A fully well-balanced scheme for the shallow-water equations with topography

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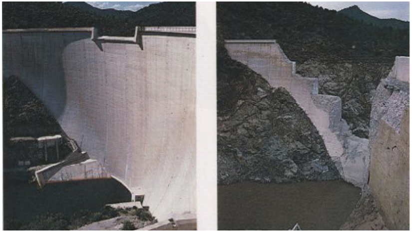
Several kinds of destructive geophysical flows



Dam failure (Malpasset, France, 1959)

Tsunami (Tōhoku, Japan, 2011)

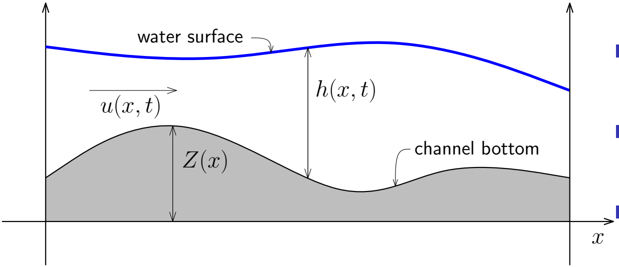
Flood (La Faute sur Mer, France, 2010)

Mudslide (Madeira, Portugal, 2010)

The shallow-water equations

$$\begin{cases} \partial_t h + \partial_x (hu) = 0 \\ \partial_t (hu) + \partial_x \left(hu^2 + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z \end{cases}$$

We can rewrite the equations as $\partial_t W + \partial_x F(W) = S(W)$, with $W = \begin{pmatrix} h \\ q \end{pmatrix}$.



- $Z(x)$ is the known topography
- g is the gravitational constant
- we label the water discharge $q := hu$

Steady state solutions

Definition: Steady state solutions

W is a steady state solution iff $\partial_t W = 0$, i.e. $\partial_x F(W) = S(W)$.

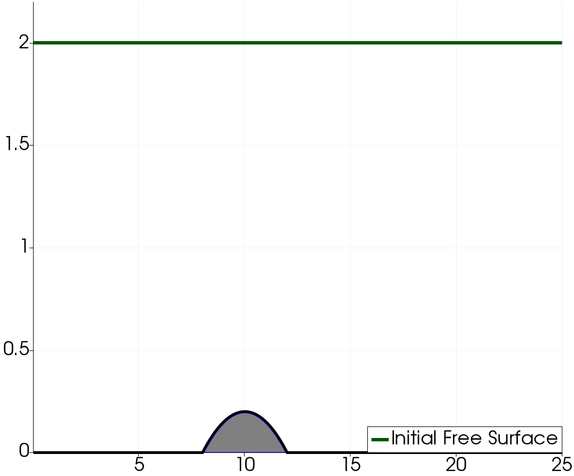
Taking $\partial_t W = 0$ in the shallow-water equations leads to

$$\begin{cases} \partial_x q = 0 \\ \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) = -gh\partial_x Z. \end{cases}$$

The smooth steady state solutions are therefore given by the following statement of Bernoulli's principle:

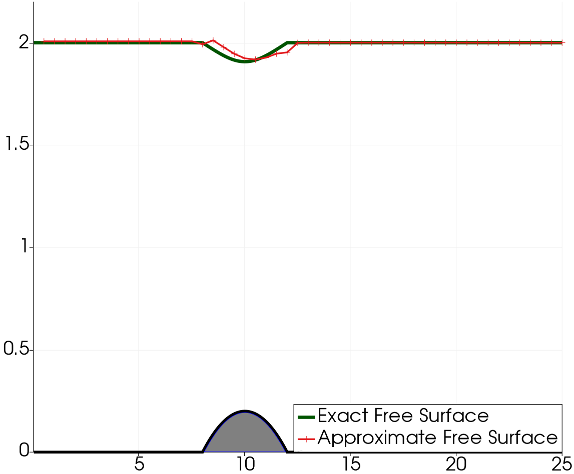
$$\begin{cases} q = \text{cst} = q_0 \\ \dfrac{q_0^2}{2} \partial_x \left(\dfrac{1}{h^2} \right) + g\partial_x (h + Z) = 0. \end{cases}$$

Steady state not captured in 1D



The initial condition is at rest; water is injected through the left boundary.

Steady state not captured in 1D



The classical HLL numerical scheme converges towards a numerical steady state which does not correspond to the physical one.

