Asymptotically accurate high-order space and time schemes for the Euler system in the low Mach regime

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SHARK-FV, 15-19 May 2017

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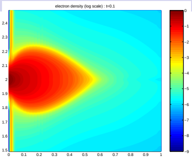
Funding : ANR MOONRISE

Outline

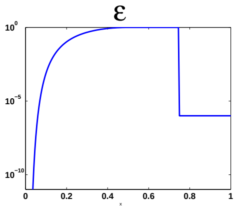
General context

Multiscale model : M_ε **depends on a parameter** ε



In the (space-time) domain ε can

- be small compared to the reference scale
- be of same order as the reference scale
- take intermediate values



When ε **is small :** $M_0 = \lim_{\varepsilon \to 0} M_\varepsilon$ asympt. model

Difficulties :

- Classical explicit schemes for M_ε : stable and consistent if the mesh resolves all the scales of ε $\Rightarrow$ huge cost
- Schemes for M_0 with meshes independent of ε

But : ➠ M_0 not valid everywhere, needs $\varepsilon \ll 1$

➠ location of the interface, moving interface

Principle of AP schemes

A possible solution : AP schemes

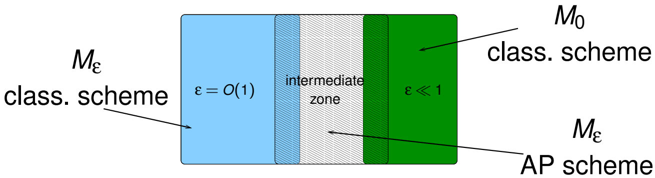
- Use the multi-scale model M_ε where you want.
- Discretize it with a scheme preserving the limit $\varepsilon \to 0$

➠ The mesh is independent of ε : **Asymptotic stability**.

➠ You recover an approximate solution of M_0 when $\varepsilon \to 0$:
Asymptotic consistency

Asymptotically stable and consistent scheme
$\Rightarrow$ Asymptotic preserving scheme (AP)
([S.Jin] kinetic $\to$ hydro)

- You can use the AP scheme only to reconnect M_ε and M_0



Outline

The multi-scale model and its asymptotic limit

➠ **Isentropic Euler system in scaled variables** $x \in \Omega \subset \mathbb{R}^d$, $t \geq 0$

$$(M_\varepsilon) \begin{cases} \partial_t \rho + \nabla \cdot (\rho\, u) = 0, & (1)_\varepsilon \\ \partial_t (\rho\, u) + \nabla \cdot (\rho\, u \otimes u) + \dfrac{1}{\varepsilon} \nabla p(\rho) = 0, & (2)_\varepsilon \end{cases}$$

Parameter : $\varepsilon = M^2 = m|\overline{u}|^2/(\gamma p(\overline{\rho})/\overline{\rho})$, $\qquad M =$Mach number

Boundary and initial conditions :

$$u \cdot n = 0, \text{ on } \partial\Omega, \quad \text{and} \quad \begin{cases} \rho(x,0) = \rho_0 + \varepsilon\, \tilde{\rho}_0(x), \\ u(x,0) = u_0(x) + \varepsilon\, \tilde{u}_0(x), \ \text{ with } \nabla \cdot u_0 = 0. \end{cases}$$

The formal low Mach number limit $\varepsilon \to 0$

$$(2)_\varepsilon \ \Rightarrow \ \nabla p(\rho) = 0, \ \Rightarrow \ \rho(x,t) = \rho(t).$$

$$(1)_\varepsilon \ \Rightarrow \ |\Omega|\, \rho'(t) + \rho(t) \int_{\partial\Omega} u \cdot n = 0, \ \Rightarrow \ \rho(t) = \rho(0) = \rho_0, \ \Rightarrow \ \nabla \cdot u = 0$$

The multi-scale model and its asymptotic limit

The asymptotic model : Rigorous limit [Klainerman, Majda, 81]

$$
(M_0)\begin{cases}
\rho = cste = \rho_0, & \\
\rho_0 \nabla \cdot u = 0, & (1)_0 \\
\rho_0 \partial_t u + \rho_0 \nabla \cdot (u \otimes u) + \nabla \pi_1 = 0, & (2)_0
\end{cases}
$$

where

$$\pi_1 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(p(\rho) - p(\rho_0)\Big).$$

