A fully well-balanced scheme for the shallow-water equations with topography

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Introduction and motivations

Several kinds of destructive geophysical flows

Dam failure (Malpasset, France, 1959)

Tsunami (Tōhoku, Japan, 2011)

Flood (La Faute sur Mer, France, 2010)

Mudslide (Madeira, Portugal, 2010)
The shallow-water equations

\[
\begin{aligned}
\partial_t h + \partial_x (hu) &= 0 \\
\partial_t (hu) + \partial_x \left( hu^2 + \frac{1}{2} gh^2 \right) &= -gh \partial_x Z
\end{aligned}
\]

We can rewrite the equations as \( \partial_t W + \partial_x F(W) = S(W) \), with \( W = \begin{pmatrix} h \\ q \end{pmatrix} \).

- \( Z(x) \) is the known topography
- \( g \) is the gravitational constant
- we label the water discharge \( q := hu \)
Steady state solutions

Definition: Steady state solutions

\( W \) is a steady state solution iff \( \partial_t W = 0 \), i.e. \( \partial_x F(W) = S(W) \).

Taking \( \partial_t W = 0 \) in the shallow-water equations leads to

\[
\begin{align*}
\partial_x q &= 0 \\
\partial_x \left( \frac{q^2}{h} + \frac{1}{2} gh^2 \right) &= -gh\partial_x Z.
\end{align*}
\]

The smooth steady state solutions are therefore given by the following statement of Bernoulli's principle:

\[
\begin{align*}
q &= \text{cst} = q_0 \\
\frac{q_0^2}{2} \partial_x \left( \frac{1}{h^2} \right) + g\partial_x (h + Z) &= 0.
\end{align*}
\]
Steady state not captured in 1D

The initial condition is at rest; water is injected through the left boundary.
Steady state not captured in 1D

The classical HLL numerical scheme converges towards a numerical steady state which does not correspond to the physical one.
Steady state not captured in 1D

The classical HLL numerical scheme converges towards a numerical steady state which does not correspond to the physical one.
A real-life simulation: the 2011 Tōhoku tsunami. The water is close to a steady state at rest far from the tsunami.
Objectives

Our goal is to derive a numerical method for the shallow-water model that exactly preserves its stationary solutions on every mesh.

To that end, we seek a numerical scheme that:

1. is well-balanced for the shallow-water equations with topography, i.e. it exactly preserves and captures the steady states without having to solve the governing nonlinear equation;

2. preserves the non-negativity of the water height;

3. can be easily extended for other source terms of the shallow-water equations (e.g. friction or breadth).
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1 Brief introduction to Godunov-type schemes

2 Derivation of a well-balanced scheme

3 Numerical simulations

4 Conclusion and perspectives
**Setting**

**Objective:** Approximate the solution $W(x, t)$ of the hyperbolic system $\partial_t W + \partial_x F(W) = S(W)$, with suitable initial and boundary conditions.

We partition $[a, b]$ in *cells*, of volume $\Delta x$ and of evenly spaced centers $x_i$, and we define:

- $x_{i-\frac{1}{2}}$ and $x_{i+\frac{1}{2}}$, the boundaries of the cell $i$;
- $W_i^n$, an approximation of $W(x, t)$, constant in the cell $i$ and at time $t^n$, which is defined as $W_i^n = \frac{1}{\Delta x} \int_{\Delta x/2}^{\Delta x/2} W(x, t^n) dx$.

![Diagram](image_url)
Using an approximate Riemann solver

As a consequence, at time $t^n$, we have a succession of Riemann problems (Cauchy problems with discontinuous initial data) at the interfaces between cells:

$$\begin{cases} \partial_t W + \partial_x F(W) = S(W) \\ W(x, t^n) = \begin{cases} W^n_i & \text{if } x < x_i + \frac{1}{2} \\ W^n_{i+1} & \text{if } x > x_i + \frac{1}{2} \end{cases} \end{cases}$$

For $S(W) \neq 0$, the exact solution to these Riemann problems is unknown or costly to compute $\rightsquigarrow$ we require an approximation.
Using an approximate Riemann solver

We choose to use an approximate Riemann solver, as follows.

