

Second order Implicit-Explicit Total Variation Diminishing schemes for the Euler system in the low Mach regime

Victor Michel-Dansac



Séminaire ANEDP, Lille, 22/03/2018

[Giacomo Dimarco](#), Univ. of Ferrara, Italy

[Raphaël Loubère](#), Univ. of Bordeaux, CNRS, France

[Marie-Hélène Vignal](#), Univ. of Toulouse, France

Funding: [ANR MOONRISE](#), [SHOM](#)

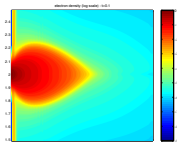
Outline

- 1 General context: multi-scale models and principle of AP schemes
- 2 An order 1 AP scheme for the Euler system in the low Mach limit
- 3 Second-order schemes in time
- 4 Second-order schemes in space and application to Euler
- 5 Work in progress and perspectives

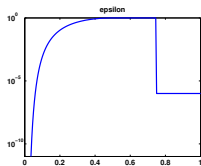
Multiscale model M_ε , depending on a parameter ε

In the (space-time) domain, ε can

- be of same order as the reference scale;
- be small compared to the reference scale;
- take intermediate values.



When ε is small: $M_0 = \lim_{\varepsilon \rightarrow 0} M_\varepsilon$ asympt. model

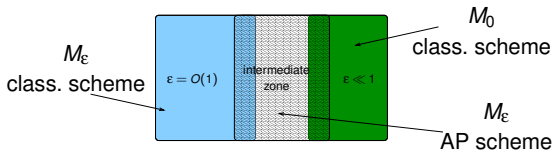


Difficulties:

- Classical explicit schemes for M_ε : they are stable and consistent if the mesh resolves all the scales of ε . \implies **very costly when $\varepsilon \rightarrow 0$**
- Schemes for $M_0 \implies$ the mesh is independent of ε
 - But:** $\implies M_0$ is not valid everywhere, it needs $\varepsilon \ll 1$
 - \implies the interface may be moving: how to locate it?

A possible solution: Asymptotic Preserving (AP) schemes

- Use the multi-scale model M_ε even for small ε .
- Discretize M_ε with a scheme preserving the limit $\varepsilon \rightarrow 0$.
 - ➡ The mesh is independent of ε : **Asymptotic stability.**
 - ➡ Recovery of an approximate solution of M_0 when $\varepsilon \rightarrow 0$:
Asymptotic consistency.
 - ➡ Asymptotically stable and consistent scheme
 \implies **Asymptotic preserving scheme (AP).**
([Jin, '99] kinetic \rightarrow hydro)
- The AP scheme may be used only to reconnect M_ε and M_0 .



Outline

- 1 General context: multi-scale models and principle of AP schemes
- 2 An order 1 AP scheme for the Euler system in the low Mach limit**
- 3 Second-order schemes in time
- 4 Second-order schemes in space and application to Euler
- 5 Work in progress and perspectives

➡ **Isentropic Euler system in scaled variables:** $x \in \Omega \subset \mathbb{R}^d$, $t \geq 0$

$$(M_\varepsilon) \begin{cases} \partial_t \rho + \nabla \cdot (\rho u) = 0 & (1)_\varepsilon \\ \partial_t (\rho u) + \nabla \cdot (\rho u \otimes u) + \frac{1}{\varepsilon} \nabla p(\rho) = 0 & (2)_\varepsilon \end{cases} \quad (\text{with } p(\rho) = \rho^\gamma)$$

Parameter: $\varepsilon = M^2 = |\bar{u}|^2 / (\gamma p(\bar{\rho}) / \bar{\rho})$, $M = \text{Mach number}$

Boundary and initial conditions:

$$u \cdot n = 0 \text{ on } \partial\Omega \quad \text{and} \quad \begin{cases} \rho(x, 0) = \rho_0 + \varepsilon \tilde{\rho}_0(x) \\ u(x, 0) = u_0(x) + \varepsilon \tilde{u}_0(x), \text{ with } \nabla \cdot u_0 = 0 \end{cases}$$

The formal low Mach number limit $\varepsilon \rightarrow 0$:

$$(2)_\varepsilon \Rightarrow \nabla p(\rho) = 0 \Rightarrow \rho(x, t) = \rho(t)$$

$$(1)_\varepsilon \Rightarrow |\Omega| \rho'(t) + \rho(t) \int_{\partial\Omega} u \cdot n = 0 \Rightarrow \rho(t) = \rho(0) = \rho_0 \Rightarrow \nabla \cdot u = 0$$

The asymptotic model: Rigorous limit [Klainerman & Majda, '81]:

$$(M_0) \begin{cases} \rho = \text{cst} = \rho_0, \\ \rho_0 \nabla \cdot u = 0, \\ \rho_0 \partial_t u + \rho_0 \nabla \cdot (u \otimes u) + \nabla \pi_1 = 0, \end{cases} \quad \begin{matrix} \\ (1)_0 \\ (2)_0 \end{matrix}$$

where

$$\pi_1 = \lim_{\varepsilon \rightarrow 0} \frac{1}{\varepsilon} \left(\rho(\rho) - \rho(\rho_0) \right).$$

Explicit eq. for π_1 : $\partial_t(1)_0 - \nabla \cdot (2)_0 \implies -\Delta \pi_1 = \rho_0 \nabla^2 : (u \otimes u)$

The pressure wave equation from M_ε :

$$\partial_t(1)_\varepsilon - \nabla \cdot (2)_\varepsilon \implies \partial_{tt} \rho - \frac{1}{\varepsilon} \Delta \rho(\rho) = \nabla^2 : (\rho u \otimes u) \quad (3)_\varepsilon$$