Explicit eq. for π_1 $\partial_t (1)_0 - \nabla \cdot (2)_0 \;\Rightarrow\; -\Delta \pi_1 = \rho_0 \nabla^2 : (u \otimes u).$

The pressure wave eq. from M_ε :

$$\partial_t (1)_\varepsilon - \nabla \cdot (2)_\varepsilon \quad \Rightarrow \quad \partial_{tt}\rho - \frac{1}{\varepsilon}\Delta p(\rho) = \nabla^2 : (\rho\, u \otimes u) \qquad (3)_\varepsilon$$

From a numerical point of view

- Explicit treatment of $(3)_\varepsilon$ $\Rightarrow$ conditional stability $\Delta t \leq \sqrt{\varepsilon}\,\Delta x$
- Implicit treatment of $(3)_\varepsilon$ $\Rightarrow$ uniform stability with respect to ε

An order 1 AP scheme in the low Mach numb. limit

Order 1 AP scheme in [Dimarco, Loubère, Vignal, SISC 2017] :
If ρ^n and u^n are known at time t^n

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^{n+1} = 0, & (1) \quad \text{(AS)} \\ \dfrac{(\rho\, u)^{n+1}-(\rho\, u)^n}{\Delta t} + \nabla\cdot(\rho\, u \otimes u)^n + \dfrac{1}{\varepsilon}\nabla p(\rho^{n+1}) = 0. & (2) \quad \text{(AC)} \end{cases}$$

$\nabla\cdot(2)$ **inserted into** (1) **:** gives an uncoupled formulation

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^n - \frac{\Delta t}{\varepsilon}\Delta p(\rho^{n+1}) - \Delta t\,\nabla^2 : (\rho\, u\otimes u)^n = 0,$$

- Results uniformly L^∞ stable if the space discretization is well chosen
- Framework of IMEX (IMplicit-EXplicit) schemes :

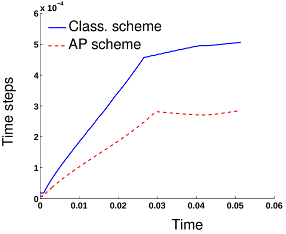
$$\partial_t \underbrace{\begin{pmatrix} \rho \\ \rho\, u \end{pmatrix}}_{W} + \nabla\cdot \underbrace{\begin{pmatrix} 0 \\ \rho\, u\otimes u \end{pmatrix}}_{F_e(W)} + \nabla\cdot \underbrace{\begin{pmatrix} \rho\, u \\ \frac{p(\rho)}{\varepsilon} Id \end{pmatrix}}_{F_i(W)} = 0.$$