- $W_{i+\frac{1}{2}}^{n}$ is an approximation of the interaction between $W_{i}^{n}$ and $W_{i+1}^{n}$ (i.e. of the solution to the Riemann problem), possibly made of several constant states separated by discontinuities.
- $\lambda_{i+\frac{1}{2}}^{L}$ and $\lambda_{i+\frac{1}{2}}^{R}$ are approximations of the wave speeds.
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Godunov-type scheme (approximate Riemann solver)

We define the time update as follows:

\[
W_i^{n+1} := \frac{1}{\Delta x} \int_{x_{i-\frac{1}{2}}}^{x_{i+\frac{1}{2}}} W^\Delta(x, t^{n+1}) \, dx.
\]

Since \(W_{i-\frac{1}{2}}^n\) and \(W_{i+\frac{1}{2}}^n\) are made of constant states, the above integral is easy to compute.
1. Brief introduction to Godunov-type schemes

2. Derivation of a well-balanced scheme

3. Numerical simulations

4. Conclusion and perspectives
The HLL approximate Riemann solver

To approximate solutions of \( \partial_t W + \partial_x F(W) = 0 \), the HLL approximate Riemann solver (Harten, Lax, van Leer (1983)) may be chosen; it is denoted by \( W^\Delta \) and displayed on the right.

The consistency condition (as per Harten and Lax) holds if:

\[
\frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} W^\Delta(\Delta t, x; W_L, W_R)dx = \frac{1}{\Delta x} \int_{-\Delta x/2}^{\Delta x/2} W_R(\Delta t, x; W_L, W_R)dx,
\]

which gives

\[
W_{HLL} = \frac{\lambda_R W_R - \lambda_L W_L}{\lambda_R - \lambda_L} - \frac{F(W_R) - F(W_L)}{\lambda_R - \lambda_L} = \left( h_{HLL}, q_{HLL} \right).
\]

Note that, if \( h_L > 0 \) and \( h_R > 0 \), then \( h_{HLL} > 0 \) for \( |\lambda_L| \) and \( |\lambda_R| \) large enough.
Modification of the HLL approximate Riemann solver

The shallow-water equations with the topography source term read as follows:

\[
\begin{align*}
\partial_t h + \partial_x q &= 0, \\
\partial_t q + \partial_x \left( \frac{q^2}{h} + \frac{1}{2}gh^2 \right) + gh\partial_x Z &= 0.
\end{align*}
\]
We can add the equation $\partial_t Z = 0$, which corresponds to the fixed geometry of the problem:

$$\begin{cases}
\partial_t h + \partial_x q = 0, \\
\partial_t q + \partial_x \left( \frac{q^2}{h} + \frac{1}{2} gh^2 \right) + gh \partial_x Z = 0, \\
\partial_t Z = 0.
\end{cases}$$
Modification of the HLL approximate Riemann solver

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\partial_t h + \partial_x q &= 0, \\
\partial_t q + \partial_x \left( \frac{q^2}{h} + \frac{1}{2} gh^2 \right) + gh \partial_x Z &= 0, \\
\partial_t Z &= 0.
\end{align*}$$

The equation $\partial_t Z = 0$ induces a stationary wave associated to the source term; we also note that $q$ is a Riemann invariant for this wave.

To approximate solutions of $\partial_t W + \partial_x F(W) = S(W)$, we thus use the approximate Riemann solver displayed on the right (assuming $\lambda_L < 0 < \lambda_R$).
Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: \( W_L^* = \begin{pmatrix} h_L^* \\ q_L^* \end{pmatrix} \) and \( W_R^* = \begin{pmatrix} h_R^* \\ q_R^* \end{pmatrix} \).
Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: \( W^*_L = \left( h^*_L \atop q^*_L \right) \) and \( W^*_R = \left( h^*_R \atop q^*_R \right) \).

- \( q \) is a 0-Riemann invariant \( \xrightarrow{\sim} \) we take \( q^*_L = q^*_R = q^* \) (relation 1)
Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: \( W^*_L = \begin{pmatrix} h^*_L \\ q^*_L \end{pmatrix} \) and \( W^*_R = \begin{pmatrix} h^*_R \\ q^*_R \end{pmatrix} \).

