Second order Implicit-Explicit Total Variation Diminishing schemes for the Euler system in the low Mach regime

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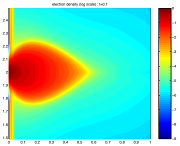
Outline

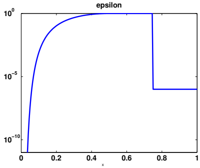
General context

Multiscale model M_ε, depending on a parameter ε



In the (space-time) domain, ε can

- be of same order as the reference scale;
- be small compared to the reference scale;
- take intermediate values.



When ε is small: $M_0 = \lim_{\varepsilon \to 0} M_\varepsilon$ asympt. model

Difficulties:

- Classical explicit schemes for M_ε: they are stable and consistent if the mesh resolves all the scales of ε. $\Longrightarrow$ very costly when $\varepsilon \to 0$
- Schemes for M_0 $\Longrightarrow$ the mesh is independent of ε

  But: ➠ M_0 is not valid everywhere, it needs $\varepsilon \ll 1$

  ➠ the interface may be moving: how to locate it?

Principle of AP schemes

A possible solution: Asymptotic Preserving (AP) schemes

- Use the multi-scale model M_ε even for small ε.
- Discretize M_ε with a scheme preserving the limit $\varepsilon \to 0$.

➠ The mesh is independent of ε: **Asymptotic stability**.

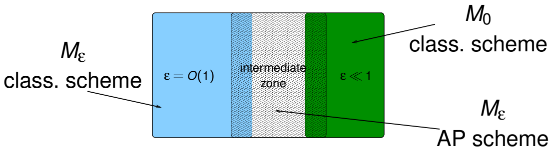
➠ Recovery of an approximate solution of M_0 when $\varepsilon \to 0$: **Asymptotic consistency**.

➠ Asymptotically stable and consistent scheme

$\Longrightarrow$ **Asymptotic preserving scheme (AP)**.

([Jin, '99] kinetic $\to$ hydro)

- The AP scheme may be used only to reconnect M_ε and M_0.



Outline

The multi-scale model and its asymptotic limit

➠ **Isentropic Euler system in scaled variables:** $x \in \Omega \subset \mathbb{R}^d$, $t \geq 0$

$$(M_\varepsilon)\begin{cases} \partial_t \rho + \nabla\cdot(\rho\, u) = 0 & (1)_\varepsilon \\ \partial_t(\rho\, u) + \nabla\cdot(\rho\, u \otimes u) + \dfrac{1}{\varepsilon}\nabla p(\rho) = 0 & (2)_\varepsilon \end{cases} \qquad (\text{with } p(\rho) = \rho^\gamma)$$

Parameter: $\varepsilon = M^2 = |\overline{u}|^2/(\gamma p(\overline{\rho})/\overline{\rho})$, $\qquad M = \text{Mach number}$

Boundary and initial conditions:

$$u\cdot n = 0 \text{ on } \partial\Omega \quad \text{and} \quad \begin{cases} \rho(x,0) = \rho_0 + \varepsilon\,\tilde{\rho}_0(x) \\ u(x,0) = u_0(x) + \varepsilon\,\tilde{u}_0(x), \ \text{ with } \nabla\cdot u_0 = 0 \end{cases}$$

The formal low Mach number limit $\varepsilon \to 0$:

$$(2)_\varepsilon \ \Rightarrow \ \nabla p(\rho) = 0 \ \Rightarrow \ \rho(x,t) = \rho(t)$$

$$(1)_\varepsilon \ \Rightarrow \ |\Omega|\,\rho'(t) + \rho(t)\int_{\partial\Omega} u\cdot n = 0 \ \Rightarrow \ \rho(t) = \rho(0) = \rho_0 \ \Rightarrow \ \nabla\cdot u = 0$$

The multi-scale model and its asymptotic limit

The asymptotic model: Rigorous limit [Klainerman & Majda, '81]:

$$(M_0)\begin{cases} \rho = \mathrm{cst} = \rho_0, & \\ \rho_0 \nabla \cdot u = 0, & (1)_0 \\ \rho_0 \partial_t u + \rho_0 \nabla \cdot (u \otimes u) + \nabla \pi_1 = 0, & (2)_0 \end{cases}$$

where

$$\pi_1 = \lim_{\varepsilon \to 0} \frac{1}{\varepsilon}\Big(p(\rho) - p(\rho_0)\Big).$$

Explicit eq. for π_1: $\partial_t (1)_0 - \nabla \cdot (2)_0 \implies -\Delta \pi_1 = \rho_0 \nabla^2 : (u \otimes u)$

The pressure wave equation from M_ε:

$$\partial_t (1)_\varepsilon - \nabla \cdot (2)_\varepsilon \implies \partial_{tt} \rho - \frac{1}{\varepsilon} \Delta p(\rho) = \nabla^2 : (\rho\, u \otimes u) \quad (3)_\varepsilon$$

From a numerical point of view

- Explicit treatment of $(3)_\varepsilon$ $\implies$ conditional stability $\Delta t \le \sqrt{\varepsilon}\, \Delta x$
- Implicit treatment of $(3)_\varepsilon$ $\implies$ uniform stability with respect to ε

Time semi-discretization: [Degond, Deluzet, Sangam & Vignal, '09], [Degond & Tang, '11], [Chalons, Girardin & Kokh, '15]

If ρ^n and u^n are known at time t^n:

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^{n+1} = 0, & (1) \quad \text{(AS)} \\ \dfrac{(\rho\, u)^{n+1}-(\rho\, u)^n}{\Delta t} + \nabla\cdot(\rho\, u\otimes u)^n + \dfrac{1}{\varepsilon}\nabla p(\rho^{n+1}) = 0. & (2) \quad \text{(AC)} \end{cases}$$