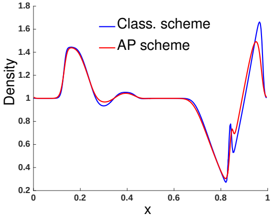
AP but diffusive results,1-D test-case

$\varepsilon = 0.99$, 300 cells

Class : 273 loops
CPU time 0.07

AP : 510 loops
CPU time 1.46

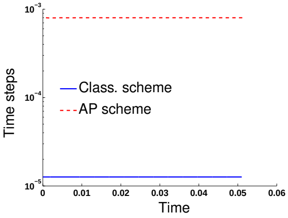


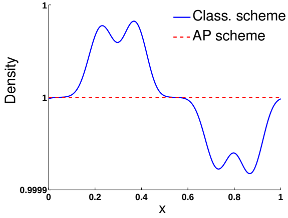


$\varepsilon = 10^{-4}$, 300 cells

Class : 4036 loops
CPU time 0.82

AP : 57 loops
CPU time 0.14

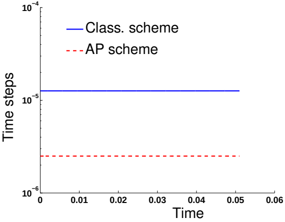


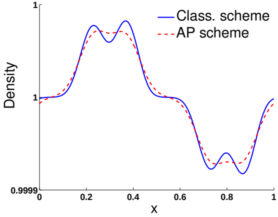


AP but diffusive results,1-D test-case

$\varepsilon = 10^{-4}$

Underlying of the viscosity





$\Downarrow$

It is necessary to use high order schemes

But they must respect the AP properties

Outline

Principle of IMEX schemes

Biblio for stiff source terms or ode pb. : Asher, Boscarino, Cafflish, Dimarco, Filbet, Gottlieb, Le Floch, Pareschi, Russo, Ruuth, Shu, Spiteri, Tadmor...

IMEX division : $\partial_t W + \nabla \cdot F_e(W) + \nabla \cdot F_i(W) = 0.$

General principle : Step n : W^n is known

- Quadrature formula with intermediate values

$$W(t^{n+1}) = W(t^n) - \underbrace{\int_{t^n}^{t^{n+1}} \nabla \cdot F_e(W(t))\, dt}\, - \underbrace{\int_{t^n}^{t^{n+1}} \nabla \cdot F_i(W(t))\, dt}$$

$$W^{n+1} = W^n \quad - \Delta t \sum_{j=1}^{s} \tilde{b}_j\, \nabla \cdot F_e(W^{n,j}) \; - \Delta t \sum_{j=1}^{s} b_j\, \nabla \cdot F_i(W^{n,j})$$

Quadratures exact on the constants : $\sum_{j=1}^{s} \tilde{b}_j = \sum_{j=1}^{s} b_j = 1$

- Intermediate values $t^{n,j} = t^n + c_j\, \Delta t$

$$W^{n,j} \approx W(t^n) + \int_{t^n}^{t^{n,j}} \partial_t W(t)\; dt = W^n + \Delta t \int_0^{c_j} \partial_t W(t^n + s\,\Delta t)\, ds$$

Principle of IMEX schemes

- Quadrature formula for intermediate values : $i = 1, \cdots, s$

$$W^{n,j} = W^n - \Delta t \sum_{k<j} \tilde{a}_{j,k} \nabla \cdot F_e(W^{n,k}) - \Delta t \sum_{k \geq j} a_{j,k} \nabla \cdot F_i(W^{n,k}),$$

Quadratures exact on the constants : $\sum_{k=1}^{s} \tilde{a}_{j,k} = \tilde{c}_j, \ \sum_{k=1}^{s} a_{j,k} = c_j$

- $W^{n+1} = W^n \quad -\Delta t \sum_{j=1}^{s} \tilde{b}_j \nabla \cdot F_e(W^{n,j}) \ -\Delta t \sum_{j=1}^{s} b_j \nabla \cdot F_i(W^{n,j})$

Butcher tableaux :

Expl. part

$$\begin{array}{c|cccc}
0 & 0 & 0 & \cdots & 0 \\
c_2 & \tilde{a}_{2,1} & 0 & \dots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_s & \tilde{a}_{s,1} & \dots & \tilde{a}_{s,s-1} & 0 \\
\hline
 & \tilde{b}_1 & \dots & \dots & \tilde{b}_s
\end{array}$$

Imp. part

$$\begin{array}{c|cccc}
c_1 & a_{1,1} & 0 & \cdots & 0 \\
c_2 & a_{2,1} & a_{2,2} & \dots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_s & a_{s,1} & \dots & a_{s,s-1} & a_{s,s} \\
\hline
 & b_1 & \dots & \dots & b_s
\end{array}$$

Conditions for 2nd order : $\sum b_j c_j = \sum b_j \tilde{c}_j = \sum \tilde{b}_j c_j = \sum \tilde{b}_j \tilde{c}_j = 1/2$

AP Order 2 scheme for Euler

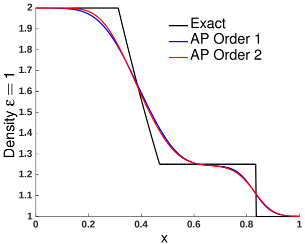
ARS scheme (Asher, Ruuth, Spiteri, 97) : "only 1" intermediate step

