A well-balanced scheme for the shallow-water equations with topography and Manning friction

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Contents

1 Introduction

2 A well-balanced scheme

3 1D numerical experiments

4 Two-dimensional and high-order extensions

5 2D numerical experiments

6 Conclusion and perspectives

1 Introduction

2 A well-balanced scheme

3 1D numerical experiments

4 Two-dimensional and high-order extensions

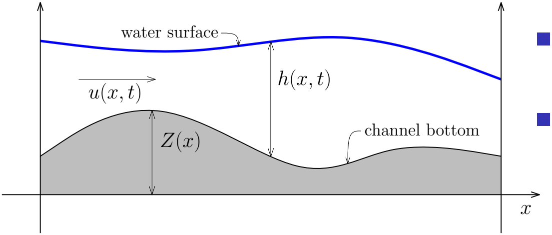
5 2D numerical experiments

6 Conclusion and perspectives

The shallow-water equations and their source terms

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{cases} \text{(with } q = hu\text{)}$$

we can rewrite the equations as $\partial_t W + \partial_x F(W) = S(W)$



- $\eta = 7/3$ and g is the gravitational constant
- $k \geq 0$ is the so-called Manning coefficient: a higher k leads to a stronger Manning friction

Steady state solutions

Definition: Steady state solutions

W is a steady state solution iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$

taking $\partial_t W = 0$ in the shallow-water equations leads to

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq|q|}{h^\eta} \end{cases}$$

the steady state solutions are therefore given by

$$\begin{cases} q = \text{cst} = q_0 \\ \partial_x \left(\dfrac{q_0^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z - \dfrac{kq_0|q_0|}{h^\eta} \end{cases}$$

Objectives

1. derive a scheme that:
   - is well-balanced for the shallow-water equations with friction and/or topography, i.e.:
     - preservation of all steady states with $k = 0$ and $Z \neq \text{cst}$
     - preservation of all steady states with $k \neq 0$ and $Z = \text{cst}$
     - preservation of steady states with $k \neq 0$ and $Z \neq \text{cst}$
   - preserves the non-negativity of the water height
   - is able to deal with wet/dry transitions
2. provide two-dimensional and high-order extensions of this scheme, while keeping the above properties

1 Introduction

2 A well-balanced scheme

3 1D numerical experiments

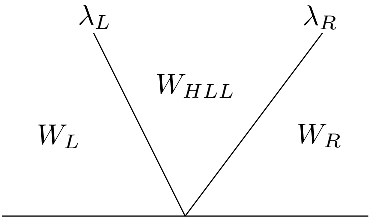
4 Two-dimensional and high-order extensions

5 2D numerical experiments

6 Conclusion and perspectives

The HLL scheme

to approximate solutions of $\partial_t W + \partial_x F(W) = 0$, we choose the HLL scheme (Harten, Lax, van Leer (1983)), which uses the approximate Riemann solver $\widetilde{W}$, to the right:



the consistency condition (as per Harten and Lax) holds if:

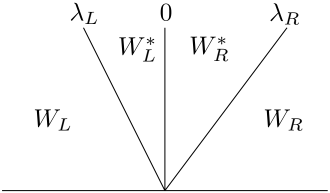
$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} \widetilde{W}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}\left(\frac{x}{\Delta t}; W_L, W_R\right) dx$$

which gives $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \begin{pmatrix} h_{HLL} \\ q_{HLL} \end{pmatrix}$

note that $h_{HLL} > 0$ for $|\lambda_L|$ and $|\lambda_R|$ large enough

Modification of the HLL scheme

to approximate solutions of $\partial_t W + \partial_x F(W) = S(W)$, we use the following approximate Riemann solver (assuming $\lambda_L < 0 < \lambda_R$):



$\rightsquigarrow$ 3 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$;

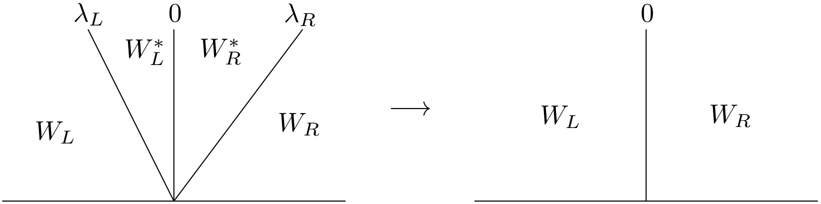
Harten-Lax consistency gives us

- $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$
- $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$ (with $\overline{S} = \overline{S}(W_L, W_R)$ approximating the mean of $S(W)$, to be determined)