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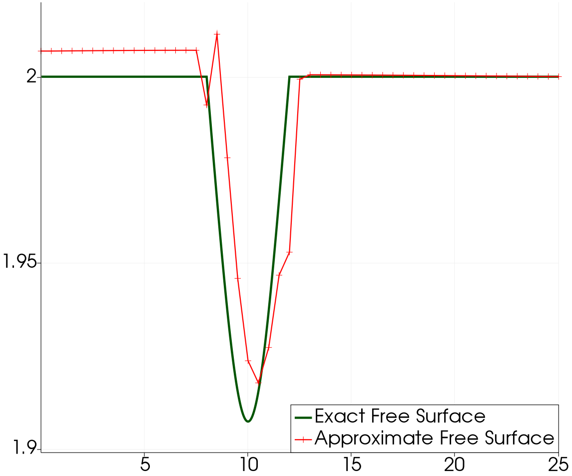
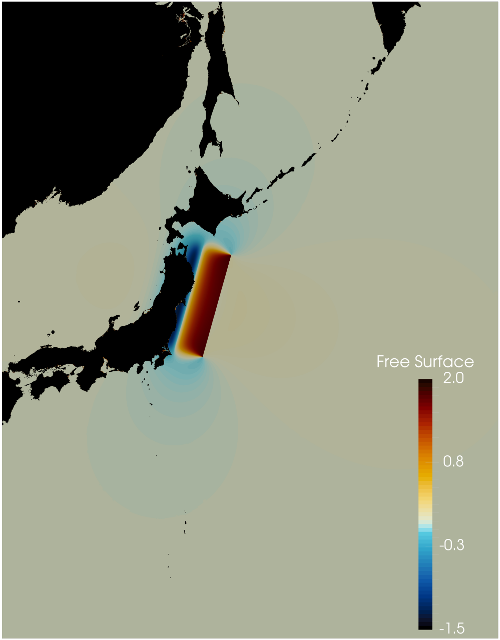


The classical HLL numerical scheme converges towards a numerical steady state which does not correspond to the physical one.

A real-life simulation: the 2011 Tōhoku tsunami. The water is close to a steady state at rest far from the tsunami.



Objectives

Our goal is to derive a numerical method for the shallow-water model that exactly preserves its stationary solutions on every mesh.

To that end, we seek a numerical scheme that:

1. is well-balanced for the shallow-water equations with topography, i.e. it exactly preserves and captures the steady states without having to solve the governing nonlinear equation;
2. preserves the non-negativity of the water height;
3. can be easily extended for other source terms of the shallow-water equations (e.g. friction or breadth).

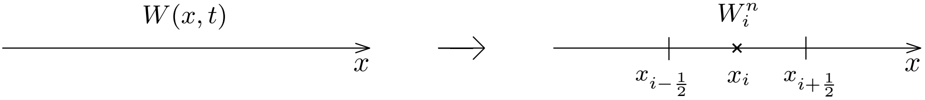
Contents

Setting

Objective: Approximate the solution $W(x,t)$ of the hyperbolic system $\partial_t W + \partial_x F(W) = S(W)$, with suitable initial and boundary conditions.

We partition $[a,b]$ in *cells*, of volume Δx and of evenly spaced centers x_i, and we define:

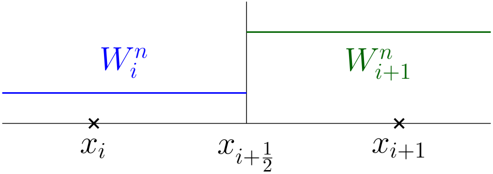
- $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$, the boundaries of the cell i;
- W_i^n, an approximation of $W(x,t)$, constant in the cell i and at time t^n, which is defined as $W_i^n = \dfrac{1}{\Delta x} \displaystyle\int_{\Delta x/2}^{\Delta x/2} W(x,t^n)dx$.



Using an approximate Riemann solver

As a consequence, at time t^n, we have a succession of Riemann problems (Cauchy problems with discontinuous initial data) at the interfaces between cells:

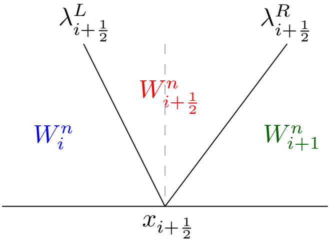
$$\begin{cases} \partial_t W + \partial_x F(W) = S(W) \\ W(x, t^n) = \begin{cases} W_i^n \text{ if } x < x_{i+\frac{1}{2}} \\ W_{i+1}^n \text{ if } x > x_{i+\frac{1}{2}} \end{cases} \end{cases}$$



For $S(W) \neq 0$, the exact solution to these Riemann problems is unknown or costly to compute $\rightsquigarrow$ we require an approximation.

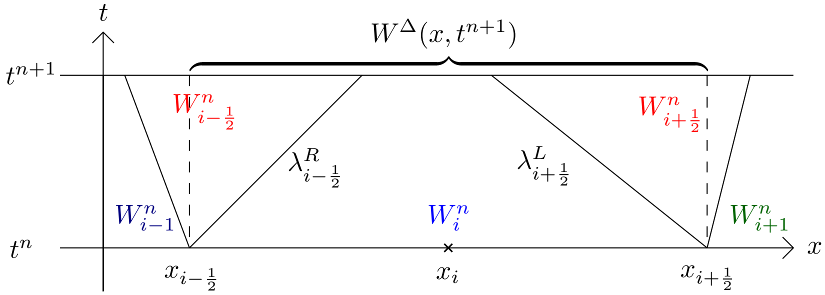
Using an approximate Riemann solver

We choose to use an approximate Riemann solver, as follows.