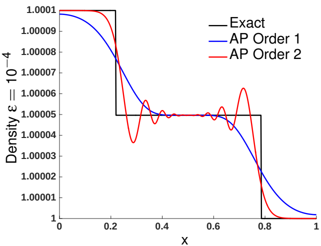
$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 \\ 1 & \beta-1 & 2-\beta & 0 \\ \hline & \beta-1 & 2-\beta & 0 \end{array} \qquad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \beta & 0 & \beta & 0 \\ 1 & 0 & 1-\beta & \beta \\ \hline & 0 & 1-\beta & \beta \end{array} \qquad \beta = 1 - \frac{1}{\sqrt{2}}$$

$$W^{n,1} = W^n$$

$$W^{n,2} = W^\star = W^n - \Delta t\, \beta\, \nabla \cdot F_e(W^n) - \Delta t\, \beta\, \nabla \cdot F_i(W^\star),$$

$$W^{n,3} = W^{n+1} = W^n - \Delta t(\beta - 1)\nabla \cdot F_e(W^n) - \Delta t(2-\beta)\nabla \cdot F_e(W^\star)$$
$$- \Delta t(1-\beta)\nabla \cdot F_i(W^\star) - \Delta t\beta\nabla \cdot F_i(W^{n+1}).$$





Better understand the oscillations

For a scalar hyperbolic eq. $\partial_t w + \partial_x f(w) = 0$

- Oscillations measured by the Total Variation and the L^∞ norm

$$TV(w^n) = \sum_j |w_{j+1}^n - w_j^n| \qquad\qquad \|w^n\|_\infty = \max |w_j^n|$$

- TVD (Total Variation Diminishing) property and L^∞ stability

$$\left\{ \begin{array}{l} TV(w^{n+1}) \leq TV(w^n) \\ \|w^{n+1}\|_\infty \leq \|w^n\|_\infty \end{array} \right. \qquad \Leftrightarrow \text{ no oscillations}$$

First idea : Find an AP order 2 scheme which satisfies these properties

Impossible

Theorem (Gottlieb,Shu,Tadmor, 01) There are no implicit Runge-Kutta schemes of order higher than one which preserves the TVD property.

A limiting procedure

Another idea : use a limited scheme

$$W^{n+1} = \theta\, W^{n+1,O2} + (1-\theta)\, W^{n+1,O1},$$

- $W^{n+1,Oj}$ = order j AP approximation
- $\theta \in [0,1]$ largest value such that W^{n+1} does not oscillate

Toy scalar equation $\quad \partial_t w + \dfrac{c_i}{\sqrt{\varepsilon}} \partial_x w + c_e \partial_x w = 0$

- Order 1 AP scheme with upwind space discretizations ($c_e, c_i > 0$)

$$w_j^{n+1,O1} = w_j^n - \frac{c_i}{\sqrt{\varepsilon}} (w_j^{n+1,O1} - w_{j-1}^{n+1,O1}) - c_e (w_j^n - w_{j-1}^n)$$

- Order 2 AP scheme ARS with the parameter $\beta = 1 - 1/\sqrt{2}$

Lemma (VMD,Loubère,Vignal) Under the CFL condition $\Delta t \leq \Delta x / c_e$

$$\theta \leq \frac{\beta}{1-\beta} \simeq 0.41 \quad \Rightarrow \begin{cases} TV(w^{n+1}) \leq TV(w^n) \\ \|w^{n+1}\|_\infty \leq \|w^n\|_\infty \end{cases}$$

A MOOD procedure

Limited AP scheme :

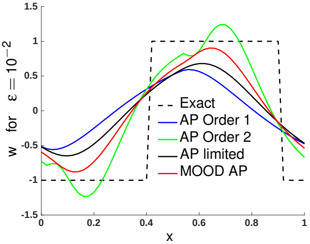
- $w^{n+1,lim} = \theta\, w^{n+1,O2} + (1-\theta)\, w^{n+1,O1}$ with $\theta = \frac{\beta}{1-\beta}$