Determination of h_L^* and h_R^*

assume that W_L and W_R define a steady state, i.e. satisfy the following discrete version of $\partial_x F(W) = S(W)$:

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S}\Delta x$$



for the steady state to be preserved, we need

$$W_L^* = W_L \text{ and } W_R^* = W_R, \text{ i.e. } h_L^* = h_L,\ h_R^* = h_R \text{ and } q^* = q_0$$

as soon as W_L and W_R define a steady state

Determination of h_L^* and h_R^*

two unknowns $\rightsquigarrow$ we need two equations

- we have $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$
- we choose $\alpha(h_R^* - h_L^*) = \overline{S}\Delta x$

where $\alpha = \dfrac{-\bar{q}^2}{h_L h_R} + \dfrac{g}{2}(h_L + h_R)$, with $\bar{q}$ to be determined

$\rightsquigarrow$ using both relations, we obtain

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)} \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)} \end{cases}$$

Correction to ensure non-negative h_L^* and h_R^*

however, these expressions of h_L^* and h_R^* do not guarantee that the intermediate heights are non-negative: instead, we use (see Audusse, Chalons, Ung (2014))

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL} \right) \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL} \right) \end{cases}$$

note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$
- the well-balance property, since it is not activated when W_L and W_R define a steady state

Summary

using a two-state approximate Riemann solver with intermediate states $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$ given by

$$\begin{cases} q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L} \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL} \right) \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\,\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL} \right) \end{cases}$$

yields a scheme that is consistent, non-negativity-preserving and well-balanced; we now need to find $\overline{S}$ and α (i.e. $\bar{q}$) according to the source term definition

The topography source term

we now consider $S(W) = S^t(W) = -gh\partial_x Z$:
discrete smooth steady states are governed by

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}[h^2] = \overline{S}^t \Delta x$$

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h+Z] = 0$$

we can exhibit an expression of q_0^2 and thus obtain

$$\overline{S}^t = -g\frac{2h_L h_R}{h_L + h_R}\frac{[Z]}{\Delta x} + \frac{g}{2\Delta x}\frac{[h]^3}{h_L + h_R}$$

but when $Z_L = Z_R$, we have $\overline{S}^t \neq \mathcal{O}(\Delta x) \rightsquigarrow$ loss of consistency with S^t (see for instance Berthon, Chalons (2015))

The topography source term

instead, we set, for some constant C,

$$\begin{cases} \overline{S}^t = -g \dfrac{2h_L h_R}{h_L + h_R} \dfrac{[Z]}{\Delta x} + \dfrac{g}{2\Delta x} \dfrac{[h]_c^3}{h_L + h_R} \\ [h]_c = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C\,\Delta x \\ \operatorname{sgn}(h_R - h_L)\, C\,\Delta x & \text{otherwise.} \end{cases} \end{cases}$$

Theorem: Well-balance for the topography source term

If W_L and W_R define a steady state, i.e. verify

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h + Z] = 0,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$.

this result holds for any $\bar{q}$: we choose $\bar{q} = q^*$

The friction source term

we consider, in this case, $S(W) = S^f(W) = -kq|q|h^{-\eta}$

the average of S^f we choose is $\overline{S}^f = -k\hat{q}|\hat{q}|\overline{h^{-\eta}}$, with

- $\hat{q}$ the harmonic mean of q_L and q_R (note that $\hat{q} = q_0$ at the equilibrium), and
- $\overline{h^{-\eta}}$ a well-chosen discretization of $h^{-\eta}$, depending on h_L and h_R, and ensuring the well-balance property

we determine $\overline{h^{-\eta}}$ using the same technique (with $\mu_0 = \mathrm{sgn}(q_0)$):

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = -k\mu_0 q_0^2 \overline{h^{-\eta}} \Delta x$$

$$-q_0^2 \frac{\left[h^{\eta-1}\right]}{\eta - 1} + g \frac{\left[h^{\eta+2}\right]}{\eta + 2} = -k\mu_0 q_0^2 \Delta x$$