From a numerical point of view

- Explicit treatment of $(3)_\varepsilon \implies$ conditional stability $\Delta t \leq \sqrt{\varepsilon} \Delta x$
- Implicit treatment of $(3)_\varepsilon \implies$ uniform stability with respect to ε

Time semi-discretization: [Degond, Deluzet, Sangam & Vignal, '09],
[Degond & Tang, '11], [Chalons, Girardin & Kokh, '15]

If ρ^n and u^n are known at time t^n :

$$\left\{ \begin{array}{l} \frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho u)^{n+1} = 0, \end{array} \right. \quad (1) \quad (\text{AS})$$

$$\left\{ \begin{array}{l} \frac{(\rho u)^{n+1} - (\rho u)^n}{\Delta t} + \nabla \cdot (\rho u \otimes u)^n + \frac{1}{\varepsilon} \nabla p(\rho^{n+1}) = 0. \end{array} \right. \quad (2) \quad (\text{AC})$$

- $\varepsilon \rightarrow 0$ gives $\nabla p(\rho^{n+1}) = 0 \implies$ consistency at the limit
- implicit treatment of the pressure wave eq. \implies uniform stability in ε

$$\frac{\rho^{n+1} - 2\rho^n + \rho^{n-1}}{\Delta t^2} - \frac{1}{\varepsilon} \Delta p(\rho^{n+1}) = \nabla^2 : (\rho U \otimes U)^n$$

$\nabla \cdot (2)$ inserted into (1): gives an uncoupled formulation

$$\frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho u)^n - \frac{\Delta t}{\varepsilon} \Delta p(\rho^{n+1}) - \Delta t \nabla^2 : (\rho u \otimes u)^n = 0$$

The scheme proposed in [Dimarco, Loubère & Vignal, '17]:

- Framework of IMEX (IMplicit-EXplicit) schemes:

$$\partial_t \underbrace{\begin{pmatrix} \rho \\ \rho u \end{pmatrix}}_W + \nabla \cdot \underbrace{\begin{pmatrix} 0 \\ \rho u \otimes u \end{pmatrix}}_{F_e(W)} + \nabla \cdot \underbrace{\begin{pmatrix} \rho u \\ \frac{\rho(p)}{\varepsilon} Id \end{pmatrix}}_{F_i(W)} = 0.$$

- The CFL condition comes from the explicit flux $F_e(W)$: in 1D, we have

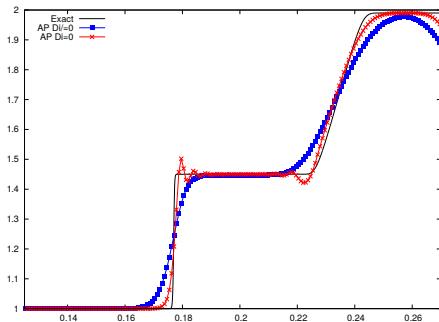
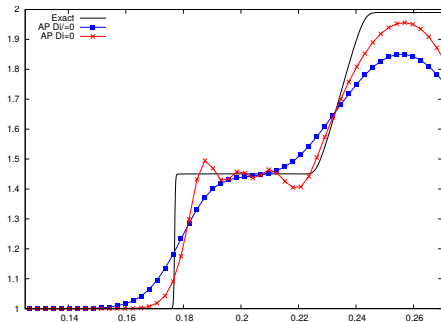
$$\Delta t^{\text{AP}} \leq \frac{\Delta x}{\lambda_j^n} = \frac{\Delta x}{2|u_j^n|}, \quad \left(\text{recall } \Delta t^{\text{class.}} \leq \frac{\Delta x \sqrt{\varepsilon}}{|u_j^n \pm \sqrt{\gamma p^{\gamma-1}}|} \right)$$

where λ_j^n are the eigenvalues of the explicit Jacobian matrix $DF_e(W_j^n)$.

- A linear stability analysis yields: if the implicit part is
 - centered $\implies L^2$ stability;
 - upwind \implies TVD and L^∞ stability.

SSP Strong Stability Preserving, [Gottlieb, Shu & Tadmor, '01]

To highlight the relevance of upwinding the implicit viscosity, we display the density ρ in the vicinity of a shock wave and a rarefaction wave ($\varepsilon = 0.99$, 45 cells in the left panel, 150 cells in the right panel).



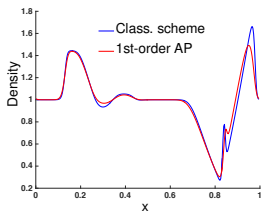
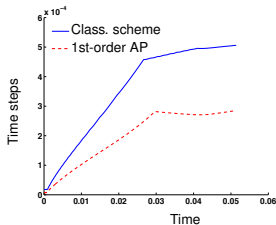
✕ : centered implicit discretization $\implies L^2$ stability and less diffusive

■ : upwind implicit discretization $\implies L^\infty$ stability but more diffusive

$\varepsilon = 0.99$, 300 cells

Class: 273 loops
CPU time 0.07

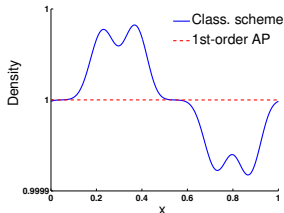
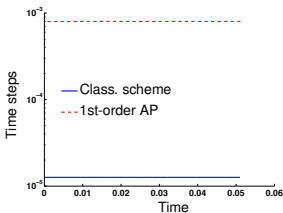
AP: 510 loops
CPU time 1.46



$\varepsilon = 10^{-4}$, 300 cells

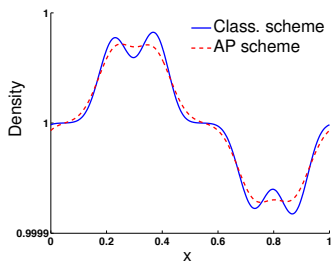
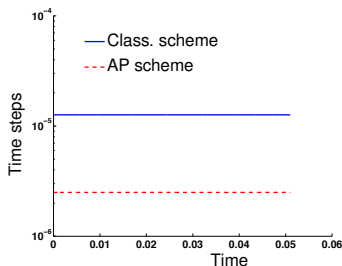
Class: 4036 loops
CPU time 0.82