- $\varepsilon \to 0$ gives $\nabla p(\rho^{n+1}) = 0 \implies$ consistency at the limit
- implicit treatment of the pressure wave eq. $\implies$ uniform stability in ε

$$\frac{\rho^{n+1}-2\rho^n+\rho^{n-1}}{\Delta t^2} - \frac{1}{\varepsilon}\Delta p(\rho^{n+1}) = \nabla^2 : (\rho U\otimes U)^n$$

$\nabla\cdot(2)$ **inserted into** (1)**:** gives an uncoupled formulation

$$\frac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^n - \frac{\Delta t}{\varepsilon}\Delta p(\rho^{n+1}) - \Delta t\nabla^2 : (\rho\, u\otimes u)^n = 0$$

An order 1 AP scheme in the low Mach limit

The scheme proposed in [Dimarco, Loubère & Vignal, '17]:

➠ Framework of IMEX (IMplicit-EXplicit) schemes:

$$\partial_t \underbrace{\begin{pmatrix} \rho \\ \rho u \end{pmatrix}}_{W} + \nabla \cdot \underbrace{\begin{pmatrix} 0 \\ \rho u \otimes u \end{pmatrix}}_{F_e(W)} + \nabla \cdot \underbrace{\begin{pmatrix} \rho u \\ \frac{p(\rho)}{\varepsilon} Id \end{pmatrix}}_{F_i(W)} = 0.$$

➠ The CFL condition comes from the explicit flux $F_e(W)$: in 1D, we have

$$\Delta t^{AP} \leq \frac{\Delta x}{\lambda_j^n} = \frac{\Delta x}{2\,|u_j^n|}, \qquad \left(\text{recall } \Delta t^{\text{class.}} \leq \frac{\Delta x \sqrt{\varepsilon}}{|u_i^n \pm \sqrt{\gamma \rho^{\gamma-1}}|} \right)$$

where λ_j^n are the eigenvalues of the explicit Jacobian matrix $DF_e(W_j^n)$.

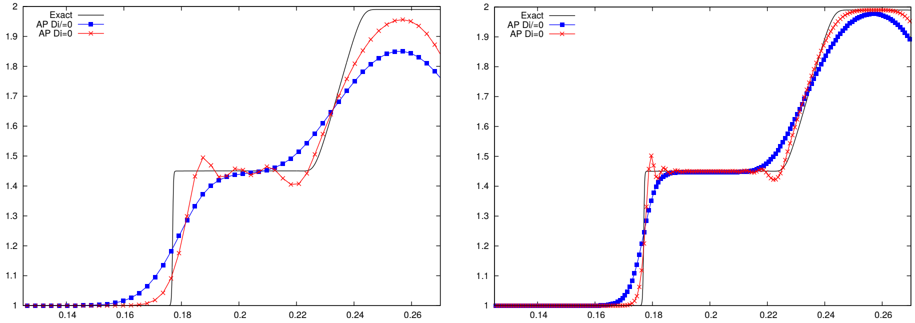
➠ A linear stability analysis yields: if the implicit part is

- centered $\Longrightarrow$ L^2 stability;
- upwind $\Longrightarrow$ TVD and L^∞ stability.

SSP Strong Stability Preserving, [Gottlieb, Shu & Tadmor, '01]

Importance of the upwind implicit viscosity

To highlight the relevance of upwinding the implicit viscosity, we display the density ρ in the vicinity of a shock wave and a rarefaction wave ($\varepsilon = 0.99$, 45 cells in the left panel, 150 cells in the right panel).



× : centered implicit discretization $\Longrightarrow$ L^2 stability and less diffusive

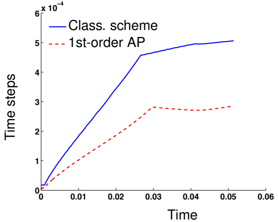
■ : upwind implicit discretization $\Longrightarrow$ L^∞ stability but more diffusive

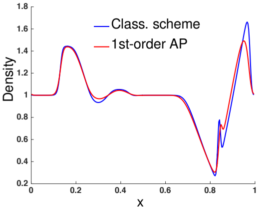
AP but diffusive results, 1D test case

$\varepsilon = 0.99$, 300 cells

Class: 273 loops
CPU time 0.07

AP: 510 loops
CPU time 1.46

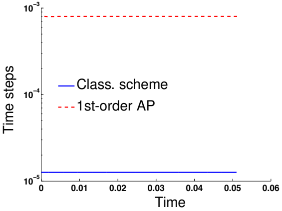


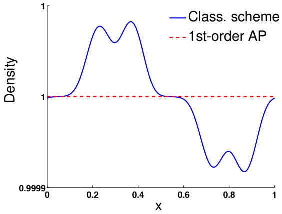


$\varepsilon = 10^{-4}$, 300 cells

Class: 4036 loops
CPU time 0.82

AP: 57 loops
CPU time 0.14

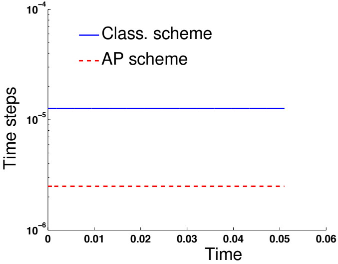


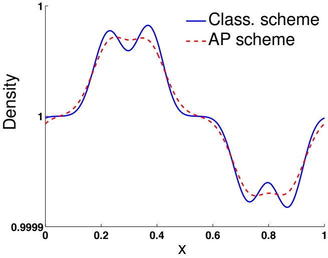


AP but diffusive results, 1D test case

$\varepsilon = 10^{-4}$

Underlying of the viscosity





$\Downarrow$

It is necessary to use high order schemes

But they must respect the AP properties

we also wish to retain the L^∞ stability

Outline

Principle of IMEX schemes

Bibliography for stiff source terms or ODE problems: Ascher, Boscarino, Cafflish, Dimarco, Filbet, Gottlieb, Happenhofer, Higueras, Jin, Koch, Kupka, LeFloch, Pareschi, Russo, Ruuth, Shu, Spiteri, Tadmor...