The friction source term

the expression for q_0^2 we obtained is now used to get:

$$\overline{h^{-\eta}} = \frac{-\mu_0}{k\Delta x}\left(\left[\frac{1}{h}\right] + \frac{[h^2]}{2}\frac{\eta+2}{[h^{\eta+2}]}\left(\frac{[h^{\eta-1}]}{\eta-1} - k\mu_0\Delta x\right)\right),$$

which gives $\overline{S}^f = -k\hat{q}|\hat{q}|\overline{h^{-\eta}}$ ($\overline{h^{-\eta}}$ is consistent with $h^{-\eta}$)

Theorem: Well-balance for the friction source term

If W_L and W_R define a steady state, i.e. verify

$$q_0^2\frac{[h^{\eta-1}]}{\eta-1} + g\frac{[h^{\eta+2}]}{\eta+2} = -kq_0|q_0|\Delta x,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$.

this result holds for any $\bar{q}$: we choose $\bar{q} = q^*$

Friction and topography source terms

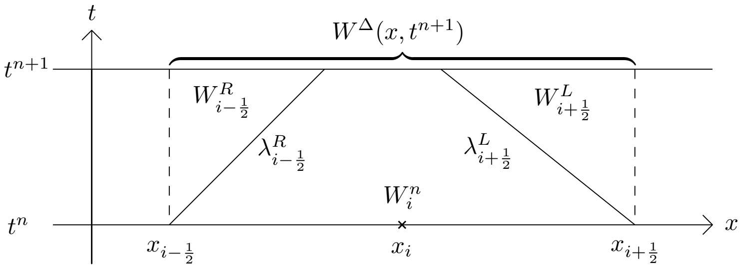
with both source terms, the scheme preserves the following discretization of the steady relation $\partial_x F(W) = S(W)$:

$$q_0^2 \left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right] = \overline{S}^t \Delta x + \overline{S}^f \Delta x$$

the intermediate states are therefore given by:

$$\begin{cases} q^* = q_{HLL} + \dfrac{(\overline{S}^t + \overline{S}^f)\Delta x}{\lambda_R - \lambda_L} \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right) \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L(\overline{S}^t + \overline{S}^f)\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right) \end{cases}$$

The full Godunov-type scheme



define $W_i^{n+1} = \dfrac{1}{\Delta x} \displaystyle\int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^{\Delta}(x, t^{n+1}) dx$: then

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left[\lambda^L_{i+\frac{1}{2}}\left(W^L_{i+\frac{1}{2}} - W_i^n\right) - \lambda^R_{i-\frac{1}{2}}\left(W^R_{i-\frac{1}{2}} - W_i^n\right)\right],$$

which can be rewritten, after straightforward computations,

$$W_i^{n+1} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}}\right) + \Delta t\left(\begin{pmatrix} 0 \\ (\mathcal{S}^t)^n_i \end{pmatrix} + \begin{pmatrix} 0 \\ (\mathcal{S}^f)^n_i \end{pmatrix}\right)$$

Summary

we have presented a scheme that:

- is consistent with the shallow-water equations with friction and topography
- is well-balanced for friction and topography steady states
- preserves the non-negativity of the water height
- is not able to correctly approximate wet/dry interfaces: we need a semi-implicitation of the friction source term

$\rightsquigarrow$ how to introduce this semi-implicitation?

Semi-implicit finite volume scheme

we use a splitting method: explicit treatment of the flux and the topography; implicit treatment of the friction

1. explicitly solve $\partial_t W + \partial_x F(W) = S^t(W)$ to get

$$W_i^{n+\frac{1}{2}} = W_i^n - \frac{\Delta t}{\Delta x}\left(\mathcal{F}^n_{i+\frac{1}{2}} - \mathcal{F}^n_{i-\frac{1}{2}}\right) + \Delta t \begin{pmatrix} 0 \\ \left(\mathcal{S}^t\right)_i^n \end{pmatrix}$$

2. implicitly solve $\partial_t W = S^f(W)$ to get

$$\begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\ \text{IVP: } \begin{cases} \partial_t q = -kq|q|(h_i^{n+1})^{-\eta} \\ q(x_i, t^n) = q_i^{n+\frac{1}{2}} \end{cases} \rightsquigarrow q_i^{n+1} \end{cases}$$