AP: 57 loops
CPU time 0.14



$$\varepsilon = 10^{-4}$$

Underlying of
the viscosity



It is necessary to use **high order schemes**

But they must respect the AP properties

we also wish to retain the L^∞ stability

Outline

- 1 General context: multi-scale models and principle of AP schemes
- 2 An order 1 AP scheme for the Euler system in the low Mach limit
- 3 Second-order schemes in time**
- 4 Second-order schemes in space and application to Euler
- 5 Work in progress and perspectives

Bibliography for stiff source terms or ODE problems: Ascher, Boscarino, Cafflish, Dimarco, Filbet, Gottlieb, Happenhofer, Higuera, Jin, Koch, Kupka, LeFloch, Pareschi, Russo, Ruuth, Shu, Spiteri, Tadmor...

IMEX division: $\partial_t W + \nabla \cdot F_e(W) + \nabla \cdot F_i(W) = 0$.

General principle: Step n: W^n is known

- Quadrature formula with intermediate values:

$$W(t^{n+1}) = W(t^n) - \Delta t \underbrace{\int_{t^n}^{t^{n+1}} \nabla \cdot F_e(W(t)) dt}_S - \Delta t \underbrace{\int_{t^n}^{t^{n+1}} \nabla \cdot F_i(W(t)) dt}_S$$
$$W^{n+1} = W^n - \Delta t \sum_{j=1}^s \tilde{b}_j \nabla \cdot F_e(W^{n,j}) - \Delta t \sum_{j=1}^s b_j \nabla \cdot F_i(W^{n,j})$$

Quadratures exact on the constants: $\sum_{j=1}^s \tilde{b}_j = \sum_{j=1}^s b_j = 1$

- Intermediate values at times $t^{n,j} = t^n + c_j \Delta t$:

$$W^{n,j} \approx W(t^{n,j}) = W(t^n) + \int_{t^n}^{t^{n,j}} \partial_t W(t) dt = W^n + \Delta t \int_0^{c_j} \partial_t W(t^n + s \Delta t) ds$$

- Quadrature formula for intermediate values: $i = 1, \dots, s$

$$W^{n,j} = W^n - \Delta t \sum_{k < j} \tilde{a}_{j,k} \nabla \cdot F_e(W^{n,k}) - \Delta t \sum_{k \leq j} a_{j,k} \nabla \cdot F_i(W^{n,k}),$$

Quadratures exact on the constants: $\sum_{k=1}^s \tilde{a}_{j,k} = \tilde{c}_j, \sum_{k=1}^s a_{j,k} = c_j$

- $W^{n+1} = W^n - \Delta t \sum_{j=1}^s \tilde{b}_j \nabla \cdot F_e(W^{n,j}) - \Delta t \sum_{j=1}^s b_j \nabla \cdot F_i(W^{n,j})$

Butcher tableaux:

| | Explicit part | | | | | Implicit part | | | |
|----------|-------------------|----------|---------------------|---------------|----------|---------------|-----------|-------------|-----------|
| 0 | 0 | 0 | ... | 0 | c_1 | $a_{1,1}$ | 0 | ... | 0 |
| c_2 | $\tilde{a}_{2,1}$ | 0 | ... | 0 | c_2 | $a_{2,1}$ | $a_{2,2}$ | ... | 0 |
| \vdots | \vdots | \ddots | \ddots | \vdots | \vdots | \vdots | \ddots | \ddots | \vdots |
| c_s | $\tilde{a}_{s,1}$ | ... | $\tilde{a}_{s,s-1}$ | 0 | c_s | $a_{s,1}$ | ... | $a_{s,s-1}$ | $a_{s,s}$ |
| | \tilde{b}_1 | ... | ... | \tilde{b}_s | | b_1 | ... | ... | b_s |

Conditions for 2nd order: $\sum b_j c_j = \sum b_j \tilde{c}_j = \sum \tilde{b}_j c_j = \sum \tilde{b}_j \tilde{c}_j = 1/2$

ARS discretization [Ascher, Ruuth & Spiteri, '97]:

“only one” intermediate step

$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \beta & \beta & 0 & 0 \\ 1 & \beta - 1 & 2 - \beta & 0 \\ \hline & \beta - 1 & 2 - \beta & 0 \end{array}$$

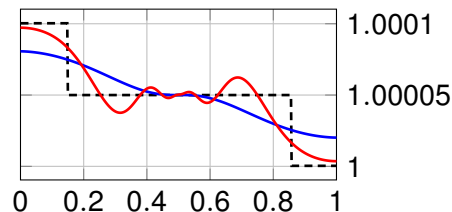
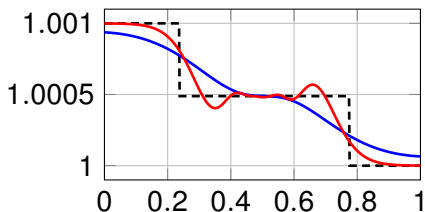
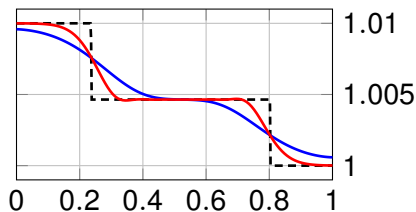
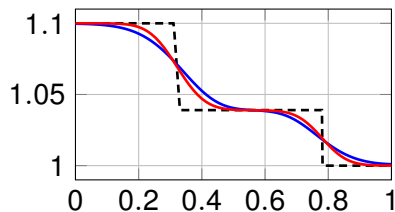
$$\begin{array}{c|ccc} 0 & 0 & 0 & 0 \\ \beta & 0 & \beta & 0 \\ 1 & 0 & 1 - \beta & \beta \\ \hline & 0 & 1 - \beta & \beta \end{array}$$