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- Harten-Lax consistency gives us the following two relations:
Modification of the HLL approximate Riemann solver

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- \( q \) is a 0-Riemann invariant \( \Rightarrow \) we take \( q^*_L = q^*_R = q^* \) (relation 1)

- Harten-Lax consistency gives us the following two relations:
  - \( \lambda_R h^*_R - \lambda_L h^*_L = (\lambda_R - \lambda_L) h_{HLL} \) (relation 2)
Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: $W_L^*(h_L^*, q_L^*)$ and $W_R^*(h_R^*, q_R^*)$.

- $q$ is a 0-Riemann invariant $\Rightarrow$ we take $q_L^* = q_R^* = q^*$ (relation 1)
- Harten-Lax consistency gives us the following two relations:
  - $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$ (relation 2)
  - $q^* = q_{HLL} + \frac{\overline{S} \Delta x}{\lambda_R - \lambda_L}$ (relation 3),

where $\overline{S} \approx \frac{1}{\Delta x} \frac{1}{\Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^\Delta t S(W_R(x,t)) \, dt \, dx$
Modification of the HLL approximate Riemann solver

We have 4 unknowns to determine: $W_L^* = \begin{pmatrix} h_L^* \\ q_L^* \end{pmatrix}$ and $W_R^* = \begin{pmatrix} h_R^* \\ q_R^* \end{pmatrix}$.

- $q$ is a 0-Riemann invariant $\Rightarrow$ we take $q_L^* = q_R^* = q^*$ (relation 1)

- Harten-Lax consistency gives us the following two relations:
  - $\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}$ (relation 2)
  - $q^* = q_{HLL} + \frac{\bar{S} \Delta x}{\lambda_R - \lambda_L}$ (relation 3),

where $\bar{S} \simeq \frac{1}{\Delta x \Delta t} \int_{-\Delta x/2}^{\Delta x/2} \int_0^{\Delta t} S(W_R(x, t)) \, dt \, dx$

- next step: obtain a fourth relation
Assume that $W_L$ and $W_R$ define a steady state, i.e. that they satisfy the following discrete version of the steady relation
\[ \partial_x F(W) = S(W) \] (where $[X] = X_R - X_L$):
\[
\frac{1}{\Delta x} \left( q_0^2 \left[ \frac{1}{h} \right] + \frac{g}{2} [h^2] \right) = \bar{S}.
\]

For the steady state to be preserved, it is sufficient to have $h^*_L = h_L$, $h^*_R = h_R$ and $q^* = q_0$.

Assuming a steady state, we easily show that $q^* = q_0$. Therefore, the additional relation should only link $h^*_L$ and $h^*_R$. 
Obtaining an additional relation

In order to determine an addition relation, we consider the discrete steady relation, satisfied when $W_L$ and $W_R$ define a steady state:

$$q_0^2 \left( \frac{1}{h_R} - \frac{1}{h_L} \right) + \frac{g}{2} ( (h_R)^2 - (h_L)^2 ) = S \Delta x.$$  

To ensure that $h_L^* = h_L$ and $h_R^* = h_R$, we impose that $h_L^*$ and $h_R^*$ satisfy the above relation, as follows:

$$q_0^2 \left( \frac{1}{h_R^*} - \frac{1}{h_L^*} \right) + \frac{g}{2} ( (h_R^*)^2 - (h_L^*)^2 ) = S \Delta x.$$  

To avoid solving a nonlinear system, we elect to use the following linearization of this relation:

$$\left( \frac{-(q^*)^2}{h_L h_R} + \frac{g}{2}(h_L + h_R) \right) (h_R^* - h_L^*) = S \Delta x.$$  

Here, $\alpha$ represents a constant factor.
Determination of $h_L^*$ and $h_R^*$

With the consistency relation between $h_L^*$ and $h_R^*$, the intermediate water heights satisfy the following linear system:

\[
\begin{cases}
\alpha(h_R^* - h_L^*) = \bar{S}\Delta x, \\
\lambda_R h_R^* - \lambda_L h_L^* = (\lambda_R - \lambda_L) h_{HLL}.
\end{cases}
\]