Semi-implicit finite volume scheme

solving the IVP yields:

$$q_i^{n+1} = \frac{(h_i^{n+1})^\eta q_i^{n+\frac{1}{2}}}{(h_i^{n+1})^\eta + k\,\Delta t \left| q_i^{n+\frac{1}{2}} \right|}$$

we use the following approximation of $(h_i^{n+1})^\eta$: this provides us with an expression of q_i^{n+1} that is equal to q_0 at the equilibrium

$$(\overline{h^\eta})_i^{n+1} = \frac{2\mu_i^{n+\frac{1}{2}}\mu_i^n}{\left(\overline{h^{-\eta}}\right)_{i-\frac{1}{2}}^{n+1} + \left(\overline{h^{-\eta}}\right)_{i+\frac{1}{2}}^{n+1}} + k\,\Delta t\,\mu_i^{n+\frac{1}{2}} q_i^n,$$

- semi-implicit treatment of the friction source term
  $\rightsquigarrow$ scheme able to model wet/dry transitions
- scheme still well-balanced and non-negativity-preserving

1 Introduction

2 A well-balanced scheme

3 1D numerical experiments

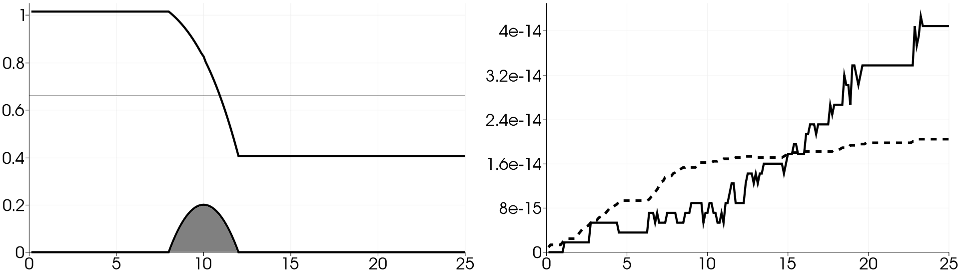
4 Two-dimensional and high-order extensions

5 2D numerical experiments

6 Conclusion and perspectives

Verification of the well-balance: topography

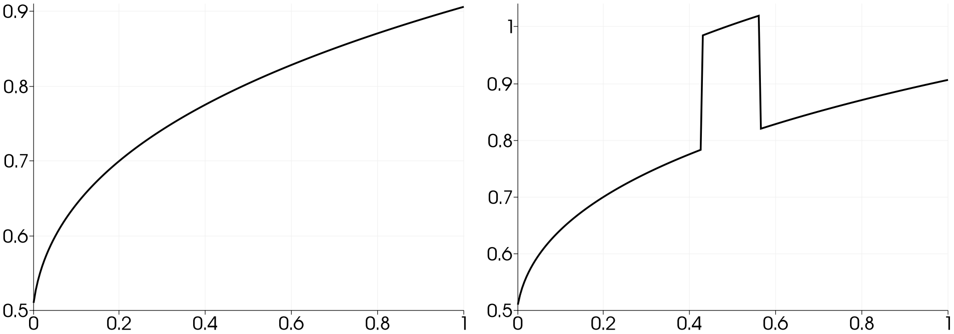
we show the so-called transcritical flow test case (see Goutal, Maurel (1997)): here, we assume $k = 0$



left panel: initial free surface and free surface for the steady state solution, obtained after a transient state

right panel: errors to the steady state (solid: h, dashed: q)

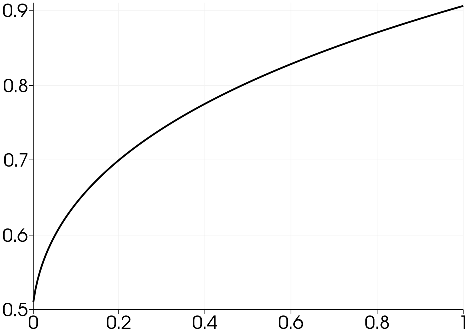
Verification of the well-balance: friction

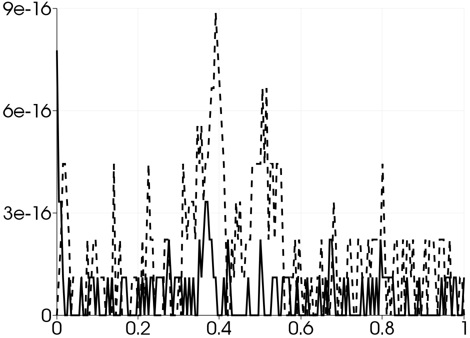


left panel: water height for the steady state solution

right panel: water height for the perturbed steady state solution

Verification of the well-balance: friction





left panel: convergence to the unperturbed steady state

right panel: errors to the steady state (solid: h, dashed: q)