- $W^n_{i+\frac{1}{2}}$ is an approximation of the interaction between W^n_i and W^n_{i+1} (i.e. of the solution to the Riemann problem), possibly made of several constant states separated by discontinuities.
- $\lambda^L_{i+\frac{1}{2}}$ and $\lambda^R_{i+\frac{1}{2}}$ are approximations of the wave speeds.

Godunov-type scheme (approximate Riemann solver)



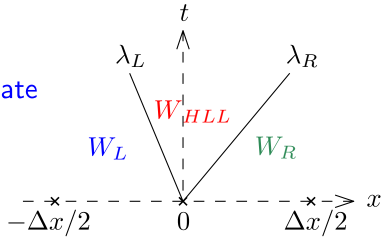
We define the time update as follows :

$$W_i^{n+1} := \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^\Delta(x, t^{n+1}) dx.$$

Since $W^n_{i-\frac{1}{2}}$ and $W^n_{i+\frac{1}{2}}$ are made of constant states, the above integral is easy to compute.

The HLL approximate Riemann solver

To approximate solutions of $\partial_t W + \partial_x F(W) = 0$, the HLL approximate Riemann solver (Harten, Lax, van Leer (1983)) may be chosen; it is denoted by W^Δ and displayed on the right.



The consistency condition (as per Harten and Lax) holds if:

$$\frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W^\Delta(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x}\int_{-\Delta x/2}^{\Delta x/2} W_{\mathcal{R}}(\Delta t, x; W_L, W_R)dx,$$

which gives $W_{HLL} = \dfrac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \dfrac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \begin{pmatrix} h_{HLL} \\ q_{HLL} \end{pmatrix}$.

Note that, if $h_L > 0$ and $h_R > 0$, then $h_{HLL} > 0$ for $|\lambda_L|$ and $|\lambda_R|$ large enough.

Modification of the HLL approximate Riemann solver

The shallow-water equations with the topography source term read as follows:

$$\begin{cases} \partial_t h + \partial_x q = 0, \\ \partial_t q + \partial_x \left(\dfrac{q^2}{h} + \dfrac{1}{2} g h^2 \right) + g h \partial_x Z = 0. \end{cases}$$

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We can add the equation $\partial_t Z = 0$, which corresponds to the fixed geometry of the problem:

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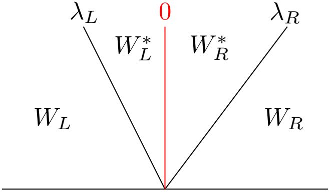
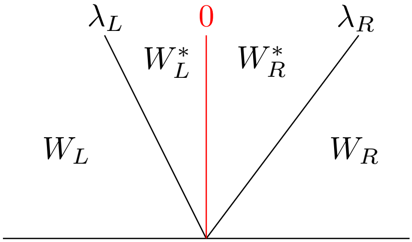
The equation $\partial_t Z = 0$ induces a stationary wave associated to the source term; we also note that q is a Riemann invariant for this wave.

To approximate solutions of $\partial_t W + \partial_x F(W) = S(W)$, we thus use the approximate Riemann solver displayed on the right (assuming $\lambda_L < 0 < \lambda_R$).



Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q_L^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q_R^* \end{pmatrix}$.



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  - $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$ (relation 2)

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  - $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$ (relation 3),

    where $\overline{S} \simeq \dfrac{1}{\Delta x}\dfrac{1}{\Delta t}\displaystyle\int_{-\Delta x/2}^{\Delta x/2}\int_0^{\Delta t} S(W_{\mathcal{R}}(x,t))\,dt\,dx$

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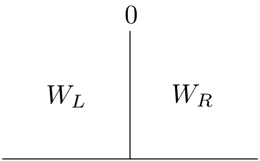
- **next step**: obtain a fourth relation

Obtaining an additional relation

Assume that W_L and W_R define a steady state, i.e. that they satisfy the following discrete version of the steady relation $\partial_x F(W) = S(W)$ (where $[X] = X_R - X_L$):

$$\frac{1}{\Delta x}\left(q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}\left[h^2\right]\right) = \overline{S}.$$

For the steady state to be preserved, it is sufficient to have $h_L^* = h_L$, $h_R^* = h_R$ and $q^* = q_0$.



Assuming a steady state, we easily show that $q^* = q_0$. Therefore, the additional relation should only link h_L^* and h_R^*.