IMEX division: $\partial_t W + \nabla \cdot F_e(W) + \nabla \cdot F_i(W) = 0.$

General principle: Step n: W^n is known

- Quadrature formula with intermediate values:

$$W(t^{n+1}) = W(t^n) - \Delta t \underbrace{\int_{t^n}^{t^{n+1}} \nabla \cdot F_e(W(t))\, dt}_{} - \Delta t \underbrace{\int_{t^n}^{t^{n+1}} \nabla \cdot F_i(W(t))\, dt}_{}$$

$$W^{n+1} = W^n \quad - \Delta t \sum_{j=1}^{s} \tilde{b}_j \nabla \cdot F_e(W^{n,j}) \; - \Delta t \sum_{j=1}^{s} b_j \nabla \cdot F_i(W^{n,j})$$

Quadratures exact on the constants: $\sum_{j=1}^{s} \tilde{b}_j = \sum_{j=1}^{s} b_j = 1$

- Intermediate values at times $t^{n,j} = t^n + c_j \Delta t$:

$$W^{n,j} \approx W(t^{n,j}) = W(t^n) + \int_{t^n}^{t^{n,j}} \partial_t W(t)\, dt = W^n + \Delta t \int_0^{c_j} \partial_t W(t^n + s\Delta t)$$

Principle of IMEX schemes

- Quadrature formula for intermediate values: $i = 1, \cdots, s$

$$W^{n,j} = W^n - \Delta t \sum_{k<j} \tilde{a}_{j,k} \nabla \cdot F_e(W^{n,k}) - \Delta t \sum_{k \le j} a_{j,k} \nabla \cdot F_i(W^{n,k}),$$

Quadratures exact on the constants: $\sum_{k=1}^{s} \tilde{a}_{j,k} = \tilde{c}_j$, $\sum_{k=1}^{s} a_{j,k} = c_j$

- $W^{n+1} = W^n - \Delta t \sum_{j=1}^{s} \tilde{b}_j \nabla \cdot F_e(W^{n,j}) - \Delta t \sum_{j=1}^{s} b_j \nabla \cdot F_i(W^{n,j})$

Butcher tableaux:

Explicit part

$$\begin{array}{c|cccc}
0 & 0 & 0 & \cdots & 0 \\
c_2 & \tilde{a}_{2,1} & 0 & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_s & \tilde{a}_{s,1} & \ldots & \tilde{a}_{s,s-1} & 0 \\
\hline
 & \tilde{b}_1 & \ldots & \ldots & \tilde{b}_s
\end{array}$$

Implicit part

$$\begin{array}{c|cccc}
c_1 & a_{1,1} & 0 & \cdots & 0 \\
c_2 & a_{2,1} & a_{2,2} & \ldots & 0 \\
\vdots & \vdots & \ddots & \ddots & \vdots \\
c_s & a_{s,1} & \ldots & a_{s,s-1} & a_{s,s} \\
\hline
 & b_1 & \ldots & \ldots & b_s
\end{array}$$

Conditions for 2nd order: $\sum b_j c_j = \sum b_j \tilde{c}_j = \sum \tilde{b}_j c_j = \sum \tilde{b}_j \tilde{c}_j = 1/2$

AP Order 2 scheme for Euler

ARS discretization [Ascher, Ruuth & Spiteri, '97]:

"only one" intermediate step

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 \\ 1 & \beta-1 & 2-\beta & 0 \\ \hline & \beta-1 & 2-\beta & 0 \end{array} \qquad \begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \beta & 0 & \beta & 0 \\ 1 & 0 & 1-\beta & \beta \\ \hline & 0 & 1-\beta & \beta \end{array} \qquad \beta = 1 - \frac{1}{\sqrt{2}}$$

$$W^{n,1} = W^n$$

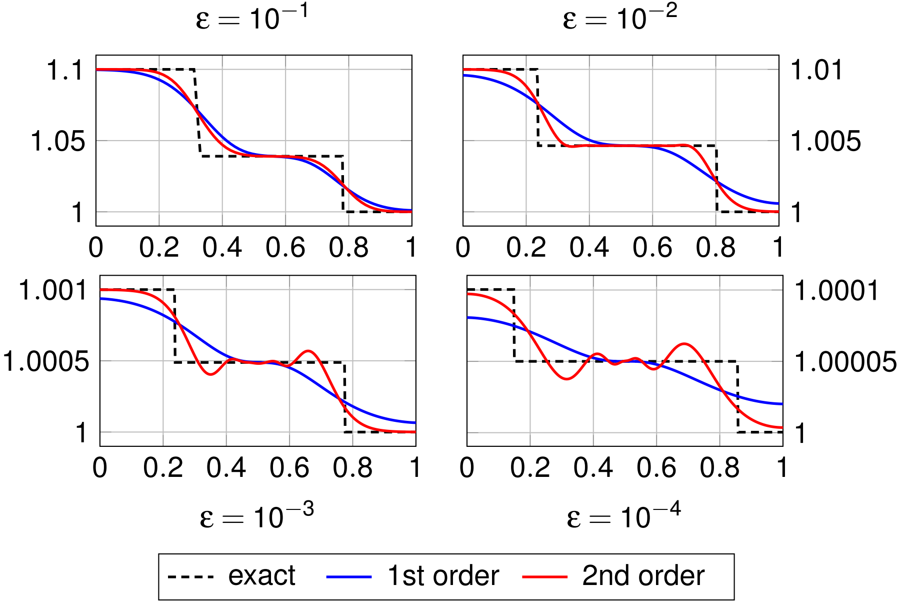
$$W^{n,2} = W^\star = W^n - \Delta t\, \beta\, \nabla \cdot F_e(W^n) - \Delta t\, \beta\, \nabla \cdot F_i(W^\star)$$

$$W^{n,3} = W^{n+1} = W^n - \Delta t(\beta - 1)\nabla \cdot F_e(W^n) - \Delta t(2-\beta)\nabla \cdot F_e(W^\star)$$

$$- \Delta t(1-\beta)\nabla \cdot F_i(W^\star) - \Delta t\beta\nabla \cdot F_i(W^{n+1})$$