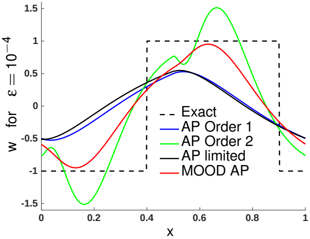
Problem : More accurate than order 1 but not order 2

Solution : MOOD procedure (Clain, Diot, Loubère, 11)

On the toy equation $w^{n+1,HO}$ **MOOD AP scheme,** CFL $\Delta t \leq \Delta x / c_e$

- Compute the order 2 approximation $w^{n+1,O2}$
- Detect if the max. principle is satisfied : $\|w^{n+1,O2}\|_\infty \leq \|w^n\|_\infty$?
- If not, compute the limited AP approximation $w^{n+1,lim}$



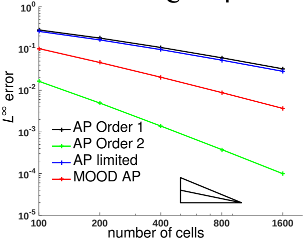


Outline

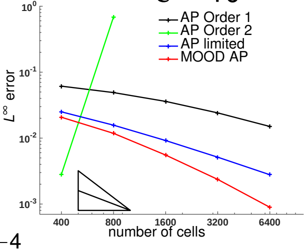
Error curves for the model problem

- Order 2 in space with MUSCL but explicit slopes for implicit fluxes
- Error curves on a smooth solution for the toy model

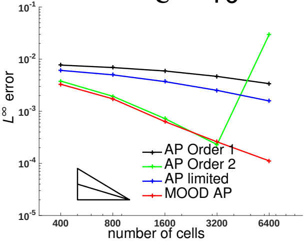
$\varepsilon = 1$



$\varepsilon = 10^{-2}$



$\varepsilon = 10^{-4}$



Second-order scheme for the Euler equations

Recall the first-order IMEX scheme for the Euler system :

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^{n+1} = 0, & (1) \\ \dfrac{(\rho\, u)^{n+1}-(\rho\, u)^n}{\Delta t} + \nabla\cdot(\rho\, u\otimes u)^n + \dfrac{1}{\varepsilon}\nabla p(\rho^{n+1}) = 0. & (2) \end{cases}$$

We apply the same convex combination procedure :

$$W^{n+1,lim} = \theta\, W^{n+1,O2} + (1-\theta)\, W^{n+1,O1}, \text{ with } \theta = \beta/(1-\beta).$$

$\rightsquigarrow$ we use the value of θ given by the study of the model problem

$\rightsquigarrow$ But we need we need a criterion to detect oscillations in the MOOD procedure !

Euler equations : MOOD procedure

The previous detector (L^∞ criterion on the solution) is irrelevant for the Euler equations, since ρ and u do not satisfy a maximum principle.

$\rightsquigarrow$ we need another detection criterion

We pick the Riemann invariants $\Phi_\pm = u \mp \frac{2}{\gamma - 1}\sqrt{\frac{1}{\varepsilon}\frac{\partial p(\rho)}{\partial \rho}}$, since they satisfy an advection equation for smooth solutions.