$$\beta = 1 - \frac{1}{\sqrt{2}}$$

$$W^{n,1} = W^n$$

$$W^{n,2} = W^* = W^n - \Delta t \beta \nabla \cdot F_e(W^n) - \Delta t \beta \nabla \cdot F_i(W^*)$$

$$\begin{aligned} W^{n,3} = W^{n+1} = & W^n - \Delta t (\beta - 1) \nabla \cdot F_e(W^n) - \Delta t (2 - \beta) \nabla \cdot F_e(W^*) \\ & - \Delta t (1 - \beta) \nabla \cdot F_i(W^*) - \Delta t \beta \nabla \cdot F_i(W^{n+1}) \end{aligned}$$

Density ρ for the ARS time discretization: (1st order in space) $\varepsilon = 10^{-1}$ $\varepsilon = 10^{-2}$  $\varepsilon = 10^{-3}$ $\varepsilon = 10^{-4}$

----- exact — 1st order — 2nd order

Consider the scalar hyperbolic equation $\partial_t w + \partial_x f(w) = 0$.

- Oscillations measured by the Total Variation and the L^∞ norm:

$$TV(w^n) = \sum_j |w_{j+1}^n - w_j^n| \quad \text{and} \quad \|w^n\|_\infty = \max_j |w_j^n|.$$

- TVD (Total Variation Diminishing) property and L^∞ stability:

$$\begin{cases} TV(w^{n+1}) \leq TV(w^n) \\ \|w^{n+1}\|_\infty \leq \|w^n\|_\infty \end{cases} \iff \text{no oscillations}$$

First idea: Find an AP order 2 scheme which satisfies these properties.

Impossible

Theorem (Gottlieb, Shu & Tadmor, '01): There are no implicit Runge-Kutta schemes of order higher than one which preserves the TVD property.

Another idea: use a limited scheme.

$$W^{n+1} = \theta W^{n+1,O2} + (1 - \theta) W^{n+1,O1}$$

- $W^{n+1,Oj}$ = order j AP approximation
- $\theta \in [0, 1]$ largest value such that W^{n+1} does not oscillate

Toy scalar equation: $\partial_t w + c_e \partial_x w + \frac{c_i}{\sqrt{\varepsilon}} \partial_x w = 0$

- Order 1 AP scheme with upwind space discretizations ($c_e, c_i > 0$):

$$w_j^{n+1,O1} = w_j^n - c_e (w_j^n - w_{j-1}^n) - \frac{c_i}{\sqrt{\varepsilon}} (w_j^{n+1,O1} - w_{j-1}^{n+1,O1}).$$

- Order 2 AP scheme: ARS with the parameter $\beta = 1 - 1/\sqrt{2}$.

Theorem (Dimarco, Loubère, M.-D., Vignal):

Under the CFL condition $\Delta t \leq \Delta x / c_e$,

$$\theta = \frac{\beta}{1 - \beta} \simeq 0.41 \quad \implies \quad \begin{cases} TV(w^{n+1}) \leq TV(w^n), \\ \|w^{n+1}\|_\infty \leq \|w^n\|_\infty. \end{cases}$$

Limited AP scheme:

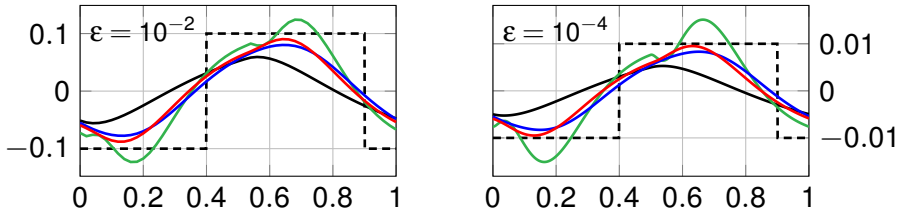
$$w^{n+1,lim} = \theta w^{n+1,O2} + (1 - \theta) w^{n+1,O1} \quad \text{with} \quad \theta = \frac{\beta}{1 - \beta}$$

Problem: More accurate than order 1 but not order 2

Solution: MOOD procedure: see [Clain, Diot & Loubère, '11]

On the toy equation: w^{n+1} **MOOD AP scheme**, CFL $\Delta t \leq \Delta x / c_e$

- Compute the order 2 approximation $w^{n+1,O2}$.
- Detect if the max. principle is satisfied: $\|w^{n+1,O2}\|_{\infty} \leq \|w^n\|_{\infty}$?
- If not, compute the limited AP approximation $w^{n+1,lim}$.



---- exact — 1st order — 2nd order — TVD-AP — AP-MOOD

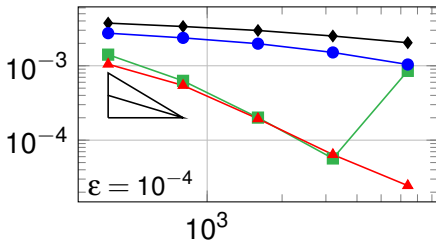
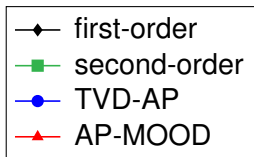
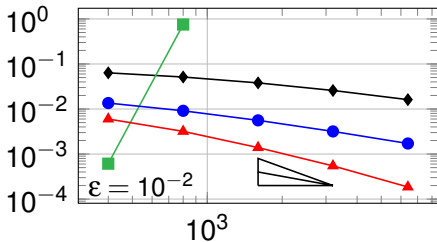
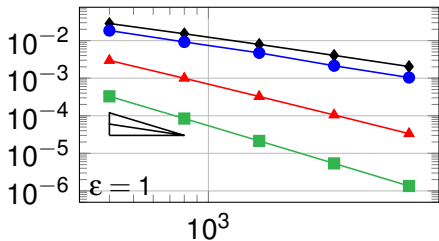
Outline