1 Introduction

2 A well-balanced scheme

3 1D numerical experiments

4 Two-dimensional and high-order extensions

5 2D numerical experiments

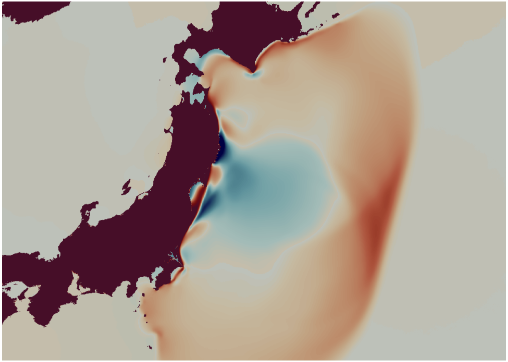
6 Conclusion and perspectives

Two-dimensional extension

2D shallow-water model: $\partial_t W + \boldsymbol{\nabla} \cdot \boldsymbol{F}(W) = \boldsymbol{S}^t(W) + \boldsymbol{S}^f(W)$

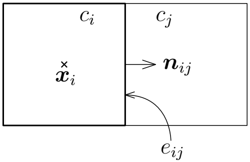
$$\begin{cases} \partial_t h + \boldsymbol{\nabla} \cdot \boldsymbol{q} = 0 \\ \partial_t \boldsymbol{q} + \boldsymbol{\nabla} \cdot \left(\dfrac{\boldsymbol{q} \otimes \boldsymbol{q}}{h} + \dfrac{1}{2} g h^2 \mathbb{I}_2 \right) = -gh \boldsymbol{\nabla} Z - \dfrac{k \boldsymbol{q} \| \boldsymbol{q} \|}{h^\eta} \end{cases}$$

to the right:
simulation of the 2011 Japan tsunami



Two-dimensional extension

the space discretization is a Cartesian mesh:



with $\mathcal{F}_{ij}^n = \mathcal{F}(W_i^n, W_j^n; \boldsymbol{n}_{ij})$, the scheme reads:

$$W_i^{n+\frac{1}{2}} = W_i^n - \Delta t \sum_{j \in \nu_i} \frac{|e_{ij}|}{|c_i|} \mathcal{F}_{ij}^n + \Delta t \sum_{j \in \nu_i} (\mathcal{S}^t)_{ij}^n$$

W_i^{n+1} is obtained from $W_i^{n+\frac{1}{2}}$ with a splitting strategy:

$$\begin{cases} \partial_t h = 0 \\ \partial_t \boldsymbol{q} = -k\, \boldsymbol{q} \|\boldsymbol{q}\| h^{-\eta} \end{cases} \rightsquigarrow \begin{cases} h_i^{n+1} = h_i^{n+\frac{1}{2}} \\ \boldsymbol{q}_i^{n+1} = \dfrac{(\overline{h^\eta})_i^{n+1} \boldsymbol{q}_i^{n+\frac{1}{2}}}{(\overline{h^\eta})_i^{n+1} + k\, \Delta t \left\| \boldsymbol{q}_i^{n+\frac{1}{2}} \right\|} \end{cases}$$

Two-dimensional extension

the 2D scheme is:

- non-negativity-preserving for the water height:
  $\forall i \in \mathbb{Z}, h_i^n \geq 0 \Longrightarrow \forall i \in \mathbb{Z}, h_i^{n+1} \geq 0$
- able to deal with wet/dry transitions thanks to the semi-implicitation with the splitting method
- well-balanced by direction for the shallow-water equations with friction and/or topography, i.e.:
  - it preserves all steady states at rest
  - it preserves friction and/or topography steady states in the x-direction and the y-direction
  - it does not preserve the fully 2D steady states

$\rightsquigarrow$ high-order extension?