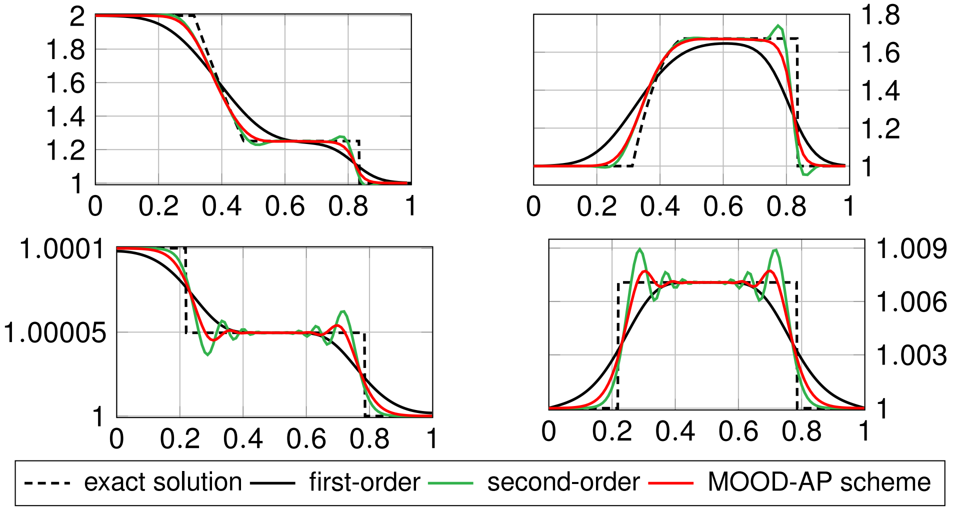
On the Euler equations $W^{n+1,HO}$ **MOOD AP scheme,** CFL $\Delta t \leq \Delta x / \lambda$

- Compute the order 2 approximation $W^{n+1,O2}$
- Detect if both Riemann invariants break the maximum principle at the same time
- If so, compute the limited AP approximation $W^{n+1,lim}$

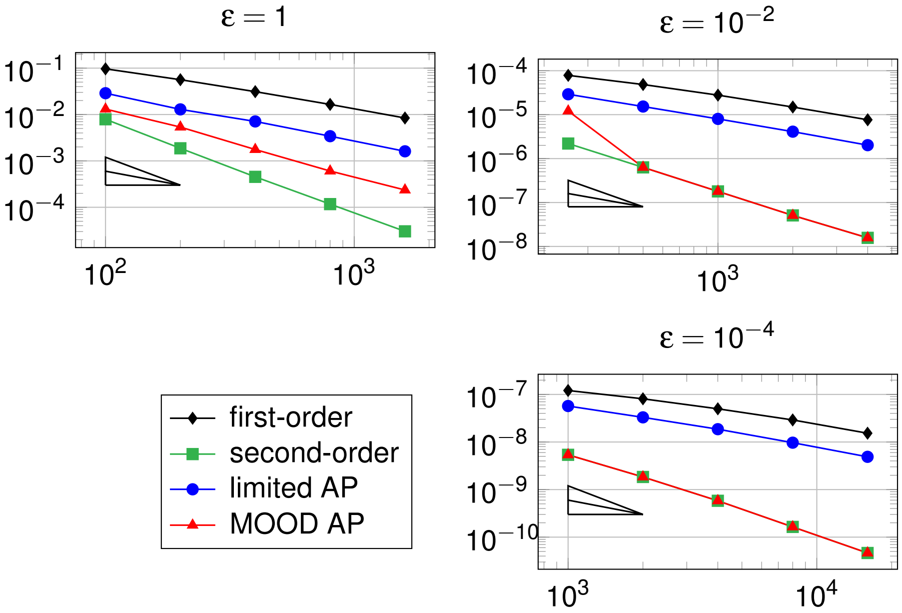
Euler equations : Numerical results

Riemann problem : left rarefaction wave, right shock ;
top curves : $\varepsilon = 1$; bottom curves : $\varepsilon = 10^{-4}$