Obtaining an additional relation

In order to determine an addition relation, we consider the discrete steady relation, satisfied when W_L and W_R define a steady state:

$$q_0^2 \left(\frac{1}{h_R} - \frac{1}{h_L} \right) + \frac{g}{2}\big((h_R)^2 - (h_L)^2\big) = \overline{S}\Delta x.$$

To ensure that $h_L^* = h_L$ and $h_R^* = h_R$, we impose that h_L^* and h_R^* satisfy the above relation, as follows:

$$q_0^2 \left(\frac{1}{h_R^*} - \frac{1}{h_L^*} \right) + \frac{g}{2}\big((h_R^*)^2 - (h_L^*)^2\big) = \overline{S}\Delta x.$$

To avoid solving a nonlinear system, we elect to use the following linearization of this relation:

$$\underbrace{\left(\frac{-(q^*)^2}{h_L h_R} + \frac{g}{2}(h_L + h_R) \right)}_{\alpha} (h_R^* - h_L^*) = \overline{S}\Delta x.$$

Determination of h_L^* and h_R^*

With the consistency relation between h_L^* and h_R^*, the intermediate water heights satisfy the following linear system:

$$\begin{cases} \alpha(h_R^* - h_L^*) = \overline{S}\Delta x, \\ \lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}. \end{cases}$$

Using both relations linking h_L^* and h_R^*, we obtain

$$\begin{cases} h_L^* = h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \\ h_R^* = h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \end{cases}$$

where $\alpha = \left(\dfrac{-(q^*)^2}{h_L h_R} + \dfrac{g}{2}(h_L + h_R) \right)$ with $q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}$.

Correction to ensure non-negative h_L^* and h_R^*

However, these expressions of h_L^* and h_R^* do not guarantee that the intermediate heights are non-negative: instead, we use the following cutoff (see Audusse, Chalons, Ung (2014)):

$$\begin{cases} h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \bar{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \bar{S} \Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right). \end{cases}$$

Note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$;
- the well-balance property, since it is not activated when W_L and W_R define a steady state.

Summary

The two-state approximate Riemann solver with intermediate states $W_L^* = \begin{pmatrix} h_L^* \\ q^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q^* \end{pmatrix}$ given by

$$\begin{cases} q^* = q_{HLL} + \dfrac{\overline{S}\Delta x}{\lambda_R - \lambda_L}, \\ h_L^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_R \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_R}{\lambda_L}\right) h_{HLL}\right), \\ h_R^* = \min\left(\left(h_{HLL} - \dfrac{\lambda_L \overline{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}\right)_+, \left(1 - \dfrac{\lambda_L}{\lambda_R}\right) h_{HLL}\right), \end{cases}$$

is consistent, non-negativity-preserving and well-balanced.

next step: determination of $\overline{S}$ according to the source term definition (topography).

The topography source term

We now consider $S(W) = S^t(W) = -gh\partial_x Z$:
the smooth steady states are governed by

$$\left.\begin{aligned} \partial_x\left(\frac{q_0^2}{h}\right) + \frac{g}{2}\partial_x(h^2) &= -gh\partial_x Z, \\ \frac{q_0^2}{2}\partial_x\left(\frac{1}{h^2}\right) + g\partial_x(h+Z) &= 0, \end{aligned}\right\} \xrightarrow[\text{discretization}]{} \left\{\begin{aligned} q_0^2\left[\frac{1}{h}\right] + \frac{g}{2}[h^2] &= \overline{S}^t\Delta x, \\ \frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h+Z] &= 0. \end{aligned}\right.$$

We can exhibit an expression of q_0^2 and thus obtain

$$\overline{S}^t = -g\frac{2h_L h_R}{h_L + h_R}\frac{[Z]}{\Delta x} + \frac{g}{2\Delta x}\frac{[h]^3}{h_L + h_R}.$$

However, when $Z_L = Z_R$, we have $\overline{S}^t \neq \mathcal{O}(\Delta x)$, i.e. a loss of consistency with S^t (see for instance Berthon, Chalons (2016)).

The topography source term

Instead, we set, for some constant $C > 0$,

$$\begin{cases} \overline{S}^t = -g \dfrac{2h_L h_R}{h_L + h_R} \dfrac{[Z]}{\Delta x} + \dfrac{g}{2\Delta x} \dfrac{[h]_c^3}{h_L + h_R}, \\ [h]_c = \begin{cases} h_R - h_L & \text{if } |h_R - h_L| \leq C\Delta x, \\ \operatorname{sgn}(h_R - h_L)\, C\Delta x & \text{otherwise.} \end{cases} \end{cases}$$

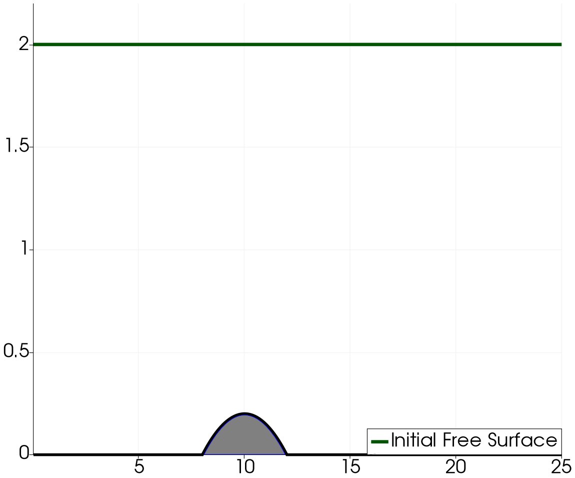
Theorem: Well-balance for the topography source term

If W_L and W_R define a smooth steady state, i.e. if they satisfy

$$\frac{q_0^2}{2}\left[\frac{1}{h^2}\right] + g[h + Z] = 0,$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced. By construction, the Godunov-type scheme using this approximate Riemann solver is consistent, fully well-balanced and positivity-preserving.