High-order extension: the polynomial reconstruction

polynomial reconstruction (Diot, Clain, Loubère (2012)):

$$\widehat{W}_i^n(x) = W_i^n + \sum_{|k|=1}^{d} \alpha_i^k \Big[(x - x_i)^k - M_i^k \Big],$$

- we have $M_i^k = \dfrac{1}{|c_i|} \displaystyle\int_{c_i} (x - x_i)^k dx$
- the polynomial coefficients α_i^k are chosen to minimize the least squares error between the reconstruction and W_j^n, for all j in the stencil S_i^d
- the conservation property is verified: $\dfrac{1}{|c_i|} \displaystyle\int_{c_i} \widehat{W}_i^n(x) dx = W_i^n$

High-order extension: the scheme

High-order space accuracy

$$W_i^{n+1} = W_i^n - \Delta t \sum_{j \in \nu_i} \frac{|e_{ij}|}{|c_i|} \sum_{r=0}^{R} \xi_r \mathcal{F}_{ij,r}^n + \Delta t \sum_{q=0}^{Q} \eta_q \left((\mathcal{S}^t)_{i,q}^n + (\mathcal{S}^f)_{i,q}^n \right)$$

- $\mathcal{F}_{ij,r}^n = \mathcal{F}(\widehat{W}_i^n(\sigma_r), \widehat{W}_j^n(\sigma_r); \boldsymbol{n}_{ij})$
- $(\mathcal{S}^t)_{i,q}^n = S^t(\widehat{W}_i^n(x_q))$ and $(\mathcal{S}^f)_{i,q}^n = S^f(\widehat{W}_i^n(x_q))$

where we have set:

- $(\xi_r, \sigma_r)_r$ quadrature on the edge e_{ij}
- $(\eta_q, x_q)_q$ quadrature on the cell c_i

time accuracy: SSPRK methods (Gottlieb, Shu (1998))

MOOD method

high-order schemes induce oscillations $\rightsquigarrow$ MOOD method to get rid of the oscillations and restore the non-negativity-preservation

MOOD loop

1. compute a candidate solution with the high-order scheme
2. test if the candidate solution satisfies several criteria:
   - PAD, to recover the non-negativity-preservation
   - DMP, to eliminate oscillations
   - u2, to detect smooth extrema
3. where necessary, decrease the degree of the reconstruction
4. compute a new candidate solution

Well-balance recovery (1D)

reconstruction procedure $\rightsquigarrow$ scheme no longer well-balanced

Well-balance recovery

we suggest a convex combination between the high-order scheme W_{HO} and the well-balanced scheme W_{WB}:

$$W_i^{n+1} = \theta_i^n (W_{HO})_i^{n+1} + (1 - \theta_i^n)(W_{WB})_i^{n+1}$$

$\theta_i^n \in [0, 1]$ parameter of the convex combination:

- $\theta_i^n = 0$: the well-balanced scheme is used
- $\theta_i^n = 1$: the high-order scheme is used

goal: derive a suitable expression for θ_i^n

Well-balance recovery (1D)

Steady state detector

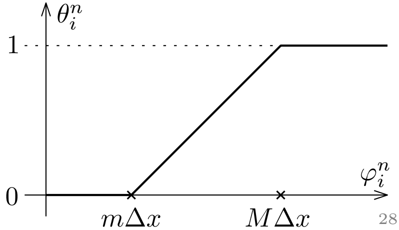
steady state solution: $\begin{cases} q_L = q_R = q_0, \\ \dfrac{q_0^2}{h_R} - \dfrac{q_0^2}{h_L} + \dfrac{g}{2}\left(h_R^2 - h_L^2\right) = (\overline{S}^t + \overline{S}^f)\Delta x \end{cases}$

steady state detector: $\varphi_i^n = \left\| \begin{pmatrix} q_i^n - q_{i-1}^n \\ \Delta \psi_{i-\frac{1}{2}}^n \end{pmatrix} \right\|_2 + \left\| \begin{pmatrix} q_{i+1}^n - q_i^n \\ \Delta \psi_{i+\frac{1}{2}}^n \end{pmatrix} \right\|_2$

$\varphi_i^n = 0$ if there is a steady state between W_{i-1}^n, W_i^n and W_{i+1}^n