AP Order 2 scheme for Euler

Density ρ for the ARS time discretization: (1st order in space)



Better understand the oscillations

Consider the scalar hyperbolic equation $\partial_t w + \partial_x f(w) = 0.$

- Oscillations measured by the Total Variation and the L^∞ norm:

$$TV(w^n) = \sum_j |w^n_{j+1} - w^n_j| \qquad \text{and} \qquad \|w^n\|_\infty = \max_j |w^n_j|.$$

- TVD (Total Variation Diminishing) property and L^∞ stability:

$$\begin{cases} TV(w^{n+1}) \leq TV(w^n) \\ \|w^{n+1}\|_\infty \leq \|w^n\|_\infty \end{cases} \iff \text{no oscillations}$$

First idea: Find an AP order 2 scheme which satisfies these properties.

Impossible

Theorem (Gottlieb, Shu & Tadmor, '01): There are no implicit Runge-Kutta schemes of order higher than one which preserves the TVD property.

A limiting procedure

Another idea: use a limited scheme.

$$W^{n+1} = \theta\, W^{n+1,O2} + (1-\theta)\, W^{n+1,O1}$$

- $W^{n+1,Oj}$ = order j AP approximation
- $\theta \in [0,1]$ largest value such that W^{n+1} does not oscillate

Toy scalar equation: $\partial_t w + c_e \partial_x w + \dfrac{c_i}{\sqrt{\varepsilon}} \partial_x w = 0$

- Order 1 AP scheme with upwind space discretizations ($c_e, c_i > 0$):

$$w_j^{n+1,O1} = w_j^n - c_e\,(w_j^n - w_{j-1}^n) - \frac{c_i}{\sqrt{\varepsilon}}\,(w_j^{n+1,O1} - w_{j-1}^{n+1,O1}).$$

- Order 2 AP scheme: ARS with the parameter $\beta = 1 - 1/\sqrt{2}$.

Theorem (Dimarco, Loubère, M.-D., Vignal):
Under the CFL condition $\Delta t \leq \Delta x / c_e$,

$$\theta = \frac{\beta}{1-\beta} \simeq 0.41 \quad \Longrightarrow \begin{cases} TV(w^{n+1}) \leq TV(w^n), \\ \|w^{n+1}\|_\infty \leq \|w^n\|_\infty. \end{cases}$$

A MOOD procedure

Limited AP scheme:

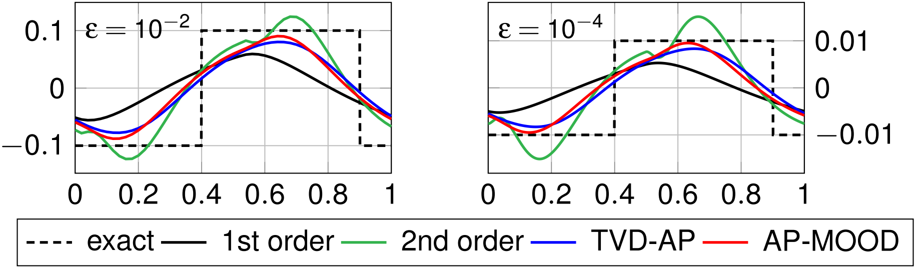
$$w^{n+1,lim} = \theta\, w^{n+1,O2} + (1-\theta)\, w^{n+1,O1} \quad \text{with} \quad \theta = \frac{\beta}{1-\beta}$$

Problem: More accurate than order 1 but not order 2

Solution: MOOD procedure: see [Clain, Diot & Loubère, '11]

On the toy equation: w^{n+1} **MOOD AP scheme,** CFL $\Delta t \leq \Delta x / c_e$

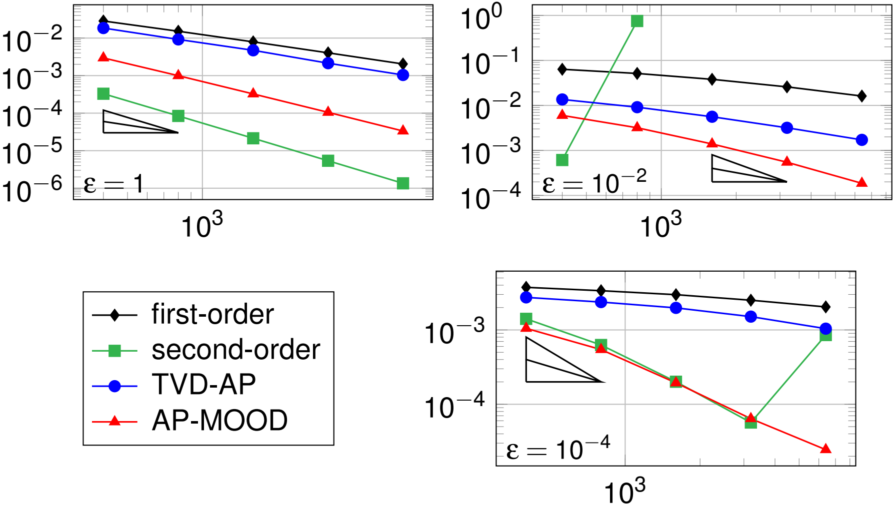
- Compute the order 2 approximation $w^{n+1,O2}$.
- Detect if the max. principle is satisfied: $\|w^{n+1,O2}\|_\infty \leq \|w^n\|_\infty$?
- If not, compute the limited AP approximation $w^{n+1,lim}$.