Verification of the well-balance: topography



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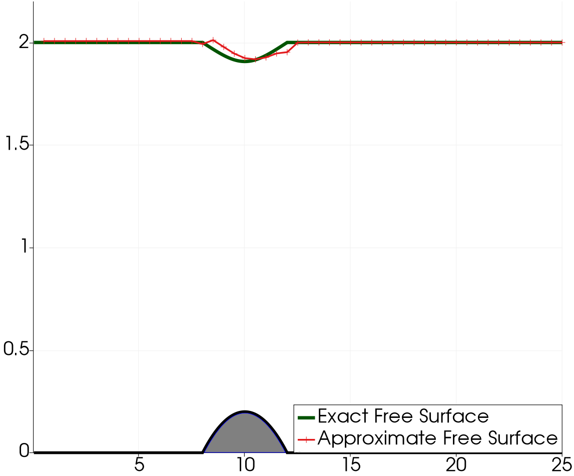

The non-well-balanced HLL scheme converges towards a numerical steady state which does not correspond to the physical one.

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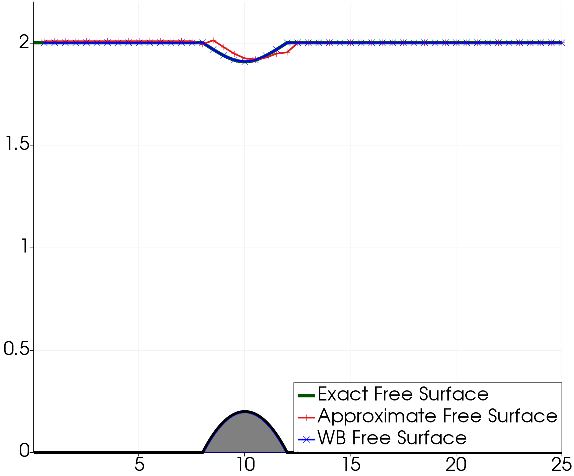

The non-well-balanced HLL scheme converges towards a numerical steady state which does not correspond to the physical one. The well-balanced scheme converges towards the physical steady state.

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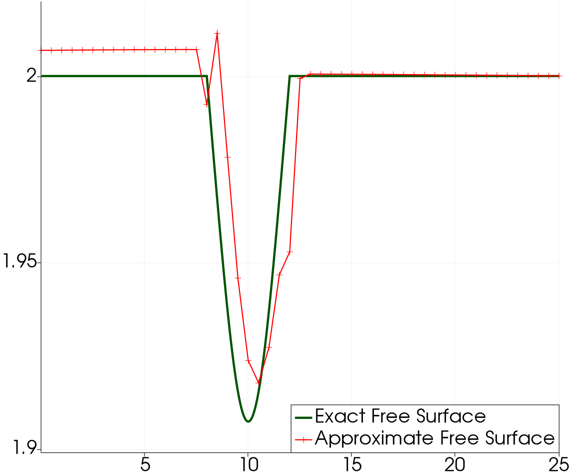

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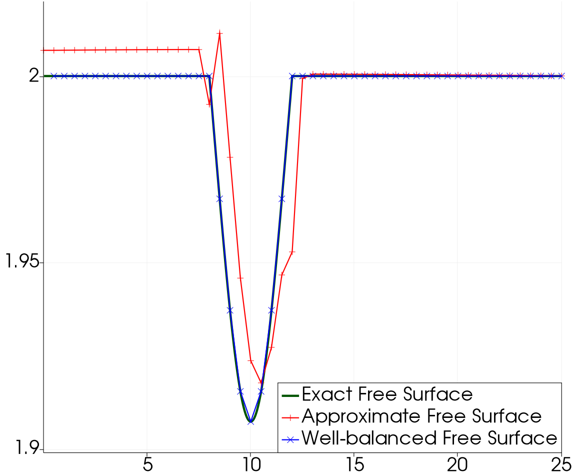

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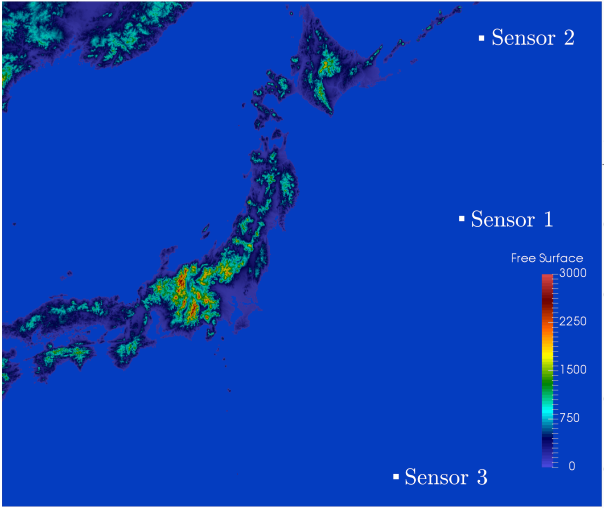
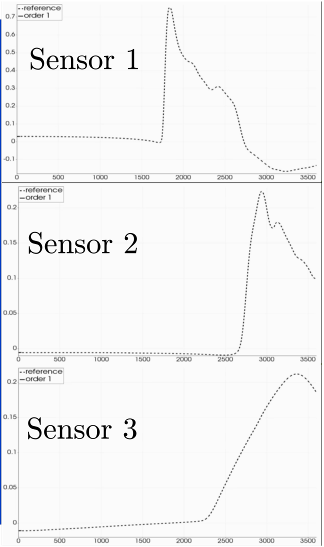
Simulation of the 2011 Tōhoku tsunami