$\rightsquigarrow$ in this case, we take $\theta_i^n = 0$

$\rightsquigarrow$ otherwise, we take $\theta_i^n > 0$



1 Introduction

2 A well-balanced scheme

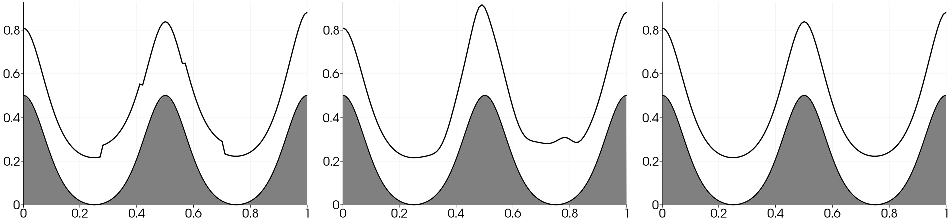
3 1D numerical experiments

4 Two-dimensional and high-order extensions

5 2D numerical experiments

6 Conclusion and perspectives

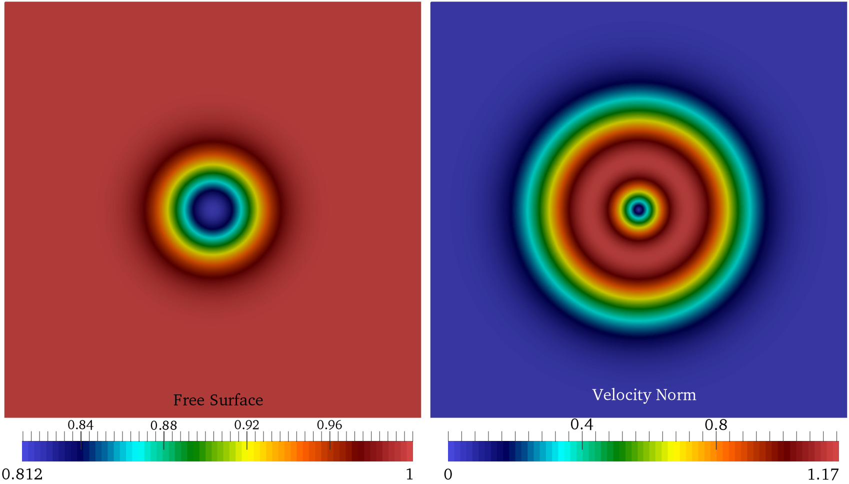
Perturbed 1D topography and friction steady state



	h			$\|\boldsymbol{q}\|$		
	L^1	L^2	L^∞	L^1	L^2	L^∞
$\mathbb{P}_0$	1.22e-15	1.71e-15	6.27e-15	2.34e-15	3.02e-15	9.10e-15
$\mathbb{P}_5$	5.01e-05	1.47e-04	1.16e-03	2.32e-04	2.63e-04	1.18e-03
$\mathbb{P}_5^{\mathrm{WB}}$	8.50e-14	1.05e-13	3.35e-13	2.82e-13	3.37e-13	6.76e-13

Order of accuracy assessment: $k = 0$

we consider only the topography (i.e. $k = 0$): 2D steady solution, not preserved by the scheme (well-balance by direction)



Order of accuracy assessment: $k = 0$

N	$h,\ L^1$		$h,\ L^2$		$h,\ L^\infty$	
900	2.04e-05	—	5.22e-05	—	7.84e-04	—
3600	3.07e-07	6.05	6.88e-07	6.25	9.94e-06	6.30
14400	3.93e-09	6.29	5.82e-09	6.88	5.53e-08	7.49
57600	5.74e-11	6.10	7.27e-11	6.32	3.30e-10	7.39

N	$\Vert\boldsymbol{q}\Vert,\ L^1$		$\Vert\boldsymbol{q}\Vert,\ L^2$		$\Vert\boldsymbol{q}\Vert,\ L^\infty$	
900	1.37e-04	—	3.46e-04	—	2.90e-03	—
3600	1.90e-06	6.17	5.27e-06	6.04	5.10e-05	5.83
14400	2.33e-08	6.35	5.33e-08	6.63	4.98e-07	6.68
57600	3.08e-10	6.24	5.76e-10	6.53	4.42e-09	6.82