Using both relations linking $h_L^*$ and $h_R^*$, we obtain

\[
\begin{align*}
h_L^* &= h_{HLL} - \frac{\lambda_R \bar{S}\Delta x}{\alpha(\lambda_R - \lambda_L)}, \\
h_R^* &= h_{HLL} - \frac{\lambda_L \bar{S}\Delta x}{\alpha(\lambda_R - \lambda_L)},
\end{align*}
\]

where $\alpha = \left( \frac{-(q^*)^2}{h_L h_R} + \frac{g}{2} (h_L + h_R) \right)$ with $q^* = q_{HLL} + \frac{\bar{S}\Delta x}{\lambda_R - \lambda_L}$. 
Correction to ensure non-negative $h^*_L$ and $h^*_R$

However, these expressions of $h^*_L$ and $h^*_R$ do not guarantee that the intermediate heights are non-negative: instead, we use the following cutoff (see Audusse, Chalons, Ung (2014)):

$$
\begin{align*}
  h^*_L &= \min \left( \left( h_{HLL} - \frac{\lambda_R \overline{S} \Delta x}{\alpha (\lambda_R - \lambda_L)} \right) + \left( 1 - \frac{\lambda_R}{\lambda_L} \right) h_{HLL} \right), \\
  h^*_R &= \min \left( \left( h_{HLL} - \frac{\lambda_L \overline{S} \Delta x}{\alpha (\lambda_R - \lambda_L)} \right) + \left( 1 - \frac{\lambda_L}{\lambda_R} \right) h_{HLL} \right).
\end{align*}
$$

Note that this cutoff does not interfere with:

- the consistency condition $\lambda_R h^*_R - \lambda_L h^*_L = (\lambda_R - \lambda_L) h_{HLL}$;
- the well-balance property, since it is not activated when $W_L$ and $W_R$ define a steady state.
Summary

The two-state approximate Riemann solver with intermediate states

\[ W^* = \begin{pmatrix} h^*_L \\ q^*_L \end{pmatrix} \quad \text{and} \quad W^*_R = \begin{pmatrix} h^*_R \\ q^*_R \end{pmatrix} \]

given by

\[
\begin{aligned}
q^*_L &= q_{HLL} + \frac{\bar{S}\Delta x}{\lambda_R - \lambda_L}, \\
\lambda_R \geq 0, \\

h^*_L &= \min\left\{ \left( h_{HLL} - \frac{\lambda_R \bar{S}\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+ + \left( 1 - \frac{\lambda_R}{\lambda_L} \right) h_{HLL} \right\}, \\

h^*_R &= \min\left\{ \left( h_{HLL} - \frac{\lambda_L \bar{S}\Delta x}{\alpha(\lambda_R - \lambda_L)} \right)_+ + \left( 1 - \frac{\lambda_L}{\lambda_R} \right) h_{HLL} \right\},
\end{aligned}
\]

is consistent, non-negativity-preserving and well-balanced.

next step: determination of \( \bar{S} \) according to the source term definition (topography).
The topography source term

We now consider \( S(W) = S^t(W) = -gh\partial_x Z \):

the smooth steady states are governed by

\[
\begin{align*}
\partial_x \left( \frac{q_0^2}{h} \right) + \frac{g}{2} \partial_x (h^2) &= -gh\partial_x Z, \\
\frac{q_0^2}{2} \partial_x \left( \frac{1}{h^2} \right) + g\partial_x (h + Z) &= 0,
\end{align*}
\]

\[
\begin{cases}
\partial_x \left( \frac{q_0^2}{h} \right) + \frac{g}{2} \partial_x (h^2) = -gh\partial_x Z, \\
\frac{q_0^2}{2} \partial_x \left( \frac{1}{h^2} \right) + g\partial_x (h + Z) = 0,
\end{cases}
\]

\[
\text{discretization} \quad \begin{align*}
q_0^2 \left[ \frac{1}{h} \right] + \frac{g}{2} [h^2] &= \bar{S}^t \Delta x, \\
\frac{q_0^2}{2} \left[ \frac{1}{h^2} \right] + g[h + Z] &= 0.
\end{align*}
\]

We can exhibit an expression of \( q_0^2 \) and thus obtain

\[
\bar{S}^t = -g \frac{2h_L h_R}{h_L + h_R} \frac{[Z]}{\Delta x} + \frac{g}{2\Delta x} \frac{[h]^3}{h_L + h_R}.
\]