Outline

Error curves for the toy scalar equation

- Order 2 in space: MUSCL (with the MC limiter) with explicit slopes for implicit fluxes.
- Error curves on a smooth solution for the toy scalar equation:



Recall the first-order IMEX scheme for the Euler system:

$$\begin{cases} \dfrac{\rho^{n+1}-\rho^n}{\Delta t} + \nabla\cdot(\rho\, u)^{n+1} = 0, & (1) \\ \dfrac{(\rho\, u)^{n+1}-(\rho\, u)^n}{\Delta t} + \nabla\cdot(\rho\, u\otimes u)^n + \dfrac{1}{\varepsilon}\nabla p(\rho^{n+1}) = 0. & (2) \end{cases}$$

We apply the same convex combination procedure:

$$W^{n+1,lim} = \theta\, W^{n+1,O2} + (1-\theta)\, W^{n+1,O1}, \quad \text{with } \theta = \frac{\beta}{1-\beta}.$$

$\rightsquigarrow$ We use the value of θ given by the study of the toy scalar equation.

$\rightsquigarrow$ But how can we detect oscillations for the MOOD procedure?

Euler equations: MOOD procedure

The previous detector (L^∞ criterion on the solution) is irrelevant for the Euler equations, since ρ and u do not satisfy a maximum principle.

$\rightsquigarrow$ we need another detection criterion

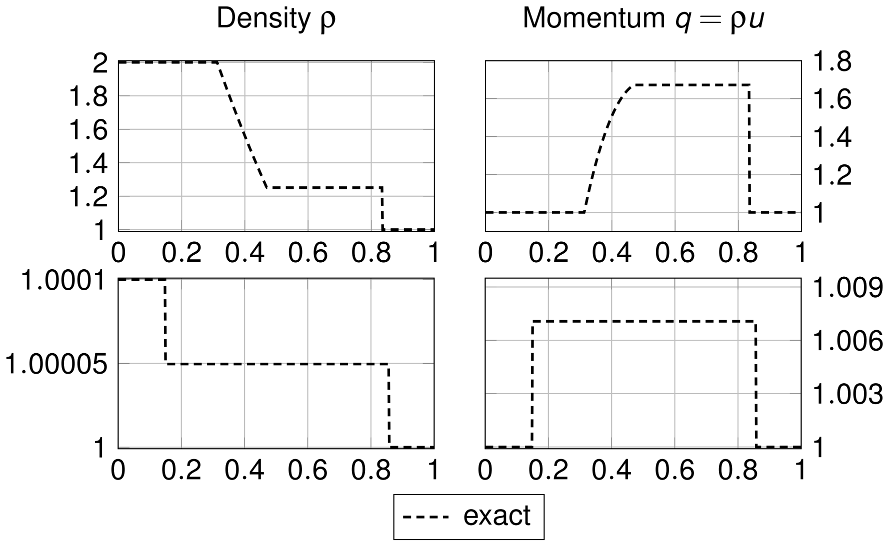
We pick the Riemann invariants $\Phi_\pm = u \mp \dfrac{2}{\gamma - 1}\sqrt{\dfrac{1}{\varepsilon}\dfrac{\partial p(\rho)}{\partial \rho}}$: in a Riemann problem, at least one of them satisfies a maximum principle. [Smoller & Johnson, '69]

On the Euler equations: W^{n+1} **MOOD AP scheme,** CFL $\Delta t \leq \Delta x / \lambda$

- Compute the order 2 approximation $W^{n+1,O2}$.
- Detect if both Riemann invariants break the maximum principle at the same time.
- If so, compute the limited AP approximation $W^{n+1,lim}$.

Euler equations: 1D Numerical results

Riemann problem: left rarefaction wave, right shock ;
top curves: $\varepsilon = 1$ (50 pts) ; bottom curves: $\varepsilon = 10^{-4}$ (500 pts)



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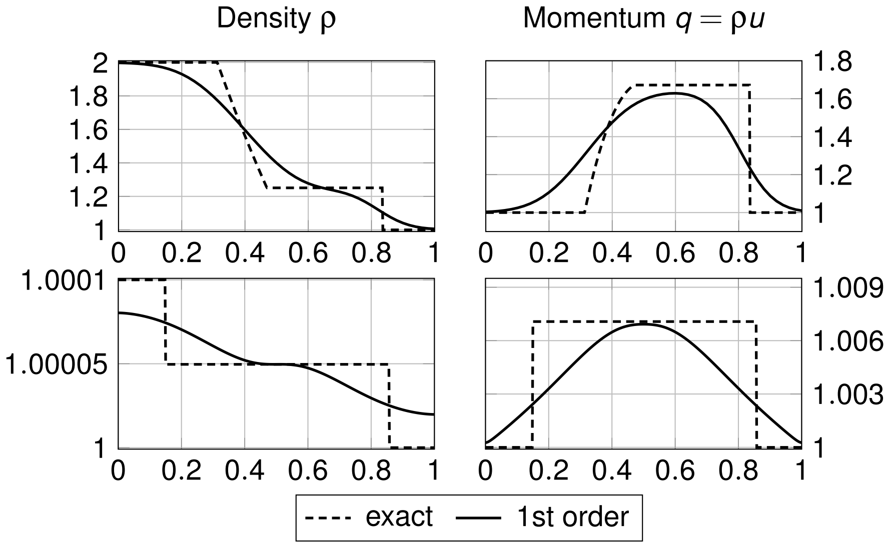