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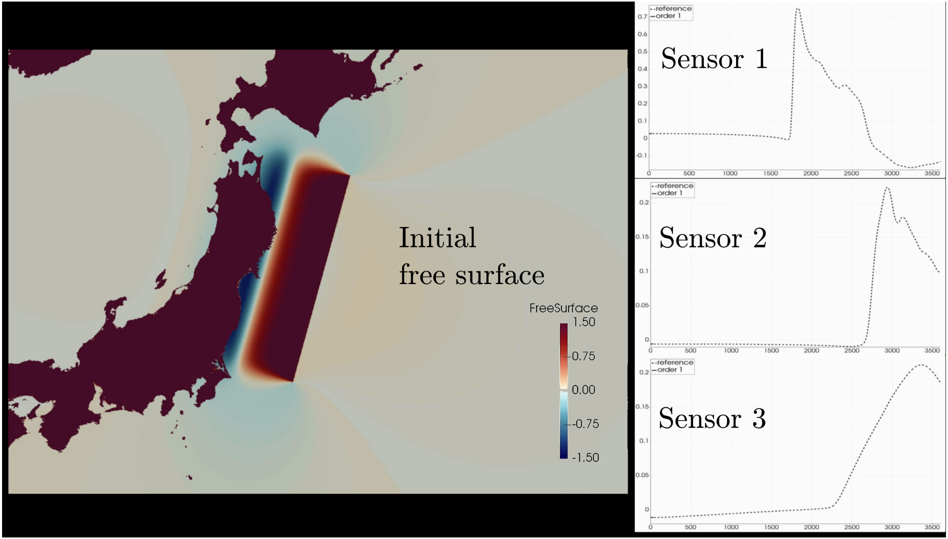

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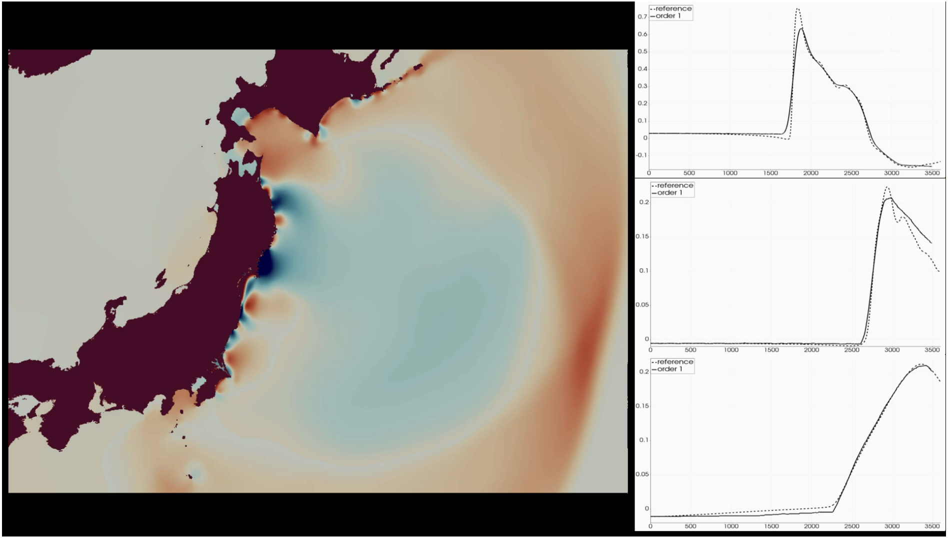


Conclusion

- We have presented a well-balanced and non-negativity-preserving numerical scheme, able to be applied to other source terms or combinations of source terms.
- We have also displayed results from a 2D well-balanced numerical method, coded in Fortran and parallelized.