- 1 General context: multi-scale models and principle of AP schemes
- 2 An order 1 AP scheme for the Euler system in the low Mach limit
- 3 Second-order schemes in time
- 4 Second-order schemes in space and application to Euler**
- 5 Work in progress and perspectives

Error curves for the toy scalar equation

17/26

- Order 2 in space: MUSCL (with the MC limiter) with explicit slopes for implicit fluxes.
- Error curves on a smooth solution for the toy scalar equation:



Recall the first-order IMEX scheme for the Euler system:

$$\left\{ \begin{array}{l} \frac{\rho^{n+1} - \rho^n}{\Delta t} + \nabla \cdot (\rho u)^{n+1} = 0, \end{array} \right. \quad (1)$$

$$\left\{ \begin{array}{l} \frac{(\rho u)^{n+1} - (\rho u)^n}{\Delta t} + \nabla \cdot (\rho u \otimes u)^n + \frac{1}{\varepsilon} \nabla p(\rho^{n+1}) = 0. \end{array} \right. \quad (2)$$

We apply the same convex combination procedure:

$$W^{n+1,lim} = \theta W^{n+1,O2} + (1 - \theta) W^{n+1,O1}, \quad \text{with } \theta = \frac{\beta}{1 - \beta}.$$

↪ We use the value of θ given by the study of the toy scalar equation.

↪ But how can we detect oscillations for the MOOD procedure?

The previous detector (L^∞ criterion on the solution) is irrelevant for the Euler equations, since p and u do not satisfy a maximum principle.

\rightsquigarrow we need **another detection criterion**

We pick the **Riemann invariants** $\Phi_{\pm} = u \mp \frac{2}{\gamma-1} \sqrt{\frac{1}{\varepsilon} \frac{\partial p(\rho)}{\partial \rho}}$: in a

Riemann problem, at least one of them satisfies a maximum principle.

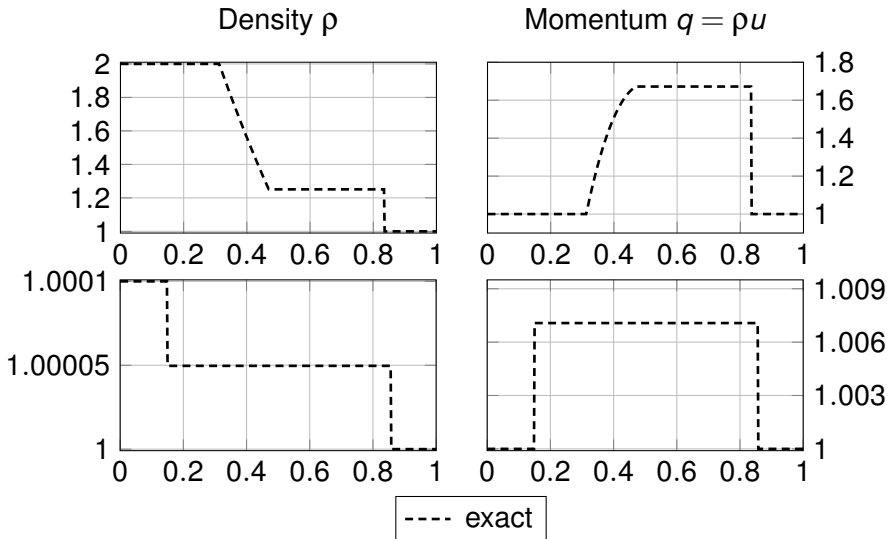
[Smoller & Johnson, '69]

On the Euler equations: W^{n+1} **MOOD AP scheme**, CFL $\Delta t \leq \Delta x / \lambda$

- Compute the order 2 approximation $W^{n+1, O2}$.
- Detect if both Riemann invariants break the maximum principle at the same time.
- If so, compute the limited AP approximation $W^{n+1, lim}$.

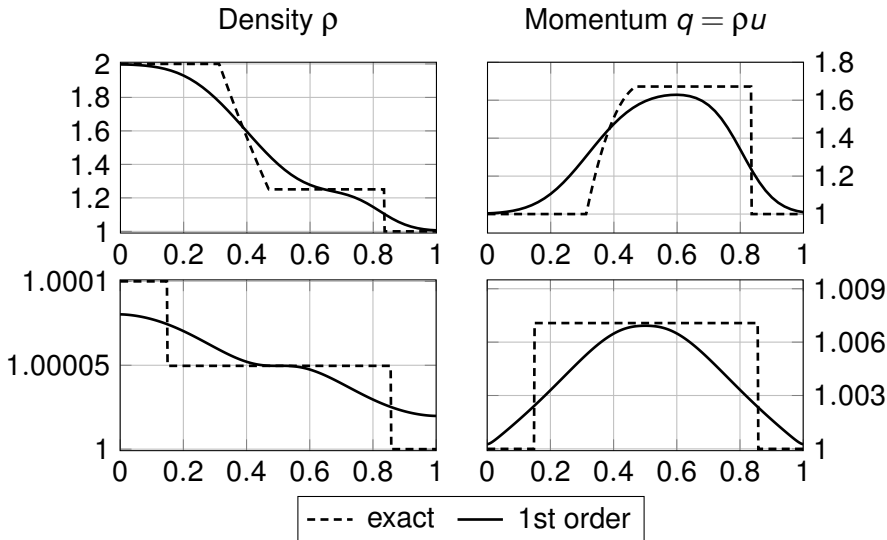
Euler equations: 1D Numerical results

Riemann problem: left rarefaction wave, right shock ;
top curves: $\varepsilon = 1$ (50 pts) ; bottom curves: $\varepsilon = 10^{-4}$ (500 pts)