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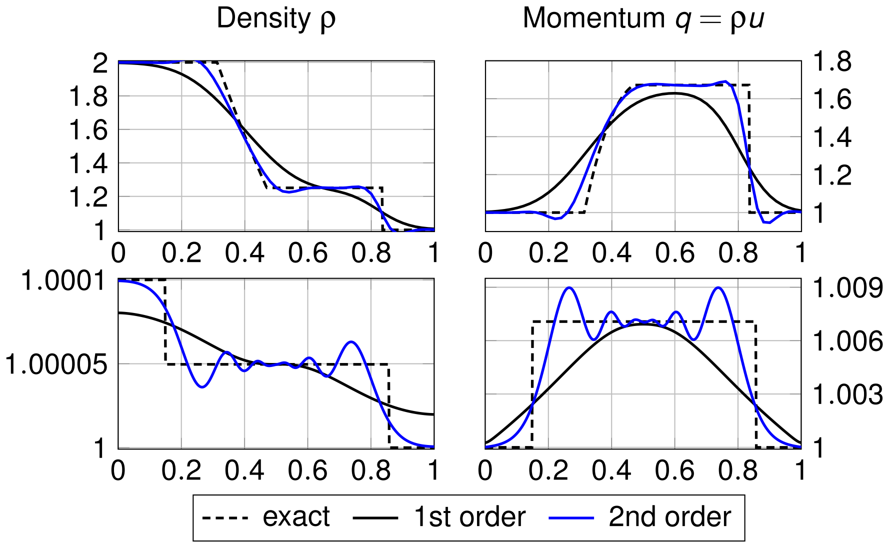

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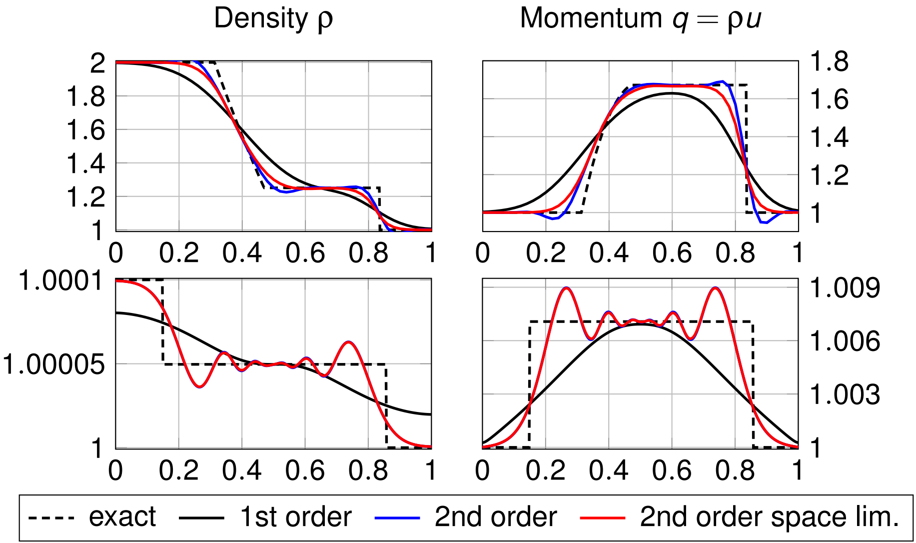

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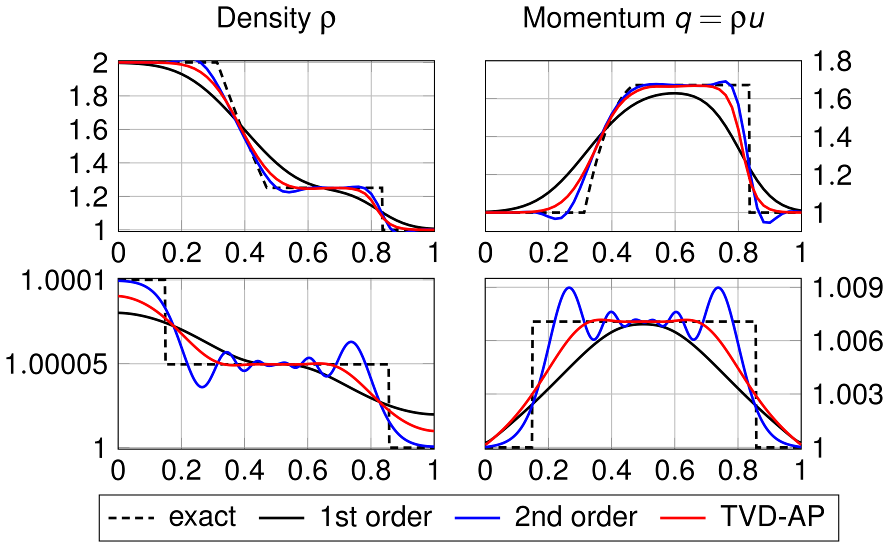