This work has been published:

V. Michel-Dansac et al. "A well-balanced scheme for the shallow-water equations with topography". In: *Comput. Math. Appl.* 72.3 (2016), pp. 568–593

V. Michel-Dansac et al. "A well-balanced scheme for the shallow-water equations with topography or Manning friction". In: *J. Comput. Phys. (accepted)* (2017)

Perspectives

Work in progress or completed

- application to other source terms:
  - friction source term (completed, article accepted)
  - Coriolis force source term (work in progress)
  - breadth variation source term (work in progress)
- high-order and 2D extensions (work in progress, collaboration with R. Loubère)

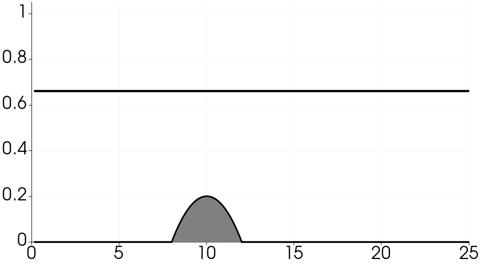
Long-term perspectives

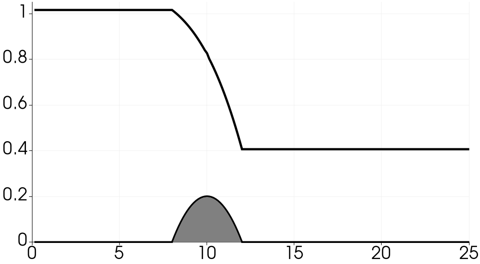
- stability of the scheme: values of C, λ_L and λ_R to ensure the entropy preservation
- ensure the entropy preservation for the high-order scheme (use of a MOOD method)

Thank you for your attention!

Verification of the well-balance: topography

transcritical flow test case (see Goutal, Maurel (1997))





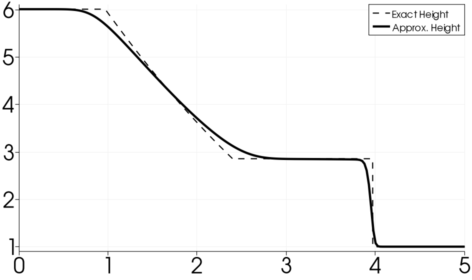
left panel: initial free surface at rest; water is injected from the left boundary

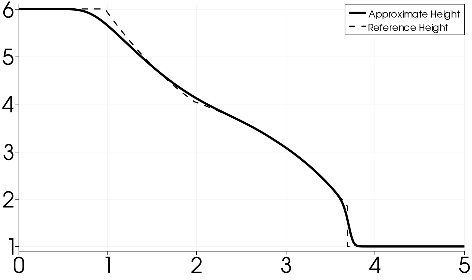
right panel: free surface for the steady state solution, after a transient state

$$\Phi = \frac{u^2}{2} + g(h+Z)$$

	L^1	L^2	L^∞
errors on q	1.47e-14	1.58e-14	2.04e-14
errors on Φ	1.67e-14	2.13e-14	4.26e-14