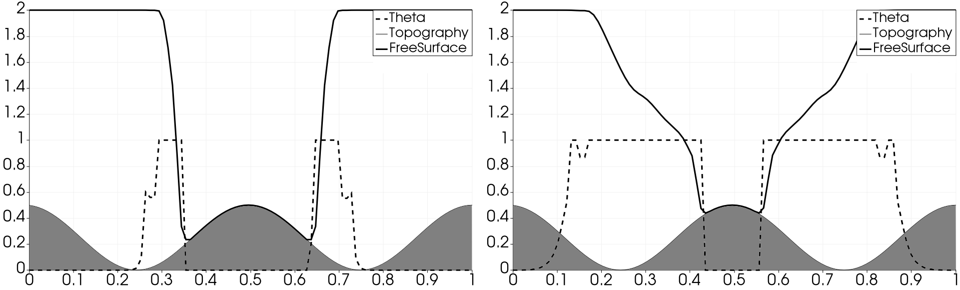
Order of accuracy assessment: $k \neq 0$

$$Z(x,y) = \frac{2k\|\boldsymbol{r}\| - 1}{2g\|\boldsymbol{r}\|^2} \quad ; \quad h(t,x,y) = 1 \quad ; \quad \boldsymbol{q}(t,x,y) = \frac{\boldsymbol{r}}{\|\boldsymbol{r}\|^2}$$

N	h		q_x		q_y	
900	2.37e-08	—	8.00e-08	—	1.12e-07	—
3600	3.77e-10	5.98	1.28e-09	5.96	1.82e-09	5.94
14400	5.89e-12	6.00	1.99e-11	6.01	2.91e-11	5.96
57600	1.24e-14	8.89	2.06e-13	6.60	1.20e-13	7.92

N	h		q_x		q_y	
900	1.04e-07	—	5.20e-07	—	5.57e-07	—
3600	1.80e-09	5.86	8.15e-09	6.00	1.02e-08	5.77
14400	3.38e-11	5.73	1.25e-10	6.02	1.71e-10	5.89
57600	8.33e-13	5.34	2.26e-12	5.79	2.59e-12	6.05

Double dry dam-break on a sinusoidal bottom



near the edges, steady state at rest $\rightsquigarrow$ well-balanced scheme

away from the edges, far from steady state $\rightsquigarrow$ high-order scheme

center, dry area $\rightsquigarrow$ well-balanced scheme

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Conclusion

1D scheme:

- well-balanced for the shallow-water equations with friction and topography
- non-negativity preservation for the height
- suitable approximation of wet/dry interfaces

2D scheme:

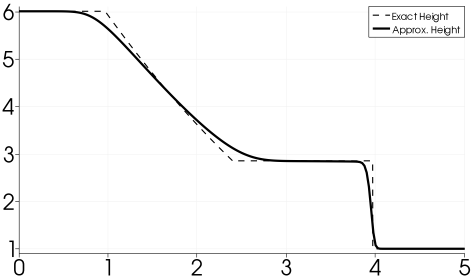
- well-balance by direction, the above properties
- high-order accuracy

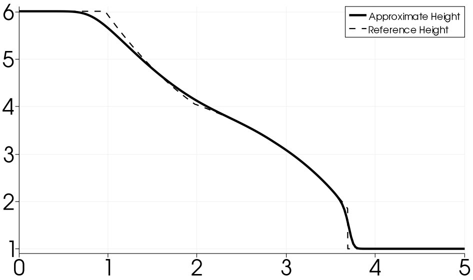
Perspectives

- real-world applications
- entropy inequality for the 1D scheme
- order of accuracy of the convex combination

Thank you for your attention!

Riemann problems between two wet areas



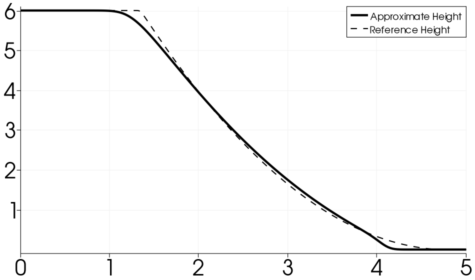


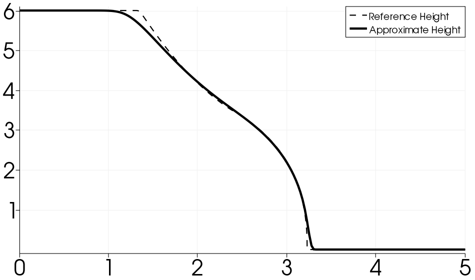
left: $k = 0$

left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

Riemann problems with a wet/dry transition





left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.15s

Double dry dam-break on a sinusoidal bottom