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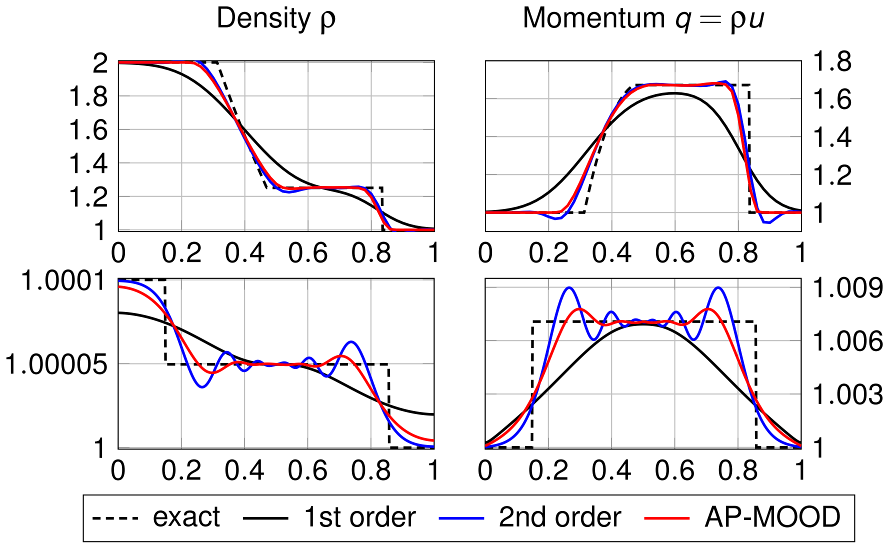
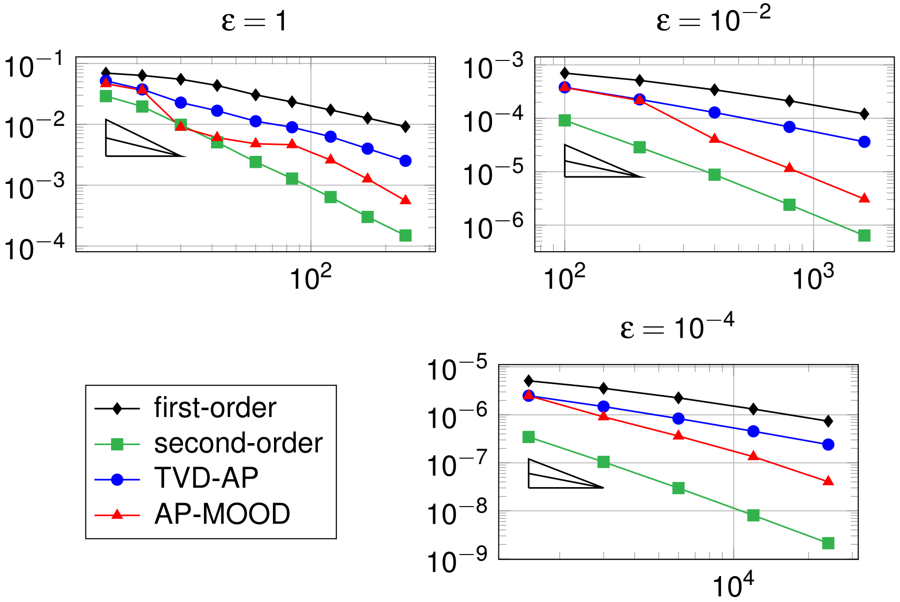


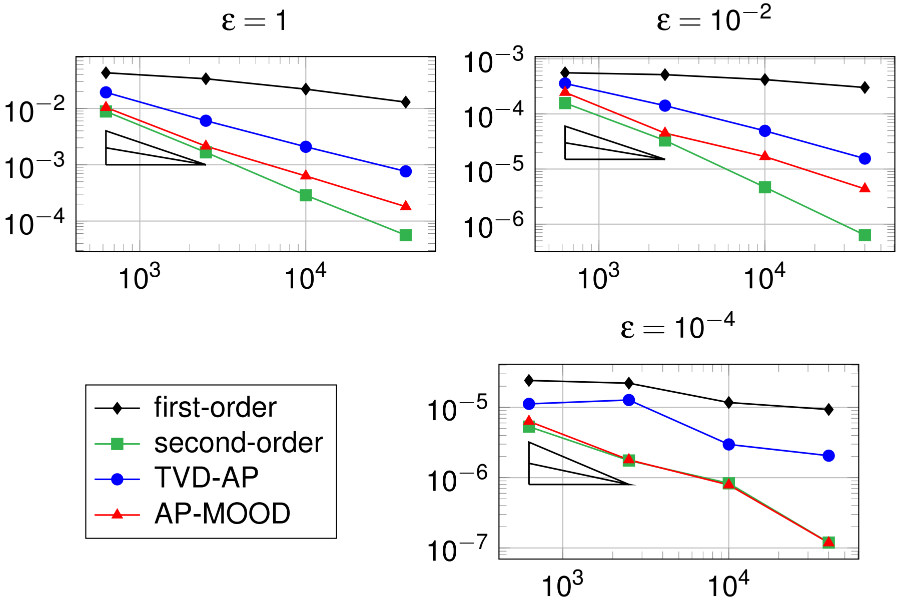
Euler equations: 1D Numerical results

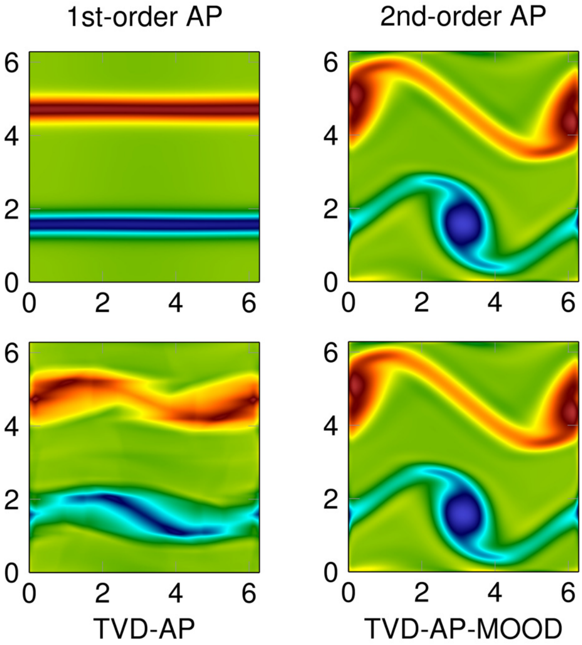
Error curves in L^∞ norm, smooth 1D solution