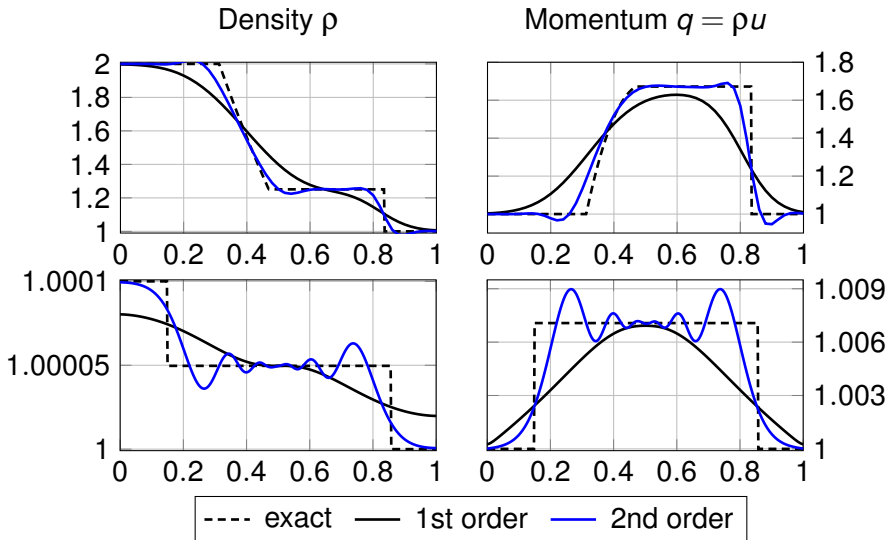
Euler equations: 1D Numerical results

Riemann problem: left rarefaction wave, right shock ;
top curves: $\varepsilon = 1$ (50 pts) ; bottom curves: $\varepsilon = 10^{-4}$ (500 pts)



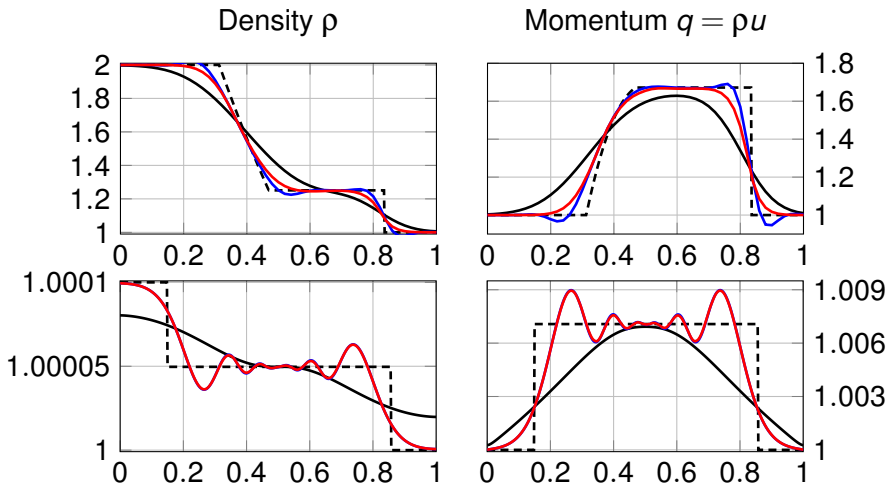
Euler equations: 1D Numerical results

Riemann problem: left rarefaction wave, right shock ;
top curves: $\varepsilon = 1$ (50 pts) ; bottom curves: $\varepsilon = 10^{-4}$ (500 pts)



Euler equations: 1D Numerical results

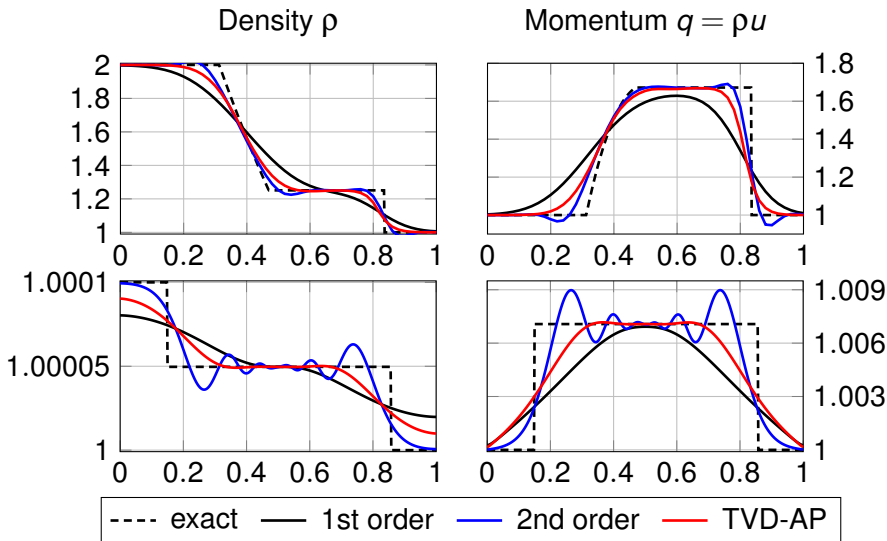
Riemann problem: left rarefaction wave, right shock ;
top curves: $\varepsilon = 1$ (50 pts) ; bottom curves: $\varepsilon = 10^{-4}$ (500 pts)



---- exact — 1st order — 2nd order — 2nd order space lim.