Riemann problems between two wet areas

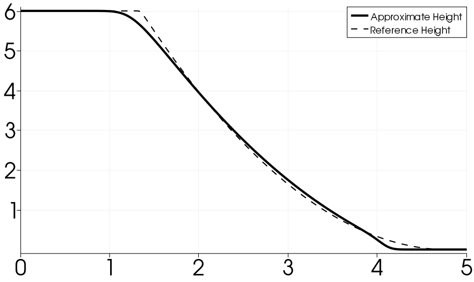


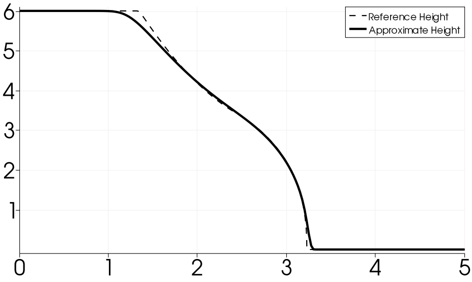


left: $k = 0$ left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

Riemann problems with a wet/dry transition



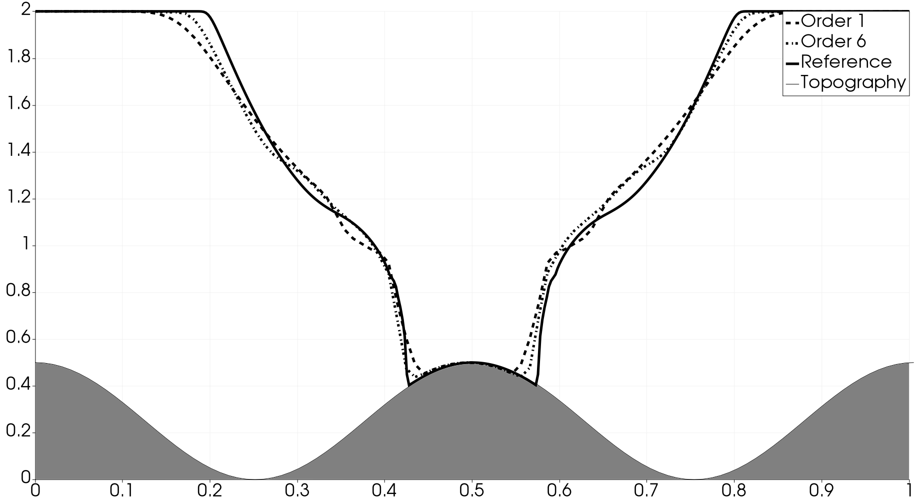


left: $k = 0$

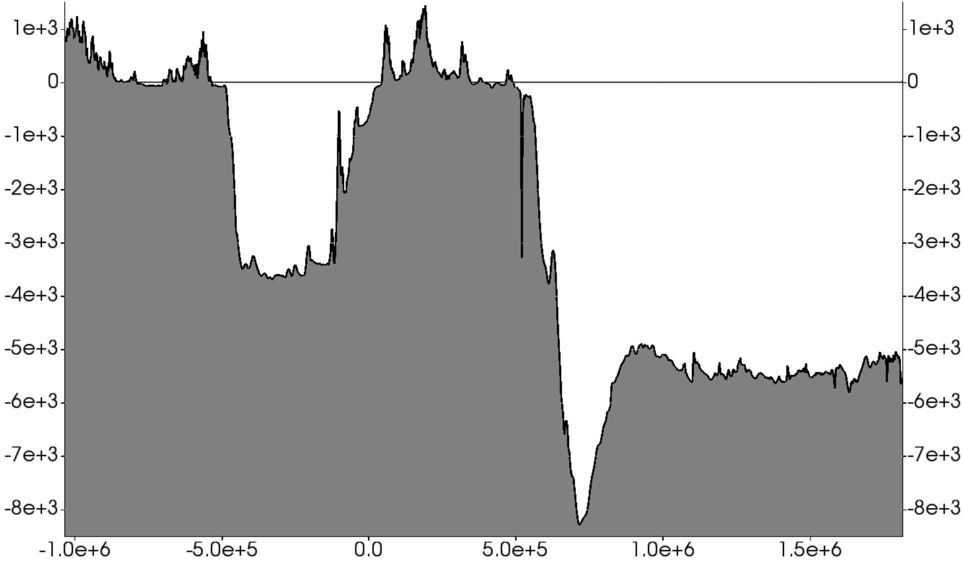
left: $k = 10$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.15s

Double dry dam-break on a sinusoidal bottom



Japan tsunami: 1D slice



Preservation of the lake at rest

A. Bermudez and M. E. Vazquez. "Upwind methods for hyperbolic conservation laws with source terms". In: *Comput. & Fluids* 23.8 (1994), pp. 1049–1071

J. M. Greenberg and A.-Y. LeRoux. "A well-balanced scheme for the numerical processing of source terms in hyperbolic equations". In: *SIAM J. Numer. Anal.* 33.1 (1996), pp. 1–16

E. Audusse et al. "A fast and stable well-balanced scheme with hydrostatic reconstruction for shallow water flows". In: *SIAM J. Sci. Comput.* 25.6 (2004), pp. 2050–2065

C. Berthon and F. Foucher. "Efficient well-balanced hydrostatic upwind schemes for shallow-water equations". In: *J. Comput. Phys.* 231.15 (2012), pp. 4993–5015

E. Audusse et al. "A simple well-balanced and positive numerical scheme for the shallow-water system". In: *Commun. Math. Sci.* 13.5 (2015), pp. 1317–1332

Fully well-balanced schemes

L. Gosse. "A well-balanced flux-vector splitting scheme designed for hyperbolic systems of conservation laws with source terms". In: *Comput. Math. Appl.* 39.9-10 (2000), pp. 135–159

M. J. Castro et al. "Well-balanced numerical schemes based on a generalized hydrostatic reconstruction technique". In: *Math. Models Methods Appl. Sci.* 17.12 (2007), pp. 2055–2113

U. S. Fjordholm et al. "Well-balanced and energy stable schemes for the shallow water equations with discontinuous topography". In: *J. Comput. Phys.* 230.14 (2011), pp. 5587–5609

Y. Xing et al. "On the advantage of well-balanced schemes for moving-water equilibria of the shallow water equations". In: *J. Sci. Comput.* 48.1-3 (2011), pp. 339–349

C. Berthon and C. Chalons. "A fully well-balanced, positive and entropy-satisfying Godunov-type method for the shallow-water equations". In: *Math. Comp.* 85.299 (2016), pp. 1281–1307

V. Michel-Dansac et al. "A well-balanced scheme for the shallow-water equations with topography". In: *Comput. Math. Appl.* 72.3 (2016), pp. 568–593

High-order well-balanced schemes and friction

M. Castro et al. "High order finite volume schemes based on reconstruction of states for solving hyperbolic systems with nonconservative products. Applications to shallow-water systems". In: *Math. Comp.* 75.255 (2006), pp. 1103–1134

M. J. Castro Díaz et al. "High order exactly well-balanced numerical methods for shallow water systems". In: *J. Comput. Phys.* 246 (2013), pp. 242–264

S. Clain and J. Figueiredo. "The MOOD method for the non-conservative shallow-water system". working paper or preprint. 2014

Q. Liang and F. Marche. "Numerical resolution of well-balanced shallow water equations with complex source terms". In: *Adv. Water Resour.* 32.6 (2009), pp. 873–884

A. Chertock et al. "Well-balanced positivity preserving central-upwind scheme for the shallow water system with friction terms". In: *Internat. J. Numer. Methods Fluids* 78.6 (2015), pp. 355–383