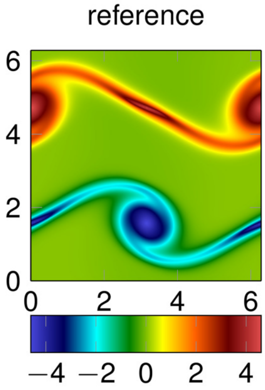
Euler equations: 2D Numerical results

Error curves in L^∞ norm, smooth 2D traveling vortex (Cartesian mesh)





reference solution
obtained solving
the vorticity formulation
$\partial_t \omega + U \cdot \nabla \omega = 0,$
with $\omega = \partial_x v - \partial_y u$



Outline

Work in progress and perspectives: the system

Extension to the full Euler system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho U) = 0, \\ \partial_t (\rho U) + \nabla \cdot (\rho U \otimes U) + \dfrac{1}{\varepsilon} \nabla p = 0, \\ \partial_t E + \nabla \cdot (U(E+p)) = 0, \end{cases} \quad \text{with} \quad p = (\gamma - 1)\left(E - \varepsilon \frac{\rho |U|^2}{2}\right).$$

In 1D, to get an AP scheme ensuring that both the explicit and the implicit parts are hyperbolic, we take:

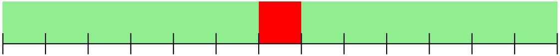
$$\frac{W^{n+1} - W^n}{\Delta t} + A_e^{n,n+1} \partial_x W^n + A_i^{n,n+1} \partial_x W^{n+1} = 0.$$

The scheme no longer takes the conservative IMEX form

$$\frac{W^{n+1} - W^n}{\Delta t} + \partial_x F_e(W^n) + \partial_x F_i(W^n) = 0.$$

Work in progress and perspectives: IMEX

1. Study a local value of θ, depending on the presence of oscillations in a given cell: how to reconcile the locality of θ with the non-locality of the implicitation?



: cell with oscillation $\implies \theta < 1$

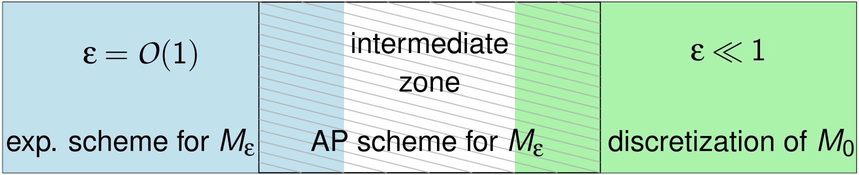
: cell without oscillation $\implies \theta = 1$ or $\theta < 1$?

2. Compute optimal values of θ for other IMEX discretizations:
   - SSPRK explicit part?
   - custom-made second-order IMEX discretization to ensure θ as close to 1 as possible?
   - higher-order discretizations?

Work in progress and perspectives: DD

Domain decomposition with respect to ε:

$$\text{Compressible Euler } (M_\varepsilon) \xrightarrow[\varepsilon \longrightarrow 0]{} \text{Incompressible Euler } (M_0)$$



- How to define the boundaries of the intermediate zones?
- How to handle interfaces in 1D with first-order schemes?
- How to extend to higher dimensions and higher-order schemes?

Thanks for your attention!

Euler equations: 2D Numerical results

To obtain a 2D reference incompressible solution, set $\omega = \partial_x v - \partial_y u$ and consider the **vorticity formulation** of the incompressible Euler equations:

$$\partial_t \omega + U \cdot \nabla \omega = 0,$$

$$\nabla \cdot U = 0 \implies \exists \text{ **stream function** } \Psi \text{ such that } \begin{cases} U &= {}^t(\partial_y \Psi, -\partial_x \Psi), \\ -\Delta \Psi &= \omega. \end{cases}$$

To get the time evolution of the vorticity from ω^n:

1. solve $-\Delta \Psi^n = \omega^n$ for Ψ^n (with periodic BC and assuming that the average of Ψ vanishes);
2. get U^n from $U^n = {}^t(\partial_y \Psi^n, -\partial_x \Psi^n)$;
3. solve $\partial_t \omega + U^n \cdot \nabla \omega^n = 0$ to get ω^{n+1}.

We get a reference incompressible vorticity $\omega(x,t)$, to be compared to the vorticity of the solution given by the compressible scheme with small ε (we take $\varepsilon = M^2 = 10^{-5}$).

Bibliography

All speed schemes

- **Preconditioning methods:** [Chorin, '65], [Choi, Merkle, '85], [Turkel, '87], [Van Leer, Lee & Roe, '91], [Li & Gu '08, '10], ...
- **Splitting and pressure correction:** [Harlow & Amsden, '68, '71], [Karki & Patankar, '89], [Bijl & Wesseling, '98], [Sewall & Tafti, '08], [Klein, Botta, Schneider, Munz & Roller '08], [Guillard, Murrone & Viozat '99, '04, '06] [Herbin, Kheriji & Latché '12, '13], ...
  - **Asymptotic preserving schemes** [Degond, Deluzet, Sangam & Vignal, '09], [Degond & Tang, '11], [Cordier, Degond & Kumbaro, '12], [Grenier, Vila & Villedieu, '13] [Dellacherie, Omnès & Raviart, '13], [Noelle, Bispen, Arun, Lukáčová & Munz, '14], [Chalons, Girardin & Kokh, '15] [Dimarco, Loubère & Vignal, '17]