Density ρ — Momentum $q = \rho u$



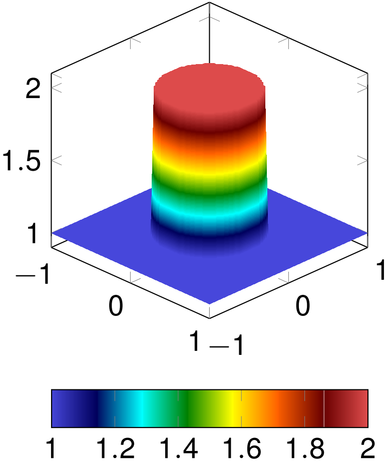
Euler equations : Numerical results

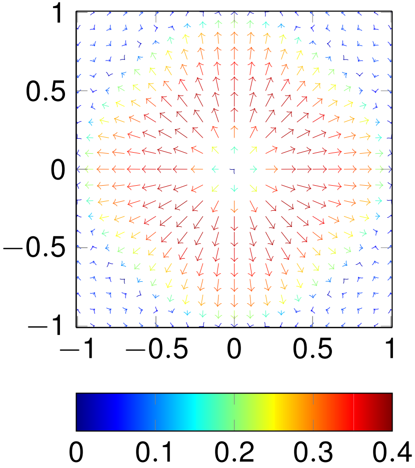


Euler equations : Numerical results

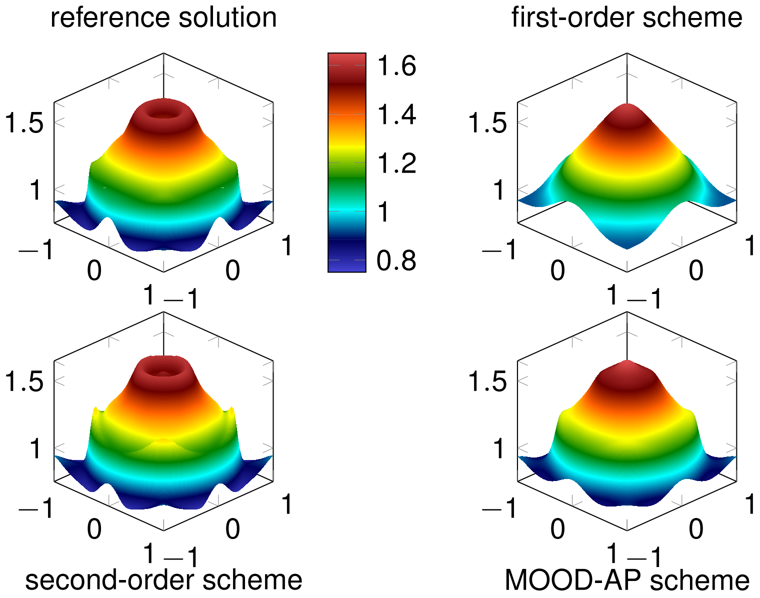
Initial data for the explosion :

- density cylinder (left) ;
- outwards velocity field (right).





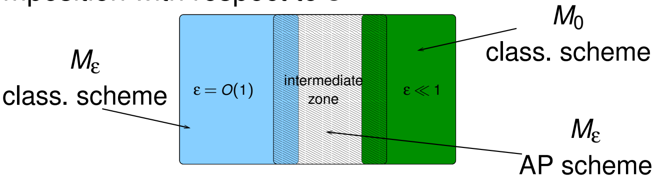
Euler equations : Numerical results



Outline

Works in progress and perspectives

- Choose the order 2 time discretization to get a θ as close as possible to 1 for the stability
- Study a local value of θ, depending on the presence of oscillations in a given cell
- Extension to full Euler (order 1 scheme exists but we have trouble decomposing between an explicit and an implicit flux)
- Simulations of physically relevant phenomena
- Domain decomposition with respect to ε



Thanks !

Bibliography

All speed schemes

- **Preconditioning methods :** [Chorin 65], [Choi, Merkle 85], [Turkel, 87], [Van Leer, Lee, Roe, 91], [Li,Gu 08,10], $\cdots$
- **Splitting and pressure correction :** [Harlow, Amsden, 68,71], [Karki, Patankar, 89], [Bijl, Wesseling, 98], [Sewall, Tafti, 08], [Klein, Botta, Schneider, Munz, Roller 08], [Guillard, Murrone, Viozat 99, 04, 06] [Herbin, Kheriji, Latché 12,13], $\cdots$
  - **Asymptotic preserving schemes** [Degond, Deluzet, Sangam, Vignal, 09], [Degond, Tang 11], [Cordier, Degond, Kumbaro 12], [Grenier, Vila, Villedieu 13] [Dellacherie, Omnes, Raviart,13], [Noelle, Bispen, Arun, Lukacova, Munz,14], [Chalons, Girardin, Kokh,15] [Dimarco, Loubère, Vignal, 17]