However, when \( Z_L = Z_R \), we have \( \bar{S}^t \neq \mathcal{O}(\Delta x) \), i.e. a loss of consistency with \( S^t \) (see for instance Berthon, Chalons (2016)).
The topography source term

Instead, we set, for some constant $C > 0$,

$$
\begin{align*}
\bar{S}^t &= -g \frac{2h_L h_R}{h_L + h_R} \frac{[Z]}{\Delta x} + \frac{g}{2\Delta x} \frac{[h]_c^3}{h_L + h_R}, \\
[h]_c &= \begin{cases} 
  h_R - h_L & \text{if } |h_R - h_L| \leq C\Delta x, \\
  \text{sgn}(h_R - h_L) C\Delta x & \text{otherwise}.
\end{cases}
\end{align*}
$$

Theorem: Well-balance for the topography source term

If $W_L$ and $W_R$ define a smooth steady state, i.e. if they satisfy

$$
\frac{q_0^2}{2} \left[ \frac{1}{h^2} \right] + g[h + Z] = 0,
$$

then we have $W_L^* = W_L$ and $W_R^* = W_R$ and the approximate Riemann solver is well-balanced. By construction, the Godunov-type scheme using this approximate Riemann solver is consistent, fully well-balanced and positivity-preserving.
1. Brief introduction to Godunov-type schemes

2. Derivation of a well-balanced scheme

3. Numerical simulations

4. Conclusion and perspectives
Verification of the well-balance: topography

The initial condition is at rest; water is injected through the left boundary.
A fully well-balanced scheme for the shallow-water equations with topography

Numerical simulations

Verification of the well-balance: topography

The non-well-balanced HLL scheme converges towards a numerical steady state which does not correspond to the physical one.
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Simulation of the 2011 Tōhoku tsunami

- Sensor 1
- Sensor 2
- Sensor 3
Simulation of the 2011 Tōhoku tsunami

Initial free surface

Sensor 1

Sensor 2

Sensor 3
Simulation of the 2011 Tōhoku tsunami
1. Brief introduction to Godunov-type schemes

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Conclusion

- We have presented a **well-balanced** and **non-negativity-preserving** numerical scheme, able to be applied to other source terms or combinations of source terms.
- We have also displayed results from a **2D well-balanced** numerical method, coded in Fortran and **parallelized**.

This work has been published:


Conclusion and perspectives

## Perspectives

### Work in progress or completed

- Application to other source terms:
  - Friction source term (completed, article accepted)
  - Coriolis force source term (work in progress)
  - Breadth variation source term (work in progress)
- High-order and 2D extensions (work in progress, collaboration with R. Loubère)

### Long-term perspectives

- Stability of the scheme: values of $C$, $\lambda_L$ and $\lambda_R$ to ensure the entropy preservation
- Ensure the entropy preservation for the high-order scheme (use of a MOOD method)
Thank you for your attention!
Verification of the well-balance: topography
transcritical flow test case (see Goutal, Maurel (1997))

left panel: initial free surface at rest; water is injected from the left boundary
right panel: free surface for the steady state solution, after a transient state

\[ \Phi = \frac{u^2}{2} + g(h + Z) \]

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<th>( L^2 )</th>
<th>( L^\infty )</th>
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Riemann problems between two wet areas

left: $k = 0$

both Riemann problems have initial data $W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix}$ and $W_R = \begin{pmatrix} 1 \\ 0 \end{pmatrix}$, on $[0, 5]$, with 200 points, and final time 0.2s

left: $k = 10$
Riemann problems with a wet/dry transition

left: \( k = 0 \)

both Riemann problems have initial data \( W_L = \begin{pmatrix} 6 \\ 0 \end{pmatrix} \) and \( W_R = \begin{pmatrix} 0 \\ 0 \end{pmatrix} \), on \([0, 5]\), with 200 points, and final time 0.15s

left: \( k = 10 \)
Double dry dam-break on a sinusoidal bottom
Japan tsunami: 1D slice
Preservation of the lake at rest


Fully well-balanced schemes


High-order well-balanced schemes and friction


