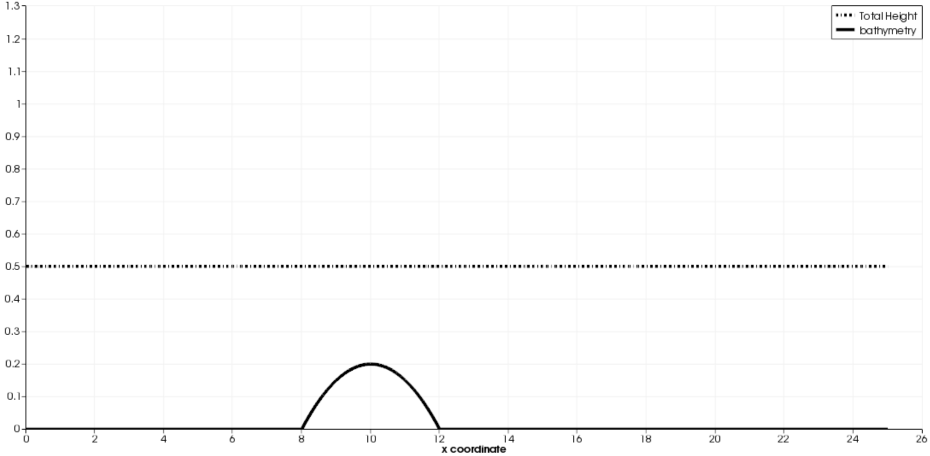
full line: topography
dashed line: free surface
results shown after 600s
top left: TF; bottom left: TFS;
right: SF

Steady test cases: discharge error

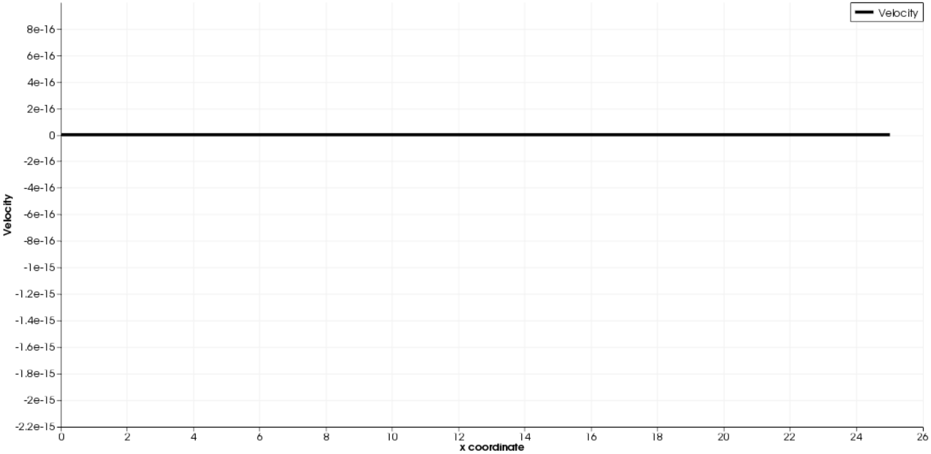
Test case	Hydrostatic upwind		SPH scheme	
	L^2 error	L^∞ error	L^2 error	L^∞ error
TF	5.98E-2	1.87E-2	5.67E-2	1.85E-2
TFS	4.68E-2	2.85E-2	5.50E-2	4.02E-2
SF	9.78E-2	2.70E-2	9.83E-2	2.74E-2

comparison between the hydrostatic upwind scheme (Berthon - Foucher, 2012) and the SPH scheme (both well-balanced)

Lake at rest without well-balanced SPH scheme



Lake at rest with well-balanced SPH scheme



Conclusion

- SPH hybrid scheme for a reformulation of the shallow-water equations
- suitable discretization of the topography source term
- scheme still conservative
- well-balancedness confirmed

Perspectives

- multiple dimensions: easy (replace W'_{ij} with ∇W_{ij})
- well-balanced SPH scheme for Euler equations with gravity

Thanks for your attention!