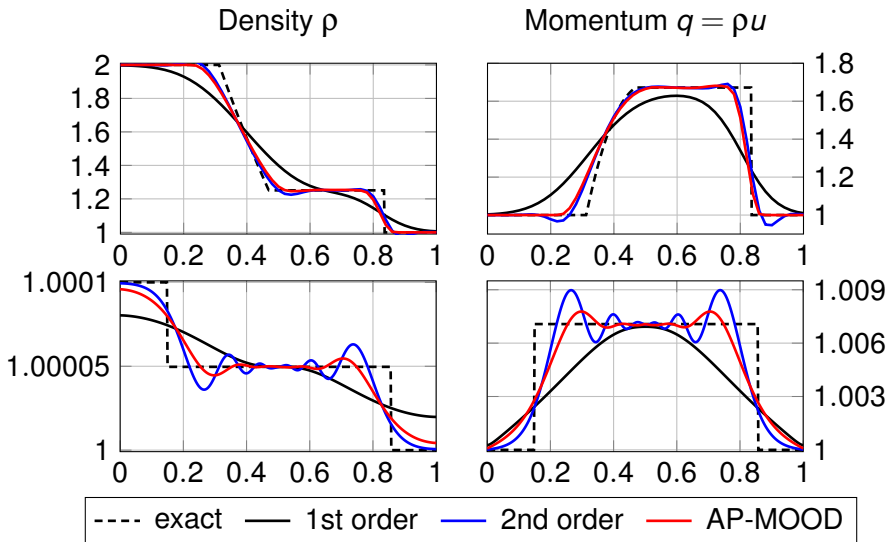
Euler equations: 1D Numerical results

Riemann problem: left rarefaction wave, right shock ;
top curves: $\varepsilon = 1$ (50 pts) ; bottom curves: $\varepsilon = 10^{-4}$ (500 pts)



Euler equations: 1D Numerical results

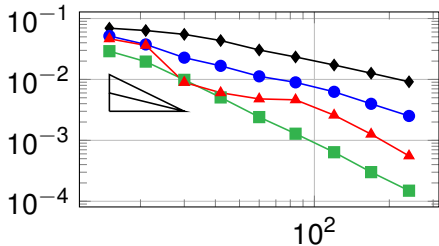
Riemann problem: left rarefaction wave, right shock ;
top curves: $\varepsilon = 1$ (50 pts) ; bottom curves: $\varepsilon = 10^{-4}$ (500 pts)



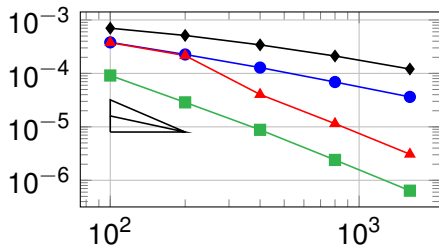
Euler equations: 1D Numerical results

Error curves in L^∞ norm, smooth 1D solution

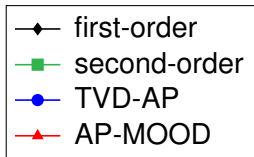
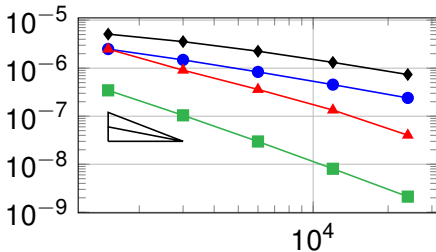
$\varepsilon = 1$



$\varepsilon = 10^{-2}$



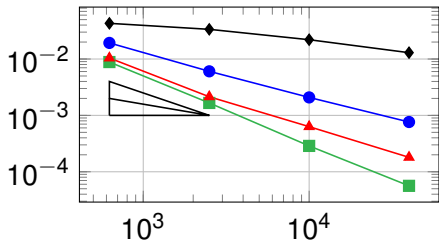
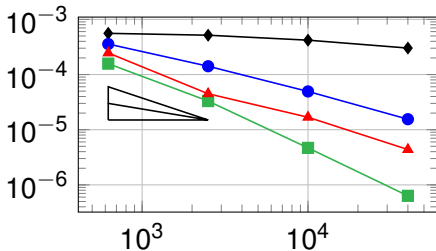
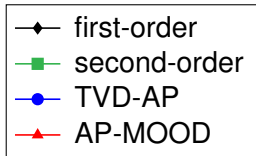
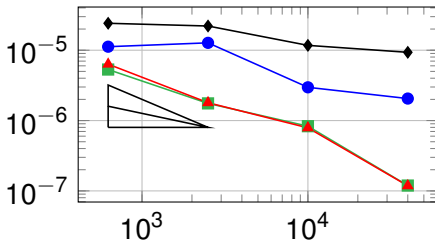
$\varepsilon = 10^{-4}$



Euler equations: 2D Numerical results

22/26

Error curves in L^∞ norm, smooth 2D traveling vortex (Cartesian mesh)

 $\varepsilon = 1$  $\varepsilon = 10^{-2}$  $\varepsilon = 10^{-4}$ 

1st-order AP

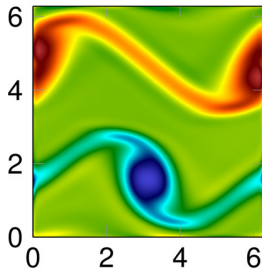
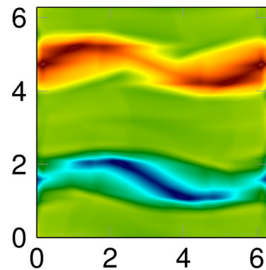
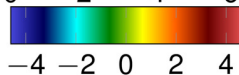
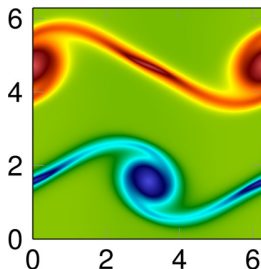
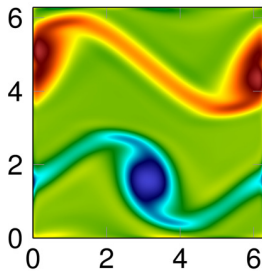
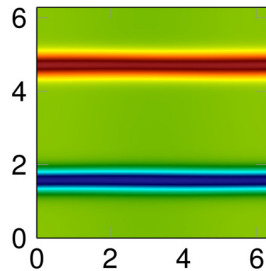
2nd-order AP

reference solution
obtained solving
the vorticity formulation

$$\partial_t \omega + U \cdot \nabla \omega = 0,$$

with $\omega = \partial_x v - \partial_y u$

reference



TVD-AP

TVD-AP-MOOD

Outline

- 1 General context: multi-scale models and principle of AP schemes
- 2 An order 1 AP scheme for the Euler system in the low Mach limit
- 3 Second-order schemes in time
- 4 Second-order schemes in space and application to Euler
- 5 Work in progress and perspectives**

Extension to the full Euler system:

$$\begin{cases} \partial_t \rho + \nabla \cdot (\rho U) = 0, \\ \partial_t (\rho U) + \nabla \cdot (\rho U \otimes U) + \frac{1}{\varepsilon} \nabla p = 0, \quad \text{with} \quad p = (\gamma - 1) \left(E - \varepsilon \frac{\rho |U|^2}{2} \right), \\ \partial_t E + \nabla \cdot (U(E + p)) = 0, \end{cases}$$

In 1D, to get an AP scheme ensuring that both the explicit and the implicit parts are hyperbolic, we take:

$$\frac{W^{n+1} - W^n}{\Delta t} + A_e^{n,n+1} \partial_x W^n + A_i^{n,n+1} \partial_x W^{n+1} = 0.$$


The scheme no longer takes the conservative IMEX form

$$\frac{W^{n+1} - W^n}{\Delta t} + \partial_x F_e(W^n) + \partial_x F_i(W^n) = 0.$$

- 1 Study a local value of θ , depending on the presence of oscillations in a given cell: how to reconcile the locality of θ with the non-locality of the implicitation?



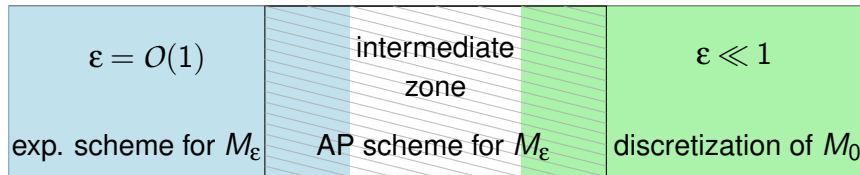
 : cell with oscillation $\implies \theta < 1$

 : cell without oscillation $\implies \theta = 1$ or $\theta < 1$?

- 2 Compute optimal values of θ for other IMEX discretizations:
 - SSPRK explicit part?
 - custom-made second-order IMEX discretization to ensure θ as close to 1 as possible?
 - higher-order discretizations?

Domain decomposition with respect to ε :

Compressible Euler (M_ε) $\xrightarrow{\varepsilon \rightarrow 0}$ Incompressible Euler (M_0)



- How to define the boundaries of the intermediate zones?
- How to handle interfaces in 1D with first-order schemes?
- How to extend to higher dimensions and higher-order schemes?

Thanks for your attention!

Euler equations: 2D Numerical results

To obtain a 2D reference incompressible solution, set $\omega = \partial_x v - \partial_y u$ and consider the **vorticity formulation** of the incompressible Euler equations:

$$\partial_t \omega + U \cdot \nabla \omega = 0,$$

$$\nabla \cdot U = 0 \implies \exists \text{ **stream function** } \Psi \text{ such that } \begin{cases} U = {}^t(\partial_y \Psi, -\partial_x \Psi), \\ -\Delta \Psi = \omega. \end{cases}$$

To get the time evolution of the vorticity from ω^n :

- 1 solve $-\Delta \Psi^n = \omega^n$ for Ψ^n (with periodic BC and assuming that the average of Ψ vanishes);
- 2 get U^n from $U^n = {}^t(\partial_y \Psi^n, -\partial_x \Psi^n)$;
- 3 solve $\partial_t \omega + U^n \cdot \nabla \omega^n = 0$ to get ω^{n+1} .

We get a reference incompressible vorticity $\omega(x, t)$, to be compared to the vorticity of the solution given by the compressible scheme with small ε (we take $\varepsilon = M^2 = 10^{-5}$).

All speed schemes

- **Preconditioning methods:** [Chorin, '65], [Choi, Merkle, '85], [Tukel, '87], [Van Leer, Lee & Roe, '91], [Li & Gu '08, '10], ...
- **Splitting and pressure correction:** [Harlow & Amsden, '68, '71], [Karki & Patankar, '89], [Bijl & Wesseling, '98], [Sewall & Tafti, '08], [Klein, Botta, Schneider, Munz & Roller '08], [Guillard, Murrone & Viozat '99, '04, '06] [Herbin, Kheriji & Latché '12, '13], ...
- ▣ **Asymptotic preserving schemes**
 - [Degond, Deluzet, Sangam & Vignal, '09], [Degond & Tang, '11], [Cordier, Degond & Kumbaro, '12], [Grenier, Vila & Villedieu, '13] [Dellacherie, Omnès & Raviart, '13], [Noelle, Bispen, Arun, Lukáčová & Munz, '14], [Chalons, Girardin & Kokh, '15] [Dimarco, Loubère & Vignal, '17]